

FOUR ROUTES TO MIXED POISSON DISTRIBUTIONS

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Declaration

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This is my original work and has never been presented for any academic award in any other learning institution.

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Dedication

*For Rodgers
who helped me all the way through
and for Kim, Limo and Sam
who put up with a lot while it was getting done*

Abstract

The objective of this work is to express mixed Poisson distributions in four ways; namely, in explicit form, in terms of special functions, in recursive form and in terms of transforms also called expectation forms.

In explicit form, a gamma function and its properties is used. Posterior distributions and posterior moments are also obtained.

Modified Bessel function of the third kind and confluent hypergeometric function with their properties are used in expressing mixed Poisson distributions in terms of special functions.

Integration by parts is used in determining recursive models for mixed Poisson distributions. To determine the corresponding differential equations for these recursive models, Wang's recursive approach is applied.

Laplace transform and j th moment of a mixing distribution are used to express Poisson mixtures in expectation forms. Factorial moments, moments about the origin and moments about the mean of the Poisson mixtures are determined in terms of probability generating functions of the mixtures. A major bottle-neck in using Laplace transform technique is to obtain its x th derivative.

Determining some mathematical identities based on Poisson mixtures is a major contribution in this research. These identities are obtained by equating results derived using explicit forms and their corresponding method of moments. Identities are also obtained by equating Poisson mixtures expressed in terms of special functions and their corresponding method of moments.

The other major contribution is use of integration by parts in determining recursive models. Other researchers obtained similar results but with certain conditions to be fulfilled. The integration by parts approach does not need these conditions.

In literature, Lindley distribution has been generalized to two parameters. A contribution in this research work is the construction of a three-parameter generalized Lindley distribution which nests the one and two parameter Lindley distributions.

The focus of this research is on constructions and properties of mixed Poisson distributions. For further research, estimations and applications could be pursued. Other approaches to constructing Poisson mixtures could also be identified and pursued.

Abbreviations and Notations

Abbreviations and notations for specific chapters can be found in those chapters. Abbreviations and notations generally used are given below:

cdf	Cumulative distribution function
pdf	Probability density function
pgf	Probability generating function
pmf	Probability mass function
PVF	Power variance function
LHS	Left hand side
RHS	Right hand side
$f(x)$	Probability mass function of a mixed Poisson distribution
$L_\lambda(t)$	Laplace transform of mixing distribution
$g(\lambda)$	Probability density function of a mixing distribution
$G(s)$	Probability generating function of mixed Poisson distribution
$\mu'_r(\lambda)$	r th raw moment of the mixing distribution
$\psi(a, c; x)$	Tricomi confluent hypergeometric function
${}_1F_1(a; c; x)$	Kummer's confluent hypergeometric function
$K_\nu(\omega)$	Modified Bessel function of third kind
$M\{\phi(x), s\}$	Mellin transform

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Chapter 1

GENERAL INTRODUCTION

1.1 Background Information

One major area of Statistics is Probability Distributions: their constructions, properties, estimation of parameters and applications.

There are various methods for constructing these probability distributions. There are those based on power series, transformations, mixtures, recursive relations in probabilities, differential equations, sums of independent random variables, hazard functions, stochastic processes, geometry and trigonometry, Lagrangian expansion, generator approach, special functions, etc.

In this work we wish to mix a Poisson distribution which is discrete with a continuous distribution, resulting in a new distribution known as continuous mixed Poisson distribution (Poisson mixture).

Poisson distribution can be constructed from an exponential power series, from a Poisson process, as a limit of binomial distribution and as a limit of negative binomial distribution.

Historical background of mixed Poisson distributions goes back to 1920 when Greenwood and Yule mixed a Poisson distribution with a gamma distribution to obtain a negative binomial distribution. This work considers many other continuous distributions to be mixed with a Poisson distribution to produce new distributions known as mixed Poisson distributions; which will be expressed in different forms.

1.2 Definitions and Terminologies

Let $f(x)$ be a function of a random variable x . If

$$f(x) \geq 0 \text{ and } \int_{-\infty}^{\infty} f(x) dx = 1$$

then $f(x)$ is called a probability density function (pdf) of a continuous random variable X .

If

$$f(x) \geq 0 \text{ and } \sum_{x=-\infty}^{\infty} f(x) = 1$$

then $f(x)$ is called a probability mass function (pmf) of a discrete random variable X .

Let $(0, t]$ be a fixed interval of time, then a Poisson distribution with parameter λt is denoted as

$$f(x|\lambda) = \frac{e^{-\lambda t} (\lambda t)^x}{x!}, x = 0, 1, 2, \dots; \lambda, t > 0 \quad (1.1)$$

which is a conditional distribution of X given $\Lambda = \lambda$, that is, λ is a value of a continuous random variable Λ whose pdf is $g(\lambda)$ which is called a mixing distribution.

The mixed Poisson distribution (Poisson mixture) is the marginal distribution $f(x)$ defined by

$$\begin{aligned} f(x) &= \int_0^{\infty} f(x|\lambda) g(\lambda) d\lambda \\ &= \int_0^{\infty} \frac{e^{-\lambda t} (\lambda t)^x}{x!} g(\lambda) d\lambda \end{aligned} \quad (1.2)$$

Since the mixing distribution $g(\lambda)$ is continuous, we call $f(x)$ a continuous Poisson mixture.

1.3 Statement of the Problem

Haight (1967), Karlis and Xekalaki (2005) are among those who reviewed works on mixed Poisson distributions. However, classifications based on ways of expressing Poisson mixtures were not examined.

In classifying the Poisson mixtures in this work, a number of issues arose and needed to be addressed. Thus, some of these issues have formed part of the problem statement described below in terms of questions.

- (i) Only a few mixed Poisson distributions are obtained by direct integration. The question therefore is; what are the other mixing distributions to be introduced so as to obtain more mixed Poisson distributions explicitly?
- (ii) Mixed Poisson distributions expressed in recursive forms were constructed under certain conditions such as in Willmot's (1993) approach. Could these conditions be relaxed?
- (iii) Some Poisson mixtures are expressed in more than one form, that is, explicit form, special function form, recursive form and expectation form. The results obtained therefore look different. Can these results be proved to be identical?
- (iv) What are other by-products (such as posterior distribution and posterior moments) of constructing Poisson mixtures?

1.4 Objectives

1.4.1 General Objectives

The main objective is to construct mixed Poisson distributions for various cases of continuous mixing distributions.

1.4.2 Specific Objectives

- (a) To construct mixed Poisson distributions via the following routes:
 - (i) explicit evaluation
 - (ii) use of special functions; (modified Bessel function of third kind and confluent hypergeometric function)
 - (iii) recursively
 - (iv) in expectation forms (transforms)
- (b) To introduce other mixing distributions which have not been considered before. These include, 3-parameter generalized Lindley and transmuted exponential distributions.
- (c) To construct mixed Poisson distributions recursively using Integration by Parts and compare the results obtained using Willmot's (1993) Approach .
- (d) To obtain moments through the various routes as in (a).
- (e) To use the constructed Poisson mixtures in explicit form to obtain Posterior distributions and the posterior moments.
- (f) To obtain identities based on the constructed Poisson mixtures.

1.5 Literature Review

Various mixed Poisson distributions can be constructed depending on the choice of the mixing distribution using several ways such as the explicit evaluation, use of recursive relations, use of the Laplace transforms of the mixing distributions and representing the mixed Poisson distributions in terms of special functions.

Among the first review on this subject was done by Haight (1967) and Karlis and Xekalaki (2005). In this work we shall re-examine what has been done and include the latest works on this subject.

1.5.1 Explicit Forms

Greenwood and Yule (1920) pioneered the derivation of mixed Poisson distributions. They mixed a gamma density and the Poisson distribution resulting in a negative binomial distribution. In this case they did not study the posterior distribution and expectation of the mixed Poisson distribution. Taking Λ to have a shifted gamma (three parameter gamma) distribution, the resulting mixed Poisson distribution is Delaporte distribution; a convolution of negative binomial distribution and Poisson distribution (Ruohonen, 1988). This model is suitable for data having a distribution with a long tail. It would be interesting to obtain other Mixed Poisson distributions which are convolutions. The Poisson-linear exponential distribution is obtained by formally mixing the Poisson distribution with the linear exponential family of distributions (Sankaran, 1970b).

The Poisson distribution is mixed with Lindley distribution resulting in the Poisson-Lindley distribution (Sankaran, 1970a). Two applications to real data suggest that the Poisson-Lindley distribution can be used as an approximation to the negative binomial distribution.

Further, Zakerzadeh and Dolati (2010) generalized the Lindley distribution to obtain a two-parameter generalized Lindley distribution. Taking this distribution as the mixing density, Mahmoudi and Zakerzadeh (2010) obtained a Poisson -generalized Lindley distribution. This distribution is among the latest mixed Poisson distributions which can be expressed explicitly. It is shown that generalized Poisson-Lindley distribution is flexible enough for the analysis of different types of count data.

The most recent mixed Poisson distribution was obtained by Bhati et al (2015) where transmuted exponential distribution was used as a mixing distribution. This mixing distribution is a finite mixture of two exponential distributions. It should be noted that in very few cases, mixed Poisson distributions are expressed explicitly. Hence there is need to find alternatives.

1.5.2 Mixed Poisson Distributions in Recursive Forms

Willmot (1993) devised a method now known as Willmot's Approach to determine mixed Poisson distributions in recursive forms. He obtained recursive formulae for the following mixing distributions:

- (i) Gamma distribution to obtain negative binomial distribution.
- (ii) Generalized inverse Gaussian distribution to obtain the Sichel distribution; Poisson - inverse Gaussian distribution is a special case.
- (iii) Beta distribution to obtain Poisson - beta
- (iv) Generalized Pareto to obtain Poisson - generalized Pareto. Poisson - Pareto is a special case.

- (v) Transformed or generalized gamma
- (vi) Transformed beta
- (vii) Inverse gamma
- (viii) Mixing distributions based on hazard functions
- (ix) Shifted and truncated mixing distributions; shifted gamma to obtain Delaporte distribution, shifted Pareto, truncated gamma, truncated normal.

Gupta and Ong (2005) obtained recursive forms of some Poisson mixtures. Sankaran (1968) obtained a recursive formula for Poisson - inverse Gaussian using differential equation in pgf.

1.5.3 Use of Generating Functions and Laplace Transforms

Probability generating function technique and Laplace transforms have been handy in determining some mixed Poisson distributions. Ruohonen (1988) obtained the Delaporte distribution in terms of a product of the pgfs of negative binomial and Poisson distributions. Gupta and Ong (2005) obtained pgfs for Poisson - generalized gamma and Poisson - generalized exponential distributions.

Power variance function (PVF) distribution is a three parameter family uniting gamma and positive stable distributions. The distribution is denoted by $PVF(\alpha, \delta, \theta)$. The Laplace transform is

$$L(s) = \exp \left\{ -\frac{\delta}{\alpha} [(\theta + s)^\alpha - \theta^\alpha] \right\} \quad (1.3)$$

according to Hougaard et al (1997).

- (i) For $\alpha \rightarrow 0$, the gamma distribution is obtained
- (ii) For $\theta = 0$, the positive stable distribution is obtained
- (iii) For $\alpha = \frac{1}{2}$, the inverse Gaussian distribution is obtained
- (iv) For $\alpha = -1$, the non-central gamma distribution of shape parameter zero is obtained.

The mixed Poisson (Poisson - Power Variance) pmf can be obtained using the formula

$$f(x) = (-1)^x \frac{L^{(x)}(1)}{x!} \quad (1.4)$$

where $L^{(x)}(s)$ denotes the x th derivative of $L(s)$.

Willmot (1986) obtained the Poisson - generalized inverse Gaussian (Sichel) distribution by considering the Laplace transform of generalized inverse Gaussian distribution. He then converted the Laplace into pgf by the relation;

$$G(s) = L_\lambda(1 - s) \quad (1.5)$$

which corresponds to $t = 1$. Hence the pmf as a coefficient of s^x . He also used the pgf to determine the recursive relation. Hougaard et al (1997) obtained $f(x)$ in terms of $L_{\Lambda}^{(x)}(s)$. Willmot(1986) used the relationship between $G(s)$ and $L_{\Lambda}(s)$ to obtain $f(x)$.

Karlis and Xekalaki (2005) gave an alternative useful method which links the probability function of a mixed Poisson distribution to the moments of the mixing distribution as

$$f(x) = \frac{1}{x!} \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} \mu'_{x+r}(\lambda) \quad (1.6)$$

where $\mu'_r(\lambda)$ is the r th raw moment of the mixing distribution.

1.5.4 Mixtures in terms of Special Functions

Some integrals that cannot be evaluated explicitly can be expressed in terms of special functions. Willmot (1993) expressed the pgf of Poisson - scaled beta distribution in terms of a confluent hypergeometric distribution. This same result was obtained by Gurland (1958) by mixing a Poisson with a parameter λp with the classical beta distribution.

1.5.5 Other Cases

Brown and Holgate (1970) found that the Poisson - lognormal distribution cannot be evaluated explicitly. Bulmer (1974) also examined the Poisson - lognormal as a model for species abundance and confirmed that there appears to be no simple form. Thus, Bulmer evaluated the model by numerical integration.

1.6 Methods

The methods used in the construction of the mixed Poisson distributions are:

- (i) Direct integration and substitution
- (ii) Special functions: Beta function, gamma function, Modified Bessel function of the third kind and Confluent hypergeometric function (Kummer's and Tricomi).
- (iii) Integration by parts
- (iv) Transforms: Generating functions, Laplace transforms and Mellin transforms

1.7 Significance of the study

Mixed Poisson distributions can be applied in many fields, such as

- (a) In Actuarial Data: Poisson-inverse Gaussian distribution was used by Tremblay (1992) in No Claims Discount Systems. Klugman et al (1998) used negative binomial distribution to fit data on number of accidents per driver of automobiles. The negative binomial distribution was also used by Greenwood and Yule (1920) for modeling accident proneness among drivers. Ruohonen (1988) considered the Delaporte distribution in modeling number of claims. Delaporte distribution is a mixed distribution of Poisson and truncated gamma which is equivalently a convolution of Poisson and negative binomial distributions - in the current context, it is being used as a mixed distribution. Lemaire (1985) used negative binomial in modeling automobile insurance data. Sankaran (1970a) applied Poisson-Lindley distribution to errors and accidents.
- (b) Health Care: Bhati et al (2015) used Poisson-transmuted exponential distribution to model epileptic seizure counts and compared the results with other Poisson mixtures.
- (c) Biological Sciences: Shanker and Fesshaye (2015) used Poisson-Lindley distribution in modeling some biological data. Bulmer (1974) used Poisson-lognormal as a model for species abundance.

In this study, posterior distributions have been obtained from the mixed Poisson distributions constructed explicitly. Posterior distributions play crucial role in Bayesian statistics and more so Bayesian inference. Mixtures adequately describe heterogeneous populations - an inherent characteristic not exhibited by simple probability distributions. This study has also made use of generalized distributions nesting other distributions. Such generalized distributions are: generalized inverse Gaussian, 3-parameter generalized Lindley and transmuted exponential. The focus in probability distributions is on generalized distributions due to their flexibility.

Mathematical identities have been deduced as a result of expressing mixed Poisson distributions in more than one route, that is explicitly, in special function form and in expectation form.

1.8 Outline of the thesis

The rest of the thesis is outlined as follows: In Chapter 2, mixed Poisson distributions have been constructed in explicit forms and their moments obtained by conditional expectation approach. In Chapter 3, mixed Poisson distributions have been expressed in terms of special functions, specifically, confluent hypergeometric functions and modified Bessel function of the third kind. Recursive relations for mixed Poisson distributions have been obtained in Chapter 4. This has been achieved through integration by parts. Differential equations for the mixed Poisson distributions are also obtained. In Chapter 5, mixed Poisson distributions and their moments are derived in terms of transforms. Specifically, Laplace transform, Mellin transform and probability generating function.

Mathematical identities based on Poisson mixtures are also deduced by equating result of mixture obtained in explicit form with that obtained by method of moments and equating result of mixture obtained in terms of special function with that obtained by the method of moments. Chapter 6 gives the summary, conclusions and recommendations of the study.

Chapter 2

MIXED POISSON DISTRIBUTIONS AND THEIR MOMENTS IN EXPLICIT FORMS

2.1 Introduction

In this chapter we consider mixed Poisson distributions in explicit form by direct integration. The moments are also obtained by conditional expectation approach. We shall specifically derive formulae for the first four moments about the origin (raw moments) and moments about the mean (central moments) of the Poisson mixtures in terms of moments of the mixing distributions. The posterior r th moment is also derived.

The following mixing distributions are used: gamma, shifted gamma, Lindley, 3-parameter generalized Lindley and transmuted exponential distribution. Some examples are given.

2.2 Mathematical Formulation of the Problem

A mixed Poisson distribution is defined by equation (1.2). The r th raw moment of the Poisson mixture is

$$\begin{aligned}\mathbb{E}(X^r) &= \mathbb{E}\mathbb{E}(X^r | \Lambda) \\ &= \mathbb{E}\left\{e^{-\Lambda t} \sum_{x=0}^{\infty} \frac{x^r (\Lambda t)^x}{x!}\right\}.\end{aligned}\tag{2.1}$$

The posterior distribution is defined by

$$\begin{aligned}
g(\lambda | x) &= \frac{f(x | \lambda) g(\lambda)}{\int_0^\infty f(x | \lambda) g(\lambda) d\lambda} \\
&= \frac{e^{-\lambda t} (\lambda t)^x g(\lambda)}{\int_0^\infty e^{-\lambda t} (\lambda t)^x g(\lambda) d\lambda} \\
&= \frac{\lambda^x e^{-\lambda t} g(\lambda)}{\int_0^\infty \lambda^x e^{-\lambda t} g(\lambda) d\lambda}
\end{aligned} \tag{2.2}$$

Moments about the origin (raw moments) and moments about the mean (central moments) of a mixed Poisson distribution in terms of moments of the mixing distribution are as follows:

Proposition 2.2.1. *The raw moments are:*

(i)

$$\mathbb{E}(X) = t\mathbb{E}(\Lambda) \tag{2.3}$$

(ii)

$$\mathbb{E}(X^2) = t^2\mathbb{E}(\Lambda^2) + t\mathbb{E}(\Lambda) \tag{2.4}$$

(iii)

$$\mathbb{E}(X^3) = t^3\mathbb{E}(\Lambda^3) + 3t^2\mathbb{E}(\Lambda^2) + t\mathbb{E}(\Lambda) \tag{2.5}$$

(iv)

$$\mathbb{E}(X^4) = t^4\mathbb{E}(\Lambda^4) + 6t^3\mathbb{E}(\Lambda^3) + 7t^2\mathbb{E}(\Lambda^2) + t\mathbb{E}(\Lambda) \tag{2.6}$$

Proof. (i) The first raw moment is obtained as

$$\begin{aligned}
\mathbb{E}(X) &= \mathbb{E} \left\{ e^{-\Lambda t} \sum_{x=0}^{\infty} \frac{x (\Lambda t)^x}{x!} \right\} \\
&= \mathbb{E} \left\{ e^{-\Lambda t} (\Lambda t) \sum_{x=1}^{\infty} \frac{(\Lambda t)^{x-1}}{(x-1)!} \right\} \\
&= \mathbb{E} \{ e^{-\Lambda t} (\Lambda t) e^{\Lambda t} \} \\
&= t\mathbb{E}(\Lambda)
\end{aligned}$$

or simply

$$\mathbb{E}(X) = \mathbb{E}\mathbb{E}(X | \Lambda) = \mathbb{E}(\Lambda t) = t\mathbb{E}(\Lambda)$$

which is (2.3).

(ii) The second raw moment is

$$\begin{aligned}
\mathbb{E}(X^2) &= \mathbb{E}\left\{e^{-\Lambda t} \sum_{x=0}^{\infty} \frac{x^2 (\Lambda t)^x}{x!}\right\} \\
&= \mathbb{E}\left\{e^{-\Lambda t} \sum_{x=1}^{\infty} \frac{(x-1+1)(\Lambda t)^x}{(x-1)!}\right\} \\
&= \mathbb{E}\left\{e^{-\Lambda t} (\Lambda t)^2 \sum_{x=2}^{\infty} \frac{(\Lambda t)^{x-2}}{(x-2)!} + e^{-\Lambda t} (\Lambda t) \sum_{x=1}^{\infty} \frac{(\Lambda t)^{x-1}}{(x-1)!}\right\} \\
&= \mathbb{E}\left\{(\Lambda t)^2 + \Lambda t\right\} \\
&= t^2 \mathbb{E}(\Lambda^2) + t \mathbb{E}(\Lambda)
\end{aligned}$$

which is (2.4).

(iii) The third raw moment is obtained as

$$\begin{aligned}
\mathbb{E}(X^3) &= \mathbb{E}\left\{e^{-\Lambda t} \sum_{x=0}^{\infty} \frac{x^3 (\Lambda t)^x}{x!}\right\} \\
&= \mathbb{E}\left\{e^{-\Lambda t} \sum_{x=1}^{\infty} \frac{(x-1+1)^2 (\Lambda t)^x}{(x-1)!}\right\} \\
&= \mathbb{E}\left\{e^{-\Lambda t} \sum_{x=1}^{\infty} \frac{[(x-1)^2 + 2(x-1) + 1] (\Lambda t)^x}{(x-1)!}\right\} \\
&= \mathbb{E}\left\{e^{-\Lambda t} \left[\sum_{x=2}^{\infty} \frac{(x-2) + 3}{(x-2)!} + \sum_{x=1}^{\infty} \frac{1}{(x-1)!}\right] (\Lambda t)^x\right\} \\
&= \mathbb{E}\left\{e^{-\Lambda t} \left[\sum_{x=3}^{\infty} \frac{1}{(x-3)!} + \sum_{x=2}^{\infty} \frac{3}{(x-2)!} + \sum_{x=1}^{\infty} \frac{1}{(x-1)!}\right] (\Lambda t)^x\right\} \\
&= \mathbb{E}\left\{e^{-\Lambda t} \left[(\Lambda t)^3 \sum_{x=3}^{\infty} \frac{(\Lambda t)^{x-3}}{(x-3)!} + 3(\Lambda t)^2 \sum_{x=2}^{\infty} \frac{(\Lambda t)^{x-2}}{(x-2)!} + (\Lambda t) \sum_{x=1}^{\infty} \frac{(\Lambda t)^{x-1}}{(x-1)!}\right]\right\} \\
&= \mathbb{E}\left\{(\Lambda t)^3 + 3(\Lambda t)^2 + (\Lambda t)\right\} \\
&= t^3 \mathbb{E}(\Lambda^3) + 3t^2 \mathbb{E}(\Lambda^2) + t \mathbb{E}(\Lambda)
\end{aligned}$$

which is (2.5).

(iv) The fourth raw moment is

$$\begin{aligned}
\mathbb{E}(X^4) &= \mathbb{E} \left\{ e^{-\Lambda t} \sum_{x=0}^{\infty} \frac{x^4 (\Lambda t)^x}{x!} \right\} \\
&= \mathbb{E} \left\{ e^{-\Lambda t} \sum_{x=1}^{\infty} \frac{(x-1+1)^3 (\Lambda t)^x}{(x-1)!} \right\} \\
&= \mathbb{E} \left\{ e^{-\Lambda t} \sum_{x=1}^{\infty} \left[(x-1)^3 + 3(x-1)^2 + 3(x-1) + 1 \right] \frac{(\Lambda t)^x}{(x-1)!} \right\} \\
&= \mathbb{E} \left\{ e^{-\Lambda t} \left[\sum_{x=2}^{\infty} \frac{(x-2+1)^2 + 3(x-2+1) + 3}{(x-2)!} + \sum_{x=1}^{\infty} \frac{1}{(x-1)!} \right] (\Lambda t)^x \right\} \\
&= \mathbb{E} \left\{ e^{-\Lambda t} \left[\sum_{x=4}^{\infty} \frac{1}{(x-4)!} + \sum_{x=3}^{\infty} \frac{6}{(x-3)!} + \sum_{x=2}^{\infty} \frac{7}{(x-2)!} + \sum_{x=1}^{\infty} \frac{1}{(x-1)!} \right] (\Lambda t)^x \right\} \\
&= t^4 \mathbb{E}(\Lambda^4) + 6t^3 \mathbb{E}(\Lambda^3) + 7t^2 \mathbb{E}(\Lambda^2) + t \mathbb{E}(\Lambda)
\end{aligned}$$

which is (2.6). □

Proposition 2.2.2. *The central moments are as follows:*

(i) *Variance*

$$\mu_2 = t^2 \text{Var}(\Lambda) + t \mathbb{E}(\Lambda) \quad (2.7)$$

(ii) *Third Moment*

$$\mu_3 = t^3 \mathbb{E}[\Lambda - \mathbb{E}(\Lambda)]^3 + 3t^2 \text{Var}(\Lambda) + t \mathbb{E}(\Lambda) \quad (2.8)$$

(iii) *Fourth Moment*

$$\begin{aligned}
\mu_4 &= t^4 \mathbb{E}[\Lambda - \mathbb{E}(\Lambda)]^4 + 6t^3 \left\{ \mathbb{E}[\Lambda - \mathbb{E}(\Lambda)]^3 + \text{Var}(\Lambda) \mathbb{E}(\Lambda) \right\} \\
&\quad + t^2 \left\{ 7 \text{Var}(\Lambda) + 3 [\mathbb{E}(\Lambda)]^2 \right\} + t \mathbb{E}(\Lambda)
\end{aligned} \quad (2.9)$$

Proof. (i) The variance is

$$\begin{aligned}
\mu_2 &= \text{Var}(X) \\
&= \mathbb{E}(X^2) - [\mathbb{E}(X)]^2 \\
&= t^2 \mathbb{E}(\Lambda^2) + t \mathbb{E}(\Lambda) - t^2 [\mathbb{E}(\Lambda)]^2 \\
&= t^2 \text{Var}(\Lambda) + t \mathbb{E}(\Lambda)
\end{aligned}$$

Alternatively,

$$\begin{aligned}
\mu_2 &= \text{Var}(X) \\
&= \text{Var}[\mathbb{E}(X | \Lambda)] + \mathbb{E}[\text{Var}(X | \Lambda)] \\
&= \text{Var}(t\Lambda) + \mathbb{E}(t\Lambda) \\
&= t^2\text{Var}(\Lambda) + t\mathbb{E}(\Lambda)
\end{aligned}$$

which is (2.7).

(ii) The third central moment is

$$\begin{aligned}
\mu_3 &= \mathbb{E}[X - \mathbb{E}(X)]^3 \\
&= \mathbb{E}(X^3) - 3\mathbb{E}(X^2)\mathbb{E}(X) + 2[\mathbb{E}(X)]^3 \\
&= t^3\mathbb{E}(\Lambda^3) + 3t^2\mathbb{E}(\Lambda^2) + t\mathbb{E}(\Lambda) - 3t^3\mathbb{E}(\Lambda^2)\mathbb{E}(\Lambda) - 3t^2[\mathbb{E}(\Lambda)]^2 + t^3[\mathbb{E}(\Lambda)]^3 \\
&= t^3\mathbb{E}[\Lambda - \mathbb{E}(\Lambda)]^3 + 3t^2\text{Var}(\Lambda) + t\mathbb{E}(\Lambda)
\end{aligned}$$

which is (2.8).

(iii) The fourth central moment is

$$\begin{aligned}
\mu_4 &= \mathbb{E}[X - \mathbb{E}(X)]^4 \\
&= \mathbb{E}(X^4) - 4\mathbb{E}(X^3)\mathbb{E}(X) + 6\mathbb{E}(X^2)[\mathbb{E}(X)]^2 - 3[\mathbb{E}(X)]^4 \\
&= t^4 \left\{ \mathbb{E}(\Lambda^4) - 4\mathbb{E}(\Lambda^3)\mathbb{E}(\Lambda) + 6\mathbb{E}(\Lambda^2)[\mathbb{E}(\Lambda)]^2 - 3[\mathbb{E}(\Lambda)]^4 \right\} \\
&\quad + 6t^3 \left\{ \mathbb{E}(\Lambda^3) - 2\mathbb{E}(\Lambda^2)\mathbb{E}(\Lambda) + [\mathbb{E}(\Lambda)]^3 \right\} \\
&\quad + t^2 \left\{ 7\mathbb{E}(\Lambda^2) - 4[\mathbb{E}(\Lambda)]^2 \right\} + t\mathbb{E}(\Lambda) \\
&= t^4\mathbb{E}[\Lambda - \mathbb{E}(\Lambda)]^4 + 6t^3 \left\{ \mathbb{E}[\Lambda - \mathbb{E}(\Lambda)]^3 + [\mathbb{E}(\Lambda^2) - (\mathbb{E}(\Lambda))^2]\mathbb{E}(\Lambda) \right\} \\
&\quad + t^2 \left\{ 7\text{Var}(\Lambda) + 3[\mathbb{E}(\Lambda)]^2 \right\} + t\mathbb{E}(\Lambda) \\
&= t^4\mathbb{E}[\Lambda - \mathbb{E}(\Lambda)]^4 + 6t^3 \left\{ \mathbb{E}[\Lambda - \mathbb{E}(\Lambda)]^3 + \text{Var}(\Lambda)\mathbb{E}(\Lambda) \right\} \\
&\quad + t^2 \left\{ 7\text{Var}(\Lambda) + 3[\mathbb{E}(\Lambda)]^2 \right\} + t\mathbb{E}(\Lambda)
\end{aligned}$$

which is (2.9). □

Proposition 2.2.3. *The posterior r th moment is*

$$\mathbb{E}(\Lambda^r | X = x) = \frac{\mathbb{E}[\Lambda^{x+r}e^{-\Lambda t}]}{\mathbb{E}[\Lambda^x e^{-\Lambda t}]} \tag{2.10}$$

and in particular the posterior mean is

$$\mathbb{E}(\Lambda | X = x) = \frac{\mathbb{E}[\Lambda^{x+1}e^{-\Lambda t}]}{\mathbb{E}[\Lambda^x e^{-\Lambda t}]} \tag{2.11}$$

Alternatively,

$$\mathbb{E}(\Lambda^r | X = x) = \frac{(x+r)! f(x+r)}{t^r x! f(x)} \quad (2.12)$$

and

$$\mathbb{E}(\Lambda | X = x) = \frac{(x+1) f(x+1)}{t f(x)}. \quad (2.13)$$

Proof. The posterior r th moment is

$$\begin{aligned} \mathbb{E}(\Lambda^r | X = x) &= \frac{\int_0^\infty \lambda^{x+r} e^{-\lambda t} g(\lambda) d\lambda}{\int_0^\infty \lambda^x e^{-\lambda t} g(\lambda) d\lambda} \\ &= \frac{\mathbb{E}[\Lambda^{x+r} e^{-\Lambda t}]}{\mathbb{E}[\Lambda^x e^{-\Lambda t}]} \end{aligned}$$

which is (2.10) and in particular the posterior mean is

$$\mathbb{E}(\Lambda | X = x) = \frac{\mathbb{E}[\Lambda^{x+1} e^{-\Lambda t}]}{\mathbb{E}[\Lambda^x e^{-\Lambda t}]}$$

which is (2.11).

Alternatively,

$$g(\lambda | x) = \frac{e^{-\lambda t} (\lambda t)^x g(\lambda)}{x! f(x)}. \quad (2.14)$$

The posterior r th moment is

$$\begin{aligned} \mathbb{E}(\Lambda^r | X = x) &= \int_0^\infty \lambda^r g(\lambda | x) d\lambda \\ &= \frac{1}{x! f(x)} \int_0^\infty \lambda^r e^{-\lambda t} (\lambda t)^x g(\lambda) d\lambda \\ &= \frac{(x+r)!}{t^r x! f(x)} \int_0^\infty \frac{e^{-\lambda t} (\lambda t)^{x+r}}{(x+r)!} g(\lambda) d\lambda \\ &= \frac{(x+r)! f(x+r)}{t^r x! f(x)} \end{aligned}$$

which is (2.12) and in particular the posterior mean is

$$\mathbb{E}(\Lambda | X = x) = \frac{(x+1) f(x+1)}{t f(x)}$$

which is (2.13). □

2.3 Examples of Mixed Poisson Distributions and Their Moments in Explicit Forms

2.3.1 Poisson-Gamma Distribution

A two - parameter gamma distribution is

$$g(\lambda) = \frac{\beta^\alpha}{\Gamma(\alpha)} e^{-\beta\lambda} \lambda^{\alpha-1}, \quad \lambda > 0; \alpha, \beta > 0 \quad (2.15)$$

Proposition 2.3.1. (a) *The mixed Poisson distribution is*

$$f(x) = \binom{\alpha + x - 1}{x} \left(\frac{t}{t + \beta}\right)^x \left(\frac{\beta}{t + \beta}\right)^\alpha, x = 0, 1, 2, \dots \quad (2.16)$$

which is the negative binomial distribution with parameters α and β (Greenwood and Yule, 1920).

(b) *The r th moment of gamma distribution is*

$$\mathbb{E}(\Lambda^r) = \frac{\Gamma(\alpha + r)}{\Gamma(\alpha) \beta^r} \quad (2.17)$$

where Γ is the gamma function. Therefore

$$\mathbb{E}(\Lambda) = \frac{\alpha}{\beta} \quad (2.18)$$

$$\text{Var}(\Lambda) = \frac{\alpha}{\beta^2} \quad (2.19)$$

$$\mathbb{E}[\Lambda - \mathbb{E}(\Lambda)]^3 = \frac{2\alpha}{\beta^3} \quad (2.20)$$

$$\mathbb{E}[\Lambda - \mathbb{E}(\Lambda)]^4 = \frac{3\alpha(\alpha + 2)}{\beta^4}. \quad (2.21)$$

(c) *Moments of the mixture (negative binomial distribution)*

$$\mathbb{E}(X) = \frac{\alpha}{\beta} t \quad (2.22)$$

$$\text{Var}(X) = \frac{\alpha}{\beta^2} t^2 + \frac{\alpha}{\beta} t \quad (2.23)$$

$$\begin{aligned} \mu_3 &= \mathbb{E}[X - \mathbb{E}(X)]^3 \\ &= \frac{2\alpha}{\beta^3} t^3 + \frac{3\alpha}{\beta^2} t^2 + \frac{\alpha}{\beta} t \end{aligned} \quad (2.24)$$

$$\begin{aligned} \mu_4 &= \mathbb{E}[X - \mathbb{E}(X)]^4 \\ &= \frac{3\alpha(\alpha + 2)}{\beta^4} t^4 + \frac{6(2 - \alpha)\alpha}{\beta^3} t^3 + \frac{(7 + 3\alpha)\alpha}{\beta^2} t^2 + \frac{\alpha}{\beta} t \end{aligned} \quad (2.25)$$

(d) *Posterior distribution is*

$$g(\lambda|x) = \frac{(t + \beta)^{\alpha+x}}{\Gamma(\alpha + x)} e^{-(t+\beta)\lambda} \lambda^{\alpha+x-1}, \lambda > 0; \alpha, \beta > 0 \quad (2.26)$$

which is Gamma($\alpha + x, t + \beta$).

The posterior r th moment is

$$\mathbb{E}(\Lambda^r | x) = \frac{\Gamma(x + \alpha + r)}{\Gamma(x + \alpha) (t + \beta)^r} \quad (2.27)$$

and in particular the posterior mean is

$$\mathbb{E}(\Lambda | x) = \frac{x + \alpha}{t + \beta} \quad (2.28)$$

which is a linear function of x .

Proof. From equation (2.15) the mixed Poisson distribution is

$$\begin{aligned}
f(x) &= \int_0^\infty \frac{e^{-\lambda t} (\lambda t)^x}{x!} \frac{\beta^\alpha}{\Gamma(\alpha)} e^{-\beta \lambda} \lambda^{\alpha-1} d\lambda \\
&= \frac{\beta^\alpha t^x}{\Gamma(\alpha) x!} \int_0^\infty \lambda^{x+\alpha-1} e^{-\lambda(t+\beta)} d\lambda \\
&= \frac{\beta^\alpha}{\Gamma(\alpha)} \frac{t^x}{x!} \frac{\Gamma(x+\alpha)}{(t+\beta)^{x+\alpha}} \\
&= \binom{x+\alpha-1}{x} \left(\frac{t}{t+\beta} \right)^x \left(\frac{\beta}{t+\beta} \right)^\alpha; \quad x = 0, 1, 2, \dots
\end{aligned}$$

which is (2.16). Its pgf is,

$$\begin{aligned}
G(s) &= \int_0^\infty e^{-\lambda t(1-s)} \frac{\beta^\alpha}{\Gamma(\alpha)} e^{-\beta \lambda} \lambda^{\alpha-1} d\lambda \\
&= \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^\infty \lambda^{\alpha-1} e^{-\lambda(t-ts+\beta)} d\lambda \\
&= \frac{\beta^\alpha}{\Gamma(\alpha)} \frac{\Gamma(\alpha)}{(t-ts+\beta)^\alpha} \\
&= \left[\frac{\beta}{t+\beta} \right]^\alpha \left[1 - \frac{t}{t+\beta} s \right]^{-\alpha}.
\end{aligned} \tag{2.29}$$

The posterior distribution is

$$\begin{aligned}
g(\lambda|x) &= \frac{f(x|\lambda) g(\lambda)}{f(x)} \\
&= \frac{e^{-\lambda t} (\lambda t)^x \beta^\alpha}{x! \Gamma(\alpha)} e^{-\beta \lambda} \lambda^{\alpha-1} \\
&= \frac{\binom{x+\alpha-1}{x} \left(\frac{t}{t+\beta} \right)^x \left(\frac{\beta}{t+\beta} \right)^\alpha}{\binom{x+\alpha-1}{x} \left(\frac{t}{t+\beta} \right)^x \left(\frac{\beta}{t+\beta} \right)^\alpha} \\
&= \frac{e^{-\lambda t} (\lambda t)^x \beta^\alpha e^{-\beta \lambda} \lambda^{\alpha-1}}{\frac{x! \Gamma(\alpha) \Gamma(\alpha+x) t^x \beta^\alpha}{x! \Gamma(\alpha) (t+\beta)^{\alpha+x}}} \\
&= \frac{(t+\beta)^{\alpha+x}}{\Gamma(\alpha+x)} e^{-(t+\beta)\lambda} \lambda^{\alpha+x-1}, \quad \lambda > 0; \alpha, \beta > 0
\end{aligned} \tag{2.30}$$

which is *Gamma* $(\alpha+x, t+\beta)$ and the posterior mean is

$$\begin{aligned}
\mathbb{E}(\Lambda | x) &= \int_0^\infty \lambda \frac{(t+\beta)^{\alpha+x}}{\Gamma(\alpha+x)} e^{-(t+\beta)\lambda} \lambda^{\alpha+x-1} d\lambda \\
&= \frac{(t+\beta)^{\alpha+x}}{\Gamma(\alpha+x)} \int_0^\infty \lambda^{(\alpha+x+1)-1} e^{-(t+\beta)\lambda} d\lambda \\
&= \frac{(t+\beta)^{\alpha+x}}{\Gamma(\alpha+x)} \frac{\Gamma(\alpha+x+1)}{(t+\beta)^{\alpha+x+1}} \\
&= \frac{\alpha+x}{t+\beta}.
\end{aligned} \tag{2.31}$$

□

2.3.2 Poisson-Shifted Gamma Distribution

Consider the shifted gamma distribution

$$g(\lambda) = \frac{\beta^\alpha}{\Gamma(\alpha)} e^{-\beta(\lambda-\mu)} (\lambda-\mu)^{\alpha-1}, \lambda > \mu; \mu, \alpha, \beta > 0 \quad (2.32)$$

where μ is the shift parameter.

Proposition 2.3.2. (a) *The mixed Poisson distribution is*

$$f(x) = \sum_{k=0}^x \left[\frac{e^{-\mu t} (\mu t)^{x-k}}{(x-k)!} \right] \binom{k+\alpha-1}{k} \left(\frac{t}{t+\beta} \right)^k \left(\frac{\beta}{t+\beta} \right)^\alpha \quad (2.33)$$

which is a convolution of Poisson distribution and a negative binomial distribution. It is called Delaporte distribution (Ruohonen, 1988).

(b) *The r th moment of the mixing distribution is*

$$\mathbb{E}(\Lambda^r) = \sum_{k=0}^r \frac{r! \mu^{r-k}}{(r-k)!} \binom{\alpha+k-1}{k} \frac{1}{\beta^k} \quad (2.34)$$

and in particular,

$$\mathbb{E}(\Lambda) = \mu + \frac{\alpha}{\beta} \quad (2.35)$$

$$\mathbb{E}(\Lambda^2) = \mu^2 + \frac{2\mu\alpha}{\beta} + \frac{\alpha(\alpha+1)}{\beta^2}$$

$$\mathbb{E}(\Lambda^3) = \mu^3 + \frac{3\alpha\mu^2}{\beta} + \frac{3\alpha(\alpha+1)\mu}{\beta^2} + \frac{\alpha(\alpha+1)(\alpha+2)}{\beta^3}$$

$$\begin{aligned} \mathbb{E}(\Lambda^4) &= \mu^4 + \frac{4\alpha\mu^3}{\beta} + \frac{6\alpha(\alpha+1)\mu^2}{\beta^2} + \frac{4\alpha(\alpha+1)(\alpha+2)\mu}{\beta^3} \\ &\quad + \frac{\alpha(\alpha+1)(\alpha+2)(\alpha+3)}{\beta^4} \end{aligned}$$

so that

$$\text{Var}(\Lambda) = \frac{\alpha}{\beta^2} \quad (2.36)$$

$$\mathbb{E}[\Lambda - \mathbb{E}(\Lambda)]^3 = \frac{2\alpha}{\beta^3} \quad (2.37)$$

$$\mathbb{E}[\Lambda - \mathbb{E}(\Lambda)]^4 = \frac{3\alpha(\alpha+2)}{\beta^4}. \quad (2.38)$$

(c) *Moments of the Delaporte distribution are*

$$\mathbb{E}(X) = \left(\mu + \frac{\alpha}{\beta} \right) t \quad (2.39)$$

$$\text{Var}(X) = \frac{\alpha}{\beta^2} t^2 + \left(\mu + \frac{\alpha}{\beta} \right) t \quad (2.40)$$

$$\begin{aligned}
\mu_3 &= \mathbb{E}[X - \mathbb{E}(X)]^3 \\
&= \frac{2\alpha}{\beta^3}t^3 + \frac{3\alpha}{\beta^2}t^2 + \left(\mu + \frac{\alpha}{\beta}\right)t
\end{aligned} \tag{2.41}$$

$$\begin{aligned}
\mu_4 &= \mathbb{E}[X - \mathbb{E}(X)]^4 \\
&= \frac{3\alpha(\alpha+2)}{\beta^4}t^4 + 6 \left[2\alpha - \left(\mu + \frac{\alpha}{\beta}\right)^3\right]t^3 \\
&\quad + \left[7\alpha + 3\left(\mu + \frac{\alpha}{\beta}\right)^2\right] \frac{t^2}{\beta^2} + \left(\mu + \frac{\alpha}{\beta}\right)t.
\end{aligned} \tag{2.42}$$

Remark: If $\mu = 0$, we obtain the results of the Poisson-Gamma distribution.

(d) The posterior distribution is

$$g(\lambda | x) = \frac{\sum_{k=0}^x \binom{x}{k} \mu^{x-k} (\lambda - \mu)^{k+\alpha-1} e^{-(t+\beta)(\lambda-\mu)}}{\sum_{k=0}^x \binom{x}{k} \mu^{x-k} \Gamma(k+\alpha) (t+\beta)^{-(k+\alpha)}} \tag{2.43}$$

The posterior r th moment is

$$\mathbb{E}(\Lambda^r | x) = \frac{\sum_{k=0}^x \left\{ \binom{x}{k} \mu^{x-k} \sum_{j=0}^r \binom{r}{j} \mu^{r-j} \Gamma(j+k+\alpha) (t+\beta)^{-(j+k+\alpha)} \right\}}{\sum_{k=0}^x \binom{x}{k} \mu^{x-k} \Gamma(k+\alpha) (t+\beta)^{-(k+\alpha)}} \tag{2.44}$$

and in particular, the posterior mean is

$$\mathbb{E}(\Lambda | x) = \frac{\sum_{k=0}^{x-1} \binom{x}{k} \mu^{x-k} \frac{\Gamma(k+\alpha)}{(t+\beta)^{k+\alpha}} \left(\mu + \frac{k+\alpha}{t+\beta}\right) + \frac{\Gamma(x+\alpha)}{(t+\beta)^{x+\alpha}} \left(\mu + \frac{x+\alpha}{t+\beta}\right)}{\sum_{k=0}^{x-1} \binom{x}{k} \mu^{x-k} \frac{\Gamma(k+\alpha)}{(t+\beta)^{k+\alpha}} + \frac{\Gamma(x+\alpha)}{(t+\beta)^{x+\alpha}}} \tag{2.45}$$

Proof. From equation (2.32) the mixed Poisson distribution is

$$\begin{aligned}
f(x) &= \int_{\mu}^{\infty} \frac{e^{-\lambda t} (\lambda t)^x}{x!} \frac{\beta^{\alpha}}{\Gamma(\alpha)} e^{-\beta(\lambda-\mu)} (\lambda - \mu)^{\alpha-1} d\lambda \\
&= \frac{t^x}{x!} \frac{\beta^{\alpha}}{\Gamma(\alpha)} \int_{\mu}^{\infty} e^{-\lambda t} \lambda^x e^{-\beta(\lambda-\mu)} (\lambda - \mu)^{\alpha-1} d\lambda
\end{aligned}$$

and making the substitution $z = \lambda - \mu$, we obtain

$$\begin{aligned}
f(x) &= \frac{t^x}{x!} \frac{\beta^{\alpha}}{\Gamma(\alpha)} e^{-\mu t} \int_0^{\infty} (z + \mu)^x z^{\alpha-1} e^{-(t+\beta)z} dz \\
&= \frac{t^x}{x!} \frac{\beta^{\alpha}}{\Gamma(\alpha)} e^{-\mu t} \sum_{k=0}^x \left\{ \binom{x}{k} \mu^{x-k} \int_0^{\infty} z^{k+\alpha-1} e^{-(t+\beta)z} dz \right\} \\
&= \frac{t^x}{x!} \frac{\beta^{\alpha}}{\Gamma(\alpha)} e^{-\mu t} \sum_{k=0}^x \binom{x}{k} \mu^{x-k} \frac{\Gamma(k+\alpha)}{(t+\beta)^{k+\alpha}} \\
&= e^{-\mu t} \sum_{k=0}^x \binom{x}{k} \frac{\Gamma(k+\alpha)}{x! \Gamma(\alpha)} (\mu t)^{x-k} \left(\frac{t}{t+\beta}\right)^k \left(\frac{\beta}{t+\beta}\right)^{\alpha} \\
&= e^{-\mu t} \left(\frac{\beta}{t+\beta}\right)^{\alpha} \sum_{k=0}^x \frac{\Gamma(k+\alpha)}{k! (x-k)! \Gamma(\alpha)} (\mu t)^{x-k} \left(\frac{t}{t+\beta}\right)^k \\
&= \sum_{k=0}^x \left[\frac{e^{-\mu t} (\mu t)^{x-k}}{(x-k)!} \right] \binom{k+\alpha-1}{k} \left(\frac{t}{t+\beta}\right)^k \left(\frac{\beta}{t+\beta}\right)^{\alpha}
\end{aligned}$$

which is (2.33).

The pgf of Delaporte distribution is,

$$\begin{aligned} G(s) &= \frac{\beta^\alpha}{\Gamma(\alpha)} \sum_{x=0}^{\infty} \frac{1}{x!} \int_{\mu}^{\infty} e^{-\lambda t} (\lambda t s)^x e^{-\beta(\lambda-\mu)} (\lambda-\mu)^{\alpha-1} d\lambda \\ &= \frac{\beta^\alpha}{\Gamma(\alpha)} \int_{\mu}^{\infty} (\lambda-\mu)^{\alpha-1} e^{-\lambda t(1-s)} e^{-\beta(\lambda-\mu)} d\lambda \end{aligned}$$

and making the substitution $z = \lambda - \mu$, we obtain

$$\begin{aligned} G(s) &= \frac{\beta^\alpha}{\Gamma(\alpha)} e^{-\mu t(1-s)} \int_0^{\infty} z^{\alpha-1} e^{-[\beta+t-ts]z} dz \\ &= \frac{\beta^\alpha}{\Gamma(\alpha)} e^{-\mu t(1-s)} \frac{\Gamma(\alpha)}{(\beta+t-ts)^\alpha} \\ &= e^{-\mu t(1-s)} \left[\frac{\beta}{t+\beta-ts} \right]^\alpha \end{aligned} \tag{2.46}$$

which is the product of the pgf of Poisson (μt) and pgf of negative binomial (α, β) and hence Delaporte distribution is a convolution of Poisson and negative binomial distributions. \square

2.3.3 Poisson-Transmuted Exponential Distribution

In general, a transmuted probability is defined as

$$F(x) = (1 + \nu) H(x) - \nu [H(x)]^2, \quad -1 \leq \nu \leq 1 \tag{2.47}$$

where $H(x)$ and $F(x)$ are old and new cdfs, respectively. If $h(x)$ and $f(x)$ are the corresponding pdfs, then by differentiation

$$f(x) = (1 + \nu) h(x) - 2\nu H(x) h(x). \tag{2.48}$$

Given an exponential pdf,

$$h(x) = \theta e^{-\theta x}, \quad x > 0; \theta > 0 \tag{2.49}$$

then

$$F(x) = (1 + \nu) [1 - e^{-\theta x}] - \nu [1 - e^{-\theta x}]^2 \tag{2.50}$$

and

$$\begin{aligned} f(x) &= (1 + \nu) \theta e^{-\theta x} - 2\nu [1 - e^{-\theta x}] \theta e^{-\theta x} \\ &= (1 + \nu) \theta e^{-\theta x} - 2\nu \theta e^{-\theta x} + 2\nu \theta e^{-2\theta x} \\ &= (1 - \nu) \theta e^{-\theta x} + 2\nu \theta e^{-2\theta x} \\ &= (1 - \nu) \theta e^{-\theta x} + \nu (2\theta e^{-2\theta x}) \quad x > 0; \theta > 0. \end{aligned} \tag{2.51}$$

Let $\nu = \alpha$ then

$$f(x) = (1 - \alpha) \theta e^{-\theta x} + 2\alpha \theta e^{-2\theta x} \tag{2.52}$$

which is a transmuted exponential distribution (Bhati et al, 2015). This is a finite mixture of an exponential distribution with parameter θ with another exponential distribution with parameter 2θ . We shall thus denote the transmuted exponential mixing distribution as

$$g(\lambda) = (1 - \alpha)\theta e^{-\theta\lambda} + 2\alpha\theta e^{-2\theta\lambda}, \lambda > 0; \theta, \alpha > 0 \quad (2.53)$$

Proposition 2.3.3. (a) *The Poisson-transmuted exponential distribution is*

$$f(x) = (1 - \alpha) \left(\frac{t}{t + \theta}\right)^x \left(\frac{\theta}{t + \theta}\right) + \alpha \left(\frac{t}{t + 2\theta}\right)^x \left(\frac{2\theta}{t + 2\theta}\right), x = 0, 1, 2, \dots; \theta > 0 \quad (2.54)$$

which is a finite mixture of geometric distributions with parameters $\frac{\theta}{t+\theta}$ and $\frac{2\theta}{t+2\theta}$.

(b) *The r th moment of the mixing distribution is*

$$\mathbb{E}(\Lambda^r) = \frac{r! [2^r - (2^r - 1)\alpha]}{2^r \theta^r}, r = 1, 2, 3, 4 \quad (2.55)$$

so that

$$\text{Var}(\Lambda) = \frac{4 - 2\alpha - \alpha^2}{4\theta^2} \quad (2.56)$$

$$\mathbb{E}[\Lambda - \mathbb{E}(\Lambda)]^3 = \frac{3(8 - 7\alpha)}{4\theta^3} - \frac{3(4 - 3\alpha)(2 - \alpha)}{4\theta^3} + \frac{(2 - \alpha)^3}{4\theta^3} \quad (2.57)$$

$$\mathbb{E}[\Lambda - \mathbb{E}(\Lambda)]^4 = \frac{3(16 - 7\alpha)}{2\theta^4} - \frac{3(8 - 7\alpha)(2 - \alpha)}{2\theta^4} + \frac{3(4 - 3\alpha)(2 - \alpha)^2}{4\theta^4} - \frac{3(2 - \alpha)^4}{16\theta^4}. \quad (2.58)$$

(c) *Raw moments of the mixture are*

$$\mathbb{E}(X) = \frac{2 - \alpha}{2\theta} t \quad (2.59)$$

$$\mathbb{E}(X^2) = \frac{4 - 3\alpha}{2\theta^2} t^2 + \frac{2 - \alpha}{2\theta} t \quad (2.60)$$

$$\mathbb{E}(X^3) = \frac{3(8 - 7\alpha)}{4\theta^3} t^3 + \frac{3(4 - 3\alpha)}{2\theta^2} t^2 + \frac{2 - \alpha}{2\theta} t \quad (2.61)$$

$$\mathbb{E}(X^4) = \frac{3(16 - 15\alpha)}{2\theta^4} + \frac{9(8 - 7\alpha)}{2\theta^3} t^3 + \frac{7(4 - 3\alpha)}{2\theta^2} t^2 + \frac{2 - \alpha}{2\theta} t. \quad (2.62)$$

Thus, variance is

$$\begin{aligned} \mu_2 &= \text{Var}(X) \\ &= \frac{(4 - 2\alpha - \alpha^2)}{4\theta^2} t^2 + \left(\frac{2 - \alpha}{2\theta}\right) t, \end{aligned} \quad (2.63)$$

third central moment is

$$\begin{aligned} \mu_3 &= \mathbb{E}[X - \mathbb{E}(X)]^3 \\ &= \frac{(8 - 3\alpha - 3\alpha^2 - \alpha^3)}{4\theta^3} t^3 + \frac{3(4 - 2\alpha - \alpha^2)}{4\theta^2} t^2 + \left(\frac{2 - \alpha}{2\theta}\right) t \end{aligned} \quad (2.64)$$

and fourth central moment is

$$\begin{aligned} \mu_4 = & \left\{ \frac{3(12 - 22\alpha + 8\alpha^2 - 5\alpha^3)}{4\theta^4} - \frac{3\alpha^4}{16\theta^4} \right\} t^4 + \frac{3(24 - 14\alpha - 6\alpha^2 + \alpha^3)}{4\theta^3} t^3 \\ & + \frac{(40 - 26\alpha - 3\alpha^2)}{4\theta^2} t^2 + \left(\frac{2 - \alpha}{2\theta} \right) t. \end{aligned} \quad (2.65)$$

(d) Posterior distribution is

$$g(\lambda | x) = \frac{e^{-\lambda t} \lambda^x \left((1 - \alpha) e^{-\theta \lambda} + 2\alpha e^{-2\theta \lambda} \right)}{x! \left[\frac{(1 - \alpha)}{(t + \theta)^{x+1}} + \frac{2\alpha}{(t + 2\theta)^{x+1}} \right]} \quad (2.66)$$

The posterior r th moment is

$$\mathbb{E}[\Lambda^r | x] = \frac{(x + r)!}{x!} \left\{ \frac{\frac{(1 - \alpha)}{(t + \theta)^{r+x+1}} + \frac{2\alpha}{(t + 2\theta)^{r+x+1}}}{\frac{(1 - \alpha)}{(t + \theta)^{x+1}} + \frac{2\alpha}{(t + 2\theta)^{x+1}}} \right\} \quad (2.67)$$

and in particular, the posterior mean is

$$\mathbb{E}[\Lambda | x] = (x + 1) \left\{ \frac{\frac{(1 - \alpha)}{(t + \theta)^{x+2}} + \frac{2\alpha}{(t + 2\theta)^{x+2}}}{\frac{(1 - \alpha)}{(t + \theta)^{x+1}} + \frac{2\alpha}{(t + 2\theta)^{x+1}}} \right\} \quad (2.68)$$

Proof. The Poisson-Transmuted Exponential distribution is obtained as;

$$\begin{aligned} f(x) &= \int_0^\infty \frac{e^{-\lambda t} (\lambda t)^x}{x!} \left[(1 - \alpha) \theta e^{-\theta \lambda} + 2\alpha \theta e^{-2\theta \lambda} \right] d\lambda \\ &= \frac{(1 - \alpha) \theta t^x}{x!} \int_0^\infty \lambda^x e^{-(t + \theta)\lambda} d\lambda + \frac{2\alpha \theta t^x}{x!} \int_0^\infty \lambda^x e^{-(t + 2\theta)\lambda} d\lambda \\ &= \frac{(1 - \alpha) \theta t^x}{x!} \frac{\Gamma(x + 1)}{(t + \theta)^{x+1}} + \frac{2\alpha \theta t^x}{x!} \frac{\Gamma(x + 1)}{(t + 2\theta)^{x+1}} \\ &= \frac{\theta t^x (1 - \alpha)}{(t + \theta)^{x+1}} + \frac{\theta t^x 2\alpha}{(t + 2\theta)^{x+1}} \\ &= (1 - \alpha) \left(\frac{t}{t + \theta} \right)^x \left(\frac{\theta}{t + \theta} \right) + \alpha \left(\frac{t}{t + 2\theta} \right)^x \left(\frac{2\theta}{t + 2\theta} \right), x = 0, 1, 2, \dots; \theta > 0 \end{aligned}$$

which is (2.54). This result was obtained by Bhati et. al (2015), when $t = 1$. \square

2.3.4 Poisson-Lindley Distribution

The pdf of Lindley distribution is

$$g(\lambda) = \frac{\theta^2}{\theta + 1} (\lambda + 1) e^{-\lambda \theta}, \lambda > 0; \theta > 0 \quad (2.69)$$

Proposition 2.3.4. (a) The Poisson-Lindley distribution is

$$f(x) = \frac{\theta^2 t^x}{(\theta + 1)} \left[\frac{x + 1 + t + \theta}{(t + \theta)^{x+2}} \right], x = 0, 1, 2, \dots; \lambda, \theta > 0 \quad (2.70)$$

When $t = 1$, we have a result similar to that of Sankaran (1970a).

(b) The r th moment of Lindley distribution is

$$\mathbb{E}(\Lambda^r) = \frac{r!}{\theta + 1} \frac{(r + 1 + \theta)}{\theta^r}, r = 1, 2, 3, 4 \quad (2.71)$$

so that

$$\text{Var}(\Lambda) = \frac{2 + 4\theta + \theta^2}{(\theta + 1)^2 \theta^2} \quad (2.72)$$

$$\mathbb{E}[\Lambda - \mathbb{E}(\Lambda)]^3 = \frac{2(8 + 6\theta + 6\theta^2 + \theta^3)}{(\theta + 1)^3 \theta^3} \quad (2.73)$$

$$\mathbb{E}[\Lambda - \mathbb{E}(\Lambda)]^4 = \frac{12(6 + 16\theta + 17\theta^2 + 8\theta^3 + \theta^4)}{(\theta + 1)^4 \theta^4}. \quad (2.74)$$

(c) Raw moments of Poisson-Lindley distribution are

$$\mathbb{E}(X) = \frac{(2 + \theta)}{(\theta + 1)\theta} t \quad (2.75)$$

$$\mathbb{E}(X^2) = \frac{2(3 + \theta)}{(\theta + 1)\theta^2} t^2 + \frac{(2 + \theta)}{(\theta + 1)\theta} t \quad (2.76)$$

$$\mathbb{E}(X^3) = \frac{6(4 + \theta)}{(\theta + 1)\theta^3} t^3 + \frac{6(3 + \theta)}{(\theta + 1)\theta^2} t^2 + \frac{(2 + \theta)}{(\theta + 1)\theta} t \quad (2.77)$$

$$\mathbb{E}(X^4) = \frac{24(5 + \theta)}{(\theta + 1)\theta^4} t^4 + \frac{36(4 + \theta)}{(\theta + 1)\theta^3} t^3 + \frac{14(3 + \theta)}{(\theta + 1)\theta^2} t^2 + \frac{(2 + \theta)}{(\theta + 1)\theta} t. \quad (2.78)$$

Thus, variance is

$$\mu_2 = \frac{(2 + 4\theta + \theta^2)}{(\theta + 1)^2 \theta^2} t^2 + \frac{(2 + \theta)}{(\theta + 1)\theta} t, \quad (2.79)$$

third central moment is

$$\mu_3 = \frac{2(8 + 6\theta + 6\theta^2 + \theta^3)}{(\theta + 1)^3 \theta^3} t^3 + \frac{3(2 + 4\theta + \theta^2)}{(\theta + 1)^2 \theta^2} t^2 + \frac{(2 + \theta)}{(\theta + 1)\theta} t \quad (2.80)$$

and fourth central moment is

$$\begin{aligned} \mu_4 = & \frac{12(6 + 16\theta + 17\theta^2 + 8\theta^3 + \theta^4)}{(\theta + 1)^4 \theta^4} t^4 + \frac{6(20 + 22\theta + 18\theta^2 + 3\theta^3)}{(\theta + 1)^3 \theta^3} t^3 \\ & + \frac{2(13 + 20\theta + 5\theta^2)}{(\theta + 1)^2 \theta^2} t^2 + \frac{(2 + \theta)}{(\theta + 1)\theta} t. \end{aligned} \quad (2.81)$$

(d) Posterior distribution is

$$g(\lambda | x) = \frac{(t + \theta)^{x+2}}{x!} \frac{\lambda^x (\lambda + 1) e^{-(t+\theta)\lambda}}{(\theta + x + 1 + t)} \quad (2.82)$$

Therefore, the r th moment of the posterior distribution is

$$\mathbb{E}(\Lambda^r | x) = \frac{(x + r)!}{x!} \frac{(x + r + 1 + t + \theta)}{(t + \theta)^r} \quad (2.83)$$

and in particular the posterior mean is

$$\mathbb{E}(\Lambda | x) = \frac{x^2 + (3 + t + \theta)x + (2 + t + \theta)}{(t + \theta)} \quad (2.84)$$

Proof. From equation (2.69) the mixed Poisson distribution is

$$\begin{aligned}
f(x) &= \int_0^\infty \frac{e^{-\lambda t} (\lambda t)^x}{x!} \frac{\theta^2}{\theta + 1} (\lambda + 1) e^{-\lambda \theta} d\lambda \\
&= \frac{t^x \theta^2}{x! (\theta + 1)} \int_0^\infty (\lambda^{x+1} + \lambda^x) e^{-(t+\theta)\lambda} d\lambda \\
&= \frac{t^x \theta^2}{x! (\theta + 1)} \left[\frac{\Gamma(x+2)}{(t+\theta)^{x+2}} + \frac{\Gamma(x+1)}{(t+\theta)^{x+1}} \right] \\
&= \frac{t^x \theta^2}{x! (\theta + 1)} \frac{\Gamma(x+1)}{(t+\theta)^{x+1}} \left\{ \frac{x+1}{t+\theta} + 1 \right\} \\
&= \frac{t^x \theta^2}{\theta + 1} \frac{1}{(t+\theta)^{x+1}} \frac{x+1+t+\theta}{t+\theta} \\
&= \frac{t^x \theta^2}{(\theta + 1)} \frac{x+1+t+\theta}{(t+\theta)^{x+2}}; \quad x = 0, 1, 2, \dots
\end{aligned}$$

which is (2.70).

The pgf of Poisson-Lindley distribution is

$$\begin{aligned}
G(s) &= \int_0^\infty e^{-\lambda t(1-s)} \frac{\theta^2}{\theta + 1} (\lambda + 1) e^{-\lambda \theta} d\lambda \\
&= \frac{\theta^2}{\theta + 1} \left\{ \int_0^\infty \lambda e^{-\lambda(\theta+t-ts)} d\lambda + \int_0^\infty e^{-\lambda(\theta+t-ts)} d\lambda \right\} \\
&= \frac{\theta^2}{\theta + 1} \left\{ \frac{\Gamma(2)}{(\theta+t-ts)^2} + \frac{1}{\theta+t-ts} \right\} \\
&= \frac{\theta^2}{\theta + 1} \frac{(1+\theta+t-ts)}{(\theta+t-ts)^2} \tag{2.85}
\end{aligned}$$

When $t = 1$ in (2.85), then

$$G(s) = \frac{\theta^2}{(\theta + 1)} \frac{(\theta + 2 - s)}{(\theta + 1 - s)^2} \tag{2.86}$$

as given by Johnson et al. (2005).

The posterior distribution is

$$\begin{aligned}
g(\lambda|x) &= \frac{e^{-\lambda t} (\lambda t)^x \theta^2 (\lambda + 1) e^{-\lambda \theta} (t + \theta)^{x+2}}{x! (\theta + 1) t^x \theta^2 (\theta + x + 1 + t)} \\
&= \frac{e^{-\lambda t} \lambda^x (\lambda + 1) e^{-\lambda \theta} (t + \theta)^{x+2}}{x! (\theta + x + 1 + t)} \\
&= \frac{(t + \theta)^{x+2}}{x! (\theta + x + 1 + t)} \lambda^x (\lambda + 1) e^{-(t+\theta)\lambda}
\end{aligned}$$

which is (2.82). The posterior mean is

$$\begin{aligned}
\mathbb{E}(\Lambda|x) &= \frac{(t+\theta)^{x+2}}{x!(\theta+x+1+t)} \int_0^\infty \lambda^{x+1} (\lambda+1) e^{-(t+\theta)\lambda} d\lambda \\
&= \frac{(t+\theta)^{x+2}}{x!(\theta+x+1+t)} \left\{ \int_0^\infty \lambda^{x+2} e^{-(t+\theta)\lambda} d\lambda + \int_0^\infty \lambda^{x+1} e^{-(t+\theta)\lambda} d\lambda \right\} \\
&= \frac{(t+\theta)^{x+2}}{x!(\theta+x+1+t)} \left\{ \frac{\Gamma(x+3)}{(t+\theta)^{x+3}} + \frac{\Gamma(x+2)}{(t+\theta)^{x+2}} \right\} \\
&= \frac{(t+\theta)^{x+2} (x+1)!}{x!(\theta+x+1+t) (t+\theta)^{x+2}} \left\{ \frac{x+2}{t+\theta} + 1 \right\} \\
&= \frac{(x+1)(\theta+x+2+t)}{(t+\theta)(\theta+x+1+t)} \\
&= \frac{x+1}{t+\theta} + \frac{x+1}{(t+\theta)(x+1+\theta+t)} \\
&= \frac{x+1}{t+\theta} + \frac{x+1+\theta+t-\theta-t}{(t+\theta)(x+1+\theta+t)} \\
&= \frac{x+2}{t+\theta} - \frac{1}{x+1+\theta+t}
\end{aligned}$$

which is (2.84). □

2.3.5 Poisson- 3 Parameter Generalized Lindley Distribution

Consider the following finite mixture

$$g(\lambda) = p_1 g_1(\lambda) + p_2 g_2(\lambda) \quad (2.87)$$

where $p_1 + p_2 = 1, p_1 > 0, p_2 > 0$. Suppose $p_1 = \frac{\theta}{\theta+\gamma}$, then $p_2 = \frac{\gamma}{\theta+\gamma}$, and hence

$$g(\lambda) = \frac{\theta}{\theta+\gamma} g_1(\lambda) + \frac{\gamma}{\theta+\gamma} g_2(\lambda). \quad (2.88)$$

If $g_1(\lambda)$ is *Gamma*(α, θ) and $g_2(\lambda)$ is *Gamma*($\alpha+1, \theta$) then (2.88) becomes

$$g(\lambda) = \frac{\theta^{\alpha+1}}{\theta+\gamma} \frac{1}{\Gamma(\alpha+1)} (\alpha+\gamma\lambda) \lambda^{\alpha-1} e^{-\theta\lambda}; \lambda > 0, \alpha, \gamma, \theta > 0 \quad (2.89)$$

which is a 3-parameter generalized Lindley distribution, with the following special cases:

- (i) $\alpha = \gamma = 1$, we have the one-parameter Lindley distribution used by Sankaran (1970a).
- (ii) $\gamma = 1$, we have a 2-parameter generalized Lindley distribution

$$g(\lambda) = \frac{\theta^{\alpha+1}}{\theta+1} \frac{1}{\Gamma(\alpha+1)} (\alpha+\lambda) \lambda^{\alpha-1} e^{-\theta\lambda}; \lambda > 0, \alpha, \theta > 0 \quad (2.90)$$

as obtained by Zakerzadeh and Dollati (2010).

(iii) $\alpha = 1$, we have a 2-parameter generalized Lindley distribution

$$g(\lambda) = \frac{\theta^2}{1 + \gamma} (1 + \gamma\lambda) e^{-\theta\lambda}; \lambda > 0; \gamma, \theta > 0 \quad (2.91)$$

as obtained by Bhati et al. (2015).

Proposition 2.3.5. (a) *The Poisson-3 parameter generalized Lindley distribution is*

$$f(x) = \frac{\Gamma(x + \alpha)}{x! \Gamma(\alpha + 1)} \left\{ \left(\alpha + \frac{\alpha t + \gamma x}{\theta + \gamma} \right) \left(\frac{t}{t + \theta} \right)^x \left(\frac{\theta}{t + \theta} \right)^{\alpha + 1} \right\} \quad (2.92)$$

with the following special cases:

(i) $\alpha = \lambda = 1$ and $t = 1$, we have

$$f(x) = \frac{\theta^2 (x + \theta + 2)}{(1 + \theta)^{x + \alpha + 2}} \quad (2.93)$$

as obtained by Sankaran (1970a).

(ii) $\gamma = 1$ and $t = 1$, we have

$$f(x) = \frac{\Gamma(x + \alpha)}{x! \Gamma(\alpha + 1)} \left\{ \alpha + \frac{x + \alpha}{1 + \theta} \right\} \frac{1}{(1 + \theta)^{x + \alpha + 1}} \quad (2.94)$$

as obtained by Mahmoudi and Zakerzadeh (2010).

(iii) $\alpha = 1$ and $t = 1$, we have

$$f(x) = \frac{\theta^2}{(\theta + \gamma)} \left\{ 1 + \frac{\gamma(x + 1)}{1 + \theta} \right\} \frac{1}{(1 + \theta)^{x + 1}}; x = 0, 1, 2, \dots \quad (2.95)$$

as obtained by Bhati et al (2015).

(b) *The r th moment of the 3-parameter generalized Lindley distribution is*

$$\mathbb{E}(\Lambda^r) = \frac{\Gamma(\alpha + r)}{\theta^r \Gamma(\alpha + 1)} \left\{ \alpha + \frac{r\gamma}{\theta + \gamma} \right\}, r = 1, 2, 3, 4 \quad (2.96)$$

Therefore mean is

$$\mathbb{E}(\Lambda) = \frac{1}{\theta} \left\{ \alpha + \frac{\gamma}{\theta + \gamma} \right\} \quad (2.97)$$

variance is

$$\text{Var}(\Lambda) = \frac{\alpha\theta^2 + 2(\alpha + 1)\gamma\theta + (\alpha + 1)\gamma^2}{\theta^2(\theta + \gamma)^2}, \quad (2.98)$$

third central moment is

$$\mathbb{E}[\Lambda - \mathbb{E}(\Lambda)]^3 = \frac{2\alpha\theta^3 + 6(\alpha + 1)\gamma\theta^2 + 6(\alpha + 1)\gamma^2\theta + 2(\alpha + 1)\gamma^3}{\theta^3(\theta + \gamma)^3} \quad (2.99)$$

and the fourth central moment is

$$\begin{aligned} \mathbb{E}(\Lambda - \mathbb{E}(\Lambda))^4 &= \frac{3\alpha(\alpha + 2)\theta^4 + 12(\alpha + 1)(\alpha + 2)\gamma\theta^3 + 6(\alpha + 1)(3\alpha + \gamma)\gamma^2\theta^2}{\theta^4(\theta + \gamma)^4} \\ &+ \frac{12(\alpha + 1)(\alpha + 3)\gamma^2\theta + 3(\alpha + 1)(\alpha + 3)}{\theta^4(\theta + \gamma)^4} \end{aligned} \quad (2.100)$$

(c) The central moments of Poisson-3-parameter generalized Lindley distribution are:

$$(i) \quad \mu_2 = \frac{[\alpha\theta^2 + 2(\alpha+1)\gamma\theta + (\alpha+1)\gamma^2]}{\theta^2(\theta+\gamma)^2}t^2 + \frac{[\alpha\theta + (\alpha+1)\gamma]}{\theta(\theta+\gamma)}t \quad (2.101)$$

When $t = 1$, then

$$\mu_2 = \frac{\alpha\theta^3 + [\alpha(\gamma+1) + \gamma(\alpha+1)]\theta^2 + (2+\gamma)(\alpha+1)\gamma\theta + (\alpha+1)\gamma^2}{\theta^2(\theta+\gamma)^2} \quad (2.102)$$

(ii)

$$\begin{aligned} \mu_3 = & \frac{[2\alpha\theta^3 + 6(\alpha+1)\gamma\theta^2 + 6(\alpha+1)\gamma^2\theta + 2(\alpha+1)\gamma^3]}{\theta^3(\theta+\gamma)^3}t^3 \\ & + \frac{3[\alpha\theta^2 + 2(\alpha+1)\gamma\theta + (\alpha+1)\gamma^2]}{\theta^2(\theta+\gamma)^2}t^2 + \frac{[\alpha\theta + (\alpha+1)\gamma]}{\theta(\theta+\gamma)}t \end{aligned} \quad (2.103)$$

When $t = 1$, then

$$\begin{aligned} \theta^3(\theta+\gamma)^3\mu_3 = & \alpha\theta^5 + (3\alpha + 3\alpha\gamma + \gamma)\theta^4 + (2\alpha + 9\alpha\gamma + 6\gamma + 3\alpha\gamma^2 + 2\gamma^2)\theta^3 \\ & + \gamma(\alpha+1)(6 + 9\gamma + \gamma^2)\theta^2 + 3(\alpha+1)(2+\gamma)\gamma^2\theta \\ & + 2(\alpha+1)\gamma^3 \end{aligned} \quad (2.104)$$

Proof. From equation (2.89), the Poisson - 3 parameter generalized Lindley distribution is

$$\begin{aligned} f(x) &= \int_0^\infty e^{-\lambda t} \frac{(\lambda t)^x}{x!} \frac{\theta^{\alpha+1}}{\theta+\gamma} \cdot \frac{1}{\Gamma(\alpha+1)} (\alpha+\gamma\lambda) \lambda^{\alpha-1} e^{-\theta\lambda} d\lambda \\ &= \frac{t^x \theta^{\alpha+1}}{x! \theta + \gamma} \frac{1}{\Gamma(\alpha+1)} \left\{ \int_0^\infty \alpha \lambda^{x+\alpha-1} e^{-(t+\theta)\lambda} d\lambda + \int_0^\infty \gamma \lambda^{x+\alpha} e^{-(t+\theta)\lambda} d\lambda \right\} \\ &= \frac{t^x \theta^{\alpha+1}}{x! \theta + \gamma} \frac{1}{\Gamma(\alpha+1)} \left\{ \frac{\alpha \Gamma(x+\alpha)}{(t+\theta)^{x+\alpha}} + \frac{\gamma \Gamma(x+\alpha+1)}{(t+\theta)^{x+\alpha+1}} \right\} \\ &= \frac{t^x \theta^{\alpha+1}}{x! \theta + \gamma} \cdot \frac{\Gamma(x+\alpha)}{\Gamma(\alpha+1)} \frac{[\alpha(t+\theta) + \gamma(x+\alpha)]}{(t+\theta)^{x+\alpha+1}} \\ &= \frac{\Gamma(x+\alpha)}{x! \Gamma(\alpha+1)} \frac{[\alpha(t+\theta) + \gamma(x+\alpha)]}{\theta+\gamma} \left(\frac{t}{t+\theta}\right)^x \left(\frac{\theta}{t+\theta}\right)^{\alpha+1} \\ &= \frac{\Gamma(x+\alpha)}{x! \Gamma(\alpha+1)} \left(\alpha + \frac{\gamma x + \alpha t}{\theta + \gamma}\right) \left(\frac{t}{t+\theta}\right)^x \left(\frac{\theta}{t+\theta}\right)^{\alpha+1}, x = 0, 1, 2, \dots; \alpha, \gamma, \theta > 0 \end{aligned}$$

which is (2.92) and its pgf is

$$\begin{aligned}
G(s) &= \int_0^\infty e^{-\lambda t(1-s)} \frac{\theta^{\alpha+1}}{\theta + \gamma} \frac{1}{\Gamma(\alpha + 1)} (\alpha + \gamma\lambda) \lambda^{\alpha-1} e^{-\theta\lambda} d\lambda \\
&= \frac{\theta^{\alpha+1}}{(\theta + \gamma) \Gamma(\alpha + 1)} \int_0^\infty (\alpha\lambda^{\alpha-1} + \gamma\lambda^\alpha) e^{-[\theta+t-ts]\lambda} d\lambda \\
&= \frac{\theta^{\alpha+1}}{(\theta + \gamma) \Gamma(\alpha + 1)} \left\{ \alpha \int_0^\infty \lambda^{\alpha-1} e^{-(\theta+t-ts)\lambda} d\lambda + \gamma \int_0^\infty \lambda^\alpha e^{-(\theta+t-ts)\lambda} d\lambda \right\} \\
&= \frac{\theta^{\alpha+1}}{(\theta + \gamma) \Gamma(\alpha + 1)} \left\{ \frac{\alpha \Gamma(\alpha)}{(\theta + t - ts)^\alpha} + \frac{\gamma \Gamma(\alpha + 1)}{(\theta + t - ts)^{\alpha+1}} \right\} \\
&= \frac{\theta^{\alpha+1}}{(\theta + \gamma)} \left\{ \frac{1}{(\theta + t - ts)^\alpha} + \frac{\gamma}{(\theta + t - ts)^{\alpha+1}} \right\} \\
&= \frac{\theta^{\alpha+1}}{\theta + \gamma} \left[\frac{\theta + t - ts + \gamma}{(\theta + t - ts)^{\alpha+1}} \right]. \tag{2.105}
\end{aligned}$$

When $t = 1$ and $\gamma = 1$ in (2.92)

$$\begin{aligned}
f(x) &= \frac{\Gamma(x + \alpha)}{x! \Gamma(\alpha + 1)} \left\{ \alpha + \frac{\alpha + x}{1 + \theta} \right\} \left(\frac{1}{1 + \theta} \right)^x \left(\frac{\theta}{1 + \theta} \right)^{\alpha+1} \\
&= \frac{\Gamma(x + \alpha)}{x! \Gamma(\alpha + 1)} \left(\alpha + \frac{\alpha + x}{1 + \theta} \right) \frac{\theta^{\alpha+1}}{(1 + \theta)^{x+\alpha+1}} \tag{2.106}
\end{aligned}$$

and in (2.105)

$$\begin{aligned}
G(s) &= \frac{\theta^{\alpha+1}}{\theta + 1} \left[\frac{\theta + 1 - s + 1}{(\theta + 1 - s)^{\alpha+1}} \right] \\
&= \frac{\theta^{\alpha+1}}{\theta + 1} \cdot \frac{\theta + 2 - s}{(\theta + 1 - s)^{\alpha+1}} \\
&= \left(\frac{\theta}{\theta - s + 1} \right)^{\alpha+1} \left(\frac{\theta - s + 2}{\theta + 1} \right) \tag{2.107}
\end{aligned}$$

as obtained by Mahmoudi and Zakerzadeh (2010).

The r th moment of 3-parameter generalized Lindley distribution is

$$\begin{aligned}
\mathbb{E}(\Lambda^r) &= \int_0^\infty \lambda^r \frac{\theta^{\alpha+1}}{(\theta + \gamma) \Gamma(\alpha + 1)} (\alpha + \gamma\lambda) \lambda^{\alpha-1} e^{-\theta\lambda} d\lambda \\
&= \frac{\theta^{\alpha+1}}{(\theta + \gamma) \Gamma(\alpha + 1)} \int_0^\infty \lambda^{r+\alpha-1} (\alpha + \gamma\lambda) e^{-\theta\lambda} d\lambda \\
&= \frac{\theta^{\alpha+1}}{(\theta + \gamma) \Gamma(\alpha + 1)} \left\{ \frac{\alpha \Gamma(r + \alpha)}{\theta^{r+\alpha}} + \frac{\gamma \Gamma(r + \alpha + 1)}{\theta^{r+\alpha+1}} \right\} \\
&= \frac{\theta^{\alpha+1}}{(\theta + \gamma) \Gamma(\alpha + 1)} \frac{\Gamma(r + \alpha)}{\theta^{r+\alpha}} \left\{ \alpha + \frac{\gamma(r + \alpha)}{\theta} \right\} \\
&= \frac{\theta^{\alpha+1} \Gamma(r + \alpha)}{(\theta + \gamma) \Gamma(\alpha + 1) \theta^{r+\alpha}} \left\{ \frac{\alpha\theta + \gamma r + \gamma\alpha}{\theta} \right\} \\
&= \frac{\theta^{\alpha+1} \Gamma(r + \alpha)}{\Gamma(\alpha + 1) \theta^{r+\alpha+1}} \left\{ \frac{\alpha(\theta + \gamma) + \gamma r}{(\theta + \gamma)} \right\} \\
&= \frac{\Gamma(r + \alpha)}{\Gamma(\alpha + 1) \theta^r} \left[\alpha + \frac{\gamma r}{\theta + \gamma} \right], r = 1, 2, 3, 4
\end{aligned}$$

which is (2.96) so that

$$\begin{aligned} \text{Var}(\Lambda) &= \mathbb{E}(\Lambda^2) - [\mathbb{E}(\Lambda)]^2 \\ &= \frac{(\alpha+1)[\alpha(\theta+\gamma)+2\gamma](\theta+\gamma)}{\theta^2} - \frac{[\alpha(\theta+\gamma)+\gamma]^2}{\theta^2(\theta+\gamma)^2}. \end{aligned}$$

Now,

$$\begin{aligned} \theta^2(\theta+\gamma)^2 \text{Var}(\Lambda) &= (\alpha+1)[\alpha(\theta+\gamma)+2\gamma](\theta+\gamma) - [\alpha(\theta+\gamma)+\gamma]^2 \\ &= \alpha(\alpha+1)(\theta+\gamma)^2 + 2(\alpha+1)\gamma(\theta+\gamma) - [\alpha^2(\theta+\gamma)^2 + 2\alpha\gamma(\theta+\gamma) + \gamma^2] \\ &= \alpha(\theta+\gamma)^2 + 2\gamma(\theta+\gamma) - \gamma^2 \\ &= (\theta+\gamma)(\alpha\theta + \alpha\gamma + 2\gamma) - \gamma^2 \\ &= \gamma\theta^2 + (2\alpha\gamma + 2\gamma)\theta + \alpha\gamma^2 + 2\gamma^2 + \gamma^2 \\ &= \alpha\theta^2 + 2(\alpha+1)\gamma\theta + (\alpha+1)\gamma^2 \end{aligned}$$

Therefore

$$\text{Var}(\Lambda) = \frac{\alpha\theta^2 + 2(\alpha+1)\gamma\theta + (\alpha+1)\gamma^2}{\theta^2(\theta+\gamma)^2}$$

which is (2.98).

Next,

$$\mathbb{E}[\Lambda - \mathbb{E}(\Lambda)]^3 = \mathbb{E}(\Lambda^3) - 3\mathbb{E}(\Lambda^2)\mathbb{E}(\Lambda) + 2[\mathbb{E}(\Lambda)]^3$$

$$\begin{aligned} \theta^3(\theta+\gamma)^3 \mathbb{E}[\Lambda - \mathbb{E}(\Lambda)]^3 &= (\alpha+1)(\alpha+2)[\alpha(\theta+\gamma)+3\gamma](\theta+\gamma)^2 \\ &\quad - 3(\alpha+1)[\alpha(\theta+\gamma)+2\gamma][\alpha(\theta+\gamma)+\gamma](\theta+\gamma) \\ &\quad + 2[\alpha(\theta+\gamma)+\gamma]^3 \\ &= \alpha(\alpha+1)(\alpha+2)(\theta+\gamma)^3 + 3(\alpha+1)(\alpha+2)\gamma(\theta+\gamma)^2 \\ &\quad - 3(\alpha+1)[\alpha^2(\theta+\gamma)^3 + 3\alpha\gamma(\theta+\gamma)^2 + 2\gamma^2(\theta+\gamma)] \\ &\quad + 2[\alpha(\theta+\gamma)+\gamma]^3 \\ &= [\alpha(\alpha+1)(\alpha+2) - 3(\alpha+1)\alpha^2](\theta+\gamma)^3 \\ &\quad + [3(\alpha+1)(\alpha+2)\gamma - 9\alpha\gamma(\alpha+1)](\theta+\gamma)^2 \\ &\quad - 6(\alpha+1)\gamma^2(\theta+\gamma) + 2[\alpha(\theta+\gamma)+\gamma]^3 \\ &= [(\alpha+1)(\alpha^2 + 2\alpha - 3\alpha^2)](\theta+\gamma)^3 \\ &\quad + [3(\alpha+1)\gamma(\alpha+2 - 3\alpha)](\theta+\gamma)^2 \\ &\quad - 6(\alpha+1)\gamma^2(\theta+\gamma) + 2[\alpha(\theta+\gamma)+\gamma]^3 \\ &= [2\alpha(\alpha+1)(1-\alpha)](\theta+\gamma)^3 + [3(\alpha+1)\gamma(2-2\alpha)](\theta+\gamma)^2 \\ &\quad - 6(\alpha+1)\gamma^2(\theta+\gamma) + 2[\alpha(\theta+\gamma)+\gamma]^3 \end{aligned}$$

Further solving yields

$$\begin{aligned}
\theta^3 (\theta + \gamma)^3 \mathbb{E} [\Lambda - \mathbb{E}(\Lambda)]^3 &= [2\alpha (1 - \alpha^2)] (\theta + \gamma)^3 + 6 (1 - \alpha^2) \gamma (\theta + \gamma)^2 \\
&\quad - 6 (\alpha + 1) \gamma^2 (\theta + \gamma) \\
&\quad + 2 [\alpha^3 (\theta + \gamma)^3 + 3\alpha^2 (\theta + \gamma)^2 \gamma + 3\alpha (\theta + \gamma) \gamma^2 + \gamma^3] \\
&= [2\alpha - 2\alpha^3 + 2\alpha^3] (\theta + \gamma)^3 + [6 (1 - \alpha^2) \gamma + 6\alpha^2 \gamma] (\theta + \gamma)^2 \\
&\quad + [6\alpha\gamma^2 - 6 (\alpha + 1) \gamma^2] (\theta + \gamma) + 2\gamma^3 \\
&= 2\alpha (\theta + \gamma)^3 + 6\gamma (\theta + \gamma)^2 - 6\gamma^2 (\theta + \gamma) + 2\gamma^3 \\
&= 2\alpha (\theta^3 + 3\theta^2\gamma + 3\theta\gamma^2 + \gamma^3) + 6\gamma (\theta^2 + 2\theta\gamma + \gamma^2) - 6\gamma^2\theta - 4\gamma^3 \\
&= 2\alpha\theta^3 + [6\alpha\gamma + 6\gamma] \theta^2 + [6\alpha\gamma^2 + 12\gamma^2] \theta \\
&\quad + 2\alpha\gamma^3 + 6\gamma^3 + 2\gamma^3 - 6\gamma^3 \\
&= 2\alpha\theta^3 + 6 (\alpha + 1) \gamma\theta^2 + (6\alpha\gamma^2 + 6\gamma^2) \theta + 2 (\alpha + 1) \gamma^3
\end{aligned}$$

Therefore the third central moment is

$$\mathbb{E} [\Lambda - \mathbb{E}(\Lambda)]^3 = \frac{2\alpha\theta^3 + 6 (\alpha + 1) \gamma\theta^2 + 6 (\alpha + 1) \gamma^2\theta + 2 (\alpha + 1) \gamma^3}{\theta^3 (\theta + \gamma)^3}$$

which is (2.99). □

2.3.6 Special Cases of Poisson-3 Parameter Generalized Lindley Distribution

When $\alpha = \gamma = 1$, then the second central moment of the mixture is

$$\mu_2 = \frac{\theta^3 + 4\theta^2 + 6\theta + 2}{\theta^2 (\theta + 1)^2}$$

a result obtained by Sankaran (1970a) and when $\gamma = 1$ then

$$\begin{aligned}
\mu_2 &= \frac{\alpha\theta^3 + (3\alpha + 1) \theta^2 + 3 (\alpha + 1) \theta + (\alpha + 1)}{\theta^2 (\theta + 1)^2} \\
&= \frac{\alpha (\theta^3 + 3\theta^2 + 3\theta + 1) + \theta^2 + 3\theta + 1}{\theta^2 (\theta + 1)^2} \\
&= \frac{\alpha (\theta + 1)^3 + \theta^2 + 3\theta + 1}{\theta^2 (\theta + 1)^2} \tag{2.108}
\end{aligned}$$

as obtained by Mahmoudi and Zakerzadeh (2010).

Also when $\gamma = 1$, the third moment about the mean for the mixture is obtained as

$$\begin{aligned}
\theta^3 (\theta + 1)^3 \mu_3 &= \alpha \theta^5 + (6\alpha + 1) \theta^4 + (14\alpha + 8) \theta^3 + 16(\alpha + 1) \theta^2 + 9(\alpha + 1) \theta + 2(\alpha + 1) \\
&= \alpha \theta [\theta^4 + (4\theta^3 + 2\theta^3) + (6\theta^2 + 8\theta^2) + (4\theta + 12\theta) + (1 + 8)] + 2\alpha + [\theta^4 + 8\theta^3 + 16\theta^2 + 9\theta + 2] \\
&= \alpha \theta [(\theta^4 + 4\theta^3 + 6\theta^2 + 4\theta + 1) + (2\theta^3 + 8\theta^2 + 12\theta + 8)] + 2\alpha + [\theta^4 + 8\theta^3 + 16\theta^2 + 9\theta + 2] \\
&= \alpha \theta (\theta + 1)^4 + \alpha \theta [2\theta^3 + 8\theta^2 + 12\theta + 8] + 2\alpha + [\theta^4 + 8\theta^3 + 16\theta^2 + 9\theta + 2] \\
&= \alpha \theta (\theta + 1)^4 + 2\alpha [\theta^4 + 4\theta^3 + 6\theta^2 + 4\theta + 1] + [\theta^4 + 8\theta^3 + 16\theta^2 + 9\theta + 2] \\
&= \alpha \theta (\theta + 1)^4 + 2\alpha (\theta + 1)^4 + [\theta^4 + 8\theta^3 + 16\theta^2 + 9\theta + 2] \\
&= \alpha (\theta + 1)^4 (\theta + 2) + [\theta^4 + 8\theta^3 + 16\theta^2 + 9\theta + 2]
\end{aligned}$$

Which on further solving becomes:

$$\begin{aligned}
\theta^3 (\theta + 1)^3 \mu_3 &= \alpha (\theta + 1)^4 (\theta + 2) + [\theta^4 + 8\theta^2 (\theta + 2) + 8\theta + (\theta + 2)] \\
&= \alpha (\theta + 1)^4 (\theta + 2) + [\theta^4 + 8\theta + (8\theta^2 + 1) (\theta + 2)] \\
&= \alpha (\theta + 1)^4 (\theta + 2) + [\theta (\theta^3 + 8) + (8\theta^2 + 1) (\theta + 2)] \\
&= \alpha (\theta + 1)^4 (\theta + 2) + [\theta (\theta + 2) (\theta^2 - 2\theta + 4) + (8\theta^2 + 1) (\theta + 2)]
\end{aligned}$$

Therefore

$$\mu_3 = \frac{\alpha (\theta + 1)^4 (\theta + 2) + [\theta^3 + 6\theta^2 + 4\theta + 1] (\theta + 2)}{\theta^3 (\theta + 1)^3} \quad (2.109)$$

as obtained by Mahmoudi and Zakerzadeh (2010).

Chapter 3

MIXED POISSON DISTRIBUTIONS IN TERMS OF SPECIAL FUNCTIONS

3.1 Introduction

In this chapter, mixed Poisson distributions and their probability generating functions have been expressed in terms of special functions. Specifically, we shall express them in terms of confluent hypergeometric functions and modified Bessel functions.

We shall first define confluent hypergeometric function and give its properties. Examples of Poisson mixtures based on this function will follow. We shall next define modified Bessel function of the third kind and give its properties. Examples of Poisson mixtures based on this function will follow.

3.2 Confluent hypergeometric functions

3.2.1 Kummer's Series

The confluent hypergeometric function, also known as Kummer's series, denoted by ${}_1F_1(a, c; x)$ is defined as:

$$\begin{aligned} {}_1F_1(a, c; x) &= 1 + \frac{a}{c} \frac{x}{1!} + \frac{a(a+1)}{c(c+1)} \frac{x^2}{2!} + \dots \\ &= 1 + \sum_{n=1}^{\infty} \frac{a(a+1)(a+2)\dots(a+n-1)}{c(c+1)(c+2)\dots(c+n-1)} \frac{x^n}{n!} \end{aligned} \quad (3.1)$$

where $c \neq 0, -1, -2, -3, \dots$

An integral representation is derived as follows:

$$\begin{aligned}
{}_1F_1(a, c; x) &= 1 + \sum_{n=1}^{\infty} \frac{(a+n-1)(a+n-2)\dots(a+2)(a+1)a\Gamma(a)\Gamma(c)x^n}{(c+n-1)(c+n-2)\dots(c+2)(c+1)c\Gamma(c)\Gamma(a)n!} \\
&= 1 + \sum_{n=1}^{\infty} \frac{\Gamma(a+n)\Gamma(c)x^n}{\Gamma(c+n)\Gamma(a)n!} \\
&= 1 + \sum_{n=1}^{\infty} \frac{\Gamma(a+n)\Gamma(c-a)}{\Gamma(c+n)} \cdot \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \frac{x^n}{n!} \\
&= 1 + \sum_{n=1}^{\infty} B(a+n, c-a) \frac{1}{B(a, c-a)} \frac{x^n}{n!}
\end{aligned}$$

On further simplification, we have

$$\begin{aligned}
{}_1F_1(a, c; x) &= 1 + \frac{1}{B(a, c-a)} \sum_{n=1}^{\infty} B(a+n, c-a) \frac{x^n}{n!} \\
&= 1 + \frac{1}{B(a, c-a)} \sum_{n=1}^{\infty} \int_0^1 t^{a+n-1} (1-t)^{c-a-1} \frac{x^n}{n!} dt \\
&= 1 + \frac{1}{B(a, c-a)} \sum_{n=1}^{\infty} \int_0^1 t^{a-1} (1-t)^{c-a-1} \frac{(xt)^n}{n!} dt \\
&= 1 + \frac{1}{B(a, c-a)} \int_0^1 t^{a-1} (1-t)^{c-a-1} \left(\sum_{n=1}^{\infty} \frac{(xt)^n}{n!} \right) dt \\
&= 1 + \frac{1}{B(a, c-a)} \int_0^1 t^{a-1} (1-t)^{c-a-1} (e^{xt} - 1) dt \\
&= \frac{1}{B(a, c-a)} \int_0^1 t^{a-1} (1-t)^{c-a-1} e^{xt} dt
\end{aligned} \tag{3.2}$$

and making the substitution $z = (1-t)$ we obtain

$$\begin{aligned}
{}_1F_1(a, c; x) &= \frac{1}{B(a, c-a)} \int_0^1 z^{c-a-1} (1-z)^{a-1} e^{x(1-z)} dz \\
&= \frac{e^x}{B(c-a, a)} \int_0^1 z^{c-a-1} (1-z)^{c-(c-a)-1} e^{-xz} dz \\
&= e^x {}_1F_1(c-a, c; -x)
\end{aligned} \tag{3.3}$$

3.2.2 Tricomi Confluent hypergeometric function

Another confluent hypergeometric function also known as Tricomi has integral representation

$$\psi(a, c; x) = \frac{1}{\Gamma(a)} \int_0^{\infty} t^{a-1} (1+t)^{c-a-1} e^{-xt} dt. \tag{3.4}$$

The following relation holds:

$$\psi(a, c; x) = x^{1-c} \psi(a-c+1, 2-c; x). \tag{3.5}$$

The connection between Tricomi and Kummer's confluent hypergeometric functions is

$$\psi(a; c; x) = \frac{\Gamma(1-c)}{\Gamma(a-c+1)} {}_1F_1(a, c; x) + \frac{\Gamma(c-1)x^{1-c}}{\Gamma(a)} {}_1F_1(a-c+1, 2-c; x) \quad (3.6)$$

where $c \neq 0, -1, -2, \dots$

3.2.3 Incomplete Gamma Function

Incomplete Gamma function is defined as

$$\gamma(a, x) = \int_0^x t^{a-1} e^{-t} dt \quad (3.7)$$

which is related to confluent hypergeometric function as shown below:

$$\begin{aligned} \gamma(a, x) &= \int_0^x t^{a-1} e^{-t} dt \\ &= \int_0^x t^{a-1} \sum_{n=0}^{\infty} \frac{(-t)^n}{n!} dt \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_0^x t^{a+n-1} dt \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{x^{a+n}}{a+n} \\ &= x^a \sum_{n=0}^{\infty} \frac{1}{a+n} \frac{(-x)^n}{n!} \end{aligned}$$

which becomes

$$\begin{aligned} \gamma(a, x) &= x^a \left\{ \frac{1}{a} + \frac{1}{a+1} \frac{(-x)}{1!} + \frac{1}{a+2} \frac{(-x)^2}{2!} + \frac{1}{a+3} \frac{(-x)^3}{3!} + \dots \right\} \\ &= \frac{x^a}{a} \left\{ 1 + \frac{a}{a+1} \frac{(-x)}{1!} + \frac{a}{a+2} \frac{(-x)^2}{2!} + \frac{a}{a+3} \frac{(-x)^3}{3!} + \dots \right\} \\ &= \frac{x^a}{a} \left\{ 1 + \frac{a}{a+1} \frac{(-x)}{1!} + \frac{a(a+1)}{(a+1)(a+2)} \frac{(-x)^2}{2!} + \frac{a(a+1)(a+2)}{(a+1)(a+2)(a+3)} \frac{(-x)^3}{3!} + \dots \right\} \\ &= \frac{x^a}{a} \left\{ 1 + \sum_{n=1}^{\infty} \frac{a(a+1)(a+2) \cdots (a+n-1)}{(a+1)(a+2) \cdots (a+n)} \frac{(-x)^n}{n!} \right\} \quad (3.8) \end{aligned}$$

therefore

$$\gamma(a, x) = \frac{x^a}{a} {}_1F_1(a, a+1; -x) \quad (3.9)$$

and using relation (3.3), we get the relation

$$\gamma(a, x) = \frac{x^a}{a} e^{-x} {}_1F_1(1, a+1; x) \quad (3.10)$$

as given by Johnson et al (2005).

3.3 Mixed Poisson distributions based on Confluent Hypergeometric Functions

3.3.1 Beta I distribution

The Beta I distribution is

$$g(\lambda) = \frac{\lambda^{\alpha-1} (1-\lambda)^{\beta-1}}{B(\alpha, \beta)}, \quad 0 < \lambda < 1; \alpha, \beta > 0 \quad (3.11)$$

Proposition 3.3.1. *The Poisson-Beta I distribution is*

$$f(x) = \frac{t^x B(x+\alpha, \beta)}{x! B(\alpha, \beta)} {}_1F_1(x+\alpha, x+\alpha+\beta; -t), \quad x = 0, 1, 2, \dots; \alpha, \beta > 0 \quad (3.12)$$

and its pgf is

$$G(s) = {}_1F_1(\alpha, \alpha+\beta; -t(1-s)) \quad (3.13)$$

Proof. The mixed Poisson distribution is

$$\begin{aligned} f(x) &= \int_0^1 \frac{e^{-\lambda t} (\lambda t)^x \lambda^{\alpha-1} (1-\lambda)^{\beta-1}}{x! B(\alpha, \beta)} d\lambda \\ &= \frac{t^x}{x! B(\alpha, \beta)} \int_0^1 \lambda^{x+\alpha-1} (1-\lambda)^{(x+\alpha)+\beta-(x+\alpha)-1} e^{-\lambda t} d\lambda \\ &= \frac{t^x B(x+\alpha, \beta)}{x! B(\alpha, \beta)} {}_1F_1(x+\alpha, x+\alpha+\beta; -t), \quad x = 0, 1, 2, \dots; \alpha, \beta > 0 \end{aligned}$$

which is (3.12) and has pgf

$$\begin{aligned} G(s) &= \frac{1}{B(\alpha, \beta)} \int_0^1 \lambda^{\alpha-1} (1-\lambda)^{\alpha+\beta-\alpha-1} e^{-[t(1-s)]\lambda} d\lambda \\ &= {}_1F_1(\alpha, \alpha+\beta; -t(1-s)) \end{aligned}$$

as obtained by Gurland (1958) and Katti (1966). □

3.3.2 Rectangular distribution

The Rectangular distribution is

$$g(\lambda) = \frac{1}{b-a}, \quad a \leq \lambda \leq b \quad (3.14)$$

Proposition 3.3.2. *The Poisson-Rectangular distribution is*

$$f(x) = \frac{t^x}{(x+1)!(b-a)} \left\{ b^{x+1} {}_1F_1(x+1, x+2; -bt) - a^{x+1} {}_1F_1(x+1, x+2; -at) \right\} \quad (3.15)$$

and its pgf is

$$G(s) = \frac{1}{(b-a)(1-s)t} \left\{ e^{-bt(1-s)} - e^{-at(1-s)} \right\} \quad (3.16)$$

Proof. The mixed Poisson distribution is

$$\begin{aligned} f(x) &= \int_a^b e^{-\lambda t} \frac{(\lambda t)^x}{x!} \frac{d\lambda}{b-a} \\ &= \frac{t^x}{x!(b-a)} \left\{ \int_0^b e^{-\lambda t} \lambda^x d\lambda - \int_0^a e^{-\lambda t} \lambda^x d\lambda \right\} \end{aligned}$$

and making the substitution $y = \lambda t$ we obtain

$$\begin{aligned} f(x) &= \frac{t^x}{x!(b-a)} \left\{ \int_0^{bt} e^{-y} \frac{y^x}{t^{x+1}} dy - \int_0^{at} e^{-y} \frac{y^x}{t^{x+1}} dy \right\} \\ &= \frac{1}{x!(b-a)t} \{ \gamma(x+1, bt) - \gamma(x+1, at) \} \\ &= \frac{1}{x!(b-a)t} \left\{ \frac{1}{x+1} (bt)^{x+1} {}_1F_1(x+1, x+2; -bt) \right\} \\ &\quad - \frac{1}{x!(b-a)t} \left\{ \frac{1}{x+1} (at)^{x+1} {}_1F_1(x+1, x+2; -at) \right\} \\ &= \frac{t^x}{(x+1)!(b-a)} \{ b^{x+1} {}_1F_1(x+1; x+2; -bt) - a^{x+1} {}_1F_1(x+1; x+2; -at) \} \end{aligned}$$

which is (3.15) and its pgf is

$$\begin{aligned} G(s) &= \int_a^b e^{-\lambda t(1-s)} \frac{d\lambda}{b-a} \\ &= \frac{1}{(b-a)(1-s)t} \{ e^{-bt(1-s)} - e^{-at(1-s)} \} \end{aligned}$$

which yields a result obtained by Bhattacharya and Holla (1965) for $t = 1$. \square

3.3.3 Beta II distribution

The beta distribution of second kind also known as inverted beta distribution is

$$g(\lambda) = \frac{\lambda^{\alpha-1}}{B(\alpha, \beta) (1+\lambda)^{\alpha+\beta}}, \quad \lambda > 0; \quad \alpha, \beta > 0 \quad (3.17)$$

Proposition 3.3.3. *The Poisson-Beta II distribution is*

$$f(x) = \frac{t^x \Gamma(x+\alpha)}{x! B(\alpha, \beta)} \psi(x+\alpha, x-\beta+1; t), \quad x = 0, 1, 2, \dots; \quad \alpha, \beta > 0 \quad (3.18)$$

and its pgf is

$$G(s) = \frac{\Gamma(\alpha)}{B(\alpha, \beta)} \psi(\alpha, 1-\beta; t(1-s)), \quad 0 < \beta < 1 \quad (3.19)$$

Proof. The mixed Poisson distribution is

$$\begin{aligned}
f(x) &= \int_0^{\infty} e^{-\lambda t} \frac{(\lambda t)^x}{x!} \frac{\lambda^{\alpha-1}}{B(\alpha, \beta) (1+\lambda)^{\alpha+\beta}} d\lambda \\
&= \frac{t^x}{x! B(\alpha, \beta)} \int_0^{\infty} \lambda^{x+\alpha-1} (1+\lambda)^{x+1-\beta-(x+\alpha)-1} e^{-\lambda t} d\lambda \\
&= \frac{t^x \Gamma(x+\alpha)}{x! B(\alpha, \beta)} \psi(x+\alpha, x-\beta+1; t)
\end{aligned}$$

which is (3.18) and its pgf is

$$\begin{aligned}
G(s) &= \frac{1}{B(\alpha, \beta)} \int_0^{\infty} \lambda^{\alpha-1} (1+\lambda)^{1-\beta-\alpha-1} e^{-\lambda t(1-s)} d\lambda \\
&= \frac{\Gamma(\alpha)}{B(\alpha, \beta)} \psi(\alpha, 1-\beta; t(1-s)), \quad 0 < \beta < 1
\end{aligned}$$

□

3.3.4 Scaled Beta distribution

Consider the classical Beta (Beta I) distribution

$$w(y) = \frac{y^{\alpha-1} (1-y)^{\beta-1}}{B(\alpha, \beta)}, \quad 0 < y < 1; \quad \alpha, \beta > 0$$

and making the substitution

$$y = \frac{\lambda}{\mu} \implies \lambda = \mu y \text{ and } \frac{dy}{d\lambda} = \frac{1}{\mu}$$

we have the scaled Beta distribution

$$g(\lambda) = \frac{\lambda^{\alpha-1} (\mu - \lambda)^{\beta-1}}{\mu^{\alpha+\beta-1} B(\alpha, \beta)}, \quad 0 < \lambda < \mu; \quad \alpha, \beta > 0 \quad (3.20)$$

Proposition 3.3.4. *The Poisson-Scaled Beta distribution is*

$$f(x) = \frac{(\mu t)^x}{x!} \frac{B(\alpha+x, \beta)}{B(\alpha, \beta)} {}_1F_1(\alpha+x, \alpha+x+\beta; -\mu t), \quad x = 0, 1, 2, \dots \quad (3.21)$$

and its pgf is

$$G(s) = {}_1F_1(\alpha, \alpha+\beta; -\mu t(1-s)) \quad (3.22)$$

Proof. The mixed Poisson distribution is

$$f(x) = \frac{t^x}{x! \mu^{\alpha+\beta-1} B(\alpha, \beta)} \int_0^{\mu} e^{-\lambda t} \lambda^{\alpha+x-1} (\mu - \lambda)^{\beta-1} d\lambda$$

and making the substitution $\lambda = \mu z$ we obtain

$$\begin{aligned} f(x) &= \frac{t^x \mu^x}{x! B(\alpha, \beta)} \int_0^1 z^{\alpha+x-1} (1-z)^{\alpha+x+\beta-(\alpha+x)-1} e^{-\mu z t} dz \\ &= \frac{(\mu t)^x B(\alpha+x, \beta)}{x! B(\alpha, \beta)} {}_1F_1(\alpha+x, \alpha+x+\beta; -\mu t), \quad x = 0, 1, 2, \dots \end{aligned}$$

Its pgf is

$$G(s) = \frac{1}{\mu^{\alpha+\beta-1} B(\alpha, \beta)} \int_0^\mu \lambda^{\alpha-1} (\mu-\lambda)^{\beta-1} e^{-\lambda t(1-s)} d\lambda$$

and using the above substitution we obtain

$$\begin{aligned} G(s) &= \frac{\mu^{\alpha-1+\beta-1+1}}{\mu^{\alpha+\beta-1} B(\alpha, \beta)} \int_0^1 z^{\alpha-1} (1-z)^{\alpha+\beta-\alpha-1} e^{-\mu t(1-s)z} dz \\ &= {}_1F_1(\alpha, \alpha+\beta; -\mu t(1-s)) \end{aligned}$$

which yields a result obtained by Willmot (1986) when $t = 1$.

A more general situation is given by letting

$$y = \frac{\lambda - \sigma}{\mu} \implies \lambda = \mu y + \sigma \text{ and } \frac{dy}{d\lambda} = \frac{1}{\mu}$$

If

$$\omega(y) = \frac{y^{\alpha-1} (1-y)^{\beta-1}}{B(\alpha, \beta)}, \quad 0 < y < 1; \quad \alpha, \beta > 0$$

then

$$g(\lambda) = \frac{(\lambda - \sigma)^{\alpha-1} ((\mu + \sigma) - \lambda)^{\beta-1}}{\mu^{\alpha+\beta-1} B(\alpha, \beta)}, \quad \sigma < \lambda < \sigma + \mu; \quad \alpha, \beta, \sigma > 0$$

and therefore

$$f(x) = \frac{t^x}{x! \mu^{\alpha+\beta-1} B(\alpha, \beta)} \int_\sigma^{\sigma+\mu} \lambda^x (\lambda - \sigma)^{\alpha-1} [(\sigma + \mu) - \lambda]^{\beta-1} e^{-\lambda t} d\lambda$$

and making the substitution

$$z = \frac{\lambda - \sigma}{\mu} \implies \lambda = \mu z + \sigma \text{ and } d\lambda = \mu dz,$$

we obtain

$$\begin{aligned} f(x) &= \frac{t^x}{x! \mu^{\alpha+\beta-1} B(\alpha, \beta)} \int_0^1 (\mu z + \sigma)^x (\mu z)^{\alpha-1} [(\sigma + \mu) - \mu z - \sigma]^{\beta-1} e^{-(\mu z + \sigma)t} \mu dz \\ &= \frac{t^x e^{-\sigma t}}{x! B(\alpha, \beta)} \int_0^1 (\mu z + \sigma)^x z^{\alpha-1} (1-z)^{\beta-1} e^{-\mu z t} dz \\ &= \frac{t^x e^{-\sigma t}}{x! B(\alpha, \beta)} \int_0^1 \left\{ \sum_{k=0}^x \binom{x}{k} \sigma^{x-k} (\mu z)^k \right\} z^{\alpha-1} (1-z)^{\beta-1} e^{-\mu z t} dz \\ &= \frac{t^x e^{-\sigma t}}{x! B(\alpha, \beta)} \sum_{k=0}^x \left\{ \binom{x}{k} \sigma^{x-k} \mu^k B(\alpha+k, \beta) {}_1F_1(\alpha+k, \alpha+k+\beta; -\mu t) \right\}. \quad (3.23) \end{aligned}$$

Its pgf is

$$G(s) = \frac{1}{\mu^{\alpha+\beta-1} B(\alpha, \beta)} \int_{\sigma}^{\sigma+\mu} e^{-\lambda t(1-s)} (\lambda - \sigma)^{\alpha-1} [(\mu + \sigma) - \lambda]^{\beta-1} d\lambda$$

and making the above substitution we obtain

$$\begin{aligned} G(s) &= \frac{e^{-\sigma t(1-s)}}{B(\alpha, \beta)} \int_0^1 z^{\alpha-1} (1-z)^{\alpha+\beta-\alpha-1} e^{-\mu t(1-s)z} dz \\ &= e^{-\sigma t(1-s)} {}_1F_1(\alpha, \alpha + \beta; -\mu t(1-s)) \end{aligned} \quad (3.24)$$

The special case when $\alpha = t = 1$ is

$$G(s) = e^{\sigma(s-1)} {}_1F_1(1, 1 + \beta; \mu(s-1)) \quad (3.25)$$

as obtained by Willmot (1986). \square

3.3.5 The Full Beta model

Kempton (1975) mixed two gamma distributions to obtain what he called Full beta model given by

$$\begin{aligned} g(\lambda) &= \int_0^{\infty} \frac{a^p}{\Gamma(p)} e^{-a\lambda} \lambda^{p-1} \cdot \frac{1}{b^q \Gamma(q)} e^{-\frac{a}{b}} a^{q-1} da \\ &= \frac{b^p}{B(p, q)} \frac{\lambda^{p-1}}{(1+b\lambda)^{p+q}}, \quad \lambda > 0; b, p, q > 0 \end{aligned} \quad (3.26)$$

This distribution can also be obtained by the following transformation:

From the Beta II pdf

$$\omega(y) = \frac{y^{p-1}}{B(p, q) (1+y)^{p+q}}, \quad y > 0; p, q > 0$$

we make the substitution

$$y = b\lambda \implies \frac{dy}{d\lambda} = b$$

and therefore the Full beta distribution is

$$g(\lambda) = \frac{b^p}{B(p, q)} \frac{\lambda^{p-1}}{(1+b\lambda)^{p+q}}, \quad \lambda > 0; b, p, q > 0$$

Proposition 3.3.5. *The Poisson-Full Beta distribution is*

$$f(x) = \left(\frac{t}{b}\right)^x \frac{\Gamma(x+p)}{B(p, q) x!} \psi\left(x+p, x+1-q; \frac{t}{b}\right), \quad x = 0, 1, 2, \dots \quad (3.27)$$

and its pgf is

$$G(s) = \frac{\Gamma(p)}{B(p, q)} \psi\left(p, 1-q; \frac{t}{b}(1-s)\right) \quad (3.28)$$

Proof. The mixed Poisson distribution is

$$f(x) = \frac{b^p}{B(p, q)} \frac{t^x}{x!} \int_0^\infty e^{-\lambda t} \lambda^{x+p-1} (1+b\lambda)^{-p-q} d\lambda$$

and making the substitution

$$z = b\lambda \implies \lambda = \frac{z}{b} \text{ and } d\lambda = \frac{dz}{b},$$

we obtain

$$\begin{aligned} f(x) &= \frac{b^p t^x}{B(p, q) x!} \frac{1}{b^{x+p}} \int_0^\infty z^{x+p-1} (1+z)^{x+1-q-(x+p)-1} e^{-\frac{t}{b}z} dz \\ &= \frac{b^p t^x}{B(p, q) x!} \frac{\Gamma(x+p)}{b^{x+p}} \psi\left(x+p, x+1-q; \frac{t}{b}\right) \\ &= \frac{1}{x!} \left(\frac{t}{b}\right)^x \frac{\Gamma(x+p)}{B(p, q)} \psi\left(x+p, x+1-q; \frac{t}{b}\right). \end{aligned}$$

and using relation (3.5) we have

$$f(x) = \left(\frac{t}{b}\right)^x \frac{\Gamma(x+p)}{B(p, q) x!} \psi\left(p+q, x+1-q; \frac{t}{b}\right), \quad x = 0, 1, 2, \dots$$

which yields the result given by Gupta and Ong (2005) when $t = 1$.

Its pgf is

$$G(s) = \frac{b^p}{B(p, q)} \int_0^\infty \lambda^{p-1} (1+b\lambda)^{-p-q} e^{-\lambda t(1-s)} d\lambda$$

and making the substitution

$$z = b\lambda \implies \lambda = \frac{z}{b} \text{ and } d\lambda = \frac{dz}{b}$$

we obtain

$$\begin{aligned} G(s) &= \frac{1}{B(p, q)} \int_0^\infty z^{p-1} (1+z)^{1-p-q-1} e^{-\frac{t}{b}(1-s)z} dz \\ &= \frac{\Gamma(p)}{B(p, q)} \psi\left(p, 1-q; \frac{t}{b}(1-s)\right) \end{aligned}$$

□

3.3.6 Pearson Type I Distribution

The Pearson Type I distribution is

$$g(\lambda) = \frac{1}{B(p, q)} \frac{(\lambda-a)^{p-1}}{(b-a)^{p-1}} \frac{(b-\lambda)^{q-1}}{(b-a)^{q-1}} \frac{1}{b-a}, \quad a \leq \lambda \leq b \quad (3.29)$$

Proposition 3.3.6. *The Poisson-Pearson Type I distribution is*

$$f(x) = \frac{(at)^x e^{-at} \Gamma(p+q)}{x! \Gamma(p)} \sum_{k=0}^x \binom{x}{k} \left(\frac{b-a}{a}\right)^k \frac{\Gamma(k+q)}{\Gamma(k+p+q)} {}_1F_1(k+p; k+p+q; -(b-a)t), \quad x = 0, 1, 2, \dots \quad (3.30)$$

and its pgf is

$$G(s) = e^{-at(1-s)} B(p, q) {}_1F_1(p, p+q; -(b-a)t(1-s)) \quad (3.31)$$

Proof. The mixed Poisson distribution is

$$\begin{aligned} f(x) &= \int_a^b e^{-\lambda t} \frac{(\lambda t)^x}{x!} \frac{1}{B(p, q)} \frac{(\lambda-a)^{p-1} (b-\lambda)^{q-1}}{(b-a)^{p-1} (b-a)^{q-1} b-a} d\lambda \\ &= \frac{t^x}{x! B(p, q)} \int_a^b e^{-\lambda t} \lambda^x \left(\frac{\lambda-a}{b-a}\right)^{p-1} \left[1 - \frac{\lambda-a}{b-a}\right]^{q-1} \frac{d\lambda}{b-a} \end{aligned}$$

and making the substitution

$$z = \frac{\lambda-a}{b-a} \implies \lambda = a + (b-a)z \text{ and } d\lambda = (b-a) dz$$

we obtain

$$\begin{aligned} f(x) &= \frac{t^x e^{-at}}{x! B(p, q)} \int_0^1 e^{-(b-a)tz} [a + (b-a)z]^x z^{p-1} (1-z)^{q-1} dz \\ &= \frac{t^x e^{-at}}{x! B(p, q)} \int_0^1 e^{-(b-a)tz} \left[\sum_{k=0}^x \binom{x}{k} a^{x-k} (b-a)^k z^k \right] z^{p-1} (1-z)^{q-1} dz \\ &= \frac{t^x e^{-at}}{x!} \sum_{k=0}^x \left\{ \binom{x}{k} a^{x-k} (b-a) \frac{B(k+p, q)}{B(p, q)} {}_1F_1(k+p, k+p+q; -(b-a)t) \right\} \\ &= \frac{(at)^x e^{-at} \Gamma(p+q)}{x! \Gamma(p)} \sum_{k=0}^x \binom{x}{k} \left(\frac{b-a}{a}\right)^k \frac{\Gamma(k+p)}{\Gamma(k+p+q)} {}_1F_1(k+p, k+p+q; -(b-a)t) \end{aligned}$$

Its pgf is

$$\begin{aligned} G(s) &= \int_0^1 e^{-[a+(b-a)z]t(1-s)} z^{p-1} (1-z)^{q-1} dz \\ &= e^{-at(1-s)} B(p, q) {}_1F_1(p, p+q; -(b-a)t(1-s)) \end{aligned}$$

□

3.3.7 Pearson Type VI Distribution

The Pearson Type VI distribution is

$$g(\lambda) = \frac{\left(\frac{\lambda-d}{d-c}\right)^{b-a-1} \frac{1}{d-c}}{B(a, b-a) \left(1 + \frac{\lambda-d}{d-c}\right)^b}, \quad \lambda > d; a, b, c, d > 0 \quad (3.32)$$

Proposition 3.3.7. *The Poisson-Pearson Type VI distribution is*

$$f(x) = \frac{(dt)^x e^{-dt}}{x! B(a, b-a)} \sum_{k=0}^x \left\{ \binom{x}{k} \left(\frac{d-c}{d}\right)^k \Gamma(k+b-a) \psi(k+b-a, k-a+1; (d-c)t) \right\} \quad (3.33)$$

and its pgf is

$$G(s) = \frac{e^{-dt(1-s)}}{B(a, b-a)} \Gamma(b-a) \psi(b-a, 1-a; (d-c)t(1-s)) \quad (3.34)$$

Proof. The mixed Poisson distribution is

$$f(x) = \int_d^\infty e^{-\lambda t} \frac{(\lambda t)^x}{x!} \frac{\left(\frac{\lambda-d}{d-c}\right)^{b-a-1} \frac{1}{d-c}}{B(a, b-a) \left(1 + \frac{\lambda-d}{d-c}\right)^b} d\lambda$$

and making the substitution

$$z = \frac{\lambda-d}{d-c} \implies \lambda = d + (d-c)z \text{ and } d\lambda = (d-c)dz$$

we obtain

$$\begin{aligned} f(x) &= \frac{t^x e^{-dt}}{x! B(a, b-a)} \int_0^\infty (d + (d-c)z)^x z^{b-a-1} (1+z)^{-b} e^{-(d-c)tz} dz \\ &= \frac{t^x e^{-dt}}{x! B(a, b-a)} \int_0^\infty \left\{ \sum_{k=0}^x \binom{x}{k} d^{x-k} (d-c)^k z^{k+b-a-1} \right\} (1+z)^{-b} e^{-(d-c)tz} dz \\ &= \frac{(dt)^x e^{-dt}}{x! B(a, b-a)} \sum_{k=0}^x \left\{ \binom{x}{k} \left(\frac{d-c}{d}\right)^k \Gamma(k+b-a) \psi(k+b-a, k-a+1; (d-c)t) \right\} \end{aligned}$$

and for $x = 0$ we have

$$\begin{aligned} f(0) &= \frac{e^{-dt}}{B(a, b-a)} \Gamma(b-a) \psi(b-a, 1-a; (d-c)t) \\ &= \frac{e^{-dt} \Gamma(b)}{\Gamma(a) \Gamma(b-a)} \Gamma(b-a) \psi(b-a, 1-a; (d-c)t) \\ &= e^{-dt} \frac{\Gamma(b)}{\Gamma(a)} \psi(b-a, 1-a; (d-c)t) \end{aligned}$$

which is a result given by Albretch (1984) when $t=1$.

Its pgf is

$$\begin{aligned}
G(s) &= \int_d^\infty \frac{e^{-\lambda t(1-s)} \left(\frac{\lambda-d}{d-c}\right)^{b-a-1}}{B(a, b-a) \left(1 + \frac{\lambda-d}{d-c}\right)^b d-c} d\lambda \\
&= \frac{e^{-dt(1-s)}}{B(a, b-a)} \int_0^\infty z^{b-a-1} (1+z)^{-a+1-(b-a)-1} e^{-(d-c)t(1-s)z} dz \\
&= \frac{e^{-dt(1-s)}}{B(a, b-a)} \Gamma(b-a) \psi(b-a, 1-a; (d-c)t(1-s))
\end{aligned} \tag{3.35}$$

□

3.3.8 Shifted Gamma (Pearson Type III) Distribution

Consider Shifted Gamma distribution given by (2.32).

Proposition 3.3.8. *The Poisson-Shifted Gamma distribution is*

$$f(x) = \frac{(\mu t)^x (\mu \beta)^\alpha e^{-\mu t}}{x!} \psi(\alpha; \alpha + x + 1; (t + \beta) \mu), \quad x = 0, 1, 2, \dots \tag{3.36}$$

and its pgf is

$$G(s) = \frac{\beta^\alpha}{\Gamma(\alpha)} e^{-\mu t(1-s)} \Gamma(\alpha) \psi(\alpha; \alpha + 1; t(1-s) + \beta) \tag{3.37}$$

Proof. The mixed Poisson distribution is

$$\begin{aligned}
f(x) &= \frac{t^x \beta^\alpha}{x! \Gamma(\alpha)} \int_\mu^\infty e^{-\lambda t} \lambda^x e^{-\beta(\lambda-\mu)} (\lambda-\mu)^{\alpha-1} d\lambda \\
&= \frac{t^x \beta^\alpha e^{-\mu t}}{x! \Gamma(\alpha)} \int_\mu^\infty \lambda^x (\lambda-\mu)^{\alpha-1} e^{-(t+\beta)(\lambda-\mu)} d\lambda
\end{aligned}$$

and making the substitution

$$z = \lambda - \mu \implies \lambda = z + \mu \text{ and } d\lambda = dz,$$

we obtain

$$f(x) = \frac{t^x \beta^\alpha e^{-\mu t}}{x! \Gamma(\alpha)} \int_0^\infty (z + \mu)^x z^{\alpha-1} e^{-(t+\beta)z} dz$$

Next, making the substitution

$$z = \mu y \implies dz = \mu dy$$

we obtain

$$\begin{aligned}
f(x) &= \frac{t^x \beta^\alpha e^{-\mu t}}{x! \Gamma(\alpha)} \int_0^\infty \mu^x (1+y)^x \mu^{\alpha-1} y^{\alpha-1} e^{-(t+\beta)\mu y} \mu dy \\
&= \frac{t^x \beta^\alpha e^{-\mu t} \mu^{x+\alpha}}{x! \Gamma(\alpha)} \int_0^\infty y^{\alpha-1} (1+y)^{\alpha+x+1-\alpha-1} e^{-(t+\beta)\mu y} dy \\
&= \frac{(\mu t)^x (\mu \beta)^\alpha e^{-\mu t}}{x!} \psi(\alpha, \alpha+x+1; (t+\beta)\mu)
\end{aligned}$$

as obtained by Rolski *et al* (1999). Using the relation (3.5), Albretcht (1984) obtained

$$f(x) = \frac{t^x \beta^\alpha e^{-\mu t}}{x!} [(t+\beta)]^{-(\alpha+x)} \psi(-x, 1-\alpha-x, (t+\beta)\mu). \quad (3.38)$$

By letting $z = \lambda - \mu$, its pgf is

$$\begin{aligned}
G(s) &= \int_\mu^\infty e^{-\lambda t(1-s)} \frac{\beta^\alpha}{\Gamma(\alpha)} e^{-\beta(\lambda-\mu)} (\lambda-\mu)^{\alpha-1} d\lambda \\
&= \frac{\beta^\alpha}{\Gamma(\alpha)} e^{-\mu t(1-s)} \int_\mu^\infty (\lambda-\mu)^{\alpha-1} e^{-(\lambda-\mu)[t(1-s)+\beta]} d\lambda \\
&= \frac{\beta^\alpha}{\Gamma(\alpha)} e^{-\mu t(1-s)} \int_0^\infty z^{\alpha-1} (1+z)^{\alpha+1-\alpha-1} e^{-z[t(1-s)+\beta]} dz \\
&= \frac{\beta^\alpha}{\Gamma(\alpha)} e^{-\mu t(1-s)} \Gamma(\alpha) \psi(\alpha, \alpha+1; t(1-s)+\beta)
\end{aligned}$$

□

3.3.9 Truncated Gamma (from above) Distribution

A two-parameter gamma is

$$h(y) = \frac{a^b}{\Gamma(b)} e^{-ay} y^{b-1}; \quad y > 0; \quad a, b > 0 \quad (3.39)$$

Consider the integral

$$I = \int_0^p e^{-ay} y^{b-1} dy, \quad p > 0$$

and making the substitution

$$x = ay \implies y = \frac{x}{a} \text{ and } dy = \frac{dx}{a}$$

we obtain

$$\begin{aligned}
I &= \int_0^{ap} e^{-x} \left(\frac{x}{a}\right)^{b-1} \frac{dx}{a} \\
&= \frac{1}{a^b} \int_0^{ap} e^{-x} x^{b-1} dx \\
&= \frac{1}{a^b} \gamma(b, ap).
\end{aligned}$$

Therefore

$$\frac{a^b}{\Gamma(b)} \int_0^p e^{-ay} y^{b-1} dy = \frac{a^b}{\Gamma(b)} \frac{1}{a^b} \gamma(b, ap) = \frac{\gamma(b, ap)}{\Gamma(b)}$$

where $\gamma(b, ap)$ is a truncated gamma function. Therefore,

$$\begin{aligned}
\frac{a^b}{\Gamma(b)} \frac{\Gamma(b)}{\gamma(b, ap)} \int_0^p e^{-ay} y^{b-1} dy &= 1 \\
\int_0^p \frac{a^b}{\gamma(b, ap)} e^{-ay} y^{b-1} dy &= 1
\end{aligned}$$

which also implies that

$$\int_0^p e^{-ay} y^{b-1} dy = \frac{\gamma(b, ap)}{a^b}$$

Thus, the mixing distribution to be considered is truncated gamma (from above)

$$g(\lambda) = \frac{a^b}{\gamma(b, ap)} e^{-a\lambda} \lambda^{b-1}, \quad 0 < \lambda < p; \quad a, b > 0 \tag{3.40}$$

where p is the truncation parameter.

Proposition 3.3.9. *The Poisson-truncated gamma (from above) distribution is*

$$f(x) = \frac{(pt)^x}{x!} \frac{b}{b+x} \frac{{}_1F_1(x+b, x+b+1; -pt-ap)}{{}_1F_1(b, b+1; -ap)} \tag{3.41}$$

and its pgf is

$$G(s) = \frac{{}_1F_1(b, b+1; -pt(1-s) - ap)}{b^{-1} (ap)^b {}_1F_1(b, b+1; -ap)} \tag{3.42}$$

Proof. The mixed Poisson distribution is

$$\begin{aligned}
f(x) &= \frac{t^x a^b}{x! \gamma(b, ap)} \int_0^p e^{-\lambda(t+a)} \lambda^{x+b-1} d\lambda \\
&= \frac{t^x a^b}{x! \gamma(b, ap)} \frac{\gamma(x+b, (t+a)p)}{(t+a)^{x+b}} \\
&= \frac{(pt)^x (ap)^b}{x! (pt+ap)^{x+b}} \frac{\gamma(x+b, (pt+ap))}{\gamma(b, ap)}
\end{aligned}$$

and using relation (3.9) we obtain

$$\begin{aligned} f(x) &= \frac{(pt)^x (ap)^b}{x! (pt+ap)^{x+b}} \frac{(x+b)^{-1} (pt+ap)^{x+b} {}_1F_1(x+b, x+b+1; -pt-ap)}{b^{-1} (ap)^b {}_1F_1(b, b+1; -ap)} \\ &= \frac{(pt)^x}{x!} \frac{b}{b+x} \frac{{}_1F_1(x+b, x+b+1; -pt-ap)}{{}_1F_1(b, b+1; -ap)} \end{aligned}$$

and using relation (3.9), we obtain

$$f(x) = \frac{(pt)^x (ap)^b}{x! (pt+ap)^{x+b}} \frac{(x+b)^{-1} (pt+ap)^{x+b}}{b^{-1} (ap)^b} \frac{e^{-pt-ap} {}_1F_1(1, x+b+1; pt+ap)}{e^{-ap} {}_1F_1(1, b+1; ap)} \quad (3.43)$$

which yields a result given by Johnson et al. (2005) for $t = 1$.

Its pgf is

$$\begin{aligned} G(s) &= \int_0^p e^{-\lambda t(1-s)} \frac{a^b e^{-a\lambda} \lambda^{b-1}}{\gamma(b, ap)} d\lambda \\ &= \frac{a^b}{[t(1-s) + a]^b} \frac{\gamma(b, (t(1-s) + a)p)}{\gamma(b, ap)} \\ &= \frac{{}_1F_1(b, b+1; -pt(1-s) - ap)}{b^{-1} (ap)^b {}_1F_1(b, b+1; -ap)}. \end{aligned}$$

□

3.3.10 Truncated Gamma (from below) Distribution

Consider gamma distribution with two parameters α and β

$$h(y) = \frac{\beta^\alpha}{\Gamma(\alpha)} e^{-\beta y} y^{\alpha-1}, \quad y > 0; \quad \alpha, \beta > 0$$

therefore

$$\begin{aligned} \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^\infty e^{-\beta y} y^{\alpha-1} dy &= 1 \\ \frac{\beta^\alpha}{\Gamma(\alpha)} \left\{ \int_0^{\lambda_0} e^{-\beta y} y^{\alpha-1} dy + \int_{\lambda_0}^\infty e^{-\beta y} y^{\alpha-1} dy \right\} &= 1 \\ \frac{\gamma(\alpha, \beta\lambda_0)}{\Gamma(\alpha)} + \frac{\beta^\alpha}{\Gamma(\alpha)} \int_{\lambda_0}^\infty e^{-\beta y} y^{\alpha-1} dy &= 1. \end{aligned}$$

Therefore

$$\begin{aligned} \int_{\lambda_0}^\infty e^{-\beta y} y^{\alpha-1} dy &= \frac{\Gamma(\alpha)}{\beta^\alpha} - \frac{\gamma(\alpha, \beta\lambda_0)}{\beta^\alpha} \\ &= \frac{1}{\beta^\alpha} \{ \Gamma(\alpha) - \gamma(\alpha, \beta\lambda_0) \} \end{aligned}$$

which implies that

$$\int_{\lambda_0}^{\infty} \frac{\beta^\alpha e^{-\beta y} y^{\alpha-1} dy}{\Gamma(\alpha) - \gamma(\alpha, \beta \lambda_0)} = 1.$$

Therefore a truncated gamma (from below) distribution is

$$g(\lambda) = \frac{\beta^\alpha e^{-\beta \lambda} \lambda^{\alpha-1}}{\Gamma(\alpha) - \gamma(\alpha, \beta \lambda_0)}, \quad \lambda > \lambda_0 \quad (3.44)$$

where

$$\gamma(\alpha, \beta \lambda_0) = \int_0^{\beta \lambda_0} e^{-y} y^{\alpha-1} dy.$$

Proposition 3.3.10. *The Poisson-truncated gamma (from below) distribution is*

$$f(x) = \frac{1}{x!} \left(\frac{\beta}{t + \beta} \right)^\alpha \left(\frac{t}{t + \beta} \right)^x \frac{\Gamma(\alpha + x) - \gamma(\alpha + x, (t + \beta) \lambda_0)}{\Gamma(\alpha) - \gamma(\alpha, \beta \lambda_0)} \quad (3.45)$$

and its pgf is

$$G(s) = \left(\frac{\beta}{\beta + t(1-s)} \right)^\alpha \frac{\Gamma(\alpha) - \gamma(\alpha, [t(1-s) + \beta] \lambda_0)}{\Gamma(\alpha) - \gamma(\alpha, \beta \lambda_0)}. \quad (3.46)$$

Proof. The mixed Poisson distribution is

$$\begin{aligned} f(x) &= \int_{\lambda_0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^x}{x!} \frac{\beta^\alpha e^{-\beta \lambda} \lambda^{\alpha-1}}{\Gamma(\alpha) - \gamma(\alpha, \beta \lambda_0)} d\lambda \\ &= \frac{1}{x!} \left(\frac{\beta}{t + \beta} \right)^\alpha \left(\frac{t}{t + \beta} \right)^x \frac{\Gamma(\alpha + x) - \gamma(\alpha + x, (t + \beta) \lambda_0)}{\Gamma(\alpha) - \gamma(\alpha, \beta \lambda_0)} \end{aligned}$$

and its pgf is

$$\begin{aligned} G(s) &= \int_{\lambda_0}^{\infty} e^{-\lambda t(1-s)} \frac{\beta^\alpha e^{-\beta \lambda} \lambda^{\alpha-1}}{\Gamma(\alpha) - \gamma(\alpha, \beta \lambda_0)} d\lambda \\ &= \left(\frac{\beta}{\beta + t(1-s)} \right)^\alpha \frac{\Gamma(\alpha) - \gamma(\alpha, [t(1-s) + \beta] \lambda_0)}{\Gamma(\alpha) - \gamma(\alpha, \beta \lambda_0)}. \end{aligned}$$

□

3.3.11 Truncated Gamma (from above and below) Distribution

Consider the integral

$$\begin{aligned} \int_a^b e^{-\beta y} y^{\alpha-1} dy &= \int_0^b e^{-\beta y} y^{\alpha-1} dy - \int_0^a e^{-\beta y} y^{\alpha-1} dy \\ &= \frac{\gamma(\alpha, \beta b)}{\beta^\alpha} - \frac{\gamma(\alpha, \beta a)}{\beta^\alpha} \end{aligned}$$

therefore

$$\int_a^b \beta^\alpha e^{-\beta y} y^{\alpha-1} dy = \gamma(\alpha, \beta b) - \gamma(\alpha, \beta a)$$

which implies

$$\int_a^b \frac{\beta^\alpha e^{-\beta y} y^{\alpha-1}}{\gamma(\alpha, \beta b) - \gamma(\alpha, \beta a)} dy = 1.$$

Hence truncated gamma (from above and below) distribution is

$$g(\lambda) = \frac{\beta^\alpha e^{-\beta \lambda} \lambda^{\alpha-1}}{\gamma(\alpha, \beta b) - \gamma(\alpha, \beta a)}, \quad 0 < a < \lambda < b < \infty; \quad \alpha, \beta > 0. \quad (3.47)$$

Proposition 3.3.11. *The Poisson - truncated gamma (from below and above) distribution is*

$$f(x) = \frac{1}{x!} \left(\frac{t}{t+\beta} \right)^x \left(\frac{\beta}{t+\beta} \right)^\alpha \frac{\gamma(x+\alpha, (t+\beta)b) - \gamma(x+\alpha, (t+\beta)a)}{[\gamma(\alpha, \beta b) - \gamma(\alpha, \beta a)]} \quad (3.48)$$

and its pgf is

$$G(s) = \left[\frac{\beta}{\beta + t(1-s)} \right]^\alpha \left\{ \frac{\gamma(\alpha, [\beta + t(1-s)]b) - \gamma(\alpha, [\beta + t(1-s)]a)}{\gamma(\alpha, \beta b) - \gamma(\alpha, \beta a)} \right\}. \quad (3.49)$$

Proof. The mixed Poisson distribution is

$$\begin{aligned} f(x) &= \int_a^b e^{-\lambda t} \frac{(\lambda t)^x}{x!} \frac{\beta^\alpha e^{-\beta \lambda} \lambda^{\alpha-1}}{\gamma(\alpha, \beta b) - \gamma(\alpha, \beta a)} d\lambda \\ &= \frac{t^x \beta^\alpha}{x! [\gamma(\alpha, \beta b) - \gamma(\alpha, \beta a)]} \int_a^b \lambda^{x+\alpha-1} e^{-\lambda(t+\beta)} d\lambda \\ &= \frac{1}{x!} \left(\frac{t}{t+\beta} \right)^x \left(\frac{\beta}{t+\beta} \right)^\alpha \frac{\gamma(x+\alpha, (t+\beta)b) - \gamma(x+\alpha, (t+\beta)a)}{[\gamma(\alpha, \beta b) - \gamma(\alpha, \beta a)]} \end{aligned}$$

and its pgf is

$$\begin{aligned} G(s) &= \int_a^b e^{-\lambda t(1-s)} \frac{\beta^\alpha e^{-\beta \lambda} \lambda^{\alpha-1}}{\gamma(\alpha, \beta b) - \gamma(\alpha, \beta a)} d\lambda \\ &= \left[\frac{\beta}{\beta + t(1-s)} \right]^\alpha \left\{ \frac{\gamma(\alpha, [\beta + t(1-s)]b) - \gamma(\alpha, [\beta + t(1-s)]a)}{\gamma(\alpha, \beta b) - \gamma(\alpha, \beta a)} \right\}. \end{aligned}$$

□

3.3.12 Truncated Pearson Type III Distribution

The Pearson differential equation is

$$\frac{1}{y} \frac{dy}{dx} = -\frac{a+x}{c_0 + c_1 x + c_2 x^2} \quad (3.50)$$

where $y = f(x)$ is a probability distribution function. Pearson Type III corresponds to the case of $c_2 = 0$ and $c_1 \neq 0$ in (3.50). Therefore

$$\begin{aligned}\frac{1}{y} \frac{dy}{dx} &= -\frac{x+a}{c_1x+c_0} \\ &= -\frac{1}{c_1} \left[\frac{x+a}{x+\frac{c_0}{c_1}} \right] \\ &= -\frac{1}{c_1} + \frac{\frac{c_0}{c_1} - a}{c_1x+c_0}\end{aligned}$$

$$\begin{aligned}\int \frac{dy}{y} &= \int \left[-\frac{1}{c_1} + \frac{\frac{c_0}{c_1} - a}{c_1x+c_0} \right] dx \\ \log y &= -\frac{x}{c_1} + (c_0c_1^{-1} - a) c_1^{-1} \log(c_1x+c_0) + \log K \\ &= -\frac{x}{c_1} + \log(c_1x+c_0)^m + \log K\end{aligned}$$

where $m = c_1^{-1} (c_0c_1^{-1} - a)$. Therefore

$$\begin{aligned}\log y &= \log e^{-\frac{x}{c_1}} + \log(c_1x+c_0)^m + \log K \\ y &= K e^{-\frac{x}{c_1}} (c_1x+c_0)^m, \quad c_1 \neq 0.\end{aligned}$$

If $c_1 > 0$, then $c_1x+c_0 > 0$ implies $x > -\frac{c_0}{c_1}$.

If $c_1 < 0$, let $c_1 = -\delta$ where $\delta > 0$ then $c_1x+c_0 > 0$ which implies that

$$-\delta x + c_0 > 0 \implies -\delta x > -c_0 \implies \delta x < c_0 \implies x < \frac{c_0}{\delta}$$

so that

$$c_1 < 0 \implies x < -\frac{c_0}{c_1}$$

a case we want to consider. Therefore

$$\begin{aligned}y &= K e^{-\frac{x}{c_1}} (c_1x+c_0)^m \\ &= K e^{\frac{x}{\delta}} (c_0 - \delta x)^m \\ &= K \delta^m e^{\frac{x}{\delta}} \left(\frac{c_0}{\delta} - x \right)^m.\end{aligned}$$

Making the substitution

$$\frac{c_0}{\delta} = 1 \text{ and } \alpha = \frac{1}{\delta}$$

we obtain

$$y = \frac{K}{\alpha^m} e^{\alpha x} (1-x)^m, \quad x < 1$$

but

$$y = f(x)$$

therefore

$$\begin{aligned}\int_0^1 f(x) dx &= \frac{K}{\alpha^m} \int_0^1 e^{\alpha x} (1-x)^m dx \\ 1 &= \frac{K}{\alpha^m} \int_0^1 e^{\alpha x} (1-x)^m dx.\end{aligned}$$

Consider the integral

$$\begin{aligned}\int_0^1 e^{\alpha x} (1-x)^m dx &= \int_0^1 x^{1-1} (1-x)^{2+m-1-1} e^{\alpha x} dx \\ &= B(1, m+1) {}_1F_1(1, m+2; \alpha)\end{aligned}\tag{3.51}$$

and making the substitution

$$\beta = m + 2 \implies \beta - 1 = m + 1 \text{ and } m = \beta - 2$$

we obtain

$$\int_0^1 e^{\alpha x} (1-x)^{\beta-2} dx = B(1, \beta-1) {}_1F_1(1, \beta; \alpha)\tag{3.52}$$

implying

$$\int_0^1 \frac{(1-x)^{\beta-2} e^{\alpha x}}{B(1, \beta-1) {}_1F_1(1, \beta; \alpha)} dx = 1.$$

Thus, the mixing distribution (Truncated Pearson Type III) under consideration is

$$g(\lambda) = \frac{(1-\lambda)^{\beta-2} e^{\alpha\lambda}}{B(1, \beta-1) {}_1F_1(1, \beta; \alpha)}, \quad 0 < \lambda < 1\tag{3.53}$$

Proposition 3.3.12. *The Poisson-Truncated Pearson Type III distribution is*

$$f(x) = t^x \frac{\Gamma(\beta)}{\Gamma(x+\beta)} \frac{{}_1F_1(x+1, x+\beta; \alpha-t)}{{}_1F_1(1, \beta; \alpha)}\tag{3.54}$$

and its pgf is

$$G(s) = \frac{{}_1F_1(1, \beta; \alpha-t+ts)}{{}_1F_1(1, \beta; \alpha)}\tag{3.55}$$

Proof. The mixed Poisson distribution is

$$\begin{aligned}f(x) &= \int_0^1 e^{-\lambda t} \frac{(\lambda t)^x}{x!} \frac{(1-\lambda)^{\beta-2} e^{\alpha\lambda}}{B(1, \beta-1) {}_1F_1(1, \beta; \alpha)} d\lambda \\ &= \frac{t^x}{x! B(1, \beta-1) {}_1F_1(1, \beta; \alpha)} \int_0^1 \lambda^{(x+1)-1} (1-\lambda)^{x+\beta-(x+1)-1} e^{(\alpha-t)\lambda} d\lambda \\ &= \frac{t^x}{x!} \frac{B(x+1, \beta-1)}{B(1, \beta-1)} \frac{{}_1F_1(x+1, x+\beta; \alpha-t)}{{}_1F_1(1, \beta; \alpha)} \\ &= t^x \frac{\Gamma(\beta)}{\Gamma(x+\beta)} \frac{{}_1F_1(x+1, x+\beta; \alpha-t)}{{}_1F_1(1, \beta; \alpha)}\end{aligned}$$

and its pgf is

$$\begin{aligned}
G(s) &= \int_0^1 e^{-\lambda t(1-s)} \frac{(1-\lambda)^{\beta-2} e^{\alpha\lambda}}{B(1, \beta-1) {}_1F_1(1, \beta; \alpha)} d\lambda \\
&= \frac{1}{B(1, \beta-1) {}_1F_1(1, \beta; \alpha)} \int_0^1 \lambda^{1-1} (1-\lambda)^{\beta-1-1} e^{[\alpha-t(1-s)]\lambda} d\lambda \\
&= \frac{{}_1F_1(1, \beta; \alpha - t + ts)}{{}_1F_1(1, \beta; \alpha)}.
\end{aligned}$$

When $\alpha = t$, we have

$$G(s) = \frac{{}_1F_1(1, \beta; ts)}{{}_1F_1(1, \beta; \alpha)}$$

a result similar to that given by Johnson et al (2005). □

3.3.13 Pareto I Distribution

The Pareto I distribution is

$$g(\lambda) = \frac{\alpha\beta^\alpha}{\lambda^{\alpha+1}}, \quad \lambda > \beta > 0; \quad \alpha > 0 \quad (3.56)$$

and is sometimes called Pareto of the first kind. Willmot (1993) calls it Shifted Pareto.

Proposition 3.3.13. *Poisson-Pareto I distribution is*

$$f(x) = \frac{\alpha(t\beta)^x e^{-\beta t}}{x!} \psi(1, x - \alpha + 1; \beta t) \quad (3.57)$$

and its pgf is

$$G(s) = \alpha e^{-\beta t(1-s)} \psi(1, 1 - \alpha; \beta t(1-s)). \quad (3.58)$$

Proof. The mixed Poisson distribution is

$$f(x) = \frac{\alpha t^x \beta^\alpha}{x!} \int_\beta^\infty e^{-\lambda t} \lambda^{x-\alpha-1} d\lambda$$

and making the substitution

$$\lambda = z + \beta \implies z = \lambda - \beta \text{ and } d\lambda = dz$$

we obtain

$$f(x) = \frac{\alpha t^x \beta^\alpha e^{-\beta t}}{x!} \int_0^\infty (z + \beta)^{x-\alpha-1} e^{-zt} dz$$

and further making the substitution

$$z = \beta y \implies dz = \beta dy$$

we obtain

$$\begin{aligned}
f(x) &= \frac{\alpha t^x \beta^\alpha e^{-\beta t}}{x!} \int_0^\infty \beta^{x-\alpha-1} (y+1)^{x-\alpha-1} e^{-\beta y t} \beta dy \\
&= \frac{\alpha (t\beta)^x e^{-\beta t}}{x!} \int_0^\infty y^{1-1} (y+1)^{x-\alpha+1-1-1} e^{-\beta y t} dy \\
&= \frac{\alpha (t\beta)^x e^{-\beta t}}{x!} \psi(1, x - \alpha + 1; \beta t).
\end{aligned}$$

Its pgf is

$$G(s) = \alpha \beta^\alpha \int_\beta^\infty \lambda^{-\alpha-1} e^{-\lambda t(1-s)} d\lambda$$

and making the substitution

$$\lambda = z + \beta \implies z = \lambda - \beta \text{ and } dz = d\lambda$$

we obtain

$$G(s) = \alpha \beta^\alpha e^{-\beta t(1-s)} \int_0^\infty (z + \beta)^{-\alpha-1} e^{-t(1-s)z} dz$$

and further making the substitution $z = \beta y \implies dz = \beta dy$ we obtain

$$\begin{aligned}
G(s) &= \alpha \beta^\alpha e^{-\beta t(1-s)} \int_0^\infty \beta^{-\alpha-1} (1+y)^{-\alpha-1} e^{-t(1-s)\beta y} \beta dy \\
&= \alpha e^{-\beta t(1-s)} \int_0^\infty y^{1-1} (1+y)^{1-\alpha-1-1} e^{-t(1-s)\beta y} dy \\
&= \alpha e^{-\beta t(1-s)} \psi(1, 1 - \alpha; \beta t(1-s)).
\end{aligned}$$

□

3.3.14 Pareto II (Lomax) Distribution

The Pareto II distribution also referred to as Lomax is

$$g(\lambda) = \frac{\alpha \beta^\alpha}{(\lambda + \beta)^{\alpha+1}}, \quad \lambda > 0; \quad \alpha, \beta > 0 \quad (3.59)$$

Proposition 3.3.14. *The Poisson-Pareto II distribution is*

$$f(x) = \alpha (\beta t)^x \psi(x + 1, x - \alpha + 1; \beta t) \quad (3.60)$$

and its pgf is

$$G(s) = \alpha \psi(1, 1 - \alpha; \beta t(1-s)) \quad (3.61)$$

Proof. The mixed Poisson distribution is

$$f(x) = \frac{t^x}{x!} \alpha \beta^\alpha \int_0^\infty \lambda^x (\lambda + \beta)^{-\alpha-1} e^{-\lambda t} d\lambda$$

and making the substitution

$$\lambda = \beta u \implies d\lambda = \beta du$$

we obtain

$$\begin{aligned} f(x) &= \frac{t^x}{x!} \alpha \beta^\alpha \int_0^\infty \beta^x u^x \beta^{-\alpha} (1+u)^{-\alpha-1} e^{-\beta t u} du \\ &= \frac{t^x}{x!} \alpha \beta^x \int_0^\infty u^{x+1-1} (1+u)^{1+x-\alpha-(x+1)-1} e^{-\beta t u} du \\ &= \alpha (\beta t)^x \psi(x+1, x-\alpha+1; \beta t). \end{aligned}$$

Its pgf is

$$\begin{aligned} G(s) &= \alpha \beta^\alpha \int_0^\infty (\lambda + \beta)^{-\alpha-1} e^{-\lambda t(1-s)} d\lambda \\ &= \alpha \beta^\alpha \int_0^\infty (\beta u + \beta)^{-\alpha-1} e^{-\beta u t(1-s)} \beta du \\ &= \alpha \int_0^\infty u^{1-1} (1+u)^{1-\alpha-1-1} e^{-\beta t(1-s)u} du \\ &= \alpha \psi(1, 1-\alpha; \beta t(1-s)). \end{aligned}$$

□

3.3.15 Generalized Pareto Distribution

The generalized Pareto distribution also known as gamma- gamma is

$$\begin{aligned} g(\lambda) &= \int_0^\infty \frac{k^\beta}{\Gamma(\beta)} e^{-k\lambda} \lambda^{\beta-1} \frac{\mu^\alpha}{\Gamma(\alpha)} e^{-\mu k} k^{\alpha-1} dk \\ &= \frac{\mu^\alpha \lambda^{\beta-1}}{B(\alpha, \beta) (\lambda + \mu)^{\alpha+\beta}}, \quad \lambda > 0; \quad \alpha, \beta, \mu > 0. \end{aligned} \tag{3.62}$$

Proposition 3.3.15. *The Poisson-Generalized Pareto distribution is*

$$f(x) = \frac{(\mu t)^x}{x! B(\alpha, \beta)} \Gamma(x + \beta) \psi(x + \beta, x - \alpha + 1; \mu t) \tag{3.63}$$

and its pgf is

$$G(s) = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)} \psi(\beta, 1 - \alpha; \mu t(1-s)) \tag{3.64}$$

Proof. The mixed Poisson distribution is

$$f(x) = \frac{t^x \mu^\alpha}{x! B(\alpha, \beta)} \int_0^\infty \lambda^{x+\beta-1} (\lambda + \mu)^{-\alpha-\beta} e^{-\lambda t} d\lambda$$

and making the substitution

$$\lambda = \mu z \implies d\lambda = \mu dz$$

we obtain

$$\begin{aligned} f(x) &= \frac{t^x \mu^\alpha}{x! B(\alpha, \beta)} \int_0^\infty \mu^{x+\beta-1-\alpha-\beta+1} z^{x+\beta-1} (1+z)^{-\alpha-\beta} e^{-\mu z t} dz \\ &= \frac{(\mu t)^x}{x! B(\alpha, \beta)} \int_0^\infty z^{x+\beta-1} (1+z)^{x+1-\alpha-(x+\beta)-1} e^{-\mu z t} dz \\ &= \frac{(\mu t)^x}{x! B(\alpha, \beta)} \Gamma(x+\beta) \psi(x+\beta, x-\alpha+1; \mu t) \end{aligned}$$

which yields a result obtained by Willmot (1993) for $t = 1$.

Its pgf is

$$\begin{aligned} G(s) &= \frac{\mu^\alpha}{B(\alpha, \beta)} \int_0^\infty \lambda^{\beta-1} (\lambda + \mu)^{-\alpha-\beta} e^{-\lambda t(1-s)} d\lambda \\ &= \frac{\mu^\alpha}{B(\alpha, \beta)} \int_0^\infty (\mu z)^{\beta-1} (\mu z + \mu)^{-\alpha-\beta} e^{-\mu z t(1-s)} \mu dz \\ &= \frac{1}{B(\alpha, \beta)} \int_0^\infty z^{\beta-1} (1+z)^{1-\alpha-\beta-1} e^{-\mu t(1-s)z} dz \\ &= \frac{1}{B(\alpha, \beta)} \Gamma(\beta) \psi(\beta, 1-\alpha; \mu t(1-s)) \\ &= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)} \psi(\beta, 1-\alpha; \mu t(1-s)). \end{aligned}$$

Using relation (3.6) we have

$$\begin{aligned} f(x) &= \frac{(\mu t)^x}{\Gamma(\alpha) \Gamma(\beta)} \frac{\Gamma(x+\beta)}{\Gamma(x+1)} \Gamma(\alpha-x) {}_1F_1(x+\beta; 1+x-\alpha; \mu t) \\ &\quad + \frac{(\mu t)^x}{\Gamma(\alpha) \Gamma(\beta)} \frac{\Gamma(x+\beta)}{\Gamma(x+1)} {}_1F_1(\alpha+\beta; 1-x+\alpha; \mu t) \end{aligned}$$

This result is similar to that of Bruno et al (2006) for $t = 1$.

The pgf becomes

$$G(s) = {}_1F_1(\beta; 1-\alpha; \mu t(1-s)) + \frac{\Gamma(-\alpha)}{B(\alpha, \beta)} [\mu t(1-s)]^\alpha {}_1F_1(\alpha+\beta; 1+\alpha; \mu t(1-s)). \quad (3.65)$$

□

3.4 Mixed Poisson Distributions based on Modified Bessel function of the third kind

In this section, mixed Poisson distributions are expressed in terms of modified Bessel function of the third kind.

3.4.1 Modified Bessel function of the third kind

The modified Bessel function of the third kind denoted by $K_v(\omega)$ is defined as

$$K_v(\omega) = \frac{1}{2} \int_0^{\infty} x^{v-1} e^{-\frac{\omega}{2}(x+\frac{1}{x})} dx \quad (3.66)$$

which is a function of ω with index v . Some properties of the Bessel function of the third kind are:

$$K_v(\omega) = K_{-v}(\omega) \quad (3.67)$$

$$K_{v+1}(\omega) = \frac{2v}{\omega} K_v(\omega) + K_{v-1}(\omega) \quad (3.68)$$

$$K'_v(\omega) = \frac{d}{d\omega} K_v(\omega) = -\frac{1}{2} [K_{v-1}(\omega) + K_{v+1}(\omega)] \quad (3.69)$$

$$K_{v+\frac{1}{2}}(\omega) = \sqrt{\frac{\pi}{2\omega}} e^{-\omega} \left\{ 1 + \sum_{i=1}^v \frac{(v+i)!(2\omega)^{-i}}{(v-i)!i!} \right\} \quad (3.70)$$

$$K_{\frac{1}{2}}(\omega) = \sqrt{\frac{\pi}{2\omega}} e^{-\omega}. \quad (3.71)$$

3.4.2 Inverse Gamma Distribution

The inverse gamma distribution is

$$g(\lambda) = \frac{\beta^\alpha}{\Gamma(\alpha)} e^{-\frac{\beta}{\lambda}} \lambda^{-\alpha-1}, \quad \lambda > 0; \alpha, \beta > 0 \quad (3.72)$$

Proposition 3.4.1. *The Poisson-inverse gamma distribution is*

$$f(x) = \frac{2}{x!} \frac{(\beta t)^{\frac{x+\alpha}{2}}}{\Gamma(\alpha)} K_{x-\alpha}(2\sqrt{\beta t}), \quad x = 0, 1, 2, \dots \quad (3.73)$$

and its pgf is

$$G(s) = \frac{2\beta^\alpha}{\Gamma(\alpha)} \left(\sqrt{\frac{\beta}{t(1-s)}} \right)^{-\alpha} K_{-\alpha}(2\sqrt{\beta t(1-s)}). \quad (3.74)$$

Proof. The mixed Poisson distribution is

$$f(x) = \frac{t^x}{x!} \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^{\infty} \lambda^{x-\alpha-1} e^{-t(\lambda+\frac{\beta}{\lambda})} d\lambda$$

and making the substitution

$$\lambda = \sqrt{\frac{\beta}{t}} z \implies d\lambda = \sqrt{\frac{\beta}{t}} dz$$

we obtain

$$\begin{aligned}
f(x) &= \frac{t^x \beta^\alpha}{x! \Gamma(\alpha)} \int_0^\infty \left(\sqrt{\frac{\beta}{t}} z \right)^{x-\alpha-1} e^{-t\sqrt{\frac{\beta}{t}}(z+\frac{1}{z})} \sqrt{\frac{\beta}{t}} dz \\
&= \frac{t^x \beta^\alpha}{x! \Gamma(\alpha)} \left(\sqrt{\frac{\beta}{t}} \right)^{x-\alpha} \int_0^\infty z^{x-\alpha-1} e^{-\frac{2\sqrt{\beta t}}{2}(z+\frac{1}{z})} dz \\
&= \frac{2}{x!} \frac{(\beta t)^{\frac{x+\alpha}{2}}}{\Gamma(\alpha)} K_{x-\alpha} \left(2\sqrt{\beta t} \right); \quad x = 0, 1, 2, \dots
\end{aligned}$$

Its pgf is

$$\begin{aligned}
G(s) &= \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^\infty \lambda^{-\alpha-1} e^{-\lambda t(1-s) - \frac{\beta}{\lambda}} d\lambda \\
&= \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^\infty \lambda^{-\alpha-1} e^{-t(1-s) \left[\lambda + \frac{\beta}{\lambda} \right]} d\lambda
\end{aligned}$$

and making the substitution

$$\lambda = \sqrt{\frac{\beta}{t(1-s)}} z \implies d\lambda = \sqrt{\frac{\beta}{t(1-s)}} dz$$

we obtain

$$\begin{aligned}
G(s) &= \frac{\beta^\alpha}{\Gamma(\alpha)} \left(\sqrt{\frac{\beta}{t(1-s)}} \right)^{-\alpha} \int_0^\infty z^{-\alpha-1} e^{-\frac{2\sqrt{\beta t(1-s)}}{2}(z+\frac{1}{z})} dz \\
&= \frac{2\beta^\alpha}{\Gamma(\alpha)} \left(\sqrt{\frac{\beta}{t(1-s)}} \right)^{-\alpha} K_{-\alpha} \left(2\sqrt{\beta t(1-s)} \right).
\end{aligned}$$

□

3.4.3 Pearson Type V Distribution

The Pearson type V distribution is

$$g(\lambda) = \frac{\beta^\alpha}{\Gamma(\alpha)} e^{-\frac{\beta}{\lambda-c}} (\lambda - c)^{-(\alpha+1)}, \quad \lambda > c; \quad \alpha, \beta > 0. \quad (3.75)$$

Proposition 3.4.2. *The Poisson-Pearson type V distribution is*

$$f(x) = \frac{2\beta^\alpha (ct)^x e^{-ct}}{x! \Gamma(\alpha)} \sum_{k=0}^x \binom{x}{k} \frac{1}{c^k} \left(\frac{\beta}{t} \right)^{\frac{k-\alpha}{2}} K_{k-\alpha} \left(2\sqrt{\beta t} \right) \quad (3.76)$$

and its pgf is

$$G(s) = 2 \frac{\beta^\alpha}{\Gamma(\alpha)} \left(\sqrt{\frac{\beta}{t(1-s)}} \right)^{-\alpha} e^{-t(1-s)c} K_{-\alpha} \left(2\sqrt{\beta t(1-s)} \right). \quad (3.77)$$

Proof. The mixed Poisson distribution is

$$f(x) = \frac{t^x \beta^\alpha}{x! \Gamma(\alpha)} \int_c^\infty \lambda^x (\lambda - c)^{-(\alpha+1)} e^{-\lambda t - \frac{\beta}{\lambda - c}} d\lambda$$

and making the substitution $z = \lambda - c$ we have

$$\begin{aligned} f(x) &= \frac{t^x \beta^\alpha}{x! \Gamma(\alpha)} e^{-ct} \int_0^\infty (z + c)^x z^{-\alpha-1} e^{-tz - \frac{\beta}{z}} dz \\ &= \frac{t^x \beta^\alpha e^{-ct}}{x! \Gamma(\alpha)} \left\{ \sum_{k=0}^x \binom{x}{k} c^{x-k} \int_0^\infty z^{k-\alpha-1} e^{-t(z + \frac{\beta}{z})} dz \right\} \end{aligned}$$

and further making the substitution

$$z = \sqrt{\frac{\beta}{t}} u \implies dz = \sqrt{\frac{\beta}{t}} du$$

we obtain

$$\begin{aligned} f(x) &= \frac{t^x \beta^\alpha e^{-ct}}{x! \Gamma(\alpha)} \left\{ \sum_{k=0}^x \binom{x}{k} c^{x-k} \int_0^\infty \left(\sqrt{\frac{\beta}{t}} \right)^{k-\alpha} u^{k-\alpha-1} e^{-\frac{2\sqrt{\beta t}}{2} \left(u + \frac{1}{u} \right)} dz \right\} \\ &= \frac{t^x \beta^\alpha e^{-ct}}{x! \Gamma(\alpha)} \sum_{k=0}^x \binom{x}{k} c^{x-k} \left(\frac{\beta}{t} \right)^{\frac{k-\alpha}{2}} 2K_{k-\alpha} \left(2\sqrt{\beta t} \right) \\ &= \frac{2\beta^\alpha (ct)^x e^{-ct}}{x! \Gamma(\alpha)} \sum_{k=0}^x \binom{x}{k} \frac{1}{c^k} \left(\frac{\beta}{t} \right)^{\frac{k-\alpha}{2}} K_{k-\alpha} \left(2\sqrt{\beta t} \right). \end{aligned}$$

When $c = 0$, we obtain the result for Poisson-Inverse Gamma distribution.

Its pgf is

$$G(s) = \frac{\beta^\alpha}{\Gamma(\alpha)} \int_c^\infty (\lambda - c)^{-(\alpha+1)} e^{-\lambda t(1-s) - \frac{\beta}{\lambda - c}} d\lambda$$

and making the substitution

$$z = \lambda - c \implies dz = d\lambda$$

we obtain

$$\begin{aligned} G(s) &= \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^\infty z^{-\alpha-1} e^{-t(1-s)(z+c) - \frac{\beta}{z}} dz \\ &= \frac{\beta^\alpha}{\Gamma(\alpha)} e^{-t(1-s)c} \int_0^\infty z^{-\alpha-1} e^{-t(1-s) \left[z + \frac{\beta}{t(1-s)z} \right]} dz \\ &= \frac{\beta^\alpha}{\Gamma(\alpha)} e^{-t(1-s)c} \int_0^\infty \left(\sqrt{\frac{\beta}{t(1-s)}} \right)^{-\alpha} y^{-\alpha-1} e^{-\frac{2\sqrt{\beta t(1-s)}}{2} \left(y + \frac{1}{y} \right)} dy \\ &= 2 \frac{\beta^\alpha}{\Gamma(\alpha)} \left(\sqrt{\frac{\beta}{t(1-s)}} \right)^{-\alpha} e^{-t(1-s)c} K_{-\alpha} \left(2\sqrt{\beta t(1-s)} \right). \end{aligned}$$

□

3.4.4 Inverse Gaussian Distribution

The inverse Gaussian distribution is

$$g(\lambda) = \left(\frac{\phi}{2\pi}\right)^{\frac{1}{2}} e^{\frac{\phi}{\mu}} \lambda^{-\frac{3}{2}} \exp\left\{-\frac{\phi\lambda}{2\mu^2} - \frac{\phi}{2\lambda}\right\}, \lambda > 0 \quad (3.78)$$

Proposition 3.4.3. *The Poisson-Inverse Gaussian distribution is*

$$f(x) = \left(\frac{2\phi}{\pi}\right)^{\frac{1}{2}} \frac{t^x e^{\sqrt{\varphi\phi}}}{x!} \left(\sqrt{\frac{\phi}{2t+\varphi}}\right)^{x-\frac{1}{2}} K_{x-\frac{1}{2}}\left(\sqrt{\phi(2t+\varphi)}\right) \quad (3.79)$$

and its pgf is

$$G(s) = \exp\left\{-\frac{\mu}{\beta} \left[\sqrt{1-2\beta t(s-1)} - 1\right]\right\}. \quad (3.80)$$

Proof. Consider inverse Gaussian distribution given by (3.78), making the substitution $\mu = \sqrt{\frac{\phi}{\varphi}}$ implying $\mu^2 = \frac{\phi}{\varphi}$, we have

$$g(\lambda) = \left(\frac{\phi}{2\pi}\right)^{\frac{1}{2}} \lambda^{-\frac{3}{2}} e^{\sqrt{\varphi\phi}} \exp\left\{-\frac{1}{2}\left(\varphi\lambda + \frac{\phi}{\lambda}\right)\right\} \quad (3.81)$$

The distribution of Poisson Inverse Gaussian can be obtained directly with the use of Bessel function of the third kind as

$$\begin{aligned} f(x) &= \left(\frac{\phi}{2\pi}\right)^{\frac{1}{2}} \frac{e^{\sqrt{\varphi\phi}} t^x}{x!} \int_0^\infty \lambda^{(x-\frac{1}{2})-1} \exp\left\{-\lambda t - \frac{1}{2}\left(\varphi\lambda + \frac{\phi}{\lambda}\right)\right\} d\lambda \\ &= \left(\frac{\phi}{2\pi}\right)^{\frac{1}{2}} \frac{e^{\sqrt{\varphi\phi}} t^x}{x!} \int_0^\infty \lambda^{(x-\frac{1}{2})-1} \exp\left\{-\left(\frac{2t+\varphi}{2}\right) \left[\lambda + \frac{\phi}{2t+\varphi} \frac{1}{\lambda}\right]\right\} d\lambda \end{aligned}$$

Let $\lambda = \sqrt{\frac{\phi}{2t+\varphi}} z$, implying that $d\lambda = \sqrt{\frac{\phi}{2t+\varphi}} dz$, then

$$\begin{aligned} f(x) &= \left(\frac{\phi}{2\pi}\right)^{\frac{1}{2}} \frac{e^{\sqrt{\varphi\phi}} t^x}{x!} \left(\sqrt{\frac{\phi}{2t+\varphi}}\right)^{x-\frac{1}{2}} \int_0^\infty z^{(x-\frac{1}{2})-1} \exp\left\{-\frac{\sqrt{\phi(2t+\varphi)}}{2} \left(z + \frac{1}{z}\right)\right\} dz \\ &= \left(\frac{2\phi}{\pi}\right)^{\frac{1}{2}} \frac{t^x e^{\sqrt{\varphi\phi}}}{x!} \left(\sqrt{\frac{\phi}{2t+\varphi}}\right)^{x-\frac{1}{2}} K_{x-\frac{1}{2}}\left(\sqrt{\phi(2t+\varphi)}\right) \end{aligned} \quad (3.82)$$

Its pgf is

$$\begin{aligned} G(s) &= \left(\frac{\phi}{2\pi}\right)^{\frac{1}{2}} e^{\frac{\phi}{\mu}} \int_0^\infty \lambda^{-\frac{1}{2}-1} \exp\left\{-\left[\frac{2\mu^2 t(1-s)+\phi}{2\mu^2}\right] \lambda - \frac{\phi}{2\lambda}\right\} d\lambda \\ &= \left(\frac{\phi}{2\pi}\right)^{\frac{1}{2}} e^{\frac{\phi}{\mu}} \int_0^\infty \lambda^{-\frac{1}{2}-1} \exp\left\{-\left(\frac{2\mu^2 t(1-s)+\phi}{2\mu^2}\right) \left[\lambda + \frac{\phi\mu^2}{2\mu^2 t(1-s)+\phi} \frac{1}{\lambda}\right]\right\} d\lambda \end{aligned}$$

and making the substitution

$$\lambda = \sqrt{\frac{\phi\mu^2}{2\mu^2 t(1-s)+\phi}} z$$

we have

$$\begin{aligned}
G(s) &= \left(\frac{\phi}{2\pi}\right)^{\frac{1}{2}} e^{\frac{\phi}{\mu}} \left(\sqrt{\frac{\phi\mu^2}{2\mu^2t(1-s)+\phi}}\right)^{-\frac{1}{2}} \int_0^\infty z^{-\frac{1}{2}-1} e^{-\frac{1}{2}\sqrt{\frac{\phi[2\mu^2t(1-s)+\phi]}{\mu^2}}(z+\frac{1}{z})} dz \\
&= \left(\frac{\phi}{2\pi}\right)^{\frac{1}{2}} e^{\frac{\phi}{\mu}} \left(\sqrt{\frac{\phi\mu^2}{2\mu^2t(1-s)+\phi}}\right)^{-\frac{1}{2}} \\
&\quad 2K_{-\frac{1}{2}}\left(\sqrt{\frac{\phi[2\mu^2t(1-s)+\phi]}{\mu^2}}\right). \tag{3.83}
\end{aligned}$$

Using Willmot's notation, $\phi = \mu^2/\beta$

$$\begin{aligned}
G(s) &= \left(\frac{\mu^2}{2\beta\pi}\right)^{\frac{1}{2}} e^{\frac{\mu}{\beta}} \left(\sqrt{\frac{\mu^4}{2\beta\mu^2t(1-s)+\mu^2}}\right)^{-\frac{1}{2}} 2K_{-\frac{1}{2}}\left(\sqrt{\frac{2\mu^2t(1-s)+\frac{\mu^2}{\beta}}{\beta}}\right) \\
&= 2\left(\frac{\mu^2}{2\beta\pi}\right)^{\frac{1}{2}} e^{\frac{\mu}{\beta}} \mu^{-\frac{1}{2}} \left(\sqrt{2\beta t(1-s)+1}\right)^{\frac{1}{2}} K_{-\frac{1}{2}}\left(\frac{\mu}{\beta}\sqrt{2\beta t(1-s)+1}\right) \\
&= \left(\frac{2\mu}{\beta\pi}\right)^{\frac{1}{2}} e^{\frac{\mu}{\beta}} \left(\sqrt{2\beta t(1-s)+1}\right)^{\frac{1}{2}} K_{\frac{1}{2}}\left(\frac{\mu}{\beta}\sqrt{2\beta t(1-s)+1}\right)
\end{aligned}$$

but $K_{\frac{1}{2}}(\omega) = \sqrt{\frac{\pi}{2\omega}}e^{-\omega}$ therefore

$$\begin{aligned}
G(s) &= \left(\frac{2\mu}{\beta\pi}\right)^{\frac{1}{2}} e^{\frac{\mu}{\beta}} \frac{1}{\left(\frac{2\mu}{\beta\pi}\right)^{\frac{1}{2}}} e^{-\frac{\mu}{\beta}\sqrt{2\beta t(1-s)+1}} \\
&= e^{\frac{\mu}{\beta}} e^{-\frac{\mu}{\beta}\sqrt{2\beta t(1-s)+1}} \\
&= \exp\left\{-\frac{\mu}{\beta}\left[\sqrt{1-2\beta t(s-1)}-1\right]\right\}.
\end{aligned}$$

□

3.4.5 Reciprocal Inverse Gaussian Distribution

The Reciprocal Inverse Gaussian distribution is

$$g(\lambda) = \left(\frac{\phi}{2\pi}\right)^{\frac{1}{2}} e^{\phi/\mu} \lambda^{-\frac{1}{2}} \exp\left\{-\frac{\phi}{2}\lambda - \frac{\phi}{2\mu^2\lambda}\right\}, \lambda > 0 \tag{3.84}$$

and making the substitution $\mu = \sqrt{\frac{\phi}{\varphi}}$; implying that $\mu^2 = \frac{\phi}{\varphi}$, we have

$$g(\lambda) = \left(\frac{\phi}{2\pi}\right)^{\frac{1}{2}} e^{\sqrt{\varphi\phi}} \lambda^{-\frac{1}{2}} \exp\left\{-\frac{\phi}{2}\lambda - \frac{\varphi}{2}\frac{1}{\lambda}\right\}, \lambda > 0. \tag{3.85}$$

Proposition 3.4.4. *The Poisson-Reciprocal Inverse Gaussian distribution is*

$$f(x) = \frac{t^x}{x!} \left(\frac{2\phi}{\pi}\right)^{\frac{1}{2}} e^{\sqrt{\varphi\phi}} \left(\sqrt{\frac{\varphi}{2t+\phi}}\right)^{x+\frac{1}{2}} K_{x+\frac{1}{2}}\left(\sqrt{\varphi(2t+\phi)}\right). \tag{3.86}$$

Proof. The mixed Poisson distribution is

$$\begin{aligned}
f(x) &= \frac{t^x}{x!} \left(\frac{\phi}{2\pi}\right)^{\frac{1}{2}} e^{\sqrt{\varphi\phi}} \int_0^{\infty} \lambda^{x-\frac{1}{2}} \exp\left\{-t\lambda - \frac{\phi}{2}\lambda - \frac{\varphi}{2\lambda}\right\} d\lambda \\
&= \frac{t^x}{x!} \left(\frac{\phi}{2\pi}\right)^{\frac{1}{2}} e^{\sqrt{\varphi\phi}} \int_0^{\infty} \lambda^{x-\frac{1}{2}} \exp\left\{-\frac{1}{2}\left[(2t+\phi)\lambda + \frac{\varphi}{\lambda}\right]\right\} d\lambda \\
&= \frac{t^x}{x!} \left(\frac{\phi}{2\pi}\right)^{\frac{1}{2}} e^{\sqrt{\varphi\phi}} \int_0^{\infty} \lambda^{x-\frac{1}{2}} \exp\left\{-\frac{(2t+\phi)}{2}\left[\lambda + \frac{\varphi}{2t+\phi}\frac{1}{\lambda}\right]\right\} d\lambda
\end{aligned}$$

and making the substitution $\lambda = \sqrt{\frac{\varphi}{2t+\phi}}z$; implying that $d\lambda = \sqrt{\frac{\varphi}{2t+\phi}}dz$, we obtain

$$\begin{aligned}
f(x) &= \frac{t^x}{x!} \left(\frac{\phi}{2\pi}\right)^{\frac{1}{2}} e^{\sqrt{\varphi\phi}} \left(\sqrt{\frac{\varphi}{2t+\phi}}\right)^{x+\frac{1}{2}} \int_0^{\infty} z^{x+\frac{1}{2}-1} \exp\left\{-\frac{\sqrt{\varphi(2t+\phi)}}{2}\left(z + \frac{1}{z}\right)\right\} dz \\
&= \frac{t^x}{x!} \left(\frac{\phi}{2\pi}\right)^{\frac{1}{2}} e^{\sqrt{\varphi\phi}} \left(\sqrt{\frac{\varphi}{2t+\phi}}\right)^{x+\frac{1}{2}} 2K_{x+\frac{1}{2}}\left(\sqrt{\varphi(2t+\phi)}\right) \\
&= \frac{t^x}{x!} \left(\frac{2\phi}{\pi}\right)^{\frac{1}{2}} e^{\sqrt{\varphi\phi}} \left(\sqrt{\frac{\varphi}{2t+\phi}}\right)^{x+\frac{1}{2}} K_{x+\frac{1}{2}}\left(\sqrt{\varphi(2t+\phi)}\right).
\end{aligned}$$

□

3.4.6 Generalized Inverse-Gaussian Distribution

The generalized inverse-Gaussian distribution is

$$g(\lambda) = \frac{\left(\frac{\varphi}{\phi}\right)^{\frac{v}{2}}}{2K_v(\sqrt{\varphi\phi})} \lambda^{v-1} \exp\left\{-\frac{1}{2}\left(\varphi\lambda + \frac{\phi}{\lambda}\right)\right\}; \lambda > 0 \quad (3.87)$$

with the parameters taking values in one of the ranges:

- (i) $\phi > 0, \varphi \geq 0$ if $v < 0$
- (ii) $\phi > 0, \varphi > 0$ if $v = 0$
- (iii) $\phi \geq 0, \varphi = 0$ if $v > 0$.

Proposition 3.4.5. *The Poisson-Generalized Inverse Gaussian Distribution is*

$$f(x) = \frac{t^x}{x!} \left(\frac{\varphi}{\phi}\right)^{\frac{v}{2}} \left(\frac{\phi}{2t+\varphi}\right)^{\frac{x+v}{2}} \frac{K_{x+v}\left(\sqrt{\phi(2t+\varphi)}\right)}{K_v(\sqrt{\varphi\phi})}, \quad x = 0, 1, 2, \dots \quad (3.88)$$

and its pgf is

$$G(s) = \left(\frac{\varphi}{2t(1-s)+\varphi}\right)^{\frac{v}{2}} \frac{K_v\left[\sqrt{2t(1-s)+\varphi}\right]}{K_v(\sqrt{\varphi\phi})}. \quad (3.89)$$

Proof. The mixed Poisson distribution is

$$f(x) = \frac{t^x}{x!} \frac{(\varphi/\phi)^{\frac{v}{2}}}{2K_v(\sqrt{\varphi\phi})} \int_0^\infty \lambda^{x+v-1} \exp\left\{-\frac{1}{2}(2t+\varphi)\left[\lambda + \frac{\phi}{(2t+\varphi)}\frac{1}{\lambda}\right]\right\} d\lambda$$

and making the substitution

$$\lambda = \sqrt{\frac{\phi}{(2t+\varphi)}} z \implies d\lambda = \sqrt{\frac{\phi}{(2t+\varphi)}} dz$$

we obtain

$$\begin{aligned} f(x) &= \frac{t^x}{x!} \left(\frac{\varphi}{\phi}\right)^{\frac{v}{2}} \left(\frac{\phi}{(2t+\varphi)}\right)^{\frac{x+v}{2}} \frac{1}{2K_v(\sqrt{\varphi\phi})} \int_0^\infty z^{x+v-1} \exp\left\{-\frac{1}{2}\sqrt{\phi(2t+\varphi)}\left(z + \frac{1}{z}\right)\right\} dz \\ &= \frac{t^x}{x!} \left(\frac{\varphi}{\phi}\right)^{\frac{v}{2}} \left(\frac{\phi}{(2t+\varphi)}\right)^{\frac{x+v}{2}} \frac{K_{x+v}\left(\sqrt{\phi(2t+\varphi)}\right)}{K_v(\sqrt{\varphi\phi})}, \quad x = 0, 1, 2, \dots \end{aligned}$$

Its pgf is

$$\begin{aligned} G(s) &= \frac{(\varphi/\phi)^{\frac{v}{2}}}{2K_v(\sqrt{\varphi\phi})} \int_0^\infty \lambda^{v-1} e^{-\frac{1}{2}[(2t(1-s)+\varphi)\lambda + \frac{\phi}{\lambda}]} d\lambda \\ &= \frac{(\varphi/\phi)^{\frac{v}{2}}}{2K_v(\sqrt{\varphi\phi})} \int_0^\infty \lambda^{v-1} e^{-\frac{1}{2}[2t(1-s)+\varphi]\left\{\lambda + \frac{\phi}{2t(1-s)+\varphi}\frac{1}{\lambda}\right\}} d\lambda \\ &= \left(\frac{\varphi}{\phi}\right)^{\frac{v}{2}} \frac{1}{2K_v(\sqrt{\varphi\phi})} \left(\sqrt{\frac{\phi}{2t(1-s)+\varphi}}\right)^v \int_0^\infty z^{v-1} e^{-\frac{1}{2}\sqrt{2t(1-s)+\varphi}\left(z + \frac{1}{z}\right)} dz \\ &= \left(\frac{\varphi}{2t(1-s)+\varphi}\right)^{\frac{v}{2}} \frac{K_v\left[\sqrt{2t(1-s)+\varphi}\right]}{K_v(\sqrt{\varphi\phi})}. \end{aligned}$$

□

3.5 Conclusion

A number of mixed Poisson distributions can be expressed in terms of special functions. This chapter has derived Poisson mixtures in terms of confluent hypergeometric functions and modified Bessel functions of the third kind for continuous mixing distributions. These expressions seem quite involving. Algorithms have also been developed by Press et al (1992) and have been used to calculate Generalized Pareto mixtures of Poisson distributions by Bruno et al (2006).

Chapter 4

MIXED POISSON DISTRIBUTIONS IN RECURSIVE FORMS AND THEIR DIFFERENTIAL EQUATIONS

4.1 Introduction

The main difficulty with the use of mixed Poisson distributions is that, with the exception of a few mixing distributions, its probability mass function is difficult to evaluate. One way of circumventing this problem is to express the mixed distributions in terms of recursive relations.

A number of methods for deriving such recursive relations have been developed: Katz (1965), Panjer (1981), Sundt and Jewell (1981), Panjer and Willmot (1982), Schroter (1990), Sundt (1992), Willmot (1993), Hesselager (1994), Wang (1994), etc.

The main objectives of this chapter are:

- (i) To review some recursive models obtained by other researchers
- (ii) To use integration by parts to obtain recursive models
- (iii) To correspond the recursive models obtained using integration by parts to Wang's (1994) model and then deduce the corresponding differential equations.

4.2 A Review of Recursive Models

4.2.1 Panjer's Class of Recursive Relations

Pearson difference equation is given by

$$\frac{f(x+1)}{f(x)} = \frac{P(x)}{Q(x)} \quad (4.1)$$

where $f(\cdot)$ is the discrete probability distribution; $P(x)$ and $Q(x)$ are polynomials.

Katz (1965) considered the difference equation

$$\frac{f(x+1)}{f(x)} = \frac{\alpha + \beta x}{1 + x}, \quad x = 0, 1, 2, \dots \quad (4.2)$$

Equation (4.2) can be rewritten as

$$\begin{aligned} f(x) &= \left(\frac{\alpha + \beta(x-1)}{x} \right) f(x-1) \\ &= \left(a + \frac{b}{x} \right) f(x-1); \quad x = 1, 2, 3, \dots \end{aligned} \quad (4.3)$$

where $a = \beta$ and $b = \alpha - \beta$.

Equation (4.3) is the Panjer's recursive relation model. By iteration or pgf technique, it can be shown that only Poisson, binomial and negative binomial distributions satisfy the Katz - Panjer model (Sundt and Jewell, 1981; Katz, 1965).

Panjer's class of order k is defined by

$$\frac{f(x+1)}{f(x)} = \frac{\alpha + \beta x}{1 + x}, \quad x = k, k+1, k+2, \dots; \quad k = 0, 1, 2, \dots \quad (4.4)$$

4.2.2 The Ratio Method for Mixtures in Explicit Form

Most Poisson mixtures expressed in explicit form can be expressed in a recursive form by taking the ratio of two consecutive probabilities as described below:

Poisson-Gamma Distribution

Using (2.16)

$$\frac{f(x+1)}{f(x)} = \left(\frac{t}{t+\beta} \right) \left(\frac{x+\alpha}{x+1} \right); \quad x = 0, 1, 2, \dots \quad (4.5)$$

with

$$f(0) = \left(\frac{\beta}{t+\beta} \right)^\alpha$$

Poisson-Lindley Distribution

Using (2.70),

$$\frac{f(x+1)}{f(x)} = \frac{t}{t+\theta} \left(\frac{x+t+\theta+2}{x+t+\theta+1} \right); x = 0, 1, 2, \dots \quad (4.6)$$

with

$$f(0) = \frac{\theta^2}{\theta+1} \frac{(1+t+\theta)}{(t+\theta)^2}$$

Poisson-Generalized Lindley Distribution

Using (2.92),

$$\frac{f(x+1)}{f(x)} = \frac{x+\alpha}{x+1} \frac{[\alpha(t+\theta) + \gamma(x+\alpha+1)]}{[\alpha(t+\theta) + \gamma(x+\alpha)]} \frac{t}{t+\theta}; x = 0, 1, 2, \dots \quad (4.7)$$

with

$$f(0) = \left(1 + \frac{t}{\theta + \gamma} \right) \left(\frac{\theta}{\theta + t} \right)^{\alpha+1}$$

4.2.3 Willmot's Recursive Model

Consider the Poisson mixture given in equation (1.2); when $t = 1$, its pgf is

$$G(s) = \int_0^\infty e^{\lambda(s-1)} g(\lambda) d\lambda \quad (4.8)$$

whose n th derivative of is

$$G^{(n)}(s) = \int_0^\infty \lambda^n e^{\lambda(s-1)} g(\lambda) d\lambda. \quad (4.9)$$

A generalization of a Pearson system, according to Ord (1972), is given by

$$\begin{aligned} \frac{d}{d\lambda} \log g(\lambda) &= \frac{\eta(\lambda)}{\varphi(\lambda)} \\ &= \frac{\sum_{n=0}^i \eta_n \lambda^n}{\sum_{n=0}^j \varphi_n \lambda^n} \end{aligned} \quad (4.10)$$

that is,

$$\frac{g'(\lambda)}{g(\lambda)} = \frac{\eta(\lambda)}{\varphi(\lambda)} \quad (4.11)$$

Willmot (1993) used this generalization to derive a recursive model.

Proposition 4.2.1. (a) *Willmot's differential equation in pgf for a Poisson mixture is given by*

$$\sum_{n=0}^k (s\varphi_n + \phi_n) G^{(n)}(s) = g(\lambda_1) \varphi(\lambda_1) e^{\lambda_1(s-1)} - g(\lambda_0) \varphi(\lambda_0) e^{\lambda_0(s-1)} \quad (4.12)$$

(b) The corresponding Willmot's recursive model is given by

$$\begin{aligned} \sum_{n=0}^k x(n+x-1)! \varphi_n f(n+x-1) + \sum_{n=0}^k (n+x)! \phi_n f(n+x) \\ = g(\lambda_1) \varphi(\lambda_1) e^{-\lambda_1} \lambda_1^x - g(\lambda_0) \varphi(\lambda_0) e^{-\lambda_0} \lambda_0^x \end{aligned} \quad (4.13)$$

where $\phi_n = (n+1)\varphi_{n+1} - \varphi_n + \eta_n$ and $\lambda_0 \leq \lambda \leq \lambda_1$.

(c) When $\lambda_0 = 0$ and $\lambda_1 = \infty$, then the recursive model becomes

$$\sum_{n=0}^k x(n+x-1)\varphi_n f(x+n-1) + \sum_{n=0}^k (n+x)\phi_n f(n+x) = 0. \quad (4.14)$$

Proof. Consider

$$\begin{aligned} \frac{d}{d\lambda} \left[e^{\lambda(s-1)} g(\lambda) \varphi(\lambda) \right] &= (s-1) e^{\lambda(s-1)} g(\lambda) \varphi(\lambda) + e^{\lambda(s-1)} \left\{ g(\lambda) \varphi'(\lambda) + g'(\lambda) \varphi(\lambda) \right\} \\ &= e^{\lambda(s-1)} g(\lambda) \left\{ (s-1) \varphi(\lambda) + \varphi'(\lambda) + \frac{g'(\lambda)}{g(\lambda)} \varphi(\lambda) \right\} \\ &= e^{\lambda(s-1)} g(\lambda) \left\{ s\varphi(\lambda) - \varphi(\lambda) + \varphi'(\lambda) + \eta(\lambda) \right\} \\ &= e^{\lambda(s-1)} g(\lambda) \{ s\varphi(\lambda) + \phi(\lambda) \} \end{aligned} \quad (4.15)$$

where

$$\begin{aligned} \phi(\lambda) &= \sum_{n=0}^k \phi_n \lambda^n \\ &= \varphi'(\lambda) - \varphi(\lambda) + \eta(\lambda) \\ &= \frac{d}{d\lambda} \sum_{n=0}^k \varphi_n \lambda^n - \sum_{n=0}^k \varphi_n \lambda^n + \sum_{n=0}^k \eta_n \lambda^n \\ &= \sum_{n=1}^k n \varphi_n \lambda^{n-1} - \sum_{n=0}^k \varphi_n \lambda^n + \sum_{n=0}^k \eta_n \lambda^n \\ &= \sum_{n=0}^{k-1} (n+1) \varphi_{n+1} \lambda^n - \sum_{n=0}^k \varphi_n \lambda^n + \sum_{n=0}^k \eta_n \lambda^n \end{aligned} \quad (4.16)$$

and therefore

$$\phi_n = (n+1)\varphi_{n+1} - \varphi_n + \eta_n, \quad n = 0, 1, 2, \dots, k. \quad (4.17)$$

Integrating (4.15) over (λ_0, λ_1) , we have

$$\int_{\lambda_0}^{\lambda_1} \frac{d}{d\lambda} \left[e^{\lambda(s-1)} g(\lambda) \varphi(\lambda) \right] d\lambda = \int_{\lambda_0}^{\lambda_1} e^{\lambda(s-1)} g(\lambda) \{ s\varphi(\lambda) + \phi(\lambda) \} d\lambda$$

that is,

$$e^{\lambda(s-1)} g(\lambda) \varphi(\lambda) \Big|_{\lambda_0}^{\lambda_1} = \int_{\lambda_0}^{\lambda_1} e^{\lambda(s-1)} g(\lambda) \left\{ s \sum_{n=0}^k \varphi_n \lambda^n + \sum_{n=0}^k \phi_n \lambda^n \right\} d\lambda.$$

Therefore

$$\begin{aligned}
e^{\lambda_1(s-1)}g(\lambda_1)\varphi(\lambda_1) - e^{\lambda_0(s-1)}g(\lambda_0)\varphi(\lambda_0) &= \sum_{n=0}^k \int_{\lambda_0}^{\lambda_1} (s\varphi_n + \phi_n) \lambda^n e^{\lambda(s-1)}g(\lambda) d\lambda \\
&= \sum_{n=0}^k (s\varphi_n + \phi_n) \int_{\lambda_0}^{\lambda_1} \lambda^n e^{\lambda(s-1)}g(\lambda) d\lambda \\
&= \sum_{n=0}^k (s\varphi_n + \phi_n) G^{(n)}(s)
\end{aligned}$$

and rearranging, we have

$$\sum_{n=0}^k (s\varphi_n + \phi_n) G^{(n)}(s) = g(\lambda_1)\varphi(\lambda_1)e^{\lambda_1(s-1)} - g(\lambda_0)\varphi(\lambda_0)e^{\lambda_0(s-1)}$$

which is a differential equation in pgf.

To obtain the corresponding recursive model, we start from (4.12), that is,

$$\sum_{n=0}^k (s\varphi_n + \phi_n) \int_0^\infty \lambda^n e^{\lambda s} e^{-\lambda} g(\lambda) d\lambda = g(\lambda_1)\varphi(\lambda_1)e^{-\lambda_1}e^{\lambda_1 s} - g(\lambda_0)\varphi(\lambda_0)e^{-\lambda_0}e^{\lambda_0 s}$$

therefore

$$\begin{aligned}
\sum_{n=0}^k (s\varphi_n + \phi_n) \int_0^\infty \lambda^n \sum_{l=0}^\infty \frac{(\lambda s)^l}{l!} e^{-\lambda} g(\lambda) d\lambda &= g(\lambda_1)\varphi(\lambda_1)e^{-\lambda_1} \sum_{l=0}^\infty \frac{(\lambda_1 s)^l}{l!} \\
&\quad - g(\lambda_0)\varphi(\lambda_0)e^{-\lambda_0} \sum_{l=0}^\infty \frac{(\lambda_0 s)^l}{l!} \\
\sum_{l=0}^\infty \left\{ \sum_{n=0}^k (s\varphi_n + \phi_n) \int_0^\infty \frac{\lambda^{n+l}}{l!} e^{-\lambda} g(\lambda) d\lambda \right\} s^l &= \sum_{l=0}^\infty \left[g(\lambda_1)\varphi(\lambda_1)e^{-\lambda_1} \frac{\lambda_1^l}{l!} \right] s^l \\
&\quad - \sum_{l=0}^\infty \left[g(\lambda_0)\varphi(\lambda_0)e^{-\lambda_0} \frac{\lambda_0^l}{l!} \right] s^l
\end{aligned}$$

On further simplification, we have

$$\begin{aligned}
\sum_{l=0}^\infty \left\{ \sum_{n=0}^k (s\varphi_n + \phi_n) \frac{(n+l)!}{l!} \int_0^\infty \frac{e^{-\lambda} \lambda^{n+l}}{(n+l)!} g(\lambda) d\lambda \right\} s^l &= \sum_{l=0}^\infty \left\{ g(\lambda_1)\varphi(\lambda_1) \frac{e^{-\lambda_1} \lambda_1^l}{l!} \right\} s^l \\
&\quad - \sum_{l=0}^\infty \left\{ g(\lambda_0)\varphi(\lambda_0) \frac{e^{-\lambda_0} \lambda_0^l}{l!} \right\} s^l
\end{aligned}$$

$$\begin{aligned}
\sum_{l=0}^\infty \left\{ \sum_{n=0}^k (s\varphi_n + \phi_n) \frac{(n+l)!}{l!} f(n+l) \right\} s^l &= \sum_{l=0}^\infty \left\{ g(\lambda_1)\varphi(\lambda_1) \frac{e^{-\lambda_1} \lambda_1^l}{l!} \right\} s^l \\
&\quad - \sum_{l=0}^\infty \left\{ g(\lambda_0)\varphi(\lambda_0) \frac{e^{-\lambda_0} \lambda_0^l}{l!} \right\} s^l
\end{aligned}$$

Therefore

$$\begin{aligned} \sum_{l=0}^{\infty} \left\{ \sum_{n=0}^k \varphi_n \frac{(n+l)!}{l!} f(n+l) \right\} s^{l+1} + \sum_{l=0}^{\infty} \left\{ \sum_{n=0}^k \phi_n \frac{(n+l)!}{l!} f(n+l) \right\} s^l \\ = \sum_{l=0}^{\infty} \left\{ g(\lambda_1) \varphi(\lambda_1) \frac{e^{-\lambda_1} \lambda_1^l}{l!} - g(\lambda_0) \varphi(\lambda_0) \frac{e^{-\lambda_0} \lambda_0^l}{l!} \right\} s^l \end{aligned}$$

Comparing the coefficient of s^x , by letting $l = x - 1$ in the first term and $l = x$ in the second term of the above equation, we get

$$\begin{aligned} \sum_{n=0}^k \varphi_n \frac{(n+x-1)!}{(x-1)!} f(n+x-1) + \sum_{n=0}^k \phi_n \frac{(n+x)!}{x!} f(n+x) \\ = g(\lambda_1) \varphi(\lambda_1) \frac{e^{-\lambda_1} \lambda_1^x}{x!} - g(\lambda_0) \varphi(\lambda_0) \frac{e^{-\lambda_0} \lambda_0^x}{x!} \end{aligned}$$

therefore

$$\begin{aligned} \sum_{n=0}^k x(n+x-1)! \varphi_n f(n+x-1) + \sum_{n=0}^k (n+x)! \phi_n f(n+x) \\ = g(\lambda_1) \varphi(\lambda_1) e^{-\lambda_1} \lambda_1^x - g(\lambda_0) \varphi(\lambda_0) e^{-\lambda_0} \lambda_0^x \end{aligned}$$

which is Willmot's recursive model. □

The following are examples of recursive models, for some Poisson mixtures, obtained using Willmot's model.

Poisson-Gamma Distribution

Consider the Gamma distribution given by (2.15), then

$$\begin{aligned} \frac{d}{d\lambda} \log g(\lambda) &= \frac{g'(\lambda)}{g(\lambda)} \\ &= \frac{(\alpha-1) - \beta\lambda}{\lambda} \\ &= \frac{\sum_{n=0}^k \eta_n \lambda^n}{\sum_{n=0}^k \varphi_n \lambda^n} \end{aligned} \tag{4.18}$$

This implies that $\frac{(\alpha-1)-\beta\lambda}{\lambda}$ is equivalent to $\frac{\eta_0+\eta_1\lambda}{\varphi_0+\varphi_1\lambda}$, therefore, $\eta_0 = \alpha - 1$, $\eta_1 = -\beta$, $\varphi_0 = 0$, and $\varphi_1 = 1$.

From (4.12), the differential equation becomes

$$\sum_{n=0}^k (s\varphi_n + \phi_n) G^{(n)}(s) = g(\lambda_1) \varphi(\lambda_1) e^{-\lambda_1(1-s)} - g(\lambda_0) \varphi(\lambda_0) e^{-\lambda_0(1-s)} \tag{4.19}$$

In this case, $\lambda_0 = 0$ and $\lambda_1 = \infty$. Hence the RHS of (4.19) is zero, since $e^{-\lambda_1(1-s)} = 0$ for $\lambda_1 = \infty$ and $g(\lambda_0) = 0$, for $\lambda_0 = 0$. Therefore, (4.19) becomes

$$(s\varphi_0 + \phi_0) G^{(0)}(s) + (s\varphi_1 + \phi_1) G^{(1)}(s) = 0$$

which is equivalent to

$$\phi_0 G(s) + (s + \phi_1) G'(s) = 0$$

since $\varphi_0 = 0$ and $\varphi_1 = 1$. But $\phi_n = (n + 1)\varphi_{n+1} - \varphi_n + \eta_n$, therefore $\phi_0 = \alpha$ and $\phi_1 = -(1 + \beta)$.

The differential equation is therefore

$$[s - (1 + \beta)] G'(s) + \alpha G(s) = 0 \quad (4.20)$$

From (4.13), the recursive model is

$$\sum_{n=0}^k x(n+x-1)! \varphi_n f(n+x-1) + \sum_{n=0}^k (n+x)! \phi_n f(n+x) = 0$$

that is,

$$\varphi_0 f(x-1) + x\varphi_1 f(x) + \phi_0 f(x) + (x+1)\phi_1 f(x+1) = 0$$

Since $\varphi_0 = 0$ and $\varphi_1 = 1$, we have

$$xf(x) + \phi_0 f(x) + (x+1)\phi_1 f(x+1) = 0$$

But $\phi_0 = \alpha$ and $\phi_1 = -(1 + \beta)$; therefore the recursive relation for Poisson-Gamma distribution is

$$(1 + \beta)(x+1)f(x+1) = (x + \alpha)f(x) \quad (4.21)$$

Poisson-Lindley Distribution

Consider the Lindley distribution whose pdf is given in (2.69), then

$$\begin{aligned} \frac{d}{d\lambda} \log g(\lambda) &= \frac{g'(\lambda)}{g(\lambda)} \\ &= \frac{(1-\theta) - \theta\lambda}{1+\lambda} \end{aligned} \quad (4.22)$$

This implies that $\frac{(1-\theta) - \theta\lambda}{1+\lambda}$ is equivalent to $\frac{\sum_{n=0}^k \eta_n \lambda^n}{\sum_{n=0}^k \varphi_n \lambda^n}$ and hence $\eta_0 = 1 - \theta$, $\eta_1 = -\theta$, $\varphi_0 = 1$ and $\varphi_1 = 1$.

Since $\lambda_0 = 0$ and $\lambda_1 = \infty$, $e^{-\lambda_1(1-s)} = 0$ and $g(\lambda_0) = \frac{\theta^2}{1+\theta}$. The differential equation (4.12) becomes

$$(s + \phi_0) G(s) + (s + \phi_1) G'(s) = 0$$

But $\phi_n = (n + 1)\varphi_{n+1} - \varphi_n + \eta_n$, therefore $\phi_0 = 1 - \theta$ and $\phi_1 = -(1 + \theta)$. The differential equation now becomes

$$(s - \theta - 1) G'(s) + (s - \theta + 1) G(s) = 0 \quad (4.23)$$

The corresponding recursive model is

$$\varphi_0 f(x-1) + x\varphi_1 f(x) + \phi_0 f(x) + (x+1)\phi_1 f(x+1) = 0$$

which, on further simplification and substitution, becomes

$$f(x-1) + (x+1-\theta)f(x) - (1+\theta)(x+1)f(x+1) = 0. \quad (4.24)$$

4.2.4 Hesselager Recursive Model

Hesselager (1994) considered the class of counting distributions which satisfy the recursive relation

$$f(x) = f(x-1) \frac{\sum_{r=0}^k a_r x^r}{\sum_{r=0}^k b_r x^r}, \quad x = 1, 2, 3, \dots$$

for some k . Therefore

$$\begin{aligned} f(x) \sum_{r=0}^k b_r x^r &= f(x-1) \sum_{r=0}^k a_r x^r \\ &= f(x-1) \sum_{r=0}^k a_r [1 + (x-1)]^r \\ &= f(x-1) \sum_{r=0}^k a_r \left\{ \sum_{l=0}^r a_l \binom{r}{l} (x-1)^l \right\}. \end{aligned}$$

Therefore the recursion becomes

$$\begin{aligned} f(x) \sum_{r=0}^k b_r x^r &= f(x-1) \sum_{r=0}^k \sum_{l=0}^r a_r \binom{r}{l} (x-1)^l \\ &= f(x-1) \sum_{l=0}^k \left\{ \sum_{r=0}^k a_r \binom{r}{l} \right\} (x-1)^l \\ &= f(x-1) \sum_{l=0}^k c_l (x-1)^l \end{aligned} \tag{4.25}$$

where $c_l = \sum_{r=0}^k a_r \binom{r}{l}$.

4.2.5 Wang's Recursive Model

Wang(1994) extended Hesselager (1994) model to

$$f(x) \sum_{i=0}^k b_i x^i = \sum_{j=1}^s \left[f(x-j) \sum_{i=0}^k a_{ji} (x-j)^i \right], \quad x = c, c+1, c+2, \dots$$

where c is a positive integer, and $f(x) = 0$ for $x < c$.

When $k = s = c = 2$, then

$$f(x) \sum_{i=0}^2 b_i x^i = \sum_{j=1}^2 \left[f(x-j) \sum_{i=0}^2 a_{ji} (x-j)^i \right].$$

Proposition 4.2.2. *When $k = s = c = 2$, then Wang's recursive model becomes*

$$\begin{aligned} (b_0 + b_1 x + b_2 x^2) f(x) &= \left[a_{10} + a_{11} (x-1) + a_{12} (x-1)^2 \right] f(x-1) \\ &+ \\ &\left[a_{20} + a_{21} (x-2) + a_{22} (x-2)^2 \right] f(x-2), \quad x = 2, 3, \dots \end{aligned} \tag{4.26}$$

The corresponding differential equation is given by

$$s^2 (b_2 - a_{12}s - a_{22}s^2) G''(s) + s [(b_1 + b_2) - (a_{11} + a_{12})s - (a_{21} + a_{22})s^2] G'(s) + (b_0 - a_{10}s - a_{20}s^2) G(s) = b_0 f(0) - [a_{10}f(0) + (-b_0 + b_1 + b_2)f(1)]s \quad (4.27)$$

Proof. The corresponding differential equation is determined by considering the following:

$$\begin{aligned} \sum_{x=2}^{\infty} (b_0 + b_1x + b_2x^2) f(x) s^x &= \sum_{x=2}^{\infty} [a_{10} + a_{11}(x-1) + a_{12}(x-1)^2] f(x-1) s^x \\ &+ \sum_{x=2}^{\infty} [a_{20} + a_{21}(x-2) + a_{22}(x-2)^2] f(x-2) s^x. \end{aligned}$$

On expansion, the above expression becomes

$$\begin{aligned} &b_0 \sum_{x=2}^{\infty} f(x) s^x + b_1 s \sum_{x=2}^{\infty} x f(x) s^{x-1} + b_2 \sum_{x=2}^{\infty} x^2 f(x) s^x \\ &= a_{10} s \sum_{x=2}^{\infty} f(x-1) s^{x-1} + a_{11} s^2 \sum_{x=2}^{\infty} (x-1) f(x-1) s^{x-2} \\ &\quad + a_{12} \sum_{x=2}^{\infty} (x-1)^2 f(x-1) s^x + a_{20} s^2 \sum_{x=2}^{\infty} f(x-2) s^{x-2} \\ &\quad + a_{21} s^3 \sum_{x=2}^{\infty} (x-2) f(x-2) s^{x-3} + a_{22} \sum_{x=2}^{\infty} (x-2)^2 f(x-2) s^x \end{aligned}$$

and therefore

$$\begin{aligned} &b_0 [G(s) - f(0) - f(1)s] + b_1 s [G'(s) - f(1)] + b_2 \sum_{x=2}^{\infty} x(x-1+1) f(x) s^x \\ &= a_{10} s [G(s) - f(0)] + a_{11} s^2 G'(s) + a_{12} \sum_{x=2}^{\infty} (x-1)(x-2+1) f(x-1) s^x \\ &\quad + a_{20} s^2 G(s) + a_{21} s^3 G'(s) + a_{22} \sum_{x=2}^{\infty} (x-2)(x-3+1) f(x-2) s^x. \end{aligned}$$

On further expansion, we have

$$\begin{aligned} &b_0 G(s) - b_0 f(0) - b_0 f(1;t)s + b_1 s G'(s) - b_1 f(1)s + b_2 s^2 \sum_{x=2}^{\infty} x(x-1) f(x) s^{x-2} + b_2 s \sum_{x=2}^{\infty} x f(x) s^{x-1} \\ &= a_{10} s G(s) - a_{10} f(0)s + a_{11} s^2 G'(s) + a_{12} s^3 \sum_{x=2}^{\infty} (x-1)(x-2) f(x) s^{x-3} \\ &\quad + a_{12} s^2 \sum_{x=2}^{\infty} (x-1) f(x-1) s^{x-2} + a_{20} s^2 G(s) + a_{21} s^3 G'(s) \\ &\quad + a_{22} s^4 \sum_{x=2}^{\infty} (x-2)(x-3) f(x-2) s^{x-4} + a_{22} s^3 \sum_{x=2}^{\infty} (x-2) f(x-2) s^{x-3} \end{aligned}$$

and therefore

$$\begin{aligned} b_0 G(s) - b_0 f(0) - b_0 f(1) s + b_1 s G'(s) - b_1 f(1) s + b_2 s^2 G''(s) + b_2 s [G'(s) - f(1)] \\ = a_{10} s G(s) - a_{10} f(0) s + a_{11} s^2 G'(s) + a_{12} s^3 G''(s) + a_{12} s^2 G'(s) \\ + a_{20} s^2 G(s) + a_{21} s^3 G'(s) + a_{22} s^4 G''(s) + a_{22} s^3 G'(s). \end{aligned}$$

Putting like terms together, we obtain the differential equation:

$$\begin{aligned} s^2 (b_2 - a_{12} s - a_{22} s^2) G''(s) + s [(b_1 + b_2) - (a_{11} + a_{12}) s - (a_{21} + a_{22}) s^2] G'(s) \\ + (b_0 - a_{10} s - a_{20} s^2) G(s) = b_0 f(0) - [a_{10} f(0) + (-b_0 + b_1 + b_2) f(1)] s. \end{aligned}$$

□

4.3 Recursive Models based on Integration by Parts and Differential Equations based on Wang's Model

A proper choice of u and dv is necessary to apply integration by parts formula:

$$\int u dv = uv - \int v du$$

In order to deduce the differential equations, the recursive relation for the Poisson mixture should be in the form of Wang's recursive model (4.26).

4.3.1 Beta I distribution

The mixing distribution used here is the Beta I given in (3.11).

Proposition 4.3.1. *The recursive relation for Poisson-Beta I distribution is*

$$(x+1)xf(x+1) = (\beta + t + x + \alpha - 1)xf(x) - t(x + \alpha - 1)f(x-1), \quad x = 1, 2, 3, \dots \quad (4.28)$$

with initial conditions

$$f(0) = {}_1F_1(\alpha, \alpha + \beta; -t)$$

and

$$f(1) = \frac{tB(\alpha + 1, \beta)}{B(\alpha, \beta)} {}_1F_1(\alpha + 1, \alpha + \beta + 1; -t).$$

Proof. The Poisson-Beta I distribution is

$$f(x) = \frac{t^x}{x!B(\alpha, \beta)} \int_0^1 e^{-\lambda t} \lambda^{x+\alpha-1} (1-\lambda)^{\beta-1} d\lambda$$

and using integration by parts, let

$$u = e^{-\lambda t} \lambda^{x+\alpha-1} \text{ and } dv = (1-\lambda)^{\beta-1} d\lambda.$$

Therefore the recursive relation is

$$(x+1)xf(x+1) = (\beta+t+x+\alpha-1)xf(x) - t(x+\alpha-1)f(x-1); x = 1, 2, 3, \dots$$

The recursive relation can be rewritten as:

$$x(x-1)f(x) = [(\alpha+\beta+t-1)(x-1) + (x-1)^2]f(x-1) - t[\alpha+x-2]f(x-2), x = 2, 3, \dots \quad (4.29)$$

Recursive relation (4.29) is equivalent to Wang's recursive model (4.26), whose coefficients are: $b_0 = 0, b_1 = -1, b_2 = 1; a_{10} = 0, a_{11} = (\alpha + \beta + t - 1), a_{12} = 1; a_{20} = -\alpha t, a_{21} = -t, a_{22} = 0$. The corresponding differential equation is therefore obtained by replacing the coefficients in equation (4.27) with the obtained values. This results in the following differential equation:

$$(1-s)G''(s) - [\alpha + \beta + t - ts]G'(s) + \alpha tG(s) = 0. \quad (4.30)$$

□

4.3.2 Rectangular distribution

The Rectangular distribution is the mixing distribution given in equation (3.14).

Proposition 4.3.2. *The recursive relation for Poisson-Rectangular distribution is*

$$f(x+1) = f(x) + \left\{ \frac{e^{-at}(at)^{x+1} - e^{-bt}(bt)^{x+1}}{t(b-a)(x+1)!} \right\}; x = 0, 1, 2, \dots \quad (4.31)$$

with initial condition

$$f(0) = \frac{e^{-at} - e^{-bt}}{t(b-a)}.$$

Proof. The Poisson-Rectangular distribution is obtained as

$$f(x) = \frac{t^x}{x!(b-a)} \left\{ \int_0^b e^{-\lambda t} \lambda^x d\lambda - \int_0^a e^{-\lambda t} \lambda^x d\lambda \right\}.$$

Let $y = \lambda t \Rightarrow \lambda = \frac{y}{t}$ and $d\lambda = \frac{dy}{t}$, then,

$$\begin{aligned} f(x) &= \frac{1}{t(b-a)x!} \left\{ \int_0^{bt} e^{-y} y^x dy - \int_0^{at} e^{-y} y^x dy \right\} \\ &= \frac{1}{t(b-a)x!} \{ \gamma(x+1, bt) - \gamma(x+1, at) \} \end{aligned}$$

where

$$\gamma(x, c) = \int_0^c y^{x-1} e^{-y} dy$$

is an incomplete gamma function.

Consider

$$\gamma(x+1, bt) = \int_0^{bt} e^{-y} y^x dy$$

making the substitution $u = y^x$ and $dv = e^{-y} dy$. we have

$$f(x+1) = f(x) + \left\{ \frac{e^{-at} (at)^{x+1} - e^{-bt} (bt)^{x+1}}{t(b-a)(x+1)!} \right\}; x = 0, 1, 2, \dots$$

□

4.3.3 Beta II distribution

Consider the Beta II given by (3.17)

Proposition 4.3.3. *The recursive relation for Poisson-Beta II distribution is*

$$(x+1)xf(x+1) = (x-\beta-t)xf(x) + t(x+\alpha-1)f(x-1); x = 1, 2, 3, \dots \quad (4.32)$$

with initial conditions

$$f(0) = \frac{\Gamma(\alpha)}{B(\alpha, \beta)} \psi(\alpha, 1-\beta; t)$$

and

$$f(1) = \frac{t\Gamma(\alpha+1)}{B(\alpha, \beta)} \psi(\alpha+1, \beta, t).$$

Proof. The Poisson-Beta II distribution is

$$f(x) = \frac{t^x}{x!B(\alpha, \beta)} \int_0^\infty \lambda^{x+\alpha-1} (1+\lambda)^{-\alpha-\beta} e^{-\lambda t} d\lambda.$$

Let $u = \lambda^{x+\alpha-1} e^{-\lambda t}$ and $dv = (1+\lambda)^{-(\alpha+\beta)} d\lambda$. Therefore the recursive relation is

$$(x+1)xf(x+1) = (x-\beta-t)xf(x) + t(x+\alpha-1)f(x-1); x = 1, 2, 3, \dots$$

The recursive relation can be rewritten as

$$x(x-1)f(x) = \left[-(\beta+t)(x-1) + (x-1)^2 \right] f(x-1) + t[\alpha+(x-2)]f(x-2), x = 2, 3, 4, \dots \quad (4.33)$$

Therefore the values for the coefficients are: $b_0 = 0, b_1 = -1, b_2 = 1; a_{10} = 0, a_{11} = -(\beta+t), a_{12} = 1; a_{20} = \alpha t, a_{21} = t, a_{22} = 0$ and the corresponding differential equation is

$$(1-s)G''(s) + [\beta+t-ts-1]G'(s) - \alpha tG(s) = 0. \quad (4.34)$$

□

4.3.4 Scaled Beta distribution

Consider the Scaled Beta distribution given by equation (3.20).

Proposition 4.3.4. *The recursive relation for Poisson-Scaled Beta distribution is*

$$x(x+1)f(x+1) = (\beta + \mu t + x + \alpha - 1)xf(x) - (x + \alpha - 1)(\mu t)f(x-1), x = 1, 2, 3, \dots \quad (4.35)$$

with initial conditions

$$f(0) = {}_1F_1(\alpha, \alpha + \beta; -\mu t)$$

and

$$f(1) = \frac{\mu t B(\alpha + 1, \beta)}{B(\alpha, \beta)} {}_1F_1(\alpha + 1, \alpha + \beta + 1; -\mu t).$$

Proof. The Poisson-Scaled Beta distribution is

$$f(x) = \frac{t^x}{x! B(\alpha, \beta) \mu^{\alpha + \beta - 1}} \int_0^\mu \lambda^{x + \alpha - 1} (\mu - \lambda)^{\beta - 1} e^{-\lambda t} d\lambda.$$

Let $\lambda = \mu z$, $\Rightarrow d\lambda = \mu dz$ and $z = \frac{\lambda}{\mu}$, therefore,

$$f(x) = \frac{(\mu t)^x}{x! B(\alpha, \beta)} \int_0^1 z^{x + \alpha - 1} (1 - z)^{\beta - 1} e^{-\mu t z} dz.$$

Put $u = e^{-\mu t z} z^{x + \alpha - 1}$ and $dv = (1 - z)^{\beta - 1} dz$, therefore the recursive relation is

$$x(x+1)f(x+1) = (\beta + \mu t + x + \alpha - 1)xf(x) - (x + \alpha - 1)(\mu t)f(x-1), x = 1, 2, 3, \dots$$

The recursive relation can be rewritten as

$$x(x-1)f(x) = [(\alpha + \beta + \mu t - 1)(x-1) + (x-1)^2]f(x-1) - \mu t[\alpha + (x-2)]f(x-2), x = 2, 3, 4, \dots \quad (4.36)$$

Therefore the values for the coefficients are: $b_0 = 0, b_1 = -1, b_2 = 1; a_{10} = 0, a_{11} = (\alpha + \beta + \mu t - 1), a_{12} = 1; a_{20} = -\alpha \mu t, a_{21} = -\mu t, a_{22} = 0$ and the corresponding differential equation is

$$(1-s)G''(s) + [\mu t s - (\alpha + \beta + \mu t)]G'(s) + \alpha \mu t G(s) = 0. \quad (4.37)$$

□

4.3.5 Full Beta Model

Consider full beta model given by (3.26)

Proposition 4.3.5. *The recursive relation for Poisson-Full Beta distribution is*

$$b^2 x(x+1)f(x+1) = [b(x-q) - t]bx f(x) + bt(x+p-1)f(x-1); x = 1, 2, 3, \dots \quad (4.38)$$

with initial conditions

$$f(0) = \frac{\Gamma(p)}{B(p, q)} \psi \left(p, 1 - q; \frac{t}{b} \right)$$

and

$$f(1) = \left(\frac{t}{b} \right)^q \frac{\Gamma(p+1)}{B(p, q)} \psi \left(p + q, q; \frac{t}{b} \right).$$

Proof. The Poisson-Full Beta distribution is

$$f(x) = \frac{t^x b^p}{x! B(p, q)} \int_0^\infty \lambda^{x+p-1} (1 + b\lambda)^{-(p+q)} e^{-\lambda t} d\lambda.$$

Let $z = b\lambda$; $dz = bd\lambda$ and $\lambda = \frac{z}{b}$, then

$$f(x) = \left(\frac{t}{b} \right)^x \frac{1}{x! B(p, q)} \int_0^\infty z^{x+p-1} (1 + z)^{-(p+q)} e^{-\frac{t}{b}z} dz.$$

Put $u = z^{x+p-1} e^{-\frac{t}{b}z}$ and $dv = (1 + z)^{-(p+q)} dz$ then the recursive relation is

$$b^2 x(x+1) f(x+1) = [b(x-q) - t] b x f(x) + b t(x+p-1) f(x-1); x = 1, 2, 3, \dots$$

The recursive relation can be rewritten as

$$b^2 x(x-1) f(x) = b \left[(bq - t)(x-1) + b(x-1)^2 \right] f(x-1) + b t [p + (x-2)] f(x-2), x = 2, 3, 4, \dots \quad (4.39)$$

Therefore the values for the coefficients corresponding to Wang's model are: $b_0 = 0, b_1 = -b^2, b_2 = b^2; a_{10} = 0, a_{11} = b(bq - t), a_{12} = b^2; a_{20} = btp, a_{21} = bt, a_{22} = 0$ and the corresponding differential equation is

$$b^2(1-s)G''(s) + [bt(1-s) - b^2(q+1)]G'(s) - btpG(s) = 0. \quad (4.40)$$

□

4.3.6 Transformed Beta Distribution

A transformed beta distribution is

$$g(\lambda) = \frac{c\mu^\alpha}{B(\alpha, \beta)} \frac{\lambda^{c\beta-1}}{(\mu + \lambda^c)^{\alpha+\beta}}, \lambda > 0 \quad (4.41)$$

Proposition 4.3.6. *The recursive relation for Poisson-Transformed Beta distribution is*

$$\begin{aligned} \mu t^c (x - c + 1) f(x - c + 1) = & t^c (x - c + c\beta) \mu f(x - c) + (x - c\alpha) \prod_{i=1}^c (x - c + i) f(x) \\ & - \prod_{i=1}^{c+1} (x - c + i) f(x + 1), x = 0, 1, 2, \dots \end{aligned} \quad (4.42)$$

Proof. The Poisson-Transformed beta distribution is

$$f(x) = \frac{c\mu^\alpha}{B(\alpha, \beta)} \cdot \frac{t^x}{x!} \int_0^\infty \lambda^{x+c\beta-1} (\mu + \lambda^c)^{-\alpha-\beta} e^{-\lambda t} d\lambda$$

and making the substitution $u = \lambda^{x-c+c\beta} e^{-\lambda t}$ and $dv = c\lambda^{c-1} (\mu + \lambda^c)^{-\alpha-\beta} d\lambda$, we have the recursive relation

$$\begin{aligned} \mu t^c (x - c + 1) f(x - c + 1) &= t^c (x - c + c\beta) \mu f(x - c) + (x - c\alpha) \prod_{i=1}^c (x - c + i) f(x) \\ &\quad - \prod_{i=1}^{c+1} (x - c + i) f(x + 1) \quad x = 0, 1, 2, \dots \end{aligned} \quad (4.43)$$

□

4.3.7 Inverse Gamma Distribution

Consider the inverse gamma distribution given by equation (3.72).

Proposition 4.3.7. *The recursive relation for Poisson-Inverse Gamma distribution is*

$$x(x+1)f(x+1) = (x-\alpha)xf(x) + \beta tf(x-1); \quad x = 1, 2, 3, \dots \quad (4.44)$$

with initial conditions

$$f(0) = \frac{2(\beta t)^{\frac{\alpha}{2}}}{\Gamma(\alpha)} K_{-\alpha}(2\sqrt{\beta t})$$

and

$$f(1) = \frac{2(\beta t)^{\frac{\alpha+1}{2}}}{\Gamma(\alpha)} K_{1-\alpha}(2\sqrt{\beta t}).$$

Proof. The Poisson-Inverse Gamma distribution is

$$f(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} \frac{t^x}{x!} \int_0^\infty \lambda^{x-\alpha-1} e^{-(\lambda t + \frac{\beta}{\lambda})} d\lambda.$$

Making the substitution $u = e^{-(\lambda t + \frac{\beta}{\lambda})}$ and $dv = \lambda^{x-\alpha-1} d\lambda$ we have the recursive relation

$$x(x+1)f(x+1) = (x-\alpha)xf(x) + \beta tf(x-1); \quad x = 1, 2, 3, \dots$$

The recursive relation can be rewritten as

$$x(x-1)f(x) = \left[-\alpha(x-1) + (x-1)^2 \right] f(x-1) + \beta tf(x-2), \quad x = 2, 3, \dots \quad (4.45)$$

Therefore the values for the coefficients corresponding to Wang's model are: $b_0 = 0, b_1 = -1, b_2 = 1; a_{10} = 0, a_{11} = -\alpha, a_{12} = 1; a_{20} = \beta t, a_{21} = 0, a_{22} = 0$ and the corresponding differential equation is

$$(1-s)G''(s) + (1-s)G'(s) - \beta tG(s) = 0. \quad (4.46)$$

□

4.3.8 Shifted Gamma Distribution

Consider a shifted gamma distribution given by (2.32).

Proposition 4.3.8. *The recursive relation for Poisson-Shifted Gamma distribution is*

$$(t + \beta)(x + 1)f(x + 1) = [x + \alpha + (t + \beta)\mu]tf(x) - \mu t^2 f(x - 1); x = 1, 2, 3, \dots \quad (4.47)$$

with initial conditions

$$f(0) = (\mu\beta)^\alpha e^{-\mu t} \psi(\alpha, \alpha + 1; (t + \beta)\mu)$$

and

$$f(1) = \mu t (\mu\beta)^\alpha e^{-\mu t} \psi(\alpha, \alpha + 2; (t + \beta)\mu).$$

Proof. The Poisson-Shifted gamma distribution is

$$f(x) = \frac{t^x \beta^\alpha}{x! \Gamma(\alpha)} \int_\mu^\infty e^{-\lambda t} \lambda^x e^{-\beta(\lambda - \mu)} (\lambda - \mu)^{\alpha - 1} d\lambda$$

and making the substitution $z = \lambda - \mu$, $\Rightarrow dz = d\lambda$ and $\lambda = \mu + z$, we have

$$f(x) = \frac{t^x \beta^\alpha}{x! \Gamma(\alpha)} \int_0^\infty e^{-(\mu + z)t} (\mu + z)^x e^{-\beta z} z^{\alpha - 1} dz.$$

On further substitution $z = \mu y$, $\Rightarrow dz = \mu dy$, we have

$$f(x) = \frac{(\mu t)^x (\mu\beta)^\alpha}{x! \Gamma(\alpha)} e^{-\mu t} \int_0^\infty y^{\alpha - 1} (1 + y)^x e^{-(t + \beta)\mu y} dy.$$

Using integration by parts, let $u = (1 + y)^x e^{-(t + \beta)\mu y}$ and $dv = y^{\alpha - 1} dy$, therefore the recursive relation is

$$(t + \beta)(x + 1)f(x + 1) = [x + \alpha + (t + \beta)\mu]tf(x) - \mu t^2 f(x - 1); x = 1, 2, 3, \dots$$

The recursive relation can be rewritten as

$$(t + \beta)xf(x) = t[\alpha + (t + \beta)\mu + (x - 1)]f(x - 1) - \mu t^2 f(x - 2), x = 2, 3, 4, \dots \quad (4.48)$$

Therefore the values for the coefficients are: $b_0 = 0, b_1 = (t + \beta), b_2 = 0; a_{10} = t[\alpha + (t + \beta)\mu], a_{11} = t, a_{12} = 0; a_{20} = -\mu t^2, a_{21} = 0, a_{22} = 0$ and the corresponding differential equation is

$$s(t + \beta - ts)G'(s) + [\mu t^2 s^2 - ts(\alpha + t\mu + t\beta)]G(s) = -ts(\alpha + t\mu + t\beta)f(0) - s(t + \beta)f(1). \quad (4.49)$$

□

4.3.9 Truncated Gamma (from below) Distribution

Consider truncated gamma (from below) distribution given by (3.44)

Proposition 4.3.9. *The recursive relation for the Poisson-Truncated gamma (from below) is*

$$(t + \beta)(x + 1)f(x + 1) = t(x + \alpha)f(x) + t^{x+1}e^{-t\lambda_0}g(\lambda_0)\frac{\lambda_0^{x+1}}{x!}, \quad x = 0, 1, 2, \dots \quad (4.50)$$

with initial condition

$$f(0) = \left(\frac{\beta}{t + \beta}\right)^\alpha \frac{\Gamma(\alpha) - \gamma(\alpha, (t + \beta)\lambda_0)}{\Gamma(\alpha) - \gamma(\alpha, \beta\lambda_0)}.$$

Proof. The Poisson-Truncated gamma (from below) distribution is

$$f(x) = \frac{\beta^\alpha t^x}{x! [1 - \gamma(\alpha, \beta\lambda_0)]} \int_{\lambda_0}^{\infty} \lambda^{x+\alpha-1} e^{-(t+\beta)\lambda} d\lambda.$$

Let $u = e^{-(t+\beta)\lambda}$ and $dv = \lambda^{x+\alpha-1} d\lambda$ then the recursive relation is

$$(t + \beta)(x + 1)f(x + 1) = t(x + \alpha)f(x) + t^{x+1}e^{-t\lambda_0}g(\lambda_0)\frac{\lambda_0^{x+1}}{x!}, \quad x = 0, 1, 2, \dots$$

□

4.3.10 Generalized Gamma Distribution

Consider the generalized gamma distribution

$$g(\lambda) = \frac{\alpha^{m-\delta} e^{-\alpha\lambda} \lambda^{m-1}}{\Gamma_\delta(m, \alpha n) (\lambda + n)^\delta}, \quad \lambda \geq 0; \quad m, \alpha, n > 0, \delta \geq 0 \quad (4.51)$$

where

$$\Gamma_\delta(m, \alpha n) = \int_0^\infty \frac{y^{m-1} e^{-y}}{(y + \alpha n)^\delta} dy.$$

Proposition 4.3.10. *The recursive relation for Poisson-Generalized Gamma distribution is*

$$\begin{aligned} (\alpha + t)x(x + 1)f(x + 1) &= [x + m - \delta - n(\alpha + t)]x(nt)f(x) \\ &\quad + (x + m - 1)(nt)^2 f(x - 1), \quad x = 1, 2, 3, \dots \end{aligned} \quad (4.52)$$

with initial conditions

$$f(0) = \frac{\alpha^{m-\delta}}{\Gamma_\delta(m, \alpha n)} \int_0^\infty e^{-(\alpha+t)\lambda} \lambda^{m-1} (n + \lambda)^{-\delta} d\lambda$$

and

$$f(1) = \frac{t\alpha^{m-\delta}}{\Gamma_\delta(m, \alpha n)} \int_0^\infty e^{-(\alpha+t)\lambda} \lambda^m (n + \lambda)^{-\delta} d\lambda.$$

Proof. The Poisson-Generalized gamma distribution is

$$f(x) = \frac{t^x \alpha^{m-\delta}}{x! \Gamma_\delta(m, \alpha n)} \int_0^\infty e^{-(\alpha+t)\lambda} \lambda^{x+m-1} (n+\lambda)^{-\delta} d\lambda$$

therefore

$$\frac{x! f(x) \Gamma_\delta(m, \alpha n)}{t^x \alpha^{m-\delta}} = \int_0^\infty e^{-(\alpha+t)\lambda} \lambda^{x+m-1} (\lambda+n)^{-\delta} d\lambda.$$

Making the substitution $\lambda = nz \Rightarrow d\lambda = ndz$ we have

$$\begin{aligned} RHS &= \int_0^\infty e^{-(\alpha+t)nz} (nz)^{x+m-1} n^{-\delta} (1+z)^{-\delta} ndz \\ &= n^{x+m-\delta} \int_0^\infty e^{-(\alpha+t)nz} z^{x+m-1} (1+z)^{-\delta} dz. \end{aligned}$$

Let $u = z^{x+m-1} e^{-(\alpha+t)nz}$ and $dv = (1+z)^{-\delta} dz$, then the recursive relation is

$$\begin{aligned} (\alpha+t)x(x+1)f(x+1) &= [x+m-\delta-n(\alpha+t)]ntxf(x) \\ &\quad + (x+m-1)(nt)^2 f(x-1), \quad x = 1, 2, 3, \dots \end{aligned}$$

a result similar to Ong (1995) for $t = 1$.

The recursive relation can be rewritten as

$$\begin{aligned} (\alpha+t)x(x-1)f(x) &= nt \left[(m-\delta-n\alpha-nt)(x-1) + (x-1)^2 \right] f(x-1) \\ &\quad + (nt)^2 [m+(x-2)] f(x-2), \quad x = 2, 3, 4, \dots \end{aligned} \quad (4.53)$$

The values for the coefficients are therefore: $b_0 = 0, b_1 = -(\alpha+t), b_2 = (\alpha+t); a_{10} = 0, a_{11} = nt(m-\delta-n\alpha-nt), a_{12} = nt; a_{20} = (nt)^2 m, a_{21} = (nt)^2, a_{22} = 0$ and the corresponding differential equation is

$$(\alpha+t-nts)G''(s) + \left[nt - nt(m-\delta-n\alpha-nt) - (nt)^2 \right] G'(s) - (nt)^2 mG(s) = 0. \quad (4.54)$$

□

4.3.11 Transformed Gamma Distribution

Consider a transformed gamma distribution

$$g(\lambda) = c \frac{\beta^\alpha}{\Gamma(\alpha)} \lambda^{\alpha-1} e^{-\beta\lambda^c}, \quad \lambda > 0; \alpha, \beta > 0, c \in \mathbb{Z}^+. \quad (4.55)$$

Proposition 4.3.11. *The recursive relation for Poisson-transformed gamma distribution is*

$$(x+c\alpha)f(x) = (x+1)f(x+1) + \frac{c\beta}{t^c} \left\{ \prod_{i=1}^c (x+i) \right\} f(x+c), \quad x = 0, 1, 2, \dots \quad (4.56)$$

Proof. The Poisson-Transformed gamma distribution is

$$f(x) = \frac{c\beta^{\alpha}t^x}{\Gamma(\alpha)x!} \int_0^{\infty} \lambda^{x+c\alpha-1} e^{-\lambda t - \beta\lambda^c} d\lambda.$$

Let $u = e^{-\lambda t - \beta\lambda^c}$ and $dv = \lambda^{x+c\alpha-1}d\lambda$, then the recursive relation is

$$(x + c\alpha) f(x) = (x + 1) f(x + 1) + \frac{c\beta}{t^c} \left\{ \prod_{i=1}^c (x + i) \right\} f(x + c), \quad x = 0, 1, 2, \dots$$

□

4.3.12 Pareto I Distribution

Consider Pareto I distribution given by (3.56)

Proposition 4.3.12. *The recursive relation for Poisson-Pareto I distribution is*

$$(x + 1) f(x + 1) = (x - \alpha) f(x - \alpha) + \frac{\alpha t^x e^{-\beta t} \beta^\alpha}{x!}, \quad x = 0, 1, 2, \dots \quad (4.57)$$

with initial condition

$$f(0) = \alpha e^{-\beta t} \psi[1, (1 - \alpha); \beta t].$$

Proof. The Poisson-Pareto I distribution is

$$f(x) = \frac{t^x}{x!} \alpha \beta^\alpha \int_{\beta}^{\infty} e^{-\lambda t} \lambda^{x-\alpha-1} d\lambda.$$

Let $u = e^{-\lambda t}$ and $dv = \lambda^{x-\alpha-1}d\lambda$, then the recursive relation is

$$(x + 1) f(x + 1) = (x - \alpha) f(x - \alpha) + \frac{\alpha t^x e^{-\beta t} \beta^\alpha}{x!}, \quad x = 0, 1, 2, \dots$$

□

4.3.13 Pareto II (Lomax) Distribution

Consider the Lomax distribution given in (3.59)

Proposition 4.3.13. *The recursive relation for Poisson-Pareto II distribution is*

$$(x + 1) f(x + 1) = (x + \beta t - \alpha) f(x) + t\beta f(x - 1); \quad x = 1, 2, \dots \quad (4.58)$$

with initial conditions

$$f(0) = \alpha \psi(1, 1 - \alpha; \beta t)$$

and

$$f(1) = \alpha \beta t \psi(2, 2 - \alpha; \beta t).$$

Proof. The Poisson-Pareto II distribution is

$$f(x) = \alpha \beta^\alpha \frac{t^x}{x!} \int_0^\infty e^{-\lambda t} \lambda^x (\lambda + \beta)^{-\alpha-1} d\lambda.$$

Let $u = e^{-\lambda t} \lambda^x$ and $dv = (\lambda + \beta)^{-\alpha-1} d\lambda$, then the recursive relation is

$$(x+1)f(x+1) = (x + \beta t - \alpha)f(x) + t\beta f(x-1); \quad x = 1, 2, 3, \dots$$

The recursive relation can be rewritten as

$$xf(x) = (\beta t - \alpha + x - 1)f(x-1) + \beta t f(x-2), \quad x = 2, 3, 4, \dots \quad (4.59)$$

Comparing equation (4.59) with equation (4.26), we have the following values for the constants: $b_0 = 0, b_1 = 1, b_2 = 0; a_{10} = (\beta t - \alpha), a_{11} = 1, a_{12} = 0; a_{20} = \beta t, a_{21} = 0, a_{22} = 0$. The corresponding differential equation is therefore obtained by replacing the given constants in equation (4.27). The resulting differential equation is:

$$(1-s)G'(s) + [\alpha - \beta t(1+s)]G(s) = (\alpha - \beta t)f(0) - f(1). \quad (4.60)$$

□

4.3.14 Generalized Pareto Type I distribution

Consider the generalized Pareto Type I distribution given by equation (3.62).

Proposition 4.3.14. *The recursive relation for Poisson-Generalized Pareto distribution is*

$$x(x+1)f(x+1) = (x - \alpha - \mu t)xf(x) + t\mu(x + \beta - 1)f(x-1), \quad x = 1, 2, 3, \dots \quad (4.61)$$

with initial conditions

$$f(0) = \frac{\Gamma(\beta)}{B(\alpha, \beta)} \psi(\beta, 1 - \alpha; \mu t)$$

and

$$f(1) = \frac{\mu t}{B(\alpha, \beta)} \Gamma(1 + \beta) \psi(1 + \beta, 2 - \alpha; \mu t).$$

Proof. The Poisson-Generalized Pareto Type I distribution is

$$f(x) = \frac{\mu^\alpha t^x}{x! B(\alpha, \beta)} \int_0^\infty e^{-\lambda t} \lambda^{x+\beta-1} (\lambda + \mu)^{-\alpha-\beta} d\lambda.$$

Let $u = e^{-\lambda t} \lambda^{x+\beta-1}$ and $dv = (\lambda + \mu)^{-\alpha-\beta} d\lambda$, then the recursive relation is

$$x(x+1)f(x+1) = (x - \alpha - \mu t)xf(x) + t\mu(x + \beta - 1)f(x-1). \quad (4.62)$$

The recursive relation can be rewritten as

$$(-x + x^2) f(x) = \left[-(\alpha + \mu t)(x - 1) + (x - 1)^2 \right] f(x - 1) + \mu t [\beta + (x - 2)] f(x - 2), x = 2, 3, 4, \dots \quad (4.63)$$

Therefore the values for the coefficients corresponding to Wang's model (4.26) are: $b_0 = 0, b_1 = -1, b_2 = 1; a_{10} = 0, a_{11} = -(\alpha + \mu t), a_{12} = 1; a_{20} = \mu t \beta, a_{21} = \mu t, a_{22} = 0$ and the corresponding differential equation is

$$(1 - s) G''(s) + [(\alpha + \mu t - 1) - \mu t s] G'(s) - \mu t \beta G(s) = 0 \quad (4.64)$$

□

4.3.15 Generalized Pareto Type II Distribution

The generalized Pareto type II distribution is

$$g(\lambda) = \frac{1}{k} \left(1 - \frac{c}{k} \lambda \right)^{\frac{1}{c} - 1}; \lambda > 0 \quad (4.65)$$

Consider the following three cases for the possible values of c :

Case (i): When $c < 0$

Let $c = -d$ where $d > 0$, therefore

$$g(\lambda) = \frac{1}{k} \left(1 + \frac{d}{k} \lambda \right)^{-\frac{1}{d} - 1}; \lambda > 0 \quad (4.66)$$

Proposition 4.3.15. *The recursive relation with respect to (4.66) is*

$$c(x + 1) f(x + 1) = (cx + kt + 1) f(x) - tkf(x - 1); c < 0, x = 1, 2, 3, \dots \quad (4.67)$$

with initial conditions

$$f(0) = \frac{1}{k} \int_0^\infty e^{-\lambda t} \left(1 + \frac{d}{k} \lambda \right)^{-\frac{1}{d} - 1} d\lambda$$

and

$$f(1) = \frac{t}{k} \int_0^\infty e^{-\lambda t} \lambda \left(1 + \frac{d}{k} \lambda \right)^{-\frac{1}{d} - 1} d\lambda.$$

Proof. The Poisson-Generalized Pareto Type II distribution is

$$f(x) = \frac{t^x}{kx!} \int_0^\infty e^{-\lambda t} \lambda^x \left(1 + \frac{d}{k} \lambda \right)^{-\frac{1}{d} - 1} d\lambda.$$

Let $u = e^{-\lambda t} \lambda^x$ and $dv = \left(1 + \frac{d}{k} \lambda \right)^{-\frac{1}{d} - 1} d\lambda$, then the recursive relation is

$$c(x + 1) f(x + 1) = (cx + kt + 1) f(x) - tkf(x - 1); c < 0, x = 1, 2, 3, \dots$$

The recursive relation can be rewritten as

$$cx f(x) = [kt + 1 + c(x - 1)] f(x - 1) - kt f(x - 2), x = 2, 3, 4, \dots \quad (4.68)$$

The values for the coefficients corresponding to Wang's model are therefore: $b_0 = 0, b_1 = c, b_2 = 0; a_{10} = (kt + 1), a_{11} = c, a_{12} = 0; a_{20} = -kt\beta, a_{21} = 0, a_{22} = 0$ and the corresponding differential equation is

$$c(1 - s) G'(s) + [kts - (kt + 1)] G(s) = -(kt + 1) f(0) - cf(1). \quad (4.69)$$

□

Case (ii): When $c \rightarrow 0$ The mixing distribution is

$$\begin{aligned} g(\lambda) &= \lim_{c \rightarrow 0} \frac{1}{k} \left(1 - \frac{c}{k} \lambda\right)^{\frac{1}{c} - 1} \\ &= \frac{1}{k} e^{-\frac{\lambda}{k}}, \lambda > 0 \end{aligned} \quad (4.70)$$

which is an exponential distribution with mean k .

Proposition 4.3.16. *The recursive relation with respect to (4.70) is*

$$f(x + 1) = \frac{tk}{kt + 1} f(x); x = 0, 1, 2, \dots \quad (4.71)$$

with initial condition

$$f(0) = \int_0^\infty e^{-\lambda t} \frac{1}{k} \left(1 + \frac{d}{k} \lambda\right)^{-\frac{1}{d} - 1} d\lambda.$$

Proof. The Poisson-Generalized Pareto Type II distribution is

$$f(x) = \frac{t^x}{kx!} \int_0^\infty e^{-\lambda(t + \frac{1}{k})} \lambda^x d\lambda.$$

Let $u = e^{-\lambda(t + \frac{1}{k})} \lambda^x$ and $dv = d\lambda$, then the recursive relation is

$$(kt + 1) f(x + 1) = tk f(x), x = 0, 1, 2, \dots$$

which can be rewritten as

$$(kt + 1) f(x) = kt f(x - 1), x = 1, 2, 3, \dots \quad (4.72)$$

The values for the coefficients corresponding to Wang's model are therefore: $b_0 = kt + 1, b_1 = b_2 = 0; a_{10} = kt, a_{11} = a_{12} = 0; a_{20} = a_{21} = a_{22} = 0$ and the corresponding differential equation is

$$G(s) = (kt - kts + 1) f(0) + (kt + 1) s f(1). \quad (4.73)$$

□

Case (iii): When $c > 0$

$$g(\lambda) = \frac{1}{k} \left(1 - \frac{c}{k}\lambda\right)^{\frac{1}{c}-1}; 0 < \lambda < \frac{k}{c} \quad (4.74)$$

Proposition 4.3.17. *The recursive relation with respect to (4.74) is*

$$c(x+1)f(x+1) = (1+tk+cx)f(x) - tkf(x-1); x = 1, 2, 3, \dots \quad (4.75)$$

with initial conditions

$$f(0) = \frac{1}{k} \int_0^{\frac{k}{c}} e^{-\lambda t} \left(1 - \frac{c}{k}\lambda\right)^{\frac{1}{c}-1} d\lambda$$

and

$$f(1) = \frac{t}{k} \int_0^{\frac{k}{c}} e^{-\lambda t} \lambda \left(1 - \frac{c}{k}\lambda\right)^{\frac{1}{c}-1} d\lambda.$$

Proof. The Poisson-Generalized Pareto Type II distribution is

$$f(x) = \frac{t^x}{kx!} \int_0^{\frac{k}{c}} e^{-\lambda t} \lambda^x \left(1 - \frac{c}{k}\lambda\right)^{\frac{1}{c}-1} d\lambda.$$

Let $u = e^{-\lambda t} \lambda^x$ and $dv = \left(1 - \frac{c}{k}\lambda\right)^{\frac{1}{c}-1} d\lambda$, then the recursive relation is

$$c(x+1)f(x+1) = (1+tk+cx)f(x) - tkf(x-1); x = 1, 2, 3, \dots$$

which is the same as equation (4.67). Therefore the corresponding differential equation will be similar to equation (4.69). \square

4.3.16 Inverse Gaussian Distribution

Consider inverse Gaussian distribution given in (3.78).

Proposition 4.3.18. *The recursive relation of Poisson-Inverse Gaussian distribution is*

$$(2\mu^2 t + \phi) x(x+1) f(x+1) = \mu^2 (2x-1) x t f(x) + \mu^2 \phi t^2 f(x-1); x = 1, 2, 3, \dots \quad (4.76)$$

with initial conditions

$$f(0) = \left(\frac{\phi}{2\pi}\right)^{\frac{1}{2}} e^{\frac{\phi}{\mu}} \left(\sqrt{\frac{\mu^2 \phi}{2\mu^2 t + \phi}}\right)^{-\frac{1}{2}} K_{-\frac{1}{2}} \left(\sqrt{\frac{(2\mu^2 t + \phi) \phi}{\mu^2}}\right)$$

and

$$f(1) = t \left(\frac{\phi}{2\pi}\right)^{\frac{1}{2}} e^{\frac{\phi}{\mu}} \left(\sqrt{\frac{\mu^2 \phi}{2\mu^2 t + \phi}}\right)^{\frac{1}{2}} K_{\frac{1}{2}} \left(\sqrt{\frac{(2\mu^2 t + \phi) \phi}{\mu^2}}\right).$$

Proof. The Poisson-Inverse Gaussian distribution is

$$f(x) = \left(\frac{\phi}{2\pi}\right)^{\frac{1}{2}} \frac{t^x}{x!} e^{\frac{\phi}{\mu}} \int_0^{\infty} \lambda^{x-\frac{3}{2}} e^{-\left(t+\frac{\phi}{2\mu^2}\right)\lambda-\frac{\phi}{2\lambda}} d\lambda$$

Let $u = e^{-\left(t+\frac{\phi}{2\mu^2}\right)\lambda-\frac{\phi}{2\lambda}}$ and $dv = \lambda^{x-\frac{3}{2}} d\lambda$, then the recursive relation is

$$(2\mu^2 t + \phi) x(x+1) f(x+1) = \mu^2 (2x-1) x t f(x) + \mu^2 \phi t^2 f(x-1), \quad x = 1, 2, 3, \dots$$

The recursive relation can be rewritten as

$$(2\mu^2 t + \phi) [-x + x^2] f(x) = 2\mu^2 t [(x-1) + (x-1)^2] f(x-1) + \mu^2 \phi t^2 f(x-2), \quad x = 2, 3, 4, \dots \quad (4.77)$$

Wang's coefficients are therefore: $b_0 = 0, b_1 = -(2\mu^2 + \phi), b_2 = (2\mu^2 + \phi); a_{10} = 0, a_{11} = 2t\mu^2, a_{12} = 2t\mu^2; a_{20} = \mu^2 \phi t^2, a_{21} = 0, a_{22} = 0$. The corresponding differential equation is therefore obtained by replacing the given constants in equation (4.27). The resulting differential equation is:

$$[2\mu^2 t + \phi - 2\mu^2 t s] G''(s) - 4\mu^2 t G'(s) - \mu^2 \phi t^2 G(s) = 0 \quad (4.78)$$

□

4.3.17 Reciprocal Inverse Gaussian Distribution

Consider reciprocal inverse Gaussian distribution given in (3.84).

Proposition 4.3.19. *The recursive relation for Poisson-Reciprocal Inverse Gaussian distribution is*

$$\mu^2 (\phi + 2t) x(x+1) f(x+1) = t\mu^2 (2x+1) x f(x) + \phi t^2 f(x-1), \quad x = 1, 2, 3, \dots \quad (4.79)$$

with initial conditions

$$f(0) = \left(\frac{2\phi}{\pi}\right)^{\frac{1}{2}} e^{\sqrt{\varphi\phi}} \left(\sqrt{\frac{\varphi}{2t+\phi}}\right)^{\frac{1}{2}} K_{\frac{1}{2}}\left(\sqrt{\varphi(2t+\phi)}\right)$$

and

$$f(1) = t \left(\frac{2\phi}{\pi}\right)^{\frac{1}{2}} e^{\sqrt{\varphi\phi}} \left(\sqrt{\frac{\varphi}{2t+\phi}}\right)^{\frac{3}{2}} K_{\frac{3}{2}}\left(\sqrt{\varphi(2t+\phi)}\right).$$

Proof. The Poisson-Reciprocal Inverse Gaussian distribution is

$$f(x) = \left(\frac{\phi}{2\pi}\right)^{\frac{1}{2}} \frac{t^x}{x!} \int_0^{\infty} \lambda^{x-\frac{1}{2}} \exp\left\{-\lambda t - \frac{\phi(1-\mu\lambda)^2}{2\mu^2\lambda}\right\} d\lambda.$$

Let $u = \exp\left\{-\frac{\phi}{2\mu^2\lambda} - \frac{(2t+\phi)\lambda}{2}\right\}$ and $dv = \lambda^{x-\frac{1}{2}} d\lambda$, then

$$\mu^2 (\phi + 2t) x(x+1) f(x+1) = t\mu^2 (2x+1) x f(x) + \phi t^2 f(x-1), \quad x = 1, 2, 3, \dots$$

The recursive relation can be rewritten as

$$\mu^2(\phi + 2t) [-x + x^2] f(x) = t\mu^2 \left[(x-1) + 2(x-1)^2 \right] f(x-1) + \phi t^2 f(x-2), \quad x = 2, 3, 4, \dots \quad (4.80)$$

The values for Wang's coefficients are therefore: $b_0 = 0, b_1 = -\mu^2(\phi + 2t), b_2 = \mu^2(\phi + 2t); a_{10} = 0, a_{11} = t\mu^2, a_{12} = 2t\mu^2; a_{20} = \phi t^2, a_{21} = 0, a_{22} = 0$. The corresponding differential equation is therefore obtained by replacing the given constants in equation (4.27). The resulting differential equation is

$$[\mu^2\phi + 2t\mu^2(1-s)] G''(s) - 3t\mu^2 G'(s) - \phi t^2 G(s) = 0. \quad (4.81)$$

□

4.3.18 Generalized Inverse Gaussian Distribution

Consider a generalized inverse Gaussian distribution given by (3.87).

Proposition 4.3.20. *The recursive relation for Poisson-generalized inverse Gaussian distribution is*

$$(2t + \varphi)x(x+1)f(x+1) = 2t(x+v)xf(x) + \phi t^2 f(x-1), \quad x = 1, 2, 3, \dots \quad (4.82)$$

with initial conditions

$$f(0) = \left(\frac{\varphi}{\phi}\right)^{\frac{v}{2}} \left(\frac{\phi}{2t+\varphi}\right)^{\frac{v}{2}} \frac{K_v(\sqrt{\phi(2t+\varphi)})}{K_v(\sqrt{\varphi\phi})}$$

and

$$f(1) = t \left(\frac{\varphi}{\phi}\right)^{\frac{v}{2}} \left(\frac{\phi}{2t+\varphi}\right)^{\frac{1+v}{2}} \frac{K_{1+v}(\sqrt{\phi(2t+\varphi)})}{K_v(\sqrt{\varphi\phi})}.$$

Proof. The Poisson-Generalized Inverse Gaussian distribution is

$$f(x) = \frac{\left(\frac{\varphi}{\phi}\right)^v t^x}{2K_v(\sqrt{\varphi\phi}) x!} \int_0^\infty e^{-\lambda t} \lambda^{x+v-1} \exp\left\{-\frac{1}{2}\left(\varphi\lambda + \frac{\phi}{\lambda}\right)\right\} d\lambda.$$

Let $u = \exp\left\{-\lambda t - \frac{1}{2}\left(\varphi\lambda + \frac{\phi}{\lambda}\right)\right\}$ and $dv = \lambda^{x+v-1} d\lambda$, then

$$(2t + \varphi)x(x+1)f(x+1) = 2t(x+v)xf(x) + \phi t^2 f(x-1), \quad x = 1, 2, 3, \dots$$

The recursive relation can be rewritten as

$$(2t + \varphi) [-x + x^2] f(x) = 2t \left[v(x-1) + (x-1)^2 \right] f(x-1) + \phi t^2 f(x-2), \quad x = 2, 3, 4, \dots \quad (4.83)$$

The values of Wang's coefficients are: $b_0 = 0, b_1 = -(2t + \varphi), b_2 = (2t + \varphi); a_{10} = 0, a_{11} = 2tv, a_{12} = 2t; a_{20} = \phi t^2, a_{21} = 0, a_{22} = 0$. The corresponding differential equation is therefore obtained by replacing the given constants in equation (4.27). The resulting differential equation is:

$$[\varphi + 2t(1-s)] G''(s) - 2t(v+1) G'(s) - \phi t^2 G(s) = 0. \quad (4.84)$$

□

4.3.19 Confluent Hypergeometric Distribution

The Confluent Hypergeometric distribution is

$$g(\lambda) = \frac{\lambda^{a-1} (1 + \lambda)^{c-a-1} e^{-k\lambda}}{\Gamma(a) \psi(a, c; k)}, \quad \lambda > 0; -\infty < a < \infty; -\infty < c < \infty \quad (4.85)$$

Proposition 4.3.21. *The recursive relation for the Poisson-Confluent Hypergeometric distribution is*

$$(k + t)x(x + 1)f(x + 1) = (c + x - 1 - k - t)xtf(x) + (x + a - 1)t^2f(x - 1), \quad x = 1, 2, 3, \dots \quad (4.86)$$

with initial conditions

$$f(0) = \int_0^\infty \frac{\lambda^{a-1} (1 + \lambda)^{c-a-1} e^{-(k+t)\lambda}}{\Gamma(a) \psi(a, c; k)} d\lambda$$

and

$$f(1) = \int_0^\infty \frac{t\lambda^a (1 + \lambda)^{c-a-1} e^{-(k+t)\lambda}}{\Gamma(a) \psi(a, c; k)} d\lambda.$$

Proof. The Poisson-Confluent Hypergeometric distribution is

$$f(x) = \int_0^\infty \frac{e^{-\lambda t} (\lambda t)^x}{x!} \cdot \frac{\lambda^{a-1} (1 + \lambda)^{c-a-1} e^{-k\lambda}}{\Gamma(a) \psi(a, c; k)} d\lambda.$$

Let $u = \lambda^{x+a-1} e^{-(k+t)\lambda}$ and $dv = (1 + \lambda)^{c-a-1} d\lambda$, then the recursive relation is

$$(k + t)x(x + 1)f(x + 1) = (c + x - 1 - k - t)xtf(x) + (x + a - 1)t^2f(x - 1), \quad x = 1, 2, 3, \dots$$

The recursive relation can be rewritten as

$$(k + t)[-x + x^2]f(x) = t[(c - k - t - 1)(x - 1) + (x - 1)^2]f(x - 1) + t^2[\alpha + (x - 2)]f(x - 2), \quad x = 2, 3, 4, \dots \quad (4.87)$$

Therefore the values for the coefficients corresponding to Wang's model (4.26) are: $b_0 = 0, b_1 = -(k + t), b_2 = (k + t); a_{10} = 0, a_{11} = t(c - k - t - 1), a_{12} = t; a_{20} = \alpha t^2, a_{21} = t^2, a_{22} = 0$ and the corresponding differential equation is

$$[k + t(1 - s)]G''(s) + [t(k - c) + t^2(1 - s)]G'(s) - \alpha t^2 G(s) = 0. \quad (4.88)$$

□

4.3.20 Half-Normal Distribution

The half normal distribution is

$$g(\lambda) = \frac{2}{\sqrt{2\pi\sigma^2}} e^{-\frac{(\lambda-\mu)^2}{2\sigma^2}}; \quad \lambda > 0; -\infty < \mu < \infty; \sigma^2 > 0. \quad (4.89)$$

Proposition 4.3.22. *The recursive relation for Poisson-Half-normal distribution is*

$$(x+2)f(x+2) = t^2\sigma^2 f(x) - (t\sigma^2 - \mu)tf(x+1), x = 0, 1, 2, \dots \quad (4.90)$$

with initial conditions

$$f(0) = \int_0^\infty \frac{2}{\sqrt{2\pi\sigma^2}} \exp\left\{-\lambda t - \frac{(\lambda - \mu)^2}{2\sigma^2}\right\} d\lambda$$

and

$$f(1) = \int_0^\infty \frac{2\lambda t}{\sqrt{2\pi\sigma^2}} \exp\left\{-\lambda t - \frac{(\lambda - \mu)^2}{2\sigma^2}\right\} d\lambda.$$

Proof. The Poisson-Half-normal distribution is

$$f(x) = \int_0^\infty \frac{e^{-\lambda t} (\lambda t)^x}{x!} \cdot \frac{2}{\sqrt{2\pi\sigma^2}} e^{-\frac{(\lambda - \mu)^2}{2\sigma^2}} d\lambda.$$

Let $u = e^{-\lambda t - \frac{(\lambda - \mu)^2}{2\sigma^2}}$ and $dv = \lambda^x d\lambda$, then the recursive relation is

$$(x+2)f(x+2) = t^2\sigma^2 f(x) - (t\sigma^2 - \mu)tf(x+1), x = 0, 1, 2, \dots$$

The recursive relation can be rewritten as

$$xf(x) = -t(t\sigma^2 - \mu)f(x-1) + t^2\sigma^2 f(x-2), x = 2, 3, 4, \dots \quad (4.91)$$

Therefore the values for the coefficients corresponding to Wang's model are: $b_0 = 0, b_1 = 1, b_2 = 0; a_{10} = -t(t\sigma^2 - \mu), a_{11} = 0, a_{12} = 0; a_{20} = t^2\sigma^2, a_{21} = 0, a_{22} = 0$ and the corresponding differential equation is

$$G'(s) + [t^2\sigma^2(1-s) - t\mu]G(s) = (t^2\sigma^2 - t\mu)f(0) - f(1) \quad (4.92)$$

□

4.4 Conclusion

In this chapter a number of recursive formulae for mixed Poisson distributions are derived using Integration by parts technique. This technique is simple and straight forward provided the choice of u and dv in the integrand is done correctly to facilitate integration.

The differential equations obtained provide a compact form of obtaining moments for the corresponding Poisson mixtures, if the differential equations are solved. However there is a limitation in generating probabilities when the initial condition is in terms of special functions, therefore a numerical approximation of the initial conditions suffice.

Chapter 5

MIXED POISSON DISTRIBUTIONS AND THEIR MOMENTS IN TERMS OF TRANSFORMS

5.1 Introduction

The objective of this chapter is to derive mixed Poisson distributions and their moments in terms of transforms. Specifically, Laplace and Mellin transforms are used in the construction and the probability generating function is used in obtaining moments.

The pgf of the mixed Poisson distribution is expressed in terms of the Laplace transform of the mixing distribution. The r th factorial moment is obtained using the pgf and is expressed in terms of the r th moment of the mixing distribution. Raw moments and central moments of a mixed Poisson distribution are obtained using pgf.

The index of dispersion of mixed Poisson distribution is also considered. The mathematical formulation follows.

5.2 Mixed Poisson Distribution and Properties based on Transforms

For a mixed Poisson distribution defined in equation (1.2), its probability generating function is

$$\begin{aligned} G(s) &= \sum_{x=0}^{\infty} f(x) s^x, \quad s \in \mathbb{R} \\ &= \mathbb{E}[s^X]. \end{aligned} \tag{5.1}$$

The Laplace transform of $\phi(x)$ is

$$L\{\phi(x)\} = \int_0^{\infty} e^{-sx} \phi(x) dx = \tau(s) \quad (5.2)$$

whenever the improper integral converges. Thus, L is an operator acting on $\phi(x)$ to produce another function, say $\tau(s)$. If $\phi(x)$ is a probability density function, then

$$L\{\phi(x)\} = \mathbb{E}[e^{-sX}] \quad (5.3)$$

Mellin transform is

$$M\{\phi(x)\} = \int_0^{\infty} x^{s-1} \phi(x) dx \quad (5.4)$$

provided the integral exists and if $\phi(x)$ is a probability density function, then

$$M\{\phi(x)\} = \mathbb{E}[X^{s-1}]. \quad (5.5)$$

Proposition 5.2.1. *Mixtures based on Laplace and Mellin transforms of mixing distribution*

(a) *The mixed Poisson distribution in terms of the Laplace transform is*

$$f(x) = (-1)^x \frac{t^x}{x!} L_{\Lambda}^{(x)}(t) \quad (5.6)$$

where

$$L_{\Lambda}(t) = f(0)$$

is the Laplace transform of the mixing distribution $g(\lambda)$ and its x th derivative is

$$\begin{aligned} L_{\Lambda}^{(x)}(t) &= \frac{d^x}{dt^x} L_{\Lambda}(t) \\ &= f^{(x)}(0) \end{aligned} \quad (5.7)$$

(b) *The mixed Poisson distribution in terms of Mellin transform is*

$$\begin{aligned} f(x) &= \frac{t^x}{x!} \sum_{r=0}^{\infty} \frac{(-t)^r}{r!} M\{g(\lambda), x+r-1\} \\ &= \frac{t^x}{x!} \sum_{r=0}^{\infty} \frac{(-t)^r}{r!} \mu'_{x+r}. \end{aligned} \quad (5.8)$$

where μ'_{x+r} is the $(x+r)$ th raw moment of mixing distribution $g(\lambda)$.

Proof. (a) The mixed Poisson distribution is

$$\begin{aligned} f(x) &= \frac{t^x}{x!} \int_0^{\infty} e^{-\lambda t} \lambda^x g(\lambda) d\lambda \\ &= \frac{t^x}{x!} \mathbb{E}[e^{-\Lambda t} \Lambda^x] \end{aligned}$$

and when $x = 0$ we have

$$\begin{aligned} f(0) &= \mathbb{E} [e^{-\Lambda t}] \\ &= L_{\Lambda}(t) \end{aligned}$$

which is the Laplace transform of the mixing distribution $g(\lambda)$. On taking the first derivative, we have

$$\begin{aligned} \frac{d}{dt} f(0) &= \frac{d}{dt} L_{\Lambda}(t) \\ &= \mathbb{E} [-\Lambda e^{-\Lambda t}], \end{aligned} \tag{5.9}$$

the second derivative is

$$\frac{d^2}{dt^2} f(0) = \mathbb{E} [(-1)^2 \Lambda^2 e^{-\Lambda t}] \tag{5.10}$$

and in general, the x th derivative is

$$f^{(x)}(0) = (-1)^x \mathbb{E} [\Lambda^x e^{-\Lambda t}] \tag{5.11}$$

hence the mixed Poisson distribution is

$$\begin{aligned} f(x) &= (-1)^x \frac{t^x}{x!} f^{(x)}(0) \\ &= (-1)^x \frac{t^x}{x!} L_{\Lambda}^{(x)}(t). \end{aligned}$$

(b) The mixed Poisson distribution in terms of Mellin transform is

$$\begin{aligned} f(x) &= \frac{t^x}{x!} \int_0^{\infty} \lambda^x e^{-\lambda t} g(\lambda) d\lambda \\ &= \frac{t^x}{x!} \int_0^{\infty} \lambda^x \sum_{r=0}^{\infty} \frac{(-\lambda t)^r}{r!} g(\lambda) d\lambda \\ &= \frac{t^x}{x!} \sum_{r=0}^{\infty} \frac{(-t)^r}{r!} \int_0^{\infty} \lambda^{(x+r+1)-1} g(\lambda) d\lambda \\ &= \frac{t^x}{x!} \sum_{r=0}^{\infty} \frac{(-t)^r}{r!} M[g(\lambda), x+r+1] \\ &= \frac{t^x}{x!} \sum_{r=0}^{\infty} \left\{ \frac{(-t)^r}{r!} \mathbb{E} [\Lambda^{x+r}] \right\} \\ &= \frac{t^x}{x!} \sum_{r=0}^{\infty} \frac{(-t)^r}{r!} \mu'_{x+r} \end{aligned}$$

or equivalently, letting $r = j - x$, we have

$$f(x) = \frac{t^x}{x!} \sum_{j=x}^{\infty} \frac{(-t)^{j-x}}{(j-x)!} \mu'_j \tag{5.12}$$

where $\mu'_j = \mathbb{E} (\Lambda^j)$ is the j th raw moment of the mixing distribution $g(\lambda)$.

□

Proposition 5.2.2.

(a) The probability generating function of the mixed Poisson distribution is expressed in terms of the Laplace transform of the mixing distribution as

$$G(s) = L_{\Lambda}[(1-s)t] \quad (5.13)$$

and therefore

$$\mathbb{E}[X(X-1)(X-2)\dots(X-r+1)] = t^r \mathbb{E}[\Lambda^r] \quad (5.14)$$

where the LHS of (5.14) is the r th factorial moment of the mixed Poisson distribution. In particular,

$$\mathbb{E}(X) = t\mathbb{E}[\Lambda] \quad (5.15)$$

and

$$\text{Var}(X) = t^2 \text{Var}(\Lambda) + t\mathbb{E}(\Lambda). \quad (5.16)$$

The index of dispersion is

$$\begin{aligned} I_X &= \frac{\text{Var}(X)}{\mathbb{E}(X)} \\ &= 1 + tI_{\Lambda}. \end{aligned} \quad (5.17)$$

(b) (i) The raw moments of Poisson mixtures in terms of pgf are:

$$\mathbb{E}(X) = G'(1) \quad (5.18)$$

$$\mathbb{E}(X^2) = G''(1) + G'(1) \quad (5.19)$$

$$\mathbb{E}(X^3) = G'''(1) + 3G''(1) + G'(1) \quad (5.20)$$

$$\mathbb{E}(X^4) = G^{(iv)}(1) + 6G'''(1) + 7G''(1) + G'(1) \quad (5.21)$$

(ii) Moments of Poisson mixtures about the mean in terms of pgf

The second central moment is

$$\mathbb{E}(X - \mu)^2 = G''(1) + G'(1) - [G'(1)]^2 \quad (5.22)$$

third central moment is

$$\begin{aligned} \mathbb{E}(X - \mu)^3 &= G'''(1) + 3[1 - G'(1)]G''(1) \\ &\quad + 2(G'(1))^3 - 3(G'(1))^2 + G'(1) \end{aligned} \quad (5.23)$$

and the fourth central moment is

$$\begin{aligned} \mathbb{E}(X - \mu)^4 &= G^{(iv)}(1) + (6 - 4G'(1))G'''(1) \\ &\quad + (7 - 12G'(1) + 6(G'(1))^2)G''(1) + G'(1) \\ &\quad - 4(G'(1))^2 + 6(G'(1))^3 - 3(G'(1))^4. \end{aligned} \quad (5.24)$$

Proof. (a) From equation (5.1) the probability generating function of a Poisson mixture is

$$\begin{aligned} G(s) &= \int_0^\infty e^{-\lambda t(1-s)} g(\lambda) d\lambda \\ &= L_\Lambda((1-s)t) \end{aligned}$$

To obtain the r th factorial moment, the pgf is differentiated r times and the value obtained when $s = 1$. The first two derivatives are:

$$G'(s) = \int_0^\infty \lambda t e^{-\lambda t} e^{\lambda t s} g(\lambda) d\lambda \quad (5.25)$$

$$G''(s) = \int_0^\infty (\lambda t)^2 e^{-\lambda t} e^{\lambda t s} g(\lambda) d\lambda \quad (5.26)$$

and in general the r th derivative is

$$G^{(r)}(s) = \int_0^\infty (\lambda t)^r e^{-\lambda t} e^{\lambda t s} g(\lambda) d\lambda. \quad (5.27)$$

The r th factorial moment is

$$\begin{aligned} \mathbb{E}(X(X-1)(X-2)\cdots(X-r+1)) &= G^{(r)}(1) \\ &= \int_0^\infty (\lambda t)^r g(\lambda) d\lambda \\ &= t^r \mathbb{E}(\Lambda^r), \quad r = 1, 2, 3, \dots \end{aligned} \quad (5.28)$$

and in particular,

$$\begin{aligned} \mathbb{E}(X) &= G'(1) \\ &= t\mathbb{E}(\Lambda) \end{aligned} \quad (5.29)$$

and

$$\begin{aligned} \text{Var}(X) &= G''(1) + G'(1) - (G'(1))^2 \\ &= t^2 \mathbb{E}(\Lambda^2) + t\mathbb{E}(\Lambda) - t^2 (\mathbb{E}(\Lambda))^2 \\ &= t^2 \text{Var}(\Lambda) + t\mathbb{E}(\Lambda). \end{aligned} \quad (5.30)$$

The index of dispersion is

$$\begin{aligned} I_X &= \frac{t\mathbb{E}(\Lambda) + t^2 \text{Var}(\Lambda)}{t\mathbb{E}(\Lambda)} \\ &= 1 + tI_\Lambda. \end{aligned} \quad (5.31)$$

(b) (i) Raw moments of Poisson mixtures

When $r = 1$, we have the first raw moment

$$\mathbb{E}(X) = G'(1)$$

and when $r = 2$, we have the second raw moment

$$\mathbb{E}(X^2) = G''(1) + G'(1).$$

When $r = 3$ we have

$$\begin{aligned} G'''(1) &= \mathbb{E}(X(X-1)(X-2)) \\ &= \mathbb{E}(X^3) - 3\mathbb{E}(X^2) + 2\mathbb{E}(X) \end{aligned}$$

and therefore the third raw moment is

$$\mathbb{E}(X^3) = G'''(1) + 3G''(1) + G'(1).$$

When $r = 4$, we have

$$\begin{aligned} G^{(iv)}(1) &= \mathbb{E}(X(X-1)(X-2)(X-3)) \\ &= \mathbb{E}(X^4) - 6\mathbb{E}(X^3) + 11\mathbb{E}(X^2) - 6\mathbb{E}(X) \end{aligned}$$

and therefore the fourth raw moment is

$$\mathbb{E}(X^4) = G^{(iv)}(1) + 6G'''(1) + 7G''(1) + G'(1)$$

(ii) Central moments

The second central moment is

$$\begin{aligned} \mathbb{E}(X - \mu)^2 &= \mathbb{E}(X^2) - 2\mu\mathbb{E}(X) + \mu^2 \\ &= \mathbb{E}(X^2) - \mu^2 \\ &= G''(1) + G'(1) - \left(G'(1)\right)^2 \end{aligned}$$

The third central moment is obtained in terms of probability generating function as

$$\begin{aligned} \mathbb{E}(X - \mu)^3 &= \mathbb{E}(X^3) - 3\mu\mathbb{E}(X^2) + 3\mu^2\mathbb{E}(X) - \mu^3 \\ &= G'''(1) + 3\left(1 - G'(1)\right)G''(1) \\ &\quad + 2\left(G'(1)\right)^3 - 3\left(G'(1)\right)^2 + G'(1) \end{aligned}$$

and the fourth central moment is

$$\begin{aligned} \mathbb{E}(X - \mu)^4 &= \mathbb{E}(X^4) - 4\mu\mathbb{E}(X^3) + 6\mu^2\mathbb{E}(X^2) - 3\mu^4 \\ &= G^{(iv)}(1) + \left(6 - 4G'(1)\right)G'''(1) \\ &\quad + \left(7 - 12G'(1) + 6\left(G'(1)\right)^2\right)G''(1) + G'(1) \\ &\quad - 4\left(G'(1)\right)^2 + 6\left(G'(1)\right)^3 - 3\left(G'(1)\right)^4 \end{aligned}$$

□

5.3 Examples of Mixed Poisson Distributions Based on Transforms

The following are examples of Mixed Poisson distributions based on Laplace and Mellin transforms of the mixing distributions.

5.3.1 Poisson-Gamma distribution

The gamma distribution given in equation (2.15) has Laplace transform

$$\begin{aligned}
 L_{\Lambda}(t) &= \mathbb{E}[e^{-\Lambda t}] \\
 &= \frac{\beta^{\alpha}}{\Gamma(\alpha)} \int_0^{\infty} \lambda^{\alpha-1} e^{-(t+\beta)\lambda} d\lambda \\
 &= \frac{\beta^{\alpha}}{\Gamma(\alpha)} \frac{\Gamma(\alpha)}{(t+\beta)^{\alpha}} \\
 &= \left(\frac{\beta}{t+\beta}\right)^{\alpha}
 \end{aligned} \tag{5.32}$$

with the first three derivatives as:

$$\begin{aligned}
 L'_{\Lambda}(t) &= -\alpha\beta^{\alpha}(t+\beta)^{-\alpha-1} \\
 L''_{\Lambda}(t) &= (-1)^2(\alpha+1)\alpha\beta^{\alpha}(t+\beta)^{-\alpha-2} \\
 L'''_{\Lambda}(t) &= (-1)^3(\alpha+2)(\alpha+1)\alpha\beta^{\alpha}(t+\beta)^{-\alpha-3}.
 \end{aligned}$$

Therefore in general the x th derivative is

$$\begin{aligned}
 L_{\Lambda}^{(x)}(t) &= (-1)^x(\alpha+x-1)(\alpha+x-2)\cdots(\alpha+1)\alpha\beta^{\alpha}(t+\beta)^{-\alpha-x} \\
 &= (-1)^x x! \binom{\alpha+x-1}{x} \beta^{\alpha}(t+\beta)^{-\alpha-x}
 \end{aligned} \tag{5.33}$$

. and from equation (5.6) the Poisson-gamma distribution is

$$\begin{aligned}
 f(x) &= \frac{(-t)^x}{x!} L_{\Lambda}^{(x)}(t) \\
 &= \frac{(-t)^x}{x!} (-1)^x x! \binom{\alpha+x-1}{x} \beta^{\alpha}(t+\beta)^{-\alpha-x} \\
 &= \binom{\alpha+x-1}{x} \left(\frac{t}{t+\beta}\right)^x \left(\frac{\beta}{t+\beta}\right)^{\alpha}; x = 0, 1, 2, \dots
 \end{aligned}$$

which is (2.16).

From equation (5.13) and (5.32) its pgf is

$$\begin{aligned}
 G(s) &= L_{\Lambda}((1-s)t) \\
 &= \left(\frac{\beta}{(1-s)t+\beta}\right)^{\alpha} \\
 &= \left(\frac{\beta}{(t+\beta)-ts}\right)^{\alpha} \\
 &= \left(\frac{\beta}{t+\beta}\right)^{\alpha} \left(1 - \frac{t}{t+\beta}s\right)^{-\alpha}
 \end{aligned} \tag{5.34}$$

and the first three derivatives are:

$$\frac{d}{ds}G(s) = \alpha \left(\frac{t}{t+\beta} \right) \left(\frac{\beta}{t+\beta} \right)^\alpha \left(1 - \frac{t}{t+\beta}s \right)^{-\alpha-1}$$

$$\frac{d^2}{ds^2}G(s) = (\alpha+1)\alpha \left(\frac{t}{t+\beta} \right)^2 \left(\frac{\beta}{t+\beta} \right)^\alpha \left(1 - \frac{t}{t+\beta}s \right)^{-\alpha-2}$$

$$\frac{d^3}{ds^3}G(s) = (\alpha+2)(\alpha+1)\alpha \left(\frac{t}{t+\beta} \right)^3 \left(\frac{\beta}{t+\beta} \right)^\alpha \left(1 - \frac{t}{t+\beta}s \right)^{-\alpha-3}$$

therefore, the r th derivative is generalized as

$$\begin{aligned} \frac{d^r}{ds^r}G(s) &= (\alpha+r-1)(\alpha+r-2)\cdots(\alpha+1)\alpha \left(\frac{t}{t+\beta} \right)^r \left(\frac{\beta}{t+\beta} \right)^\alpha \left(1 - \frac{t}{t+\beta}s \right)^{-\alpha-r} \\ &= r! \binom{\alpha+r-1}{r} \left(\frac{t}{t+\beta} \right)^r \left(\frac{\beta}{t+\beta} \right)^\alpha \left(1 - \frac{t}{t+\beta}s \right)^{-\alpha-r} \end{aligned}$$

and making the substitution $s = 1$, we have

$$\begin{aligned} \frac{d^r}{ds^r}G(1) &= r! \binom{\alpha+r-1}{r} \left(\frac{t}{t+\beta} \right)^r \left(\frac{\beta}{t+\beta} \right)^\alpha \left(\frac{t+\beta}{\beta} \right)^{\alpha+r} \\ &= r! \binom{\alpha+r-1}{r} \left(\frac{t}{\beta} \right)^r \end{aligned}$$

that is

$$\mathbb{E}(X(X-1)(X-2)\cdots(X-r+1)) = t^r \frac{r!}{\beta^r} \binom{\alpha+r-1}{r} \quad (5.35)$$

and from equation (5.28)

$$\begin{aligned} \mathbb{E}(\Lambda^r) &= \frac{r!}{\beta^r} \binom{\alpha+r-1}{r} \\ &= \frac{\Gamma(\alpha+r)}{\Gamma(\alpha)\beta^r}. \end{aligned} \quad (5.36)$$

When $r = 1$, we have the first moment

$$\mathbb{E}(X) = \frac{t}{\beta}\alpha. \quad (5.37)$$

When $r = 2$, we have the second factorial moment

$$\begin{aligned} \mathbb{E}(X(X-1)) &= \frac{2t^2}{\beta^2} \binom{\alpha+1}{2} \\ &= \frac{t^2(\alpha+1)\alpha}{\beta^2}, \end{aligned}$$

therefore the variance is

$$\begin{aligned} \text{Var}(X) &= \mathbb{E}(X(X-1)) + \mathbb{E}(X) - (\mathbb{E}(X))^2 \\ &= \frac{t^2(\alpha+1)\alpha}{\beta^2} + \frac{t\alpha}{\beta} - \frac{t^2\alpha^2}{\beta^2} \\ &= \frac{t\alpha}{\beta} \left(\frac{t}{\beta} + 1 \right) \end{aligned} \quad (5.38)$$

and the index of dispersion is

$$I_X = \frac{t}{\beta} + 1. \quad (5.39)$$

The j th moment of the gamma distribution is

$$\begin{aligned} \mathbb{E}(\Lambda^j) &= \int_0^\infty \lambda^j \frac{\beta^\alpha}{\Gamma(\alpha)} e^{-\beta\lambda} \lambda^{\alpha-1} d\lambda \\ &= \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^\infty \lambda^{j+\alpha-1} e^{-\beta\lambda} d\lambda \\ &= \frac{\Gamma(j+\alpha)}{\Gamma(\alpha) \beta^j} \end{aligned} \quad (5.40)$$

and the Poisson-gamma distribution by method of moments is

$$f(x) = \sum_{j=x}^{\infty} \frac{t^x (-t)^{j-x} \Gamma(j+\alpha)}{(j-x)! x! \Gamma(\alpha) \beta^j}. \quad (5.41)$$

Let $k = j - x \Rightarrow j = k + x$, therefore

$$\begin{aligned} f(x) &= \sum_{k=0}^{\infty} \frac{t^x (-t)^k \Gamma(k+x+\alpha)}{k! x! \Gamma(\alpha) \beta^{k+x}} \\ &= \binom{\alpha+x-1}{x} \left(\frac{t}{\beta}\right)^x \sum_{k=0}^{\infty} (-1)^k \binom{x+\alpha+k-1}{k} \left(\frac{t}{\beta}\right)^k \\ &= \binom{\alpha+x-1}{x} \left(\frac{t}{\beta}\right)^x \sum_{k=0}^{\infty} \binom{-(x+\alpha)}{k} \left(\frac{t}{\beta}\right)^k \\ &= \binom{\alpha+x-1}{x} \left(\frac{t}{\beta}\right)^x \left(1 + \frac{t}{\beta}\right)^{-(x+\alpha)} \\ &= \binom{\alpha+x-1}{x} \left(\frac{\beta}{\beta+t}\right)^\alpha \left(\frac{t}{\beta+t}\right)^x \end{aligned}$$

5.3.2 Special cases of Poisson-Gamma Distribution

Exponential distribution

When $\alpha = 1$ we have:

$$g(\lambda) = \beta e^{-\beta\lambda}; \lambda > 0, \beta > 0 \quad (5.42)$$

$$L_\Lambda(t) = \frac{\beta}{t+\beta} \quad (5.43)$$

$$L_\Lambda^{(x)}(t) = (-1)^x x! \frac{\beta}{(t+\beta)^{x+1}}$$

Therefore,

$$f(x) = \left(\frac{t}{t+\beta}\right)^x \left(\frac{\beta}{t+\beta}\right); x = 0, 1, 2, \dots \quad (5.44)$$

which is a Geometric distribution with parameter $\frac{\beta}{t+\beta}$.

$$\begin{aligned} G(s) &= \frac{\beta}{(1-s)t+\beta} \\ &= \frac{\beta}{t+\beta} \left(1 - \frac{t}{t+\beta}s\right)^{-1} \end{aligned} \quad (5.45)$$

$$\begin{aligned}\mu'_r &= \mathbb{E}[\Lambda^r] \\ &= \frac{r!}{\beta^r}\end{aligned}$$

then

$$\mathbb{E}(X) = \frac{t}{\beta} \quad (5.46)$$

$$\text{Var}(X) = \frac{t(t+\beta)}{\beta^2} \quad (5.47)$$

and

$$I_X = 1 + \frac{t}{\beta}. \quad (5.48)$$

One parameter Gamma distribution

When $\beta = 1$, we have

$$g(\lambda) = \frac{e^{-\lambda}\lambda^{\alpha-1}}{\Gamma(\alpha)}; \lambda > 0, \alpha > 0 \quad (5.49)$$

$$L_\Lambda(t) = \left(\frac{1}{t+1}\right)^\alpha \quad (5.50)$$

$$L_\Lambda^{(x)}(t) = (-1)^x x! \binom{\alpha+x-1}{x} (t+1)^{-\alpha-x}$$

Therefore,

$$f(x) = \binom{\alpha+x-1}{x} \left(\frac{t}{t+1}\right)^x \left(\frac{1}{t+1}\right)^\alpha; x = 0, 1, 2, \dots \quad (5.51)$$

which is *NBD* $(\alpha, 1)$.

$$\begin{aligned}G(s) &= \left[\frac{1}{(1-s)t+1}\right]^\alpha \\ &= \left(\frac{1}{t+1}\right)^\alpha \left(1 - \frac{1}{t+1}s\right)^{-\alpha}\end{aligned} \quad (5.52)$$

$$\mathbb{E}(\Lambda^r) = \frac{\Gamma(r+\alpha)}{\Gamma(\alpha)} \quad (5.53)$$

then,

$$\mathbb{E}(\Lambda) = \alpha$$

$$\mathbb{E}(\Lambda^2) = \alpha(\alpha+1)$$

$$\text{Var}(\Lambda) = \alpha \quad (5.54)$$

$$\mathbb{E}(X) = t\alpha$$

$$\text{Var}(X) = \alpha t(t+1) \quad (5.55)$$

and

$$I_X = 1 + t. \quad (5.56)$$

Chi-Squared distribution

When $\alpha = \frac{n}{2}$, $\beta = \frac{1}{2}$, where n is a fixed positive integer, we have:

$$g(\lambda) = \frac{1}{2^{\frac{n}{2}} \Gamma(\frac{n}{2})} e^{-\frac{\lambda}{2}} \lambda^{\frac{n}{2}-1}, \lambda > 0 \quad (5.57)$$

$$L_{\Lambda}(t) = \left(\frac{1}{1+2t} \right)^{\frac{n}{2}} \quad (5.58)$$

$$L_{\Lambda}^{(x)}(t) = (-1)^x x! \binom{\frac{n}{2} + x - 1}{x} \left(\frac{1}{1+2t} \right)^{\frac{n}{2}} \left(\frac{2}{1+2t} \right)^x$$

Therefore,

$$f(x) = \binom{\frac{n}{2} + x - 1}{x} \left(\frac{2t}{2t+1} \right)^x \left(\frac{1}{2t+1} \right)^{\frac{n}{2}}; x = 0, 1, 2, \dots \quad (5.59)$$

which is $NBD\left(\frac{n}{2}, \frac{1}{x+1}\right)$ with pgf

$$G(s) = \left(\frac{1}{2t+1} \right)^{\frac{n}{2}} \left[1 - \frac{2t}{2t+1} s \right]^{-\frac{n}{2}} \quad (5.60)$$

$$\mathbb{E}(\Lambda^r) = \frac{2^r \Gamma(\frac{n}{2} + r)}{\Gamma(\frac{n}{2})} \quad (5.61)$$

$\mathbb{E}(\Lambda) = n$, $\mathbb{E}(\Lambda^2) = (n+2)n$ and $Var(\Lambda) = 2n$.

$$\mathbb{E}(X) = nt \quad (5.62)$$

$$Var(X) = nt(1+2t) \quad (5.63)$$

and

$$I_X = 1 + 2t. \quad (5.64)$$

Scaled Chi-Squared distribution

When $\alpha = \frac{n}{2}$ and $\beta = \frac{1}{2\sigma^2}$, we have:

$$g(\lambda) = \frac{1}{(2\sigma^2)^{\frac{n}{2}} \Gamma(\frac{n}{2})} e^{-\frac{\lambda}{2\sigma^2}} \lambda^{\frac{n}{2}-1}; \lambda > 0 \quad (5.65)$$

$$\begin{aligned} L_{\Lambda}(t) &= \left(\frac{1}{2\sigma^2 t + 1} \right)^{\frac{n}{2}} \\ &= (1 + 2\sigma^2 t)^{-\frac{n}{2}} \end{aligned} \quad (5.66)$$

$$L_{\Lambda}^{(x)}(t) = (-1)^x x! \binom{\frac{n}{2} + x - 1}{x} \left(\frac{1}{1+2\sigma^2 t} \right)^{\frac{n}{2}} \left(\frac{2\sigma^2}{1+2\sigma^2 t} \right)^x$$

Therefore

$$f(x) = \binom{\frac{n}{2} + x - 1}{x} \left(\frac{2\sigma^2 t}{1+2\sigma^2 t} \right)^x \left(\frac{1}{1+2\sigma^2 t} \right)^{\frac{n}{2}}; x = 0, 1, 2, \dots \quad (5.67)$$

which is $NBD\left(\frac{n}{2}, \frac{1}{1+2\sigma^2 t}\right)$.

$$G(s) = \left(\frac{1}{1+2\sigma^2 t}\right)^{\frac{n}{2}} \left[1 - \frac{2\sigma^2 t}{1+2\sigma^2 t} s\right]^{-\frac{n}{2}} \quad (5.68)$$

$$\mathbb{E}(\Lambda^r) = \frac{(2\sigma^2)^r \Gamma\left(r + \frac{n}{2}\right)}{\Gamma\left(\frac{n}{2}\right)} \quad (5.69)$$

then $\mathbb{E}(\Lambda) = n\sigma^2$, $\mathbb{E}(\Lambda^2) = n(n+2)\sigma^4$ and $Var(\Lambda) = 2n\sigma^4$.

$$\mathbb{E}(X) = nt\sigma^2 \quad (5.70)$$

$$Var(X) = nt(1+2\sigma^2 t)\sigma^2 \quad (5.71)$$

and

$$I_X = 1 + 2\sigma^2 t. \quad (5.72)$$

5.3.3 Poisson-3 Parameter Generalized Lindley Distribution

Consider the generalized 3-parameter Lindley distribution given in (2.89), its Laplace transform is

$$\begin{aligned} L_\Lambda(t) &= \int_0^\infty \frac{e^{-t\lambda} \theta^2 (\theta\lambda)^{\alpha-1} (\alpha + \gamma\lambda) e^{-\theta\lambda}}{(\theta + \gamma)\Gamma(\alpha + 1)} d\lambda \\ &= \frac{\theta^{\alpha+1}}{(\theta + \gamma)\Gamma(\alpha + 1)} \left\{ \alpha \int_0^\infty \lambda^{\alpha-1} e^{-(t+\theta)\lambda} d\lambda + \gamma \int_0^\infty \lambda^{(\alpha+1)-1} e^{-(t+\theta)\lambda} d\lambda \right\} \\ &= \frac{\theta^{\alpha+1}}{(\theta + \gamma)\Gamma(\alpha + 1)} \left\{ \frac{\alpha\Gamma(\alpha)}{(t + \theta)^\alpha} + \frac{\gamma\Gamma(\alpha + 1)}{(t + \theta)^{\alpha+1}} \right\} \\ &= \frac{\theta^{\alpha+1}}{\theta + \gamma} \left[(t + \theta)^{-\alpha} + \gamma(t + \theta)^{-\alpha-1} \right] \end{aligned} \quad (5.73)$$

The first three derivatives of the laplace transform (5.73) are:

$$L'_\Lambda(t) = \frac{\theta^{\alpha+1}}{\theta + \gamma} \left\{ (-1)\alpha(t + \theta)^{-\alpha-1} + \gamma(-1)(\alpha + 1)(t + \theta)^{-\alpha-2} \right\}$$

$$L''_\Lambda(t) = \frac{\theta^{\alpha+1}}{\theta + \gamma} \left\{ (-1)^2 \alpha(\alpha + 1)(t + \theta)^{-\alpha-2} + \gamma(-1)^2 (\alpha + 1)(\alpha + 2)(t + \theta)^{-\alpha-3} \right\}$$

$$\begin{aligned} L'''_\Lambda(t) &= \frac{\theta^{\alpha+1}}{\theta + \gamma} \left\{ (-1)^3 \alpha(\alpha + 1)(\alpha + 2)(t + \theta)^{-\alpha-3} + \gamma(-1)^3 (\alpha + 1)(\alpha + 2)(\alpha + 3)(t + \theta)^{-\alpha-4} \right\} \\ &= \frac{\theta^{\alpha+1}}{\theta + \gamma} \left\{ (-1)^3 \frac{\Gamma(\alpha + 3)}{\Gamma(\alpha)} (t + \theta)^{-\alpha-3} + \gamma(-1)^3 \frac{\Gamma(\alpha + 4)}{\Gamma(\alpha + 1)} (t + \theta)^{-\alpha-4} \right\} \end{aligned}$$

Therefore, the x th derivative is

$$L_\Lambda^{(x)}(t) = \frac{\theta^{\alpha+1}}{\theta + \gamma} \left\{ (-1)^x \frac{\Gamma(\alpha + x)}{\Gamma(\alpha)} (t + \theta)^{-\alpha-x} + \gamma(-1)^x \frac{\Gamma(\alpha + x + 1)}{\Gamma(\alpha + 1)} (t + \theta)^{-\alpha-(x+1)} \right\} \quad (5.74)$$

and the Poisson-Generalized Lindley is obtained via laplace transform is

$$\begin{aligned}
f(x) &= \frac{(-t)^x}{x!} L_{\Lambda}^{(x)}(t) \\
&= \frac{\theta^{\alpha+1} t^x}{\theta + \gamma} \left\{ \frac{\Gamma(\alpha + x)}{x! \Gamma(\alpha)} (t + \theta)^{-\alpha-x} + \frac{\gamma \Gamma(\alpha + x + 1)}{x! \Gamma(\alpha + 1)} (t + \theta)^{-\alpha-x-1} \right\} \\
&= \frac{\theta^{\alpha+1} t^x}{\theta + \gamma} \left\{ \binom{\alpha + x - 1}{x} \left(\frac{1}{t + \theta} \right)^{\alpha+x} + \gamma \binom{\alpha + x}{x} \left(\frac{1}{t + \theta} \right)^{\alpha+x+1} \right\}. \tag{5.75}
\end{aligned}$$

Making the substitution $\alpha = \gamma = 1$ in equation (5.75) we have

$$\begin{aligned}
f(x) &= \frac{\theta^2 t^x}{\theta + 1} \left\{ \left(\frac{1}{t + \theta} \right)^{x+1} + (x + 1) \left(\frac{1}{t + \theta} \right)^{x+2} \right\} \\
&= \frac{\theta^2 t^x}{(\theta + 1)(t + \theta)^{x+2}} (t + \theta + x + 1) \tag{5.76}
\end{aligned}$$

which is Poisson-Lindley distribution.

The pgf for Poisson-3-parameter generalized Lindley distribution is

$$G(s) = \frac{\theta^{\alpha+1}}{\theta + \gamma} \left(((1-s)t + \theta)^{-\alpha} + \gamma ((1-s)t + \theta)^{-\alpha-1} \right) \tag{5.77}$$

and the first three derivatives are:

$$\begin{aligned}
\frac{d}{ds} G(s) &= \frac{\theta^{\alpha+1}}{\theta + \gamma} \left(\frac{t\alpha}{((1-s)t + \theta)^{\alpha+1}} + \frac{\gamma(\alpha+1)t}{((1-s)t + \theta)^{\alpha+2}} \right) \\
\frac{d^2}{ds^2} G(s) &= \frac{\theta^{\alpha+1}}{\theta + \gamma} \left(\frac{t^2\alpha(\alpha+1)}{((1-s)t + \theta)^{\alpha+2}} + \frac{\gamma(\alpha+1)(\alpha+2)t^2}{((1-s)t + \theta)^{\alpha+3}} \right) \\
\frac{d^3}{ds^3} G(s) &= \frac{\theta^{\alpha+1}}{\theta + \gamma} \left(\frac{t^3\alpha(\alpha+1)(\alpha+2)}{((1-s)t + \theta)^{\alpha+3}} + \frac{\gamma(\alpha+1)(\alpha+2)(\alpha+3)t^3}{((1-s)t + \theta)^{\alpha+4}} \right)
\end{aligned}$$

therefore the r th derivative is generalized as

$$\frac{d^r}{ds^r} G(s) = \frac{\theta^{\alpha+1}}{\theta + \gamma} \left(\frac{t^r \alpha (\alpha + r - 1) (\alpha + r - 2) \cdots (\alpha + 1) \alpha}{((1-s)t + \theta)^{\alpha+r}} + \frac{\gamma (\alpha + r) (\alpha + r - 1) \cdots (\alpha + 1) t^r}{((1-s)t + \theta)^{\alpha+r+1}} \right).$$

Making the substitution $s = 1$, we have

$$\frac{d^r}{ds^r} G(1) = \frac{\theta^{\alpha+1} t^r}{\theta + \gamma} \left(\frac{\alpha (\alpha + r - 1) (\alpha + r - 2) \cdots (\alpha + 1) \alpha}{\theta^{\alpha+r}} + \frac{\gamma (\alpha + r) (\alpha + r - 1) \cdots (\alpha + 1)}{\theta^{\alpha+r+1}} \right)$$

that is

$$\mathbb{E}(X(X-1)(X-2)\cdots(X-r+1)) = \frac{t^r r!}{\theta + \gamma} \left(\frac{1}{\theta^{r-1}} \binom{\alpha + r - 1}{r} + \frac{\gamma}{\theta^r} \binom{\alpha + r}{r} \right) \tag{5.78}$$

and from equation (5.28),

$$\mathbb{E}(\Lambda^r) = \frac{1}{\theta + \gamma} \left(\frac{\Gamma(\alpha + r)}{\theta^{r-1} \Gamma(\alpha)} + \frac{\gamma \Gamma(\alpha + r + 1)}{\theta^r \Gamma(\alpha + 1)} \right), \quad r = 1, 2, 3, \dots \tag{5.79}$$

When $r = 1$, we have the first moment

$$\mathbb{E}(X) = \frac{t}{\theta(\theta + \gamma)} (\alpha\theta + \gamma(\alpha + 1)). \quad (5.80)$$

When $r = 2$, we have the second factorial moment

$$\mathbb{E}(X(X - 1)) = \frac{t^2}{\theta^2(\theta + \gamma)} (\alpha(\alpha + 1)\theta + \gamma(\alpha + 2)(\alpha + 1)). \quad (5.81)$$

The j th moment of the 3-parameter generalized Lindley distribution is

$$\begin{aligned} \mathbb{E}(\Lambda^j) &= \int_0^\infty \lambda^j \frac{\theta^{\alpha+1}}{(\theta + \gamma)\Gamma(\alpha + 1)} (\alpha + \gamma\lambda) \lambda^{\alpha-1} e^{-\theta\lambda} d\lambda \\ &= \frac{\theta^{\alpha+1}}{(\theta + \gamma)\Gamma(\alpha + 1)} \left\{ \alpha \int_0^\infty \lambda^{j+\alpha-1} e^{-\theta\lambda} d\lambda + \gamma \int_0^\infty \lambda^{(j+\alpha+1)-1} e^{-\theta\lambda} d\lambda \right\} \\ &= \frac{\theta^{\alpha+1}}{(\theta + \gamma)\Gamma(\alpha + 1)} \left\{ \frac{\alpha\theta\Gamma(j + \alpha)}{\theta^{j+\alpha+1}} + \frac{\gamma(j + \alpha)\Gamma(j + \alpha)}{\theta^{j+\alpha+1}} \right\} \\ &= \frac{\Gamma(j + \alpha)}{\Gamma(\alpha + 1)} \left\{ \frac{\alpha\theta + j\gamma + \gamma\alpha}{\theta^j(\theta + \gamma)} \right\} \end{aligned}$$

which simplifies to

$$\begin{aligned} \mathbb{E}(\Lambda^j) &= \frac{\Gamma(j + \alpha)}{\Gamma(\alpha + 1)} \left\{ \frac{\alpha(\theta + \gamma) + \gamma j}{(\theta + \gamma)\theta^j} \right\} \\ &= \frac{\Gamma(j + \alpha)}{\Gamma(\alpha + 1)} \left\{ \alpha + \frac{\gamma j}{\theta + \gamma} \right\} \frac{1}{\theta^j} \end{aligned} \quad (5.82)$$

Therefore, the Poisson-generalized Lindley distribution is obtained by the method of moments as

$$\begin{aligned} f(x) &= \sum_{j=x}^\infty \frac{t^x (-t)^{j-x} \Gamma(j + \alpha)}{(j - x)! x! \Gamma(\alpha + 1)} \left\{ \alpha + \frac{\gamma j}{\theta + \gamma} \right\} \frac{1}{\theta^j} \\ &= \sum_{j=x}^\infty \frac{j! t^x (t)^{j-x} (-1)^{j-x} \Gamma(j + \alpha)}{(j - x)! x! j! \Gamma(\alpha + 1)} \left\{ \alpha + \frac{\gamma j}{\theta + \gamma} \right\} \frac{1}{\theta^j} \\ &= \sum_{j=x}^\infty (-1)^{j-x} \binom{j}{x} \binom{j + \alpha - 1}{j} \left(1 + \frac{\gamma}{\alpha(\theta + \gamma)^j} \right) \left(\frac{t}{\theta} \right)^j \\ &= \sum_{j=x}^\infty (-1)^{j-x} \binom{j}{x} \binom{\alpha + j - 1}{j} \left(\frac{t}{\theta} \right)^j \\ &\quad + \frac{\gamma}{\alpha(\theta + \gamma)} \sum_{j=x}^\infty (-1)^{j-x} \binom{j}{x} \binom{\alpha + j - 1}{j} j \left(\frac{t}{\theta} \right)^j \end{aligned} \quad (5.83)$$

But

$$\begin{aligned}
\sum_{j=x}^{\infty} (-1)^{j-x} \binom{j}{x} \binom{\alpha+j-1}{j} \left(\frac{t}{\theta}\right)^j &= \sum_{j=x}^{\infty} \frac{(-1)^{j-x} \Gamma(\alpha+j)}{x!(j-x)!\Gamma(\alpha)} \left(\frac{t}{\theta}\right)^j \\
&= \frac{\Gamma(x+\alpha) t^x}{x!\Gamma(\alpha) \theta^x} \sum_{j=x}^{\infty} \frac{(-1)^{j-x} \Gamma(x+\alpha+j-x)}{(j-x)!\Gamma(x+\alpha)} \left(\frac{t}{\theta}\right)^{j-x} \\
&= \frac{\Gamma(x+\alpha)}{x!\Gamma(\alpha)} \left(\frac{t}{\theta}\right)^x \sum_{k=0}^{\infty} (-1)^k \binom{x+\alpha+k-1}{k} \left(\frac{t}{\theta}\right)^k \\
&= \frac{\Gamma(x+\alpha)}{x!\Gamma(\alpha)} \left(\frac{t}{\theta}\right)^x \sum_{k=0}^{\infty} \binom{-(x+\alpha)}{k} \left(\frac{t}{\theta}\right)^k \\
&= \frac{\Gamma(x+\alpha)}{x!\Gamma(\alpha)} \left(\frac{t}{\theta}\right)^x \left(\frac{\theta}{\theta+t}\right)^{x+\alpha} \\
&= \frac{\Gamma(x+\alpha)}{x!\Gamma(\alpha)} \frac{t^x \theta^\alpha}{(\theta+t)^{x+\alpha}} \tag{5.84}
\end{aligned}$$

Next, consider

$$\begin{aligned}
&\frac{\gamma}{\alpha(\theta+\gamma)} \sum_{j=x}^{\infty} (-1)^{j-x} \binom{j}{x} \binom{\alpha+j-1}{j} j \left(\frac{t}{\theta}\right)^j \\
&= \frac{\gamma}{\alpha(\theta+\gamma)} \sum_{j=x}^{\infty} \frac{(-1)^{j-x} \Gamma(\alpha+j)}{x!(j-x)!\Gamma(\alpha)} j \left(\frac{t}{\theta}\right)^j \\
&= \frac{\gamma\Gamma(x+\alpha)}{(\theta+\gamma)\Gamma(\alpha+1)} \sum_{j=x}^{\infty} \frac{(-1)^{j-x} \Gamma(x+\alpha+j-x)}{x! (j-x)!\Gamma(x+\alpha)} \\
&\quad (j-x+x) \left(\frac{t}{\theta}\right)^{j-x+x} \\
&= \frac{\gamma\Gamma(x+\alpha)}{(\theta+\gamma)\Gamma(\alpha+1)} \left(\frac{t}{\theta}\right)^x \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma(x+\alpha+k)}{k!\Gamma(x+\alpha)} (k+x) \left(\frac{t}{\theta}\right)^k
\end{aligned}$$

Further simplification yields the following

$$\begin{aligned}
& \frac{\gamma}{\alpha(\theta + \gamma)} \sum_{j=x}^{\infty} (-1)^{j-x} \binom{j}{x} \binom{\alpha + j - 1}{j} j \left(\frac{t}{\theta}\right)^j \\
&= \frac{\gamma \Gamma(x + \alpha)}{(\theta + \gamma) \Gamma(\alpha + 1)} \left(\frac{t}{\theta}\right)^x \left\{ \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma(x + \alpha + k)}{(k - 1)! \Gamma(x + \alpha)} \left(\frac{t}{\theta}\right)^k + x \sum_{k=0}^{\infty} (-1)^k \binom{x + \alpha + k - 1}{k} \left(\frac{t}{\theta}\right)^k \right\} \\
&= \frac{\gamma \Gamma(x + \alpha)}{(\theta + \gamma) \Gamma(\alpha + 1)} \left(\frac{t}{\theta}\right)^x \left\{ - \left[\frac{t(x + \alpha)}{\theta} \right] \sum_{k=1}^{\infty} \frac{(-1)^{k-1} \Gamma(x + \alpha + k)}{(k - 1)! \Gamma(x + \alpha + 1)} \left(\frac{t}{\theta}\right)^{k-1} + x \left(\frac{\theta}{\theta + t}\right)^{x+\alpha} \right\} \\
&= \frac{\gamma \Gamma(x + \alpha)}{(\theta + \gamma) \Gamma(\alpha + 1)} \left(\frac{t}{\theta}\right)^x \left\{ - \left[\frac{t(x + \alpha)}{\theta} \right] \sum_{k=1}^{\infty} (-1)^{k-1} \binom{x + \alpha + 1 + k - 1}{k - 1} \left(\frac{t}{\theta}\right)^{k-1} + x \left(\frac{\theta}{\theta + t}\right)^{x+\alpha} \right\} \\
&= \frac{\gamma \Gamma(x + \alpha)}{(\theta + \gamma) \Gamma(\alpha + 1)} \left(\frac{t}{\theta}\right)^x \left\{ - \left[\frac{t(x + \alpha)}{\theta} \right] \left(\frac{\theta}{\theta + t}\right)^{x+\alpha+1} + x \left(\frac{\theta}{\theta + t}\right)^{x+\alpha} \right\} \\
&= \frac{\gamma \Gamma(x + \alpha)}{(\theta + \gamma) \Gamma(\alpha + 1)} \frac{t^x \theta^\alpha}{(\theta + t)^{x+\alpha}} \left\{ - \frac{t(x + \alpha)}{\theta + t} + x \right\} \\
&= \frac{\gamma \Gamma(x + \alpha)}{(\theta + \gamma) \Gamma(\alpha + 1)} \frac{t^x \theta^\alpha}{(\theta + t)^{x+\alpha}} \{ \theta x - t \alpha \} \tag{5.85}
\end{aligned}$$

Now, replacing (5.84) and (5.85) in (5.83) we obtain the following expression

$$\begin{aligned}
& \sum_{j=x}^{\infty} (-1)^{j-x} \binom{j}{x} \binom{\alpha + j - 1}{j} \left(1 + \frac{\gamma j}{\alpha(\theta + \gamma)}\right) \left(\frac{t}{\theta}\right)^j \\
&= \frac{\Gamma(x + \alpha)}{x! \Gamma(\alpha)} \frac{t^x \theta^\alpha}{(\theta + t)^{x+\alpha}} + \frac{\gamma \Gamma(x + \alpha)}{(\theta + \gamma) \Gamma(\alpha + 1)} \frac{t^x \theta^\alpha}{(\theta + t)^{x+\alpha}} \{ \theta x - t \alpha \} \\
&= \frac{\Gamma(x + \alpha) t^x \theta^\alpha}{x! \Gamma(\alpha + 1) (\theta + t)^{x+\alpha}} \left\{ \frac{\alpha(\theta + \gamma)(\theta + t) + \gamma(\theta x - t \alpha)}{(\theta + \gamma)(\theta + t)} \right\} \\
&= \frac{\Gamma(x + \alpha) t^x \theta^\alpha}{x! \Gamma(\alpha + 1) (\theta + \gamma) (\theta + t)^{x+\alpha+1}} \{ (\alpha \theta + \alpha \gamma)(\theta + t) + \gamma \theta x - \gamma t \alpha \} \\
&= \frac{\Gamma(x + \alpha) t^x \theta^{\alpha+1} [\alpha(\theta + \gamma) + \alpha t + \gamma x]}{x! (\theta + \gamma) \Gamma(\alpha + 1) (\theta + t)^{x+\alpha+1}} \\
&= \frac{\Gamma(x + \alpha)}{x! \Gamma(\alpha + 1)} \left(\alpha + \frac{\alpha t + \gamma x}{\theta + \gamma} \right) \left(\frac{\theta}{\theta + t}\right)^{\alpha+1} \left(\frac{t}{\theta + t}\right)^x
\end{aligned}$$

5.3.4 Poisson-Transmuted Exponential Distribution

Consider the transmuted exponential distribution given in (2.53), its Laplace transform is

$$\begin{aligned}
L_\Lambda(t) &= \int_0^\infty e^{-\lambda t} \left[(1 - \alpha) \theta e^{-\theta \lambda} + 2\alpha \theta e^{-2\theta \lambda} \right] d\lambda \\
&= (1 - \alpha) \theta \int_0^\infty e^{-(t+\theta)\lambda} d\lambda + 2\alpha \theta \int_0^\infty e^{-(t+2\theta)\lambda} d\lambda \\
&= (1 - \alpha) \theta (t + \theta)^{-1} + 2\alpha \theta (t + 2\theta)^{-1}, \tag{5.86}
\end{aligned}$$

the first three derivatives of (5.86) are

$$L'_\Lambda(t) = (1 - \alpha) \theta (-1) (t + \theta)^{-2} + 2\alpha \theta (-1) (t + 2\theta)^{-2}$$

$$L_{\Lambda}''(t) = (1 - \alpha)\theta(-1)^2 2!(t + \theta)^{-3} + 2\alpha\theta(-1)^2 2!(t + 2\theta)^{-3}$$

$$L_{\Lambda}'''(t) = (1 - \alpha)\theta(-1)^3 3!(t + \theta)^{-4} + 2\alpha\theta(-1)^3 3!(t + 2\theta)^{-4}$$

and in general, the x th derivative is

$$L_{\Lambda}^{(x)}(t) = (1 - \alpha)\theta(-1)^x x!(t + \theta)^{-(x+1)} + 2\alpha\theta(-1)^x x!(t + 2\theta)^{-(x+1)}. \quad (5.87)$$

The Poisson-transmuted exponential distribution is obtained via laplace transform as

$$\begin{aligned} f(x) &= \frac{(-t)^x}{x!} L_{\Lambda}^{(x)}(t) \\ &= \frac{(1 - \alpha)\theta t^x}{(t + \theta)^{x+1}} + \frac{2\alpha\theta t^x}{(t + 2\theta)^{x+1}} \\ &= (1 - \alpha) \left(\frac{\theta}{t + \theta}\right) \left(\frac{t}{t + \theta}\right)^x + \alpha \left(\frac{2\theta}{t + 2\theta}\right) \left(\frac{t}{t + 2\theta}\right)^x, \quad x = 0, 1, 2, \dots \end{aligned} \quad (5.88)$$

The j th moment of transmuted exponential distribution is

$$\begin{aligned} \mathbb{E}(\Lambda^j) &= \int_0^{\infty} \lambda^j [(1 - \alpha)\theta e^{-\theta\lambda} + 2\alpha\theta e^{-2\theta\lambda}] d\lambda \\ &= (1 - \alpha)\theta \int_0^{\infty} \lambda^j e^{-\theta\lambda} d\lambda + 2\alpha\theta \int_0^{\infty} \lambda^j e^{-2\theta\lambda} d\lambda \\ &= (1 - \alpha)\theta \frac{\Gamma(j + 1)}{\theta^{j+1}} + 2\alpha\theta \frac{\Gamma(j + 1)}{(2\theta)^{j+1}} \\ &= \frac{j!}{\theta^j} \left((1 - \alpha) + \frac{\alpha}{2^j} \right) \end{aligned} \quad (5.89)$$

therefore, by method of moments, the Poisson-transmuted exponential distribution is

$$\begin{aligned} f(x) &= \sum_{j=x}^{\infty} \frac{t^x (-t)^{j-x} j!}{(j-x)! x! \theta^j} \left((1 - \alpha) + \frac{\alpha}{2^j} \right) \\ &= \sum_{j=x}^{\infty} (-1)^{j-x} \binom{j}{x} \left((1 - \alpha) + \frac{\alpha}{2^j} \right) \left(\frac{t}{\theta}\right)^j \\ &= (1 - \alpha) \sum_{j=x}^{\infty} (-1)^{j-x} \binom{j}{x} \left(\frac{t}{\theta}\right)^j + \alpha \sum_{j=x}^{\infty} (-1)^{j-x} \binom{j}{x} \left(\frac{t}{2\theta}\right)^j \\ &= (1 - \alpha) \sum_{j=x}^{\infty} (-1)^k \binom{x+k}{k} \left(\frac{t}{\theta}\right)^{x+k} + \alpha \sum_{j=x}^{\infty} (-1)^k \binom{x+k}{k} \left(\frac{t}{2\theta}\right)^{x+k} \\ &= (1 - \alpha) \left(\frac{t}{\theta}\right)^x \sum_{k=0}^{\infty} (-1)^k \binom{x+1+k-1}{k} \left(\frac{t}{\theta}\right)^k \\ &\quad + \alpha \left(\frac{t}{2\theta}\right)^x \sum_{k=0}^{\infty} (-1)^k \binom{x+1+k-1}{k} \left(\frac{t}{2\theta}\right)^k \end{aligned}$$

On further solving, the expression becomes

$$\begin{aligned}
f(x) &= (1 - \alpha) \left(\frac{t}{\theta}\right)^x \sum_{k=0}^{\infty} (-1)^k \binom{-(x+1)}{k} \left(\frac{t}{\theta}\right)^k \\
&\quad + \alpha \left(\frac{t}{2\theta}\right)^x \sum_{k=0}^{\infty} (-1)^k \binom{-(x+1)}{k} \left(\frac{t}{2\theta}\right)^k \\
&= (1 - \alpha) \left(\frac{t}{\theta}\right)^x \left(\frac{\theta}{\theta+t}\right)^{x+1} + \alpha \left(\frac{t}{2\theta}\right)^x \left(\frac{2\theta}{2\theta+t}\right)^{x+1} \\
&= \frac{\theta t^x (1 - \alpha)}{(t + \theta)^{x+1}} + \frac{\theta t^x 2\alpha}{(t + 2\theta)^{x+1}} \\
&= (1 - \alpha) \left(\frac{t}{t + \theta}\right)^x \left(\frac{\theta}{t + \theta}\right) + \alpha \left(\frac{t}{t + 2\theta}\right)^x \left(\frac{2\theta}{t + 2\theta}\right)
\end{aligned}$$

5.3.5 Poisson-Inverse Gamma Distribution

Consider the inverse gamma distribution given by (3.72), its Laplace transform is

$$\begin{aligned}
L_{\Lambda}(t) &= \mathbb{E} \left[e^{-\lambda t} \right] \\
&= \int_0^{\infty} e^{-\lambda t} \frac{\beta^{\alpha}}{\Gamma(\alpha)} e^{-\frac{\beta}{\lambda}} \lambda^{-\alpha-1} d\lambda \\
&= \frac{\beta^{\alpha}}{\Gamma(\alpha)} \int_0^{\infty} \lambda^{-\alpha-1} e^{-\lambda t - \frac{\beta}{\lambda}} d\lambda \\
&= \frac{\beta^{\alpha}}{\Gamma(\alpha)} \int_0^{\infty} \lambda^{-\alpha-1} e^{t(\lambda + \frac{\beta}{t}\frac{1}{\lambda})} d\lambda
\end{aligned} \tag{5.90}$$

and making the substitution $\lambda = \sqrt{\frac{\beta}{t}}z$, implying $d\lambda = \sqrt{\frac{\beta}{t}}dz$, we have

$$\begin{aligned}
L_{\Lambda}(t) &= \frac{\beta^{\alpha}}{\Gamma(\alpha)} \int_0^{\infty} \left(\sqrt{\frac{\beta}{t}}z\right)^{-\alpha-1} e^{-t\sqrt{\frac{\beta}{t}}(z + \frac{1}{z})} \sqrt{\frac{\beta}{t}} dz \\
&= \frac{\beta^{\alpha}}{\Gamma(\alpha)} \left(\sqrt{\frac{\beta}{t}}\right)^{-\alpha} \int_0^{\infty} z^{-\alpha-1} e^{-2\sqrt{\frac{\beta t}{2}}(z + \frac{1}{z})} dz \\
&= \frac{\beta^{\alpha}}{\Gamma(\alpha)} \left(\sqrt{\frac{t}{\beta}}\right)^{\alpha} 2K_{-\alpha}(2\sqrt{\beta t}) \\
&= \frac{\beta^{\alpha}}{\Gamma(\alpha)} \frac{t^{\frac{\alpha}{2}}}{\beta^{\frac{\alpha}{2}}} 2K_{-\alpha}(2\sqrt{\beta t}) \\
&= 2 \frac{(\beta t)^{\frac{\alpha}{2}}}{\Gamma(\alpha)} K_{-\alpha}(2\sqrt{\beta t}).
\end{aligned} \tag{5.91}$$

The pgf is

$$G(s) = \frac{2[\beta(1-s)t]^{\frac{\alpha}{2}}}{\Gamma(\alpha)} K_{-\alpha}(2\sqrt{\beta(1-s)t}). \tag{5.92}$$

To obtain derivatives of $L_{\Lambda}(t)$ take

$$L_{\Lambda}(t) = 2 \frac{(\beta t)^{\frac{\alpha}{2}}}{\Gamma(\alpha)} K_{\alpha}(2\sqrt{\beta t}).$$

The first derivative is

$$\begin{aligned} L'_\Lambda(t) &= \frac{d}{dt} L_\lambda(t) \\ &= \frac{2}{\Gamma(\alpha)} \left\{ \frac{\alpha}{2} (\beta t)^{\frac{\alpha}{2}-1} \beta K_\alpha(2\sqrt{\beta t}) + (\beta t)^{\frac{\alpha}{2}} \frac{d}{dt} K_\alpha(2\sqrt{\beta t}) \right\}, \end{aligned}$$

but

$$\begin{aligned} K'_v(\omega) &= \frac{d}{d\omega} K_v(\omega) \\ &= -\frac{1}{2} [K_{v-1}(\omega) + K_{v+1}(\omega)] \end{aligned}$$

and

$$K_{v+1}(\omega) = \frac{2v}{\omega} K_v(\omega) + K_{v-1}(\omega),$$

therefore

$$\begin{aligned} K'_v(\omega) &= -\frac{1}{2} \left\{ K_{v-1}(\omega) + \frac{2v}{\omega} K_v(\omega) + K_{v-1}(\omega) \right\} \\ &= -\frac{1}{2} \left\{ 2K_{v-1}(\omega) + \frac{2v}{\omega} K_v(\omega) \right\} \end{aligned}$$

and

$$K'_v(\omega) = -K_{v-1}(\omega) - \frac{v}{\omega} K_v(\omega).$$

Now, the first derivative is

$$\begin{aligned} L'_\Lambda &= \frac{2}{\Gamma(\alpha)} \left\{ \frac{\alpha\beta}{2} (\beta t)^{\frac{\alpha}{2}-1} K_\alpha(2\sqrt{\beta t}) \right\} \\ &\quad + \frac{2}{\Gamma(\alpha)} (\beta t)^{\frac{\alpha}{2}} \left[-K_{\alpha-1}(2\sqrt{\beta t}) - \frac{\alpha}{2\sqrt{\beta t}} K_\alpha(2\sqrt{\beta t}) \right] (\beta t)^{-\frac{1}{2}} \beta \\ &= \frac{2}{\Gamma(\alpha)} \left[\frac{\alpha\beta}{2} (\beta t)^{\frac{\alpha}{2}-1} K_\alpha(2\sqrt{\beta t}) - (\beta t)^{\frac{\alpha}{2}} \beta (\beta t)^{-\frac{1}{2}} K_{\alpha-1}(2\sqrt{\beta t}) \right] \\ &\quad - \frac{2}{\Gamma(\alpha)} \left[(\beta t)^{\frac{\alpha}{2}} \beta (\beta t)^{-\frac{1}{2}} \frac{\alpha}{2\sqrt{\beta t}} K_\alpha(2\sqrt{\beta t}) \right] \\ &= \frac{2}{\Gamma(\alpha)} \left\{ \frac{\alpha\beta}{2} (\beta t)^{\frac{\alpha}{2}-1} K_\alpha(2\sqrt{\beta t}) - \beta (\beta t)^{\frac{\alpha-1}{2}} K_{\alpha-1}(2\sqrt{\beta t}) \right\} \\ &\quad - \frac{2}{\Gamma(\alpha)} \cdot \frac{\alpha\beta}{2} (\beta t)^{\frac{\alpha}{2}-1} K_\alpha(2\sqrt{\beta t}) \\ &= \frac{2}{\Gamma(\alpha)} \left[-\beta (\beta t)^{\frac{\alpha-1}{2}} K_{\alpha-1}(2\sqrt{\beta t}) \right] \\ &= -\frac{2\beta}{\Gamma(\alpha)} (\beta t)^{\frac{\alpha-1}{2}} K_{\alpha-1}(2\sqrt{\beta t}), \end{aligned}$$

the second derivative is

$$\begin{aligned}
L''_{\Lambda}(t) &= -\frac{2\beta}{\Gamma(\alpha)} \left(\frac{\alpha-1}{2} \right) (\beta t)^{\frac{\alpha-1}{2}-1} \beta K_{\alpha-1} (2\sqrt{\beta t}) \\
&\quad - \frac{2\beta}{\Gamma(\alpha)} (\beta t)^{\frac{\alpha-1}{2}} \left[-K_{\alpha-2} (2\sqrt{\beta t}) - \frac{\alpha-1}{2\sqrt{\beta t}} K_{\alpha-1} (2\sqrt{\beta t}) \right] 2 \cdot \frac{1}{2} (\beta t)^{-\frac{1}{2}} \beta \\
&= -\frac{2\beta}{\Gamma(\alpha)} \left[\frac{(\alpha-1)\beta}{2} (\beta t)^{\frac{\alpha-1}{2}-1} K_{\alpha-1} (2\sqrt{\beta t}) \right] \\
&\quad - \frac{2\beta}{\Gamma(\alpha)} (\beta t)^{\frac{\alpha-1}{2}} \beta (\beta t)^{-\frac{1}{2}} \left[-K_{\alpha-2} (2\sqrt{\beta t}) - \frac{\alpha-1}{2\sqrt{\beta t}} \cdot K_{\alpha-1} (2\sqrt{\beta t}) \right],
\end{aligned}$$

therefore,

$$\begin{aligned}
L''_{\Lambda}(t) &= -\frac{2\beta}{\Gamma(\alpha)} \left\{ \frac{(\alpha-1)}{2} \beta (\beta t)^{\frac{\alpha-1}{2}-1} K_{\alpha-1} (2\sqrt{\beta t}) + (\beta t)^{\frac{\alpha-1}{2}} \beta (\beta t)^{-\frac{1}{2}} \cdot -K_{\alpha-2} (2\sqrt{\beta t}) \right\} \\
&\quad - \frac{2\beta}{\Gamma(\alpha)} (\beta t)^{\frac{\alpha-1}{2}} \beta (\beta t)^{-\frac{1}{2}} \cdot -\frac{\alpha-1}{2\sqrt{\beta t}} K_{\alpha-1} (2\sqrt{\beta t}) \\
&= -\frac{2\beta}{\Gamma(\alpha)} \left\{ \frac{(\alpha-1)}{2} \beta (\beta t)^{\frac{\alpha-1}{2}-1} K_{\alpha-1} (2\sqrt{\beta t}) - \beta (\beta t)^{\frac{\alpha-2}{2}} K_{\alpha-2} (2\sqrt{\beta t}) \right\} \\
&\quad + \frac{2\beta}{\Gamma(\alpha)} \frac{(\alpha-1)\beta}{2} (\beta t)^{\frac{\alpha-1}{2}-1} K_{\alpha-1} (2\sqrt{\beta t}) \\
&= -\frac{2\beta}{\Gamma(\alpha)} \left\{ -\beta (\beta t)^{\frac{\alpha-2}{2}} K_{\alpha-2} (2\sqrt{\beta t}) \right\} \\
&= (-1)^2 \frac{2\beta^2}{\Gamma(\alpha)} (\beta t)^{\frac{\alpha-2}{2}} K_{\alpha-2} (2\sqrt{\beta t})
\end{aligned}$$

and the third derivative is

$$\begin{aligned}
L'''_{\Lambda}(t) &= (-1)^2 \frac{2\beta^2}{\Gamma(\alpha)} \left\{ \frac{d}{dt} (\beta t)^{\frac{\alpha-2}{2}} K_{\alpha-2} (2\sqrt{\beta t}) \right\} \\
&= \frac{2}{\Gamma(\alpha)} (-1)^2 \beta^2 \left\{ \left(\frac{\alpha-2}{2} \right) (\beta t)^{\frac{\alpha-2}{2}-1} \beta K_{\alpha-2} (2\sqrt{\beta t}) + (\beta t)^{\frac{\alpha-2}{2}} \frac{d}{dt} K_{\alpha-2} (2\sqrt{\beta t}) \right\} \\
&= \frac{2}{\Gamma(\alpha)} (-1)^2 \beta^2 \left\{ \frac{(\alpha-2)\beta}{2} (\beta t)^{\frac{\alpha-2}{2}-1} K_{\alpha-2} (2\sqrt{\beta t}) \right\} \\
&\quad + \frac{2}{\Gamma(\alpha)} (-1)^2 \beta^2 \left\{ (\beta t)^{\frac{\alpha-2}{2}} \left[-K_{\alpha-3} (2\sqrt{\beta t}) - \frac{\alpha-2}{2\sqrt{\beta t}} K_{\alpha-2} (2\sqrt{\beta t}) \right] 2 \cdot \frac{1}{2} \frac{1}{\sqrt{\beta t}} \beta \right\} \\
&= \frac{2}{\Gamma(\alpha)} (-1)^2 \beta^2 \left\{ \frac{\alpha-2}{2} \beta (\beta t)^{\frac{\alpha-2}{2}-1} K_{\alpha-2} (2\sqrt{\beta t}) - \beta (\beta t)^{\frac{\alpha-2}{2}-\frac{1}{2}} K_{\alpha-3} (2\sqrt{\beta t}) \right\} \\
&\quad - \frac{2}{\Gamma(\alpha)} (-1)^2 \beta^2 \left\{ \beta (\beta t)^{\frac{\alpha-2}{2}-1} \frac{\alpha-2}{2} K_{\alpha-2} (2\sqrt{\beta t}) \right\} \\
&= \frac{2}{\Gamma(\alpha)} (-1)^2 \beta^2 \left\{ \frac{\alpha-2}{2} \beta (\beta t)^{\frac{\alpha-2}{2}-1} K_{\alpha-2} (2\sqrt{\beta t}) - \beta (\beta t)^{\frac{\alpha-3}{2}} K_{\alpha-3} (2\sqrt{\beta t}) \right\} \\
&\quad - \frac{2}{\Gamma(\alpha)} (-1)^2 \beta^2 \left\{ \frac{(\alpha-2)}{2} \beta (\beta t)^{\frac{\alpha-2}{2}-1} K_{\alpha-2} (2\sqrt{\beta t}) \right\}
\end{aligned}$$

therefore

$$L'''_{\Lambda}(t) = \frac{2}{\Gamma(\alpha)} (-1)^3 \beta^3 (\beta t)^{\frac{\alpha-3}{2}} K_{\alpha-3} (2\sqrt{\beta t}).$$

By induction

$$L_{\Lambda}^{(x)}(t) = \frac{2}{\Gamma(\alpha)} (-1)^x \beta^x (\beta t)^{\frac{\alpha-x}{2}} K_{\alpha-x} \left(2\sqrt{\beta t} \right) \quad (5.93)$$

and therefore the mixed Poisson distribution is

$$\begin{aligned} f(x) &= \frac{(-t)^x}{x!} L_{\Lambda}^{(x)}(t) \\ &= \frac{(-t)^x}{x!} \cdot \frac{2}{\Gamma(\alpha)} (-1)^x \beta^x (\beta t)^{\frac{\alpha-x}{2}} K_{\alpha-x} \left(2\sqrt{\beta t} \right) \\ &= \frac{2}{\Gamma(\alpha)} \frac{(\beta t)^x}{x!} (\beta t)^{\frac{\alpha-x}{2}} K_{\alpha-x} \left(2\sqrt{\beta t} \right) \\ &= \frac{2}{\Gamma(\alpha)} \frac{(\beta t)^{\frac{x+\alpha}{2}}}{x!} K_{\alpha-x} \left(2\sqrt{\beta t} \right); \quad x = 0, 1, 2, \dots \end{aligned} \quad (5.94)$$

The j th moment of inverse gamma is

$$\begin{aligned} \mathbb{E}(\Lambda^j) &= \int_0^{\infty} \lambda^j \frac{\beta^{\alpha}}{\Gamma(\alpha)} e^{-\frac{\beta}{\lambda}} \lambda^{-\alpha-1} d\lambda \\ &= \frac{\beta^{\alpha}}{\Gamma(\alpha)} \int_0^{\infty} \lambda^{j-\alpha-1} e^{-\frac{\beta}{\lambda}} d\lambda \end{aligned}$$

and making the substitution $z = \frac{\beta}{\lambda}$, implying $\lambda = \frac{\beta}{z}$ and $dz = -\frac{\beta}{z^2} dz$ we have

$$\begin{aligned} \mathbb{E}(\Lambda^j) &= \frac{\beta^{\alpha}}{\Gamma(\alpha)} \int_0^{\infty} \left(\frac{\beta}{z} \right)^{j-\alpha-1} e^{-z} \frac{\beta}{z^2} dz \\ &= \frac{\beta^{\alpha}}{\Gamma(\alpha)} \beta^{j-\alpha-1} \cdot \beta \int_0^{\infty} z^{-j+\alpha+1-2} e^{-z} dz \\ &= \frac{\beta^j}{\Gamma(\alpha)} \int_0^{\infty} z^{-(j-\alpha)-1} e^{-z} dz \\ &= \frac{\beta^j}{\Gamma(\alpha)} \int_0^{\infty} z^{\alpha-j-1} e^{-z} dz \\ &= \frac{\beta^j}{\Gamma(\alpha)} \Gamma(\alpha - j); \quad \alpha > j. \end{aligned} \quad (5.95)$$

Now, the variance of inverse gamma distribution is

$$\begin{aligned} Var(\Lambda) &= \frac{\beta^2}{(\alpha-1)(\alpha-2)} - \frac{\beta^2}{(\alpha-1)^2} \\ &= \frac{\beta^2}{\alpha-1} \left[\frac{1}{\alpha-2} - \frac{1}{\alpha-1} \right] \\ &= \frac{\beta^2}{(\alpha-1)^2(\alpha-2)}, \end{aligned}$$

the mean of mixed Poisson distribution is

$$\mathbb{E}(X) = \frac{t\beta}{\alpha-1}, \quad (5.96)$$

the variance is

$$\begin{aligned}
Var(X) &= t\mathbb{E}(\Lambda) + t^2Var(\Lambda) \\
&= \frac{t\beta}{\alpha-1} + \frac{t^2\beta^2}{(\alpha-1)^2(\alpha-2)} \\
&= \frac{t\beta}{\alpha-1} \left[1 + \frac{t\beta}{(\alpha-1)(\alpha-2)} \right], \tag{5.97}
\end{aligned}$$

and the index of dispersion is

$$\begin{aligned}
I_X &= 1 + \frac{t\beta^2}{(\alpha-1)^2(\alpha-2)} \cdot \frac{\alpha-1}{\beta} \\
&= 1 + \frac{t\beta}{(\alpha-1)(\alpha-2)}. \tag{5.98}
\end{aligned}$$

The Poisson-inverse gamma distribution is therefore obtained by method of moments as

$$\begin{aligned}
f(x) &= \sum_{j=x}^{\infty} \frac{t^x (-t)^{j-x} \beta^j \Gamma(\alpha-j)}{(j-x)! x! \Gamma(\alpha)} \\
&= \frac{1}{x! \Gamma(\alpha)} \sum_{j=x}^{\infty} \frac{(-1)^{j-x} (\beta t)^j}{(j-x)!} \int_0^{\infty} z^{\alpha-j-1} e^{-z} dz \\
&= \frac{1}{x! \Gamma(\alpha)} \int_0^{\infty} \left\{ \sum_{j=x}^{\infty} \frac{(-1)^{j-x}}{(j-x)!} \left(\frac{\beta t}{z} \right)^j \right\} z^{\alpha-1} e^{-z} dz \\
&= \frac{1}{x! \Gamma(\alpha)} \int_0^{\infty} \left\{ \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \left(\frac{\beta t}{z} \right)^{x+k} \right\} z^{\alpha-1} e^{-z} dz \\
&= \frac{1}{x! \Gamma(\alpha)} \int_0^{\infty} \left(\frac{\beta t}{z} \right)^x \left[\sum_{k=0}^{\infty} \left(\frac{-\beta t}{z} \right)^k \frac{1}{k!} \right] z^{\alpha-1} e^{-z} dz \\
&= \frac{1}{x! \Gamma(\alpha)} \int_0^{\infty} \left(\frac{\beta t}{z} \right)^x e^{-\frac{\beta t}{z}} z^{\alpha-1} e^{-z} dz \\
&= \frac{1}{x! \Gamma(\alpha)} (\beta t)^x \int_0^{\infty} z^{\alpha-x-1} e^{-(z+\frac{\beta t}{z})} dz
\end{aligned}$$

and making the substitution $z = \sqrt{\beta t}y$, implying $dz = \sqrt{\beta t}dy$, we have

$$\begin{aligned}
f(x) &= \frac{(\beta t)^x}{x! \Gamma(\alpha)} \int_0^{\infty} (\sqrt{\beta t}y)^{\alpha-x-1} e^{-\sqrt{\beta t}(y+\frac{1}{y})} \sqrt{\beta t} dy \\
&= \frac{(\beta t)^x (\sqrt{\beta t})^{\alpha-x}}{x! \Gamma(\alpha)} \int_0^{\infty} y^{\alpha-x-1} e^{-\sqrt{\beta t}(y+\frac{1}{y})} dy \\
&= \frac{2(\beta t)^{\frac{\alpha+x}{2}}}{x! \Gamma(\alpha)} \left[\frac{1}{2} \int_0^{\infty} y^{\alpha-x-1} e^{-\frac{2\sqrt{\beta t}}{2}(y+\frac{1}{y})} dy \right] \\
&= \frac{2(\beta t)^{\frac{\alpha+x}{2}}}{x! \Gamma(\alpha)} K_{\alpha-x} (2\sqrt{\beta t})
\end{aligned}$$

5.3.6 Special Cases of Poisson-Inverse Gamma Distribution

Inverse Exponential distribution

When $\alpha = 1$, we have

$$g(\lambda) = \beta e^{-\frac{\beta}{\lambda}} \lambda^{-2}; \lambda > 0, \beta > 0 \quad (5.99)$$

and its Laplace transform is

$$L_{\Lambda}(t) = 2(\beta t)^{\frac{1}{2}} K_{-1}\left(2\sqrt{\beta t}\right). \quad (5.100)$$

The pgf is

$$G(s) = 2[\beta(1-s)t]^{-\frac{1}{2}} K_{-1}\left(2\sqrt{\beta(1-s)t}\right). \quad (5.101)$$

The x th derivative of the Laplace transform is

$$L_{\Lambda}^{(x)} = 2(-\beta)^x (\beta t)^{\frac{1-x}{2}} K_{1-x}\left(2\sqrt{\beta t}\right) \quad (5.102)$$

$$\mu_r' = \beta^r \Gamma(1-r) \quad (5.103)$$

$$\mathbb{E}(\Lambda^r) = \frac{\beta}{0} = \infty \quad \mathbb{E}(\Lambda^2) = \infty, \quad \text{Var}(\Lambda) = \infty.$$

$$\mathbb{E}(X) = \infty, \quad \text{Var}(X) = \infty, \quad I_X = \infty.$$

The mixed Poisson distribution is

$$f(x) = 2(\beta t)^{\frac{x+1}{2}} K_{1-x}\left(2\sqrt{\beta t}\right) \quad (5.104)$$

Inverse Chi-Squared distribution

When $\alpha = \frac{n}{2}$, and $\beta = 2$, we have

$$g(\lambda) = \frac{2^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)} e^{-\frac{2}{\lambda}} \lambda^{-\frac{n}{2}-1}; \lambda > 0 \quad (5.105)$$

and its Laplace transform is

$$L_{\Lambda}(t) = \frac{2(2t)^{\frac{n}{4}}}{\Gamma\left(\frac{n}{2}\right)} K_{-\frac{n}{2}}\left(2\sqrt{2t}\right) \quad (5.106)$$

with the x th derivative

$$L_{\Lambda}^{(x)} = \frac{(-1)^x}{\Gamma\left(\frac{n}{2}\right)} 2^{\frac{x}{2} + \frac{n}{4} + 1} t^{\frac{n}{4} - \frac{x}{2}} K_{\frac{n}{2} - x}\left(2\sqrt{2t}\right) \quad (5.107)$$

The mixed Poisson distribution is

$$f(x) = \frac{2}{\Gamma\left(\frac{n}{2}\right)} \frac{(2t)^{\frac{1}{2}(x + \frac{n}{2})}}{x!} K_{\frac{n}{2} - x}\left(2\sqrt{2t}\right), \quad (5.108)$$

the r th raw moment of inverse chi-squared distribution is

$$\mathbb{E}(\Lambda^r) = \frac{2^r}{\Gamma\left(\frac{n}{2}\right)} \Gamma\left(\frac{n}{2} - r\right),$$

$$\begin{aligned}
\mathbb{E}(\Lambda) &= \frac{2}{\Gamma\left(\frac{n}{2}\right)} \Gamma\left(\frac{n}{2} - 1\right) \\
&= \frac{2}{\frac{n}{2} - 1} \\
&= \frac{4}{n - 2}
\end{aligned}$$

and variance

$$\begin{aligned}
\text{Var}(\Lambda) &= \frac{2^2}{\left(\frac{n}{2} - 1\right)^2 \left(\frac{n}{2} - 2\right)} \\
&= \frac{4}{\frac{(n-2)^2 (n-4)}{4}} \\
&= \frac{32}{(n-2)^2 (n-4)}.
\end{aligned}$$

The mean of Poisson mixture is

$$\begin{aligned}
\mathbb{E}(X) &= \frac{2t}{\frac{n}{2} - 1} \\
&= \frac{4t}{n - 2},
\end{aligned}$$

variance is

$$\begin{aligned}
\text{Var}(X) &= \frac{2t}{\frac{n}{2} - 1} \left[1 + \frac{2t}{\left(\frac{n}{2} - 1\right) \left(\frac{n}{2} - 2\right)} \right] \\
&= \frac{4t}{n - 2} \left[1 + \frac{8t}{(n - 2)(n - 4)} \right]
\end{aligned} \tag{5.109}$$

and the index of dispersion is

$$\begin{aligned}
I_X &= 1 + \frac{2t}{\left(\frac{n}{2} - 1\right) \left(\frac{n}{2} - 2\right)} \\
&= 1 + \frac{8t}{(n - 2)(n - 4)}.
\end{aligned} \tag{5.110}$$

The pgf is

$$G(s) = \frac{2 [2(1-s)t]^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)} K_{-\frac{n}{2}} \left(2\sqrt{2(1-s)t} \right).$$

Scaled Inverse Chi-Squared distribution

For $\alpha = \frac{n}{2}$, $\beta = 2\sigma^2$,

$$g(\lambda) = \frac{(2\sigma^2)^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)} e^{-\frac{2\sigma^2}{\lambda}} \lambda^{-\frac{n}{2}-1} \tag{5.111}$$

$$L_\Lambda(t) = \frac{2}{\Gamma\left(\frac{n}{2}\right)} \left(\sqrt{2\sigma^2 t}\right)^{\frac{n}{2}} K_{-\frac{n}{2}} \left(2\sqrt{2\sigma^2 t}\right) \tag{5.112}$$

$$f(x) = \frac{2 \left(\sqrt{2\sigma^2 t}\right)^{\frac{x}{2} + \frac{n}{4}}}{x! \Gamma\left(\frac{n}{2}\right)} K_{\frac{n}{2}-x} \left(2\sqrt{2\sigma^2 t}\right); x = 0, 1, 2, \dots \tag{5.113}$$

$$G(s) = \frac{2}{\Gamma\left(\frac{n}{2}\right)} \left[\sqrt{2\sigma^2(1-s)t} \right]^{\frac{n}{2}} K_{-\frac{n}{2}} \left(2\sqrt{2\sigma^2(1-s)t} \right)$$

$$\mu'_r = \frac{(2\sigma^2)^r}{\Gamma\left(\frac{n}{2}\right)} \Gamma\left(\frac{n}{2} - r\right) \quad (5.114)$$

$$\mathbb{E}(X(X-1)(X-2)\cdots(X-r+1)) = \frac{(2\sigma^2 t)^r}{\Gamma\left(\frac{n}{2}\right)} \Gamma\left(\frac{n}{2} - r\right)$$

$$\mathbb{E}(X) = \frac{2\sigma^2 t}{\frac{n}{2} - 1}$$

$$Var(X) = \frac{2\sigma^2 t}{\frac{n}{2} - 1} \left[\frac{2\sigma^2 t}{\left(\frac{n}{2} - 1\right)\left(\frac{n}{2} - 2\right)} + 1 \right] \quad (5.115)$$

$$I_X = \frac{2\sigma^2 t}{\left(\frac{n}{2} - 1\right)\left(\frac{n}{2} - 2\right)} + 1 \quad (5.116)$$

5.3.7 Poisson-Hougaard Distribution

To determine Poisson-Hougaard distribution through Laplace transform, we use equation (5.6), implying that

$$f(0) = L_\Lambda(t)$$

$$= \exp\left\{-\frac{\delta}{\alpha}[(\theta+t)^\alpha - \theta^\alpha]\right\}, \alpha \in (0, 1).$$

The first derivative of the Laplace transform is

$$L'_\Lambda(t) = (-1) \delta (0+t)^{\alpha-1} f_0(t)$$

second derivative is

$$L''_\Lambda(t) = (-1) \delta \left\{ (\alpha-1)(\theta+t)^{\alpha-2} f(0) + (\theta+t)^{\alpha-1} f'(0) \right\}$$

$$= (-1) \delta \left\{ (\alpha-1)(\theta+t)^{\alpha-2} + (\theta+t)^{\alpha-1} (-1) \delta (\theta+t)^{\alpha-1} \right\} f(0)$$

$$= (-1)^2 \left\{ (1-\alpha) \delta (\theta+t)^{\alpha-2} + \delta^2 (\theta+t)^{2\alpha-2} \right\} f(0)$$

$$= (-1)^2 \left\{ (1-\alpha) \frac{\Gamma(1-\alpha)}{\Gamma(1-\alpha)} \delta (\theta+t)^{\alpha-2} + \delta^2 (\theta+t)^{2\alpha-2} \right\} f(0)$$

$$= (-1)^2 \left\{ \frac{\Gamma(2-\alpha)}{\Gamma(1-\alpha)} \delta (\theta+t)^{\alpha-2} + \delta^2 (\theta+t)^{2\alpha-2} \right\} f(0)$$

$$= (-1)^2 \left\{ c_{2,1}(\alpha) \delta (\theta+t)^{\alpha-2} + c_{2,2}(\alpha) \delta^2 (\theta+t)^{2\alpha-2} \right\} f(0)$$

$$= (-1)^2 \sum_{i=1}^2 c_{2,i}(\alpha) \delta^i (\theta+t)^{i\alpha-2} f(0) \quad (5.117)$$

where $c_{2,1}(\alpha) = \frac{\Gamma(2-\alpha)}{\Gamma(1-\alpha)}$ and $c_{2,2}(\alpha) = 1$.

The third derivative is obtained as follows:

$$\begin{aligned}
L_{\Lambda}'''(t) &= (-1)^2 \sum_{i=1}^2 c_{2,i}(\alpha) \delta^i \left\{ (i\alpha - 2)(\theta + t)^{i\alpha-3} f(0) + (\theta + t)^{i\alpha-2} f'(0) \right\} \\
&= (-1)^2 \sum_{i=1}^2 c_{2,i}(\alpha) \delta^i \left\{ (i\alpha - 2)(\theta + t)^{i\alpha-3} + (\theta + t)^{i\alpha-2} (-1) \delta(\theta + t)^{\alpha-1} \right\} f(0) \\
&= (-1)^3 \sum_{i=1}^2 c_{2,i}(\alpha) \delta^i \left\{ (2 - i\alpha)(\theta + t)^{i\alpha-3} + (\theta + t)^{i\alpha-2} (-1) \delta(\theta + t)^{\alpha-1} \right\} f(0) \\
&= (-1)^3 \sum_{i=1}^2 c_{2,i}(\alpha) \left\{ (2 - i\alpha) \delta^i (\theta + t)^{i\alpha-3} + \delta^{i+1} (\theta + t)^{(i+1)\alpha-3} \right\} f(0)
\end{aligned}$$

On solving further, then

$$\begin{aligned}
L_{\Lambda}'''(t) &= (-1)^3 \sum_{i=1}^2 \left\{ (2 - i\alpha) c_{2,i}(\alpha) \delta^i (\theta + t)^{i\alpha-3} + c_{2,i}(\alpha) \delta^{i+1} (\theta + t)^{(i+1)\alpha-3} \right\} f(0) \\
&= (-1)^3 \left\{ (2 - \alpha) c_{2,1}(\alpha) \delta (\theta + t)^{\alpha-3} + c_{2,1}(\alpha) \delta^2 (\theta + t)^{2\alpha-3} \right\} f(0) + \\
&\quad (-1)^3 \left\{ (2 - 2\alpha) c_{2,2}(\alpha) \delta^2 (\theta + t)^{2\alpha-3} + c_{2,2}(\alpha) \delta^3 (\theta + t)^{3\alpha-3} \right\} f(0) \\
&= (-1)^3 \left\{ (2 - \alpha) \frac{\Gamma(2 - \alpha)}{\Gamma(1 - \alpha)} \delta (\theta + t)^{\alpha-3} + c_{2,1}(\alpha) \delta^2 (\theta + t)^{2\alpha-3} \right\} f(0) + \\
&\quad (-1)^3 \left\{ 2(1 - \alpha) c_{2,2}(\alpha) \delta^2 (\theta + t)^{2\alpha-3} + c_{2,2}(\alpha) \delta^3 (\theta + t)^{3\alpha-3} \right\} f(0) \\
&= (-1)^3 \left\{ \frac{\Gamma(3 - \alpha)}{\Gamma(1 - \alpha)} \delta (\theta + t)^{\alpha-3} + [c_{2,1}(\alpha) + (2 - 2\alpha) c_{2,2}(\alpha)] \delta^2 (\theta + t)^{2\alpha-3} \right\} f(0) + \\
&\quad (-1)^3 \left\{ c_{2,2}(\alpha) \delta^3 (\theta + t)^{3\alpha-3} \right\} f(0) \\
&= (-1)^3 \left\{ c_{3,1}(\alpha) \delta (\theta + t)^{\alpha-3} + c_{3,2}(\alpha) \delta^2 (\theta + t)^{2\alpha-3} + c_{3,3}(\alpha) \delta^3 (\theta + t)^{3\alpha-3} \right\} f(0)
\end{aligned}$$

therefore,

$$L_{\Lambda}'''(t) = (-1)^3 \sum_{i=1}^3 c_{3,i}(\alpha) \delta^i (\theta + t)^{i\alpha-3} f(0)$$

where

$$c_{3,1}(\alpha) = \frac{\Gamma(3 - \alpha)}{\Gamma(1 - \alpha)}$$

$$c_{3,2}(\alpha) = c_{2,1}(\alpha) + c_{2,2}(\alpha) \{2 - 2\alpha\}$$

and

$$c_{3,3}(\alpha) = c_{2,2}(\alpha) = 1.$$

Next,

$$\begin{aligned}
L_{\Lambda}^{(iv)}(t) &= (-1)^3 \sum_{i=1}^3 c_{3,i}(\alpha) \delta^i \left\{ (i\alpha - 3)(\theta + t)^{i\alpha-4} f(0) + (\theta + t)^{i\alpha-3} f'(0) \right\} \\
&= (-1)^3 \sum_{i=1}^3 c_{3,i}(\alpha) \delta^i \left\{ (i\alpha - 3)(\theta + t)^{i\alpha-4} + (-1) \delta(\theta + t)^{(i+1)\alpha-4} \right\} f(0) \\
&= (-1)^4 \sum_{i=1}^3 c_{3,i}(\alpha) \delta^i \left\{ (3 - i\alpha)(\theta + t)^{i\alpha-4} + \delta(\theta + t)^{(i+1)\alpha-4} \right\} f(0) \\
&= (-1)^4 \sum_{i=1}^3 \left\{ (3 - i\alpha) c_{3,i}(\alpha) \delta^i (\theta + t)^{i\alpha-4} + c_{3,i}(\alpha) \delta^{i+1} (\theta + t)^{(i+1)\alpha-4} \right\} f(0)
\end{aligned}$$

that is,

$$\begin{aligned}
L_{\Lambda}^{(iv)}(t) &= (-1)^4 \sum_{i=1}^3 \left\{ (3 - i\alpha) c_{3,i}(\alpha) \delta^i (\theta + t)^{i\alpha-4} + c_{3,i}(\alpha) \delta^{i+1} (\theta + t)^{(i+1)\alpha-4} \right\} f(0) \\
&= (-1)^4 \left\{ (3 - \alpha) c_{3,1}(\alpha) \delta (\theta + t)^{\alpha-4} + c_{3,1}(\alpha) \delta^2 (\theta + t)^{2\alpha-4} \right\} f(0) \\
&\quad + (-1)^4 \left\{ (3 - 2\alpha) c_{3,2}(\alpha) \delta^2 (\theta + t)^{2\alpha-4} + c_{3,2}(\alpha) \delta^3 (\theta + t)^{3\alpha-4} \right\} f(0) \\
&\quad + (-1)^4 \left\{ (3 - 3\alpha) c_{3,3}(\alpha) \delta^3 (\theta + t)^{3\alpha-4} + c_{3,3}(\alpha) \delta^4 (\theta + t)^{4\alpha-4} \right\} f(0) \\
&= (-1)^4 \left\{ c_{4,1}(\alpha) \delta (\theta + t)^{\alpha-4} + [c_{3,1}(\alpha) + (3 - 2\alpha) c_{3,2}(\alpha)] \delta^2 (\theta + t)^{2\alpha-4} \right\} f(0) \\
&\quad + (-1)^4 \left\{ [c_{3,2}(\alpha) + (3 - 3\alpha) c_{3,3}(\alpha)] \delta^3 (\theta + t)^{3\alpha-4} \right\} f(0) \\
&\quad + (-1)^4 \left\{ c_{3,3}(\alpha) \delta^4 (\theta + t)^{4\alpha-4} \right\} f(0).
\end{aligned}$$

Therefore,

$$L_{\Lambda}^{(iv)}(t) = (-1)^4 \sum_{i=1}^4 c_{4,i}(\alpha) \delta^i (\theta + t)^{i\alpha-4} f(0)$$

where $c_{4,1}(\alpha) = \frac{\Gamma(4-\alpha)}{\Gamma(1-\alpha)}$; $c_{4,2}(\alpha) = c_{3,1}(\alpha) + c_{3,2}(\alpha) \{3 - 2\alpha\}$; $c_{4,3}(\alpha) = c_{3,2}(\alpha) + c_{3,3}(\alpha) \{3 - 3\alpha\}$ and $c_{4,4}(\alpha) = c_{3,3}(\alpha) = 1$.

In general the x th derivative is

$$L_{\Lambda}^{(x)}(t) = (-1)^x \sum_{i=1}^x c_{x,i}(\alpha) \delta^i (\theta + t)^{i\alpha-x} f(0) \quad (5.118)$$

where $c_{x,1}(\alpha) = \frac{\Gamma(x-\alpha)}{\Gamma(1-\alpha)}$, $c_{x,i}(\alpha) = c_{x-1,i-1}(\alpha) + c_{x-1,i}(\alpha) \{(x-1) - i\alpha\}$ for $i = 2, 3, \dots, x-1$ and $c_{x,x}(\alpha) = 1$.

The Poisson-Hougaard distribution is therefore

$$f(x) = \frac{t^x}{x!} \sum_{i=1}^x c_{x,i}(\alpha) \delta^i (\theta + t)^{i\alpha-x} f(0) \quad (5.119)$$

where

$$f(0) = \exp \left\{ -\frac{\delta}{\alpha} [(\theta + t)^{\alpha} - \theta^{\alpha}] \right\}, \quad \alpha \leq 1.$$

Remark: Hougaard distribution is a case where the Laplace of a mixing distribution is relatively easier to handle than the pdf. We shall therefore not obtain the pgf, factorial moments, the index of dispersion and Poisson mixture in terms of Mellin transform.

5.4 Mixed Poisson distributions by method of moments

In this section, we shall obtain Poisson mixtures by method of moments only. We shall also derive the Laplace transform of the mixing distribution and hence the pgf of the mixture. We shall not however obtain the x th derivative of the Laplace transform.

5.4.1 Lindley Distribution

Consider Lindley distribution given in (2.69), its Laplace transform is

$$\begin{aligned} L_{\Lambda}(t) &= f(0) \\ &= \frac{\theta^2(\theta + t + 1)}{(\theta + 1)(t + \theta)^2} \end{aligned} \quad (5.120)$$

and therefore the pgf of Poisson-Lindley distribution is

$$G(s) = \frac{\theta^2(\theta + (1-s)t + 1)}{(\theta + 1)((1-s)t + \theta)^2} \quad (5.121)$$

The j th moment of Lindley distribution is

$$\begin{aligned} \mathbb{E}(\Lambda^j) &= \int_0^{\infty} \lambda^j \frac{\theta^2}{\theta + 1} (\lambda + 1) e^{-\theta\lambda} d\lambda \\ &= \frac{\theta^2}{\theta + 1} \int_0^{\infty} (\lambda^{j+2-1} + \lambda^{j+1-1}) e^{-\theta\lambda} d\lambda \\ &= \frac{\theta^2}{\theta + 1} \left[\frac{\Gamma(j+2)}{\theta^{j+2}} + \frac{\Gamma(j+1)}{\theta^{j+1}} \right] \\ &= \frac{\theta^2}{\theta + 1} \frac{\Gamma(j+1)}{\theta^{j+1}} \left(\frac{j+1}{\theta} + 1 \right) \\ &= \frac{j!}{(\theta + 1)\theta^j} (j + 1 + \theta) \end{aligned} \quad (5.122)$$

The Poisson-Lindley distribution is therefore obtained by the method of moments as

$$\begin{aligned}
f(x) &= \sum_{j=x}^{\infty} \frac{t^x (-t)^{j-x}}{(j-x)!x!} \frac{j!}{(\theta+1)\theta^j} (j+1+\theta) \\
&= \frac{1}{\theta+1} \sum_{j=x}^{\infty} \left(\frac{-t}{\theta}\right)^j (j+1+\theta) (-1)^x \binom{j}{x} \\
&= \frac{1}{\theta+1} \sum_{k=0}^{\infty} \frac{(-1)^k}{(\theta)^{k+x}} t^{k+x} \binom{j}{x} (k+x+1+\theta) \\
&= \frac{t^x}{(\theta+1)\theta^x} \sum_{k=0}^{\infty} (-1)^k \binom{x+1+k-1}{k} (k+x+1+\theta) \left(\frac{t}{\theta}\right)^k \\
&= \frac{t^x}{(\theta+1)\theta^x} \sum_{k=0}^{\infty} \binom{-(x+1)}{k} \left(\frac{t}{\theta}\right)^k (k+x+1+\theta) \\
&= \frac{t^x}{(\theta+1)\theta^x} \left\{ (x+1+\theta) \sum_{k=0}^{\infty} \binom{-(x+1)}{k} \left(\frac{t}{\theta}\right)^k \right\} \\
&\quad + \frac{t^x}{(\theta+1)\theta^x} \left\{ \sum_{k=0}^{\infty} k \binom{-(x+1)}{k} \left(\frac{t}{\theta}\right)^k \right\} \\
&= \frac{t^x}{(\theta+1)\theta^x} \left\{ (x+1+\theta) \left(1 + \frac{t}{\theta}\right)^{-(x+1)} \right\} \\
&\quad + \frac{t^x}{(\theta+1)\theta^x} \left\{ \sum_{k=0}^{\infty} k (-1)^k \binom{x+1+k-1}{k} \left(\frac{t}{\theta}\right)^k \right\}
\end{aligned}$$

which becomes

$$\begin{aligned}
f(x) &= \frac{t^x}{(\theta+1)\theta^x} \left\{ (x+1+\theta) \left(\frac{\theta}{\theta+t}\right)^{x+1} \right\} \\
&\quad + \frac{t^x}{(\theta+1)\theta^x} \left\{ \sum_{k=0}^{\infty} k (-1)^k \binom{x+k}{k} \left(\frac{t}{\theta}\right)^k \right\} \\
&= \frac{t^x}{(\theta+1)\theta^x} \left\{ (x+1+\theta) \left(\frac{\theta}{\theta+t}\right)^{x+1} \right\} \\
&\quad + \frac{t^x}{(\theta+1)\theta^x} \left\{ \left(\frac{-t}{\theta}\right) (x+1) \sum_{k=1}^{\infty} \binom{x+k}{k-1} \left(\frac{-t}{\theta}\right)^{k-1} \right\} \\
&= \frac{t^x}{(\theta+1)\theta^x} \left\{ (x+1+\theta) \left(\frac{\theta}{\theta+t}\right)^{x+1} \right\} \\
&\quad - \frac{t^x}{(\theta+1)\theta^x} \left\{ \left(\frac{t(x+1)}{\theta}\right) \sum_{k=1}^{\infty} (-1)^{k-1} \binom{x+2+k-1-1}{k-1} \left(\frac{t}{\theta}\right)^{k-1} \right\}
\end{aligned}$$

therefore;

$$\begin{aligned}
f(x) &= \frac{t^x}{(\theta+1)\theta^x} \left\{ (x+1+\theta) \left(\frac{\theta}{\theta+t} \right)^{x+1} \right\} \\
&\quad - \frac{t^x}{(\theta+1)\theta^x} \left\{ \left(\frac{t(x+1)}{\theta} \right) \sum_{k=1}^{\infty} \binom{-(x+2)}{k-1} \left(\frac{t}{\theta} \right)^{k-1} \right\} \\
&= \frac{t^x}{(\theta+1)\theta^x} \left\{ (x+1+\theta) \left(\frac{\theta}{\theta+t} \right)^{x+1} \right\} \\
&\quad - \frac{t^x}{(\theta+1)\theta^x} \left\{ \left(\frac{t(x+1)}{\theta} \right) \left(\frac{\theta}{\theta+t} \right)^{x+2} \right\}
\end{aligned}$$

On solving further, we have

$$\begin{aligned}
f(x) &= \frac{t^x \theta^{x+1}}{\theta^x (\theta+1) (\theta+t)^{x+1}} \left[\frac{(x+1+\theta)(\theta+t) - t(x+1)}{(\theta+t)} \right] \\
&= \frac{t^x \theta^2}{(\theta+1) (\theta+t)^{x+2}} (x+1+\theta+t)
\end{aligned}$$

5.4.2 Beta I Distribution

Consider Beta I distribution whose pdf is given in (3.11), then its Laplace transform is

$$L_{\Lambda}(t) = {}_1F_1(\alpha, \alpha + \beta; -t) \quad (5.123)$$

and the pgf of Poisson-Beta I distribution is

$$G(s) = {}_1F_1(\alpha, \alpha + \beta; -t(1-s))$$

The j th moment of Beta I distribution is obtained as

$$\begin{aligned}
\mathbb{E}(\Lambda^j) &= \int_0^1 \frac{\lambda^{j+\alpha-1} (1-\lambda)^{\beta-1}}{B(\alpha, \beta)} d\lambda \\
&= \frac{B(j+\alpha, \beta)}{B(\alpha, \beta)}
\end{aligned} \quad (5.124)$$

Therefore, Poisson-Beta I distribution obtained in terms of moments is

$$\begin{aligned}
f(x) &= \frac{t^x}{x!} \sum_{j=x}^{\infty} \frac{(-t)^{j-x}}{(j-x)!} \frac{B(j+\alpha, \beta)}{B(\alpha, \beta)} \\
&= \frac{t^x}{x! B(\alpha, \beta)} \sum_{j=x}^{\infty} \frac{(-t)^{j-x}}{(j-x)!} \int_0^1 y^{j+\alpha-1} (1-y)^{\beta-1} dy \\
&= \frac{t^x}{x! B(\alpha, \beta)} \int_0^1 \left\{ y^{\alpha-1} (1-y)^{\beta-1} \left[\sum_{j=x}^{\infty} \frac{(-t)^{j-x}}{(j-x)!} y^j \right] \right\} dy \\
&= \frac{t^x}{x! B(\alpha, \beta)} \int_0^1 \left\{ y^{\alpha-1} (1-y)^{\beta-1} \left[\sum_{k=0}^{\infty} \frac{(-t)^k}{k!} y^{k+x} \right] \right\} dy \\
&= \frac{t^x}{x! B(\alpha, \beta)} \int_0^1 \left\{ y^{x+\alpha-1} (1-y)^{\beta-1} \sum_{k=0}^{\infty} \frac{(-yt)^k}{k!} \right\} dy \\
&= \frac{t^x}{x! B(\alpha, \beta)} \int_0^1 y^{x+\alpha-1} (1-y)^{(x+\alpha+\beta)-(x+\alpha)-1} e^{-ty} dy
\end{aligned}$$

Further simplification yields

$$\begin{aligned}
f(x) &= \frac{t^x B(x+\alpha, \beta)}{x! B(\alpha, \beta)} \int_0^1 \frac{y^{x+\alpha-1} (1-y)^{(x+\alpha+\beta)-(x+\alpha)-1} e^{-ty}}{B(x+\alpha, \beta)} dy \\
&= \frac{t^x B(x+\alpha, \beta)}{x! B(\alpha, \beta)} {}_1F_1(x+\alpha; x+\alpha+\beta; -t)
\end{aligned}$$

5.4.3 Rectangular Distribution

Consider the Rectangular distribution given by equation (3.14), then its Laplace transform is

$$L_{\Lambda}(t) = \frac{1}{(b-a)t} (e^{-bt} - e^{-at}) \quad (5.125)$$

and the pgf of Poisson-Rectangular distribution is

$$G(s) = \frac{1}{(b-a)(1-s)t} (e^{-bt(1-s)} - e^{-at(1-s)}).$$

The j th moment of Rectangular distribution is

$$\begin{aligned}
\mathbb{E}(\Lambda^j) &= \int_a^b \frac{\lambda^j}{b-a} d\lambda \\
&= \frac{1}{b-a} \left(\frac{b^{j+1} - a^{j+1}}{j+1} \right).
\end{aligned} \quad (5.126)$$

The Poisson-Rectangular distribution is therefore obtained by method of moments as

$$\begin{aligned}
f(x) &= \frac{t^x}{x!} \sum_{j=x}^{\infty} \frac{(-t)^{j-x}}{(j-x)!} \frac{1}{b-a} \left(\frac{b^{j+1} - a^{j+1}}{j+1} \right) \\
&= \frac{t^x}{x!(b-a)} \sum_{j=x}^{\infty} \frac{(-t)^{j-x}}{(j-x)!} \int_a^b z^j dz \\
&= \frac{t^x}{x!(b-a)} \sum_{j=x}^{\infty} \frac{(-t)^{j-x}}{(j-x)!} \left\{ \int_0^b z^j dz - \int_0^a z^j dz \right\} \\
&= \frac{t^x}{x!(b-a)} \int_0^b \sum_{j=x}^{\infty} \frac{(-t)^{j-x}}{(j-x)!} z^{j-x+x} dz - \int_0^a \sum_{j=x}^{\infty} \frac{(-t)^{j-x}}{(j-x)!} z^{j-x+x} dz \\
&= \frac{t^x}{x!(b-a)} \int_0^b z^x \sum_{j=x}^{\infty} \frac{(-tz)^{j-x}}{(j-x)!} dz - \int_0^a z^x \sum_{j=x}^{\infty} \frac{(-tz)^{j-x}}{(j-x)!} dz \\
&= \frac{t^x}{x!(b-a)} \int_0^b z^x e^{-tz} dz - \int_0^a z^x e^{-tz} dz
\end{aligned}$$

and making the substitution $tz = y$, implying $z = \frac{y}{t}$ and $dz = \frac{dy}{t}$, we have

$$\begin{aligned}
f(x) &= \frac{t^x}{x!(b-a)} \int_0^{bt} \left(\frac{y}{t}\right)^x e^{-y} \frac{dy}{t} - \int_0^{at} \left(\frac{y}{t}\right)^x e^{-y} \frac{dy}{t} \\
&= \frac{t^x}{x!(b-a)} \frac{1}{t^{x+1}} \left\{ \int_0^{bt} y^{(x+1)-1} e^{-y} dy - \int_0^{at} y^{(x+1)-1} e^{-y} dy \right\} \\
&= \frac{t^x}{x!(b-a)} \frac{1}{t^{x+1}} \{ \gamma(x+1, bt) - \gamma(x+1, at) \}
\end{aligned}$$

Formula (3.9) gives the relationship between incomplete gamma function and confluent hypergeometric function; that is

$$\gamma(a, x) = \frac{x^a}{a} {}_1F_1(a; a+1; -x)$$

therefore

$$\begin{aligned}
f(x) &= \frac{t^x}{x!(b-a)} \frac{1}{t^{x+1}} \left\{ \frac{(bt)^{x+1}}{x+1} {}_1F_1(x+1; x+2; -bt) - \frac{(at)^{x+1}}{x+1} {}_1F_1(x+1; x+2; -at) \right\} \\
&= \frac{t^x}{(x+1)!(b-a)} (b^{x+1} {}_1F_1(x+1; x+2; -bt) - a^{x+1} {}_1F_1(x+1; x+2; -at))
\end{aligned}$$

5.4.4 Beta II Distribution

Consider Beta II distribution given in equation (3.17), then its Laplace transform is

$$L_{\Lambda}(t) = \frac{\Gamma(\alpha)}{B(\alpha, \beta)} \psi(\alpha, 1 - \beta; t) \quad (5.127)$$

and the pgf of Poisson-Beta II distribution is

$$G(s) = \frac{\Gamma(\alpha)}{B(\alpha, \beta)} \psi(\alpha, 1 - \beta; t(1-s)).$$

The j th moment of Beta II distribution is

$$\begin{aligned}
\mathbb{E}(\Lambda^j) &= \int_0^\infty \frac{\lambda^{j+p-1}}{B(p, q) (1 + \lambda)^{p+q}} d\lambda \\
&= \frac{1}{B(p, q)} \int_0^\infty \frac{\lambda^{j+p-1}}{(1 + \lambda)^{(j+p)+(q-j)}} d\lambda \\
&= \frac{B(p + j, q - j)}{B(p, q)}
\end{aligned} \tag{5.128}$$

Therefore, the Poisson-Beta II distribution by method of moments is

$$\begin{aligned}
f(x) &= \frac{t^x}{x!} \sum_{j=x}^\infty \frac{(-t)^{j-x}}{(j-x)!} \frac{B(p + j, q - j)}{B(p, q)} \\
&= \frac{t^x}{x! B(p, q)} \sum_{j=x}^\infty \frac{(-t)^{j-x}}{(j-x)!} \int_0^\infty \frac{z^{p+j-1}}{(1+z)^{(p+j)+(q-j)}} dz \\
&= \frac{t^x}{x! B(p, q)} \sum_{j=x}^\infty \frac{(-t)^{j-x}}{(j-x)!} \int_0^\infty \frac{z^{p+j-1}}{(1+z)^p + q} dz \\
&= \frac{t^x}{x! B(p, q)} \int_0^\infty \frac{z^{p-1}}{(1+z)^{p+q}} \sum_{j=x}^\infty \frac{(-t)^{j-x}}{(j-x)!} z^j dz \\
&= \frac{t^x}{x! B(p, q)} \int_0^\infty \frac{z^{x+p-1}}{(1+z)^{p+q}} \sum_{j=x}^\infty \frac{(-tz)^{j-x}}{(j-x)!} dz \\
&= \frac{t^x}{x! B(p, q)} \int_0^\infty \frac{z^{x+p-1}}{(1+z)^{p+q}} e^{-tz} dz
\end{aligned}$$

Further simplification yields

$$\begin{aligned}
f(x) &= \frac{t^x}{x! B(p, q)} \int_0^\infty z^{x+p-1} (1+z)^{(x-q)-(x+p)} e^{-tz} dz \\
&= \frac{t^x}{x! B(p, q)} \int_0^\infty z^{x+p-1} (1+z)^{(x-q+1)-(x+p)-1} e^{-tz} dz \\
&= \frac{t^x}{x! B(p, q)} \Gamma(x+p) \int_0^\infty \frac{z^{x+p-1}}{\Gamma(x+p)} (1+z)^{(x-q+1)-(x+p)-1} e^{-tz} dz \\
&= \frac{t^x}{x! B(p, q)} \Gamma(x+p) \psi(x+p, x-q+1; t)
\end{aligned}$$

5.4.5 Scaled Beta Distribution

Consider the scaled Beta distribution given in (3.20), then its Laplace transform is

$$L_\Lambda(t) = {}_1F_1(\alpha, \alpha + \beta; -\mu t) \tag{5.129}$$

and the pgf of Poisson-Scaled Beta distribution is

$$G(s) = {}_1F_1(\alpha, \alpha + \beta; -\mu t(1-s)).$$

The j th moment of Scaled Beta distribution is

$$\begin{aligned}\mathbb{E}(\Lambda^j) &= \int_0^\mu \frac{\lambda^{j+\alpha-1} (\mu - \lambda)^{\beta-1}}{\mu^{\alpha+\beta-1} B(\alpha, \beta)} d\lambda \\ &= \frac{1}{\mu^{\alpha+\beta-1} B(\alpha, \beta)} \int_0^\mu \lambda^{j+\alpha-1} (\mu - \lambda)^{\beta-1} d\lambda\end{aligned}$$

Let $\lambda = \mu z$ implying that $z = \frac{\lambda}{\mu}$ and $\mu dz = d\lambda$, therefore

$$\begin{aligned}\mathbb{E}(\Lambda^j) &= \frac{1}{\mu^{\alpha+\beta-1} B(\alpha, \beta)} \int_0^1 (\mu z)^{j+\alpha-1} (\mu - \mu z)^{\beta-1} \mu dz \\ &= \frac{\mu^{j+\alpha+\beta-1}}{\mu^{\alpha+\beta-1} B(\alpha, \beta)} \int_0^1 z^{j+\alpha-1} (1 - z)^{\beta-1} dz \\ &= \frac{\mu^j}{B(\alpha, \beta)} B(j + \alpha, \beta)\end{aligned}\tag{5.130}$$

Therefore, Poisson-Scaled Beta distribution by method of moments is given as

$$\begin{aligned}f(x) &= \frac{t^x}{x!} \sum_{j=x}^{\infty} \frac{(-t)^{j-x} \mu^j B(j + \alpha, \beta)}{(j - x)! B(\alpha, \beta)} \\ &= \frac{(\mu t)^x}{x!} \sum_{j=x}^{\infty} \frac{(-\mu t)^{j-x} B(j + \alpha, \beta)}{(j - x)! B(\alpha, \beta)} \\ &= \frac{(\mu t)^x}{x! B(\alpha, \beta)} \sum_{j=x}^{\infty} \frac{(-\mu t)^{j-x}}{(j - x)!} \int_0^1 z^{j+\alpha-1} (1 - z)^{\beta-1} dz \\ &= \frac{(\mu t)^x}{x! B(\alpha, \beta)} \int_0^1 \sum_{j=x}^{\infty} \frac{(-\mu t z)^{j-x}}{(j - x)!} z^{x+\alpha-1} (1 - z)^{\beta-1} dz \\ &= \frac{(\mu t)^x}{x! B(\alpha, \beta)} \int_0^1 z^{x+\alpha-1} (1 - z)^{(x+\alpha)+\beta-(x+\alpha)-1} e^{-\mu t z} dz \\ &= \frac{(\mu t)^x B(x + \alpha, \beta)}{x! B(\alpha, \beta)} \int_0^1 \frac{z^{x+\alpha-1} (1 - z)^{(x+\alpha)+\beta-(x+\alpha)-1}}{B(x + \alpha, \beta)} e^{-\mu t z} dz \\ &= \frac{(\mu t)^x B(x + \alpha, \beta)}{x! B(\alpha, \beta)} {}_1F_1(x + \alpha; x + \alpha + \beta; -\mu t)\end{aligned}$$

5.4.6 Full Beta Distribution

Consider the Full Beta model given in equation (3.26), then its Laplace transform is

$$L_\Lambda(t) = \frac{\Gamma(p)}{B(p, q)} \psi\left(p, 1 - q; \frac{t}{b}\right)\tag{5.131}$$

and the pgf of Poisson-Full Beta distribution is

$$G(s) = \frac{\Gamma(p)}{B(p, q)} \psi\left(p, 1 - q; \frac{t}{b}(1 - s)\right).$$

The j th moment of Poisson-Full Beta distribution is

$$\mathbb{E}(\Lambda^j) = \frac{b^p}{B(p, q)} \int_0^\infty \frac{\lambda^{j+p-1}}{B(p, q) (1 + b\lambda)^{p+q}}$$

Let $b\lambda = z$ implying $\lambda = \frac{z}{b}$ and $d\lambda = \frac{dz}{b}$, therefore

$$\begin{aligned}\mathbb{E}(\Lambda^j) &= \frac{b^p}{B(p, q)} \int_0^\infty \frac{z^{j+p-1}}{b^{j+p-1} (1+z)^{p+q}} \frac{dz}{b} \\ &= \frac{1}{b^j B(p, q)} \int_0^\infty \frac{z^{j+p-1}}{(1+z)^{p+q-j}} dz \\ &= \frac{B(p+j, q-j)}{b^j B(p, q)}\end{aligned}\tag{5.132}$$

Therefore, Poisson-Full Beta distribution is obtained by the method of moments as

$$\begin{aligned}f(x) &= \frac{t^x}{x!} \sum_{j=x}^\infty \frac{(-t)^{j-x}}{(j-x)!} \frac{B(p+j, q-j)}{b^j B(p, q)} \\ &= \frac{1}{x!} \left(\frac{t}{b}\right)^x \sum_{j=x}^\infty \left(\frac{-t}{b}\right)^{j-x} \frac{1}{(j-x)!} \frac{B(p+j, q-j)}{B(p, q)} \\ &= \frac{1}{x! B(p, q)} \left(\frac{t}{b}\right)^x \sum_{j=x}^\infty \left(\frac{-t}{b}\right)^{j-x} \frac{1}{(j-x)!} \int_0^\infty \frac{z^{p+j-1}}{(1+z)^{p+q}} dz \\ &= \frac{1}{x! B(p, q)} \left(\frac{t}{b}\right)^x \int_0^\infty \frac{z^{x+p-1}}{(1+z)^{p+q}} \sum_{j=x}^\infty \frac{\left(\frac{-t}{b}z\right)^{j-x}}{(j-x)!} dz \\ &= \frac{1}{x! B(p, q)} \left(\frac{t}{b}\right)^x \int_0^\infty z^{x+p-1} (1+z)^{(x-q)-(x+p)} e^{-\frac{t}{b}z} dz \\ &= \frac{1}{x! B(p, q)} \left(\frac{t}{b}\right)^x \int_0^\infty z^{x+p-1} (1+z)^{(x-q+1)-(x+p)-1} e^{-\frac{t}{b}z} dz \\ &= \frac{\Gamma(x+p)}{x! B(p, q)} \left(\frac{t}{b}\right)^x \int_0^\infty \frac{z^{x+p-1}}{\Gamma(x+p)} (1+z)^{(x-q+1)-(x+p)-1} e^{-\frac{t}{b}z} dz \\ &= \frac{\Gamma(x+p)}{x! B(p, q)} \left(\frac{t}{b}\right)^x \psi\left(x+p, x+1-q; \frac{t}{b}\right)\end{aligned}\tag{5.133}$$

5.4.7 Pearson Type I Distribution

Consider the Pearson Type I mixing distribution given in (3.29), then its Laplace transform is

$$L_\Lambda(t) = e^{-at} B(p, q) {}_1F_1(p, p+q; -(b-a)t)\tag{5.134}$$

and the pgf of Poisson-Pearson Type I distribution is

$$G(s) = e^{-at(1-s)} B(p, q) {}_1F_1(p, p+q; -(b-a)t(1-s)).$$

The j th moment of Pearson Type I distribution is

$$\mathbb{E}(\Lambda^j) = \int_a^b \lambda^j \frac{1}{B(p, q)} \frac{(\lambda-a)^{p-1}}{(b-a)^{p-1}} \frac{(b-\lambda)^{q-1}}{(b-a)^{q-1}} \frac{1}{b-a} d\lambda$$

Let $\frac{\lambda-a}{b-a} = z$, therefore $\lambda = a + (b-a)z$ and $d\lambda = (b-a) dz$

$$\begin{aligned}\mathbb{E}(\Lambda^j) &= \int_0^1 \frac{[a + (b-a)z]^j}{B(p, q)} z^{p-1} (1-z)^{q-1} dz \\ &= \int_0^1 \frac{1}{B(p, q)} \sum_{k=0}^j \binom{j}{k} a^{j-k} (b-a)^k z^{k+p-1} (1-z)^{q-1} dz \\ &= \sum_{k=0}^j \binom{j}{k} a^{j-k} (b-a)^k \frac{B(k+p, q)}{B(p, q)}\end{aligned}$$

The j th moment is therefore

$$\mathbb{E}(\Lambda^j) = \sum_{i=0}^j \binom{j}{i} a^{j-i} (b-a)^i \frac{B(i+p, q)}{B(p, q)} \quad (5.135)$$

The Poisson-Pearson Type I distribution by method of moments is

$$\begin{aligned}f(x) &= \frac{t^x}{x!} \sum_{j=x}^{\infty} \frac{(-t)^{j-x}}{(j-x)!} \sum_{i=0}^j \binom{j}{i} a^{j-i} (b-a)^i \frac{B(i+p, q)}{B(p, q)} \\ &= \frac{t^x}{x! B(p, q)} \sum_{j=x}^{\infty} \left\{ \frac{(-t)^{j-x}}{(j-x)!} \sum_{i=0}^j \binom{j}{i} a^{j-i} (b-a)^i \int_0^1 z^{i+p-1} (1-z)^{q-1} dz \right\} \\ &= \frac{t^x}{x! B(p, q)} \sum_{j=x}^{\infty} \left\{ \frac{(-t)^{j-x}}{(j-x)!} a^j \int_0^1 \left[\sum_{i=0}^j \binom{j}{i} a^{j-i} \left(\frac{b-a}{a} z \right)^i z^i \right] z^{p-1} (1-z)^{q-1} dz \right\} \\ &= \frac{t^x}{x! B(p, q)} \sum_{j=x}^{\infty} \left\{ \frac{(-t)^{j-x}}{(j-x)!} a^j \int_0^1 \left(1 + \frac{b-a}{a} z \right)^j z^{p-1} (1-z)^{q-1} dz \right\} \\ &= \frac{t^x}{x! B(p, q)} \sum_{j=x}^{\infty} \left\{ \frac{(-t)^{j-x}}{(j-x)!} \int_0^1 [a + (b-a)z]^j z^{p-1} (1-z)^{q-1} dz \right\} \\ &= \frac{t^x}{x! B(p, q)} \int_0^1 \left\{ \sum_{j=x}^{\infty} \frac{(-t)^{j-x}}{(j-x)!} [a + (b-a)z]^j \right\} z^{p-1} (1-z)^{q-1} dz \\ &= \frac{t^x}{x! B(p, q)} \int_0^1 \left\{ \sum_{j=x}^{\infty} \frac{[-t(a + (b-a)z)]^{j-x}}{(j-x)!} \right\} [a + (b-a)z]^x z^{p-1} (1-z)^{q-1} dz\end{aligned}$$

Further simplification yields

$$\begin{aligned}
f(x) &= \frac{t^x}{x!B(p, q)} \int_0^1 e^{-t[a+(b-a)z]} [a + (b-a)z]^x z^{p-1} (1-z)^{q-1} dz \\
&= \frac{t^x}{x!B(p, q)} \int_0^1 \left\{ e^{-at} e^{-(b-a)tz} \sum_{k=0}^x \binom{x}{k} a^{x-k} (b-a)^k z^{k+p-1} (1-z)^{q-1} \right\} dz \\
&= \frac{t^x}{x!B(p, q)} e^{-at} \sum_{k=0}^x \binom{x}{k} a^{x-k} (b-a)^k \int_0^1 z^{k+p-1} (1-z)^{q-1} e^{-(b-a)tz} dz \\
&= \frac{t^x}{x!B(p, q)} e^{-at} \sum_{k=0}^x \binom{x}{k} a^{x-k} (b-a)^k B(k+p, q) \int_0^1 \frac{z^{k+p-1} (1-z)^{k+q+p-k-p-1}}{B(k+p, q)} e^{-(b-a)tz} dz \\
&= \frac{t^x}{x!B(p, q)} e^{-at} \sum_{k=0}^x \binom{x}{k} a^{x-k} (b-a)^k B(k+p, q) {}_1F_1(k+p; k+p+q; -(b-a)t) \\
&= \frac{(at)^x e^{-at} \Gamma(p+q)}{x! \Gamma(p)} \\
&\quad \sum_{k=0}^x \binom{x}{k} \left(\frac{b-a}{a}\right)^k \frac{\Gamma(k+q)}{\Gamma(k+p+q)} {}_1F_1(k+p, k+p+q; -(b-a)t)
\end{aligned}$$

5.4.8 Pearson Type VI Distribution

Consider Pearson Type VI distribution given by (3.32), then its Laplace transform is

$$L_{\Lambda}(t) = \frac{e^{-dt}}{B(a, b-a)} \Gamma(b-a) \psi(b-a, 1-a; (d-c)t) \quad (5.136)$$

and the pgf of Poisson-Pearson Type VI distribution is

$$G(s) = \frac{e^{-dt(1-s)}}{B(a, b-a)} \Gamma(b-a) \psi(b-a, 1-a; (d-c)t(1-s)).$$

The j th moment of Pearson Type VI distribution is

$$\mathbb{E}(\Lambda^j) = \int_d^{\infty} \frac{\lambda^j \left(\frac{\lambda-d}{d-c}\right)^{b-a-1} \frac{1}{d-c}}{B(a, b-a) \left(1 + \frac{\lambda-d}{d-c}\right)^b} d\lambda$$

Let $\frac{\lambda-d}{d-c} = z$ implying $\lambda = d + (d-c)z$ and $d\lambda = (d-c)dz$, therefore

$$\begin{aligned}
\mathbb{E}(\Lambda^j) &= \int_0^{\infty} \frac{[d + (d-c)z]^j z^{b-a-1}}{B(a, b-a) (1+z)^b} dz \\
&= \int_0^{\infty} \frac{\sum_{i=0}^j \binom{j}{i} d^{j-i} (d-c)^i z^{i+b-a-1}}{B(a, b-a) (1+z)^b} dz \\
&= \sum_{i=0}^j \left\{ \frac{\binom{j}{i} d^j \left(\frac{d-c}{d}\right)^i}{B(a, b-a)} \int_0^{\infty} \frac{z^{i+b-a-1}}{(1+z)^{(i+b-a)+(a-i)}} dz \right\} \\
&= \sum_{i=0}^j \frac{\binom{j}{i} d^j \left(\frac{d-c}{d}\right)^i}{B(a, b-a)} B(i+b-a, a-i). \quad (5.137)
\end{aligned}$$

Therefore, by method of moments, Poisson-Pearson Type VI is

$$\begin{aligned}
f(x) &= \frac{t^x}{x!} \sum_{j=x}^{\infty} \frac{(-t)^{j-x}}{(j-x)!} \left\{ \sum_{i=0}^j \frac{\binom{j}{i} d^j \left(\frac{d-c}{d}\right)^i}{B(a, b-a)} B(i+b-a, a-i) \right\} \\
&= \frac{t^x}{x! B(a, b-a)} \sum_{j=x}^{\infty} \frac{(-t)^{j-x}}{(j-x)!} \left\{ \sum_{i=0}^j \binom{j}{i} \left(\frac{d-c}{d}\right)^i d^j \int_0^{\infty} \frac{z^{(i+b-a)-1}}{(1+z)^b} dz \right\} \\
&= \frac{t^x}{x! B(a, b-a)} \sum_{j=x}^{\infty} \frac{(-t)^{j-x}}{(j-x)!} \left\{ \int_0^{\infty} \left[\sum_{i=0}^j \binom{j}{i} \left(\frac{d-c}{d} z\right)^i \right] d^j \frac{z^{b-a-1}}{(1+z)^b} dz \right\} \\
&= \frac{t^x}{x! B(a, b-a)} \sum_{j=x}^{\infty} \frac{(-t)^{j-x}}{(j-x)!} \int_0^{\infty} \left(1 + \frac{d-c}{d} z\right)^j d^j \frac{z^{b-a-1}}{(1+z)^b} dz \\
&= \frac{t^x}{x! B(a, b-a)} \int_0^{\infty} \sum_{j=x}^{\infty} \frac{(-t)^{j-x}}{(j-x)!} [d + (d-c)z]^j \frac{z^{b-a-1}}{(1+z)^b} dz \\
&= \frac{t^x}{x! B(a, b-a)} \int_0^{\infty} \sum_{j=x}^{\infty} \frac{\{-t[d + (d-c)z]\}^{j-x}}{(j-x)!} [d + (d-c)z]^x \frac{z^{b-a-1}}{(1+z)^b} dz \\
&= \frac{t^x}{x! B(a, b-a)} \int_0^{\infty} e^{-t[d+(d-c)z]} [d + (d-c)z]^x \frac{z^{b-a-1}}{(1+z)^b} dz
\end{aligned}$$

Further solving yields

$$\begin{aligned}
f(x) &= \frac{t^x}{x! B(a, b-a)} \int_0^{\infty} [d + (d-c)z]^x \frac{z^{b-a-1}}{(1+z)^b} e^{-td} e^{-(d-c)tz} dz \\
&= \frac{t^x e^{-td}}{x! B(a, b-a)} \int_0^{\infty} \sum_{k=0}^x \binom{x}{k} d^{x-k} (d-c)^k \frac{z^{k+b-a-1}}{(1+z)^b} e^{-(d-c)tz} dz \\
&= \frac{t^x e^{-td}}{x! B(a, b-a)} \sum_{k=0}^x \binom{x}{k} d^{x-k} (d-c)^k \int_0^{\infty} z^{k+b-a-1} (1+z)^{1+k-a-(k+b-a)-1} e^{-(d-c)tz} dz \\
&= \frac{t^x e^{-td}}{x! B(a, b-a)} \sum_{k=0}^x \binom{x}{k} d^{x-k} (d-c)^k \Gamma(k+b-a) \psi(k+b-a, k-a+1; (d-c)t) \\
&= \frac{(dt)^x e^{-td}}{x! B(a, b-a)} \sum_{k=0}^x \binom{x}{k} \left(\frac{d-c}{d}\right)^k \Gamma(k+b-a) \psi(k+b-a, k-a+1; (d-c)t)
\end{aligned}$$

5.4.9 Shifted Gamma Distribution

Consider the Shifted-Gamma distribution given in (2.32), then its Laplace transform is

$$L_{\Lambda}(t) = \frac{\beta^{\alpha}}{\Gamma(\alpha)} e^{-\mu t} \Gamma(\alpha) \psi(\alpha, \alpha+1; t+\beta) \quad (5.138)$$

and the pgf of Pisson-Shifted Gamma distribution is

$$G(s) = \frac{\beta^{\alpha}}{\Gamma(\alpha)} e^{-\mu t(1-s)} \Gamma(\alpha) \psi(\alpha; \alpha+1; t(1-s)+\beta).$$

The j th moment of Shifted Gamma distribution is

$$\begin{aligned}
\mathbb{E}(\Lambda^j) &= \int_{\mu}^{\infty} \lambda^j \frac{\beta^\alpha}{\Gamma(\alpha)} e^{-\beta(\lambda-\mu)} (\lambda-\mu)^{\alpha-1} d\lambda \\
&= \frac{\beta^\alpha}{\Gamma(\alpha)} \int_{\mu}^{\infty} (\lambda-\mu+\mu)^j e^{-\beta(\lambda-\mu)} (\lambda-\mu)^{\alpha-1} d\lambda \\
&= \frac{\beta^\alpha}{\Gamma(\alpha)} \int_{\mu}^{\infty} \sum_{i=0}^j \binom{j}{i} \mu^{j-i} (\lambda-\mu)^i e^{-\beta(\lambda-\mu)} (\lambda-\mu)^{\alpha-1} d\lambda \\
&= \frac{\beta^\alpha}{\Gamma(\alpha)} \sum_{i=0}^j \int_{\mu}^{\infty} \binom{j}{i} \mu^{j-i} e^{-\beta(\lambda-\mu)} (\lambda-\mu)^{i+\alpha-1} d\lambda \\
&= \frac{\beta^\alpha}{\Gamma(\alpha)} \sum_{i=0}^j \left\{ \binom{j}{i} \mu^{j-i} \int_{\mu}^{\infty} e^{-\beta(\lambda-\mu)} (\lambda-\mu)^{i+\alpha-1} d\lambda \right\}
\end{aligned}$$

and making the substitution $z = \lambda - \mu \Rightarrow dz = d\lambda$, we have

$$\begin{aligned}
\mathbb{E}(\Lambda^j) &= \frac{\beta^\alpha}{\Gamma(\alpha)} \sum_{i=0}^j \left\{ \binom{j}{i} \mu^{j-i} \int_0^{\infty} z^{i+\alpha-1} e^{-\beta z} dz \right\} \\
&= \frac{\beta^\alpha}{\Gamma(\alpha)} \sum_{i=0}^j \binom{j}{i} \mu^{j-i} \frac{\Gamma(i+\alpha)}{\beta^{i+\alpha}} \\
&= \sum_{i=0}^j \binom{j}{i} \mu^{j-i} \frac{\Gamma(i+\alpha)}{\Gamma(\alpha) \beta^i}.
\end{aligned} \tag{5.139}$$

The Poisson-Shifted Gamma distribution is therefore obtained in terms of moments as

$$\begin{aligned}
f(x) &= \sum_{j=x}^{\infty} \sum_{i=0}^j \frac{t^x (-t)^{j-x}}{(j-x)! x!} \binom{j}{i} \mu^{j-i} \frac{\Gamma(i+\alpha)}{\Gamma(\alpha) \beta^i} \\
&= \frac{t^x}{x! \Gamma(\alpha)} \sum_{j=x}^{\infty} \frac{(-t)^{j-x}}{(j-x)!} \sum_{i=0}^j \binom{j}{i} \frac{\mu^{j-i}}{\beta^i} \int_0^{\infty} z^{i+\alpha-1} e^{-z} dz \\
&= \frac{t^x}{x! \Gamma(\alpha)} \sum_{j=x}^{\infty} \frac{(-t)^{j-x}}{(j-x)!} \int_0^{\infty} \sum_{i=0}^j \binom{j}{i} \left(\frac{z}{\mu\beta}\right)^i \mu^j z^{\alpha-1} e^{-z} dz \\
&= \frac{t^x}{x! \Gamma(\alpha)} \sum_{j=x}^{\infty} \frac{(-t)^{j-x}}{(j-x)!} \int_0^{\infty} \left(1 + \frac{z}{\mu\beta}\right)^j \mu^j z^{\alpha-1} e^{-z} dz \\
&= \frac{t^x}{x! \Gamma(\alpha)} \int_0^{\infty} \sum_{j=x}^{\infty} \frac{(-t)^{j-x}}{(j-x)!} \left(\mu + \frac{z}{\beta}\right)^j z^{\alpha-1} e^{-z} dz \\
&= \frac{t^x}{x! \Gamma(\alpha)} \int_0^{\infty} \sum_{j=x}^{\infty} \frac{\left[-t \left(\frac{\mu\beta+z}{\beta}\right)\right]^{j-x}}{(j-x)!} \left(\frac{\mu\beta+z}{\beta}\right)^x z^{\alpha-1} e^{-z} dz \\
&= \frac{t^x}{x! \Gamma(\alpha)} \int_0^{\infty} e^{-t \left(\frac{\mu\beta+z}{\beta}\right)} \left(\frac{\mu\beta+z}{\beta}\right)^x z^{\alpha-1} e^{-z} dz
\end{aligned}$$

Let $z = \mu\beta y$ then $dz = \mu\beta dy$, therefore

$$\begin{aligned}
f(x) &= \frac{t^x}{x!\Gamma(\alpha)} \int_0^\infty e^{-\mu t(1+y)} \mu^x (1+y)^x (\mu\beta)^{\alpha-1} y^{\alpha-1} e^{-\mu\beta y} \mu\beta dy \\
&= \frac{t^x}{x!\Gamma(\alpha)} e^{-\mu t} \mu^x (\mu\beta)^\alpha \int_0^\infty y^{\alpha-1} (1+y)^x e^{-\mu(t+\beta)y} dy \\
&= \frac{(\mu t)^x e^{-\mu t} (\mu\beta)^\alpha}{x!} \int_0^\infty \frac{y^{\alpha-1} (1+y)^{x+\alpha+1-\alpha-1}}{\Gamma(\alpha)} e^{-\mu(t+\beta)y} dy \\
&= \frac{(\mu t)^x e^{-\mu t} (\mu\beta)^\alpha}{x!} \psi(\alpha, x + \alpha + 1; \mu(t + \beta))
\end{aligned}$$

5.4.10 Truncated Gamma (from above) Distribution

Consider a Truncated Gamma (from above) distribution whose pdf is given in (3.40), then its Laplace transform is

$$L_\Lambda(t) = \frac{{}_1F_1(b; b+1; -pt - ap)}{b^{-1} (ap)^b {}_1F_1(b; b+1; -ap)} \quad (5.140)$$

and the pgf of Poisson-Truncated Gamma (from above) distribution is

$$G(s) = \frac{{}_1F_1(b; b+1; -pt(1-s) - ap)}{b^{-1} (ap)^b {}_1F_1(b; b+1; -ap)}.$$

The j th moment of Truncated Gamma (from above) distribution is

$$\begin{aligned}
\mathbb{E}(\Lambda^j) &= \int_0^p \frac{\lambda^j a^b}{\gamma(b, ap)} e^{-a\lambda} \lambda^{b-1} d\lambda \\
&= \frac{a^b}{\gamma(b, ap)} \int_0^p \lambda^{j+b-1} e^{-a\lambda} d\lambda
\end{aligned}$$

Let $z = a\lambda$, then $\lambda = \frac{z}{a}$ and $d\lambda = \frac{dz}{a}$. Therefore

$$\begin{aligned}
\mathbb{E}(\Lambda^j) &= \frac{a^b}{\gamma(b, ap)} \int_0^{ap} \left(\frac{z}{a}\right)^{j+b-1} e^{-z} \frac{dz}{a} \\
&= \frac{a^b}{\gamma(b, ap)} \frac{1}{a^{j+b}} \int_0^{ap} z^{j+b-1} e^{-z} dz \\
&= \frac{\gamma(j+b, ap)}{a^j \gamma(b, ap)} \quad (5.141)
\end{aligned}$$

The Poisson-Gamma (Truncated from above) distribution is therefore obtained by method of mo-

ments as

$$\begin{aligned}
f(x) &= \frac{t^x}{x!} \sum_{j=x}^{\infty} \frac{(-t)^{j-x}}{(j-x)!} \frac{\gamma(j+b, ap)}{a^j \gamma(b, ap)} \\
&= \frac{t^x}{x!} \sum_{j=x}^{\infty} \frac{(-t)^{j-x}}{(j-x)!} \frac{(ap)^{j+b}}{(j+b)} \frac{{}_1F_1(j+b, j+b+1; -ap)}{\frac{a^j (ap)^b}{{}_1F_1(b, b+1; -ap)}} \\
&= \frac{t^x}{x!} \sum_{j=x}^{\infty} \frac{(-t)^{j-x}}{(j-x)!} \frac{b}{j+b} \frac{p^j}{{}_1F_1(b, b+1; -ap)} \frac{{}_1F_1(j+b, j+b+1; -ap)}{{}_1F_1(b, b+1; -ap)} \\
&= \frac{t^x}{{}_1F_1(b, b+1; -ap)} \sum_{j=x}^{\infty} \frac{(-t)^{j-x}}{(j-x)!} \frac{b}{j+b} p^j \frac{(ap)^{j+b}}{(ap)^{j+b}} {}_1F_1(j+b, j+b+1; -ap) \\
&= \frac{t^x}{{}_1F_1(b, b+1; -ap)} \sum_{j=x}^{\infty} \frac{(-t)^{j-x}}{(j-x)!} b \frac{p^j}{(ap)^{j+b}} \frac{(ap)^{j+b}}{j+b} {}_1F_1(j+b, j+b+1; -ap) \\
&= \frac{t^x}{{}_1F_1(b, b+1; -ap)} \sum_{j=x}^{\infty} \frac{(-t)^{j-x}}{(j-x)!} \frac{b}{a^{j+b} p^b} \gamma(j+b, ap) \\
&= \frac{t^x}{{}_1F_1(b, b+1; -ap)} \sum_{j=x}^{\infty} \frac{(-t)^{j-x}}{(j-x)!} \frac{b}{a^{j+b} p^b} \int_0^{ap} z^{j+b-1} e^{-z} dz \\
&= \frac{t^x}{{}_1F_1(b, b+1; -ap)} \int_0^{ap} \left[\sum_{j=x}^{\infty} \frac{(-t)^{j-x}}{(j-x)!} \frac{z^j}{a^j} \right] \frac{b}{(ap)^b} z^{b-1} e^{-z} dz
\end{aligned}$$

Solving further, we have

$$\begin{aligned}
f(x) &= \frac{t^x}{{}_1F_1(b, b+1; -ap)} \int_0^{ap} \sum_{j=x}^{\infty} \frac{(-\frac{tz}{a})^{j-x}}{(j-x)!} \left(\frac{z}{a}\right)^x \frac{b}{(ap)^b} z^{b-1} e^{-z} dz \\
&= \frac{t^x}{{}_1F_1(b, b+1; -ap)} \int_0^{ap} e^{-\frac{tz}{a}} \frac{z^x b}{a^{x+b} p^b} z^{b-1} e^{-z} dz \\
&= \frac{t^x}{{}_1F_1(b, b+1; -ap)} \frac{b}{a^{x+b} p^b} \int_0^{ap} z^{x+b-1} e^{-\frac{(a+t)}{a} z} dz
\end{aligned}$$

Put $\frac{a+t}{a}z = y$, this implies that $z = \frac{a}{a+t}y$ and $dz = \frac{a}{a+t}dy$. Therefore,

$$\begin{aligned}
f(x) &= \frac{t^x}{{}_1F_1(b, b+1; -ap)} \frac{b}{a^{x+b} p^b} \int_0^{(a+t)p} \left(\frac{a}{a+t}y\right)^{x+b-1} e^{-y} \frac{a}{a+t} dy \\
&= \frac{t^x}{{}_1F_1(b, b+1; -ap)} \frac{b}{a^{x+b} p^b} \left(\frac{a}{a+t}\right)^{x+b} \int_0^{(a+t)p} y^{x+b-1} e^{-y} dy \\
&= \frac{t^x}{{}_1F_1(b, b+1; -ap)} \frac{b}{p^b} \frac{1}{(a+t)^{x+b}} \gamma(x+b, (a+t)p) \\
&= \frac{t^x}{{}_1F_1(b, b+1; -ap)} \frac{b}{p^b} \frac{1}{(a+t)^{x+b}} \frac{[(a+t)p]^{x+b}}{x+b} {}_1F_1(x+b, x+b+1; -(a+t)p) \\
&= \frac{(pt)^x}{x!} \frac{b}{x+b} \frac{{}_1F_1(x+b, x+b+1; -(a+t)p)}{{}_1F_1(b, b+1; -ap)}
\end{aligned}$$

5.4.11 Truncated Gamma (from below) Distribution

Consider a truncated gamma (from below) distribution given in (3.44), then its Laplace transform is

$$L_{\Lambda}(t) = \left(\frac{\beta}{\beta+t} \right)^{\alpha} \frac{\Gamma(\alpha) - \gamma(\alpha, (t+\beta)\lambda_0)}{\Gamma(\alpha) - \gamma(\alpha, \beta\lambda_0)} \quad (5.142)$$

and the pgf of Poisson-truncated gamma (from below) distribution is

$$G(s) = \left(\frac{\beta}{\beta+t(1-s)} \right)^{\alpha} \frac{\Gamma(\alpha) - \gamma(\alpha, (t(1-s)+\beta)\lambda_0)}{\Gamma(\alpha) - \gamma(\alpha, \beta\lambda_0)}.$$

The j th moment of truncated gamma (from below) distribution is obtained as

$$\begin{aligned} \mathbb{E}(\Lambda^j) &= \int_{\lambda_0}^{\infty} \frac{\lambda^j \beta^{\alpha} e^{-\beta\lambda} \lambda^{\alpha-1}}{\Gamma(\alpha) - \gamma(\alpha, \beta\lambda_0)} d\lambda \\ &= \frac{\beta^{\alpha}}{\Gamma(\alpha) - \gamma(\alpha, \beta\lambda_0)} \int_{\lambda_0}^{\infty} \lambda^{j+\alpha-1} e^{-\beta\lambda} d\lambda \\ &= \frac{\beta^{\alpha}}{\Gamma(\alpha) - \gamma(\alpha, \beta\lambda_0)} \left\{ \int_0^{\infty} \lambda^{j+\alpha-1} e^{-\beta\lambda} d\lambda - \int_0^{\lambda_0} \lambda^{j+\alpha-1} e^{-\beta\lambda} d\lambda \right\} \\ &= \frac{\beta^{\alpha}}{\Gamma(\alpha) - \gamma(\alpha, \beta\lambda_0)} \left\{ \frac{\Gamma(j+\alpha)}{\beta^{j+\alpha}} - \frac{\gamma(j+\alpha, \beta\lambda_0)}{\beta^{j+\alpha}} \right\} \\ &= \frac{\Gamma(j+\alpha) - \gamma(j+\alpha, \beta\lambda_0)}{\beta^j [\Gamma(\alpha) - \gamma(\alpha, \beta\lambda_0)]}. \end{aligned} \quad (5.143)$$

The Poisson-Gamma (Truncated from below) distribution is therefore obtained by the method of moments as

$$\begin{aligned} f(x) &= \frac{t^x}{x!} \sum_{j=x}^{\infty} \frac{(-t)^{j-x}}{(j-x)!} \frac{\Gamma(j+\alpha) - \gamma(j+\alpha, \beta\lambda_0)}{\beta^j (\Gamma(\alpha) - \gamma(\alpha, \beta\lambda_0))} \\ &= \frac{t^x}{x! (\Gamma(\alpha) - \gamma(\alpha, \beta\lambda_0))} \sum_{j=x}^{\infty} \frac{(-t)^{j-x}}{(j-x)!} \left\{ \int_0^{\infty} \frac{z^{j+\alpha-1}}{\beta^j} e^{-z} dz - \int_0^{\beta\lambda_0} \frac{z^{j+\alpha-1}}{\beta^j} e^{-z} dz \right\} \\ &= \frac{t^x}{x! (\Gamma(\alpha) - \gamma(\alpha, \beta\lambda_0))} \int_0^{\infty} \sum_{j=x}^{\infty} \frac{(-t)^{j-x}}{(j-x)!} \frac{z^j}{\beta^j} z^{\alpha-1} e^{-z} dz \\ &\quad - \int_0^{\beta\lambda_0} \sum_{j=x}^{\infty} \frac{(-t)^{j-x}}{(j-x)!} \frac{z^j}{\beta^j} z^{\alpha-1} e^{-z} dz \\ &= \frac{t^x}{x! (\Gamma(\alpha) - \gamma(\alpha, \beta\lambda_0))} \int_0^{\infty} \sum_{j=x}^{\infty} \frac{\left(-\frac{tz}{\beta}\right)^{j-x}}{(j-x)!} \frac{z^{x+\alpha-1}}{\beta^x} e^{-z} dz \\ &\quad - \int_0^{\beta\lambda_0} \sum_{j=x}^{\infty} \frac{\left(-\frac{tz}{\beta}\right)^{j-x}}{(j-x)!} \frac{z^{x+\alpha-1}}{\beta^x} e^{-z} dz \\ &= \frac{t^x}{x! (\Gamma(\alpha) - \gamma(\alpha, \beta\lambda_0))} \frac{1}{\beta^x} \int_0^{\infty} e^{-\frac{tz}{\beta}} z^{x+\alpha-1} e^{-z} dz - \frac{1}{\beta^x} \int_0^{\beta\lambda_0} e^{-\frac{tz}{\beta}} z^{x+\alpha-1} e^{-z} dz \\ &= \frac{t^x}{x! (\Gamma(\alpha) - \gamma(\alpha, \beta\lambda_0))} \frac{1}{\beta^x} \left\{ \int_0^{\infty} z^{x+\alpha-1} e^{-\left(\frac{t}{\beta}+1\right)z} dz - \int_0^{\beta\lambda_0} z^{x+\alpha-1} e^{-\left(\frac{t}{\beta}+1\right)z} dz \right\} \end{aligned}$$

$$\begin{aligned}
f(x) &= \frac{t^x}{x! (\Gamma(\alpha) - \gamma(\alpha, \beta\lambda_0))} \frac{1}{\beta^x} \left\{ \frac{\Gamma(x+\alpha)}{\left(\frac{t}{\beta} + 1\right)^{x+\alpha}} - \frac{\gamma(x+\alpha, (t+\beta)\lambda_0)}{\left(\frac{t}{\beta} + 1\right)^{x+\alpha}} \right\} \\
&= \frac{1}{x!} \left(\frac{\beta}{t+\beta}\right)^\alpha \left(\frac{t}{t+\beta}\right)^x \frac{\Gamma(x+\alpha) - \gamma(x+\alpha, (t+\beta)\lambda_0)}{\Gamma(\alpha) - \gamma(\alpha, \beta\lambda_0)}
\end{aligned}$$

5.4.12 Truncated Gamma (from below and above) Distribution

Consider a truncated gamma (from below and above) distribution given in (3.47), then its Laplace transform is

$$L_\Lambda(t) = \left(\frac{\beta}{\beta+t}\right)^\alpha \frac{\gamma(\alpha, (\beta+t)b) - \gamma(\alpha, (\beta+t)a)}{\gamma(\alpha, \beta b) - \gamma(\alpha, \beta a)} \quad (5.144)$$

and the pgf of truncated gamma (from below and above) distribution is

$$G(s) = \left(\frac{\beta}{\beta+t(1-s)}\right)^\alpha \frac{\gamma(\alpha, (\beta+t(1-s))b) - \gamma(\alpha, (\beta+t(1-s))a)}{\gamma(\alpha, \beta b) - \gamma(\alpha, \beta a)}.$$

The j th moment of truncated gamma (from below and above) is obtained as

$$\begin{aligned}
\mathbb{E}(\Lambda^j) &= \int_a^b \frac{\lambda^j \beta^\alpha e^{-\beta\lambda} \lambda^{\alpha-1}}{\gamma(\alpha, \beta b) - \gamma(\alpha, \beta a)} d\lambda \\
&= \frac{\beta^\alpha \left(\int_0^b \lambda^{j+\alpha-1} e^{-\beta\lambda} d\lambda - \int_0^a \lambda^{j+\alpha-1} e^{-\beta\lambda} d\lambda \right)}{\gamma(\alpha, \beta b) - \gamma(\alpha, \beta a)}
\end{aligned}$$

and making the substitution $\beta\lambda = z$ implying $\lambda = \frac{z}{\beta}$ and $d\lambda = \frac{dz}{\beta}$, we have

$$\begin{aligned}
\mathbb{E}(\Lambda^j) &= \frac{\beta^\alpha \left(\int_0^{\beta b} \frac{z^{j+\alpha-1} e^{-z}}{\beta^{j+\alpha}} dz - \int_0^{\beta a} \frac{z^{j+\alpha-1} e^{-z}}{\beta^{j+\alpha}} dz \right)}{\gamma(\alpha, \beta b) - \gamma(\alpha, \beta a)} \\
&= \frac{\gamma(j+\alpha, \beta b) - \gamma(j+\alpha, \beta a)}{\beta^j (\gamma(\alpha, \beta b) - \gamma(\alpha, \beta a))} \quad (5.145)
\end{aligned}$$

The Poisson-Truncated Gamma (from below and above) distribution is therefore obtained by method of moments as

$$\begin{aligned}
f(x) &= \frac{t^x}{x!} \sum_{j=x}^{\infty} \frac{(-t)^{j-x}}{(j-x)!} \left\{ \frac{\gamma(j+\alpha, \beta b) - \gamma(j+\alpha, \beta a)}{\beta^j (\gamma(\alpha, \beta b) - \gamma(\alpha, \beta a))} \right\} \\
&= \frac{t^x}{x! (\gamma(\alpha, \beta b) - \gamma(\alpha, \beta a))} \sum_{j=x}^{\infty} \frac{(-t)^{j-x}}{(j-x)! \beta^j} \left\{ \int_0^{\beta b} z^{j+\alpha-1} e^{-z} dz - \int_0^{\beta a} z^{j+\alpha-1} e^{-z} dz \right\} \\
&= \frac{t^x}{x! (\gamma(\alpha, \beta b) - \gamma(\alpha, \beta a))} \frac{1}{\beta^x} \sum_{j=x}^{\infty} \left(\frac{-t}{\beta}\right)^{j-x} \left\{ \int_0^{\beta b} z^{j+\alpha-1} e^{-z} dz - \int_0^{\beta a} z^{j+\alpha-1} e^{-z} dz \right\} \\
&= \frac{t^x}{x! (\gamma(\alpha, \beta b) - \gamma(\alpha, \beta a))} \frac{1}{\beta^x} \left\{ \int_0^{\beta b} \sum_{j=x}^{\infty} \frac{\left(\frac{-t}{\beta} z\right)^{j-x}}{(j-x)!} z^{x+\alpha-1} e^{-z} dz - \int_0^{\beta a} \sum_{j=x}^{\infty} \frac{\left(\frac{-t}{\beta} z\right)^{j-x}}{(j-x)!} z^{x+\alpha-1} e^{-z} dz \right\} \\
&= \frac{t^x}{x! (\gamma(\alpha, \beta b) - \gamma(\alpha, \beta a))} \frac{1}{\beta^x} \left\{ \int_0^{\beta b} z^{x+\alpha-1} e^{-\left(\frac{t}{\beta} + 1\right)z} dz - \int_0^{\beta a} z^{x+\alpha-1} e^{-\left(\frac{t}{\beta} + 1\right)z} dz \right\} \\
&= \frac{t^x}{x! (\gamma(\alpha, \beta b) - \gamma(\alpha, \beta a))} \frac{1}{\beta^x} \left\{ \int_0^{\beta b} z^{x+\alpha-1} e^{-\left(\frac{t+\beta}{\beta}\right)z} dz - \int_0^{\beta a} z^{x+\alpha-1} e^{-\left(\frac{t+\beta}{\beta}\right)z} dz \right\}
\end{aligned}$$

making the substitution $\left(\frac{t+\beta}{\beta}\right) z = y$ implying $z = \left(\frac{\beta}{t+\beta}\right) y$ and $dz = \left(\frac{\beta}{t+\beta}\right) dy$, we have

$$\begin{aligned}
f(x) &= \frac{t^x}{x! (\gamma(\alpha, \beta b) - \gamma(\alpha, \beta a)) \beta^x} \left\{ \int_0^{(t+\beta)b} \left(\frac{\beta}{t+\beta} y\right)^{x+\alpha-1} e^{-y} \frac{\beta}{t+\beta} dy - \int_0^{(t+\beta)a} \left(\frac{\beta}{t+\beta} y\right)^{x+\alpha-1} e^{-y} \frac{\beta}{t+\beta} dy \right\} \\
&= \frac{t^x}{x! (\gamma(\alpha, \beta b) - \gamma(\alpha, \beta a)) \beta^x} \left\{ \int_0^{(t+\beta)b} y^{x+\alpha-1} e^{-y} dy - \int_0^{(t+\beta)a} y^{x+\alpha-1} e^{-y} dy \right\} \left(\frac{\beta}{t+\beta}\right)^{x+\alpha} \\
&= \frac{t^x \beta^\alpha}{x! (t+\beta)^{x+\alpha}} \frac{\gamma(x+\alpha, (t+\beta)b) - \gamma(x+\alpha, (t+\beta)a)}{(\gamma(\alpha, \beta b) - \gamma(\alpha, \beta a))} \\
&= \frac{1}{x!} \left(\frac{t}{t+\beta}\right)^x \left(\frac{\beta}{t+\beta}\right)^\alpha \frac{\gamma(x+\alpha, (t+\beta)b) - \gamma(x+\alpha, (t+\beta)a)}{\gamma(\alpha, \beta b) - \gamma(\alpha, \beta a)}
\end{aligned}$$

5.4.13 Truncated Pearson Type III Distribution

Consider the Truncated Pearson Type III distribution given by equation (3.53), then its Laplace transform is

$$L_\Lambda(t) = \frac{{}_1F_1(1, \beta; \alpha - t)}{{}_1F_1(1, \beta; \alpha)} \quad (5.146)$$

and the pgf of Poisson-Truncated Pearson Type III is

$$G(s) = \frac{{}_1F_1(1, \beta; \alpha - t + ts)}{{}_1F_1(1, \beta; \alpha)}$$

The j th moment of Truncated Pearson Type III is obtained as

$$\begin{aligned}
\mathbb{E}(\Lambda^j) &= \int_0^1 \frac{\lambda^j (1-\lambda)^{\beta-2} e^{\alpha\lambda}}{B(1, \beta-1) {}_1F_1(1; \beta; \alpha)} d\lambda \\
&= \int_0^1 \frac{\lambda^{(j+1)-1} (1-\lambda)^{(j+\beta)-(j+1)-1} e^{\alpha\lambda}}{B(1, \beta-1) {}_1F_1(1; \beta; \alpha)} d\lambda \\
&= \frac{B(j+1, \beta-1) {}_1F_1(j+1; j+\beta; \alpha)}{B(1, \beta-1) {}_1F_1(1; \beta; \alpha)} \quad (5.147)
\end{aligned}$$

The Poisson-Truncated Pearson Type III distribution is therefore obtained by the method of mo-

ments as

$$\begin{aligned}
f(x) &= \frac{t^x}{x!} \sum_{j=x}^{\infty} \frac{(-t)^{j-x}}{(j-x)!} \frac{B(j+1, \beta-1)}{B(1, \beta-1)} \frac{{}_1F_1(j+1; j+\beta; \alpha)}{{}_1F_1(1; \beta; \alpha)} \\
&= \frac{t^x}{x! B(1, \beta-1) {}_1F_1(1; \beta; \alpha)} \sum_{j=x}^{\infty} \frac{(-t)^{j-x}}{(j-x)!} B(j+1, \beta-1) \int_0^1 \frac{z^{(j+1)-1} (1-z)^{(j+\beta)-(j+1)-1} e^{\alpha z}}{B(j+1, \beta-1)} dz \\
&= \frac{t^x}{x! B(1, \beta-1) {}_1F_1(1; \beta; \alpha)} \sum_{j=x}^{\infty} \frac{(-t)^{j-x}}{(j-x)!} \int_0^1 z^j (1-z)^{\beta-2} e^{\alpha z} dz \\
&= \frac{t^x}{x! B(1, \beta-1) {}_1F_1(1; \beta; \alpha)} \int_0^1 \sum_{j=x}^{\infty} \frac{(-tz)^{j-x}}{(j-x)!} z^x (1-z)^{\beta-2} e^{\alpha z} dz \\
&= \frac{t^x}{x! B(1, \beta-1) {}_1F_1(1; \beta; \alpha)} \int_0^1 z^{(x+1)-1} (1-z)^{(x+\beta-(x+1)-1)-1} e^{(\alpha-t)z} dz \\
&= \frac{t^x B(x+1, \beta-1)}{x! B(1, \beta-1) {}_1F_1(1; \beta; \alpha)} {}_1F_1(x+1; x+\beta; \alpha-t) \\
&= t^x \frac{\Gamma(\beta)}{\Gamma(x+\beta)} \frac{{}_1F_1(x+1, x+\beta; \alpha-t)}{{}_1F_1(1, \beta; \alpha)}
\end{aligned}$$

5.4.14 Pareto I Distribution

Consider the pdf of Pareto I distribution given in (3.56), then its Laplace transform is

$$L_{\Lambda}(t) = \alpha e^{-\beta t} \psi(1, 1-\alpha; \beta t) \quad (5.148)$$

and the pgf of Poisson-Pareto I distribution is

$$G(s) = \alpha e^{-\beta t(1-s)} \psi(1, 1-\alpha; \beta t(1-s)).$$

The j th moment of Pareto I distribution is obtained as

$$\begin{aligned}
\mathbb{E}(\Lambda^j) &= \alpha \beta^{\alpha} \int_{\beta}^{\infty} \lambda^{j-\alpha-1} d\lambda \\
&= \frac{\alpha \beta^j}{\alpha - j} \quad (5.149)
\end{aligned}$$

The Poisson-Pareto I distribution is therefore obtained by method of moments as

$$\begin{aligned}
f(x) &= \frac{t^x}{x!} \sum_{j=x}^{\infty} \frac{(-t)^{j-x}}{(j-x)!} \frac{\alpha \beta^j}{\alpha - j} \\
&= \frac{\alpha t^x}{x!} \sum_{j=x}^{\infty} \frac{(-t)^{j-x}}{(j-x)!} \beta^{j-x} \beta^x \int_0^1 z^{\alpha-j-1} dz \\
&= \frac{\alpha t^x}{x!} \beta^x \int_0^1 \sum_{j=x}^{\infty} \frac{(-\beta t)^{j-x}}{(j-x)!} \left(\frac{1}{z}\right)^j z^{\alpha-1} dz \\
&= \frac{\alpha t^x}{x!} \beta^x \int_0^1 \sum_{j=x}^{\infty} \frac{\left(-\frac{\beta}{z} t\right)^{j-x}}{(j-x)!} z^{\alpha-x-1} dz \\
&= \frac{\alpha t^x}{x!} \beta^x \int_0^1 e^{-\frac{\beta}{z} t} z^{\alpha-x-1} dz
\end{aligned}$$

and making the substitution $z = \frac{1}{y}$ implying $dz = -\frac{dy}{y^2}$, we have

$$\begin{aligned} f(x) &= \frac{\alpha t^x}{x!} \beta^x \int_{\infty}^1 e^{-\beta t y} \frac{1}{y^{\alpha-x-1}} \left(-\frac{dy}{y^2}\right) \\ &= \frac{\alpha t^x}{x!} \beta^x \int_1^{\infty} y^{x-\alpha-1} e^{-\beta t y} dy \end{aligned}$$

Making the substitution $\omega = y - 1$, implying $y = 1 + \omega$, we have

$$\begin{aligned} f(x) &= \frac{\alpha t^x}{x!} \beta^x \int_0^{\infty} (1 + \omega)^{x-\alpha-1} e^{-\beta t(1+\omega)} d\omega \\ &= \frac{\alpha t^x e^{-\beta t}}{x!} \beta^x \int_0^1 \omega^{1-1} (1 + \omega)^{(x-\alpha+1)-1-1} e^{-\beta t \omega} d\omega \\ &= \frac{\alpha (t\beta)^x e^{-\beta t}}{x!} \psi(1, x - \alpha + 1; \beta t) \end{aligned}$$

5.4.15 Pareto II Distribution

Consider Pareto II distribution given by (3.59), its Laplace transform is

$$L_{\Lambda}(t) = \alpha \psi(1, 1 - \alpha; \beta t) \quad (5.150)$$

and the pgf of Poisson-Pareto II distribution is

$$G(s) = \alpha \psi(1, 1 - \alpha; \beta t(1 - s)).$$

The j th moment of Pareto II distribution is

$$\mathbb{E}(\Lambda^j) = \alpha \beta^{\alpha} \int_0^{\infty} \frac{\lambda^j}{(\lambda + \beta)^{\alpha+1}} d\lambda$$

Making the substitution $\lambda = \beta z$ implying $d\lambda = \beta dz$, we have

$$\begin{aligned} \mathbb{E}(\Lambda^j) &= \alpha \beta^{\alpha} \int_0^{\infty} \frac{\beta^j z^j}{\beta^{\alpha+1} (1 + z)^{\alpha+1}} \beta dz \\ &= \alpha \beta^j \int_0^{\infty} \frac{z^{(j+1)-1}}{(1 + z)^{(j+1)+(\alpha-j)}} dz \\ &= \alpha \beta^j B(j + 1, \alpha - j) \end{aligned} \quad (5.151)$$

Poisson-Pareto II distribution is therefore obtained by method of moments as

$$\begin{aligned}
f(x) &= \frac{t^x}{x!} \sum_{j=x}^{\infty} \frac{(-t)^{j-x}}{(j-x)!} \alpha \beta^j B(j+1, \alpha-j) \\
&= \frac{\alpha t^x}{x!} \sum_{j=x}^{\infty} \frac{(-t)^{j-x}}{(j-x)!} \beta^j \int_0^{\infty} \frac{z^{(j+1)-1}}{(1+z)^{(j+1)+(\alpha-j)}} dz \\
&= \frac{\alpha t^x}{x!} \int_0^{\infty} \sum_{j=x}^{\infty} \frac{(-t)^{j-x}}{(j-x)!} (\beta z)^j \frac{1}{(1+z)^{\alpha+1}} dz \\
&= \frac{\alpha t^x}{x!} \int_0^{\infty} \sum_{j=x}^{\infty} \frac{(-\beta t z)^{j-x}}{(j-x)!} \frac{(\beta z)^x}{(1+z)^{\alpha+1}} dz \\
&= \frac{\alpha t^x}{x!} \beta^x \int_0^{\infty} z^{(x+1)-1} (1+z)^{x+1-\alpha-(x+1)-1} e^{-\beta t z} dz \\
&= \alpha (\beta t)^x \psi(x+1, x-\alpha+1; \beta t)
\end{aligned}$$

5.4.16 Generalized Pareto Distribution

Consider the Generalized Pareto pdf given in (3.62), then its Laplace transform is

$$L_{\Lambda}(t) = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)} \psi(\beta, 1 - \alpha; \mu t) \quad (5.152)$$

and the pgf of Poisson-Generalized Pareto distribution is

$$G(s) = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)} \psi(\beta, 1 - \alpha; \mu t(1 - s)).$$

The j th moment of generalized Pareto distribution is

$$\mathbb{E}(\Lambda^j) = \frac{\mu^\alpha}{B(\alpha, \beta)} \int_0^{\infty} \frac{\lambda^{j+\beta-1}}{(\lambda + \mu)^{\alpha+\beta}} d\lambda$$

Let $\lambda = \mu z$ then $d\lambda = \mu dz$

$$\begin{aligned}
\mathbb{E}(\Lambda^j) &= \frac{\mu^\alpha}{B(\alpha, \beta)} \int_0^{\infty} \frac{\mu^{j+\beta} z^{j+\beta-1}}{\mu^{\alpha+\beta} (1+z)^{\alpha+\beta}} dz \\
&= \frac{\mu^j}{B(\alpha, \beta)} \int_0^{\infty} \frac{z^{j+\beta-1}}{(1+z)^{(j+\beta)+(\alpha-j)}} dz \\
&= \mu^j \frac{B(j+\beta, \alpha-j)}{B(\alpha, \beta)}
\end{aligned} \quad (5.153)$$

Poisson-Generalized Pareto distribution is therefore obtained by method of moments as

$$\begin{aligned}
f(x) &= \frac{t^x}{x!} \sum_{j=x}^{\infty} \frac{(-t)^{j-x}}{(j-x)!} \frac{\mu^j B(j+\beta, \alpha-j)}{B(\alpha, \beta)} \\
&= \frac{t^x}{x! B(\alpha, \beta)} \sum_{j=x}^{\infty} \frac{(-t)^{j-x}}{(j-x)!} \mu^j \int_0^{\infty} \frac{z^{j+\beta-1}}{(1+z)^{j+\beta+\alpha-j}} dz \\
&= \frac{t^x}{x! B(\alpha, \beta)} \int_0^{\infty} \sum_{j=x}^{\infty} \frac{(-\mu t z)^{j-x}}{(j-x)!} \frac{(\mu z)^x z^{\beta-1}}{(1+z)^{\alpha+\beta}} dz \\
&= \frac{(\mu t)^x}{x! B(\alpha, \beta)} \int_0^{\infty} z^{x+\beta-1} (1+z)^{x-\alpha+1-x-\beta-1} e^{-\mu t z} dz \\
&= \frac{(\mu t)^x}{x! B(\alpha, \beta)} \Gamma(x+\beta) \psi(x+\beta, x-\alpha+1; \mu t)
\end{aligned}$$

5.4.17 Pearson Type V Distribution

Consider Pearson Type V distribution given by equation (3.75), its Laplace transform is

$$L_{\Lambda}(t) = 2 \frac{\beta^{\alpha}}{\Gamma(\alpha)} \left(\sqrt{\frac{\beta}{t}} \right)^{-\alpha} e^{-tc} K_{-\alpha} \left(2\sqrt{\beta t} \right) \quad (5.154)$$

and pgf of the mixed Poisson distribution is

$$G(s) = 2 \frac{\beta^{\alpha}}{\Gamma(\alpha)} \left(\sqrt{\frac{\beta}{t(1-s)}} \right)^{-\alpha} e^{-t(1-s)c} K_{-\alpha} \left(2\sqrt{\beta t(1-s)} \right).$$

The j th moment of Pearson Type V distribution is

$$\mathbb{E}(\Lambda^j) = \frac{\beta^{\alpha}}{\Gamma(\alpha)} \int_c^{\infty} \lambda^j (\lambda - c)^{-\alpha-1} e^{-\frac{\beta}{\lambda-c}} d\lambda$$

and making the substitution $z = \lambda - c$, implying $\lambda = z + c$ and $d\lambda = dz$ we have

$$\mathbb{E}(\Lambda^j) = \frac{\beta^{\alpha}}{\Gamma(\alpha)} \int_0^{\infty} (z+c)^j \frac{1}{z^{\alpha+1}} e^{-\frac{\beta}{z}} dz$$

Let $y = \frac{1}{z}$ implying $z = \frac{1}{y}$ and $dz = -\frac{dy}{y^2}$, therefore

$$\begin{aligned}
\mathbb{E}(\Lambda^j) &= \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^\infty \left(\frac{1}{y} + c\right)^j y^{\alpha+1} e^{-\beta y} \frac{dy}{y^2} \\
&= \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^\infty (1 + cy)^j y^{\alpha-j-1} e^{-\beta y} dy \\
&= \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^\infty \sum_{i=0}^j \binom{j}{i} (cy)^i y^{\alpha-j-1} e^{-\beta y} dy \\
&= \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^\infty \sum_{i=0}^j \binom{j}{i} c^i y^{\alpha-(j-i)-1} e^{-\beta y} dy \\
&= \frac{\beta^\alpha}{\Gamma(\alpha)} \sum_{i=0}^j \binom{j}{i} c^i \int_0^\infty y^{\alpha-(j-i)-1} e^{-\beta y} dy \\
&= \frac{\beta^\alpha}{\Gamma(\alpha)} \sum_{i=0}^j \binom{j}{i} c^i \frac{\Gamma(\alpha - (j - i))}{\beta^{\alpha-(j-i)}} \\
&= \frac{1}{\Gamma(\alpha)} \sum_{i=0}^j \binom{j}{i} c^i \beta^{j-i} \Gamma(\alpha - (j - i))
\end{aligned} \tag{5.155}$$

Poisson-Pearson Type V is therefore obtained by method of moments as

$$\begin{aligned}
f(x) &= \frac{t^x}{x!} \sum_{j=x}^\infty \frac{(-t)^{j-x}}{(j-x)!} \frac{1}{\Gamma(\alpha)} \left\{ \sum_{i=0}^j \binom{j}{i} c^i \beta^{j-i} \Gamma(\alpha - (j - i)) \right\} \\
&= \frac{t^x}{\Gamma(\alpha) x!} \sum_{j=x}^\infty \frac{(-t)^{j-x}}{(j-x)!} \left\{ \sum_{i=0}^j \binom{j}{i} \left(\frac{c}{\beta}\right)^i \beta^j \int_0^\infty z^{\alpha-j+i-1} e^{-z} dz \right\} \\
&= \frac{t^x}{\Gamma(\alpha) x!} \sum_{j=x}^\infty \frac{(-t)^{j-x}}{(j-x)!} \left\{ \int_0^\infty \left[\sum_{i=0}^j \binom{j}{i} \left(\frac{cz}{\beta}\right)^i \right] \beta^j z^{\alpha-j-1} e^{-z} dz \right\} \\
&= \frac{t^x}{\Gamma(\alpha) x!} \sum_{j=x}^\infty \frac{(-t)^{j-x}}{(j-x)!} \left\{ \int_0^\infty \left(1 + \frac{cz}{\beta}\right)^j \beta^j z^{\alpha-j-1} e^{-z} dz \right\} \\
&= \frac{t^x}{\Gamma(\alpha) x!} \sum_{j=x}^\infty \frac{(-t)^{j-x}}{(j-x)!} \int_0^\infty \left(\frac{\beta + cz}{z}\right)^j z^{\alpha-1} e^{-z} dz \\
&= \frac{t^x}{\Gamma(\alpha) x!} \int_0^\infty \sum_{j=x}^\infty \frac{(-t)^{j-x}}{(j-x)!} \left(\frac{\beta + cz}{z}\right)^j z^{\alpha-1} e^{-z} dz \\
&= \frac{t^x}{\Gamma(\alpha) x!} \int_0^\infty \sum_{j=x}^\infty \frac{[-t(c + \frac{\beta}{z})]^{j-x}}{(j-x)!} \left(c + \frac{\beta}{z}\right)^x z^{\alpha-1} e^{-z} dz
\end{aligned}$$

Solving further,

$$\begin{aligned}
f(x) &= \frac{t^x}{\Gamma(\alpha) x!} \int_0^\infty e^{-t(c+\frac{\beta}{z})} \left(c + \frac{\beta}{z}\right)^x z^{\alpha-1} e^{-z} dz \\
&= \frac{t^x}{\Gamma(\alpha) x!} e^{-ct} \int_0^\infty \sum_{k=0}^x \binom{x}{k} c^{x-k} \left(\frac{\beta}{z}\right)^k z^{\alpha-1} e^{-z-\frac{\beta t}{z}} dz \\
&= \frac{t^x}{\Gamma(\alpha) x!} c^x e^{-ct} \int_0^\infty \sum_{k=0}^x \binom{x}{k} \frac{\beta^k}{c^k} z^{-k+\alpha-1} e^{-z-\frac{\beta t}{z}} dz \\
&= \frac{t^x}{\Gamma(\alpha) x!} c^x e^{-ct} \sum_{k=0}^x \binom{x}{k} \frac{\beta^k}{c^k} \int_0^\infty z^{-(k-\alpha)-1} e^{-z-\frac{\beta t}{z}} dz
\end{aligned}$$

Let $z = \sqrt{\beta t} y$, implying $dz = \sqrt{\beta t} dy$ then

$$\begin{aligned}
f(x) &= \frac{t^x}{\Gamma(\alpha) x!} c^x e^{-ct} \sum_{k=0}^x \binom{x}{k} \frac{\beta^k}{c^k} \int_0^\infty (\sqrt{\beta t} y)^{-(k-\alpha)-1} e^{-\sqrt{\beta t}(y+\frac{1}{y})} \sqrt{\beta t} dy \\
&= \frac{t^x}{\Gamma(\alpha) x!} c^x e^{-ct} \sum_{k=0}^x \binom{x}{k} \frac{\beta^k (\sqrt{\beta t})^{-(k-\alpha)}}{c^k} \int_0^\infty y^{-(k-\alpha)-1} e^{-\frac{2\sqrt{\beta t}}{2}(y+\frac{1}{y})} dy \\
&= \frac{2t^x}{\Gamma(\alpha) x!} c^x e^{-ct} \sum_{k=0}^x \binom{x}{k} \frac{(\sqrt{\beta})^{2k}}{c^k} (\sqrt{\beta})^{-k+\alpha} (\sqrt{t})^{-k+\alpha} K_{-(k-\alpha)}(2\sqrt{\beta t}) \\
&= \frac{2t^x}{\Gamma(\alpha) x!} c^x e^{-ct} \sum_{k=0}^x \binom{x}{k} \frac{(\sqrt{\beta})^{k-\alpha+2\alpha}}{c^k} \frac{1}{(\sqrt{t})^{k-\alpha}} K_{-(k-\alpha)}(2\sqrt{\beta t}) \\
&= \frac{2t^x}{\Gamma(\alpha) x!} c^x (\sqrt{\beta})^{2\alpha} e^{-ct} \sum_{k=0}^x \binom{x}{k} \frac{(\sqrt{\beta})^{k-\alpha}}{c^k (\sqrt{t})^{k-\alpha}} K_{-(k-\alpha)}(2\sqrt{\beta t}) \\
&= \frac{2\beta^\alpha (ct)^x e^{-ct}}{\Gamma(\alpha) x!} \sum_{k=0}^x \binom{x}{k} \frac{1}{c^k} \left(\sqrt{\frac{\beta}{t}}\right)^{k-\alpha} K_{k-\alpha}(2\sqrt{\beta t})
\end{aligned}$$

5.4.18 Inverse Gaussian Distribution

Consider inverse Gaussian distribution given by (3.78), its Laplace transform is

$$\begin{aligned}
L_\Lambda(t) &= \mathbb{E}(e^{-t\Lambda}) \\
&= \int_0^\infty e^{-t\lambda} \left(\frac{\phi}{2\pi\lambda^3}\right)^{\frac{1}{2}} \exp\left\{-\frac{\phi(\lambda-\mu)^2}{2\lambda\mu^2}\right\} d\lambda \\
&= \left(\frac{\phi}{2\pi}\right)^{\frac{1}{2}} \int_0^\infty \lambda^{-\frac{3}{2}} \exp\left\{-t\lambda - \frac{[\phi\lambda^2 - 2\phi\lambda\mu + \phi\mu^2]}{2\lambda\mu^2}\right\} d\lambda \\
&= \left(\frac{\phi}{2\pi}\right)^{\frac{1}{2}} e^{\frac{\phi}{\mu}} \int_0^\infty \lambda^{-\frac{3}{2}} \exp\left\{-t\lambda - \frac{\phi}{2\mu^2}\lambda - \frac{\phi}{2\lambda}\right\} d\lambda \\
&= \left(\frac{\phi}{2\pi}\right)^{\frac{1}{2}} e^{\frac{\phi}{\mu}} \int_0^\infty \lambda^{-\frac{3}{2}} \exp\left\{-\left(\frac{2\mu^2 t + \phi}{2\mu^2}\right)\lambda - \frac{\phi}{2\lambda}\right\} d\lambda \\
&= \left(\frac{\phi}{2\pi}\right)^{\frac{1}{2}} e^{\frac{\phi}{\mu}} \int_0^\infty \lambda^{-\frac{3}{2}} \exp\left\{-\frac{1}{2}\left(\frac{2\mu^2 t + \phi}{\mu^2}\right)\left[\lambda + \frac{\phi\mu^2}{2\mu^2 t + \phi}\frac{1}{\lambda}\right]\right\} d\lambda
\end{aligned}$$

and making the substitution $\lambda = \sqrt{\frac{\phi\mu^2}{2\mu^2t+\phi}}z$, implying that $d\lambda = \sqrt{\frac{\phi\mu^2}{2\mu^2t+\phi}}dz$ we have

$$\begin{aligned}
L_{\Lambda}(t) &= \left(\frac{\phi}{2\pi}\right)^{\frac{1}{2}} e^{\frac{\phi}{\mu}} \left(\sqrt{\frac{\phi\mu^2}{2\mu^2t+\phi}}\right)^{-\frac{1}{2}} \int_0^{\infty} z^{-\frac{1}{2}-1} \exp\left\{-\frac{1}{2}\frac{2\mu^2t+\phi}{\mu^2}\sqrt{\frac{\phi\mu^2}{2\mu^2t+\phi}}\left[z+\frac{1}{z}\right]\right\} dz \\
&= \left[\frac{\phi}{2\pi}\sqrt{\frac{2\mu^2t+\phi}{\phi\mu^2}}\right]^{\frac{1}{2}} e^{\frac{\phi}{\mu}} \int_0^{\infty} z^{-\frac{1}{2}-1} \exp\left\{-\frac{1}{2}\sqrt{\frac{\phi(2\mu^2t+\phi)}{\mu^2}}\left(z+\frac{1}{z}\right)\right\} dz \\
&= \left[\sqrt{\frac{\phi(2\mu^2t+\phi)}{(2\pi\mu)^2}}\right]^{\frac{1}{2}} e^{\frac{\phi}{\mu}} 2K_{-\frac{1}{2}}\left(\sqrt{\frac{\phi(2\mu^2t+\phi)}{\mu^2}}\right) \\
&= \left[\frac{\sqrt{\phi(2\mu^2t+\phi)}}{2\pi\mu}\right]^{\frac{1}{2}} e^{\frac{\phi}{\mu}} 2K_{-\frac{1}{2}}\left(\sqrt{\frac{\phi(2\mu^2t+\phi)}{\mu^2}}\right) \\
&= \left[\frac{2}{\pi\mu}\sqrt{\phi(2\mu^2t+\phi)}\right]^{\frac{1}{2}} e^{\frac{\phi}{\mu}} K_{-\frac{1}{2}}\left(\sqrt{\frac{\phi(2\mu^2t+\phi)}{\mu^2}}\right). \tag{5.156}
\end{aligned}$$

Using Willmot's (1986) notations, let $\phi = \frac{\mu^2}{\beta}$ therefore the Laplace transform for the inverse Gaussian distribution is

$$\begin{aligned}
L_{\Lambda}(t) &= \left[\frac{2}{\pi\mu}\sqrt{\frac{\mu^2}{\beta}\left(2\mu^2t+\frac{\mu^2}{\beta}\right)}\right]^{\frac{1}{2}} e^{\frac{\mu^2}{\beta\mu}} K_{-\frac{1}{2}}\left(\sqrt{\frac{\mu^2}{\beta}\left(\frac{2\mu^2t+\frac{\mu^2}{\beta}}{\mu^2}\right)}\right) \\
&= \left[\frac{2}{\pi\mu}\sqrt{\frac{\mu^2}{\beta^2}(2\mu^2\beta t+\mu^2)}\right]^{\frac{1}{2}} e^{\frac{\mu}{\beta}} K_{-\frac{1}{2}}\left(\sqrt{\frac{\mu^2}{\beta^2}\left(\frac{2\mu^2t+\mu^2}{\mu^2}\right)}\right) \\
&= \left[\frac{2}{\pi\mu}\frac{\mu}{\beta}\sqrt{2\mu^2\beta t+\mu^2}\right]^{\frac{1}{2}} e^{\frac{\mu}{\beta}} K_{-\frac{1}{2}}\left(\frac{\mu}{\beta}\sqrt{2\beta t+1}\right) \\
&= \left[\frac{2\mu}{\pi\beta}\sqrt{2\beta t+1}\right]^{\frac{1}{2}} e^{\frac{\mu}{\beta}} K_{-\frac{1}{2}}\left(\frac{\mu}{\beta}\sqrt{2\beta t+1}\right).
\end{aligned}$$

But by (Jorgensen, 1982), $K_{-\frac{1}{2}}(\omega) = K_{\frac{1}{2}}(\omega) = \sqrt{\frac{\pi}{2\omega}}e^{-\omega}$ therefore,

$$\begin{aligned}
L_{\Lambda}(t) &= \left[\frac{2\mu}{\pi\beta}\sqrt{2\beta t+1}\right]^{\frac{1}{2}} e^{\frac{\mu}{\beta}} \left[\frac{\pi}{\frac{2\mu}{\beta}\sqrt{2\beta t+1}}\right]^{\frac{1}{2}} e^{-\frac{\mu}{\beta}\sqrt{2\beta t+1}} \\
&= \left[\frac{2\mu}{\pi\beta}\sqrt{2\beta t+1}\frac{\pi\beta}{2\mu}\frac{1}{\sqrt{2\beta t+1}}\right]^{\frac{1}{2}} e^{\frac{\mu}{\beta}} e^{-\frac{\mu}{\beta}\sqrt{2\beta t+1}} \\
&= \exp\left\{\frac{\mu}{\beta}\left[1-\sqrt{2\beta t+1}\right]\right\} \\
&= \exp\left\{-\frac{\mu}{\beta}\left[\sqrt{1+2\beta t}-1\right]\right\} \tag{5.157}
\end{aligned}$$

as obtained by Willmot (1986). The pgf of the Poisson mixture is

$$G(s) = \exp\left\{-\frac{\mu}{\beta}\left[\sqrt{1+2\beta(1-s)t}-1\right]\right\} \tag{5.158}$$

The j th moment of inverse Gaussian distribution is

$$\mathbb{E}(\Lambda^j) = \left(\frac{\phi}{2\pi}\right)^{\frac{1}{2}} e^{\sqrt{\varphi\phi}} \int_0^\infty \lambda^{(j-\frac{1}{2})-1} \exp\left\{-\frac{1}{2}\left(\varphi\lambda + \frac{\phi}{\lambda}\right)\right\}$$

and making the substitution $\lambda = \sqrt{\frac{\phi}{\varphi}}z$, implying that $d\lambda = \sqrt{\frac{\phi}{\varphi}}dz$, we have

$$\begin{aligned} \mathbb{E}(\Lambda^j) &= \left(\frac{\phi}{2\pi}\right)^{\frac{1}{2}} e^{\sqrt{\varphi\phi}} \int_0^\infty \left(\sqrt{\frac{\phi}{\varphi}}\right)^{j-\frac{1}{2}} z^{(j-\frac{1}{2})-1} \exp\left\{-\frac{\varphi}{2}\sqrt{\frac{\phi}{\varphi}}\left(z + \frac{1}{z}\right)\right\} dz \\ &= \left(\frac{\phi}{2\pi}\right)^{\frac{1}{2}} e^{\sqrt{\varphi\phi}} \left(\sqrt{\frac{\phi}{\varphi}}\right)^{j-\frac{1}{2}} \int_0^\infty z^{(j-\frac{1}{2})-1} \exp\left\{-\frac{\sqrt{\varphi\phi}}{2}\left(z + \frac{1}{z}\right)\right\} dz \\ &= \left(\frac{2\phi}{\pi}\right)^{\frac{1}{2}} e^{\sqrt{\varphi\phi}} \left(\sqrt{\frac{\phi}{\varphi}}\right)^{j-\frac{1}{2}} K_{j-\frac{1}{2}}(\sqrt{\varphi\phi}) \end{aligned} \quad (5.159)$$

The Poisson-Inverse Gaussian distribution is obtained by the method of moments as

$$\begin{aligned} f(x) &= \sum_{j=x}^\infty \frac{(-t)^{j-x} t^x}{(j-x)!x!} \left(\frac{2\phi}{\pi}\right)^{\frac{1}{2}} e^{\sqrt{\varphi\phi}} \left(\sqrt{\frac{\phi}{\varphi}}\right)^{j-\frac{1}{2}} K_{j-\frac{1}{2}}(\sqrt{\varphi\phi}) \\ &= \left(\frac{2\phi}{\pi}\right)^{\frac{1}{2}} \left(\sqrt{\frac{\phi}{\varphi}}\right)^{-\frac{1}{2}} e^{\sqrt{\varphi\phi}} \sum_{j=x}^\infty \frac{(-1)^{j-x}}{(j-x)!x!} \left(t\sqrt{\frac{\phi}{\varphi}}\right)^j K_{j-\frac{1}{2}}(\sqrt{\varphi\phi}) \\ &= \frac{t^x}{x!} e^{\sqrt{\varphi\phi}} \left(\frac{2\phi}{\pi}\right)^{\frac{1}{2}} \sum_{j=x}^\infty \frac{(-t)^{j-x}}{(j-x)!} \left(\sqrt{\frac{\phi}{\varphi}}\right)^{j-\frac{1}{2}} K_{j-\frac{1}{2}}(\sqrt{\varphi\phi}) \\ &= \frac{t^x}{x!} e^{\sqrt{\varphi\phi}} \left(\frac{2\phi}{\pi}\right)^{\frac{1}{2}} \frac{1}{2} \sum_{j=x}^\infty \frac{(-t)^{j-x}}{(j-x)!} \left(\sqrt{\frac{\phi}{\varphi}}\right)^{j-\frac{1}{2}} \int_0^\infty z^{j-\frac{1}{2}-1} e^{-\frac{\sqrt{\varphi\phi}}{2}\left(z+\frac{1}{z}\right)} dz \\ &= \frac{t^x}{x!} e^{\sqrt{\varphi\phi}} \left(\frac{2\phi}{\pi}\right)^{\frac{1}{2}} \frac{1}{2} \int_0^\infty \sum_{j=x}^\infty \frac{(-t)^{j-x}}{(j-x)!} \left(\sqrt{\frac{\phi}{\varphi}}\right)^j \left(\sqrt{\frac{\phi}{\varphi}}\right)^{-\frac{1}{2}} z^j z^{-\frac{1}{2}-1} e^{-\frac{\sqrt{\varphi\phi}}{2}\left(z+\frac{1}{z}\right)} dz \\ &= \frac{t^x}{x!} e^{\sqrt{\varphi\phi}} \left(\frac{2\phi}{\pi}\right)^{\frac{1}{2}} \frac{1}{2} \int_0^\infty \sum_{j=x}^\infty \frac{[-t\sqrt{\frac{\phi}{\varphi}}z]^{j-x}}{(j-x)!} \left(\sqrt{\frac{\phi}{\varphi}}\right)^{x-\frac{1}{2}} z^{x-\frac{1}{2}-1} e^{-\frac{\sqrt{\varphi\phi}}{2}\left(z+\frac{1}{z}\right)} dz \\ &= \frac{t^x}{x!} e^{\sqrt{\varphi\phi}} \left(\frac{2\phi}{\pi}\right)^{\frac{1}{2}} \frac{1}{2} \left(\sqrt{\frac{\phi}{\varphi}}\right)^{x-\frac{1}{2}} \int_0^\infty z^{(x-\frac{1}{2})-1} e^{-\frac{1}{2}\left\{[2t\frac{\sqrt{\phi}}{\sqrt{\varphi}}+\sqrt{\varphi\phi}]z+\frac{\sqrt{\varphi\phi}}{z}\right\}} dz \end{aligned}$$

On solving further, we have

$$\begin{aligned} f(x) &= \frac{t^x}{x!} e^{\sqrt{\varphi\phi}} \left(\frac{2\phi}{\pi}\right)^{\frac{1}{2}} \frac{1}{2} \left(\sqrt{\frac{\phi}{\varphi}}\right)^{x-\frac{1}{2}} \int_0^\infty z^{(x-\frac{1}{2})-1} \exp\left\{-\frac{1}{2}\left\{\left[\frac{2t\sqrt{\phi}+\sqrt{\varphi^2\phi}}{\sqrt{\varphi}}\right]z+\frac{\sqrt{\varphi\phi}}{z}\right\}\right\} dz \\ &= \frac{t^x}{x!} e^{\sqrt{\varphi\phi}} \left(\frac{2\phi}{\pi}\right)^{\frac{1}{2}} \frac{1}{2} \left(\sqrt{\frac{\phi}{\varphi}}\right)^{x-\frac{1}{2}} \int_0^\infty z^{(x-\frac{1}{2})-1} \exp\left\{-\frac{1}{2}\left[\frac{\sqrt{\phi}(2t+\varphi)}{\sqrt{\varphi}}z+\frac{\sqrt{\varphi\phi}}{z}\right]\right\} dz \\ &= \frac{t^x}{x!} e^{\sqrt{\varphi\phi}} \left(\frac{2\phi}{\pi}\right)^{\frac{1}{2}} \frac{1}{2} \left(\sqrt{\frac{\phi}{\varphi}}\right)^{x-\frac{1}{2}} \int_0^\infty z^{(x-\frac{1}{2})-1} \exp\left\{-\frac{1}{2}\frac{\sqrt{\phi}(2t+\varphi)}{\sqrt{\varphi}}\left[z+\frac{\varphi}{(2t+\varphi)z}\right]\right\} dz \end{aligned}$$

Let $z = \sqrt{\frac{\varphi}{2t+\varphi}}y$ implying $dz = \sqrt{\frac{\varphi}{2t+\varphi}}dy$, therefore

$$\begin{aligned}
f(x) &= \frac{t^x}{x!} e^{\sqrt{\varphi\phi}} \left(\frac{2\phi}{\pi}\right)^{\frac{1}{2}} \frac{1}{2} \left(\sqrt{\frac{\phi}{\varphi}}\right)^{x-\frac{1}{2}} \left(\sqrt{\frac{\varphi}{2t+\varphi}}\right)^{x-\frac{1}{2}} \int_0^\infty y^{(x-\frac{1}{2})-1} \exp\left\{\frac{\sqrt{\phi(2t+\varphi)}}{2}\left(y+\frac{1}{y}\right)\right\} dy \\
&= \frac{t^x}{x!} e^{\sqrt{\varphi\phi}} \left(\frac{2\phi}{\pi}\right)^{\frac{1}{2}} \left(\sqrt{\frac{\phi}{\varphi}}\right)^{x-\frac{1}{2}} \left(\sqrt{\frac{\varphi}{2t+\varphi}}\right)^{x-\frac{1}{2}} K_{(x-\frac{1}{2})}\left(\sqrt{\phi(2t+\varphi)}\right) \\
&= \frac{t^x}{x!} e^{\sqrt{\varphi\phi}} \left(\frac{2\phi}{\pi}\right)^{\frac{1}{2}} \left(\sqrt{\frac{\phi}{2t+\varphi}}\right)^{x-\frac{1}{2}} K_{(x-\frac{1}{2})}\left(\sqrt{\phi(2t+\varphi)}\right)
\end{aligned}$$

5.4.19 Reciprocal Inverse Gaussian Distribution

Consider the pdf of reciprocal inverse Gaussian distribution given by equation (3.84), its Laplace transform is

$$\begin{aligned}
L_\Lambda(t) &= \mathbb{E}(e^{-t\Lambda}) \\
&= \int_0^\infty \left(\frac{\phi}{2\pi}\right)^{\frac{1}{2}} e^{\phi/\mu} \lambda^{-\frac{1}{2}} \exp\left\{-t\lambda - \frac{\phi}{2}\lambda - \frac{\phi}{2\mu^2\lambda}\right\} d\lambda \\
&= \left(\frac{\phi}{2\pi}\right)^{\frac{1}{2}} e^{\phi/\mu} \int_0^\infty \lambda^{-\frac{1}{2}} \exp\left\{-\frac{2t+\phi}{2}\left[\lambda + \frac{\phi}{(2t+\phi)\mu^2}\frac{1}{\lambda}\right]\right\} d\lambda
\end{aligned}$$

and making the substitution $\lambda = \sqrt{\frac{\phi}{\mu^2(2t+\phi)}}z$ implying that $d\lambda = \sqrt{\frac{\phi}{\mu^2(2t+\phi)}}dz$, we have

$$\begin{aligned}
L_\Lambda(t) &= 2 \left(\frac{\phi}{2\pi}\right)^{\frac{1}{2}} e^{\frac{\phi}{\mu}} \left(\sqrt{\frac{\phi}{\mu^2(2t+\phi)}}\right)^{\frac{1}{2}} \frac{1}{2} \int_0^\infty z^{\frac{1}{2}-1} \exp\left\{-\sqrt{\frac{\phi}{\mu^2(2t+\phi)}}\left(z+\frac{1}{z}\right)\right\} dz \\
&= \left[\left(\frac{2\phi}{\pi}\right) \sqrt{\frac{\phi}{\mu^2(2t+\phi)}}\right]^{\frac{1}{2}} e^{\frac{\phi}{\mu}} K_{\frac{1}{2}}\left(\sqrt{\frac{\phi}{\mu^2(2t+\phi)}}\right)
\end{aligned}$$

But $K_{\frac{1}{2}}(\omega) = \sqrt{\frac{\pi}{2\omega}}e^{-\omega}$, then

$$\begin{aligned}
L_\Lambda(t) &= \left[\left(\frac{2\phi}{\pi}\right) \sqrt{\frac{\phi}{\mu^2(2t+\phi)}}\right]^{\frac{1}{2}} e^{\frac{\phi}{\mu}} \left[\frac{\pi}{2\sqrt{\frac{\phi}{\mu^2(2t+\phi)}}}\right]^{\frac{1}{2}} e^{-\sqrt{\frac{\phi}{\mu^2(2t+\phi)}}} \\
&= \left[\left(\frac{2\phi}{\pi}\right) \sqrt{\frac{\phi}{\mu^2(2t+\phi)}} \frac{\pi}{2\sqrt{\frac{\phi}{\mu^2(2t+\phi)}}}\right]^{\frac{1}{2}} e^{\frac{\phi}{\mu}} e^{-\sqrt{\frac{\phi}{\mu^2(2t+\phi)}}} \\
&= \left[\phi \sqrt{\frac{\phi}{\mu^2(2t+\phi)}} \frac{\phi}{\mu^2(2t+\phi)}\right]^{\frac{1}{2}} e^{\frac{\phi}{\mu}} e^{-\sqrt{\frac{\phi}{\mu^2(2t+\phi)}}} \\
&= \left[\frac{\phi}{2t+\phi}\right]^{\frac{1}{2}} e^{\frac{\phi}{\mu}} e^{-\sqrt{\frac{\phi}{\mu^2(2t+\phi)}}}
\end{aligned}$$

Therefore

$$\begin{aligned}
L_{\Lambda}(t) &= \left(\frac{\phi}{\phi+2t}\right)^{\frac{1}{2}} \exp\left\{-\left[\sqrt{\frac{\phi}{\mu^2}(2t+\phi)} - \frac{\phi}{\mu}\right]\right\} \\
&= \left(\frac{\phi}{\phi+2t}\right)^{\frac{1}{2}} \exp\left\{-\frac{\phi}{\mu}\left[\sqrt{\frac{2t+\phi}{\phi}} - 1\right]\right\}
\end{aligned} \tag{5.160}$$

Parameterization 1: Let $\mu = \sqrt{\frac{\phi}{\varphi}}$ implying $\mu^2 = \frac{\phi}{\varphi}$, then

$$L_{\Lambda}(t) = \left(\frac{\phi}{\phi+2t}\right)^{\frac{1}{2}} \exp\left\{-\sqrt{\phi\varphi}\left[\sqrt{\frac{2t+\phi}{\phi}} - 1\right]\right\} \tag{5.161}$$

Parameterization 2: (Willmot's, 1986). Let $\phi = \frac{\mu^2}{\beta}$, then

$$\begin{aligned}
L_{\Lambda}(t) &= \left[\frac{\mu^2}{\beta\left(\frac{\mu^2}{\beta} + 2t\right)}\right]^{\frac{1}{2}} \exp\left\{-\frac{\mu}{\beta}\left[\sqrt{\frac{\beta\left(2t + \frac{\mu^2}{\beta}\right)}{\mu^2}} - 1\right]\right\} \\
&= \left(\frac{\mu^2}{\mu^2 + 2\beta t}\right)^{\frac{1}{2}} \exp\left\{-\frac{\mu}{\beta}\left[\sqrt{\frac{2\beta t}{\mu^2} + 1} - 1\right]\right\}
\end{aligned} \tag{5.162}$$

The pgf of the mixture is

$$G(s) = \left(\frac{\phi}{\phi + 2(1-s)t}\right)^{\frac{1}{2}} \exp\left\{-\sqrt{\phi\varphi}\left[\sqrt{\frac{2(1-s)t + \phi}{\phi}} - 1\right]\right\} \tag{5.163}$$

Remark: The Laplace transform of a reciprocal inverse Gaussian distribution is a product of the Laplace transform of a Gamma distribution and the Laplace transform of an inverse Gaussian distribution.

The j th moment of reciprocal inverse Gaussian distribution is obtained as

$$\begin{aligned}
\mathbb{E}(\Lambda^j) &= \int_0^{\infty} \lambda^j \left(\frac{\phi}{2\pi}\right)^{\frac{1}{2}} e^{\sqrt{\varphi\phi}\lambda - \frac{1}{2}} \exp\left\{-\frac{\phi}{2}\lambda - \frac{\varphi}{2}\frac{1}{\lambda}\right\} \\
&= \left(\frac{\phi}{2\pi}\right)^{\frac{1}{2}} e^{\sqrt{\varphi\phi}} \int_0^{\infty} \lambda^{j-\frac{1}{2}} \exp\left\{-\frac{\phi}{2}\lambda - \frac{\varphi}{2}\frac{1}{\lambda}\right\}
\end{aligned}$$

and making the substitution $\lambda = \sqrt{\frac{\varphi}{\phi}}z$ implying $d\lambda = \sqrt{\frac{\varphi}{\phi}}dz$, we have

$$\begin{aligned}
\mathbb{E}(\Lambda^j) &= \left(\frac{\phi}{2\pi}\right)^{\frac{1}{2}} e^{\sqrt{\varphi\phi}} \left(\sqrt{\frac{\varphi}{\phi}}\right)^{j+\frac{1}{2}} \int_0^{\infty} z^{j+\frac{1}{2}-1} e^{-\frac{\varphi\phi}{2}\left(z+\frac{1}{z}\right)} dz \\
&= 2\left(\frac{\phi}{2\pi}\right)^{\frac{1}{2}} e^{\sqrt{\varphi\phi}} \left(\sqrt{\frac{\varphi}{\phi}}\right)^{j+\frac{1}{2}} K_{j+\frac{1}{2}}\left(\sqrt{\varphi\phi}\right) \\
&= \left(\frac{2\phi}{\pi}\right)^{\frac{1}{2}} e^{\sqrt{\varphi\phi}} \left(\sqrt{\frac{\varphi}{\phi}}\right)^{j+\frac{1}{2}} K_{j+\frac{1}{2}}\left(\sqrt{\varphi\phi}\right)
\end{aligned} \tag{5.164}$$

Poisson-Reciprocal Inverse Gaussian distribution is therefore obtained by method of moments as

$$\begin{aligned}
f(x) &= \frac{t^x}{x!} \left(\frac{2\phi}{\pi}\right)^{\frac{1}{2}} e^{\sqrt{\varphi\phi}} \sum_{j=x}^{\infty} \frac{(-t)^{j-x}}{(j-x)!} \left(\sqrt{\frac{\varphi}{\phi}}\right)^{j+\frac{1}{2}} K_{j+\frac{1}{2}}(\sqrt{\varphi\phi}) \\
&= \frac{t^x}{x!} \left(\frac{2\phi}{\pi}\right)^{\frac{1}{2}} e^{\sqrt{\varphi\phi}} \sum_{j=x}^{\infty} \frac{(-t)^{j-x}}{(j-x)!} \left(\sqrt{\frac{\varphi}{\phi}}\right)^{j-x} \left(\sqrt{\frac{\varphi}{\phi}}\right)^{x+\frac{1}{2}} K_{j+\frac{1}{2}}(\sqrt{\varphi\phi}) \\
&= \frac{t^x}{x!} \left(\frac{2\phi}{\pi}\right)^{\frac{1}{2}} e^{\sqrt{\varphi\phi}} \left(\sqrt{\frac{\varphi}{\phi}}\right)^{x+\frac{1}{2}} \sum_{j=x}^{\infty} \frac{(-t\sqrt{\frac{\varphi}{\phi}})^{j-x}}{(j-x)!} \frac{1}{2} \int_0^{\infty} z^{(j+\frac{1}{2})-1} \exp\left\{-\frac{\sqrt{\varphi\phi}}{2} \left(z + \frac{1}{z}\right)\right\} dz \\
&= \frac{t^x}{x!} \left(\frac{2\phi}{\pi}\right)^{\frac{1}{2}} e^{\sqrt{\varphi\phi}} \left(\sqrt{\frac{\varphi}{\phi}}\right)^{x+\frac{1}{2}} \frac{1}{2} \int_0^{\infty} \sum_{j=x}^{\infty} \frac{(-t\sqrt{\frac{\varphi}{\phi}}z)^{j-x}}{(j-x)!} z^{(x+\frac{1}{2})-1} \exp\left\{-\frac{\sqrt{\varphi\phi}}{2} \left(z + \frac{1}{z}\right)\right\} dz \\
&= \frac{t^x}{x!} \left(\frac{2\phi}{\pi}\right)^{\frac{1}{2}} e^{\sqrt{\varphi\phi}} \left(\sqrt{\frac{\varphi}{\phi}}\right)^{x+\frac{1}{2}} \frac{1}{2} \int_0^{\infty} z^{(x+\frac{1}{2})-1} \exp\left\{-t\sqrt{\frac{\varphi}{\phi}}z - \frac{\sqrt{\varphi\phi}}{2}z - \frac{\sqrt{\varphi\phi}}{2z}\right\} dz
\end{aligned}$$

Solving further, we obtain

$$\begin{aligned}
f(x) &= \frac{t^x}{x!} \left(\frac{2\phi}{\pi}\right)^{\frac{1}{2}} e^{\sqrt{\varphi\phi}} \left(\sqrt{\frac{\varphi}{\phi}}\right)^{x+\frac{1}{2}} \frac{1}{2} \int_0^{\infty} z^{(x+\frac{1}{2})-1} \exp\left\{-\frac{1}{2} \left[\frac{(2t+\phi)\varphi}{\sqrt{\phi}}z + \frac{\sqrt{\varphi\phi}}{z}\right]\right\} dz \\
&= \frac{t^x}{x!} \left(\frac{2\phi}{\pi}\right)^{\frac{1}{2}} e^{\sqrt{\varphi\phi}} \left(\sqrt{\frac{\varphi}{\phi}}\right)^{x+\frac{1}{2}} \frac{1}{2} \int_0^{\infty} z^{(x+\frac{1}{2})-1} \exp\left\{-\frac{\sqrt{\varphi}}{2} \frac{(2t+\phi)}{\sqrt{\phi}} \left[z + \frac{\phi}{2t+\phi} \frac{1}{z}\right]\right\} dz
\end{aligned}$$

Let $z = \sqrt{\frac{\phi}{2t+\phi}}y$ implying that $dz = \sqrt{\frac{\phi}{2t+\phi}}dy$, therefore

$$\begin{aligned}
f(x) &= \frac{t^x}{x!} \left(\frac{2\phi}{\pi}\right)^{\frac{1}{2}} e^{\sqrt{\varphi\phi}} \left(\sqrt{\frac{\varphi}{\phi}}\right)^{x+\frac{1}{2}} \left(\sqrt{\frac{\phi}{2t+\phi}}\right)^{x+\frac{1}{2}} \frac{1}{2} \int_0^{\infty} y^{(x+\frac{1}{2})-1} \exp\left\{-\frac{\sqrt{\varphi(2t+\phi)}}{2} \left(y + \frac{1}{y}\right)\right\} dy \\
&= \frac{t^x}{x!} \left(\frac{2\phi}{\pi}\right)^{\frac{1}{2}} e^{\sqrt{\varphi\phi}} \left(\sqrt{\frac{\varphi}{2t+\phi}}\right)^{x+\frac{1}{2}} K_{x+\frac{1}{2}}(\sqrt{\varphi(2t+\phi)})
\end{aligned}$$

5.4.20 Generalized Inverse Gaussian Distribution

Consider the generalized inverse Gaussian (GIG) distribution given by equation (3.87), its Laplace transform is

$$\begin{aligned}
L_{\Lambda}(t) &= \int_0^{\infty} e^{-t\lambda} \frac{\left(\frac{\varphi}{\phi}\right)^{\frac{v}{2}}}{2K_v(\sqrt{\varphi\phi})} \lambda^{v-1} \exp\left\{-\frac{1}{2} \left(\varphi\lambda + \frac{\phi}{\lambda}\right)\right\} d\lambda \\
&= \frac{\left(\frac{\varphi}{\phi}\right)^{\frac{v}{2}}}{2K_v(\sqrt{\varphi\phi})} \int_0^{\infty} \lambda^{v-1} \exp\left\{-\left(t + \frac{1}{2}\varphi\right)\lambda - \frac{1}{2}\frac{\phi}{\lambda}\right\} d\lambda \\
&= \frac{\left(\frac{\varphi}{\phi}\right)^{\frac{v}{2}}}{2K_v(\sqrt{\varphi\phi})} \int_0^{\infty} \lambda^{v-1} \exp\left\{-\left(\frac{2t+\varphi}{2}\right)\left[\lambda + \frac{\phi}{2t+\varphi}\frac{1}{\lambda}\right]\right\} d\lambda
\end{aligned}$$

and making the substitution $\lambda = \sqrt{\frac{\phi}{2t+\varphi}}z$ implying $d\lambda = \sqrt{\frac{\phi}{2t+\varphi}}dz$ we have

$$\begin{aligned} L_{\Lambda}(t) &= \frac{\left(\frac{\varphi}{\phi}\right)^{\frac{v}{2}}}{2K_v(\sqrt{\varphi\phi})} \int_0^{\infty} \left(\sqrt{\frac{\phi}{2t+\varphi}}\right)^v z^{v-1} \exp\left\{-\frac{\sqrt{\phi(2t+\varphi)}}{2}\left(z+\frac{1}{z}\right)\right\} dz \\ &= \frac{\left(\frac{\varphi}{\phi} \cdot \frac{\phi}{2t+\varphi}\right)^{\frac{v}{2}}}{2K_v(\sqrt{\varphi\phi})} \int_0^{\infty} z^{v-1} \exp\left\{-\frac{\sqrt{\phi(2t+\varphi)}}{2}\left(z+\frac{1}{z}\right)\right\} dz \\ &= \left(\frac{\varphi}{2t+\varphi}\right)^{\frac{v}{2}} \frac{K_v\left(\sqrt{\phi(2t+\varphi)}\right)}{K_v(\sqrt{\varphi\phi})} \end{aligned}$$

Using Willmot's (1986) notations, let $\phi = \frac{\mu^2}{\beta}$ and $\varphi = \frac{1}{\beta}$, therefore

$$\begin{aligned} L_{\Lambda}(t) &= \left(\frac{1}{2\beta t+1}\right)^{\frac{v}{2}} \frac{K_v\left(\sqrt{\frac{\mu^2}{\beta}\left(2t+\frac{1}{\beta}\right)}\right)}{K_v\left(\sqrt{\frac{1}{\beta}\frac{\mu^2}{\beta}}\right)} \\ &= \left(\frac{1}{2\beta t+1}\right)^{\frac{v}{2}} \frac{K_v\left(\frac{\mu}{\beta}\sqrt{2\beta t+1}\right)}{K_v\left(\frac{\mu}{\beta}\right)} \\ &= (1+2\beta t)^{-\frac{v}{2}} \frac{K_v\left[\mu\beta^{-1}(1+2\beta t)^{\frac{1}{2}}\right]}{K_v(\mu\beta^{-1})} \end{aligned} \quad (5.165)$$

and

$$G(s) = (1+2\beta(1-s)t)^{-\frac{v}{2}} \frac{K_v\left[\mu\beta^{-1}(1+2\beta(1-s)t)^{\frac{1}{2}}\right]}{K_v(\mu\beta^{-1})} \quad (5.166)$$

as given by Willmot (1986).

The j th moment of generalized inverse Gaussian distribution is

$$\begin{aligned} \mathbb{E}(\Lambda^j) &= \frac{(\varphi/\phi)^{\frac{v}{2}}}{2K_v(\sqrt{\varphi\phi})} \int_0^{\infty} \lambda^{j+v-1} \exp\left\{-\frac{1}{2}\left(\varphi\lambda+\frac{\phi}{\lambda}\right)\right\} d\lambda \\ &= \frac{(\varphi/\phi)^{\frac{v}{2}}}{2K_v(\sqrt{\varphi\phi})} \int_0^{\infty} \lambda^{j+v-1} \exp\left\{-\frac{\varphi}{2}\left(\lambda+\frac{\phi}{\varphi\lambda}\right)\right\} d\lambda \end{aligned}$$

and making the substitution $\lambda = \sqrt{\frac{\phi}{\varphi}}z$ implying $d\lambda = \sqrt{\frac{\phi}{\varphi}}dz$, we have

$$\begin{aligned} \mathbb{E}(\Lambda^j) &= \frac{(\varphi/\phi)^{\frac{v}{2}}}{2K_v(\sqrt{\varphi\phi})} \left(\sqrt{\frac{\phi}{\varphi}}\right)^{j+v} \int_0^{\infty} z^{j+v-1} \exp\left\{-\frac{\sqrt{\varphi\phi}}{2}\left(z+\frac{1}{z}\right)\right\} dz \\ &= \frac{(\varphi/\phi)^{\frac{v}{2}}}{K_v(\sqrt{\varphi\phi})} \left(\sqrt{\frac{\phi}{\varphi}}\right)^{j+v} K_{j+v}(\sqrt{\varphi\phi}) \end{aligned} \quad (5.167)$$

Therefore, Poisson-generalized inverse Gaussian distribution is obtained by the method of moments

as

$$\begin{aligned}
f(x) &= \frac{t^x (\varphi/\phi)^{\frac{v}{2}}}{x! K_v(\sqrt{\varphi\phi})} \sum_{j=x}^{\infty} \frac{(-t)^{j-x}}{(j-x)!} \left(\sqrt{\frac{\phi}{\varphi}}\right)^{j+v} K_{j+v}(\sqrt{\varphi\phi}) \\
&= \frac{t^x (\varphi/\phi)^{\frac{v}{2}}}{x! K_v(\sqrt{\varphi\phi})} \sum_{j=x}^{\infty} \frac{(-t)^{j-x}}{(j-x)!} \left(\sqrt{\frac{\phi}{\varphi}}\right)^{j-x} \left(\sqrt{\frac{\phi}{\varphi}}\right)^{x+v} K_{j+v}(\sqrt{\varphi\phi}) \\
&= \frac{t^x (\varphi/\phi)^{\frac{v}{2}}}{x! K_v(\sqrt{\varphi\phi})} \left(\sqrt{\frac{\phi}{\varphi}}\right)^{x+v} \sum_{j=x}^{\infty} \frac{(-t\sqrt{\frac{\phi}{\varphi}})^{j-x}}{(j-x)!} K_{j+v}(\sqrt{\varphi\phi}) \\
&= \frac{t^x (\varphi/\phi)^{\frac{v}{2}}}{x! K_v(\sqrt{\varphi\phi})} \left(\sqrt{\frac{\phi}{\varphi}}\right)^{x+v} \sum_{j=x}^{\infty} \frac{(-t\sqrt{\frac{\phi}{\varphi}})^{j-x}}{(j-x)!} \frac{1}{2} \int_0^{\infty} z^{j+v-1} \exp\left\{-\frac{\sqrt{\varphi\phi}}{2}\left(z+\frac{1}{z}\right)\right\} dz \\
&= \frac{t^x (\varphi/\phi)^{\frac{v}{2}}}{x! K_v(\sqrt{\varphi\phi})} \left(\sqrt{\frac{\phi}{\varphi}}\right)^{x+v} \frac{1}{2} \int_0^{\infty} \sum_{j=x}^{\infty} \frac{(-t\sqrt{\frac{\phi}{\varphi}})^{j-x}}{(j-x)!} z^{x+v-1} \exp\left\{-\frac{\sqrt{\varphi\phi}}{2}\left(z+\frac{1}{z}\right)\right\} dz
\end{aligned}$$

Solving further, we obtain

$$\begin{aligned}
f(x) &= \frac{t^x (\varphi/\phi)^{\frac{v}{2}}}{x! K_v(\sqrt{\varphi\phi})} \left(\sqrt{\frac{\phi}{\varphi}}\right)^{x+v} \frac{1}{2} \int_0^{\infty} e^{-t\sqrt{\frac{\phi}{\varphi}}z} z^{x+v-1} \exp\left\{-\frac{\sqrt{\varphi\phi}}{2}z - \frac{\sqrt{\varphi\phi}}{2z}\right\} dz \\
&= \frac{t^x (\varphi/\phi)^{\frac{v}{2}}}{x! K_v(\sqrt{\varphi\phi})} \left(\sqrt{\frac{\phi}{\varphi}}\right)^{x+v} \frac{1}{2} \int_0^{\infty} z^{x+v-1} \exp\left\{-\frac{1}{2}\left[\frac{2t\sqrt{\phi}}{\sqrt{\varphi}} + \sqrt{\varphi\phi}\right]z + \frac{\sqrt{\varphi\phi}}{2z}\right\} dz \\
&= \frac{t^x (\varphi/\phi)^{\frac{v}{2}}}{x! K_v(\sqrt{\varphi\phi})} \left(\sqrt{\frac{\phi}{\varphi}}\right)^{x+v} \frac{1}{2} \int_0^{\infty} z^{x+v-1} \exp\left\{-\frac{1}{2}\left[\frac{(2t+\varphi)\sqrt{\phi}}{\sqrt{\varphi}}z + \frac{\sqrt{\varphi\phi}}{2z}\right]\right\} dz \\
&= \frac{t^x (\varphi/\phi)^{\frac{v}{2}}}{x! K_v(\sqrt{\varphi\phi})} \left(\sqrt{\frac{\phi}{\varphi}}\right)^{x+v} \frac{1}{2} \int_0^{\infty} z^{x+v-1} \exp\left\{-\frac{(2t+\varphi)\sqrt{\phi}}{2\sqrt{\varphi}}\left[z + \frac{\varphi}{2t+\varphi}\frac{1}{z}\right]\right\} dz
\end{aligned}$$

Let $z = \sqrt{\frac{\varphi}{2t+\varphi}}y$ implying that $dz = \sqrt{\frac{\varphi}{2t+\varphi}}dy$, therefore

$$\begin{aligned}
f(x) &= \frac{t^x (\varphi/\phi)^{\frac{v}{2}}}{x! K_v(\sqrt{\varphi\phi})} \left(\sqrt{\frac{\phi}{\varphi}}\right)^{x+v} \left(\sqrt{\frac{\varphi}{2t+\varphi}}\right)^{x+v} \frac{1}{2} \int_0^{\infty} y^{x+v-1} \exp\left\{-\frac{\sqrt{\phi(2t+\varphi)}}{2}\left(y+\frac{1}{y}\right)\right\} dy \\
&= \frac{t^x (\varphi/\phi)^{\frac{v}{2}}}{x! K_v(\sqrt{\varphi\phi})} \left(\sqrt{\frac{\phi}{2t+\varphi}}\right)^{x+v} K_{x+v}(\sqrt{\phi(2t+\varphi)})
\end{aligned}$$

5.4.21 Special Cases of Generalized Inverse Gaussian Distribution

Case (i): When $v = -\frac{1}{2}$

Using the formula (5.165), then

$$L_{\Lambda}(t) = \left(\sqrt{1+2\beta t}\right)^{\frac{1}{2}} \frac{K_{-\frac{1}{2}}\left[\mu\beta^{-1}(1+2\beta t)^{\frac{1}{2}}\right]}{K_{-\frac{1}{2}}(\mu\beta^{-1})}$$

But $K_{-\frac{1}{2}}(\omega) = K_{\frac{1}{2}}(\omega) = \sqrt{\frac{\pi}{2\omega}}e^{-\omega}$, therefore

$$\begin{aligned}
L_{\Lambda}(t) &= \left(\sqrt{1+2\beta t}\right)^{\frac{1}{2}} \frac{\sqrt{\frac{\pi}{2\mu\beta^{-1}(1+2\beta t)^{\frac{1}{2}}}}e^{-\mu\beta^{-1}(1+2\beta t)^{\frac{1}{2}}}}{\sqrt{\frac{\pi}{2\mu\beta^{-1}}}}e^{-\mu\beta^{-1}} \\
&= \exp\left\{\mu\beta^{-1} - \mu\beta^{-1}(1+2\beta t)^{\frac{1}{2}}\right\} \\
&= \exp\left\{-\frac{\mu}{\beta}\left[\sqrt{1+2\beta t} - 1\right]\right\}
\end{aligned} \tag{5.168}$$

as given in (5.157).

The pgf is

$$G(s) = (1+2\beta(1-s)t)^{\frac{1}{4}} \frac{K_{-\frac{1}{2}}\left[\mu\beta^{-1}(1+2\beta(1-s)t)^{\frac{1}{2}}\right]}{K_{-\frac{1}{2}}(\mu\beta^{-1})} \tag{5.169}$$

From (5.167),

$$\begin{aligned}
\mathbb{E}(\Lambda^j) &= \frac{\left(\sqrt{\frac{\varphi}{\phi}}\right)^{-\frac{1}{2}} \left(\sqrt{\frac{\phi}{\varphi}}\right)^{j-\frac{1}{2}} K_{j-\frac{1}{2}}(\sqrt{\varphi\phi})}{K_{-\frac{1}{2}}(\sqrt{\varphi\phi})} \\
&= \frac{\left(\sqrt{\frac{\varphi}{\phi}}\right)^{-\frac{1}{2}} \left(\sqrt{\frac{\phi}{\varphi}}\right)^{j-\frac{1}{2}} K_{j-\frac{1}{2}}(\sqrt{\varphi\phi})}{\sqrt{\frac{\pi}{2\sqrt{\varphi\phi}}}e^{-\sqrt{\varphi\phi}}} \\
&= \sqrt{\frac{2\phi}{\pi}}e^{\sqrt{\varphi\phi}} \left(\sqrt{\frac{\phi}{\varphi}}\right)^{j-\frac{1}{2}} K_{j-\frac{1}{2}}(\sqrt{\varphi\phi})
\end{aligned} \tag{5.170}$$

as given in (5.159).

$$\begin{aligned}
f(x) &= \frac{t^x \left(\sqrt{\frac{\varphi}{\phi}}\right)^{-\frac{1}{2}}}{x! K_{-\frac{1}{2}}(\sqrt{\varphi\phi})} \sum_{j=x}^{\infty} \frac{(-t)^{j-x}}{(j-x)!} \left(\sqrt{\frac{\phi}{\varphi}}\right)^{j-\frac{1}{2}} K_{j-\frac{1}{2}}(\sqrt{\varphi\phi}) \\
&= \frac{t^x \left(\sqrt{\frac{\varphi}{\phi}}\right)^{-\frac{1}{2}}}{x!} \frac{\sum_{j=x}^{\infty} \frac{(-t)^{j-x}}{(j-x)!} \left(\sqrt{\frac{\phi}{\varphi}}\right)^{j-\frac{1}{2}} K_{j-\frac{1}{2}}(\sqrt{\varphi\phi})}{\sqrt{\frac{\pi}{2\sqrt{\varphi\phi}}}e^{-\sqrt{\varphi\phi}}} \\
&= \frac{t^x \sqrt{\frac{2\phi}{\pi}}e^{\sqrt{\varphi\phi}}}{x!} \sum_{j=x}^{\infty} \frac{(-t)^{j-x}}{(j-x)!} \left(\sqrt{\frac{\phi}{\varphi}}\right)^{j-\frac{1}{2}} K_{j-\frac{1}{2}}(\sqrt{\varphi\phi})
\end{aligned} \tag{5.171}$$

Thus when $v = -\frac{1}{2}$, we get similar results as those for Inverse Gaussian and Poisson-Inverse Gaussian distributions.

Case (ii): When $v = \frac{1}{2}$

From (5.165),

$$\begin{aligned}
L_{\Lambda}(t) &= \left(\sqrt{1+2\beta t}\right)^{-\frac{1}{2}} \frac{K_{\frac{1}{2}}\left[\mu\beta^{-1}(1+2\beta t)^{\frac{1}{2}}\right]}{K_{\frac{1}{2}}(\mu\beta^{-1})} \\
&= \left(\sqrt{1+2\beta t}\right)^{-\frac{1}{2}} \frac{\sqrt{\frac{\pi}{2\mu\beta^{-1}\sqrt{(1+2\beta t)}}} e^{-\mu\beta^{-1}(1+2\beta t)^{\frac{1}{2}}}}{\sqrt{\frac{\pi}{2\mu\beta^{-1}}} e^{-\mu\beta^{-1}}} \\
&= \frac{1}{\sqrt{1+2\beta t}} \exp\left\{\mu\beta^{-1}\left[1-\sqrt{(1+2\beta t)}\right]\right\} \\
&= \left(\frac{1}{1+2\beta t}\right)^{\frac{1}{2}} \exp\left\{-\frac{\mu}{\beta}\left[\sqrt{1+2\beta t}-1\right]\right\} \\
&= \left(\frac{\frac{1}{2\beta}}{\frac{1}{2\beta}+t}\right)^{\frac{1}{2}} \exp\left\{-\frac{\mu}{\beta}\left[\sqrt{1+2\beta t}-1\right]\right\} \tag{5.172}
\end{aligned}$$

which is a product of Laplace Transform of a Gamma distribution and Inverse gamma distribution.

The pgf is

$$G(s) = (1+2\beta(1-s)t)^{-\frac{1}{4}} \frac{K_{\frac{1}{2}}\left[\mu\beta^{-1}(1+2\beta(1-s)t)^{\frac{1}{2}}\right]}{K_{\frac{1}{2}}(\mu\beta^{-1})} \tag{5.173}$$

From (5.167),

$$\begin{aligned}
\mathbb{E}(\Lambda^j) &= \frac{\left(\sqrt{\frac{\varphi}{\phi}}\right)^{\frac{1}{2}} \left(\sqrt{\frac{\phi}{\varphi}}\right)^{j+\frac{1}{2}} K_{j+\frac{1}{2}}(\sqrt{\phi\varphi})}{K_{\frac{1}{2}}(\sqrt{\varphi\phi})} \\
&= \frac{\left(\sqrt{\frac{\varphi}{\phi}}\right)^{\frac{1}{2}} \left(\sqrt{\frac{\phi}{\varphi}}\right)^{j+\frac{1}{2}} K_{j+\frac{1}{2}}(\sqrt{\phi\varphi})}{\sqrt{\frac{\pi}{2\sqrt{\varphi\phi}}} e^{-\sqrt{\varphi\phi}}} \\
&= \sqrt{\frac{2\varphi}{\pi}} e^{\sqrt{\varphi\phi}} \left(\sqrt{\frac{\phi}{\varphi}}\right)^{j+\frac{1}{2}} K_{j+\frac{1}{2}}(\sqrt{\varphi\phi}) \tag{5.174}
\end{aligned}$$

and therefore,

$$f(x) = \frac{t^x}{x!} \frac{\left(\sqrt{\frac{\varphi}{\phi}}\right)^{\frac{1}{2}}}{K_{\frac{1}{2}}(\sqrt{\varphi\phi})} \sum_{j=x}^{\infty} \frac{(-t)^{j-x}}{(j-x)!} \left(\sqrt{\frac{\phi}{\varphi}}\right)^{j+\frac{1}{2}} K_{j+\frac{1}{2}}(\sqrt{\varphi\phi}) \tag{5.175}$$

This is the case of Reciprocal Inverse Gaussian and Poisson-Reciprocal-Inverse Gaussian distributions.

5.5 Identities based on Poisson mixtures and by method of moments

As a consequence of identifying the routes to mixed Poisson distributions, in this section we deduce mathematical identities based on Poisson mixtures. Specifically, we equate the result of a mixture

obtained in explicit form with that obtained by method of moments. We also equate the result obtained in terms of a special function with that obtained by method of moments.

5.5.1 Explicit form Identities

Poisson-Gamma Distribution

$$\sum_{j=x}^{\infty} \frac{t^x (-t)^{j-x} \Gamma(j+\alpha)}{(j-x)! x! \Gamma(\alpha) \beta^j} = \binom{x+\alpha-1}{x} \left(\frac{t}{t+\beta}\right)^x \left(\frac{\beta}{t+\beta}\right)^\alpha \quad (5.176)$$

Poisson-Shifted Gamma Distribution

$$\sum_{j=x}^{\infty} \sum_{l=0}^j \frac{t^x (-t)^{j-x}}{(j-x)! x!} \binom{j}{l} \mu^{j-l} \frac{\Gamma(l+\alpha)}{\Gamma(\alpha) \beta^l} = \sum_{k=0}^x \frac{e^{-\mu t} (\mu t)^{x-k}}{(x-k)!} \frac{\Gamma(k+\alpha)}{\Gamma(k+1) \Gamma(\alpha)} \left(\frac{\beta}{\beta+t}\right)^\alpha \left(\frac{t}{\beta+t}\right)^k \quad (5.177)$$

Poisson-Lindley Distribution

$$\frac{1}{\theta+1} \sum_{j=x}^{\infty} \left(\frac{-t}{\theta}\right)^j (j+1+\theta) (-1)^x \binom{j}{x} = \frac{t^x \theta^2 (x+t+\theta+1)}{(t+\theta)^{x+2} (1+\theta)} \quad (5.178)$$

Poisson-3-parameter Generalized Lindley Distribution

$$\sum_{j=x}^{\infty} (-1)^{j-x} \binom{j}{x} \binom{\alpha+j-1}{j} \left(1 + \frac{\gamma j}{\alpha(\theta+\gamma)}\right) \left(\frac{t}{\theta}\right)^j = \frac{\Gamma(x+\alpha)}{x! \Gamma(\alpha+1)} \left(\alpha + \frac{\alpha t + \gamma x}{\theta + \gamma}\right) \left(\frac{t}{t+\theta}\right)^x \left(\frac{\theta}{t+\theta}\right)^{\alpha+1} \quad (5.179)$$

Poisson-Transmuted Exponential Distribution

$$\sum_{j=x}^{\infty} (-1)^{j-x} \binom{j}{x} \left(\bar{\alpha} + \frac{\alpha}{2j}\right) \left(\frac{t}{\theta}\right)^j = \frac{\theta t^x \bar{\alpha}}{(t+\theta)^{x+1}} + \frac{\theta t^x 2\alpha}{(t+2\theta)^{x+1}} \quad (5.180)$$

5.5.2 Confluent hypergeometric function Identities

Poisson-Beta I Distribution

$$\sum_{j=x}^{\infty} \frac{(-t)^{j-x}}{(j-x)!} B(j+\alpha, \beta) = B(x+\alpha, \beta) {}_1F_1(x+\alpha; x+\alpha+\beta; -t) \quad (5.181)$$

Poisson-Rectangular Distribution

$$\sum_{j=x}^{\infty} \frac{(-t)^{j-x}}{(j-x)!} \left[\frac{b^{j+1} - a^{j+1}}{j+1} \right] = \frac{1}{(x+1)} \{ b^{x+1} {}_1F_1(x+1; x+2; -bt) - a^{x+1} {}_1F_1(x+1; x+2; -at) \} \quad (5.182)$$

Poisson-Beta II Distribution

$$\sum_{j=x}^{\infty} \frac{(-t)^{j-x}}{(j-x)!} B(p+j, q-j) = \Gamma(x+p) \psi(x+p, x-q+1; t) \quad (5.183)$$

Poisson-Scaled Beta Distribution

$$\sum_{j=x}^{\infty} \frac{(-\mu t)^{j-x}}{(j-x)!} B(j+\alpha, \beta) = B(x+\alpha, \beta) {}_1F_1(x+\alpha; x+\alpha+\beta; -\mu t) \quad (5.184)$$

Poisson-Full Beta Distribution

$$\sum_{j=x}^{\infty} \left(\frac{-t}{b}\right)^{j-x} \frac{1}{(j-x)!} B(p+j, q-j) = \Gamma(x+p) \psi\left(x+p, x+1-q; \frac{t}{b}\right) \quad (5.185)$$

Poisson-Pearson Type I Distribution

$$\begin{aligned} & \sum_{j=x}^{\infty} \frac{(-t)^{j-x}}{(j-x)!} \sum_{i=0}^j \binom{j}{i} a^{j-i} (b-a)^i B(i+p, q) \\ &= e^{-at} \sum_{k=0}^x \left\{ \binom{x}{k} a^{x-k} (b-a)^k B(k+p, q) {}_1F_1(k+p; k+p+q; -(b-a)t) \right\} \end{aligned} \quad (5.186)$$

Poisson-Pearson Type VI Distribution

$$\begin{aligned} & \sum_{j=x}^{\infty} \frac{(-t)^{j-x}}{(j-x)!} \left\{ \sum_{i=0}^j \binom{j}{i} \left(\frac{d-c}{d}\right)^i d^j B(i+b-a, a-i) \right\} \\ &= d^x e^{-dt} \sum_{k=0}^x \binom{x}{k} \left(\frac{d-c}{d}\right)^k \Gamma(k+b-a) \psi(k+b-a, k-a+1; (d-c)t) \end{aligned} \quad (5.187)$$

Poisson-Shifted Gamma Distribution

$$\sum_{j=x}^{\infty} \frac{(-t)^{j-x}}{(j-x)!} \sum_{i=0}^j \binom{j}{i} \frac{\mu^{j-i} \Gamma(i+\alpha)}{\beta^i} = \mu^x (\mu\beta)^\alpha e^{-\mu t} \Gamma(\alpha) \psi(\alpha, \alpha+x+1; (t+\beta)\mu) \quad (5.188)$$

Poisson-Truncated Gamma (from above) Distribution

$$\sum_{j=x}^{\infty} \frac{(-t)^{j-x}}{(j-x)!} \frac{b}{j+b} p^j {}_1F_1(j+b; j+b+1; -ap) = \frac{b}{x+b} p^x {}_1F_1(x+b; x+b+1; -(a+t)p) \quad (5.189)$$

Poisson-Truncated Gamma (from below) Distribution

$$\sum_{j=x}^{\infty} \frac{(-t)^{j-x}}{(j-x)!} \frac{[\Gamma(j+\alpha) - \gamma(j+\alpha, \beta\lambda_0)]}{\beta^j} = \frac{\beta^\alpha}{(t+\beta)^{x+\alpha}} [\Gamma(x+\alpha) - \gamma(x+\alpha, (t+\beta)\lambda_0)] \quad (5.190)$$

Poisson-Truncated Gamma (from below and above) Distribution

$$\sum_{j=x}^{\infty} \frac{(-t)^{j-x}}{(j-x)!} \left\{ \frac{\gamma(j+\alpha, \beta b) - \gamma(j+\alpha, \beta a)}{\beta^j} \right\} = \frac{\beta^\alpha}{(t+\beta)^{x+\alpha}} \{ \gamma(x+\alpha, (t+\beta)b) - \gamma(x+\alpha, (t+\beta)a) \} \quad (5.191)$$

Poisson-Truncated Pearson Type III Distribution

$$\sum_{j=x}^{\infty} \frac{(-t)^{j-x}}{(j-x)!} B(j+1, \beta-1) {}_1F_1(j+1; j+\beta; \alpha) = B(x+1, \beta-1) {}_1F_1(x+1; x+\beta; \alpha-t) \quad (5.192)$$

Poisson-Pareto I Distribution

$$\sum_{j=x}^{\infty} \frac{(-t)^{j-x}}{(j-x)!} \frac{\beta^j}{\alpha-j} = \beta^x e^{-\beta t} \psi(1, x-\alpha+1; \beta t) \quad (5.193)$$

Poisson-Pareto II Distribution

$$\sum_{j=x}^{\infty} \frac{(-t)^{j-x}}{(j-x)!} \beta^j B(j+1, \alpha-j) = \beta^x \Gamma(x+1) \psi(x+1, x-\alpha+1; \beta t) \quad (5.194)$$

Poisson-Generalized Pareto Distribution

$$\sum_{j=x}^{\infty} \frac{(-t)^{j-x}}{(j-x)!} \mu^j B(j+\beta, \alpha-j) = \mu^x \Gamma(x+\beta) \psi(x+\beta, x-\alpha+1; \mu t) \quad (5.195)$$

5.5.3 Bessel function of third kind Identities

Poisson-Inverse Gamma Distribution

$$\sum_{j=x}^{\infty} \frac{t^x (-t)^{j-x} \beta^j \Gamma(\alpha-j)}{(j-x)!} = 2(\beta t)^{\frac{x+\alpha}{2}} K_{x-\alpha}(2\sqrt{\beta t}) \quad (5.196)$$

Poisson-Pearson Type V Distribution

$$\sum_{j=x}^{\infty} \frac{(-t)^{j-x}}{(j-x)!} \left\{ \sum_{i=0}^j \binom{j}{i} \left(\frac{c}{\beta}\right)^i \beta^j \Gamma(\alpha-(j-i)) \right\} = 2\beta^\alpha c^x e^{-ct} \sum_{k=0}^x \binom{x}{k} \frac{1}{c^k} \left(\sqrt{\frac{\beta}{t}}\right)^{k-\alpha} K_{k-\alpha}(2\sqrt{\beta t}) \quad (5.197)$$

Poisson-Inverse Gaussian Distribution

$$\sum_{j=x}^{\infty} \frac{(-t)^{j-x}}{(j-x)!} \left(\sqrt{\frac{\phi}{\varphi}}\right)^{j-\frac{1}{2}} K_{j-\frac{1}{2}}(\sqrt{\varphi\phi}) = \left(\sqrt{\frac{\phi}{2t+\varphi}}\right)^{x-\frac{1}{2}} K_{x-\frac{1}{2}}(\sqrt{\phi(2t+\varphi)}) \quad (5.198)$$

Poisson-Reciprocal Inverse Gaussian Distribution

$$\sum_{j=x}^{\infty} \frac{(-t)^{j-x}}{(j-x)!} \left(\sqrt{\frac{\varphi}{\phi}}\right)^{j+\frac{1}{2}} K_{j+\frac{1}{2}}(\sqrt{\varphi\phi}) = \left(\frac{\varphi}{2t+\phi}\right)^{x+\frac{1}{2}} K_{x+\frac{1}{2}}(\sqrt{\varphi(2t+\phi)}) \quad (5.199)$$

Poisson-Generalized Inverse Gaussian Distribution

$$\sum_{j=x}^{\infty} \frac{(-t)^{j-x}}{(j-x)!} \left(\sqrt{\frac{\phi}{\varphi}} \right)^{j+v} K_{j+v}(\sqrt{\varphi\phi}) = \left(\sqrt{\frac{\phi}{2t+\varphi}} \right)^{x+v} K_{x+v}(\sqrt{\phi(2t+\varphi)}) \quad (5.200)$$

Chapter 6

CONCLUSIONS AND RECOMMENDATIONS

6.1 Summary of Results and Challenges

The objective of this research was to construct mixed Poisson distributions via four routes, namely, explicit, special functions, recursive and transform routes.

Explicit Route

By explicit route, mixed Poisson distributions were obtained using the following mixing distributions: Gamma, Shifted Gamma, Transmuted Exponential, Lindley and 3-parameter Generalized Lindley distributions.

Moments about the origin and moments about the mean of the Poisson mixtures were obtained in terms of moments of the mixing distributions. Posterior distributions, posterior r th moments and posterior means were also obtained.

Remark 7.1: Explicit route was achieved by direct integration. Very few cases however follow this route.

Remark 7.2: Transmuted Exponential and 3-parameter generalized Lindley distributions are finite mixtures used as mixing distributions. The 3-parameter generalized Lindley distribution nests one parameter Lindley distribution and two types of 2-parameter generalized Lindley distributions.

Special Functions Route

The mixing distributions leading to Poisson mixtures expressed in terms of Kummer's and Tricomi's confluent hypergeometric functions are: Beta I, Rectangular, Beta II, Scaled Beta, Full Beta, Pearson Type I, Pearson Type VI, Shifted Gamma (Pearson Type III), Gamma truncated from above,

Gamma truncated from below, Gamma truncated from above and below, truncated Pearson Type III, Pareto I, Pareto II (Lomax) and generalized Pareto distributions. The corresponding pgfs of the mixtures were also expressed in terms of confluent hypergeometric functions.

For Poisson mixtures in terms of modified Bessel function of the third kind, the following mixing distributions were used: Inverse Gamma, Inverse Gaussian, Reciprocal Inverse Gaussian, Pearson Type V and Generalized Inverse Gaussian distributions. The pgfs were also obtained in terms of modified Bessel function of the third kind.

Recursive Route

The following mixing distributions were used to obtain mixed Poisson distributions using integration by parts: Beta I, Rectangular, Beta II, Scaled Beta, Full Beta, transformed Beta, Inverse Gamma, Shifted Gamma, Gamma truncated from below, Generalized Gamma, transformed (Generalized) Gamma, Pareto I, Pareto II (Lomax), Generalized Pareto, Generalized Pareto Type II, Inverse Gaussian, Reciprocal Inverse Gaussian, Generalized Inverse Gaussian, Confluent hypergeometric and Half-Normal distributions.

Remark 7.3: Recursive models obtained were similar to those obtained by other methods. The disadvantage of the integration by parts technique is that it does not have a general formula for differential equation in pgf. For each case, a differential equation has to be derived.

Transform Route

Some Poisson mixtures can be determined through x th derivatives of Laplace transforms of mixing distributions such as Gamma, 3-parameter generalized Lindley, Transmuted Exponential, Inverse Gamma and Hougaard.

In particular, it is tedious to find the x th derivative of Laplace transform of Hougaard distribution.

In the Mellin transform approach, the r th moment of mixing distribution was used to determine Poisson mixture.

The pgf technique was used to determine factorial moments of the mixtures. The pgf is expressible in terms of Laplace transform.

Identities based on Poisson Mixtures

By comparing results obtained by explicit and by method of moments, we were able to deduce mathematical identities. Also by comparing results obtained by special function and by method of moments, other identities were deduced.

6.2 Recommendations

The following recommendations are suggested.

Explicit Form

More mixing distributions leading to mixed Poisson distributions in explicit forms could be identified. In particular, Generalized Lindley and Transmuted Exponential distributions are mixing distributions of finite mixtures. More finite mixtures of this nature could be identified to obtain Poisson mixtures in Explicit form.

Special Functions

Confluent hypergeometric and Bessel functions have been used to construct mixing distributions. There are other special functions such as Laguerre Polynomials which could be explored.

Recursive Models

Recursion is one way of numerical or approximation methods. Other techniques, such as Taylor's series could be explored. In this research, we have used integration by parts technique to obtain recursive models. In obtaining the corresponding differential equations in probability generating functions, we have made use of Wang's (1994) recursive model. Other existing differential equations could be used and compared.

Expectation Forms

Mathematical identities based on Poisson mixtures have been derived by equating results obtained in explicit forms and those in terms of special functions with those results obtained by method of moments. More identities could be derived.

Using Laplace transform technique, mixed Poisson distributions have been obtained. However for Poisson mixtures in terms of special functions, obtaining many differentiations of the Laplace transforms is quite involving. Patterns of differentiations need to be identified. Further work therefore needs to be done in this area.

Other routes

Further work could be on identifying other routes to obtaining mixed Poisson distributions, such as numerical integration given by Bulmer (1974).

Properties

In this research, we have concentrated on constructing posterior distributions from Poisson mixtures and hence obtained posterior moments. We have also obtained the general formula for factorial moments, moments about the origin and about the mean in terms of derivatives of probability generating functions. However, other properties have not been looked at, such as identifiability, infinite divisibility, compound distributions, etc. Extensive works in this area would be worthwhile.

Inference on Parameters and Applications

The focus in this research is on constructions and properties of mixed Poisson distributions. Estimations, testing of hypotheses and applications of Poisson mixtures are definitely major areas for further research.

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