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Noncommutative Gröbner Basis on Quiver Algebras

Research Report in Mathematics, Number 11, 2017

DANIEL KARIUKI WAWERU

July 2017

Submitted to the School of Mathematics in partial fulfilment for a degree in Master of Science in Pure Mathematics

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Abstract

For a quiver Q, we define a path algebra KQ as a span of all the paths of positive length. We study left (respective right) sided ideals and their Gröbner bases. We introduce the two-sided ideals, two-sided division algorithm for elements of KQ and study the two-sided Gröbner bases. For a two-sided ideal I with a finite Gröbner basis, we attempt to study its quotient algebra $KQ \setminus I$ and its ideals.

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Declaration and Approval

I the undersigned declare that this project report is my original work and to the best of my knowledge, it has not been submitted in support of an award of a degree in any other university or institution of learning.

Signature	Date

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In my capacity as a supervisor of the candidate, I certify that this report has my approval for submission.

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Dedication

This project is dedicated to my dear lovely parents and siblings.

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Daniel Waweru

Nairobi, 2017.

1 Introduction

Why Path Algebras?

To each finite dimensional algebra over an algebraically closed field K corresponds a graphical structure called a quiver [ASS06]. Conversely to each quiver there corresponds an associative K-algebra, which has an identity element and is finite dimensional under some conditions. We call this K-algebra a path algebra. We can study a diagram of linear maps between vector spaces by studying the corresponding path algebra. For example let V_1, V_2, V_3 and V_4 be four vector space over some field K, and α, β and γ be linear maps as shown in figure 1. This figure suggests a directed graph or a quiver Q with vertices $1,2,3$ and 4 and directed edges a, b and c each with a source and a target.

$$
= \begin{tikzcd}
 & & V_3 \\
V_1 \arrow[r, "\alpha"] & V_2 \arrow[ur, "\beta"] \arrow[dr, "\gamma"] & \\
 & & V_4
\end{tikzcd}
$$

Figure 1. Linear Diagram of Vector Spaces

$$
Q = \begin{tikzcd}
 & & 3 \\
1 \arrow[r, "a"] & 2 \arrow[ur, "b"] \arrow[dr, "c"] & \\
 & & 4
\end{tikzcd}
$$

Figure 2. The Quiver Representation

This graph gives a "ring" or more formally a path algebra denoted by KQ, whose basis are the paths of length $l \geq 0$. Corresponding to each vertex i is a trivial path e_i of length zero. a, b and c are paths of length one, while ab and ac are paths of length two. Multiplication of basis vectors are as follows

$$
e_i p = \begin{cases} p, & \text{if source of path } p \text{ is vertex } i \\ 0, & \text{otherwise} \end{cases}
$$

$$pe_i = \begin{cases} p, & \text{if target of the path } p \text{ is vertex } i \\ 0, & \text{otherwise} \end{cases}$$

$$pq = \begin{cases} p, & \text{if source of path } q \text{ is the target of the path } p \\ 0, & \text{otherwise} \end{cases}$$

So that we have a ring structure of this path algebra,

$$KQ = Ke_1 \oplus Ke_2 \oplus Ke_3 \oplus Ke_4 \oplus Ka \oplus Kb \oplus Kc \oplus Kab \oplus Kac$$

How does this ring help us study diagrams such as in Figure 1 ? The linear maps in figure 1 corresponds precisely to right KQ-modules. Let $M = V_1 \oplus V_2 \oplus V_3 \oplus V_4$ then M has a vector space structure. We define scalar multiplication by letting the paths in quiver Q act on M on the right. For any vector $(v_1, v_2, v_3, v_4) \in M$

i). Paths of length zero e_i act on M by projecting onto V_i. e.g

$$(v_1, v_2, v_3, v_4)e_1 = (v_1, 0, 0, 0).$$

ii). Paths of length one a, b and c act on M by the linear maps α, β and γ respectively. e.g

$$(v_1, v_2, v_3, v_4)a = (0, \alpha(v_1), 0, 0).$$

iii). Paths of length two ab and ac act on M by the composition of linear maps $\beta\alpha$ and $\gamma\alpha$ respectively.

Gröbner Basis

Let A be a free associative algebra over a field K, S a set of elements in A and $\langle S \rangle$ an ideal of A generated by S. One of the fundamental problems in the theory of abstract algebras is the reduction problem of a given element $f \in A$ with the respect to the elements of S. The most common approach to this problem is to find the set of generators G, of the elements of S. With G available there exist a criterion by which one can determine whether an element $f \in A$ is reducible with respect to S.

Gröbner bases are generating sets of ideals in commutative polynomial ring $K[x_1, \ldots, x_n]$, (where K is coefficient ring), that help solve polynomial system of equations and describe quotient algebra effectively. Buchberger [Buc70] found a criterion and an algorithm of computing Gröbner basis for the ideals in commutative polynomial ring. To date many improvement have been made extending the now known as commutative Gröbner basis theory to other coefficient rings such as valuation rings [MS15]. Algorithm for computing commutative Gröbner basis in polynomial ring are presented and we see that the Dickson's Lemma for monomial ideals and the Hilbert Basis Theorem ensures that this algorithm always terminates producing a finite Gröbner basis.

This theory fails for noncommutative polynomial rings, for in such ring there is no version of Dickson's Lemma for noncommutative monomial ideals. However an analogous theory of computing the Gröbner basis for ideals generated by finite sets in noncommutative polynomial ring $K[x_1,...,x_n]$ have been developed and with this we are able to describe quotient algebras effectively. [Mor86] first describes an algorithm for computing noncommutative Göbner basis in associative algebras. This algorithm do not always terminate and hence not all Gröbner basis will be finite for any choice of finite initial generating set. In the paper [FFG93], the theory of Gröbner basis is extended to the path algebras with the realization that free associative algebra are indeed special case of path algebras. By defining an element of a path algebra KQ as a polynomial, ideals in KQ will be those generated by a finite set of polynomials. Algorithm for computing noncommutative Gröbner basis on one sided ideal (i.e both right ideal and left ideal) in a path algebra are presented. While a two sided ideal is one which is both a right ideal and a left ideal, an algorithm for computing Gröbner basis in a two sided ideal is developed.

Chapter three is a brief introduction to polynomial ring, monomial ordering and polynomial division in both commutative and noncommutative point of view. In chapter four we give a brief introduction to commutative Gröbner basis theory giving the one Buchberger's Algorithm of finding a Gröbner basis. Here we also introduce noncommutative Gröbner basis theory according to [Mor86] and give his noncommutative version of Buchbergers's Algorithm on free associative algebras.

We dedicate chapter five to a brief but subtle introduction to path algebras. With examples, we illustrate few properties of path algebras that contribute to their Gröbner basis theory. We define a two sided ideal and point out the condition on which such an ideal shall have Gröbner basis theory. In chapter six we study briefly respective left and right Gröbner basis theory on a path algebras. In chapter seven we present our main work where we extend and encompass left and right Buchberger's algorithms into one algorithm for two sided ideals on a path algebras.

2 Literature Review

In 1965 inspired by Gröbner, [Buc65, Buc70] Buchberger found a criterion and an algorithm of computing such a set generators of the ideals for commutative algebra. This set of generators of ideal is now referred to as a Gröbner Basis and it is today an important tool in the study of commutative algebra. In 1978 [Ber78], Bergman developed the theory of Gröbner bases for associative algebras by proving the Bergman-Diamond lemma. His idea is a generalization of Gröbner basis theory in associative algebras. Other authors expounded on this theory.

Path algebras plays a major role in the representation theory of finite dimensional algebras. In 1986 [Mor86], Mora adapted the Buchberger's algorithm to ideals in the free associative algebras as an implementation of Bergman's Diamond lemma. In 1993 [FFG93], Farka, Feustel and Green extended the theory of Gröbner basis to the path algebras with the realization that free associative algebra are indeed special case of path algebras. In 2004 [Lea06], Leamer expounded on the theory of Gröbner basis on path algebras and pointed out the conditions under which an ideal in a path algebra has a finite Gröbner basis.

In 2006 [AM10] Attan and Mialebama studies respective left and right Gröbner bases and their syzygies. They develop a modification of Buchberger algorithms in [Mor86, FFG93] which successively help them accomplish the study Gröbner basis for left and right ideals from an initial finite generating set of polynomials in a path algebras. By defining an ideal (two-sided ideal) as an ideal which is both a left and a right ideal, the aim of this paper is to extend the theory of Gröbner basis in [AM10] to two-sided ideals in a path algebra.

3 Preliminaries

In this chapter, we recall some algebraic concepts that will be used extensively in the following chapters. In particular, we will introduce polynomial rings, ideals and algebras which are the main objects of study in this dissertation. We follow the approach of [CLD92] and [ASS06].

3.1 Rings

Definition 3.1.1. *A ring is a set A with two binary operations $+$ and $\times$, known as addition and multiplication, such that addition has an identity element 0, called zero, and the following axioms hold.*

a. A is an abelian group with respect to addition.

b. $r_1 \times r_2 \in A$ for all $r_1, r_2 \in A$

c. $(r_1 \times r_2) \times r_3 = r_1 \times (r_2 \times r_3)$ for all $r_1, r_2, r_3 \in A$ (multiplication is associative).

d. $r_1 \times (r_2 + r_3) = r_1 \times r_2 + r_1 \times r_3$ and $(r_1 + r_2) \times r_3 = r_1 \times r_3 + r_2 \times r_3$ for all $r_1, r_2, r_3 \in A$. (the distributive laws hold).

Definition 3.1.2. *Let A be a ring;*

i) A is a ring with identity, if it contains a unique element 1, called the unit element, such that $1 \neq 0$ and $1 \times r = r = r \times 1$ for all $r \in A$.

ii) A is commutative if multiplication is commutative, i.e $r_1 \times r_2 = r_2 \times r_1$ for all $r_1, r_2 \in A$.

iii) A is noncommutative if $r_1 \times r_2 \neq r_2 \times r_1$ for some $r_1, r_2 \in A$.

iv) If S is a subset of a ring A that is itself a ring under the same binary operations of addition and multiplication, then S is a subring of A.

v) A is a division ring if every nonzero element $a \in A$ has a multiplicative inverse a^{-1}.

vi) A is a field if it is a commutative division ring.

3.2 K-Algebras

Definition 3.2.1. *Let K be a field and A a ring with unity. An algebra over K is an associative ring A with unit, together with ring homomorphism $\phi : K \to A$ whose image is a copy of K in the center of A and whose unit element coincides with that of A.*

Hence we may view a K-algebra is a K-vector space together with an associative product $A \times A \to A$ which is K-linear, with respect to which it has a unit.

Definition 3.2.2. *Let A be an algebra over K. $H \subseteq A$ is called a subalgebra of A if for all $x, y \in H$ and $\lambda \in K$*

a) $x + y \in H$

b) $xy \in H$

c) $\lambda x \in H$

Definition 3.2.3. *i) An algebra A is unital if it has a unit element.*

ii) An algebra A is said to be associative if for all $x, y, z \in A$ we have that $(xy)z = x(yz) \in A$.

iii) Let A be an algebra and $\{e_1, \ldots, e_m\}$ be a complete set of primitive orthogonal idempotents i.e ($1 = e_1 + e_2 + \cdots + e_m$). Then A is called a basic algebra if e_iA is not isomorphic to e_jA for all $i \neq j$.

Definition 3.2.4. *Let A be an algebra and $I \subset A$.*

i) I is called left ideal of A if:

- *a)* $x + y \in I; \quad x, y \in I.$
- *b)* $wx \in I; \quad x \in I, \quad w \in A.$
- *c)* $\lambda x \in I; \quad \lambda \in k, \quad x \in I.$

ii) I is called right ideal of A if:

- *a)* $x + y \in I; \quad x, y \in I.$
- *b)* $xz \in I; \quad x \in I, \quad z \in A.$
- *c)* $\lambda x \in I; \quad \lambda \in k, \quad x \in I.$

iii) A $two-sided$ ideal is a subspace I, which is both a right and a left ideal of A.

Example 3.2.5. *Let K be a field,* *i) a polynomial ring $K[X]$ with one indeterminate X is a one dimensional polynomial-algebra.*

ii) a noncommutative polynomial ring $K[X,Y]$ with two noncommuting indeterminates X and Y, is a two dimensional free associative algebra.

3.3 Polynomial Rings

Definition 3.3.1. *A monomial in $x_1,\ldots,x_n$ is a product of the form*

$$x_1^{\alpha_1}\cdot x_2^{\alpha_2}\cdots x_n^{\alpha_n},$$

in which all the exponents $\alpha_1,\ldots,\alpha_n$ are nonnegative integers. The total degree of this monomial is the sum $\alpha_1+\cdots+\alpha_n$. Each monomial may be represented in terms of its exponents only, as a multi-degree $\alpha=(\alpha_1,\ldots,\alpha_n)$, so that a monomial may be written as multi-set x^α over the set $\{x_1,\ldots,x_n\}$ and denote its total-degree as $|\alpha|=\alpha_1+\cdots+\alpha_n$. We write,

$$x^\alpha = x_1^{\alpha_1}\cdot x_2^{\alpha_2}\cdots x_n^{\alpha_n}.$$

Definition 3.3.2. *A polynomial f in $x_1,\ldots,x_n$ variables is a finite linear combination, with coefficients in a ring K, of monomials. We will write a polynomial f in the form*

$$f=\sum_\alpha a_\alpha x^\alpha, \quad a_\alpha\in K.$$

We denote the set of all polynomials in $x_1,\ldots,x_n$ with coefficients in K by $K[x_1,\ldots,x_n]$.

Definition 3.3.3. *Let $K[x_1,\ldots,x_n]$ denote the set of all functions $f:\mathbb{N}^n\to K$ such that each function f represents a polynomial in n variables $x_1,\ldots,x_n$ with coefficients in a ring K. Given two functions $f,g\in K[x_1,\ldots,x_n]$, let us define the functions $f+g$ and $f\times g$ as follows;*

$$(f+g)(\alpha)=f(\alpha)+g(\alpha) \quad \textit{for all} \quad \alpha\in\mathbb{N}^n$$

$$(f\times g)(\alpha)=\sum_{\beta+\gamma=\alpha} f(\beta)\times g(\gamma) \quad \textit{for all} \quad \alpha\in\mathbb{N}^n.$$

Then the set $K[x_1,\ldots,x_n]$ becomes a ring, known as the polynomial ring in n variables over K, with the functions corresponding to the zero and constant polynomials being the respective zero and unit elements of the ring.

Remark 3.3.4. *From now on, unless otherwise stated, all coefficient rings of polynomial rings will be fields.*

Definition 3.3.5. *$A=K[x_1,\ldots,x_n]$ is said to be commutative polynomial ring if the variables $x_1,\ldots,x_n$ are commuting. Otherwise $A=K[x_1,\ldots,x_n]$ is noncommutative polynomial ring.*

Remark 3.3.6. *Let $A = K[x_1, \dots, x_n]$ be a noncommutative polynomial ring.*

a) A nontrivial polynomial f in n- noncommuting variables $x_1, \dots, x_n$ is written as a sum;

$$f = \sum_{i=1}^{s} a_i w_i, \qquad s \in \mathbb{N}_{>0}, \quad a_i \in K.$$

where a monomial w_i is a word over the alphabets $\{x_1, \dots, x_n\}$.

b) The product $w_1 w_2$ of two monomials $w_1, w_2 \in \langle x_1, \dots, x_n \rangle$ is given by concatenation. For example in $K[x_1, x_2]$, let $w_1 = x_1x_2x_1$ and $w_2 = x_2x_2$. Then the product $w_1w_2 = x_1x_2x_1x_2x_2$.

c) The respective constant polynomials $f = 0$ and $f = 1$ are the polynomials $f = 0\varepsilon$ and $f = 1\varepsilon$ where ε is the empty word in $\langle x_1, \dots, x_n \rangle$.

Lemma 3.3.7. *[CLD92] If $f_1, \dots, f_s \in A = K[x_1, x_2, \dots, x_n]$ then $\langle f_1, \dots, f_s \rangle$ is an ideal of $A = K[x_1, x_2, \dots, x_n]$ generated by $f_1, \dots, f_s$.*

3.4 Monomial ordering

3.4.1 Commutative Monomial Ordering

Through out this subsection $A = K[x_1, \dots, x_n]$ is taken to be a commutative polynomial ring.

Definition 3.4.1. *A monomial ordering on $A = K[x_1, \dots, x_n]$ is any relation $\prec$ on the set of monomials $x^\alpha, \alpha \in \mathbb{Z}^n_{\geq 0}$ satisfying;*

a) $\prec$ is a total ordering on $\mathbb{Z}^n_{\geq 0}$.

b) if $\alpha \prec \beta$ and $\gamma \in \mathbb{Z}^n_{\geq 0}$ then $\alpha + \gamma \prec \beta + \gamma$.

c) $\prec$ is a well ordering on $\mathbb{Z}^n_{\geq 0}$.

A monomial ordering requires an ordering on the variables in our chosen polynomial ring. If $A = K[x_1, \dots, x_n]$ is our polynomial ring, we will assume this order to be $x_1 \prec x_2 \prec \cdots \prec x_n$. Now we shall consider the most frequently used monomial orderings.

Example 3.4.2. *i) Lexicographic Order (Lex order): Let $\alpha = (\alpha_1, \dots, \alpha_n)$ and $\beta = (\beta_1, \dots, \beta_m)$ be in $\mathbb{Z}^n_{\geq 0}$. We say that α is greater than β with respect to lexicographic order and denote $\alpha \succ_{lex} \beta$ if in the vector difference $\alpha - \beta \in \mathbb{Z}^n_{\geq 0}$, the leftmost nonzero entry is positive. We write $x^\alpha \succ_{lex} x^\beta$ when $\alpha \succ_{lex} \beta$. For instance; let $A = \mathbb{Q}[x, y, z]$ with $x \succ y \succ z$, $\alpha = (1, 3, 5)$ and $\beta = (0, 3, 7)$. We have that $\alpha \succ_{lex} \beta$ or $(x^1y^3z^5 \succ `_{lex} y^3z^7)$ since $\alpha - \beta = (1, 0, -2)$.*

ii) *Graded Lexicographic ordering: Let* $\alpha, \beta \in \mathbb{Z}^n_{\geq 0}$. *We say that* α *is greater than* β *with respect to graded lexicographic order and denote* $\alpha \succ_{grlex} \beta$ *if* $|\alpha| = \sum_{i=1}^n \alpha_i > |\beta| = \sum_{i=1}^n \beta_i$ *or* $|\alpha| = |\beta|$ *and* $\alpha \succ_{lex} \beta$. *For instance,* $(1,2,3) \succ_{grlex} (3,1,1)$ *since* $|(1,2,3)| = 6 > |(3,1,1)| = 5$

iii) *Graded Reverse Lexicographic order: Let* $\alpha, \beta \in \mathbb{Z}^n_{\geq 0}$. *We say that* α *is greater than* β *with respect to graded reverse lexicographic order and denote* $\alpha \succ_{grevlex} \beta$ *if* $|\alpha| = \sum_{i=1}^n \alpha_i > |\beta| = \sum_{i=1}^n \beta_i$ *or* $|\alpha| = |\beta|$ *and the rightmost non zero entry of* $\alpha - \beta$ *is negative. For instance* $(1,6,2) \succ_{grevlex} (4,2,3)$ *since* $|(1,6,2)| = |(4,2,3)| = 9$ *and* $(1,6,2) - (4,2,3) = (-3,4,-1)$.

Definition 3.4.3. *Let* $f = \sum_\alpha a_\alpha x^\alpha$ *be non zero polynomial in* $A = K[x_1, \ldots, x_n]$ *and let* $\prec$ *be a monomial ordering:*

i) *The multidegree of* f *is* $multideg(f) = max(\alpha \in \mathbb{Z}^n_{\geq 0} : a_\alpha \neq 0)$ *where the maximum is taken with respect* $\prec$.

ii) *The leading coefficient of* f *is* $LC(f) = a_{multideg(f)} \in K$.

iii) *The leading monomial of* f *is* $LM(f) = x^{multideg(f)}$.

iv) *The leading term of* f *is* $LT(f) = LC(f).LM(f)$.

v) *Let* $J \subset A = K[x_1, \ldots, x_n]$. $\quad LT(J) = \{LT(f) : f \in J\}$

Example 3.4.4. *Let* $f = 19x^2y^3z^2 - 13x^5 - 27z^3$. *With respect to lex ordering* $x \succ_{lex} y \succ_{lex} z$ *we have that;*

$$multideg(f) = (5,0,0), \quad LC(f) = -13, \quad LM(f) = x^5 \quad and \quad LT(f) = -13x^5$$

Definition 3.4.5. *A monomial ordering* $\prec$ *is admissible when;*

a) $x \succ 1$ *for all monomials* $x \neq 1$, *and*

b) $x \succ y \Rightarrow wxz \succ wyz$ *for all monomials* w, x, y, z.

By convention, a polynomial will be written in descending order, with respect to a given monomial ordering, so that the leading term of the polynomial, (with associated leading coefficient and leading monomial), always comes first.

3.4.2 Noncommutative Monomial Ordering

In the noncommutative case, because we use words or (paths) and not multidegrees to represent monomials, our definitions for the lexicographically based orderings will have to be adapted slightly. All other definitions and conventions will stay the same. Here we take $A = K[x_1, \dots, x_n]$ to be a noncommutative polynomial ring.

Definition 3.4.6. *A relation $\prec$ is said to be a noncommutative monomial ordering on set of monomials M if it satisfies;*

i) $\prec$ *is a total order on* M.

ii) $x \succ 1, \forall x \in M$.

iii) $x \succ y \Rightarrow wxz \succ wyz, \quad \forall x, y, w, z \in M$.

Example 3.4.7. *Lexicographic Order: Let w_1 and w_2 be monomials. Define $w_2 \succ w_1$ if, working left-to-right, the first (say $i-th$) letter on which w_1 and w_2 differ is such that the $i-th$ letter of w_1 is lexicographically less than the $i-th$ letter of w_2 in the variable ordering. (This ordering is not admissible). For instance, let $K[x, y]$ be a nonommutative polynomial ring. If $x \succ y$ is the variable ordering, then $w_1 = x \prec w_2 = xy$ but $w_3 = x^2 \prec w_4 = xyx$.*

Definition 3.4.8. *Let $A = K[x_1, \dots, x_n]$ be a noncommutative polynomial ring. Every element $f \in A$ has a unique form $f = \sum_{i=1}^{n} a_i w_i, \quad a_i \in K, \quad w_i \in \langle x_1, \dots, x_n \rangle$. Furthermore for a noncommutative monomial ordering $\prec$, we have the following;*

a) w is called the leading monomial of $f \in A$ and we denote $w = LM(f)$ if w occurs in f and $w \succ m$ for all monomials $m \in Mon(f)$.

b) The coefficient of $LM(f)$ is called the leading coefficient and is denoted by $LC(f)$.

c) The term $LT(f) = LC(f)LM(f)$ is called the leading term of f.

d) Let $J \subset A$ then we define $LT(J) = \{LT(g) : g \in J\}$

3.5 Polynomial Division

Let $A = K[x_1, \dots, x_n]$ be a polynomial ring and $\prec$ be an admissible ordering. Given two nonzero polynomials $f, g \in A$, we say that f divides g if the leading term of f divides some term in g. For commutative polynomial ring we have $x = LM(f)x'$ for some monomial x', while in noncommutative polynomial ring $x = x_l LM(f) x_r$ for some monomials x_l and x_r. Division removes an appropriate multiple of f from g in order to cancel off $LT(f)$ with the term involving x in g. We perform division as follows:

$$\text{In commutative case} \quad g - \frac{\lambda}{LC(f)} f x'.$$

In noncommmutatie case $\quad g-\dfrac{\lambda}{LC(f)}x_l f x_r.$

For a fixed admissible monomial ordering on a set $F=\{f_1,\dots,f_s\}$. Any $f\in A$ can be written as $f=a_1f_1+\cdots+a_sf_s+r$ where $a_i,r,f_i\in A$ and either $r=0$ or r is a linear combination, with coefficients in K, of monomials which are not divisible by $LT(f_i)$. We call r a remainder of f by division by F and denote by $r=Red_F(f)$ a reduction of f with respect to the set F. Moreover if $a_if_i\neq 0$ then we have that ($multideg(f)\geq multideg(a_if_i)$ for commutative case) respective ($LM(f)\succeq LM(a_if_i)$ for noncommutative case).

Algorithm 1: Commutative Division Algorithm [CLD92]

Input :$A=K[x_1,\dots,x_n]; f; F=\{f_1,\dots,f_n\}$ such that $f,f_i\in A\setminus\{0\}$; an admissible order $\prec$.

Output :$Red_F(f)=r$

$r=0$

while $f\neq 0$ **do**

  $u=LM(f); \lambda=LC(f); i=1$; found= false

  while *$i\leq n$ and found=false* **do**

    if *$LM(f_i)$ divides u* **then**

      found=true; $u'=\frac{u}{LM(f_i)}$; $f=f-\frac{\lambda}{LC(f_i)}f_iu'$

    else

      $i=i+1$

    end

  end

  if *found = = false* **then**

    $r=r+LT(f); f=f-LT(f)$

  end

end

Return r.

To divide a nonzero polynomial f with respect to nonzero polynomials $F=\{f_1,\dots,f_n\}$ where $f,f_i\in A=K[x_1,\dots,x_n]$ a noncommutative ring, we apply Algorithm 1 with the following changes

Algorithm 2: Noncommutative Division Algorithm

a.) In the input $A = K[x_1, \ldots, x_n]$ is taken to be noncommutative.

b.) We change the first **if** condition to read:

if $LM(f_i)$ *divides* u **then**

found=true; choose u_l and u_r such that $u = u_l LM(f) u_r$; $f = f - \frac{\lambda}{LC(f_i)} u_l f_i u_r$

else

$i = i+1$

end

Example 3.5.1. *Let $A = \mathbb{Q}[x,y]$ commutative. We divide $f = xy^2 + 1$ by $F = \{f_1 = xy + 1, f_2 = y + 1\}$ using the lexicographic ordering with $x \succ y$. We have $LT(f) = xy^2, LT(f_1) = xy$ and $LT(f_2) = y$. First we note that $LT(f) = LT(f_1)y$ so we replace f by $f' = f - f_1 y = -y + 1$. Since $LT(f') = -1LT(f_2)$, we replace f' by $f'' = f' - (-1)f_2 = 2$. Neither $LT(f_1)$ nor $LT(f_2)$ divides 2, therefore we stop and the remainder $r = f'' = 2$. Thus we write $f = yf_1 - f_2 + 2$.*

4 Introduction to Gröbner Bases Theory

In this chapter we review the theory of commutative Gröbner basis briefly. Most of these deliberations may be found in [CLD92]. We also review briefly the theory of noncommutative Gröbner basis according to [Mor86].

4.1 Commutative Gröbner basis

Given an initial basis F generating an ideal I in a ring A, Gröbner basis theory uses F to find a basis G for I with the property that for any $f \in A$, division of f by G has a unique remainder. Through out this section $A = K[x_1, \ldots, x_n]$ is taken to be a commutative polynomial ring.

Definition 4.1.1. *The S-polynomial of two distinct polynomials f and g in A is given by*

$$S(f,g) = \frac{lcm(LM(f), LM(g))}{LT(f)} \cdot f - \frac{lcm(LM(f), LM(g))}{LT(g)} \cdot g$$

4.1.1 Dickson's Lemma and Hilbert Basis Theorem

Definition 4.1.2. *An ideal $I \subset A$ is a monomial ideal if it is generated by a subset of monomials in A.*

Lemma 4.1.3 (Dickson's Lemma [CLD92])**.** *A monomial ideal $I = \langle x^{\alpha} : \alpha \in \mathbb{Z}^n_{\geq 0} \rangle \subset A$ is finitely generated.*

Proposition 4.1.4. *[CLD92] Let $I \subset A$ be an ideal.*

i) *$\langle LT(I) \rangle$ is a monomial ideal.*

ii) *There are $g_1, \ldots, g_t \in I$ such that $\langle LT(I) \rangle = \langle LT(g_1), \ldots, LT(g_t) \rangle$*

Proof

i) *If $g \in I \setminus \{0\}$ the leading monomial of g generate an ideal $\langle LM(g) : g \in I \setminus \{0\} \rangle$. But $LM(g)$ and $LT(g)$ differ by nonzero constant thus we assert that $\langle LM(g) : g \in I \setminus \{0\} \rangle = \langle LT(g) : g \neq 0 \rangle = \langle LT(I) \rangle$.*

ii) Since $\langle LT(I)\rangle$ is generated by $LT(g)$ for $g \in I \setminus \{0\}$, by Dickson's lemma $\langle LT(I)\rangle = \langle LT(g_1), \dots, LT(g_t)\rangle$.

Theorem 4.1.5 (Hilbert Basis Theorem [CLD92]). *Every ideal $I \subset A$ is finitely generated.*

Proof. If $I = 0$ then the generating set is $\{0\}$. Suppose $I \neq 0$, by proposition 4.1.4 for I, the monomial ideal $LT(I)$ is finitely generated and $\langle LT(I)\rangle = \langle LT(g_1), \dots, LT(g_t)\rangle$ for some $g_1, \dots, g_t \in I$. We claim that $I = \langle g_1, \dots, g_t\rangle$
Since $g_i \in I$ then $\langle g_1, \dots, g_t\rangle \subset I. \cdots (*)$
If $f \in I$ is a polynomial, dividing f by $\langle g_1, \dots, g_t\rangle$ gives $f = a_1 g_1 + \cdots + a_t g_t + r$. Suppose $r \neq 0$ then $r = f - (a_1 g_1 + \cdots + a_t g_t) \in I$. Thus $LT(r) \in \langle LT(I)\rangle = \langle LT(g_1), \dots, LT(g_t)\rangle$, is possible only if $LT(r)$ divides $LT(g_i)$ which is a contradiction for any remainder r. Thus $r = 0$ and $f = a_1 g_1 + \cdots + a_t g_t \in \langle g_1, \dots, g_t\rangle$ i.e $I \subset \langle g_1, \dots, g_t\rangle \cdots (**)$
From $(*)$ and $(**)$ $I = \langle g_1, \dots, g_t\rangle$, for some $g_1, \dots, g_t \in I$

Definition 4.1.6. *A finite subset $G = \{g_1, \dots, g_t\}$ of an ideal I is said to be a Gröbner basis if $\langle LT(g_1), \dots, LT(g_t)\rangle = \langle LT(I)\rangle$.*

Corollary 4.1.7. *[CLD92] Fix a monomial order. Then every ideal $I \subset A$ other than $\{0\}$ has a Gröbner basis. Furthermore any Gröbner basis for an ideal I is a basis for I.*

4.1.2 Buchbergers Algorithm

The algorithm used to compute a Gröbner Basis is known as Buchberger's Algorithm. Bruno Buchberger was a student of Wolfgang Gröbner at the University of Innsbruck, Austria, and the publication of his PhD thesis in 1965 [Buc70] marked the start of Gröbner Basis theory.

Definition 4.1.8 (Buchberger's Criterion). *Let I be a polynomial ideal. Then a basis $G = \{g_1, \dots, g_t\}$ is a Gröbner basis for I if and only if for all pairs $i \neq j$, the remainder on division of $S(g_i, g_j)$ by G is Zero.*

Theorem 4.1.9. *[CLD92] Let $I = \langle f_1, \dots, f_s\rangle \neq \{0\}$ be a polynomial ideal. Then a Gröbner basis for I can be constructed in finite number of steps by the algorithm 3.*

Algorithm 3: Commutative Buchberger's Algorithm

Input : $F = \{f_1, \dots, f_s\}$.
Output : A Gröbner basis $G = \{g_1, \dots, g_t\}$ for I.
$G = F$
repeat
  $G' = G$. For each pair $p, q \in G'$, $p \neq q$, find $S(p,q) \neq 0$ over G'. Then
  $G = G' \cup \{S(p,q)\}$
until $G = G'$;

Example 4.1.10. *Let $F = \{f_1, f_2\} = \{x^3 - 2xy, x^2y - 2y^2 + x\}$ generate an ideal over the commutative polynomial ring $Q[x,y]$, and let the monomial ordering be Graded Reverse Lex order. Running Algorithm 3 on F, there is only one S-polynomial to consider initially, namely $S(f_1, f_2) = y(f_1) - x(f_2) = -x^2$. We have that $LT(S(f_1, f_2)) = S(f_1, f_2) = -x^2 \notin \langle LT(f_1), LT(f_2)\rangle = \langle x^3, x^2y\rangle$. So $S(f_1, f_2) \notin I = \langle f_1, f_2\rangle$. Thus F is not a Gröbner basis for I. We add $f_3 = -x^2$ to F and set $F = \{f_1, f_2, f_3\}$. Then $S(f_1, f_2) = f_3$ is reduced to zero by F. Proceeding we have $S(f_1, f_3) = f_1 - (-x)(f_3) = -2xy$. We add $f_4 = -2xy$ and set $F = \{f_1, f_2, f_3, f_4\}$. $S(f_1, f_4) = y(f_1) - (-\frac{x^2}{2})f_4 = -2xy^2 = yf_4$. So that $S(f_1, f_4)$ is reduced to zero by F. $S(f_2, f_3) = f_2 - (-y)(f_3) = -2y^2 + x$. We add $f_5 = -2y^2 + x$ and set $F = \{f_1, f_2, f_3, f_4, f_5\}$. It is a routine check up that all $S(f_i, f_j), i \neq j$ reduce to zero by $F = \{f_1, f_2, f_3, f_4, f_5\}$. Therefore the Algorithm terminates with $G = \{f_1, f_2, f_3, f_4, f_5\}$ as an output Gröbner basis for I.*

Definition 4.1.11. *A Gröbner basis G for a polynomial ideal I is said to be reduced if:*

i) $LC(g) = 1$ for all $g \in G$ and

ii) No term in any polynomial $g \in G$ is divisible by any $LT(G - \{g\})$.

Proposition 4.1.12. *Let $I \neq 0$ be a polynomial ideal. Then for a given monomial ordering, I has a unique reduced Gröbner basis.*

4.2 Noncommutative Gröbner basis

In 1986, Teo Mora published a paper [Mor86] giving an algorithm for constructing a noncommutative Gröbner Basis. This work built upon the work of George Bergman in particular his diamond lemma for ring theory [Ber78]. Mora's algorithm and the theory behind it, in many ways gives a noncommutative version of the Gröbner Basis theory as in the previous section. This means that concepts from the previous section will have to be duplicated with slight variant in the definition of S-polynomial which we wish to point out in this section. Therefore through out this section $A = K[x_1, \ldots, x_n]$ is a noncommutative polynomial ring.

How we obtain that Gröbner Basis remains the same as in commutative case where we add nonzero S-polynomials to an initial basis. The difference comes in the definition of an S-polynomial. Since the purpose of S-polynomial $S(f,g)$ for each pair of nonzero polynomials $f, g \in A$ is to ensure that any polynomial $h \in A$ reducible by both f and g has a unique remainder when divided by a set of polynomials containing both f and g, in commutative case there is only one way to divide h by f and g giving the reduction $(h - x_1 f)$ or $(h - x_2 g)$ respectively, where x_1 and x_2 are terms. Thus there is only one S-polynomial for each pair of polynomials.

In noncommutative case however, a polynomial may divide another in many different ways. Therefore we do not have a fixed number of S-polynomials for each pair of polynomials in A. The number of S-polynomials depend on the number of overlaps between the leading monomials of f and g.

Definition 4.2.1. *Let x and y be monomials in a set of monomials M in A. A x-y overlap occurs when one can find factors $x = x_1 z, y = z y_1$ where $x \neq x_1$ and $y \neq y_1$. Different factorization in M gives different overlaps.*

Definition 4.2.2. *Let $f, g \in A$ and the leading monomials of f and g overlap such that $x_1 LM(f) y_1 = x_2 LM(g) y_2$, where $x_1, x_2, y_1, y_2 \in M$ are chosen so that at least one of x_1 and x_2 and at least one of y_1 and y_2 is equal to unit monomial. Then the S-polynomial associated with this overlap is given by*

$$S(f,g) = \lambda_1 x_1 \cdot f \cdot y_1 - \lambda_2 x_2 \cdot g \cdot y_2$$

where $\lambda_1 = \frac{LC(x_1)}{LC(f)}$ when $x_1 \neq 1$ or $\lambda_1 = \frac{LC(y_1)}{LC(f)}$ when $y_1 \neq 1$ and $\lambda_2 = \frac{LC(x_2)}{LC(g)}$ when $x_2 \neq 1$ or $\lambda_2 = \frac{LC(y_2)}{LC(g)}$ when $y_2 \neq 1$.

4.2.1 Mora's Algorithm

In commutative Gröbner basis theory Dickson's Lemma and Hilbert's Basis Theorem assures termination of the Buchberge's Algorithm for all possible inputs. However there is no analogous Dickson's Lemma for noncommutative monomial ideals, hence Mora's Algorithm does not terminates for all possible inputs. It is therefore possible to have infinite Gröbner basis for some finitely generated ideal.

Algorithm 4: Noncommutative Mora's Algorithm [Mor86]

Input : A basis $F = \{f_1, \ldots, f_n\}$ for ideal I over a non commutative polynomial ring $A = K[x_1, \ldots, x_n]$ and an admissible order $\prec$.

Output : A Gröbner basis $G = \{g_1, \ldots, g_t\}$ for I (In the case of termination)

Let $G = F$ and let $B = \emptyset$. for each pair $(g_i, g_j) \in G, i \leq j)$ add an S-polynomial $s(g_i, g_j)$ to B for each overlap $x_1 LM(g_i) y_1 = x_2 LM(g_j) y_2$ between the leading monomials $LM(g_i)$ and $LM(g_j)$.

while $B \neq \emptyset$ **do**

  Remove the first entry s_1 from B. $s_1' = Red_G(s_1)$

  if $s_1' \neq 0$ **then**

    Add s_1' to G and then for all $g_i \in G$ add all $S(g_i, g_j)$ to B.

  end

end

Return G

Proposition 4.2.3. *Not all noncommutative monomial ideals are finitely generated.*

Proof. Assume to the contrary that all noncommutative monomial ideals are finitely generated, and consider an ascending chain of such ideals $J_1 \subset J_2 \subset \cdots$. Then $J = \cup J_i$ is finitely generated and there is some $d \geq 1$ such that $J_d = J_{d+1} = \cdots$. For a counterexample, let $A = K[x,y]$ be a noncommutative polynomial ring, and define J_i for $(i > 1)$ to be the ideal in A generated by the set of monomials $\{xyx, xy^2x, \ldots, xy^ix\}$. Thus we have an ascending chain of such ideals $J_1 \subset J_2 \subset \cdots$. However because no member of this set is a multiple of any other member of the set, it is clear that there cannot be a $d \geq 1$ such that $J_d = J_{d+1} = \cdots$, because $xy^{d+1}x \in J_{d+1}$ and $xy^{d+1}x \notin J_d$ for all $d > 1$. $\square$

Now we are ready to study the core object of this dissertation, a noncommutative free associative algebras called path algebras. For a path algebra we will give concrete example of overlap relations and noncommutative Gröbner basis.

5 Path Algebras

5.1 Quivers

Definition 5.1.1. *A quiver is a quadruple $Q = (Q_0, Q_1, s, t)$ consisting of two sets: Q_0 whose elements are called points or vertices, say $\{1, 2, 3, ..., n, ...\}$, Q_1 whose elements are called arrows, say $\{\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n, \ldots\}$ and two maps: $s, t : Q_1 \longmapsto Q_0$ which associates to each arrow $\alpha \in Q_1$ its source $s(\alpha) \in Q_0$ and its target $t(\alpha) \in Q_0$ respectively.*

Definition 5.1.2. *An arrow $\alpha \in Q_1$ of source $s(\alpha) = 1$ and target $t(\alpha) = 2$ is usually denoted by $\alpha : 1 \longmapsto 2$. A path x, of length $l > 1$, with a source a and target b is a sequence of arrows $\alpha_1, \alpha_2, \alpha_3, ..., \alpha_n$ with $a = s(\alpha_1)$ and $b = t(\alpha_n)$ where $\alpha_k \in Q_1$ for all $1 \leq k \leq n$ and with $t(\alpha_k) = s(\alpha_{k+1})$ for $1 \leq k < n$. Such a path x is denoted by $x = \alpha_1\alpha_2\alpha_3....\alpha_n$ and visualized as:*

$$a = 1 \xrightarrow{\alpha_1} 2 \xrightarrow{\alpha_2} 3 \xrightarrow{\alpha_3} .... \xrightarrow{\alpha_n - 1} n \xrightarrow{\alpha_n} n+1 = b$$

Definition 5.1.3. *The length of a path x, denoted by $l = l(x)$ is the number of arrows in it. An arrow $\alpha : 1 \longmapsto 2$ is a path of length 1. A trivial path denote by v_i is a path of length zero associated with each vertex i. A path of length $l \geq 1$ is called a cycle whenever its source and target coincide. A loop is a cycle of $l = 1$. A quiver is said to be acyclic if it has no cycles. A quiver is said to be finite if Q_0 and Q_1 are both finite sets.*

5.2 Path Algebras

Definition 5.2.1. *Let Q be a quiver and K an arbitrary field. The path algebra KQ of Q is the K-algebra whose underlying K-vector space has as its basis the set of all paths of length $l \geq 0$ in Q, and such that the product of two basis vectors namely $x = \alpha_1\alpha_2\alpha_3....\alpha_n$ and $y = \beta_1\beta_2\beta_3....\beta_k$ is defined by*

$$xy = \begin{cases} \alpha_1\alpha_2\alpha_3...\alpha_n\beta_1\beta_2\beta_3...\beta_k, & if \quad s(y) = t(x) \\ 0, & otherwise \end{cases}$$

i.e the product xy is a concatenation or zero otherwise, so that $Q \bigcup \{0\}$ is closed under multiplication. Multiplication as defined above is also distributive K-linearly in $Q \bigcup \{0\}$. Addition in KQ is the usual K-vector space addition where Q is a K-basis for KQ.

Remark 5.2.2. *i) Let Q be finite. The set $\{v_1, v_2, v_3, ..., v_n\}$ of the trivial paths corresponding to the vertices $\{1, 2, \ldots, n\}$ is a complete set of primitive orthogonal idempotents. Thus $1 = v_1 + v_2 + v_3 + ... + v_n = \sum_{i=1}^{n} v_i$ is the called the identity element of KQ.*

ii) For each arrow $\alpha : 1 \longmapsto 2$ *we have the following defining relations;*

- $v_i^2 = v_i v_i = v_i$ *for* $i = 1, 2$.
- $v_1 \alpha = \alpha$
- $\alpha v_2 = \alpha$
- $v_1 v_2 = 0$.

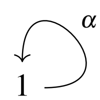
iii) Let Q *denote the set of all paths of length* $l \geq 0$*, then the above product extend to all elements of* KQ *and there is a direct sum*

$$KQ = KQ_1 \oplus KQ_2 \oplus \cdots \oplus KQ_i \oplus \ldots$$

Where KQ_i *is subspace of* KQ *generated by the set* Q_i*, where* Q_i *is the set of all paths of length* i*, over* K*. Since the product of path of length* n *with path of length* m *is zero or a path of length* $n+m$ *then the above decomposition defines a grading on* KQ*. Hence* KQ *is a graded* K*-algebra.*

Definition 5.2.3. *An element* $f \in KQ$*;* $\quad (f = \sum \lambda_i x_i, \quad \lambda_i \in K)$*, is a linear combination of paths* $x_i \in Q$ *over* K*. Elements of* KQ *will be called polynomials. The paths* $x_i \in Q$ *appearing in each polynomials will be called monomials. We shall denote by* $Mon(f)$ *the set of all monomials* x_i *appearing in the polynomial* f*.*

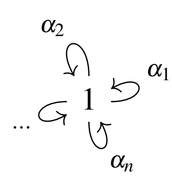
Example 5.2.4. *i) If* Q *consist of one vertex and one loop* α*.*

$Q =$ 

The defining basis of the path algebra KQ *is* $\{v_1, \alpha, \alpha^2, \alpha^3, \ldots, \alpha^n, \ldots\}$ *therefore* $KQ \cong K[X]$*. The isomorphism is induced by the linear maps*

$$v_1 \longmapsto 1 \qquad \textit{and} \qquad \alpha \longmapsto X.$$

ii) If Q *consist of one vertex and* n *loops,* $\alpha_1, \alpha_2 \ldots \alpha_n$*, then* $KQ \cong K[X_1, X_2, \ldots, X_n]$*.*

$Q =$ 

The isomorphism is induced by the K*-linear maps*

$$v_1 \longmapsto 1, \quad \alpha_1 \longmapsto X_1, \quad \alpha_2 \longmapsto X_2, \quad \ldots \quad \alpha_n \longmapsto X_n.$$

iii) Let Q be the quiver

$$Q = \ 1 \xrightarrow{\alpha} 2$$

The basis of the path algebra is $\{v_1, v_2, \alpha\}$ with the multiplication table

$\times$	v_1	v_2	α
v_1	v_1	*0*	α
v_2	*0*	v_2	*0*
α	*0*	α	*0*

Therefore

$$KQ \cong M_2(K) = \begin{bmatrix} K & 0 \\ K & K \end{bmatrix} = \{ \begin{bmatrix} a & 0 \\ b & c \end{bmatrix} a, b, c \in K\}$$

Where the isomorphism is induced by the K-linear maps

$$v_1 \longmapsto \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad v_2 \longmapsto \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad \alpha \longmapsto \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

iv) If the quiver is

$$Q = \ 1 \xrightarrow{\alpha_1} 2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_n} n+1$$

Where there is at most one arrow between two vertices then

$$KQ \cong \{M_n(K) : a_{i,j} = 0 \quad \textit{if there is no path from j to i.}\}$$

Lemma 5.2.5. *[ASS06] Let Q be a quiver and KQ be its path algebra. Then*

a) KQ is an associative algebra.

b) KQ has an identity element if and only if Q is finite.

c) KQ is finite dimensional if and only if Q is finite and acyclic.

Proof

a) This follows from the definition of multiplication, because the product of basis vectors is the composition of paths which is associative.

b) *Each trivial paths is an idempotent of KQ. Thus if Q_0 is finite $\sum_{a\in Q_0} v_a$ is the identity for KQ. Conversely suppose to the contrary that $1 = \sum_{i=1}^{m} \lambda_i x_i$ is an identity element of KQ, $\lambda_i \in K$ and $x_i \in Q$. The set Q_0' of sources of x_i has at most m elements and hence finite. For if $a \in Q_0/Q_0'$ then $v_a.1 = 0$ is a contradiction.*

c) *If Q is infinite then so is the basis of KQ, which is therefore infinite dimensional. If $W = \alpha_1\alpha_2\alpha_3\ldots\alpha_l$ is a cycle in Q, then for each $t \geq 0$ we have $W^t = (\alpha_1\alpha_2\alpha_3\ldots\alpha_l)^t$ is also a basis vector. So KQ is again infinite dimensional. conversely if Q is finite and acyclic, it contain only finitely many paths so KQ is finite dimensional.*

Remark 5.2.6. *For the rest of the sections of this chapter Q will contain paths of finite length l. By convection we write a path $\alpha_1\alpha_2\alpha_3\ldots\alpha_n$ from left to right such that $t(\alpha_i) = s(\alpha_{i+1})$. For path $x = \alpha_1\alpha_2\alpha_3\ldots\alpha_n$ we denote its length $l(x) = n$.*

5.3 Basics to Noncommutative Gröbner Basis in a Path Algebra

Definition 5.3.1. a) *A subset L of KQ is called a left ideal if:*

i) $0 \in L$

ii) $x + y \in L$ *for all* $x, y \in L$

iii) $xy \in L$ *for all* $x \in KQ$ *and all* $y \in L$

b) *A subset R of KQ is called a right ideal if:*

i) $0 \in R$

ii) $x + y \in R$ *for all* $x, y \in R$

iii) $xy \in R$ *for all* $x \in R$ *and all* $y \in KQ$

c) *In general a subset I of KQ is called a two-sided ideal or simply an ideal, if it is both a left and a right ideal.*

Remark 5.3.2. *In general an ideal I in the path algebra KQ has a Gröbner basis depending on the ordering of the path in Q. On the other hand an ideal I in KQ with some path ordering has a Gröbner basis whenever the path ordering is admissible.*

Definition 5.3.3 (Path Ordering)**.** *By a path ordering we refer to the noncommutative ordering as defined in section 3.4.2. In addition we arbitrary order the vertices $v_1 \prec v_2 \prec v_3 \prec \cdots \prec v_k$ and arbitrary order the arrows all larger than given vertex say v_k as $v_k \prec \alpha_1 \prec \alpha_2 \prec \alpha_3 \prec\prec \alpha_r$.*

Definition 5.3.4. *A path order $\prec$ is said to be admissible order if:*

a) *Whenever $x \neq y$ either $x \prec y$ or $x \succ y$.*

b) *Every nonempty set of paths has a least element.*

c) $x \prec y \Rightarrow xz \prec yz$, *whenever* $xz \neq 0$ *and* $yz \neq 0$.

d) *Also* $x \prec y \Rightarrow wx \prec wy$, *whenever* $wx \neq 0$ *and* $wy \neq 0$.

e) $x = yz$ *implies* $x \succeq y$ *and* $x \succeq z$.

Remark 5.3.5. *Conditions* a *through* b *makes* $\prec$ *a right admissible order. Condition* a, b *and* d *make* $\prec$ *left admissible ordering whilst condition* b *say that an admissible ordering is a well ordering.*

While in section 3.4.3 we defined a noncommutative monomial ordering, those monomial ordering were not necessarily admissible. In the following examples we use appropriate monomial (path) ordering to construct admissible ordering for paths in KQ.

Example 5.3.6. i) *Left Lexicographic order: Let* $x = \alpha_1 \dots \alpha_n$ *and* $y = \beta_1 \dots \beta_m$ *be paths. We say that* x *is less than* y *with respect to left lexicographic order and denote* $x \prec_{llex} y$ *if there exist a path* z *(otherwise we set* $z = 1$*), such that* $x = z\alpha_k \dots \alpha_n, \quad y = z\beta_s \dots \beta_m$ *and* $\alpha_k \prec \beta_s$. *Left lexicographic order is not a left admissible ordering since it is not a well ordering. For example let* Q *be*

$$Q = \begin{tikzcd} 1 \arrow[loop left, "\alpha"] \arrow[r, "\beta"] & 2 \end{tikzcd}$$

with $\alpha \prec \beta$. *We have* $(\alpha\beta \succ_{llex} \alpha^2\beta \succ_{llex} \alpha^3\beta \dots)$. *Then the subset* $\{\alpha^n\beta : n \in \mathbb{N} - \{0\}\} \subset Q$ *does not have a least element.*

ii) *Length left lexicographic order: Let* $x = \alpha_1 \dots \alpha_n$ *and* $y = \beta_1 \dots \beta_m$ *be paths. We say that* x *is less than* y *with respect to length left lexicographic order and denote* $x \prec_{Lex} y$ *if* $l(x) < l(y)$ *or* $l(x) = l(y)$ *and* $x \prec_{llex} y$. *Length Left lexicographic order is a left admissible order.*

iii) *Right lexicographic order: Let* $x = \alpha_1 \dots \alpha_n$ *and* $y = \beta_1 \dots \beta_m$ *be two paths in* Q. *We say that* x *is less than* y *with respect to right lexicographic order and denote* $x \prec_{rlex} y$ *if there exist a path* z *(otherwise we set* $z = 1$*), such that* $x = \alpha_1 \dots \alpha_k z, \quad y = \beta_1 \dots \beta_s z$ *and* $\alpha_k \prec \beta_s$. *This ordering is not a well ordering and hence not admissible.*

iv) *Length right lexicographic order: Let* $x = \alpha_1 \dots \alpha_n$ *and* $y = \beta_1 \dots \beta_m$ *be paths. We say that* x *is less than* y *with respect to length right lexicographic order and denote* $x \prec_{rLex} y$ *if* $l(x) < l(y)$ *or* $l(x) = l(y)$ *and* $x \prec_{rlex} y$. *Length right lexicographic order is a right admissible order.*

v) *Left weight-lexicographic order: Define the weight function W on the set of paths Q. For a fixed set of positive integers $\{n_\alpha \in \mathbb{N} | \alpha \in Q_1\}$ let $W : Q_1 \to \mathbb{N}$. Define $W : Q \to \mathbb{N}$, such that $W(\alpha_1 \dots \alpha_r) = \sum_{i=1}^{r} W(\alpha_i)$. Next, order the vertices and assign $W(v_i) = 0$ for all $v_i \in Q_0$. Order the arrows so that $\alpha_i \prec \alpha_j$ whenever $W(\alpha_i) < W(\alpha_j)$. Finally define $x \prec y$, if $W(x) < W(y)$ or $W(x) = W(y)$ and $x \prec_{llex} y$. The length lexicographic order is a special case of the weight lexicographic order, where all the arrows are assigned the same weight. Similarly we can define the right weight-lexicographic order.*

vi) *The total lexicographic order: Order the arrows arbitrarily $\alpha_1 \prec \dots \prec \alpha_m$ and also order the vertices. The vertices will be less than all paths of positive length. Let $x, y \in Q$. Then $x \prec y$, if there exists i such that $\forall j < i \quad \alpha_j$'s occurs in x and y the same number of times, and α_i occurs in x less than it occurs in y. If x and y have the same number of each arrow then $x \prec_{llex} y \Rightarrow x \prec y$.*

vii) *The dual-weighted-left-lexicographic order: Let Q be a graph such that no path on Q passes through two intermediate cycles. Let Q_1' be the set of arrows which are not on an intermediate cycle. Fix a set of positive integers $\{n_\alpha \in \mathbb{N} | \alpha \in Q_1\}$. Let $W : Q_1 \longrightarrow \mathbb{N} \oplus \mathbb{N}$ be such that for $\alpha \in Q_1'$ we have $W(\alpha) = (0, n_\alpha)$ and for $\alpha \in Q_1 - Q_1'$ we have $W(\alpha) = (n_\alpha, 0)$. Define $W : Q \longrightarrow \mathbb{N}$ such that $W(\alpha_1 \dots \alpha_r) = \sum_{i=1}^{r} W(\alpha_i)$, with componentwise addition. Order the set $\mathbb{N} \oplus \mathbb{N}$ so that $(n, m) < (n', m')$ whenever $n < n'$ or when $n = n'$ and $m < m'$. Next, order the vertices and let $W(v_i) = (0, 0)$ for all $v_i \in Q_0$. Thus we define $x \prec y$ if $W(x) < W(y)$; or $W(x) = W(y)$ and $x \prec_{llex} y$. The dual-weighted-left-lexicographic order satisfies the four conditions in definition 5.3.4 which make it an admissible order.*

Definition 5.3.7. *Let $\prec$ be an admissible ordering and $f \in KQ \setminus \{0\}, f = \sum_i \lambda_i x_i$. We have the following definitions;*

a) *The leading monomial of f denoted by $LM(f)$ is the largest monomial occurring in f with respect to $\prec$.*

b) *The leading coefficient of f denoted by $LC(f)$ is the coefficient of $LM(f)$ in f.*

c) *The leading term $LT(f) = LC(f)LM(f)$.*

d) *Let X be a subset of KQ. We define $LM(X) = \{LM(f) : f \in X\}$.*

Definition 5.3.8. *Let x and y be paths.*

i) *We say that x left divide y if $y = wx$ for some path $w \in Q$.*

ii) *Similarly x right divide y if $y = xz$ for some path $z \in Q$.*

iii) *Otherwise we will say that x divides y if $y = wxz$ for some paths $w, z \in Q$.*

iv) An element $f \in KQ \setminus \{0\}$ is said to be an uniform if there exist vertices u and v such that $f = uf = fv = ufv$.

Proposition 5.3.9. *[Lea06] All elements of KQ are uniform.*

Proof. $f = \sum_{i=1}^{n} \lambda_i x_i$ is uniform since for each monomial x_i, which is a sequence of arrows, has a source vertex say u_i and a target vertex say v_i and hence $x_i = u_i x_i v_i$. Therefore f is sum if uniform elements $f = \sum_{i,j=1}^{n} u_i f v_j$. $\square$

Definition 5.3.10. *Let H be a subset of KQ and $g \in KQ$. We say that g can be reduced by H if for some $x \in Mon(g)$ there exist $h \in H$ such that $LM(h)$ divides x, i.e $x = pLM(h)q$ for some monomials $p, q \in KQ$.*

Remark 5.3.11. *i) For $g \in KQ$, the reduction of g by H is given by $g - \lambda phq$ where $h \in H$, $p, q \in Q$ and $\lambda \in K \setminus \{0\}$ such that $\lambda pLM(h)q$ is a term in g. λ is uniquely determined by $\lambda = \frac{LC(g)}{LC(h)}$.*

ii) A total reduction of g by H is an element resulting from a sequence of reductions that cannot be further reduced by H.

iii) We say that an element $g \in KQ$ reduces to 0 by H if there is a total reduction of g by H which is 0. In general two total reductions need not be the same.

iv) A set $H \subset KQ$ is said to be a reduced set if for all $g \in H$, g cannot be reduced by $H - \{g\}$.

Our next goal will be to look at the left and right division algorithms in a path algebras. Since these algorithm will be an indirect entry in the respective left and right Buchberger's Algorithm which inturn produces respective left and right Gröbner basis, we will first introduce the onesided Gröbner basis and give all one-sided algorithms in the next chapter.

6 One-side Gröbner Bases in Path Algebra

6.1 Left Gröbner Bases in Path Algebra

Definition 6.1.1. *Let L be a left ideal of KQ with a left admissible order $\prec$. We say that a set $G_L \subset L$ is a left Gröbner basis for L when for all $f \in L \setminus \{0\}$ there exist $g \in G_L$ such that $LM(g)$ left divides $LM(f)$. Equivalently we say that a set $G_L \subset L$ is a left Gröbner basis for L with respect to a left admissible order $\prec$ if $\langle LM(G_L)\rangle = \langle LM(L)\rangle$.*

Theorem 6.1.2. *[AM10] Let $\prec$ be a left admissible ordering and $S = \{f_1, \ldots, f_n\}$ be a set of nonzero polynomials in KQ. For $g \in KQ \setminus \{0\}$ there exist a unique determined expression $g = \sum_{i=1}^{n} g_i f_i + h$ where $h, g_1, \ldots, g_n \in KQ$ satisfying:*

A. *For any path p occurring in each g_i, $\quad t(p) = s(LM(f_i))$.*

B. *For $i > j$, no term $g_i LT(f_i)$ is left divisible by $LT(f_j)$.*

C. *No path in h is left divisible by $LM(f_i)$ for all $1 \leq i \leq n$.*

Remark 6.1.3. *The expression $g = \sum_{i=1}^{n} g_i f_i + h$ in theorem 6.1.2 is called $left\ standard\ representation$ of $g \in KQ$ with respect to the set S. Algorithm 5 gives as an output h, a remainder of g with left division by S. Let $LRed_S(g) = h$ denote the particular remainder of g by the set S produced by division algorithm with respect to a fixed admissible ordering.*

Algorithm 5: Left Division Algorithm

Input : $g, S = \{f_1, \ldots, f_n\}$ $g, f_i \in KQ \setminus \{0\}$ and left admissible order $\prec$ on KQ

Output : $g_i, \ldots, g_n, h \in KQ$ such that $g = \sum\limits_{i=1}^{n} g_i f_i + h$

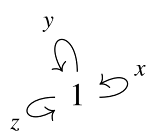
a) For any multiple v_{1i} of $LT(f_1)$ occurring in g with $(1 \le i \le r_1)$, find for each i a term h_{1i} such that $v_{1i} = h_{1i}LT(f_1)$. Afterwards do the same for any multiple v_{2i} of $LT(f_2)$ occurring in g such that $v_{2i} = h_{2i}LT(f_2)$ with $1 \le i \le r_2$. Continue in this way for any multiple v_{ki} of $LT(f_k)$ such that $v_{ki} = h_{ki}LT(f_k)$ with $1 \le i \le r_k$ and $k \in 3, \ldots, n$

b) Write $g = \sum\limits_{j=1}^{n} (\sum\limits_{i=1}^{r_j} h_{ji})LT(f_j) + h_1$ and set $g^1 = g - (\sum\limits_{j=1}^{n} (\sum\limits_{i=1}^{r_j} h_{ji})f_j + h_1)$

c) If $g^1 = 0$ then we are done and $g = \sum\limits_{j=1}^{n} g_j LT(f_j) + h_1$ where $g_j = \sum\limits_{i=1}^{r_j} h_{ji}$ and $h_1 = h$.

d) If $g^1 \neq 0$, go back to a and continue the process with $g = g^1$

Example 6.1.4. *Let Q be the quiver with one vertex and three loops over the field of rationals.*

$Q =$ 

With a left length lexicographic ordering $z \prec y \prec x$. We find the standard representation of $g = zxxyz + xyxxy - xyz$ with respect to the set $\{f_1 = xyz - zy, f_2 = xxy - yx\}$. We first note that $LM(f_1) = xyz$ and $LM(f_2) = xxy$. Initializing we get $g = zxLM(f_1) + xyLM(f_2) - LM(f_1)$. We replace g by $g^1 = g - (zxf_1 + xyf_2 - f_1) = zxzy + xyyx + zy$. Neither $LM(f_1)$ and $Lm(f_2)$ left divides $zxzy + xyyx + zy$, so we set $h = zxzy + xyyx + zy$ and $zxzy + xyyx + zy$ is replaced by 0 and the algorithm stops. Thus the standard representation of g is $g = zxf_1 + xyf_2 - f_1 + h$.

Proof of Theorem 6.1.2 Existence: First the algorithm removes any multiple of f_1 from g. Then removes any multiple of f_2 and continue in this way until any multiple of any of f_k has been removed. In this case if $g = \sum_{j=1}^{n}\sum_{i=1}^{r_j} h_{ji}LT(f_i) + h_1$ is the resulting standard representation of g, we have either $g^1 = g - (\sum_{j=1}^{n}\sum_{i=1}^{r_j} h_{ji}(f_i) + h_1) = 0$ Or $LM(g^1) \prec LM(g)$. Since the path ordering $\prec$ is well ordering, by recursion the algorithm produces a standard representation for g^1, $g^1 = \sum_{j=1}^{n}\sum_{i=1}^{r_j} h^1_{ji}(f_i) + h^1$ satisfying conditions A, B and C. Thus $g = \sum_{j=1}^{n}\sum_{i=1}^{r_j}(h_{ji} + h^1_{ji})(f_i) + (h_1 + h^1)$ is a representation for g satisfying the conditions A, B and C.

Uniqueness: For $g \in L \setminus \{0\}$ let $g = g_1 f_1 + \cdots + g_n f_n + h$. Then the three conditions A, B and C implies that the terms $LT(g_i f_i) = LT(g_i)LT(f_i)$ and $LT(h)$ do not divide each other to the left. Otherwise these terms cancels with each other into zero polynomial. Therefore the representation $g = \sum_{i=1}^{n} g_i f_i + h$ is unique.

Termination: The algorithm produces elements $g, g^1, g^2, \ldots, g^k$ so that at each k^{th} iteration $LM(g^{k+1}) \prec LM(g^k)$. Since $\prec$ is a well ordering, the algorithm terminates at some $g^k = 0$ satisfying the conditions of the theorem.

Given a finite generating set $S = \{f_1, \ldots, f_n\}$. For a left admissible order $\prec$, the following algorithm gives as an output $R_L = R_L(S)$, a left reduction of S.

Algorithm 6: Set Left Reduction Algorithm

Input :$S = \{f_1, \ldots, f_n\}$ $f_i \neq 0$ and a left admissible ordering $\prec$

Output :R_L a left reduction of the set S

a) $R_L = \emptyset$

b) Find the maximal element f_k of S with respect to $<$, for $1 \leq k \leq n$.

c) Write $S = S - \{f_k\}$

d) Do $f'_k = LRed_{S \cup R_L}(f_k)$

e) If $f'_k \neq 0$ then $R_L = R_L \cup \{\frac{f'_k}{LM(f'_k)}\}$

f) If $f_k \neq f'_k$ then $S = S \cup R_L$; Go back to a and continue with the process.

Proposition 6.1.5. *[AM10] Let $G = \{f_1, \ldots, f_n\} \subset KQ$ be a left Gröbner basis for the ideal $L = \langle f_1, \ldots, f_n \rangle \subset KQ$. If $g = \sum_{i=1}^{n} g_i f_i + h$ is a left standard expression of $g \in KQ \setminus \{0\}$ then $g \in L$ if and only if $h = 0$.*

Proof. If $h = 0$ clearly $g \in L$. Conversely if $g \in L$ then $h \in L \Longrightarrow LM(h) \in \langle LM(f_1), \cdots, LM(f_n) \rangle$ which is impossible by the theorem 6.1.2.

Definition 6.1.6 (Left S-Polynomial)**.** *Let $f, g \in KQ \setminus \{0\}$ and $\prec$ be a left admissible ordering. Let p, q be paths such that $pLM(f) = qLM(g)$, the left S-polynomial $S_L(f, g)$ is defined as*

$$S_L(f, g) = \frac{p}{LC(f)} \cdot f - \frac{q}{LC(g)} \cdot g$$

Theorem 6.1.7 (Left Buchberger's Criterion)**.** *Let $f_1, \ldots, f_n \in KQ \setminus \{0\}$ and $\prec$ be a left admissible ordering. Let $S_L(f_i, f_j) = \sum_{k=1}^{n} g_k f_k + h_{ij}$ be a left a standard expression of $S_L(f_i, f_j)$*

for each pair (i, j). $f_1, \ldots, f_n$ form a left Gröbner basis for $L = \langle f_1, \ldots, f_n \rangle$ if and only if all the remainders h_{ij} are zero.

Proof. See Attan and Mialebama [AM10].

Algorithm 7: Left Buchberger's Algorithm

Input : $L = \langle f_1, \ldots, f_n \rangle \subset KQ$ and a left admissible order $\prec$.

Output : A reduced left Gröbner basis G_m for L.

a) $m = 0$; $G_0 = \emptyset$; $G_1 = R_L(\{f_1, \ldots, f_n\})$

b) While $G_m \neq G_{m+1}$, $m = m + 1$

c) For all $g, h \in G_m$ find all $S_L(g, h) \neq 0$

d) Write $G'_m = G'_m \cup \{S_L(g, h)\}$

e) $G_{m+1} = R_L(G'_m)$

6.2 Right Gröbner Basis in a Path Algebra

Definition 6.2.1. *Let R be a right ideal of KQ with a right admissible order $\prec$. We say that a set $G_R \subset R$ is a right Gröbner basis for R when for all $f \in R \setminus \{0\}$ there exist $g \in G_R$ such that $LM(g)$ right divides $LM(f)$. Equivalently we say that a set $G_R \subset R$ is a right Gröbner basis for R with respect to a right admissible order $\prec$ if $\langle LM(G_R) \rangle = \langle LM(R) \rangle$.*

Theorem 6.2.2. *[AM10] Let $\prec$ be a right admissible ordering and $S = \{f_1, \ldots, f_n\}$ be a set of non zero polynomials in KQ. For $g \in KQ \setminus \{0\}$ there exist a unique determined expression $g = \sum_{i=1}^{n} f_i g_i + h$ where $h, g_1, \ldots, g_n \in KQ$ satisfying:*

A2. For any path p occurring in each g_i, $\quad s(p) = t(LM(f_i))$.

B2. For $i > j$, no term $LT(f_i)g_i$ is right divisible by $LT(f_j)$.

C2. No path in h is right divisible by $LM(f_i)$ for all $1 \leq i \leq n$.

Then algorithm 8 gives as an output h, a remainder of g with right division by S. Let $Rred_S(g) = h$ denote the particular remainder of g by the set S produced by division algorithm with respect to a fixed right admissible ordering.

Algorithm 8: Right Division Algorithm

Input : $g, S = \{f_1, f_2, \ldots, f_n\}$ and right admissible order $\prec$ on KQ

Output : $g_i, \ldots, g_n, h \in KQ$ such that $g = \sum_{i=1}^{n} g_i f_i + h$

a) For any multiple v_{1i} of $LT(f_1)$ occurring in g with $(1 \leq i \leq r_1)$, find for each i a term h_{1i} such that $v_{1i} = LT(f_1)h_{1i}$. Afterwards do the same for any multiple v_{2i} of $LT(f_2)$ occurring in g such that $v_{2i} = LT(f_2)h_{2i}$ with $1 \leq i \leq r_2$. Continue in this way for any multiple v_{ki} of $LT(f_k)$ such that $v_{ki} = LT(f_k)h_{ki}$ with $1 \leq i \leq r_k$ and $k \in 3, \ldots, n$

b) Write $g = \sum_{j=1}^{n} \sum_{i=1}^{r_j} LT(f_j)h_{ji} + h_1$ and set $g^1 = g - (\sum_{j=1}^{n} \sum_{i=1}^{r_j} f_j h_{ji} + h_1)$

c) If $g^1 = 0$ then we are done and $g = \sum_{j=1}^{n} LT(f_j)g_j + h_1$ where $g_j = \sum_{i=1}^{r_j} h_{ji}$ and $h_1 = h$.

d) If $g^1 \neq 0$, go back to a and continue the process with $g = g^1$

Given a finite generating set $S = \{f_1, \ldots, f_n\}$. For a left admissible order $\prec$, the following algorithm gives as an output $R_R = R_R(S)$, a right reduction of S.

Algorithm 9: Set Right Reduction Algorithm

Input : $S = \{f_1, \ldots, f_n\}$ $f_i \neq 0$ and a right admissible ordering $\prec$

Output : R_R a right reduction of the set S

a) $R_R = \emptyset$

b) Find the maximal element f_k of S with respect to $<$, for $1 \leq k \leq n$.

c) Write $S = S - \{f_k\}$

d) $f'_k = Rred_{S \cup R_R}(f_k)$

e) If $f'_k \neq 0$ then $R_R = R_R \cup \{\frac{f'_k}{LM(f'_k)}\}$

f) If $f_k \neq f'_k$ then $S = S \cup R_R$; go back to a and continue with the process.

Proposition 6.2.3. *[AM10] Let $G = \{f_1, \ldots, f_n\} \subset KQ$ be a right Gröbner basis for that ideal $R = \langle f_1, \ldots, f_n \rangle$ in KQ. If $g = \sum_{i=1}^{n} f_i g_i + h$ is a right standard expression of $g \in KQ \setminus \{0\}$ then $g \in R$ if and only if $h = 0$.*

Proof. The proof is similar to that in left case.

Definition 6.2.4 (Right S-Polynomial). *Let $f, g \in KQ \setminus \{0\}$ and $\prec$ be a right admissible ordering. Let p, q be paths such that $LM(f)p = LM(g)q$ the right S-polynomial $S_R(f,g)$ is defined as*

$$S_R(f,g) = \frac{f}{LC(f)} \cdot p - \frac{g}{LC(g)} \cdot q$$

Theorem 6.2.5 (Right Buchberger's Criterion:). *Let $f_1, \ldots, f_n \in KQ \setminus \{0\}$ and $\prec$ be a right admissible ordering. Let $S_R(f_i, f_j) = \sum_{k=1}^{n} f_k g_k + h_{ij}$ be a right a standard expression of $S_R(f_i, f_j)$ for each pair (i, j). $f_1, \ldots, f_n$ form a right Gröbner basis for $R = \langle f_1, \ldots, f_n \rangle$ if and only if all the remainders h_{ij} are zero.*

Algorithm 10: Right Buchberger's Algorithm

Input : $R = \langle f_1, \ldots, f_n \rangle \subset KQ$ and a right admissible order $\prec$.

Output: A reduced right Gröbner basis G_m for R.

a) $m = 0$; $G_0 = \emptyset$; $G_1 = R_R(\{f_1, \ldots, f_n\})$

b) While $G_m \neq G_{m+1}$, $m = m + 1$

c) For all $g, h \in G_m$ find all $S_R(g,h) \neq 0$

d) Write $G'_m = G'_m \cup \{S_R(g,h)\}$

e) $G_{m+1} = R_L(G'_m)$

Example 6.2.6. *Let Q be the quiver;*

$$Q = \begin{tikzcd} 1 \arrow[r, "z"] \arrow[dr, "y"] & 2 \arrow[d, "t"] \\ & 3 \arrow[loop right, "x"] \end{tikzcd}$$

Let $F = \{f_1 = ztx^3, f_2 = zt + y\}$ be a subset of KQ with respect to the right length lexicographic ordering $v_1 \prec v_2 \prec v_3 \prec t \prec z \prec y \prec x$. Running Algorithm 10, we note that $LM(f_1) = ztx^3$ and $LM(f_2) = zt$ and they only factor each other to the right in one way namely $LM(f_1)v_3 = LM(f_2)x^3$. Thus we have one right S-polynomial $S_R(f_1, f_2) = f_1 v_3 - f_2 x^3 = -yx^3$. Neither $LM(f_1)$ nor $LM(f_2)$ right divide $-yx^3$ so we add $f_3 = -yx^3$ to F. Now every right S-polynomial reduces to zero by F. Thus $F = \{f_1, f_2, f_3\}$ is a right Gröbner basis for the ideal $R = \langle f_1, f_2 \rangle$.

7 Twosided Gröbner Bases

Definition 7.0.1. *Let I be an ideal of KQ with an admissible order $\prec$. We say that a set $G \subset I$ is a Gröbner basis for I when for all $f \in I \setminus \{0\}$ there exist $g \in G$ such that $LM(g)$ divides $LM(f)$.*

Remark 7.0.2. *Equivalently we say that a set $G \subset I$ is a Gröbner basis for I with respect to an order $\prec$ if $\langle LM(G) \rangle = \langle LM(I) \rangle$.*

Proposition 7.0.3. *If G is a Gröbner basis for the ideal I, then G is a generating set for the elements of I and also G reduces elements of I to 0.*

Proof. Let KQ be a path algebra with an admissible ordering $\prec$. Let I be an ideal and let G be a Gröbner basis for I. Let $f_i \in I$ $i = 1, \ldots, n, \ldots$, for every $f_n \in I$ such that $f_n \neq 0 \quad \exists \quad g \in G$ such that $LM(g)$ divides $LM(f_n)$. Let $f_{n+1} = f_n - \frac{LC(f_n)}{LC(g)} xgy$ be a reduction of f_n by g. Then $LM(f_{n+1}) \prec LM(f_n)$. But $g, f_n \in I \implies f_{n+1} \in I$. Repeating this reduction on f_i to produce f_{i+1} yields a decreasing sequence $LM(f_1) \succ LM(f_2) \succ \ldots \ldots$, which terminates only if $f_n = 0$. Since $\prec$ is an admissible order, every set of paths has a least element hence the sequence must terminate with $f_n = 0$. □

7.1 Division Algorithms

Theorem 7.1.1. *Let $\prec$ be an admissible ordering and $S = \{f_1, \ldots, f_n\}$ be a set of non zero polynomials in KQ. For $g \in KQ \setminus \{0\}$ there exist a unique determined expression $g = \sum_{i=1}^{n} w_i f_i z_i + h$ where $h, w_1, \ldots, w_n, z_1 \ldots, z_n \in KQ$ satisfying:*

A3. For any path p occurring in each w_i, $t(p) = s(LM(f_i))$ and for any path q occurring in z_i, $t(LT(f_i)) = s(q)$.

B3. For $i > j$ no term $w_i LT(f_i) z_i$ is divisible by $LT(f_j)$.

C3. No path in h is divisible by $LM(f_i)$ for all $1 \leq i \leq n$.

Algorithm 11: Twosided Division Algorithm

Input : $g, S = \{f_1, ..., f_n\}$ and an admissible order $\prec$ on elements of KQ.

Output : $w_1, ..., w_n, z_1, ..., z_n, h \in KQ$ such that $g = \sum_{i=1}^{n} w_i f_i z_i + h$.

a) For any multiple O_{1i} of $LT(f_i)$ occurring in g with $1 \leq i \leq r_1$, find for each i the terms u_{1i} and v_{1i} such that $O_{1i} = u_{1i} LT(f_1) v_{1i}$. Following this do the same for any multiple O_{2i} of $LT(f_2)$ occurring in g such that $O_{2i} = u_{2i} LT(f_2) v_{2i}$ with $1 \leq i \leq r_2$. Continue in this way for any multiple O_{ki} of f_k such that $O_{ki} = u_{ki} LT(f_k) v_{ki}$ with $1 \leq i \leq r_k$ and $k \in \{3, ..., n\}$.

b) Write $g = \sum_{j=1}^{n} \sum_{i=1}^{r_j} u_{ji} LT(f_j) v_{ji} + h_1$ and set $g^1 = g - (\sum_{j=1}^{n} \sum_{i=1}^{r_j} u_{ji} f_j v_{ji} + h_1)$

c) If $g^1 = 0$ then we are done and $g = \sum_{j=1}^{n} w_j f_j z_j + h_1$ where $w_j = \sum_{i=1}^{r_j} u_{ji}$, $z_j = \sum_{i=1}^{r_j} v_{ji}$ and $h = h_1$

d) If $g^1 \neq 0$, go back to a and proceed with $g = g^1$.

Proof Let $Red_S(g) = h$ denote the particular total reduction of an element g by a set S produced by the algorithm 11 with respect to a fixed admissible ordering.

existence : This algorithm finds a standard representation of g as follows. First it removes any multiple of f_1 in g. Afterwards removes any multiples of f_2. Continue in this way until any multiple of any $f_k, k \in \{3, 4, \ldots, n\}$ has been removed. Hence if $g = \sum_{j=1}^{n} \sum_{i=1}^{r_j} u_{ji} LT(f_i) v_{ji} + h_1$ is the resulting representation of g then either $g^1 = g - (\sum_{j=1}^{n} \sum_{i=1}^{r_j} u_{ji}(f_i) v_{ji} + h_1)$ equal to zero and we are done, or $LM(g) \succ lm(g^1)$. Since $\prec$ is a well ordering then the algorithm finds a representation $g^1 = \sum_{j=1}^{n} \sum_{i=1}^{r_j} u_{ji}^1 f_i v_{ji}^1 + h^1$ satisfying conditions $A3$, $B3$ and $C3$ so that $g = \sum_{j=1}^{n} \sum_{i=1}^{r_j} (u_{ji}^1 + u_{ji}) f_i (v_{ji}^1 + v_{ji} + (h^1 + h_1)$ is the standard representation of g satisfying conditions $A3, B3$ and $C3$.

Uniqueness : Given g and conditions $A3$, $B3$ and $C3$, no term $LT(w_i f_i z_i)$ for all $1 \leq i \leq n$ divides $LT(h)$. Therefore the algorithm produces a unique standard representation $g = \sum_{i-1}^{n} w_i f_i z_i + h$ where w_i or z_i may be unit monomials.

termination : Note that the algorithm produces elements $g, g^1, g^2, ..., g^k$ such that at each k^{th} iteration $LM(g^k) \succ LM(g^{k+1})$ and the algorithm must terminate at some k where $g^k = \sum_{j=1}^{n} \sum_{i=1}^{r_j} u_{ji}(f_i) v_{ji} + h_k = 0$ and every monomial occurring in the final h_k is not divisible by $LM(f_i), 1 \leq i \leq n$. $\square$

Example 7.1.2. *Consider the quiver;*

$$Q = \begin{tikzcd} 1 \arrow[loop above left, "y"] \arrow[loop above right, "x"] \arrow[loop below, "z"] \end{tikzcd}$$

Let $\prec$ the length lexicographic order with $x \succ y \succ z$. Let's divide $f_1 = xy - x, f_2 = xx - xz$ into $f = zxxyx$. Note that the $LM(f_1) = xy$ and $LM(f_2) = xx$. Beginning the algorithm 11, we see that $zxxyx = (zx)LM(f_1)(x)$. Thus $p_1 = zx, q_1 = x$ and we replace $zxxyx$ by $zxxyx - zx(f_1)x = zxxx$. Now $LM(f_1)$ does not divide $zxxx$. Continuing, $LM(f_2)$ does. There are two ways to divide $zxxx$ by xx and for the algorithm to be precise we must choose one. Say we choose the "left most" division. Then $zxxx = z(LM(f_2))x$ and we let $p_2 = z, q_2 = x$ and replace $zxxx$ by $zxxx - z(f_2)x = zxzx$. Neither $LM(f_1)$ nor $LM(f_2)$ divide $zxzx$ so we let $r = zxzx$ and $zxzx$ is replaced by 0 and the algorithm stops. We have $zxxyx = (zx)f_1(x) + (z)f_2(x) + zxzx$. The remainder is $zxzx$.

Given a finite generating set $S = \{f_1, \dots, f_n\}$. For an ideal $I \subset KQ$ and an admissible order $\prec$, the following algorithm gives as an output $R(S)$ a finite monic reduced generating set for I.

Algorithm 12: Set Reduction Algorithm

Input :$S = \{f_1, \dots, f_n\}$ $f_i \neq 0$ and an admissible ordering $\prec$.
Output :$R = R(S)$ a reduction of elements of S

a) $R = \emptyset$

b) Find the maximal element f_k of S with respect to $\prec$.

c) Write $S = S - \{f_k\}$

d) Do $f'_k = Red_{S \cup R}(f_k)$

e) If $f'_k \neq 0$ then $R = R \bigcup \{\frac{f'_k}{LM(f'_k)}\}$

f) If $f_k \neq f'_k$ then $S = S \cup R$; Go back to a and continue with the process.

Proposition 7.1.3. *Given an ideal I in KQ and admissible order $\prec$, there is a unique Gröbner basis G such that G is a reduced set and the coefficient of the leading monomials of the polynomials in G are all 1.*

Proof. Let KQ be a path algebra, I an ideal and $\prec$ an admissible order. Let G and G' be Gröbner bases for I. Suppose G and G' are both reduced monic sets. Since $G \subset I$, for every $g_1 \in G$ there exist $g' \in G'$ such that $LM(g')$ divides $LM(g_1)$. Also since $G' \subset I$ there exist $g_2 \in G$ such that $LM(g_2)$ divides $LM(g')$. Thus $LM(g_2)$ divides $LM(g_1)$. But G is a reduced set hence we must have that $g_2 = g_1$ so that $LM(g_1) = LM(g') = LM(g_2)$. So there is a bijection correspondence between elements of G and the elements of G' with the same leading monomials. Thus g' cannot be reduced by $G - \{g_1\}$. Hence $g' - g_1$ cannot be reduced by G, since $g' - g_1 \in I$. Thus $g' - g_1 = 0 \Longrightarrow g' = g_1$ hence $G' = G$. $\square$

Definition 7.1.4. *We call the unique reduced monic Gröbner basis the reduced Gröbner basis.*

Proposition 7.1.5. *The reduced Gröbner basis G is minimal in the sense that for any other reduced Gröbner basis G' for the same ideal with the same admissible order, we have $LM(G') \subset LM(G)$.*

7.2 Twosided S-Polynomial

While noncommutative S-polynomials for each pair of polynomials $f, g \in KQ$ may be different due to different factorizations in the set of monomials in KQ, for onesided case these S-polynomials are finitely many. However we may have ambiguity while dealing with twosided S-polynomials due to possible different choices of right and left factors of each overlap of $LM(f)$ and $LM(g)$ Therefore a condition namely $l(p) \leq l(LM(g))$ whenever $LM(f) \cdot p = q \cdot LM(g)$, is added to the definition of two sided S-polynomial to eliminate such ambiguity so that we define;

Definition 7.2.1. *Let $f, g \in KQ$ with an admissible order $\prec$ on elements of KQ. An $(f-g)$ overlap is said to occur if there are paths p and q of positive length such that $LM(f)p = qLM(g)$ where $l(p) \leq l(LM(g))$. Thus an f and g are said to have an overlap relation or a twosided S-polynomial denoted by $S(f,g)$ and defined as;*

$$S(f,g,p,q) = \frac{1}{LC(f)} f \cdot p - \frac{1}{LC(g)} q \cdot g.$$

Remark 7.2.2. *Given elements $f, g \in KQ$ such that $LM(f)p = qLM(g)$ where $l(p) \leq l(LM(g))$, monomials p and q will not necessarily be unique. Consequently the same two elements f and g may still have multiple S-polynomials. In addition an element may have an S-polynomial with itself i.e $S(f,f)$ will be a possible.*

Example 7.2.3. *a) Let Q be*

$$Q = \begin{tikzcd} 1 \arrow[loop above, "x"] \arrow[loop below, "y"'] \end{tikzcd}$$

Let $x \prec y$ with respect to the length lexicographic order. Let $f = 5yyxyx - 2xx$ and $g = xyxy - 7y$. We see that $LM(f) = yyxyx$ and $LM(g) = xyxy$. The following are the S-polynomials among f and g are:

i)

$$S(f,g,y,yy) = \frac{1}{5}fy - yyg = \frac{1}{5}(5yyxyx - 2xx)y - yy(xyxy - 7y) = -\frac{2}{5}xxy + 7yyy$$

ii)

$$S(f,g,yxy,yyxy) = \frac{1}{5}fy - yyg = \frac{1}{5}(5yyxyx - 2xx)yxy - yyxy(xyxy - 7y) = -\frac{2}{5}xxyxy + 7yyxyy$$

iii)

$$S(g,g,xy,xy) = gxy - xyg = (xyxy - 7y)xy - xy(xyxy - 7y) = -7yxy + 7xyy$$

b) Let Q be

$$\begin{tikzcd} 1 \arrow[r, bend left, "x"] \arrow[r, bend right, "y"'] & 2 \end{tikzcd}$$

Let $f = 2xyxyxy - 4xy$ and $g = xyxyx - x$. We have the following S-polynomials

i)

$$S(f,g,x,xy) = \frac{1}{2}(2xyxyxy - 4xy)x - xy(xyxyx - x) = -xyx$$

ii)

$$S(f.g,xyx,xyxy) = \frac{1}{2}(2xyxyxy - 4xy)xyx - xyxy(xyxyx - x) = xyxyx$$

iii)

$$S(g,f,yxy,xy) = (xyxyx - x)yxy - \frac{xy}{2}(2xyxyxy - 4xy) = xyxy$$

iv)

$$S(g,f,yxyxy,xyxy) = (xyxyx - x)yxyxy - \frac{xyxy}{2}(2xyxyxy - 4xy) = xyxyxy$$

Lemma 7.2.4 (Bergman's Diamond, [Ber78])**.** *Let G be a set of uniform elements that form a generating set for the ideal $I \subset KQ$, such that for all $g, g_1 \in G$, $LM(g) \nmid LM(g_1)$. If for each $f \in I$ and $g \in G$ every S-polynomial $S(f,g,p,q)$ is reduced to 0 by G, then G is a Gröbner basis for I.*

7.3 The Main Theorem

Theorem 7.3.1. *Given a path algebra KQ, an admissible order $\prec$ and a finite generating set $\{f_1, f_2, \dots, f_m\}$ for an ideal I the following algorithm gives a reduced Gröbner basis for I in the limit.*

Algorithm 13: Twosided Buchberger's Algorithm

Input : $I = \langle f_1, \dots, f_n \rangle$, $f_i \neq 0$ and an admissible order $\prec$.

Output : A reduced Gröbner basis G_m for I.

a) $m = 0; G_0 = \emptyset; G_1 = R(\{f_1, f_2, \dots, f_n\})$

b) For $G_m \neq G_{m+1}; m = m+1$

c) For all pairs $(g_i, g_j) \in G_m$ and all $1 \leq i \leq j \leq n$, find $S(g_i, g_j, p, q) \neq 0$

d) Do $G'_m = G_m \bigcup \{S(g_i, g_j, p, q)\}$

e) $G_{m+1} = R(G'_m)$

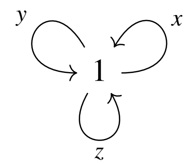
Proof Let G_m be the output of the algorithm 13. Thus if this algorithm terminates on the set m^{th} iteration. The set G_m is reduced Gröbner basis.

I) We first show by induction on m that for each m^{th} iteration every S-polynomial has a standard representation $S(g_i, g_j, p, q) = \sum_{k=1}^{n} w_k f_k z_k + h_{ij}$. Consider $m = 1; G_1 = R(\{f_1, f_2, \dots, f_n\})$; the algorithm produces $f = S(g_i, g_j, p, q) = \sum_{k=1}^{n} w_k f_k z_k + h_{ij}$ as a reduced of $S(g_i, g_j, p, q)$ with respect to $S \cup R$. If $h_{ij} \neq 0$, then $h_{ij} \in G_2$ and again f has a standard representation with respect to G_2. Suppose that the hypothesis hold true for m. We now prove for $m+1$. If the algorithm terminates at $m+1$ then $G_{m+2} = G_{m+1} = G_m$and hence $f = S(g_i, g_j, p, q)$ has a standard representation with respect to $G_{m+2} = G_{m+1}$. If the algorithm does not terminate at $m+1$, $G_{m+1} = R(G_m \cup \{S(g_i, g_j, p, q)\})$ so that the algorithm reduces $f = S(g_i, g_j, p, q)$ to h_{ij}. This ensures that f has a standard representation with respect to G_{m+2}. Hence the hypothesis hold true for $m+1$. By induction the statement hold true for all m.

II) We now show that the algorithm terminates at $m+1$ if and only if G_m is a finite Gröbner basis of I: If the algorithm terminates at some $m+1$ then all $S(g_i, g_j, p, q) = 0$ and $G_{m+1} = R(G_m) = G_m$, for G_m is a reduced set at every step. Since $\langle G_m \rangle = I$ then we conclude that G_m is a finite reduced Gröbner basis for I. Conversely if G_m is a finite reduced Gröbner basis of I, then $R(G_m) = G_m$ and for each pair $(g_i, g_j) \in G_m$, $f = S(g_i, g_j, p, q)$ is reduced to zero by G_m. Therefore the algorithm terminates at G_{m+1}.

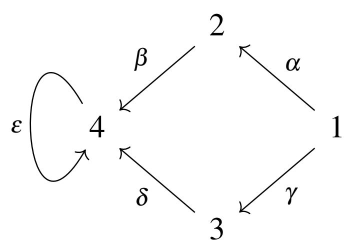
III) If the algorithm never terminates, Let $G = \cup_{m=1}^{\infty} G_m$, then for m sufficiently large every S-polynomial $S(g_i, g_j, p, q)$ has a standard representation with respect to $G_{m+1} \subset G$. Obviously $\langle G \rangle = I$ and hence G is an infinite Gröbner basis of I.

Example 7.3.2. *i) Let Q be the quiver;*

$Q =$ 

Let $\prec$ be the length lexicographic order with $v_1 \prec x \prec y \prec z$. Consider the ideal $I = \langle xy - 2xy, zx, zy \rangle$, with $f = xy - 2xy, \quad g = zx \quad$ and $\quad h = zy$. We see that $LM(f) = yx \nmid LM(g) = zx \nmid LM(h) = zy$. Hence for each pair of elements in $G = \{f, g, h\}$, every S-polynomial will reduce to 0. Thus the set $G = \{f, g, h\}$ is reduced and uniform and hence a Gröbner basis of I.

ii) Let Q be

$Q =$ 

with the length lexicographic ordering $v_1 \prec \cdots \prec v_4 \prec \varepsilon \prec \beta \prec \delta \prec \alpha \prec \gamma$. Let $f = \alpha\beta - \gamma\delta$, $g = \beta\varepsilon$ and $h = \varepsilon^3$. We see that $LM(f) = \gamma\delta, \quad LM(g) = \beta\varepsilon$ and $LM(h) = \varepsilon^3$. $LM(f) \nmid LM(h)$ and $LM(f) \nmid LM(h)$. The only possible S-polynomial is $S(g, h, \varepsilon^2, \beta) = 0$. Thus the set G=$\{f, g, h\}$ is the Gröbner basis since all the S-polynomial reduces to 0.

On the other hand if we consider another admissible order $v_1 \prec \cdots \prec v_4 \prec \varepsilon \prec \beta \prec \delta \prec \gamma \prec \alpha$. We now see that $LM(f) = \alpha\beta, \quad LM(g) = \beta\varepsilon$ and $LM(h) = \varepsilon^3$. In this case the only S-polynomial possible is

$$S(f, g, \varepsilon, \alpha) = (\alpha\beta - \gamma\delta)\varepsilon - \alpha(\beta\varepsilon) = -\gamma\delta\varepsilon.$$

$LM(S(f, g, \varepsilon, \alpha)) = \gamma\delta\varepsilon \notin \langle LM(F), LM(g), LM(h) \rangle$, Thus $G = \{f, g, h\}$ is not a Gröbner basis for $I = \langle G \rangle$. We add $r = \gamma\delta\varepsilon$ to G, and we set $G = \{f, g, h, r\}$. Therefore $S(f, g, \varepsilon, \alpha) = r$ and there are no further possible S-polynomial relations. Thus $R(G) = G = \{f, g, h, r\}$ is a Gröbner basis for I.

iii) Let Q be

$$Q = \begin{tikzcd}
 & & 2 \arrow[dl, "b"'] & \\
5 & 4 \arrow[l, "e"'] & & 1 \arrow[ul, "a"'] \arrow[dl, "c"] \\
 & & 3 \arrow[ul, "d"] &
\end{tikzcd}$$

with an admissible order $v_1 \prec \cdots \prec v_5 \prec e \prec d \prec c \prec b \prec a$. Let $I = \langle ab - cd, be \rangle$. There are the following S-polynomials possible:

$$S(ab - cd, be, e, a) = (ab - cd)e - a(be) = -cde$$

We see that $LM(S(ab - cd, be, e, a)) = cde \notin \langle LM(ab - cd), LM(be) \rangle$. Therefore $G = \{ab - cd, be\}$ is not a Gröbner basis for I. We add $(-cde)$ to G. We thus have; $S(ab - cd, be, e, a) = -cde \in G$ is the only possible S-polynomial. Hence $G = \{ab - cd, be, cde\}$ is a Gröbner basis for I.

If we change the order to $v_5 \prec v_4 \prec \cdots \prec v_1 \prec a \prec b \prec c \prec d \prec e$. $LM(ab - cd) = cd$ and $LM(be) = be$, we have $LM(ab - cd) \nmid LM(be)$. Therefore the set $G = \{ab - cd, be\}$ is reduced and its element are uniform. By Bergman Diamond Lemma G is the Gröbner basis of I.

8 Applications

Let $I \subset KQ$ be a two-sided ideal and $G = \{g_1, \ldots, g_k\}$ its Gröbner basis for a fixed admissible ordering. We define $KQ \setminus I = KQ \subset \langle G \rangle$ to be the factor algebra of KQ if all $g_1, \ldots, g_k$ vanishes in $KQ \setminus I$ as elements of KQ. In this section we attempt to extend the Gröbner basis theory to the residue class algebra $KQ \setminus I$. For simplicity we denote $A = KQ \setminus I$ and assume G to be finite Gröbner basis for ideal $I \subset KQ$ with respect to a fixed path ordering $\prec$. The elements of A are the equivalence classes. For a nonzero polynomial $f \in KQ$, the residue class of f denoted by $\bar{f}$ is the class of all those polynomials of KQ that are equivalent to f modulo I.For f, G and $\prec$ applying theorem 7.1.1, algorithm 11 produces representatives of the equivalence class of f, i.e $\bar{f} = Red_G(f)$.

Our next goal is to show that there is an initial basis for the algebra A. Macualay's Basis Theorem tells us that if there is an order ideal in KQ then its equivalence classes is a basis for A as a K-vector.

Lemma 8.0.1. *The set $O_\prec(I) = Mon(KQ) \setminus LT(I)$ is an order ideal of I with respect to $\prec$.*
Proof: *We show that if $x \in O_\prec(I)$ and $x = x_1 \cdot x_2$ then $x_1 \in O_\prec(I)$ and $x_2 \in O_\prec(I)$. Suppose $x_1, x_2 \in O_\prec(I)$ then either $x_1 \cdot x_2$ is a concatenation or zero. But x=0 is a contradiction.* □

Theorem 8.0.2 (Macualay's Basis Theorem for KQ [AK91].)**.** *Let $I \subset KQ$ be an ideal. The residue classes of elements of $O_\prec(I)$ forms a basis of the K-vector space A.*

Proof. Take $B = Span_K\{\bar{x} \in A | x \in O_\prec(I)\}$. Obviously $B \subseteq A$. Assume that $B \subset A$, since $\prec$ is a well ordering on $Mon(KQ)$ there exist $f \in KQ \setminus \{0\}$ satisfying $f \notin I$, $\bar{f} \notin B$ and having minimal leading term $LT(f)$ with respect to $\prec$. If $LT(f) \in LT(I)$ then there exist $g \in I$ such that $LT(f) = LT(g)$. Thus by theorem 7.1.1 we obtain that $f' = f - \frac{LC(f)}{LC(G)} \cdot g$, satisfying $f' \notin I$, $f' \notin B$ and having smaller leading term than f, a contradiction. Therefore $LT(f) \in O_\prec(I)$. However, we obtain a polynomial $f'' = f - \frac{1}{LC(f)} \cdot f$ satisfying $f'' \notin I$, $f'' \notin B$ and having smaller leading term than f, again in contradiction with the choice of f. Finally suppose that $f = \sum_{i=1}^{k} \lambda_i \bar{x}_i = \bar{0}$. For $k \geq 1$, $\lambda_i \in K \setminus \{0\}$, $x_i \in O_\prec(I)$ then $\sum_{i=1}^{k} \lambda_i x_i \in I$. Let $LT(f) = \lambda_f x_f$, we have that $\lambda_f x_f \in LT(I) \cap O_\prec(I) = \emptyset$ a contradiction. Hence the elements of $O_\prec(I)$ are linearly independent.

Theorem 8.0.3 ([Hun74].)**.** *Let $I \subset KQ$ be two-sided ideal. Then there is one to one correspondence between the set of all two-sided (or one sided) ideals in KQ containing I and the set of all two-sided (or one sided) ideals in $KQ \setminus I$ given by $J \to J \setminus I$.*

Proposition 8.0.4 ([Nor01]). *Let I, J be two-sided ideals of KQ, $I \subset J$. Let G_I be the reduced Gröbner basis for I and G_J a reduced Gröbner basis for J. Then for every $g \in G_J$ either $\bar{g} = Red_I(g)$ or $\bar{g} = g_i \in G_I$.*

Proof. For every $f \in KQ$, either $\bar{f} = Red_I(f)$ or there exist $g \in G_I$ such that $LT(f)$ is a multiple of $LT(g)$. Since $\langle G_I \rangle \subset J$ then $g \in G_J$ and hence there exist $g_i \in G_I$ such that $LT(g)$ is a multiple of $LT(g_i)$. Since $g_i \in J$, then there exist $g_j \in G_J$ such that $LT(g)$ is a multiple of $LT(g_J)$. Now $LT(g_j)$ divides $LT(g)$ and $LT(g_i)$ divides $LT(g)$ and G_J is a reduced, we must have $g_j = g$ and $LT(g) = LT(g_i)$.

Definition 8.0.5. *Let $J \subset KQ$ be a two-sided ideal containing I and $G_J \subseteq J$ be a set of nonzero reduced polynomial modulo I with respect to $\prec$. Then the set G_J is called a Gröbner basis for the two-sided ideal $J \setminus I$, if for every $f \in J \setminus I$ there exist $g \in G_J$ such that $LT(g)$ divides $LT(\bar{f})$ where $\bar{f} = Red_I(f)$.*

Proposition 8.0.6. *Let $J \subset KQ$ be a two sided ideal containing I, and $G_J \subseteq J$ be a set of nonzero reduced polynomial modulo I with respect to $\prec$. Then the following conditions are equivalent:*

a) The set G_J is Gröbner basis of the ideal $(J \setminus I) \subset (KQ \setminus I)$.

b) The set $G_J \cup G_I$ is a Gröbner basis for the ideal J.

c) Every reduced polynomial $f \in J \setminus I$ with respect $\prec$ has a standard representation $f = \sum_{j=1}^{s} \lambda_j x_j g_j x'_j + h$ where $\lambda_j \in K \setminus \{0\}$, x_j, $x'_j \in Mon(KQ)$, $g_j \in G_J$ $h \in I$ such that $LT(f) \succeq LT(x_j g_j x'_j) = x_j LT(g_j) x'_j$ for all $j = \{1, \dots, s\}$.

Proof. i) $a \Rightarrow b$. Let $f \neq 0 \in J$. Then $\bar{f} = Red_I(f) = Red_{G_I}(f)$. If $\bar{f} = 0$ 0r $\bar{f} \neq 0$ and $LT(\bar{f}) \neq LT(f)$ by theorems 7.1.1 and 7.3.1 there exist $g \in G_I$ such that $LT(f)$ is a multiple of $LT(g)$. If $\bar{f} = 0$ 0r $\bar{f} \neq 0$ and $LT(\bar{f}) = LT(f)$ then by definition 8.0.5 there exist $g \in G_I$ such that $LT(f)$ is a multiple of $LT(g)$. Therefore $G_J \cup G_I$ is a Gröbner Basis for J.

ii) $b \Rightarrow a$. Let $f \in J \setminus \{0\}$. We have that $\bar{f} = Red_I(f) \in J$ and there exist a polynomial $g \in G_J \cup G_I$ such that $LT(\bar{f})$ is a multiple of $LT(g)$. Since G_I is a reduced basis for I and $\bar{f}$ is reduced, we must have that $g \notin G_I$. Therefore $g \in G_J$. Hence G_J is a reduced Gröbner basis for $J \setminus I$.

iii) $b \Rightarrow c$, follows from theorem 7.1.1.

In [Ufn89] suggests that one can construct a quiver $U(A)$ referred to as ufnarovskii graph for A. An analogous procedure to that in theorem 7.3.1 may be developed to compute the Gröbner basis for such ideals $J \setminus I$. However it is not clear whether or when the Gröbner basis of any given ideal $J \setminus I \subset KQ \setminus I$ is finite or not.

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