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Research Report in Mathematics, Number 03, 2018

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August 2018

Submitted to the School of Mathematics in partial fulfilment for a degree in Master of Science in Applied Mathematics

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Hydrodynamic Stability of Taylor-Couette Flow

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Master of Science Project

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Abstract

The stability of Couette flow between two cylinders with the outer one stationary and the inner one rotating and also moving with a constant axial velocity is investigated. Both axisymmetric and non-axisymmetric disturbances are considered. The perturbations equations governing the marginal stability state of the flow are derived and solved analytically for the situation when the gap spacing between the cylinders is small compared to the radius.The equations of motion governing flow are also studied.

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Declaration and Approval

I the undersigned declare that this project report is my original work and to the best of my knowledge, it has not been submitted in support of an award of a degree in any other university or institution of learning.

Signature Date

NANDWA CHITERI VINCENT
Reg No. I56/89489/2016

In my capacity as a supervisor of the candidate, I certify that this report has my approval for submission.

Signature Date

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Dedication

This project is dedicated to my son Einstein and parents Jackson Chiteri and Mary Chiteri.

Contents

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First and foremost, I thank the Almighty God for granting me strength, zeal, power and determination to proceed successfully with this thesis. Secondly, I would like to thank The University of Nairobi and EAUMP-ICTP for offering me a scholarship which enabled me undertake my studies comfortably. I appreciate my lecturers in the School of Mathematics for equipping me with the knowledge that has enabled me face this thesis with ease. I would like in a more special way to thank my wise and able supervisor; Dr Charles Nyandwi for the hard work he did guiding me on all the aspects of this thesis to see that my thesis was of high quality. I could not have managed to come this far without his support. I also give credit to Prof Ogana and Dr Katende; who supported me to access some useful materials which were not directly available online.

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Orio muno sana mama!

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Nandwa Chiteri Vincent

Nairobi, 2018.

1 Introduction

This chapter gives information on hydrodynamic stability and history on hydrodynamic stability of Couette flow.We also define some important terms that will be used in the preceding chapters.

1.1 Preliminary

The analysis of stability on Couette flow was started by Taylor(1921). Other investigators were DiPrima(1981),Drazin(1982),Kataoka(1986),Krueger (1966),Marques (1997),Weisberg(1997), and the area is still being pursued by many other researchers.

1.2 The Concept of Stability

Stability theory has become one of dominant importance in the study of dynamical systems. It has many applications in basic fields like meteorology, oceanography, astrophysics and geophysics- to mention few of them. The concept of stability was developed very early in the eighteenth century and was specialized in mechanics for equilibrium studies. In precise mathematical terms, the equilibrium of a particle, subjected to some forces, is stable when least perturbations makes particle not change near the equilibrium point.
Next great advance came in hydrodynamic stability which laid foundations of the stability theory in fluid mechanics. Hydrodynamic stability has been recognized as important subject in mechanics.
In recent years the theoretical developments in the studies of instabilities and turbulence have been as profound as the developments in experimental methods.Linear stability is significant in which effect of less fluctuation away from a solution to the equations is examined as a function of a parameter such as the Reynolds number. Study of stability problems is relevant to the study of structure of a physical system. It is particularly important when it is not possible to probe into its interior and obtain information on its structure by direct method.

1.3 Fundamental Concepts of Hydrodynamic Stability

Choice of suitable equations describing flow are often difficult tasks, but we suppose here that the equations and their solution are completely known, even though minor features of the observed flow may be neglected or only an appropriate solution found.
Disturbance may die away, persist as a disturbance of similar magnitude or grow so much

that the basic flow becomes a different laminar or turbulent flow.Broadly speaking, we call such disturbances (asymptotically) stable, neutrally stable or unstable respectively. All possible slight disturbances are likely to be excited in some degree by small irregularities or vibrations of the basic flow in practice, so it will persist only if it is stable to all slight disturbances. In seeking more precise definitions of stability we may be guided by the considerable mathematical literature of stability, but must frame the definitions to further our physical understanding. The choice of useful definitions of 'disturbed slightly', 'die away' and 'disturbance of similar magnitude' is usually clear unless the basic flow is unsteady or nonlinearity is significant.

Liapounov definition may be unsatisfactory when the norm of the basic flow itself decreases or increases substantially in time. Then a time-dependent norm may have to be carefully chosen to represent what the experimentalist or observer means intuitively by stability. Other perturbations that might lead to instability arise from small changes in the boundary conditions due to irregularities in nature or imperfections of laboratory equipment. The mathematical treatment of these perturbations is closely related to that of a small initial disturbance of the basic flow.

Also, it must be recognized that an unstable basic flow free of any disturbance cannot instantaneously be set up in the laboratory or arise in nature. Rather a stable basic flow evolves in space or time until it becomes unstable, and the nature of the instability may be affected by the means of evolution.

The method of separation of variables and Laplace Transforms suggests that in general the solutions of the system can be expressed as the real parts of integrals of components, each component changing with time like e^{st} for some complex number $s = \sigma + i\omega$. **The linear system** will determine the values of s and the spatial variation of corresponding components as eigenvalues and eigenfunctions. If the basic flow has some simple symmetry, the linear system may be transformed with respect to some of the space variables as well as time. For example, **Poiseuille flow** has basic velocity and pressure respectively given by

$$U = V(1 - \frac{r^2}{a^2})i,$$

$$P = p_0 - \frac{4\rho\nu V x}{a^2} \quad for \quad 0 \le r \le a, 0 \le \theta \le 2\pi, -\infty < x < \infty$$

1.4 Hydrodynamic Stability

The essential problems of hydrodynamic stability were realized by Helmholtz, Reynolds, Kelvin and Rayleigh.Small disturbance may upset equilibrium of the external forces,inertia and viscous stresses. The tendency of fluid to move down pressure gradients may amplify disturbances of certain flows and thereby create instability.Viscosity has great stabilizing influence.

Thermal conductivity or molecular diffusion of heat has also some effects similar to those of viscosity or molecular diffusion of momentum and has usually a stabilizing influence.Centrifugal and Coriolis forces are regarded as external forces in the case of rotation of the whole system in which the fluid moves.

The problem of hydrodynamic instability originated in the differentiation between stable and unstable patterns of permissible flows. Analysis of dynamic instabilities dates back to the work of Helmholtz and Reynolds. Helmholtz (1890) has analyzed the stability of wave motion along surfaces of discontinuity assuming sharp changes in wind and density along the verticals and showed that the over-all surface is unstable under sufficiently large perturbations. He has also shown that a finite discontinuity in the wind will result in reduced stability. Later Rayleigh (1913) studied the stability.
Thus he formulated the result as follows: "Parallel flows of an inviscid fluid are stable if the velocity profile has no point of inflection." This is known as **Rayleigh's theorem.** The theorem gives a sufficient condition for stability for inviscid fluids. Later Tollmien (1936) showed that this condition is also sufficient for velocity distributions of certain types. A physical mechanism for interpreting this result was derived by Lin (1945), using an acceleration formula derived on the basis of von Karman's (1934) mechanism of vorticity redistribution. Rossby (1949) applied these ideas to the motion of polar air masses, fundamental in atmospheric process. A stronger form of Rayleigh's theorem was obtained later by Fjortoft (1950), who proved that for instability the value of vorticity of the primary flow must have a maximum in the domain of flow. This theorem also gives only a necessary condition for instability.

1.5 Nonlinear Stability

Nonlinear stability analysis is necessary when one investigate the development of secondary flows and the onset of higher instabilities. Reynold (1883) has appreciated the importance of nonlinear disturbances of Poiseuille flow in a pipe and Bhor (1909), Noether (1921) and Heisenberg (1951) treated them theoretically for special problems. The main concepts of the theory of nonlinear hydrodynamic stability are due to Landau (1944). Hopf (1948) has developed similar ideas on turbulence , through repeated bifurcation to solution representing flow. One of the specific methods is **the energy method,** which originated in the early work of Reynolds (1895) and Orr (1907). In the global theory of stability the energy methods have an important place. This method leads to a variational problem and a definite criterion for the stability of basic flow. In fact, any method based on a variational problem can be considered as energy method in generalized sense. This aspect of subject has been extensively studied by Serrin (1959) and a fuller account of this method till that date has been given by Joseph (1976). The significance of this method is that it provides rigorous criteria for stability with respect to arbitrary disturbances whereas the linear theory provides criteria for instability. At the end of the last century the celebrated Russian Mathematician Liapunov (1892) elaborated a general method for investigating stability :

$$\frac{dx_i}{dt} = f_i(t, x_1, ..., x_n)$$

Zubov (1957) and Movchan (1959) have generalized the method in order to apply to continuous systems, though it has been used for over sixty years to determine stability of system of ordinary differential equations before them. Pritchard (1968) has derived some criteria for the nonlinear stability of **Bernard convection and Couette flow** between rotating cylinders.

1.6 Definition of Terms

Viscosity
Is the property of the fluid by which it offers resistance to shear or angular deformation. The resistance to flow because of internal friction is called **viscous resistance.**

Free Vortex Flow
A vortex flow is characterized by a flow pattern where in the streamlines are curved.

Laminar and Turbulent Flows
A laminar flow is characterized by a smooth flow of one lamina of fluid over another. Fluid elements move in well-defined paths and they retain the same relative position at successive cross-sections of flow. The laminar flow is also called **the streamline or viscous flow.** This type of flow occurs in smooth pipes when the velocity of flow is low, and also in liquid having a high viscosity.
In turbulent, flow, the fluid elements move in erratic and unpredictable paths. Individual fluid particles are subjected to fluctuating transverse velocities so that the motion is eddying and sinuous rather than rectilinear. The random eddying motion is called **turbulence.**

Steady and Unsteady Flow
Motion of a fluid is said to be steady when the fluid parameters at any point in the flow field remain constant with respect to time; the parameters may, however, be different at different cross-sections of the flow passage. This means that quantities like velocity, pressure, temperature and density etc., are functions only of location and do not vary with time.
Flow is unsteady when conditions vary with respect to time; unsteadiness refers to changing flow pattern with the passage of time at a position in the flow.

Uniform and Non-Uniform Flow

Flow is uniform in character if the parameters like pressure, velocity, density, viscosity and temperature remain constant throughout the flow field at any given time. Flow is non-uniform if there is a change in the flow parameters from one section to another.

Compressible and Incompressible Flow

Flow is incompressible if the density changes due to pressure and temperature variations, are insignificant in the flow field. When the density changes are appreciable, the flow is called compressible.

One,Two and Three-Dimensional Flows

In one-dimensional flow, the fluid parameters (velocity, pressure, temperature and thus density and viscosity) remain constant throughout any cross-section normal to flow direction. In a two-dimensional flow, the flow velocity and other fluid parameters vary along two-directions. A three-dimensional flow stipulates that the flow properties vary in all the three directions; the stream lines are space curves.

2 Chapter Two Background of the Problem

Objectives

In this chapter, we illustrate the problem statement,the main and specific objectives.We also give the history of our problem under study.

Taylor-Couette is a flow between rotating concentric cylinders. More importantly, the flow instabilities that arises in the TCF and the related theoretical framework to describe these instabilities have provided valuable insight into the commonly used no-sip boundary condition, linear stability analysis, low dimension bifurcation phenomena, chaotic advection, absolute and convective instabilities and a host of other fundamental physical phenomenon and analytic methods. The flow is frequently studied because it is easy to produce in small closed systems, demonstrates a fundamental fluid flow phenomenon that can be mathematically predicted from basic principles and is simple and beautiful to observe.

It was discovered by R.A Mallock(1880) and M.M. Couette(1890).

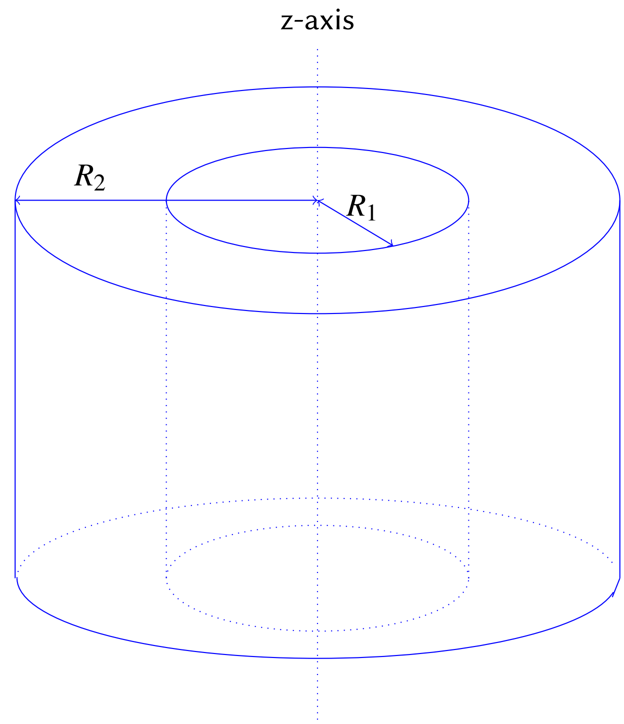
2.0.1 Problem Statement

Our main task is to determine the stability of the flow of a viscous incompressible fluid between two concentric rotating cylinders using **the Method of Normal Modes**. The equations describing the flow are the Navier-Stokes Equations **(NSE)** and the Continuity equation:

$$\begin{cases} \frac{\partial V}{\partial t} + \frac{1}{\rho}\nabla P = \nu \triangle V + F \\ \nabla . V = 0 \end{cases} \tag{1}$$

Subject to the boundary conditions:

$$v_r = v_z = 0, \quad v_\theta = \Omega_j R_j \quad \text{at } r = R_j, j = 1,2$$



2.0.2 Main Objective

To determine the stability of flow that exists between rotating concentric cylinders .

2.0.3 Specific Objectives

(a) To study and understand the flows between parallel plates and co-rotating cylinders.

(b) To understand, derive and apply the Navier-Stokes Equations (NSE).

(c) To understand hydrodynamic stability of The Taylor-Couette Flow.

2.1 Literature Review

The problem of hydrodynamic stability originated in the differentiation between stable and unstable patterns of permissible flows.Thermal conductivity or molecular diffusion of heat has also some effects similar to those of viscosity or molecular diffusion of momentum and has usually a stabilizing influence.Centrifugal and Coriolis forces are regarded as external forces in the case of rotation of the whole system in which the fluid moves.Magnetic field can inhibit the motion of an electrically conducting fluid across the magnetic lines of force and thereby stabilizes flows.Small disturbance may upset the equilibrium.The tendency of fluid to move down pressure gradients may amplify disturbances of certain flows and thereby create instability.Viscosity has great stabilizing influence.

The essential problems of hydrodynamic stability were formulated by Helmholtz, Reynolds, Kelvin and Reynolds in 19^{th} century.Analysis of dynamic instabilities dates back to the work of Helmholtz and Reynolds. Helmholtz (1890) has analysed the stability of wave motion along surfaces of discontinuity assuming sharp changes in wind and density along the verticals and showed that the over-all surface is unstable under sufficiently large perturbations. He has also shown that a finite discontinuity in the wind will result in reduced stability.Later Rayleigh (1913) studied the stability.Thus he formulated result as follows: *"Parallel flows of an inviscid fluid are stable if the velocity profile has no point of inflection."* This is known as Rayleigh's theorem.The theorem gives a sufficient condition for stability for inviscid fluids.

Later Tollmien (1936) showed that this condition is also sufficient for velocity distributions of certain types.A physical mechanism for interpreting this result was derived by Lin (1945), using an acceleration formula derived on the basis of Von Karman's (1934) mechanism of vorticity redistribution. Rossby (1949) applied these ideas to the motion of polar air masses, fundamental in atmospheric process. A stronger form of Rayleigh's theorem was obtained later by Fjortoft (1950), who proved that for instability the value of vorticity of the primary flow must have a maximum in the domain of flow. This theorem also gives only a necessary condition for instability.Some of the instabilities which arise from different causes are Taylor-Rayleigh instability,Helmholtz instability (Chandrasekhar-1961).
In Rayleigh-Taylor instability, the qualitative observations have been made by Lewis (1950) and others. The method has been applied by Pramod (1989) to study interfacial waves.

Mallock (1880) and Couette (1890) independently studied flow in two differentially rotating concentric cylinders, now known as a Taylor-Couette cell (Mallock 1880,Couette 1890).Couette rotated outer cylinder while he kept inner cylinder stationary/fixed, which is the basis for the modern viscometer, thus avoiding the vertical structure and obtaining an accurate measurement of viscosity of various fluids.Mallock performed similar experiments to Couette, but in addition he rotated inner cylinder and kept outer one stationary/fixed.He found anomalous results in this case because Taylor vortices occurred.In fact,Mallock's experiment prompted Lord Kelvin to write a letter to Lord Rayleigh in 1895 bringing the instability to his attention (Donnelly 1991).While Rayleigh's eventual analysis in 1916 explained the physical origin of the vertical structure.Taylor(1923)investigation became a key development in the modern study of fluid mechanics because velocity of a particle in contact with a wall moves at the same velocity as the wall moves at the same velocity as the wall.Although this concept has become a fundamental tenet for the study of fluid flow, it was questioned until Taylor used it with such success in his analysis of the stability of TCF.It offered convincing proof that the NSE indeed accurately describe the flow of a Newtonian fluid,not just at the base flow level, but at a level that permitted the analysis of secondary flows and instabilities.It was the first successful application of linear stability analysis.

3 Chapter 3
Equations of Motion

Objectives
In this chapter, we discuss and derive the equations of motion, The Navier-Stokes Equations **(NSE)**.

Since 2000,proving the Navier-Stokes smoothness and boundedness in $\mathbb{R}^3$ has been made one of the 7 millenium prize problems. in 1934, the French mathematician Jean Leray proved the existence of the so called **weak solutions** of NSE, satisfying equations in the mean value,not pointwise.In the 1960's , proof has been given about the smoothness and boundedness of the 2-dimensional Navier-Stokes Equations. Early in 2014, a Kazakh mathematician, Mukhtarbay Otelbayev, claimed he had solved this problem in 3-dimension,but very recently, Terence Tao has shown that Otelbayev's proof was wrong. Shortly after, Tao has published a paper proving that the 3-dimensional incompressible Navier-Stokes Equations admit solutions blowing up in a finite time.

3.1 Navier Stoke's Equations

Most of the incompressible fluid mechanics (dynamics) problems are described by simple Navier-Stokes Equations for incompressible fluid velocity, which can be written with a form

$$\frac{\partial U}{\partial t} = (-U\nabla)U - \nabla\varphi + \nu\nabla^2 U + g$$

where φ is defined as the relation of pressure to density:

$$\varphi = \frac{p}{\rho}$$

Satya [1966], studied exact solutions of NSE of viscous liquid motion in spherical polar coordinates (r, θ, ϕ) with axial symmetry, the line OZ $(i.e.\theta = 0)$ being the axis of symmetry in the annulus of a convergent tunnel bounded by two porous coaxial cones with variable suction and injection and the results were found to be in agreement with those discussed in Schlichting's book (Schlichling [1960]) and the solution discussed by Agarwal [1957].
The investigation of the axially symmetric flow of a viscous liquid through a convergent tunnel bounded by a porous wall $\theta = \alpha$ and $\theta = \beta(0 < \beta < \alpha < \frac{\pi}{2})$ between the sections $r = a$ and $r = b$, where $0 < b < a$ and a is finite since $r \neq 0$ and $\eta = cos\theta \neq \mp 1$. The conclusion was that the flow of viscous liquid with axial symmetry along a plane boundary

$(\theta = \frac{\pi}{2})$ which ejects liquid with velocity kr^2, and in which the velocity along symmetry is zero. Velocity, pressure distribution were

$$v_r = -k_1 r^2 cos\theta sin\theta$$

$$v_\theta = k_1 r^2 sin^3\theta$$

$$p = C - 4k_1 vrcos\theta$$

Gupta and Goyal [1970], studied **Plane Couette flow uniform suction** at stationary plate, on pressure, longitudinal and transverse velocity as independent of x and by introducing the non-dimensional quantities in such a way that the results of the plane Couette flow without suction can directly be obtained by taking λ equal to zero.
Sinha and Chaudhary [1965]; attempted to get exact solution for NSE for coaxial porous cylinder rotating with constant angular velocities. A solution was obtained under the assumption of uniform conditions along the axis of cylinders. The cylinder being porous, a hyperbolic radial velocity distribution has been superimposed over the circumferential velocity produced due to rotation. There is a Bernoulli type pressure variation in radial direction.If inner cylinder was at rest, shearing stress at it and the torque transmitted to it decreases as $\sigma(= \frac{v_0 y_1}{v})$ increases.Singh(2007) solved exact solution of NSE on Hydro magnetic by the application of Laplace transform and analytical expression was obtained. Further analysis showed the velocity profile decreases as the Hartmann number increases. He suggested that a similar approach can be used to solve some of the meteorological problems which involve differential equation and are difficult to solve directly by applying boundary conditions.
The equations of viscous incompressible fluid flow, called NSE named after Frenchman (Claude Louis Marie Henri Navier) and Englishman (George Gabriel Stokes) who proposed them in the early to mid$-$19th century, is:

$$\frac{Dq}{Dt} = F - \frac{1}{\rho}\nabla p + \frac{\mu}{\rho}\nabla^2 q$$

$$\nabla . q = 0$$

where ρ = fluid density (constant); $q \equiv (u_1, u_2, u_3)$ is the velocity vector which will often be written as $(u, v, w)^T$; p = air pressure; μ = viscosity, $F = Xi + Yj + Zk$ is the body force.

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}$$

= material derivative or substantial derivative expressing the Lagrangian, or total acceleration of the fluid particle; ∇^2 is the Laplacian, i.e.

$$\nabla^2 \equiv i\frac{\partial^2}{\partial x^2} + j\frac{\partial^2}{\partial y^2} + k\frac{\partial^2}{\partial z^2}$$

and $\nabla.$ is the divergence operator.
NSE in Cartesian is:

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}=X-\frac{1}{\rho}\frac{\partial p}{\partial x}+\frac{\mu}{\rho}\nabla^2 u$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}=Y-\frac{1}{\rho}\frac{\partial p}{\partial y}+\frac{\mu}{\rho}+\nabla^2 v$$

$$\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}=Z-\frac{1}{\rho}\frac{\partial p}{\partial z}+\frac{\mu}{\rho}\nabla^2 w$$

These equations have been widely accepted as an excellent model of the macroscopic motions of most real fluids, including air and water, and are used by countless engineers, physicists, chemists, mathematicians, meteorologists, oceanographers, geologists and biologists.
They don't actually tell us what the values of the variables are, they talk about the relationships between rates of change. So far we (Mathematicians) haven't been able to actually solve the Navier-stokes equations in a way that gives us a useful closed-form solution.
There, however, notable and useful models of fluids whose motions are not governed by the Navier-Stokes equations. For example there non-Newtonian fluids which are governed by a non-linear stress tensor, and visco-elastic fluids in which the stress depends on the strain as well as on the rate of strain of the fluid and retains a *'memory'* of previous deformation; Lloyd[1981]. An exact solution may seem to be more or less than a solution, because either a given set of fields q, p for given ρ, μ and body force F satisfies the governing equations or it does not. By exact solution we mean a solution which has a simple explicit form, usually an expression in finite terms of elementary or other well-known special functions. Sometimes an exact solution is taken to be one which can be reduced to a solution of ordinary differential equations. Barely we go even further, and take an exact solution to be the solution of a partial differential equation, provided that the equation has fewer independent variables than the Navier-Stokes equations themselves. This is in contrast with to an approximate solution which is taken to be a field, simple or complicated, which approximates a solution either in numerical sense or asymptotic limit, for example vanishingly small viscosity thus the logical distinctions between solutions are blurred, but in practice the distinctions made are usually clear and useful.
The exact solutions are, essentially, a subset of solutions of NSE which happen to have relatively simple mathematical expressions and which are, mostly simple physically. The essence of this account, then, is the explicitness and relatively simplicity of the expression of the solutions. Many exact solutions of NSE are unstable therefore unobservable in practice. In the early decades of development of mathematical theory of motion of fluid which is viscous, exact solutions was the only solutions available. Researchers solved what problems they could, rather than solving the practical problems in hand. Inevitably

the solvable problems were the simple ones, usually idealized with a strong symmetry. From the mid-nineteenth century, and early twentieth century, asymptotic method were developed, and thereafter numerical method. Nevertheless, the exact solution remain a valuable and irreplaceable resource. The immediately convey more physical insight than a numerical table.

3.1.1 Systems of Coordinates for Navier-Stokes Equations

NSE for a fluid which is incompressible occurs in three most common co-ordinates systems, notably, the Cartesian co-ordinates, the cylindrical polar co-ordinates and Spherical polar co-ordinates.

(i) **Cartesian co-ordinate:**

In Cartesian co-ordinates (x,y,z), $q = ui + vj + wk$ is the velocity. The Navier-Stokes Equations and Continuity equations are:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + X + \nu\nabla^2 u$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + Y + \nu\nabla^2 v$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + Z + \nu\nabla^2 w$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

where $F = (X,Y,Z), \nabla^2$represent the three dimensional Laplacian operator.

(ii) **Cylindrical Polar Co-ordinate:**

We define cylindrical polar co-ordinates (r,θ,z) such that

$$x = rcos\theta,$$

$$y = rsin\theta,$$

$$r \geq 0, 0 \leq \theta \leq 2\pi,$$

with corresponding velocity components $v = (v_r, v_\theta, v_z) = v_r r + v_\theta \theta + v_z z$,vorticity and body force components $\omega = (\omega_r, \omega_\theta, \omega_z), F = (v_r, v_\theta, v_z)$ respectively. The components of Navier-Stokes equations are, then,

$$\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} + v_z \frac{\partial v_r}{\partial z} - \frac{v_\theta^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + F_r + \nu(\nabla^2 v_r - \frac{v_r}{r^2} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}),$$

$$\frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial \theta} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + v_z \frac{\partial v_\theta}{\partial z} + \frac{v_r v_\theta}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial \theta} + F_\theta + \nu(\nabla^2 v_\theta + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta}{r^2}),$$

$$\frac{\partial v_z}{\partial t} + v_r \frac{\partial v_z}{\partial \theta} + \frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta} + v_z \frac{\partial v_z}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + F_z + \nu \nabla^2 v_z,$$

with continuity equation

$$\frac{1}{r}\frac{\partial}{\partial r}(r v_r) + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z} = 0.$$

The components of vorticity are given by

$$\omega_r = \frac{1}{r}\frac{\partial v_r}{\partial \theta} - \frac{\partial v_\theta}{\partial z}$$

$$\omega_\theta = \frac{\partial v_r}{\partial z} - \frac{\partial v_r}{\partial r}$$

$$\omega_z = \frac{1}{r}\frac{\partial}{\partial r}(r v_\theta) - \frac{1}{r}\frac{\partial v_\theta}{\partial \theta}.$$

For a rotationally symmetric flow, independent of θ, we introduce a different stream function ψ such that with

$$v_r = -\frac{1}{r}\frac{\partial \psi}{\partial z}$$

$$v_z = \frac{1}{r}\frac{\partial \psi}{\partial r}$$

The continuity equation is satisfied identically.

(iii) **Spherical Polar Co-ordinates:**

We define spherical polar co-ordinates (r, θ, ϕ) such that

$$x = r sin\theta cos\phi,$$

$$y = r sin\theta sin\phi$$

$$z = rcos\theta$$
$$r \geq 0, 0 \leq \theta \leq 2\pi, 0 \leq \phi \leq 2\pi.$$

With corresponding velocity components $v = (v_r, v_\theta, v_\phi) = v_r r + v_\theta \theta + v_\phi \phi$, vorticity and body-force components $\omega = (\omega_r, \omega_\theta, \omega_\phi), F = (F_r, F_\theta, F_\phi)$ respectively. The components of Navier-Stokes equations are, then,

$\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} + \frac{v_\phi}{rsin\theta}\frac{\partial v_r}{\partial \phi} - \frac{v_\theta^2 + v_\phi^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + F_r + \nu(\nabla^2 v_r - \frac{2v_r}{r^2} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta} - \frac{2v_\theta cot\theta}{r^2} - \frac{2}{r^2 sin\theta}\frac{\partial v_\phi}{\partial \phi})$

$\frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_\phi}{rsin\theta}\frac{\partial v_\theta}{\partial \phi} + \frac{v_r v_\theta}{r} - \frac{v_\phi^2 cot\theta}{r} = -\frac{1}{\rho}\frac{1}{r}\frac{\partial p}{\partial \theta} + F_\theta + \nu(\nabla^2 v_\theta + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta}{r^2 sin^2\theta} - \frac{2}{r^2}\frac{cos\theta}{sin^2\theta}\frac{\partial v_\phi}{\partial \phi})$

$\frac{\partial v_\phi}{\partial t} + v_r \frac{\partial v_\phi}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\phi}{\partial \theta} + \frac{v_\phi}{rsin\theta}\frac{\partial v_\phi}{\partial \phi} + \frac{v_\phi v_r}{r} + \frac{v_\theta v_\phi}{r} = -\frac{1}{\rho}\frac{1}{rsin\theta}\frac{\partial p}{\partial \phi} + F_\phi + \nu(\nabla^2 v_\phi - \frac{v_\phi}{r^2 sin^2\theta} + \frac{2}{r^2 sin\theta}\frac{\partial v_r}{\partial \phi} + \frac{2}{r^2}\frac{cos\theta}{sin^2\theta}\frac{\partial v_\theta}{\partial \phi}),$

with continuity equation

$$\omega_r = \frac{1\omega}{rsin\theta}(\frac{\partial}{\partial \theta}(v_\theta sin\theta) - \frac{\partial v_\theta}{\partial \phi})$$
$$\omega_\theta = \frac{1}{rsin\theta}\frac{\partial v_r}{\partial \phi} - \frac{1}{r}\frac{\partial}{\partial r}(rv_\theta)$$
$$\omega_\phi = \frac{1}{r}\frac{\partial}{\partial r}(rv_\theta) - \frac{1}{r}\frac{\partial v_r}{\partial \theta}$$

The Stokes stream function, for a rotationally symmetric flow independent of ϕ, is now defined such that

$$v_r = \frac{1}{r^2 sin\theta}\frac{\partial \psi}{\partial \theta},$$
$$v_\theta = \frac{1}{rsin\theta}\frac{\partial \psi}{\partial r}.$$

3.1.2 Derivation of Navier Stoke's Equation

Let there be a viscous fluid occupying a certain region.In this region let V be the volume enclosed by a surface S that moves with the fluid and so contains the same fluid particles at all times.Let dV be the volume element and dS be the surface element surrounding the fluid particle P of density ρ. The mass ρdV of this particle remains constant throughout. Then if q is the velocity then the momentum M of the particle is given by

$$M = \int\int\int_V \rho q dV$$

where the integral has been carried out over the entire volume V.
Let p the normal pressure force which has an outward unit normal n.
The surface force due to pressure p is therefore

$$-\int\int_A pndS = -\int\int\int_V \nabla pdV$$

[By Gauss Divergence Theorem.]
The frictional force is

$$\int\int\int_V \mu\nabla^2 qdV$$

Again let F be the external force acting to fluid, so that the total force in space S at any time is

$$\int\int\int_V F\rho dV$$

Thus the total force (of Euler's equations for perfect fluid) will be

$$\int\int\int_V F\rho dV - \int\int\int_V \nabla pdV + \int\int\int_V \mu\nabla^2 qdV$$

By Newton's 2^{nd} law, we have

$$\frac{DM}{Dt} = \int\int\int_V (\rho F - \nabla p + \mu\nabla^2 q)dV$$

$$\int\int\int_V \frac{Dq}{Dt}\rho dV + \int\int\int_V q\frac{D}{Dt}(\rho dV) = \int\int\int_V (\rho F - \nabla p + \mu\nabla^2 q)dV$$

$\frac{D}{Dt}(\rho dV)$ being zero since ρdV is constant.
Now since volume V can be taken as arbitrary volume of the fluid in the region considered;

$$\rho\frac{Dq}{Dt} = \rho F - \nabla p + \mu\nabla^2 q$$

Or

$$\frac{Dq}{Dt} = F - \frac{1}{\rho}\nabla p + \nu\nabla^2 q$$

where $\nu = \frac{\mu}{\rho}$ is taken to be the kinematic of viscosity.
These are NSE in vector form.
If $q = ui + vj + wk$ and $F = Xi + Yj + Zk$, then the Navier-Stokes Equations take the form:

$$\frac{D}{Dt}[ui + vj + wk] = (Xi + Yj + Zk) - \frac{1}{\rho}[i\frac{\partial p}{\partial x} + j\frac{\partial p}{\partial y} + k\frac{\partial p}{\partial z}] + \nu[i\nabla^2 u + j\nabla^2 v + k\nabla^2 w]$$

So that

$$\frac{Du}{Dt} = X - \frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\nabla^2 u$$
$$\frac{Dv}{Dt} = Y - \frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\nabla^2 v$$
$$\frac{Dw}{Dt} = Z - \frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\nabla^2 w$$

These are NSE Cartesian coordinates.
Using the definition of material derivatives;

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}$$

above equations maybe written as:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = X - \frac{\partial p}{\partial x} + \nu\nabla^2 u$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = Y - \frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\nabla^2 v$$
$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = Z - \frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\nabla^2 w$$

These are the Navier-Stokes Equations in a simplified Cartesian form.

3.1.3 Importance of Terms Related to Navier-Stoke's Equation

(i) **Body Force terms** F
Body force because of gravity is important in flow problems in which free liquid surface exists or when the fluid is non-homogeneous, i.e. its density changes from one point to another so that there exists a density gradient. If a fluid is rotating about an axis, body force due to centripetal action must be considered. In case of homogeneous fluid flow within closed boundaries, there is an equilibrium between the weight of a fluid and the buoyant force acting on it. In such a case, body force due to gravity does not influence the fluid motion and hence can be neglected from Navier-Stokes Equations.

(ii) **Viscous Terms** $[\nu\nabla^2 u, \textbf{etc.}]$
The no-slip condition between the fluid and the solid boundary requires that the fluid velocity must be equal to that of boundary (i.e. zero for a stationary boundary). In other words, both normal and tangential velocity components must be zero. Two

independent boundary conditions must, therefore, be fulfilled, which require a partial differential equation (PDE) of second order. For that, it is not permissible to ignore the viscous terms in the PDE, even for very small values of v, if true behaviors of the viscous fluid is to be determined in the vicinity of the boundary.

(iii) **Pressure Terms** $[(\frac{\partial P}{\partial x}),$ **etc.**$]$
The pressure gradient terms is incorporated in Navier-Stokes Equations to show the pressure distribution across a fluid flow, subjected to different boundary terms.

(iv) **Inertia Terms** $[(\frac{\partial u}{\partial x}),$ **etc.**$]$
For high Reynolds number flow, inertia terms dominate over the viscous terms and hence, the viscous terms can be neglected to lead a fair approximation. However, in very low Reynolds number flow (known as creep flow), the velocity components are very small and higher order inertia terms can be neglected, which converts the Navier-Stokes Equations into a linear PDE, which is much easier to solve.

3.2 Limiting cases of the Navier Stoke's Equations

(i) **Potential Flow Case**
In potential flow, viscous forces tend to zero. For incompressible flow,

$$u = \frac{\partial \varnothing}{\partial x}$$

$$v = \frac{\partial \varnothing}{\partial y}$$

$$w = \frac{\partial \varnothing}{\partial z}$$

where $\varnothing(x,y,z)$ =velocity potential function.
Using the equation of continuity for incompressible flow,

$$\nabla . q = 0$$

and since $q = ui + vj + wk$, then the Navier-Stokes Equations of motion reduces to the Euler's equations of motion:

$$\nabla . q = (i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}).(i\frac{\partial \varnothing}{\partial x} + j\frac{\partial \varnothing}{\partial y} + k\frac{\partial \varnothing}{\partial z}) = \frac{\partial^2 \varnothing}{\partial x^2} + \frac{\partial^2 \varnothing}{\partial y^2}\frac{\partial^2 \varnothing}{\partial z^2} = 0$$

$$\Longrightarrow \nabla^2 \varnothing = 0$$

Viscous components of NSE are

$$v\nabla^2 q = 0.$$

As in case of incompressible flow, viscous components in NSE are zero and become viscous term independent, i.e.

$$\frac{Dq}{Dt} = F - \frac{1}{\rho}\nabla p$$

This is *the Euler's equation of motion.*

(ii) **Creep Flow Case**
Creep flow occurs at very low Reynolds numbers. At these Reynolds numbers (i.e., at very small velocity, small linear dimensions of the body or of the flow passage and large viscosity of fluid), the inertia forces are much smaller than the viscous forces. We know that for steady flow, inertia force= $\rho L^2 V^2$ and viscous force= $\mu A(\frac{\partial u}{\partial x})$.
As for creep flow, q is very small, hence, we neglect higher order terms of the inertia force, like $u(\frac{\partial u}{\partial x}, v(\frac{\partial v}{\partial y}, etc)$. Then

$$\frac{Dq}{Dt} = F - \frac{1}{\rho}\nabla p + \frac{\mu}{\rho}\nabla^2 q$$

becomes;

$$(i\frac{\partial u}{\partial t} + j\frac{\partial v}{\partial t} + k\frac{\partial w}{\partial t}) = F - \frac{1}{\rho}\nabla p + \nu\nabla^2 q$$

$$\Longrightarrow \frac{\partial q}{\partial t} = F - \frac{1}{\rho}\nabla p + \nu\nabla^2 q$$

Creep flow analysis is of potential importance for laminar flow in pipes and open channels, for seepage flow of water and oil underground, for motion of very small bodies such bodies such as spheres in highly viscous fluid and in the theory of lubrication.

3.3 Exact Solutions for Navier-Stoke's Equations

Basic difficulty in solving Navier-Stokes equations arises due to presence of nonlinear (quadratic) inertia terms [i.e $u\frac{\partial u}{\partial x}, v\frac{\partial v}{\partial y},$ etc] on the L.H.S of the Navier-Stokes equations in a simplified Cartesian form. However, there are some trivial solutions to NSE in which non-linear inertia terms are usually zero, Akshoy [2005]. One such flow is parallel flow, in which only one velocity term is trivial and all the fluid particles move in one direction only.

3.3.1 Steady Flow Between Parallel Planes

For a viscous incompressible fluid in steady flow, the Navier-Stokes equation with negligible body forces, are

$$\frac{dq}{dt} = \frac{-\nabla p}{\rho} + \frac{\mu}{\rho}\nabla^2 q = \frac{-\nabla p}{\rho} + \nu\nabla^2 q, \nu = \frac{\mu}{\rho}$$

In Cartesian coordinates; these are

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2})$$

This is because for steady case,

$$\frac{du}{dt} = \frac{\partial u}{\partial t} + (q.\nabla)u = (q.\nabla)u, \frac{\partial}{\partial t} = 0$$

$$= (u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z})u$$

$$= (u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z})$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2})$$

$$u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2})$$

The equation of continuity for incompressible flow is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

since

$$\nabla.q = 0$$

We have the conditions

$$v = 0, w = 0, \frac{\partial}{\partial z} \equiv 0$$

For the continuity equation $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$, we have

$$\frac{\partial u}{\partial x} = 0 \Rightarrow u = u(y)$$

The second equation of the NSE in Cartesian coordinates gives

$$-\frac{\partial p}{\partial y} = 0$$

$$\Rightarrow p = p(x)$$

Third equation in the NSE in Cartesian coordinates is identically satisfied and the NSE in Cartesian coordinates gives

$$0 = -\frac{1}{\rho}\frac{dp}{dx} + \nu\frac{d^2u}{dy^2} \Rightarrow \frac{dp}{dx} = \mu\frac{d^2u}{dy^2}$$

since

$$\frac{\mu}{\rho} = \nu$$

Since u is a function of y only, so $\frac{dp}{dx}$ is either a function of y or a constant. But from

$$-\frac{\partial p}{\partial y} = 0$$

$$\Rightarrow p = p(x),$$

p is in terms of x alone. Hence

$$\frac{dp}{dx}$$

is constant. i.e. pressure gradient is constant.
Integrating $0 = -\frac{1}{\rho}\frac{dp}{dx} + \nu\frac{d^2u}{dy^2} \Rightarrow \frac{dp}{dx} = \mu\frac{d^2u}{dy^2}$ with respect to y twice, we get the general solution to be

$$u = \frac{1}{\mu}\frac{dp}{dx}\frac{y^2}{2} + Ay + B$$

Now we take the following particular cases:

(i) Couette's Flow

It is the flow between planes which are parallel (flat plates) one at rest and other in motion with velocity U parallel to the fixed plate. Here, the constants A and B in $u = \frac{1}{\mu}\frac{dp}{dx}\frac{y^2}{2} + Ay + B$ are determined from the rules

$$u = 0, y = 0,$$

$$u = U, y = h$$

Using these conditions, we get

$$B = 0, U = \frac{1}{\mu}(\frac{dp}{dx}\frac{h^2}{2} + Ah)$$

$$\Rightarrow A = \frac{U}{h} - \frac{h}{2\mu}(\frac{dp}{dx}), B = 0$$

Therefore, the solution $u = \frac{1}{\mu}\frac{dp}{dx}\frac{y^2}{2} + Ay + B$ becomes

$$u = \frac{1}{\mu}(\frac{dp}{dx})\frac{y^2}{2} + y[\frac{U}{h} - \frac{h}{2\mu}(\frac{dp}{dx})]$$

$$= \frac{y^2 - hy}{2\mu}(\frac{dp}{dx}) + \frac{Uy}{h}$$

$$\frac{U}{h}y - \frac{h^2}{2\mu}\frac{dp}{dx}\frac{y}{h}(1 - \frac{y}{h})$$

We note that this represents a parabolic curve. This equation is known as *the equation of Couette's flow.* Thus the velocity profile for Couette's flow is parabolic.The flow Q per unit breadth is given by

$$Q = \int_0^h u dy = \int_0^h [\frac{1}{m}\frac{dp}{dx}\frac{y^2}{2} + y(\frac{U}{h} - \frac{h}{2\mu}\frac{dp}{dx})]dy$$

$$= \frac{hU}{2} - \frac{h^3}{12\mu}\frac{dp}{dx}$$

$$= \frac{hU}{2} + \frac{h^3}{12\mu}P, P = -\frac{dp}{dx}$$

In non-dimensional form,

$$\frac{U}{h}y - \frac{h^2}{2\mu}\frac{dp}{dx}\frac{y}{h}(1 - \frac{y}{h})$$

will be

$$\frac{u}{U} = \frac{y}{h} + \alpha\frac{y}{h}(1 - \frac{y}{h})$$

where

$$\alpha = \frac{h^2}{2\mu U}(-\frac{dp}{dx})$$

α is the non-dimensional pressure gradient. If $\alpha > 0$, the pressure is decreasing in the direction of flow and the velocity is positive between the plates. If $\alpha < 0, \frac{u}{U} = \frac{y}{h} + \alpha\frac{y}{h}(1 - \frac{y}{h})$ can be put as

$$\frac{u}{U} = \frac{y}{h}(1+\alpha) - \frac{\alpha y^2}{h^2}$$

The pressure is increasing as flow and the reverse of flow begins when $\alpha < -1$
y^2 is neglected because y is small i.e.,
If $\alpha = 0 (i.e. \frac{dp}{dx} = 0)$, then the particular case is called *simple Couette's flow*, the velocity is

$$\frac{u}{U} = \frac{y}{h}$$

which gives $u = 0$ where $y = 0$ i.e. on the stationary plane.

Average and Extreme Values of Velocity

The average velocity of a Couette's flow between plates which are parallel is

$$u_0 = \frac{1}{h}\int_0^h u dy$$

because $u = u(y)$
Using the value of u from $\frac{u}{U} = \frac{y}{h} + \alpha\frac{y}{h}(1 - \frac{y}{h})$, we get

$$u_0 = \frac{1}{h}\int_0^h [\frac{Uy}{h} + U\alpha\frac{y}{h}(1 - \frac{y}{h})]dy$$

$$= \frac{Uy^2}{2h^2} + U\alpha(\frac{y^2}{2h^2} - \frac{y^3}{3h^3})$$

$$= \frac{U}{2} + \frac{U\alpha}{6} = (\frac{1}{2} + \frac{\alpha}{6})U$$

$$= \frac{U}{2} - \frac{\mu^2}{12\mu}\frac{dp}{dx} = \frac{U}{2} + \frac{h^2}{12\mu}P, P = -\frac{dp}{dx}$$

For simple Couette's flow, velocity goes up from zero on stationary plate up to U on the moving plate such that the average velocity is $\frac{U}{2}$. When the non-dimensional pressure gradient is $\alpha = -3$, then from $= \frac{U}{2} + \frac{U\alpha}{6} = (\frac{1}{2} + \frac{\alpha}{6})U$, we get u_0. This means that there is no flow because pressure gradient is equalized with viscous force.
For maximum and minimum values of u, we have

$$\frac{du}{dy} = 0 \Rightarrow \frac{U}{h} + U\alpha(\frac{1}{h} - \frac{2y}{h^2}) = 0$$

$$\Rightarrow y = (\frac{1+\alpha}{2\alpha})h$$

From here,

$$\frac{y}{h} = 1 \quad \text{when} \alpha = 1$$

and

$$\frac{y}{h} = 0 \quad \text{when} \alpha = -1$$

so from $\frac{u}{U} = \frac{y}{h} + \alpha\frac{y}{h}(1 - \frac{y}{h})$, we get

$$u = [\frac{1+\alpha}{2\alpha} + \alpha(\frac{1+\alpha}{2\alpha}(1 - \frac{1+\alpha}{2\alpha})]U$$

$$= \frac{(1+\alpha)^2}{4\alpha}U$$

and thus u is maximum for $\alpha \geq 1$ and minimum for $\alpha \leq -1$

(ii) On Plane-Poiseuille Flow

A flow between stationary plates which are parallel; a Plane Poiseuille Flow.
The origin is taken on the line midway between the plates which are placed at a distance h and x-axis is along this line.
The conditions to be used in this problem are $u = 0$ when $y = \pm\frac{h}{2}$
Using these conditions in $u = \frac{1}{\mu}\frac{dp}{dx}\frac{y^2}{2} + Ay + B$, we get

$$A = 0, B = \frac{1}{\mu}(-\frac{dp}{dx})\frac{h^2}{8}$$

and thus the solution $u = \frac{1}{\mu}\frac{dp}{dx}\frac{y^2}{2} + Ay + B$ is

$$u = \frac{1}{\mu}(\frac{dp}{dx})(\frac{y^2}{2} - \frac{h^2}{8})$$

This represents a parabola and thus the laminar flow in a Plane Poiseuille Flow is parabolic.

Average and Maximum velocity

For extreme values of u, we have $\frac{du}{dy} = 0$ and thus from $u = \frac{1}{\mu}(\frac{dp}{dx})(\frac{y^2}{2} - \frac{h^2}{8})$, we get

$$\frac{1}{\mu}(\frac{dp}{dx}y = 0 \quad \Rightarrow y = 0)$$

Therefore,

$$U_{max} = \frac{h^2}{8\mu}(-\frac{dp}{dx})$$

The average velocity in the plane Poiseuille flow is defined by

$$u_0 = \frac{1}{h}\int_{-h/2}^{h/2} u dy$$

Using the value of u from $u = \frac{1}{\mu}(\frac{dp}{dx})(\frac{y^2}{2} - \frac{h^2}{8})$, we get

$$u_0 = \frac{1}{h}\int_{-h/2}^{h/2} \frac{-h^2}{8\mu}\frac{dp}{dx}(1 - \frac{4y^2}{h^2})dy$$

$$\frac{2}{3}(\frac{-h^2}{8\mu}\frac{dp}{dx}) = \frac{2}{3}U_{max}$$

From $U_{max} = \frac{h^2}{8\mu}(-\frac{dp}{dx})$ and $\frac{2}{3}(\frac{-h^2}{8\mu}\frac{dp}{dx}) = \frac{2}{3}U_{max}$, decrease in the pressure is given by

$$\frac{dp}{dx} = -\frac{8\mu}{h^2}U_{max} = \frac{-8\mu}{h^2}\frac{3}{2}u_0 = \frac{-12\mu}{h^2}u_0$$

This further shows that $\frac{dp}{dx}$ is a negative constant.

4 Taylor-Couette Flow

Objectives
In this chapter, we fully discuss about the Taylor-Couette Flow.

Taylor-Couette Flow is a canonical flow geometry that has long been a subject of interest in the fluid mechanics community. The study of Taylor-Couette Flow is useful in many research and industrial application such as water purification and desalination (Dutta and Ray,2004; Sengupta etc. et al. 2001, Wereley and Leuptow, 1998) and bisectors (Bo and Vigil,2013; Curran and Black,2005; Haut etc et al. 2003)

4.1 The Background of Taylor-Couette Problem

Many researchers have made significant contributions to the understanding of flow transition and turbulence by studying this canonical flow system (Taylor,1923; Davey,1962; Coles,1965; Gollub, & Freilich,1976;Walden & Donnelly,1979; Gorman and Swinney,1982;Wereley & Leuptow,1994; Smith & Matsoukas, 1998.) Because of its interest in the fluid mechanics community, Taylor-Couette Flow has been studied over a 100 years. The study of Taylor-Couette Flow began as early as the 1890 and still continues on today due to its importance to various areas of fundamental and applied research.Before the development of laser-based measurement techniques, the flow was studied by visual observations such as Couette 1890 described a series of experiments in which he measured the viscosities of water and air using the concentric cylinder apparatus of his own design, Taylor(1923) used ink visualization and presented for the first time measurements of patterns in the unstable flow, and Taylor (1936) reported the series of measurements of the torque on the cylinder of Couette flow apparatus due to the rotation of the other for a variety of radius ratios and Reynolds numbers, and by intrusive electrical measurements such as Taylor (1936) described the velocity profile of Taylor-Couette flow observed and measured by pitot tube; outer cylinder rotated while inner cylinder was fixed, Wendt(1933) reported measurements of velocity and pressure distributions inside the gap between the inner and outer cylinders of Taylor-Couette flow apparatus, Bagnold (1954) designed a Couette rheometer to measure the shear and normal stresses, Hollis-Hallett and W.J. Heikkila (1955) adapted the Hollis-Halliet's viscometer to study the Couette-Taylor flow at very low Reynolds number, and Coles (1965) used hot-wire measurement in air to investigate the changes in Couette-Taylor flow and showed patterns with alternating laminar & turbulent flow. Development for laser Doppler velocimetry was a big advantage in the study of Taylor-Couette flow as it allowed for the non-intrusive measurement of velocity.

4.2 Flow Description

A simple Taylor-Couette flow is a flow which is steady and created between rotating infinitely long cylinders which are coaxial. Since the cylinder lengths are infinitely long, the flow is essentially unidirectional in steady state. If inner cylinder with radius R_1 rotate at constant angular velocity Ω_1 & outer cylinder with radius R_2 rotates at constant angular velocity Ω_2, then azimuthal velocity component is given by

$$v_\theta = Ar + \frac{B}{r}$$

Proof

We take flow between cylinders which are concentric and rotating with radii $r_1, r_2 (r_2 > r_1)$ having viscous fluid between them. We assume that the flow is circular such that only the tangential component of velocity exists. Let Ω_1 & Ω_2 be angular velocity of inner & outer cylinders, respectively. The continuity equation in cylindrical coordinates (r, θ, z) reduces to

$$\frac{\partial q_\theta}{\partial \theta} = 0 \Longrightarrow q_\theta = q_\theta(r)$$

where $q_r = q_z = 0$

Now, the NSE of viscous incompressible fluid in cylindrical coordinates:

$$\rho(\frac{dq_r}{dt} - \frac{q_\theta^2}{r}) = \rho X_r - \frac{\partial p}{\partial r} + \mu(\nabla^2 q_r - \frac{q_r}{r^2} - \frac{2}{r^2}\frac{\partial q_\theta}{\partial \theta})$$

$$\rho(\frac{dq_\theta}{dt} + \frac{q_r q_\theta}{r}) = \rho X_\theta - \frac{1}{r}\frac{\partial p}{\partial \theta} + \mu(\nabla^2 q_\theta + \frac{2}{r^2} - \frac{2}{r^2}\frac{\partial q_r}{\partial \theta} - \frac{q_\theta}{r^2})$$

$$\rho\frac{dq_r}{dt} = \rho X_z - \frac{\partial p}{\partial z} + \mu\nabla^2 q_z$$

Here,

$$q_r = q_z = 0; (X_r, X_\theta, X_z) = 0, q_\theta = q_\theta(r)$$

From the last two equations, we have:

$$\frac{\partial p}{\partial z} = 0, -\frac{1}{r}\frac{\partial p}{\partial \theta} + \mu(\nabla^2 q_\theta - \frac{q_\theta}{r^2} = 0) \quad (2)$$

and the first equation gives

$$\rho\frac{q_\theta^2}{r} = \frac{\partial p}{\partial r}$$

The LHS of this equation is in terms of r and thus p is a component of r only. i.e.

$$\frac{\partial p}{\partial \theta} = 0$$

Therefore equation (2) reduces to

$$\nabla^2 q_\theta - \frac{q_\theta}{r^2} = 0$$

where

$$\nabla^2 \equiv \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}$$

$$\Longrightarrow \frac{d^2 q_\theta}{dr^2} + \frac{1}{r}\frac{dq_\theta}{dr} - \frac{q_\theta}{r^2} = 0$$

$$\Longrightarrow \frac{d^2 q_\theta}{dr^2} + \frac{d}{dr}(\frac{q_\theta}{r}) = 0$$

Integrating, we get

$$\frac{dq_\theta}{dr} + \frac{q_\theta}{r} = 2A$$

$$\Rightarrow r\frac{dq_\theta}{dr} + q_\theta = 2Ar$$

$$\frac{d}{dr}(rq_\theta) = 2Ar$$

Integrating, we get

$$rq_\theta = Ar^2 + B$$

$$\Longrightarrow q_\theta = Ar + \frac{B}{r} \quad \textbf{(3)}$$

Or

$$v_\theta = Ar + \frac{B}{r}$$

Hence proved.

The boundary conditions are
$q_\theta = r_1\Omega_1$, when $r = r_1$
and
$q_\theta = r_2\Omega_2$, when $r = r_2$
There on surface

$$v = r\frac{d\theta}{dt} \Rightarrow v = r\Omega$$

$$l = r\theta \Rightarrow \frac{dl}{dt} = r\frac{dl}{dt} \Rightarrow v = r\Omega$$

Using these equations in $q_\theta = Ar + \frac{B}{r}$, we obtain

$$A = \frac{\Omega_1 r_1^2 - \Omega_2 r_2^2}{r_1^2 - r_2^2} \quad \textbf{(4)}$$

$$B = \frac{r_1^2 r_2^2(\Omega_1 - \Omega_2)}{r_2^2 - r_1^2} \quad \textbf{(5)}$$

Thus the solution of (3) in the present case is

$$q_\theta = \frac{1}{r_2^2 - r_1^2}[(r_2^2\Omega_2 - r_1^2\Omega_1)r - \frac{r_1^2 r_2^2(\Omega_2 - \Omega_1)}{r}]$$

In particular, if the inner cylinder is at rest, i.e.

$$\Omega_1 = 0, \Omega_2 = \Omega(say), r_1 = a, r_2 = b,$$

then the solution becomes

$$q_\theta = \frac{\Omega b^2}{b^2 - a^2}(r - \frac{a^2}{r})$$

The radial pressure, given by $\rho\frac{q_\theta^2}{r} = \frac{\partial p}{\partial r}$, is

$$\frac{dp}{dr} = \rho\frac{q_\theta^2}{r} = \frac{\rho}{r}(A_1^2 r^2 + \frac{B_2^2}{r^2} + 2AB)$$

$$= \rho(A^2 r + \frac{B^2}{r^3} + \frac{2AB}{r})$$

Integrating with respect to r, we get

$$p = \rho[\frac{A^2 r^2}{2} - \frac{B^2}{2r^2} + 2AB\log r] + C$$

If $p = p_1$ when $r = r_1$, then

$$p_1 = \rho[\frac{A^2 r_1^2}{2} - \frac{B^2}{2r_1^2} + 2AB\log r_1] + C$$

$$\Rightarrow C = p_1 - \rho[\frac{A^2 r_1^2}{2} - \frac{B^2}{2r_1^2} + 2AB\log r_1]$$

Hence the pressure is given by

$$p = p_1 - \rho[A^2(\frac{r^2 - r_1^2}{2}) - \frac{B^2}{2}(\frac{1}{r^2} - \frac{1}{r_1^2}) + 2AB\log\frac{r}{r_1}]$$

where A and B are as given by (4) and (5). The formula for shearing stress is

$$\sigma_{r\theta} = \mu[\frac{dq_\theta}{dr} - \frac{q_\theta}{r}] = \mu[r\frac{d}{dr}(\frac{q_\theta}{r})]$$

$$= \mu[r\frac{d}{dr}(\frac{Ar + B/r}{r})]$$

$$= \mu . r \frac{d}{dr}(A + B/r) = \mu(-\frac{2B}{r^3})$$

$$= -\frac{2\mu B}{r^2} = \frac{-2\mu r_1^2 r_2^2(\Omega_1 - \Omega_2)}{r^2(r_2^2 - r_1^2)}$$

The expressions for shearing stress on the outer and the inner cylinder are

$$(\sigma_{r\theta})_r = r_2 = \frac{2\mu(\Omega_2 - \Omega_1)r_1^2}{r_2^2 - r_1^2}$$

$$(\sigma_{r\theta})_r = r_1 = \frac{2\mu(\Omega_2 - \Omega_1)r_2^2}{r_2 - r_1^2}$$

Also A and B are constants related to the boundary conditions set by the rotation speeds of each cylinder Ω_1 and Ω_2 and may be expressed as:

$$A = \frac{\Omega_1(\eta^2 - \mu)}{1 - \eta^2}$$

and

$$B = \Omega_1 R_1^2 \frac{1 - \mu}{1 - \eta^2}$$

where

$$\mu = \frac{\Omega_2}{\Omega_1} \text{and} \eta = \frac{R_1}{R_2}$$

4.3 Volumetric flow Between Co-axial Circular Cylinders

We suppose steady flow of fluid which is viscous parallel to the axis in the annular space between cylinders which are coaxial of radii r_1 & r_2 ($r_2 > r_1$). The velocity of such a flow is

$$q_z = \frac{1}{4\mu}(\frac{dp}{dz})r^2 + Alog \quad r + B \tag{6}$$

Boundary conditions are

$$q_z = 0 \quad \text{at} r = r_1 \quad \text{and} r = r_2$$

Applying these conditions in (6), we get

$$A = \frac{1}{4\mu}(\frac{dp}{dz})\frac{r_2^2 - r_1^2}{log \quad r_1/r_2} = -\frac{1}{4\mu}(\frac{dp}{dz})\frac{(n^2 - 1)r_1^2}{log \quad n} \tag{7}$$

,

$$n = r_2/r_1$$

and

$$B = \frac{1}{4\mu}(\frac{dp}{dz})[\frac{(n^2-1)r_1^2}{log \quad n} log \quad r_1 - r_1^2] \tag{8}$$

Thus the velocity distribution in space between cylinders which are coaxial is

$$q_z = -\frac{1}{4\mu}(\frac{dp}{dz})[(r_1^2 - r^2) + \frac{(n^2-1r_1^2)}{log \quad n} log(r/r_1)] \tag{9}$$

The volumetric flow in this case is

$$Q = \int_0^{2\pi}\int_{r_1}^{r_2} q_z r dr d\theta$$

$$= \int_0^{2\pi}\int_{r_1}^{r_2} -\frac{1}{4\mu}(\frac{dp}{dz})[(r_1^2 - r^2) + \frac{(n^2-1)r_1^2}{log \quad n} log(r/r_1)] r dr d\theta$$

$$= -\frac{2\pi}{4\mu}(\frac{dp}{dz})[r_1^2\frac{r^2}{2} - \frac{r^4}{4} + \frac{(n^2-1)r_1^2}{log \quad n}(\frac{r^2}{2}log(r/r_1) - \frac{r^2}{4})]_{r_1}^{nr_1}$$

$$= -\frac{\pi}{2\mu}(\frac{dp}{dz})[\frac{n^2r_1^4}{2} - \frac{r_1^4}{2} - \frac{n^4r_1^4}{4} + \frac{r_1^4}{4} + \frac{(n^2-1)r_1^2}{log \quad n}(log \quad n - 1/2)\frac{n^2r_1^2}{2} + \frac{r_1^2}{4}]$$

$$= -\frac{\pi r_1^4}{8\mu}(\frac{dp}{dz})[2n^2 - 2 - n^4 + 1 + \frac{n^2-1}{log \quad n}(2log \quad n - 1)n^2 + 1]$$

$$-\frac{\pi r_1^4}{8\mu}(\frac{dp}{dz})[2n^2 - n^4 - 1 + 2n^4 - 2n^2 - \frac{(n^2-1)^2}{log \quad n}]$$

$$= -\frac{\pi r_1^4}{8\mu}(\frac{dp}{dz})[(n^4-1) - \frac{(n^2-1)^2}{log \quad n}]$$

4.4 Expression for Shearing Stress

We consider flow between two concentric cylinders which are rotating with radii $r_1, r_2, (r_2 > r_1)$ having viscous fluid in between them. We assume that the flow is circular such that only the tangential component of velocity exists. Let Ω_1 & Ω_2 be angular velocity of inner & outer cylinders, respectively.

The continuity equation in cylindrical co-ordinates (r, θ, z) reduces to

$$\frac{\partial q_\theta}{\partial \theta} = 0, \quad \Rightarrow q_\theta = q_\theta(r) \tag{10}$$

where

$$q_r = q_z = 0$$

Now NSE for fluid which is viscous and incompressible in cylindrical co-ordinates are

$$\rho(\frac{dq_r}{dt} - \frac{q_\theta^2}{r}) = \rho X_r - \frac{\partial p}{\partial r} + \mu(\nabla^2 q_r - \frac{q_r}{r^2} - \frac{2}{r^2}\frac{\partial q_\theta}{\partial \theta})$$

$$\rho(\frac{dq_\theta}{dt} + \frac{q_r q_\theta}{r}) = \rho X_\theta - \frac{1}{r}\frac{\partial p}{\partial \theta} + \mu(\nabla^2 q_\theta + \frac{2}{r^2}\frac{\partial q_r}{\partial \theta} - \frac{q_\theta}{r^2})$$

$$\rho(\frac{dq_z}{dt} = \rho X_z - \frac{\partial p}{\partial z} + \mu \nabla^2 q_z$$

Here,

$$q_r = q_z = 0; X = (X_r, X_\theta, X_z) = 0, q_\theta = q_\theta(r)$$

From the last two equations, we have

$$\frac{\partial p}{\partial z} = 0, -\frac{1}{r}\frac{\partial p}{\partial \theta} + \mu(\nabla^2 q_\theta - \frac{q_\theta}{r^2}) = 0$$

and the first equation gives

$$\rho\frac{q_\theta^2}{r} = \frac{\partial p}{\partial r}$$

The LHS of this equation is a term of r and thus p is a term of r only, i.e.

$$\frac{\partial p}{\partial \theta} = 0$$

Therefore, $\frac{\partial p}{\partial z} = 0, -\frac{1}{r}\frac{\partial p}{\partial \theta} + \mu(\nabla^2 q_\theta - \frac{q_\theta}{r^2}) = 0$ reduces to

$$\nabla^2 q_\theta - \frac{q_\theta}{r^2} = 0,$$

$$\nabla^2 \equiv \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}$$

$$\Rightarrow \frac{d^2 q_\theta}{dr^2} + \frac{1}{r}\frac{dq_\theta}{dr} - \frac{q_\theta}{r^2} = 0$$

$$\Rightarrow \frac{d^2 q_\theta}{dr^2} + \frac{d}{dr}(\frac{q_\theta}{r}) = 0$$

Integrating, we get

$$\frac{dq_\theta}{dr} + \frac{q_\theta}{r} = 2C_1$$

$$r\frac{dq_\theta}{dr}+q_\theta=2C_1r \quad \Rightarrow \frac{d}{dr}(rq_\theta)=2C_1r$$

Integrating, we get

$$rq_\theta=C_1r^2+C_2 \quad C_1r+\frac{C_2}{r}$$

which is the general solution.
Boundary conditions are

$$q_\theta=r_1\Omega-1, \quad \text{when} r=r_1$$

and

$$q_\theta=r_2\Omega_2, \quad \text{when} r=r_2$$

Because on the surface, $v=r\frac{d\theta}{dt} \quad v=r\Omega$
$l=r\theta \quad \Rightarrow \frac{dl}{dt}=r\frac{d\theta}{dt}, i.e. v=r\Omega$
Using $rq_\theta=C_1r^2+C_2 \quad C_1r+\frac{C_2}{r}$, we obtain

$$C_1=\frac{\Omega_1r_1^2-\Omega_2^2}{r_1^2-r_2^2}$$

$$C_2=\frac{r_1^2r_2^2(\Omega_1-\Omega_2)}{r_2^2-r_1^2}$$

Thus the solution of $rq_\theta=C_1r^2+C_2 \quad C_1r+\frac{C_2}{r}$ in the present case is

$$q_\theta=\frac{1}{r-2^2-r_1^2}[(r_2^2\Omega_2-r_1^2\Omega_1)r-\frac{r_1^2r_2^2(\Omega_2-\Omega_1)}{r}]$$

In particular, if the inner cylinder is at rest, i.e. $\Omega_1=0, \Omega_2=\Omega(say), r_1=a, r_2=b$, then the solution becomes

$$q_\theta=\frac{\Omega b^2}{b^2-a^2}(r-\frac{a^2}{r})$$

The radial pressure given by $\rho\frac{q_\theta^2}{r}=\frac{\partial p}{\partial r}$, is

$$\frac{dp}{dr}=\rho\frac{q_\theta^2}{r}$$

$$=\frac{\rho}{r}(C_1^2r^2+\frac{C_2^2}{r^2}+2C_1C_2)$$

$$=\rho(C_1^2r+\frac{C_2^2}{r^3}+\frac{2C_1C_2}{r})$$

Integrating with respect to r we get

$$p=\rho[\frac{C_1^2r^2}{2}-\frac{C_2^2}{2r^2}+2C_1C_2log \quad r]+C_3$$

If $p = p_1$ when $r = r_1$, then

$$p_1 = \rho[\frac{C_1^2 r_1^2}{2} - \frac{C_2^2}{2r_1^2} + 2C_1C_2 log \quad r_1] + C_3$$

$$\Rightarrow C_3 = p_1 - \rho[\frac{C_1^2 r_1^2}{2} - \frac{C_2^2}{2r_1^2} + 2C_1C_2 log \quad r_1]$$

Hence the pressure is given by

$$p = p_1 + \rho[C_1^2(\frac{r^2 - r_1^2}{2}) - \frac{C_2^2}{2}(\frac{1}{r^2} - \frac{1}{r_1^2} + 2C_1C_2 log(r/r_1))]$$

where C_1 and C_2 are given by

$$C_1 = \frac{\Omega_1 r_1^2 - \Omega_2^2}{r_1^2 - r_2^2}$$

$$C_2 = \frac{r_1^2 r_2^2(\Omega_1 - \Omega_2)}{r_2^2 - r_1^2}$$

The formula for shearing stress is

$$\sigma_{r\theta} = \mu[\frac{dq_\theta}{dr} - \frac{q_\theta}{r}] = \mu[d\frac{d}{dr}(\frac{q_\theta}{r})]$$

$$= \mu[r\frac{d}{dr}(\frac{C_1 r + C_2/r}{r})]$$

$$= \mu r\frac{d}{dr}(C_1 + \frac{C_2}{r^2})$$

$$= \mu r(-\frac{2C_2}{r^3})$$

$$= \frac{-2\mu C_2}{r^2}$$

$$= \frac{-2\mu r_1^2 r_2^2(\Omega_1 - \Omega_2)}{r^2(r_2^2 - r_1^2)}$$

The expressions for the shearing stress on the outer and the inner cylinders are

$$(\sigma_{r\theta})_{r=r_2} = \frac{2\mu(\Omega_2 - \Omega_1)r_1^2}{r_2^2 - r_1^2}$$

$$(\sigma_{r\theta})_{r=r_1} = \frac{2\mu(\Omega_2 - \Omega_1)r_2^2}{r_2^2 - r_1^2}$$

4.5 Normal Modes

Modes-harmonic (sinusoidal) motions.
Normal-independent of each other.
A normal mode is a motion where all parts of the system are moving sinusoidally with the same frequency and in phase.
All observed configurations of a system may be generated from its normal modes.Each normal mode has a characteristic frequency, its eigenvalue.

4.5.1 Characteristics of Normal Mode Motion

1. Each normal mode acts like a simple harmonic oscillator.
2. A normal mode is concerted motion of many atoms.
3. Centre of mass does not move.
4. All atom pass through their equilibrium positions at the same time.
5. Normal modes are orthogonal to each other; they resonate independently.
6. Directly related to vibrational spectroscopy.

4.6 Couette Flow and Perturbation

To determine the flow stability that exists between cylinders which are concentric and rotating; we take the following considerations:
(i) We take the basic Couette flow

$$V(r) = Ar + \frac{B}{r} \tag{11}$$

for Navier-Stokes Equations of a viscous fluid.
Where we take rigid cylinders $r = R_1, R_2$ with angular velocities Ω_1, Ω_2 respectively with

$$A = \frac{\Omega_2 R_2^2 - \Omega_1 R_1^2}{R_2^2 - R_1^2}, B = \frac{\Omega_1 - \Omega_2}{R_1^- 2 - R_2^- 2} \tag{12}$$

(ii)We linearize the NSE and boundary conditions for small perturbations of basic flow.
(iii) We choose dimensionless variables and parameters.
(iv) We take the normal modes:

$$u'(x,t) = u(r)e^{st+i(n\theta+kz)}$$

(v)We then derive an ordinary differential eigenvalue problem to find $s,$ for given real wavenumber k and integral wavenumber n.

Since the resulting numerical problem is too difficult to solve; we will simplify our problem based on the following assumptions:
(i)Most unstable perturbations are axisymmetric and so $n = o$.
(ii)Exchange of stabilities Principle holds, i.e. $Im(s) = 0$ at the onset of instability, and so $s = 0$ thus we seek dimensionless parameters which give the margin of instability.
(iii) There is a narrow gap between the cylinders, i.e. $R_2 - R_1 << R_2$.

Let u_r, u_θ and w be the velocity components r, θ & z directions, respectively.
Then governing relations are: are:

$$\frac{1}{r}\frac{\partial}{\partial r}(ru_r) + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial w}{\partial z} = 0$$

$$\frac{\partial u_r}{\partial t} + u_r\frac{\partial u_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial u_r}{\partial \theta} + w\frac{\partial u_r}{\partial z} - \frac{v_\theta^2}{r} = -\frac{\partial p}{\partial r} + \frac{1}{Re}(\nabla^2 u_r - \frac{u_r}{r^2} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}),$$

$$\frac{\partial v_\theta}{\partial t} + u_r\frac{\partial u_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + w\frac{\partial v_\theta}{\partial z} + \frac{u_r v_\theta}{r} = -\frac{1}{r}\frac{\partial p}{\partial \theta} + \frac{1}{Re}(\nabla^2 v_\theta - \frac{v_\theta}{r^2} + \frac{2}{r}\frac{\partial u_r}{\partial \theta}),$$

$$\frac{\partial w}{\partial t} + u_r\frac{\partial w}{\partial r} + \frac{v_\theta}{r}\frac{\partial w}{\partial \theta} + w\frac{\partial w}{\partial z} = -\frac{\partial p}{\partial r} + \frac{1}{Re}\nabla^2 w.$$

where

$$\nabla^2 \equiv \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}$$

All variables have been nondimensionalized with respect to the gap width, $l = R_2 - R_1$ and a characteristic velocity U_0; the Reynolds number,$Re = \frac{U_0 l}{\nu}$ and the density, ρ is a constant.
Now the solution for the mean flow is one that is $\theta-$component and a term of r only or $V = V(r)$. The mean pressure , P, is taken as only a function of r. Thus,

$$V = Ar + \frac{B}{r}$$

is the solution that meets these requirements and the pressure,P, can be obtained from the relation

$$\frac{1}{\rho}\frac{dP}{dr} = \frac{V^2}{r}.$$

The coefficients A and B are fixed by requiring the value of V to be that of the cylinders at $r = R_1$ and R_2 and are found to be

$$A = \frac{(\frac{R_2}{R_1})^2 \frac{\Omega_2}{\Omega_1} - 1}{(\frac{R_2}{R_1})^2 - 1}$$

$$B = -A,$$

when the cylinders rotate in the similar way. If rotation is in reverse ways, simply we replace Ω_1 by $-\Omega_1$.

Next, small perturbations are introduced and the governing equations are linearized. In this way, with the notation for the velocity as $(u_r, V + v_\theta.w)$ and $P + p$ for the pressure, the linearized equations are:

$$\frac{1}{r}\frac{\partial}{\partial r}(ru_r) + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial w}{\partial z} = 0$$

$$\frac{\partial u_r}{\partial t} + \frac{V}{r}\frac{\partial u_r}{\partial \theta} - \frac{2V}{r}v_\theta = -\frac{\partial p}{\partial r} + \frac{1}{Re}(\nabla^2 u_r - \frac{u_r}{r^2} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta})$$

$$\frac{\partial v_\theta}{\partial t} + \frac{V}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{dV}{dr}u_r = -\frac{1}{r}\frac{\partial p}{\partial \theta} + \frac{1}{Re}(\nabla^2 v_\theta - \frac{v_\theta}{r^2} - \frac{2}{r^2}\frac{\partial u_r}{\partial \theta})$$

$$\frac{\partial w}{\partial t} + \frac{V}{r}\frac{\partial w}{\partial \theta} = -\frac{\partial p}{\partial z} + \frac{1}{Re}\nabla^2 w$$

The solutions of these equations must satisfy the B.C that all the 3 perturbation components of velocity die on both inner and outer cylinder end walls.

For obtaining general solution:

(i) Only axisymmetric disturbances will be treated hence the $\theta-$ dependence is omitted.

(ii) The cylinders are fixed in such a way that the gap width, l, is small or, in the sense of the nondimensional variables,

$$\frac{l}{r} << 1.$$

When axisymmetry ($\frac{\partial}{\partial \theta} = 0$) is incorporated into the set of the above equations and the pressure is eliminated, then the pair of coupled equations:

$$[\frac{\partial}{\partial t} - \frac{1}{Re}(\nabla^2 - \frac{1}{r^2})](\nabla^2 u_r - \frac{u_r}{r^2}) = 2(\frac{V}{r})\frac{\partial^2 v_\theta}{\partial z^2}$$

and

$$\frac{\partial v_\theta}{\partial t} - \frac{1}{Re}(\nabla^2 v_\theta - \frac{v_\theta}{r^2} = -\frac{dV}{dr}u_r)$$

result.

The solutions for u_r and v_θ can be obtained in terms of normal modes or

$$u_r(r,z,t) = u'(r)e^{\sigma t + i\lambda z}$$

$$v_\theta(r,z,t) = v'(r)e^{\sigma t + i\lambda z}$$

As a result of this form for the solutions, the above equations now become coupled ordinary differential equations. Of course, in this form, even the general set of the linearized equations where the assumption of axisymmetry is not used,we express in eigenfunctions,Bessel functions. The resulting eigenvalue problem that determines the stability will be given as

$$F(\sigma, \lambda, \frac{R_2}{R_1}, \frac{\Omega_2}{\Omega_1}, Re) = 0$$

The net result of axisymmetry, the small gap approximation and the normal mode solutions reduces the coupled equations to:

$$[\frac{1}{Re}(\frac{d^2}{dr^2} - \lambda^2) - \sigma](\frac{d^2u'}{dr^2} - \lambda^2 u') = 2\lambda^2(\frac{V}{r}v')$$

and

$$\frac{1}{Re}(\frac{d^2v'}{dr^2} - \lambda^2 v') - \sigma v' = -\frac{dV}{dr}u'$$

with B.C.

$$u' = u'' = v' = 0$$

at cylinder walls.

The small gap approximation also affects the variation of the mean velocity profile. Specifically, the part of mean flow that varies as $\frac{1}{r}$ can be neglected just as it was for the operators in the perturbation equations. Thus, $V(r) = Ar$, a linear variation, and is the reason that the flow has been referred to as Couette flow.

If we define $\varepsilon = Re^-1$ and expand the immediate previous equations so that all terms involving the fourth and second derivatives as well as the dependent variables can be grouped.

These reordered equations are then multiplied by the respective complex conjugates of u' and v'. Once this has been done, the product equation relations are integrated over r from R_1 to R_2.

The imaginary relations for σ_i can be combined by recognizing the relations of complex conjugate pairs. This means the integrals on the RHS of (26) and (28) differ only by a minus sign and

$$\sigma_i[\frac{2\lambda(V/r)}{V'} - \frac{(I_1^2 + \lambda^2 I_0^2)}{J_1^2}] = 0$$

results.
For this to be true, either $\sigma_i = 0$ or the bracketed terms must balance.
If $\sigma_i \neq 0$, then the disturbances are periodic in time. In turn, for this to occur, the ratio

$$2\lambda(V/r)/V' > 0$$

$\Rightarrow$ that the mean profile must increase outward. Certainly, if $V > 0, V' > 0$. Let us call the necessary relation as

$$2\lambda(V/r)/V' = P > 0.$$

Now, the expression for the real parts of σ_r can be used. In fact, it is found that

$$\sigma_r = -\frac{(E + B/P)}{(D + A/P)}$$

where

$$A = (I_1^2 + \lambda^2 I_0^2,)$$
$$B = \varepsilon(I_2^2 + 2\lambda^2 I_1^2 + \lambda^4 I_0^2)$$
$$J_1^2, E = \varepsilon(I_1^2 + \lambda^2 J_0^2)$$

Thus, $\sigma_r < 0$ and consequently, a disturbance that is periodic in time must decay. The combination indicates that neutral stability corresponds to $\sigma = \sigma_r + i\sigma_r = 0$. We call it exchange of stabilities principle. Again, unlike the results for parallel or almost parallel mean flows, neutrality is tantamount to a steady state. By using the principle just established, the neutral solution for the Taylor problem can now be determined.Now the reduced normal modes solution becomes after setting $\sigma = 0$;

$$(D^2 - \lambda^2)^2 u' = 2\lambda^2 (\frac{V}{r}) Rev'$$

$$(D^2 - \lambda^2) v' = -V' Reu'$$

where the Reynolds number has been moved to the RHS for those relations. And, strictly speaking, for the narrow gap approximation, $V/r = V' = A$ with A an angular velocity in dimensional terms. These two equations can be combined to form one for u', namely

$$(D^2 - \lambda^2)^3 u' = -2\lambda^2 Re^2 (\frac{V}{R}) V'$$

Now, the Taylor number emerges and is defined as

$$Ta = -2Re^2 (\frac{V}{r}) V' >> 1$$

and, for the Taylor problem, $V' < 0$.Fof small gap approximation, the above equation can be solved exactly. When the cylinders rotate in opposite directions, then $V = 0$ for some value of r between R_1 and R_2 and the solution must account for this fact and this has been done.$[CJ03]$

4.7 Analytical Discussion of the Stability of Inviscid Couette Flow

We now discuss about the analytic solution and the stability of inviscid Couette flow.

4.8 On Viscous Couette Flow

We now turn our attention to stationary viscous flow between two rotating cylinders. In cylindrical polar coordinates, the Navier-Stokes equations for viscous incompressible fluids take the forms:

$$\frac{\partial u_r}{\partial t}+(u.\nabla)u_r-\frac{u_\theta^2}{r}=-\frac{\partial}{\partial r}(\frac{p}{\rho})+\nu(\nabla^2 u_r-\frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta}-\frac{u_r}{r^2})$$

$$\frac{\partial u_\theta}{\partial t}+(u.\nabla)u_\theta+\frac{u_r u_\theta}{r}=-\frac{1}{r}\frac{\partial}{\partial \theta}(\frac{p}{\rho})+\nu(\nabla^2 u_\theta+\frac{2}{r^2}\frac{\partial u_r}{\partial \theta}-\frac{u_\theta}{r^2})$$

$$\frac{\partial u_z}{\partial t}+(u.\nabla)u_z=-\frac{\partial}{\partial z}(\frac{p}{\rho})+\nu\nabla^2 u_z$$

where

$$u.\nabla \equiv u_r\frac{\partial}{\partial r}+\frac{u_\theta}{r}\frac{\partial}{\partial \theta}+u_z\frac{\partial}{\partial z}$$

and

$$\nabla^2=\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}+\frac{\partial^2}{\partial z^2}$$

We have also the equation of continuity,

$$\frac{\partial u_r}{\partial r}+\frac{u_r}{r}+\frac{1}{r}\frac{\partial u_\theta}{\partial \theta}+\frac{\partial u_z}{\partial z}=0$$

These equations allow a stationary solution of the form

$$u_r=u_z=0, u_\theta=V(r)$$

provided

$$\frac{d}{dr}(\frac{p}{\rho}=\frac{V^2}{r})$$

and

$$\nu(\nabla^2 V-\frac{V}{r^2})=\nu\frac{d}{dr}(\frac{d}{dr}+\frac{1}{r})V=0$$

From the equation above, it is apparent that the most general form of$V(r)$ which is compatible with $v \neq 0$ is

$$V = Ar + \frac{B}{r}$$

where$A$and B are two arbitrary constants. The corresponding expression for the angular velocity is

$$\Omega = A + \frac{B}{r^2}$$

The constants A and B can, therefore be related to the angular velocities Ω_1 and Ω_2 with which the two cylinders are related.We have as before;

$$A = -\Omega_1 \eta^2 \frac{1-\mu/\eta^2}{1-\eta^2}$$

and

$$B = \Omega_1 \frac{R_1^2(1-\mu)}{1-\eta^2}$$

where

$$\mu = \frac{\Omega_2}{\Omega_1}$$

and

$$\eta = \frac{R_1}{R_2}$$

As we have seen before, Rayleigh's criterion for the stability of inviscid Couette flow applied to the distribution $\Omega = A + \frac{B}{r^2}$ yields

$$\mu > \eta^2$$

Taylor found an explicit analytical expression for the criterion; and he was able to confirm by experiments that the marginal state is stationary and exhibits a break-up of the basic flow into a cellular pattern.

4.8.1 The Perturbation Equations

Suppose

$$\frac{d}{dr}(\frac{p}{\rho}) = \frac{v^2}{r}$$

and

$$V = Ar + \frac{B}{r}$$

Let the perturbed state be characterized by

$$u_r, V + u_\theta, u_z, \frac{\delta p}{\rho} = \varpi$$

The linearized equations are:

$$\frac{\partial u_r}{\partial t} - 2\frac{V}{r}u_\theta = \frac{\partial \varpi}{\partial r} + \nu(\nabla^2 u_r - \frac{u_r}{r^2})$$

$$\frac{\partial u_\theta}{\partial t} + (\frac{dV}{dr} + \frac{V}{r})u_r = \nu(\nabla^2 u_\theta - \frac{u_\theta}{r^2})$$

and

$$\frac{\partial u_z}{\partial t} = -\frac{\partial \varpi}{\partial z} + \nu\nabla^2 u_z$$

where

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}$$

Also, for axisymmetric motions, the equation of continuity reduces to

$$\frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{\partial u_z}{\partial z} = 0$$

By analyzing the disturbance into *normal modes,* we look for solution of the above equations which are of the forms

$$u_r = u(r)e^{pt}coskz$$

$$u_\theta = v(r)e^{pt}coskz$$

$$u_z = w(r)e^{pt}sinkz$$

$$\varpi = \varpi(r)e^{pt}coskz$$

For solutions of the form as above,the linearized equations become:

$$\nu(DD_* - k^2 - \frac{p}{\nu})u + 2\frac{V}{r}v = \frac{d\varpi}{dr}$$

$$\nu(DD_* - k^2 - \frac{p}{\nu})v - (D_*V)u = 0$$

$$\nu(DD_* - k^2 - \frac{p}{\nu})w = -k\varpi$$

$$\nabla^2 = (\frac{d}{dr} + \frac{1}{r})\frac{d}{dr} - k^2 = D_*D - k^2 = DD_* + \frac{1}{r^2} - k^2$$

and

$$D_*u = -kw$$

Eliminating w in the equations above, we have

$$\frac{\nu}{k^2}(D_*D - k^2 - \frac{p}{\nu})D_*u = \varpi$$

Inserting this expression for ϖ in $\nu(DD_* - k^2 - \frac{p}{\nu})u + 2\frac{V}{r}v = \frac{d\varpi}{dr}$, we find after some rearranging,

$$\frac{\nu}{k^2}(D_*D - k^2 - \frac{p}{\nu})(DD_* - k^2)u = 2\frac{V}{r}v$$

This equation must be considered together with

$$\nu(D_*D - k^2 - \frac{p}{\nu})v = (D_*V)u$$

These equations are general and do not depend on any particular form of $V(r)$.Measuring r in units of radius R_2 of the outer cylinder and writing

$$k^2 = \frac{a^2}{R_2^2}$$

and

$$\sigma = pR_2^2\nu$$

Equations $\frac{\nu}{k^2}(D_*D - k^2 - \frac{p}{\nu})(DD_* - k^2)u = 2\frac{V}{r}v$ and $\nu(D_*D - k^2 - \frac{p}{\nu})v = (D_*V)u$ become (when $V(r)$ has the particular form $V = Ar + \frac{B}{r}$)

$$(DD_* - a^2 - \sigma)(DD_* - a^2)u = a^2\frac{2B}{\nu}(\frac{1}{r^2} + \frac{AR_2^2}{B})v$$

and

$$(DD_* - a^2 - \sigma)v = \frac{2A}{\nu}R_2^2u$$

For convenience we make transformation

$$\frac{AR_2^2}{\nu}u \mapsto u$$

The equations then take the more convenient forms

$$(DD_* - a^2 - \sigma)(DD_* - a^2)u = -Ta^2(\frac{1}{r^2} - \kappa)v$$

and

$$(DD_* - a^2 - \sigma)v = u$$

where

$$T = -\frac{4AB}{\nu^2}R_2^2 = \frac{4\Omega_1^2R_1^4}{\nu^2}\frac{(1-\mu)(1-\mu/\eta^2)}{(1-\eta^2)^2}$$

and

$$\kappa = -\frac{AR_2^2}{B} = \frac{1-\mu/\eta^2}{1-\mu}$$

Solutions of $(DD_* - a^2 - \sigma)(DD_* - a^2)u = -Ta^2(\frac{1}{r^2} - \kappa)v$ and $(DD_* - a^2 - \sigma)v = u$ must be sought which satisfy the boundary conditions appropriate for no-slip on the cylindrical walls at $r = 1$ and η.

4.8.2 The Stability of the Flow for $\mu > \eta^2$

We shall now show when Rayleigh's criterion$\mu > \eta^2$ is satisfied, the flow is indeed stable. First, we may notice certain elementary integral properties of the operator DD_*. If $f(r)$ and $g(r)$ are any two functions and if one of them, say $f(r)$, vanishes at the limits of integration,

$$\int rfDD_*gdr = -\int (r\frac{df}{dr}\frac{dg}{dr} + \frac{fg}{r})dr$$

and if the derivative of f also vanishes at the limits,

$$\int rfDD_*gdr = \int rgDD_*fdr$$

These relations follow by successive integrations by parts. Thus, by writing

$$\int rfDD_*gdr = \int [f\frac{d}{dr}(r\frac{dg}{dr}) - \frac{fg}{r}]dr,$$

The truth of $\int rfDD_*gdr = -\int (r\frac{df}{dr}\frac{dg}{dr} + \frac{fg}{r})dr$ becomes self evident and a further integration by parts leads to $\int rfDD_*gdr = \int rgDD_*fdr$
Now returning to $(DD_* - a^2 - \sigma)(DD_* - a^2)u = -Ta^2(\frac{1}{r^2} - \kappa)v$
and $(DD_* - a^2 - \sigma)v = u$,multiply $(DD_* - a^2 - \sigma)(DD_* - a^2)u = -Ta^2(\frac{1}{r^2} - \kappa)v$ by ru^* and integrate over the range of r.
We have

$$\int_\eta^1 ru^*[(DD_* - a^2)^2u - \sigma(DD_* - a^2)u]dr = -\Upsilon a^2 \int_\eta^1 r\phi(r)vu^*dr$$

where, for brevity, we have written

$$\Upsilon = \frac{T}{1-\mu} = \frac{4\Omega_1^2 R_1^4}{\nu^2}\frac{(1-\mu/\eta^2)}{(1-\eta^2)^2}$$

and

$$\phi = (1-\mu)(1/r^2 - \kappa)$$

Since u and its derivative vanish at $r=1$ and η, the integrals on the left-hand side of $\int_\eta^1 ru^*[(DD_*-a^2)^2u-\sigma(DD_*-a^2)u]dr=-\Upsilon a^2\int_\eta^1 r\phi(r)vu^*dr$ can be transformed to positive definite forms by making use of $\int rfDD_*gdr=-\int(r\frac{df}{dr}\frac{dg}{dr}+\frac{fg}{r})dr$ and $\int rfDD_*gdr=\int rgDD_*fdr$. Thus, $\int_\eta^1 ru^*[(DD_*-a^2)^2u-\sigma(DD_*-a^2)u]dr=\int_\eta^1 r(\mid(DD_*-a^2)u\mid)^2dr+\sigma\int_\eta^1[r(\mid\frac{du}{dr}\mid)^2+(\frac{1}{r}+a^2r)(\mid u\mid)^2]dr$

Next, substituting for u^* (from $(DD_*-a^2-\sigma)v=u$) in the integrand on the right-hand side of equation $\int_\eta^1 ru^*[(DD_*-a^2)^2u-\sigma(DD_*-a^2)u]dr=-\Upsilon a^2\int_\eta^1 r\phi(r)vu^*dr$, we obtain

$$\int_\eta^1 r\phi(r)vu^*dr=\int_\eta^1 r\phi(r)v(DD_*-a^2-\sigma^*)v^*dr$$

$$=-(a^2+\sigma^*)\int_\eta^1\phi(r)r\mid v\mid^2dr+\int_\eta^1 r\phi(r)vDD_*V^*dr$$

Again, by making use of $\int rfDD_*gdr=-\int(r\frac{df}{dr}\frac{dg}{dr}+\frac{fg}{r})dr$, we have

$$\int_\eta^1 r\phi(r)vDD_*v^*dr=-\int_\eta^1\phi(r)(r\mid\frac{dv}{dr}\mid^2+\frac{\mid v\mid^2}{r})dr+2(1-\mu)\int_\eta^1\frac{v}{r^2}\frac{dv^*}{dr}dr$$

Now combining equations above, we obtain

$$\sigma I_1+I_2=\Upsilon a^2[(a^2+\sigma^*)I_3+I_4]$$

where

$$I_1=\int_\eta^1[r\mid\frac{du}{dr}\mid^2+(\frac{1}{r}+a^2r)\mid u\mid^2]dr$$

$$I_2=\int_\eta^1\mid(DD_*-a^2)u\mid^2rdr$$

$$I_3=\int_\eta^1\phi(r)r\mid v\mid^2dr$$

and

$$I_4=\int_\eta^1\phi(r)(r\mid\frac{dv}{dr}\mid^2+\frac{\mid v\mid^2}{r})dr-2(1-\mu)\int_\eta^1\frac{v}{r^2}\frac{dv^*}{dr}dr$$

The integrals I_1 and I_2 are clearly positive definite. For $\mu>0,\phi(r)>0$; so that in this case, I_3 is also positive definite. The first of the two integrals included in I_4 is positive for $\mu>0$;but the second is complex. However, the real part of I_4 is positive definite for $\mu>0$;in fact,

$$re(I_4)=\int_\eta^1 r\phi(r)\mid\frac{dv}{dr}-\frac{v}{r}\mid^2dr$$

For, expanding the integrand in $re(I_4)=\int_\eta^1 r\phi(r)\mid\frac{dv}{dr}-\frac{v}{r}\mid^2 dr$, we have

$$\int_\eta^1 r\phi(r)\mid\frac{dv}{dr}-\frac{v}{r}\mid^2 dr=\int_\eta^1\phi(r)(r\mid\frac{dv}{dr}\mid^2+\frac{\mid v\mid^2}{r})dr-\int_\eta^1\phi(r)\frac{d\mid v\mid^2}{dr}dr$$

But

$$\int_\eta^1\phi(r)\frac{d\mid v\mid^2}{dr}dr=(1-\mu)\int_\eta^1(\frac{1}{r^2}-\kappa)\frac{d\mid v\mid^2}{dr}dr=(1-\mu)\int_\eta^1\frac{1}{r^2}\frac{d\mid v\mid^2}{dr}dr$$

Therefore, the right- hand side of $\int_\eta^1 r\phi(r)\mid\frac{dv}{dr}-\frac{v}{r}\mid^2 dr=\int_\eta^1\phi(r)(r\mid\frac{dv}{dr}\mid^2+\frac{|v|^2}{r})dr-\int_\eta^1\phi(r)\frac{d|v|^2}{dr}dr$ is, indeed, the real part of I_4 Returning to $\sigma I_1+I_2=\Upsilon a^2[(a^2+\sigma^*)I_3+I_4]$ and equating the real parts of this equation, we obtain

$$re(\sigma)(I_1-\Upsilon a^2 I_3)+I_2-\Upsilon a^2[a^2 I_3+re(I_4)]=0$$

When $\mu>\eta^2$; $\Upsilon<0$ and the coefficient of $re(\sigma)$in $re(\sigma)(I_1-\Upsilon a^2 I_3)+I_2-\Upsilon a^2[a^2 I_3+re(I_4)]=0$
Therefore,

$$re(\sigma)<0 \quad \text{for} \quad \mu>\eta^2$$

And the flow is stable; this result is entirely to be expected on physical grounds. Nevertheless, it appears to be the only one which can be established by general analytical arguments.By equating the imaginary parts of$\sigma I_1+I_2=\Upsilon a^2[(a^2+\sigma^*)I_3+I_4]$, we obtain

$$im(\sigma)(I_2+\Upsilon a^2 I_3)=-2\Upsilon a^2 im\int_\eta^1\frac{v}{r^2}\frac{dv^*}{dr}dr$$

and no general conclusions can be drawn from this equation; when $\mu<0$,even I_3 is not positive definite!

5 Hydrodynamic Stability Analysis

Objectives

Here,we study about the Hydrodynamic Stability Analysis.

5.1 Linear Stability Analysis

To determine whether the flow is stable or unstable, one often employs the method of linear stability analysis. In this type of analysis, the governing equations and boundary conditions are linearized. This is based on the fact that the concept of 'stable' or 'unstable' is based on an infinitely small disturbance. For such disturbances, it is reasonable to assume that disturbances of different wavelengths evolve independently. (A nonlinear governing equation will allow disturbances of different wavelengths to interact with each other).

Analysing Flow Stability

5.1.1 Bifurcation Theory

This is a useful way to discuss stability of given flow bifurcation with changes that occur in the structure of a given system in the case of hydrodynamic stability; this is a series of differential equations and their solutions. A bifurcation occurs when a small change in the parameters of the system causes a qualitative change in its behavior, the parameter that is being changed in the case of hydrodynamic stability is the Reynolds number. It can be shown that the occurrence of bifurcations falls in line with the occurrence of instabilities.

5.1.2 Laboratory and Computational Experimentation

Laboratory experiments are a very useful way of gaining information about a given flow without having to use more complex; mathematical techniques. Sometimes physically seeing the change in the flow overtime is just as useful as a numerical approach and any findings from these experiments can be related back to the underlying theory. Experimental analysis is also useful because it allows one to vary the governing parameters very easily and their effects will be visible. When dealing with more complicated mathematical theories such as Bifurcation theory and weakly nonlinear theory, numerically solving such problems becomes very difficult and time consuming but with the help of computers this process becomes much easier and quicker. Since the 1980s computational analysis has become more useful, the improvement of algorithms which can solve the governing equations, such as the NSE means that they can be integrated more accurately for various types of flow.

Applications

5.1.3 Kelvin-Helmholtz Instability(KHI)

Kelvin-Helmoltz instability is an application of hydrodynamic stability that can be seen in nature. It occurs when there are two fluids flowing at different velocities. The difference in velocity at the interface of the two layers. The shear velocity of one fluid moving induces a shear stress on the other which, if greater than the restraining surface tension, then results in an instability along the interface between them. This motion causes the appearance of a series of overturning ocean waves, a characteristic of the KHI. Indeed, the apparent ocean wave-like nature is an example of vortex formation, which are formed when a fluid is rotating about some axis, and is often associated with this phenomenon. The KHI can be seen in the bands in planetary atmospheres such as Saturn and Jupiter, for example in the giant red spot vortex.

5.1.4 Rayleigh-Taylor Instability(RTI)

Rayleigh-Taylor instability is another application of hydrodynamic stability and also occurs between two fluids but this time the densities are different. Due to difference in densities, the two fluids will try to reduce their combined potential energy. The less dense fluid will do this by trying to force its way upwards and the more dense fluid will try to force its way downwards. Therefore, there are two possibilities; if the lighter fluid is on top the interface is said to be stable, but if the heavier fluid is on top, then the equilibrium of the system is unstable to any disturbances of the interface. If this is the case then both fluids will begin to mix.

In the linear phase, equations can be linearized and the amplitude of perturbations is growing exponentially with time. In the non-linear phase, perturbation amplitude is too large for the non-linear terms to be neglected. In general, the density disparity between the fluids determines the structure of the subsequent non-linear RTI flows (assuming other variables such as surface tension and viscosity are negligible here).

This process is evident not only in many terrestrial examples, from salt domes to weather inversions, but also in astrophysics and electro hydrodynamics. RTI structure is also evident in the Crab Nebula, in which the expanding pulsar wind nebula powered by the Crab pulsar is sweeping up ejected material from the supernova explosion 1000 years ago. The RTI has also recently been discovered in the sun's outer atmosphere, or solar corona, when a relatively dense solar prominence overlies a less dense plasma bubble.

The inviscid two-dimensional RTI provides an excellent springboard into the mathematical study of stability because of the simple nature of the base state. This is the equilibrium state that exists before any perturbation is added to the system, and is described by velocity field $U(x,z) = W(x,z) = 0$, where the gravitational field is $g = -gz$. An interface at $z = 0$ separates the fluids of densities ρG in the upper region, and ρL in the lower region. In this section it is shown that when the heavy fluid sits on top, the growth of a small perturbation at the interface is exponential, and takes place at the rate

$$exp(\Upsilon t), \quad \text{with} \quad \Upsilon = \sqrt{Ag\alpha} \quad \text{and} \quad \frac{\rho_{heavy} - \rho_{light}}{\rho_{heavy} + \rho_{light}}$$

where Υ is the temporal growth rate, $\alpha$is the spatial wavenumber and A is the Atwood number.

5.1.5 Details of Linear Stability analysis

The perturbation introduced to the system is described by a velocity field of infinitesimally small amplitude, $(u'(x,z,t), w'(x,z,t))$. Because fluid is assumed incompressible, this velocity field has the stream function representation

$$u' = (u'(x,z,t), w'(x,z,t)) = (\psi_z, \psi_x),$$

where subscripts indicate partial derivatives. Moreover, in an initially stationary incompressible fluid, there is no vorticity, and the fluid stays irrotational, hence$\nabla \times u' = 0$. In the stream function representation,$\nabla^2 \psi = 0$. Next,

$$\psi(x,z,t) = e^{i\alpha(x-ct)}\Psi(z)$$

where α is a spatial wavenumber. Thus, the problem reduces to solving the equation

$$(D^2 - \alpha^2)\Psi_j = 0, \quad D = \frac{d}{dz}, \quad j = L, G$$

The domain of the problem is the following:
The fluid with label 'L' lives in the region $-\infty < z \leq 0$, while the fluid with label 'G' in the upper half-plane $0 \leq z < \infty$. To specify the solution fully, it is necessary to fix conditions at the boundaries and interface. This determines the wave speed, which in turn determines the stability.First of these conditions is provided by details at the boundary. The perturbation velocities w_i' should satisfy a no-flux condition, so that fluid does not

leak out at the boundaries $z = \pm\infty$. Thus, $w'_L = 0$ on $z = -\infty$,and $w'_G = 0$ on $z = \infty$. In terms of the stream function, this is

$$\Psi_L(-\infty) = 0, \quad \Psi_G(\infty) = 0$$

The other three conditions are provided by details at the interface $z = \eta(x,t)$.

Continuity of Vertical Velocity

At $z = \eta$, the vertical velocities match, $w'_L = w'_G$.Using the streamfuction representation, this gives

$$\Psi_L(\eta) = \Psi_G(\eta)$$

Expanding about $z = 0$ gives

$$\Psi_L(0) = \Psi_G(0) + H.O.T.$$

where H.O.T means higher order terms.
This equation is the required interfacial condition.

The Free-surface Condition

$$\frac{\partial \eta}{\partial t} + u' \frac{\partial \eta}{\partial x} = w'(\eta)$$

Linearizing, this is simply,

$$\frac{\partial \eta}{\partial t} = w'(0),$$

where the velocity $w'(\eta)$ is linearized on to the surface $z = 0$. Using the normal mode and stream function representation, this condition is $c\eta = \Psi$, the second interfacial condition.

Pressure Relation Across the Interface

For the case with surface tension, the pressure difference over the interface at $z = \eta$ is given by the Young-Laplace equation:

$$p_G(Z = \eta) - p_G(z = \eta) = \sigma\kappa$$

where σ is the surface tension and κ is the curvature of the interface, which in a linear approximation is

$$\kappa = \nabla^2 \eta = \eta_{xx}$$

Thus,

$$p_G(z=\eta) - p_L(z=\eta) = \sigma\eta_{xx}$$

However, this condition refers to as the total pressure (base perturbed), thus

$$[P_G(\eta) + p'_G(0)] - [P_L(\eta) + p'_L(0)] = \sigma\eta_{xx}$$

(As usual, the perturbed quantities can be linearized onto the surface $= 0$). Using hydrostatic balance,in the form

$$P_L = -\rho_L g z + p_0$$

$$P_G = -\rho_G g z + p_0$$

This becomes

$$p'_G - p'_L = g\eta(\rho_G - \rho_L) + \sigma\eta_{xx}$$

on $z = 0$.

The perturbed pressures are evaluated in terms of stream functions, using the horizontal momentum equation of the linearized Euler equations for the perturbations,

$$\frac{\partial u'_i}{\partial t} = -\frac{1}{\rho_i}\frac{\partial p'_i}{\partial x} \quad \text{with} \quad i = L, G$$

to yield

$$p'_i = \rho_i c D\Psi_i, \quad i = L, G.$$

Putting this last equation and the jump condition on $p'_G - p'_L$ together,

$$c(\rho_G D\Psi_G - \rho_L D\Psi_L) = g\eta(\rho_G - \rho_L) + \sigma\eta_{xx},$$

Substituting the second interfacial condition $c\eta = \Psi$ and using the normal mode representation, this relation becomes

$$c^2(\rho_G D\Psi_G - \rho_L D\Psi_L) = g\Psi(\rho_G - \rho_L) - \sigma\alpha^2\Psi,$$

where there is no need to label Ψ (only its derivatives) because

$$\Psi_L = \Psi_G \quad \text{at} \quad z = 0.$$

Solution

Now the model of stratified flow has been set up, the solution is at hand. The stream function equation

$$(D^2 - \alpha^2)\Psi_i = 0,$$

with the boundary conditions $\Psi(\pm\infty)$ has the solution

$$\Psi_L = A_L e^{\alpha z}$$

$$\Psi_G = A_G e^{-\alpha z}.$$

The first interfacial condition states that $\Psi_L = \Psi_G \quad \text{at} \quad z = 0$, which forces $A_L = A_G = A$.The third interfacial condition states that

$$c^2(\rho_G D\Psi_G - \rho_L D\Psi_L) = g\Psi(\rho_G - \rho_L) - \sigma\alpha^2\Psi,$$

Plugging the solution into this equation gives the relation

$$Ac^2\alpha(-\rho_G - \rho_L) = Ag(\rho_G - \rho_L) - \sigma\alpha^2 A.$$

The A cancels from both sides and we are left with

$$c^2 = \frac{g}{\alpha}\frac{\rho_L - \rho_G}{\rho_L + \rho_G} + \frac{\sigma\alpha}{\rho_L + \rho_G}.$$

To understand the implications of this result in full, it is helpful to consider the case of zero surface tension. Then,

$$c^2 = \frac{g}{\alpha}\frac{\rho_L - \rho_G}{\rho_L + \rho_G}, \quad \sigma = 0,$$

and clearly,
(i) If $\rho_G < \rho_L, c^2 < 0$ and c is real. This happens if lighter fluid is on top.
(ii) If $\rho_G > \rho_L, c^2 > 0$ and c is purely imaginary. This happens if the heavier fluid is on top.
Now =,when heavier fluid is on top, $c^2 < 0$, and

$$c = \pm\sqrt{\frac{gA}{\alpha}}$$

,

$$A = \frac{\rho_G - \rho_L}{\rho_G + \rho_L}$$

where A is the Atwood number.
By taking the positive solution, we see that the solution is

$$\Psi(x,z,t) = ae^{-\alpha|z|}exp[\alpha(x - ct)] = aexp(\alpha\sqrt{\frac{gA}{\alpha}}t)exp(i\alpha x - \alpha \mid z \mid)$$

and this is associated to the interface position by

$$c\eta = \Psi.$$

Now define $B = a/c$. The time evolution of the interface elevation $z = \eta(x,t)$, initially at $\eta(x,0) = \Re Bexp(i\alpha x)$, is given by:

$$\eta = \Re Bexp(\sqrt{Ag\alpha t})exp(i\alpha x)$$

which grows exponentially in time.Here B is the amplitude of the initial perturbation and $\Re$. represents real part of complex valued expression .
In general, condition for linear stability is that the imaginary part of the "wave speed" c is positive. Finally,restoring the surface tension makes c^2 less negative and therefore stabilizing.Indeed, there is a range of short waves for which the surface tension stabilizes the system and prevents the instability.

5.2 Centrifugal Instabilities

Flows with curved streamlines, such as those sketched below, can be unstable due to the centrifugal effects of rotation. Here we focus on centrifugal instabilities in inviscid fluids. Our main focus is Rayleigh's criterion for the instability of a basic swirling flow with an arbitrary dependence of $\Omega(r)$ on r . This states that

$$\Phi(r) < o$$

for instability, where

$$\Phi = \frac{1}{r^3}\frac{d}{dr}(r^4\Omega^2)$$

In the first case we motivate the above equation using a physical argument and in the second, we prove it via linear stability analysis. Lastly, we apply it to flow between concentric cylinders. Note an analogy between these curvature driven instabilities and the thermal instabilities and the thermal instabilities discussed above. Fluid elements are forced outward by centrifugal effects in one case and upwards by their bouyancy in the other. The governing equations are as follows. For inviscid fluids, in the absence of body forces, the Navier-Stokes equations reduces to Euler's equations:
continuity $\nabla.u = 0$
Momentum balance $\rho[\partial_t u + (u.\nabla)u] = -\nabla p$
We use cylindrical coordinates throughout this section. We consider axisymmetric flows, which can depend on r and z but not θ. Componentwise we then have
continuity $\frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{\partial u_z}{\partial z} = 0$
momentum balance $\frac{Du_r}{Dt} - \frac{u_\theta^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r}$

$$\frac{Du_\theta}{Dt} + \frac{u_r u_\theta}{r} = 0$$

$$\frac{Du_z}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial z}$$

In which the material derivative

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u_r \frac{\partial}{\partial r} + u_z \frac{\partial}{\partial z}$$

5.3 Rayleigh's Criterion For Inviscid Stability

Consider an initially laminar azimuthal flow

$$u = u_\theta(r)\theta$$

with an arbitrary dependence of azimuthal velocity $u_\theta = r\Omega(r)$ on r. Rayleigh provided criterion to distinguish between stable and unstable distributions of the angular velocity $\Omega(r)$ using a simple physical argument, which we now describe. Noting that θ component of momentum balance $\frac{Du_\theta}{Dt} + \frac{u_r u_\theta}{r} = 0$,can be written as

$$\frac{D}{Dt}(ru_\theta) = 0$$

We see that the quantity $H = ru_\theta$, which is the angular momentum, is conserved for each material element. This is to be expected in the absence of viscous dissipation. Associated with the azimuthal motion is a kinetic energy per unit volume of

$$\frac{1}{2}\rho u_\theta^2 = \frac{1}{2}\frac{\rho H^2}{r^2}$$

Now consider two volume elements of equal volumes dV at radial locations $r = r_1$ and $r = r_2$ with $r_2 > r_1$. Their combined kinetic energy is

$$E = \frac{1}{2}\rho(\frac{H_1^2}{r_1^2} + \frac{H_2^2}{r_2^2})dV$$

Now imagine that these elements swap positions. By virtue of $\frac{D}{Dt}(ru_\theta) = 0$, each keeps its own angular momentum. After the swap, their combined energy is thus

$$E_{new} = \frac{1}{2}\rho(\frac{H_1^2}{r_1^2} + \frac{H_2^2}{r_2^2})dV$$

So the swap has resulted in an energy change

$$\Delta E \propto (H_2^2 - H_1^2)(\frac{1}{r_1^2} - \frac{1}{r_2^2})$$

If the swap has released energy ($\triangle E < 0, H_1^2 > H_2^2$),the laminar base flow will be unstable to such swaps. Thus, the criterion for instability is that H^2 decreases with r:

$$\frac{d}{dr}H^2 < 0 \quad \text{for instability}$$

Recalling that $H = ru_\theta = r^2\Omega$, the condition for instability is finally seen to be

$$\frac{d}{dr}(r^4\Omega^2) < 0 \quad \text{for instability}$$

This is consistent with our original statement that

$$\Phi = \frac{1}{r^3}\frac{d}{dr}(r^4\Omega^2)$$

5.4 Proof via Linear Stability Analysis

Governing Equations and Boundary Conditions

We apply the governing equations of inviscid axisymmetric flow to the flow between two impermeable boundaries located at $r = r_1, r_2$.

Base State

For the base state we consider a laminar swirling azimuthal flow

$$u_B = (u_r, u_\theta, u_z)^T = (0, r\Omega(r), 0)^T$$

with an arbitrary dependence of the angular velocity $\Omega(r)$ on radius, r.

Small Perturbation

We now subject the base state to small perturbation, still assuming axisymmetric ($\partial_\theta \ldots = 0$)

$$u = (0, r\Omega(r), 0) + \delta(u'(r,z,t), v'(r,z,t), w'(r,z,t)$$

with an analogous expression for the pressure p (δ is small).

Linearize the Equations

We substitute $u = (0, r\Omega(r), 0) + \delta(u'(r,z,t), v'(r,z,t), w'(r,z,t)$ into governing equations and expand in powers of δ. Neglecting terms $O(\delta^2)$ and higher, we get the linearized equations: Continuity

$$(\partial_r + \frac{1}{r})u' + \partial_z w' = 0$$

Momentum balance

$$\partial_t u' - 2\Omega(r)v' = -\frac{1}{\rho}\partial_r p'$$
$$\partial_t v' + u'\partial_r(\Omega(r)r)u' = 0$$
$$\partial_t w' = -\frac{1}{\rho}\partial_z p'$$

Solution of the linearized equations using Normal Modes

We now express the perturbation as a sum of normal modes:

$$[u'(r,z,t), v'(r,z,t), w'(r,z,t)] = \sum [u'(r), v'(r), w'(r)] exp(ikz + st)$$

with an analogous expression for the pressure. k is the wavevector in the axial direction. The equations

$$(\partial_r + \frac{1}{r})u' + \partial_z w' = 0$$

$$\partial_t u' - 2\Omega(r)v' = -\frac{1}{\rho}\partial_r p'$$
$$\partial_t v' + u'\partial_r(\Omega(r)r)u' = 0$$
$$\partial_t w' = -\frac{1}{\rho}\partial_z p'$$

then become

Continuity

$$(\frac{d}{dr} + \frac{1}{r})u' + ikw' = 0$$

Momentum balance

$$su' - 2\Omega(r)v' = -\frac{1}{\rho}\frac{dp}{dr}$$

$$sv' + u'\frac{d}{dr}[\Omega(r)r] + \Omega(r)u' = 0$$

$$sw' = -\frac{ik}{\rho}p'$$

The strategy now is to progressively eliminate p', v' and w' leaving a single equation for u'. We will then use this to distinguish between stable ($s_r < 0$) and unstable ($s_r > 0$) perturbations. First we eliminate p' by getting the difference between $ik \times (su' - 2\Omega(r)v' = -\frac{1}{\rho}\frac{dp}{dr})$ *and* $\frac{d}{dr} \times (sw' = -\frac{ik}{\rho}p')$ to get

$$ik[su' - 2\Omega v'] - s\frac{dw'}{dr} = 0$$

This leaves$[(\frac{d}{dr} + \frac{1}{r})u' + ikw' = 0, sv' + u'\frac{d}{dr}[\Omega(r)r] + \Omega(r)u' = 0, ik[su' - 2\Omega v'] - s\frac{dw'}{dr} = 0]$in u', v', w'. From $sv' + u'\frac{d}{dr}[\Omega(r)r] + \Omega(r)u' = 0$, we have

$$v' = -\frac{1}{s}[2\Omega + r\frac{d\Omega}{dr}]u'$$

which can be substituted into $ik[su' - 2\Omega v'] - s\frac{dw'}{dr} = 0$ to give

$$iksu' + \frac{2k^2\Omega}{s}[2\Omega + r\frac{d\Omega}{dr}]u' = s\frac{dw'}{dr}$$

We have now eliminated v' leaving $((\frac{d}{dr} + \frac{1}{r})u' + ikw' = 0$,$iksu' + \frac{2k^2\Omega}{s}[2\Omega + r\frac{d\Omega}{dr}]u' = s\frac{dw'}{dr}$) for u', w'. Multiplying $iksu' + \frac{2k^2\Omega}{s}[2\Omega + r\frac{d\Omega}{dr}]u' = s\frac{dw'}{dr}$ by ik/s,we get

$$-k^2u' - \frac{2k^2\Omega}{s^2}[2\Omega + r\frac{d\Omega}{dr}]u' = ik\frac{dw'}{dr}$$

This can finally be combined with $\frac{d}{dr} \times ((\frac{d}{dr} + \frac{1}{r})u' + ikw' = 0)$ to eliminate w', leaving a single equation in u'

$$\frac{d}{dr}[\frac{d}{dr} + \frac{1}{r}]u' - k^2u' - \frac{2k^2\Omega}{s^2}[2\Omega + r\frac{d\Omega}{dr}]u' = 0$$

Defining Φ as $\Phi = \frac{1}{r^3}\frac{d}{dr}(r^4\Omega^2)$, this can be written in simpler form

$$\frac{d}{dr}[\frac{1}{r}\frac{d}{dr}(ru')] - k^2u' = \frac{k^2}{s^2}\Phi(r)u'$$

Now multiplying across by $u'^{(c)}$, where (c) denotes complex conjugate, and integrating from r_1 *to* r_2, we get

$$\int_{r_1}^{r_2} ru'(c)\frac{d}{dr}[\frac{1}{r}\frac{d}{dr}(ru')]dr - k^2\int_{r_1}^{r_2} r \mid u' \mid^2 dr = \frac{k^2}{s^2}\int_{r_1}^{r_2} \Phi(r)r \mid u' \mid^2$$

Integrating the first term by parts we get

$$[ru'^{(c)}\frac{1}{r}\frac{d}{dr}(ru')]_{r_1}^{r_2} - \int_{r_1}^{r_2} \frac{1}{r} \mid \frac{d}{dr}(ru') \mid^2 dr - k^2 \int_{r_1}^{r_2} r \mid u' \mid^2 dr = \frac{k^2}{s^2} \int_{r_1}^{r_2} \Phi(r) r \mid u' \mid^2 dr$$

The first term is zero, because the boundaries at r_1, r_2 are impermeable. Hence the above equation has the form

$$-I_1 - k^2 I_2 = \frac{k^2}{s^2} \int_{r_1}^{r_2} \Phi(r) r \mid u' \mid^2 dr$$

In which $I_1 > 0, I_2 > 0$.The characteristic values $\frac{k^2}{s^2}$ are therefore all negative if $\Phi > 0$ throughout the interval $r_1 < r < r_2$.In contrast, if $\Phi < 0$ in some region then we can have $s^2 > 0$ and $s_r > 0$, denoting linear instability. This is in accordance with the Rayleigh'r criterion that

$$\Phi = \frac{1}{r^3}\frac{d}{dr}(r^4\Omega^2)$$

If Φ is positive everywhere, however, note that we still cannot conclude stability without considering non-axisymmetric disturbances well.We do not pursue that issue further.

5.4.1 Taylor Vortices

We now apply Rayleigh's criterion $\Phi = \frac{1}{r^3}\frac{d}{dr}(r^4\Omega^2)$ to Couette flow between infinitely long concentric cylinders. In particular, we are interested whether a basic swirl solution

$$u_B = (u_r, u_\theta, u_z) = (0, v(r), 0) = (0, r\Omega(r), 0)$$

is stable with respect to axisymmetric perturbations.The main task here is to derive the basic flow $v(r) = r\Omega(r)$.This can then be plugged directly into Rayleigh's criterion to determine stability/instability.

As shown earlier, the possible basic solution has the form

$$v(r) = A + \frac{B}{r}$$

To determine A and B, we apply boundary conditions at

$$r = R_1, \Omega = \Omega_1 = A + \frac{B}{R_1^2}$$

at

$$r = R_2, \Omega = \Omega_2 = A + \frac{B}{R_2^2}$$

Solving these gives

$$B = \frac{(\Omega_1 - \Omega_2)}{(\frac{1}{R_1^2} - \frac{1}{R_2^2})} = \Omega_1 R_1^2 \frac{(1 - \frac{\Omega_2}{\Omega_1})}{(1 - \frac{R_1^2}{R_2^2})}$$

Now let

$$\mu = \frac{\Omega_2}{\Omega_1}, \eta = \frac{R_1}{R_2} < 1$$

This gives

$$B = \Omega_1 R_1^2 \frac{(1-\mu)}{(1-\eta^2)}$$

and

$$A = -\Omega_1 \frac{(\eta^2 - \mu)}{(1-\eta^2)}$$

So

$$v(r) = A + \frac{B}{r}$$

and

$$B = \Omega_1 R_1^2 \frac{(1-\mu)}{(1-\eta^2)}, A = -\Omega_1 \frac{(\eta^2-\mu)}{(1-\eta^2)}$$

together give the laminar base flow. We now examine the linear stability of this base flow using Rayleigh's criterion. First we need to calculate

$$\Phi = \frac{1}{r^3} \frac{d}{dr} (r^4 \Omega^2)$$

$$= \frac{1}{r^3} \frac{d}{dr} (r^4 [A^2 + \frac{2AB}{r^2} + \frac{B^2}{r^4}])$$

$$= 4A^2 (1 + \frac{B}{Ar^2})$$

Recall that the condition for instability is $\Phi < 0$. Now $4A^2 > 0$ always, so the condition for instability is just the quantity $1 + \frac{B}{Ar^2}$ is negative. Expanding this expression with actual values of A and B we get

$$\Phi(r) = 4A^2 (1 - \frac{(1-\mu) R_1^2}{(\eta^2 - \mu) r^2})$$

We consider values of $\mu > 0$, corresponding to both cylinders rotating in the same sense (Ω_1 and Ω_2 having the same sign). In this case, it can be shown that

$$\Phi > 0 \quad \text{(giving stability) if } \mu > \eta^2$$

$$\Phi < 0 \quad \text{(giving instability) if } \mu < \eta^2$$

This is shown by the solid line in the figure overleaf. The dashed line shows the stabilizing effect of a non-zero viscosity, though we do not calculate that result here, as can be seen, the flow is linearly stable if only the outer cylinder rotates $(\Omega_1 = 0)$

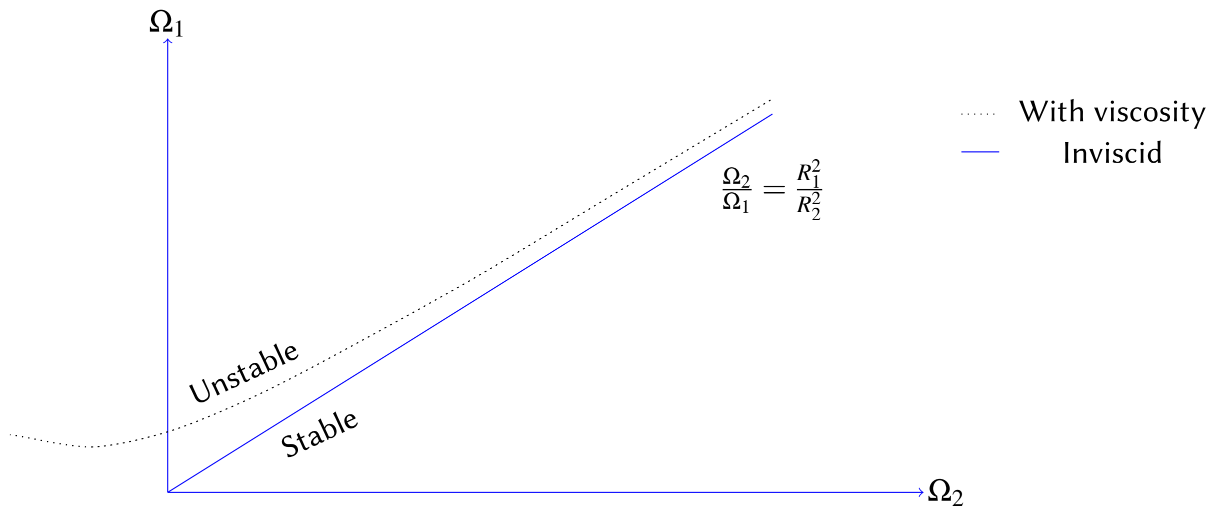


Figure: linear stability/instability of axisymmetric Couette flow for different (co)rotation rates of the outer and

6 Conclusion

In this project we have been able to determine the stability of the Taylor-Couette flow using the method of normal modes.The dynamics are parametrized by the three nondimensional numbers, the Reynolds number, determining the shear rate,the rotation number, describing the effect of the system rotation and the associated Coriolis force and the Taylor number, for determining the stability of the flow.The objective of the project was to determine the stability of the Taylor-Couette flow .We have also derived the equations of motion;the Navier-Stokes equations and solved simple cases of the Navier-Stokes equations;steady flow between parallel plates,Couette flow and Plane Poiseuille flow.
In order for us to determine the stability of the flow that exists between two concentric rotating cylinders; we took the following considerations:
(i) We took the basic Couette flow

$$V(r) = Ar + \frac{B}{r}$$

for the Navier-Stokes Equations of a viscous fluid.
Where we take rigid cylinders $r = R_1, R_2$ with angular velocities Ω_1, Ω_2 respectively with

$$A = \frac{\Omega_2 R_2^2 - \Omega_1 R_1^2}{R_2^2 - R_1^2}, B = \frac{\Omega_1 - \Omega_2}{R_1^- 2 - R_2^- 2}$$

(ii)We linearized the Navier-Stokes equations and the boundary conditions for small perturbations of the basic flow.
(iii) We chose dimensionless variables and parameters.
(iv) We took the normal modes of the form

$$u'(x,t) = u(r)e^{st+i(n\theta+kz)}$$

(v)We then derived an ordinary differential eigenvalue problem to find s, for given real wavenumber k and integral wavenumber n.

Since the resulting numerical problem was too difficult to solve; we simplified our problem based on the following assumptions:
(i)Most unstable perturbations are axisymmetric and so $n = 0$.
(ii)The principle of exchange of stabilities is valid, i.e. $Im(s) = 0$ at the onset of instability, and so $s = 0$ thus we sought dimensionless parameters which give the margin of instability.
(iii) There is a narrow gap between the cylinders, i.e. $R_2 - R_1 << R_2$.

We have also studied hydrodynamic stability ;linear stability analysis by analysing the stability using bifurcation theory.
Our hope is that this preliminary study will help to increase understanding of the subject of Hydrodynamics and stability of Taylor-Couette Flow.

According to Rayleigh's criterion, the flow is unstable if the inner cylinder rotates and the outer cylinder is at rest, and stable if the outer cylinder rotates and the inner is at rest. If the cylinders rotate in opposite directions, however, the circulation decreases outward in at least part of the flow field, and the flow is unstable.
The current study concludes some preliminary investigations on the stability of incompressible Taylor-Couette flow. Still their are many other interesting aspects that are needed to be further examined, such as the effects of axial flow, gap width, finite axial length and other possible influential factors.Small gap limitation was successively removed to allow for the determination of critical Taylor numbers for finite gap width.In all of these investigations, the stability of the basic flow has been considered only with respect to axisymmetric disturbances. However, it is known from experiments that non-axisymmetric disturbances play an important role in the instability of Taylor vortices.It has been indicated that, when the ratio of the rotational speeds of the outer and inner cylinders is sufficiently negative, the critical speed for Taylor-Couette flow may occur for axisymmetric disturbances.For example Coles noted that for counter-rotating cylinder a weak spiral configuration is quite typical of the Taylor instability boundary, except at low Reynolds numbers for the outer cylinder.Furthermore, it has been found from experiments by Snyder that the lowest mode of instability is in a non axisymmetric form, depicting a weak helical motion similar to that observed by Coles.

We can use the knowledge of our study in the purification of water through a process called Taylor vortex photocatalytic reactor for water purification.Heterogeneous photocatalysis on semiconductor particles has shown to be an effective means of removing toxic organic pollutants from water.Unsteady Taylor-Couette flow between two coaxial cylinders, where the inner cylinder (coated with TiO_2 catalyst) is rotated to achieve the desired instability.The main advantage of considering this type of flow pattern is its creation of wavy vortex flow in the laminar-flow regime.Significant transfer of fluid between neighboring vortices occurs in a cyclic fashion along certain wave, and net axial flow also occurs in which fluids wind around the vortices.Our future research is on the large-scale purification of water using our knowlegde on the Taylor-Couette flow.

Bibliography

[AB07] Avila M,Belisle M,J, et al *Mode Competition in Modulated Talor-Couette Flow* 2007.

[Sag01] Bruce E. Sagan. *The Symmetric Group,Representations, Combinatorial Algorithms, and Symmetric Functions, Graduate Texts in Mathematics*, 2001.

[Ful14] William Fulton. *Young Tableaux, with Applications to Representation Theory and Geometry, Landon Mathematical Society Student Texts, 2014.*

[BJ89] Barenghi F, Jonnes, *Modulated Taylor-Couette Flow* 1989.

[CT81] Carmi S. and Tustaniwskyj J. *Stability of Modulated Finite-Gap Cylindrical Couette Flow* 1981.

[CJ03] Criminale W.O, Jackson T.L, et al :*Theory and Computation of Hydrodynamic Stability,* 2003.

[CH61] Chandrasekhar S. *Hydrodynamic and Hydronagnetic Stability* 1961

[CH61] Charru F. *Hydrodynamic Instabilities* 1961

[CO65] Coles D. *Transition in Circular Couette Flow* 1965

[CL07] Czarny O.and Leuptow *Time Scales for Transition in Taylor-Couette Flow* 2007.

[DO90] Donnelly R.J. *Externally Modulated Hydrodynamic Systems in Nonlinear Evolution of Spatio-Temporal Structures in Dissipative Continuous Systems* 1990

[DR02] Drazin P.G. *Introduction to Hydrodynamic Stability.* 2002.

[DR04] Drazin P.G and Reid W.H *Hydrodynamic Stability,* 2004.

[EC65] Eckhaus W. *Studies in Nonlinear Stability Theory,* 1965.

[EG96] Esser A. and Grossmann S. *Analytical Expression for Taylor-Couette Stability Boundary* 1996

[HA75] Hall P. *The Stability of Unsteady Cylinder Flow,* 1975.

[LA98] Linek M. and Ahlers G *Boundary Limitation of Wavenumbers in Taylor-Vortex Flow,* 1998.

[LO20] Lord Rayleigh *On The Dynamics of Resolving Fluids,* 1920.

[MA99] M.A Ali *The Stability of Couette Flow with an Inner Cylinder Rotating and Moving with a Constant Axial Velocity.* 1999.

[NA90] Ning L, Ahler G et al. *Wave number Selection at Finite Amplitude in Rotating Couette Flow,* 1990.

[SA85] Savas O. *On Flow Visualization using reflexive flakes,* 1985.

[SN69] Snyder H.A. *Wave number Selection at Finite Amplitude in Rotating Couette Flow,* 1965

[TA94] Tagg R. *The Couette-Taylor Problem,* 1994.

[SA86] Swinney H.I, Andereck C.D. et al. *Flow Regimes in a Circular Couette System with Independently Rotating Cylinders,* 1986.

[SG75] Swinney and Gollub J.P. *Onset to Turbulence in a Rotating Fluid,* 1975.

[TA79] Takeuchi D.I. *A Numerical and Experimental Investigation of the Stability of Spiral Poiseuille Flow,PhD Thesis.* 1979.

[TJ81] Takeuchi D.I. and Jankowski D.F. *A Numerical and Experimental Investigation of the Stability of Spiral Poiseuille Flow,* 1981.

[TA23] Taylor G.I. *Stability of a Viscous Liquid Contained Between Two Rotating Cylinders,* 1923.

[TH68] Thompson R.J. *Instabilities of Some Time-dependent Flow,PhD Thesis.* 1968

[TB90] Tuckerman L.S. and Barkley D. *Bifurcation Analysis of the Eckhaus Instability,* 1990.

[WD88] Walsh T.J. and Donnelly R.J *Taylor-Couette Flow with Periodically Corrugated and Counter Rotated Cylinders,* 1988.

[WE96] Weisberg A.Y. *Control of Transition in Taylor-Couette Flow with Axial Modulation of the inner Cylinder,* 1996.

[YW03] Youd A.J.,Willis A.P. et al. *Reversing and Non-reversing Modulated Taylor-Couette Flow,* 2003.

[YB05] Youd A.J, Barenghi C.F., et al. *Non-reversing Modulated Taylor-Couette Flows,* 2005.