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On Equivalence Of Some Operators And Topology Of Invariant Subspaces

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Wilfred Gitonga

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Submitted to the School of Mathematics in partial fulfilment for a degree in Master of Science in Pure Mathematics

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Master of Science Project

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Abstract

In this project we investigate the invariant and hyper-invariant subspace lattices of some operators. We give a lattice theoretic description of the lattice of hyper-invariant subspace of an operator in terms of its lattice of invariant subspace. We also study the structure of these lattices for operators in certain equivalence classes.

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Declaration and Approval

I the undersigned declare that this project report is my original work and to the best of my knowledge, it has not been submitted in support of an award of a degree in any other university or institution of learning.

Signature Date

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In my capacity as a supervisor of the candidate, I certify that this report has my approval for submission.

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Dedication

I dedicate this work to my son Rashford and my wife Betty.

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Acknowledgments

Am grateful to the Almighty God for the strength and sound health in each and every step of this long and tiresome journey.
My very sincere gratitude goes to my very able supervisors Dr. Nzimbi and Prof. Khalagai for their continued support, endurance, push and shove that made this project a success. May God bless you abundantly. I would also not fail to recognize the very important role played by my lecturers throughout the course in entirety; Prof. Weke, Prof. Pokhariyal, Dr. Were, Dr. Maingi, Mr. Achola, Dr. Muriuki, Dr. Luketero and Dr. Katende.
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Wilfred Gitonga

Nairobi, 2018.

1 PRELIMINARIES

1.1 Introduction

This chapter outlines the historical background, notations, terminologies and definitions that shall be used throughout this work.

1.1.1 Historical Background

Functional analysis is a branch of mathematics which studies the analytical structures of a vector space endowed with a topology. Functional analysis emerged as a distinct field in the 20th century when it was realized that diverse mathematical processes from arithmetic to calculus procedures, exhibit very similar properties. One of the branches of functional analysis is operator theory. Operator theory found the limelight in 1900 when David Hilbert held a conference in Paris and posed 23 famous problems.A linear operator is a linear transformation from a vector space to itself. We can therefore confirm that a linear operator is a transformation which maps linear subspaces to linear subspaces. In studying the lattices of operators, we are going to view our operators as matrices which is the model for operator theory. Toeplitz [20] found out that every linear operator can be represented by a matrix for easier operations on these operators. In this work we are going to investigate the lattice invariant subspace of some operators which is a branch of operator theory. The invariant subspaces of an operator, their classification and description play a central role in operator theory. They are analogue of the eigenvectors of a linear operator. Reducing subspaces are useful in direct sum decomposition of an operator. The basic motivation for the study of invariant subspace comes from the interest in the structures of the operators and from approximation theory. The knowledge of hyper-lat $T \in B(\mathscr{H})$ give information about the structure of the commutant of T. The ordinary sum decomposition of an operator $T \in B(\mathscr{H})$ that is, $T = T_1 + T_2$ is important in operator theory, but will not be used in the sequel, since, unlike the direct sum decomposition, it misses a crucial feature of failing to transfer invariant subspace from the parts, (the direct summands), to the operator itself, (the direct sum). In other words, if a subspace is invariant under the direct sum, then is invariant under the direct summand.

1.1.2 Notations,Terminologies and Definitions

Notations

In this work we will denote by
$\mathscr{H}$: Complex separable Hilbert space.
$B(\mathscr{H})$: Banach Algebra of bounded linear operator.
T^* : Adjoint of an operator $T \in B(\mathscr{H})$.
$Ker(T)$: Kernel of an operator $T \in B(\mathscr{H})$.
$Ran(T)$: Range of an operator $T \in B(\mathscr{H})$.
$\mathscr{M}$: Closure of a closed subspace $\mathscr{M}$ of $\mathscr{H}$.
$\mathscr{M}^\perp$: an orthogonal complement of a closed subspace of $\mathscr{H}$.
$< x,y >$: The inner product of x and y on the Hilbert space $\mathscr{H}$
$Lat(T)$: The subspace lattice of all invariant subspace of $T \in B(\mathscr{H})$.
$Red(T)$: The subspace lattice of all reducing subspace of $T \in B(\mathscr{H})$
$Hyperlat(T)$: The subspace lattice of all hyper invariant subspaces of $T \in B(\mathscr{H})$.
$W(T)$: The numerical range of an operator $T \in B(\mathscr{H})$.
$P_{\mathscr{M}}$: Orthogonal Projection onto a closed subspace $\mathscr{M} \subseteq \mathscr{H}$.
$W^*(T)$: The weakly closed von Neumann algebra generated by T.

Terminologies and Definitions

Definition 1.1.1. *An operator $T \in B(\mathscr{H})$ is said to be:*
normal if $TT^* = T^*T$
unitary if $TT^* = T^*T = I$
projection if $T^2 = T$ and $T^* = T$
scalar if $T = \alpha I$ for some α a scalar and identity operator I.
compact if for each bounded set $\mathscr{M} \subseteq \mathscr{H}$ the closure of the image $T(\mathscr{M})$ is compact.

Definition 1.1.2. *An operator$T \in B(\mathscr{H}, \mathscr{K})$ is said to be quasi-affinity if it is injective and with dense range.*

Definition 1.1.3. *Two operators $A \in B(\mathscr{H}), B \in B(\mathscr{K})$ are said to be:*
Similar; if there exists an invertible operator $N \in B(\mathscr{H}, \mathscr{K})$ such that $NA = BN$.
unitarily equivalents; if there exist a invertible unitary operator $U \in B(\mathscr{H}, \mathscr{K})$ such that $UA = BU$.
quasi-similar; if there exist quasi-affinities $X \in B(\mathscr{H}, \mathscr{K})$ and $Y \in B(\mathscr{K}, \mathscr{H})$ such that $XA = BX$ and $AY = YB$.
quasi-affine transform of each other; if there exists a quasi-affinity $X \in B(\mathscr{H}, \mathscr{K})$ such that $XA = BX$.

metrically equivalent*; if* $A^*A = B^*B$.
almost similar*; if there exists an invertible operator* N *such that the following conditions are satisfied;* $A^*A = N^{-1}\ (B^*B)\ N$ *and* $A^* + A = N^{-1}\ (B^* + B)\ N$.

Definition 1.1.4. *A quasi-affinity* X *is said to have the hereditary property with respect to an operator* $T \in B(\mathscr{H})$ *if* $X \in \{T\}^{'}$ *and* $\overline{X(\mathscr{M})} = \mathscr{M} \forall\ \mathscr{M} \in Hyperlat(T)$.

Definition 1.1.5. *A lattice is a partially ordered set say* X, *in which every pair of elements has a least upper bound and a greatest lower bound.*

If every pair $\{x,y\}$ of elements of X is bounded above, then X is a *directed set* (or the set X is said to be directed upward). If every pair $\{x,y\}$ is bounded below, then X is said to be directed downward.Every lattice is directed both upward and downward. If every bounded subset of X has a supremum and an infimum, then X is a *boundedly complete lattice*. If every subset of X has a supremum and an infimum, then X is a *complete lattice*. Then we have the following chain of implications:

$$complete\ lattice \Rightarrow boundedly\ complete\ lattice \Rightarrow lattice \Rightarrow directed\ set$$

Definition 1.1.6. *A subspace* $\mathscr{M} \subseteq \mathscr{H}$ *is said to be:*
• *invariant under* $T \in B(\mathscr{H})$ *if* $T\mathscr{M} \subseteq \mathscr{M}$. *In this case , we say that the subspace* $\mathscr{M}$ *is* $T-$*invariant.*
•*reducing under* $T \in B(\mathscr{H})$ *if it is invariant under* T *and* T^*. *Equivalently an operator is said to be reducible if it has a nontrivial reducing subspace. A subspace that is not reducible is said to be irreducible. This means that an operator is irreducible if it has no reducing subspace other than* $\{0\}$ *and* $\mathscr{H}$.
• *hyperinvariant under* $T \in B(\mathscr{H})$ *if* $S\mathscr{M} \subseteq \mathscr{M} \forall$
$S \in \{T\}^{`}$.
• *hyper-reducing under* $T \in B(\mathscr{H})$ *if it reduces every operator in the* $\{T\}^{`}$

We use $Lat(T)$, $Red(T)$, $Hyperlat(T)$ *and* $HyperRed(T)$ the collection of all subspaces invariant, reducing, hyperinvariant and hyper-reducing for $T \in B(\mathscr{H})$. Clearly

$$HyperRed(T) \subseteq Lat(T).[12].$$

The concept of hyper-reducibility of a subspace of a Hilbert space was introduced by Moore[15]. We will prove in Chapter 3 that

$$HyperRed(T) = Red(\{T\}^{'}) = Lat(\{T\}^{'}) \cap Lat(\{T^*\}^{'}).$$

An operator $T \in B(\mathscr{H})$ is said to be reducible if it has a nontrivial reducing subspace (equivalently, if it has a proper nonzero direct summand-that is, if there exists a subspace $\mathscr{M}$ of $\mathscr{H}$ such that $\mathscr{M}$ and $\mathscr{M}^{\perp}$ are nonzero and $T-invariant$(see[14]). This is equivalent to saying that if $\mathscr{M}$ is nontrivial and invariant under T and T^*. A subspace that is not reducible is said to be irreducible. This means that an operator is irreducible if it has no reducing subspace other than $\{0\}$ and $\mathscr{H}$. It has been shown in[11] that an operator $T \in B(\mathscr{H})$ is reducible if and only if there exists a non-scalar operator L such that $LT = TL$ and $T^*L = LT^*$. That is if and only if there exists a non-scalar operator $L \in \{T\}^{'} \bigcap \{T^*\}^{'}$ Equivalently, T is reducible if and only if both T and T^* lie in $\{L\}^{'}$for some non-scalar operator L.

Definition 1.1.7. *A subspace lattice on a Hilbert space $\mathscr{H}$ is a family of subspaces of $\mathscr{H}$ which is closed under the formation of arbitrary intersections and arbitrary linear spans and which contains the zero subspace $\{0\}$ and $\mathscr{H}$.*

Remark 1.1.8. *The set of all invariant subspaces of $T \in B(\mathscr{H})$ is a lattice. The subspace lattice of all invariant, reducing and hyperinvariant subspaces of T are complete, in the sense that intersections and closed linear spans of subspaces are also in these lattices. Since T commutes with itself, we have that:*

$$Hyperlat(T) \subseteq Lat(T)$$

and

$$Red(T) \subseteq Lat(T).$$

If T_1 and T_2 are quasisimilar and there exists an implementing pair (X,Y) of quasiaffinities such that XY has the hereditary property with respect to T_1 and YX has the hereditary property with respect to T_2 then we say that T_1 is hyperquasisimilar to T_2. This is denoted by $T_1 \approx T_2$. The notion of hyper-quasisimilarity was introduced by C. Foias etal [6]. We note that hyper-quasisimilarity is strictly stronger than quasisimilarity . In fact the following inclusion of operator equivalences is true;

$$Similar \subset Hyper-quasisimilar \subset Quasisimilar.$$

1.2 Some Properties Of Invariant Subspaces

Theorem 1.2.1. *Suppose $T \in B(\mathscr{H})$ is normal and $\mathscr{M}$ is a subspace of finite dimensional Hilbert space $\mathscr{H}$ that is invariant under T. Then $\mathscr{M}^{\perp}$ is invariant under T and hence $\mathscr{M}$ reduces T.*

Proof. Let $e_1,...,e_m$ be the orthonormal basis of $\mathscr{M}$. Extend to an orthonormal basis $\beta = (e_1,...e_m, f_1...f_n)$ of $\mathscr{H}$. Thus the matrix of T with respect to the basis of $\mathscr{H}$ is of the form
$[T]_\beta = \begin{bmatrix} A & B \\ 0 & C \end{bmatrix}$
since T is normal,we have:

$$\sum_{j=1}^{n} \| Te_j \|^2 = \sum_{j=1}^{n} \| T^* e_j \|^2$$

Thus B is an operator of all zeros ,hence the results. □

Definition 1.2.2. *A subspace $\mathscr{M} \subseteq \mathscr{H}$ is said to be inaccessible if the only continuous mapping Ψ of the closed interval $[0,l]$ into $Lat(T)$ with $\Psi(0) = \mathscr{M}$ is the constant map $\Psi(t) \cong \mathscr{M}$.*

Corollary 1.2.3. *[4]If $T \in B(\mathscr{H})$ and $\mathscr{M}$ is an inaccessible invariant subspace for T, then M is hyperinvariant subspace for T.*

Corollary 1.2.4. *[4]If $\mathscr{M}$ is an isolated point of $Lat(T)$, then $\mathscr{M}$ is hyperinvariant subspace for T.*

Definition 1.2.5. *We say that a subspace $\mathscr{M} \in Lat(T)$ commutes with $Lat(T)$ if $P_{\mathscr{M}}$ commutes with $P_{\mathscr{N}}$ $\forall$ $\mathscr{N} \in Lat(T)$, and that $\mathscr{M}$ is a pinch point of $Lat(T)$ if $\forall$ $\mathscr{N} \in Lat(T)$ either $\mathscr{N} \subset \mathscr{M}$ or $\mathscr{M} \subset \mathscr{N}$.*

Theorem 1.2.6. *[4]If T is an operator on a finite-dimensional Hilbert space, then there exists an integer $k \geq 1$ and nilpotent operators $N_1...N_k$ on finite-dimensional spaces such that $Lat(T)$ is homeomorphic to the product space*

$$Lat(N_1) \times ... \times Lat(N_k).$$

Corollary 1.2.7. *If T is any operator on a finite-dimensional Hilbert space, then the isolated points of $Lat(T)$ can be specified exactly.*

Proof. We know that there exists an integer $k \geq 1$ and nilpotent operators $N_1...N_k$ on finite-dimensional spaces such that $Lat(T)$ is homeomorphic to the product space

$$Lat(N_1) \times ... \times Lat(N_k)$$

Since the isolated invariant subspaces of the operators $N_i (1 \leqslant i \leqslant k)$ are completely determined, clearly a subspace $\mathscr{M} \in Lat(T)$will be isolated if and only if each of the subspaces $\mathscr{M}_i \in Lat(N_i)$ is an isolated point in $Lat(N_i)$. □

Proposition 1.2.8. *If $T \in B(\mathscr{H})$ and $p \in P(F)$, then null $p(T)$ is invariant under T.*

Proof Suppose $T \in B(\mathscr{H})$ and $p \in P(F)$. Let $v \in null\ p(T)$. Then $p(T)v = 0$. Thus $(p(T))(Tv) = T(p(T)v) = T(0) = 0$, and hence $Tv \in null\ p(T)$. Thus null $p(T)$ is invariant under T, as desired.

1.2.1 Invariant Subspaces And Local Spectral Theory

Definition 1.2.9. *An operator $T \in B(\mathscr{H})$ is said to be decomposable, provided that, for each open cover $\mathbf{C} = U \cup V$ of the complex plane $\mathbb{C}$, there exist $Y, Z \in Lat(T)$ for which*

$$\mathscr{H} = Y + Z, \sigma(T_{|Y}) \subseteq U and\ \sigma(T_{|Z}) \subseteq V$$

where σ is the spectrum.

This simple definition is equivalent to the original notion of decomposability, as introduced by Foias[3]and discussed in the classical books by Colojoarva and Foias. The theory of decomposable operators is now richly developed with many interesting applications and connections. Evidently, the class of decomposable operators contains all normal operators on Hilbert spaces and more generally, all spectral operators in the sense of Dunford on Banach spaces. Moreover, a simple application of the Riesz functional calculus shows that all operators with totally disconnected spectrum are decomposable. In particular, all compact and algebraic operators are decomposable.
Given an arbitrary operator $T \in B(\mathscr{H})$, let $\sigma_T(x) \subseteq \mathbb{C}$ denote the local spectrum of T at the point $x \in \mathscr{H}$ i.e. the complement of the set $\sigma_T(x) \forall \lambda \in C$ for which there exist an open neighborhood U of λ in $\mathbb{C}$ and an analytic function $f : U \to X$ such that $(T - \mu)f(\mu) = x$ holds for all $\mu \in U$. For every closed subset $\mathscr{M}$ of $\mathbb{C}$, let $X_T(\mathscr{M}) = \{x \subseteq \mathscr{H} : \sigma_T(x) \subseteq \mathscr{M}\}$ denote the corresponding analytic spectral subspace of T.

Definition 1.2.10. *An operator $T \in B(\mathscr{H})$ is said to have Dunford's property if $X_T(\mathscr{M})$ is closed for every closed subspace $\mathscr{M} \subseteq \mathbb{C}$.*

Theorem 1.2.11. *Let $T \in B(\mathscr{H})$ be a bounded linear operator on a Hilbert space $\mathscr{H}$ of dimension greater than 1. If $T \in B(\mathscr{H})$ has both Dunford's property and decomposition property, then T has a non-trivial invariant closed linear subspace.*

`Proof.` Suppose that $T \in B(\mathscr{H})$ has both Dunford‘s property and decomposition property on a Hilbert space $\mathscr{H}$ of dimension greater than 1. Then $T$is decomposable. At first, we know that if $\sigma(T)$ contains at least two points, then T has a non-trivial hyperinvariant closed linear subspace. Since T is decomposable, it follows from that T has a non-trivial hyperinvariant closed linear subspace. It remains to consider the case of operator $T \in B(\mathscr{H})$ such that $\mathscr{H}$ is at least two-dimensional and $\sigma(T)$ is a singleton. Then it follows that $T = \lambda I + N$ for some $\lambda \in \mathbb{C}$ and some nilpotent operator $N \in B(\mathscr{H})$. Let $p \in \mathbb{Z}$ be the smallest integer for which $N^p = 0$, and choose an $x \in \mathscr{H}$ for which $N^{p-1} \neq 0$. The linear subspace generated by $N^{p-1}x$ is a one-dimensional $T-$ invariant linear subspace of $\mathscr{H}$. This completes the proof. $\square$

Corollary 1.2.12. *Every generalized scalar operator on a Banach space of dimension greater than 1 has a non-trivial invariant closed linear subspace.*

2 LITERATURE REVIEW

The invariant subspace problem asks whether every operator on a complex separable Hilbert space has a nontrivial invariant subspace. This problem has its origin approximately when, (according to Aronszaj and Smith), J. von Neumann [17]proved that every compact operator on a separable infinite-dimensional complex Hilbert space has a nontrivial invariant subspace.

Aronszaj and Smith[2] proved that every compact operator on any Hilbert space of dimension at least two has non-trivial invariant subspace. Hoover [10] studied hyper invariant subspaces and proved that if and are quasi similar operators and if S has a hyper invariant subspace then so does T. if in addition, S is normal, then the lattice of hyper invariant subspaces for T contains a sub-lattice which is lattice isomorphic to the lattice of spectral projections for S. Similar results for hyper-invariance have been studied by Kubrusly [11] and has shown that similarity preserves nontrivial subspaces while quasisimilarity preserves hyperinvariant subspaces. Lomonosov [12] proved that an operator has a non-trivial invariant subspace if it commutes with a non-scalar operator that commutes with a non-zero compact operator. Brown [16] proved that every subnormal operator has a non-trivial invariant subspaces. Atzmon [5] gave an example of an operator without invariant sub-spaces on a nuclear Frechet space. Brown, Chevreau and Pearcy [1] showed that every contraction on a Hilbert space whose spectrum contains the unit circle has a nontrivial invariant subspace. Brown [6] proved that every hyponormal operator whose spectrum has nonempty interior has a non-trivial invariant subspace. Herrero [9] proved that the structure of the hyperlattice of an operator is not preserved under quasi similarity. Gamal [7] proved that the lattices of invariant subspaces remain isomorphic under quasiaffine transforms. Nagy and Foias [21] proved that a contraction T has a non-trivial invariant sub-space. In addition Nzimbi etal[18] introduced the concept of metric equivalence of operators. The author explained metric equivalence relation and closely related relations on some classes of operators. He also described the spectral picture of metrically equivalent operators and gave some conditions when metric equivalence of operators implies unitary equivalence.

However, there is very little in the literature about relationship between invariant,reducing and hyper-invariant subspace lattices for operators in some equivalence relations and that this thesis sets out to show some of these relationships. For example, we wish to investigate if given the lattices of two operators say A and B, we can determine the structure and properties of the operators and how they are related.

3 OPERATOR EQUIVALENCE AND LATTICES

In this chapter we are going to study the lattices of operators in some equivalence classes.

3.1 Isomorphism Of Lattices

We now present some results on isomorphism of invariant subspace lattices.

Definition 3.1.1. *(Isomorphism of Hilbert spaces). Two Hilbert spaces $\mathscr{V}$ and $\mathscr{W}$ over the same field $\mathbb{F}$ are isomorphic if there is a bijection $T : \mathscr{V} \mapsto \mathscr{W}$ which preserves addition and scalar multiplication.*

Theorem 3.1.2. *Suppose that $T \in B(\mathscr{H})$, where H is a finite dimensional Hilbert space and $\varphi : B(\mathscr{H})(\mathscr{H})$ defined by $T \mapsto \varphi(T)$ is a linear map. Then the following statements are equivalent.*
(i) $Lat(T) \cong Lat(\varphi(T))$.
(ii) $Hyperlat(T) \cong Hyperlat(\varphi(T))$.
(iii) $Red(T) \cong Red(\varphi(T))$.

From above theorem, we can conclude that $Lat(T) \cong Lat(cT)$,$Hyperlat(T) \cong Hyperlat(cT)$ and $Red(T) \cong Red(cT)$. where $0 \neq c \in \mathbb{C}$.

Theorem 3.1.3. *Similarity of operators preserves non-trivial invariant and non-trivial hyperinvariant subspaces.*

Proof. We prove the case for invariance and the proof for hyperinvariance can be proved similarly.Suppose $A, B \in B(H)$ are such that $A = X^{-1}BX$;That is, $XA = BX$. Thus $BX\mathscr{M} = XA\mathscr{M} \subseteq X\mathscr{M}$.
Since $\mathscr{M}$ is non-trivial and X is invertible, we conclude that XM is a non-trivial invariant subspace for B. Thus $\mathscr{M}$ is $A-invariant$ if and only if $\mathscr{M}$ is $B-invariant$. □

It has been proved([9], [10]) that if A and B are quasisimilar and one has a nontrivial hyperinvariant subspace, then so does the other. However, similar (quasisimilar) operators need not have isomorphic invariant (hyperinvariant) lattices. An example is given in Herrero[9] of two quasisimilar nilpotent operators of the same order but with non-isomorphic hyperlattices. This shows that structure of the hyperlattice of an operator is not preserved under quasisimilarity.

Example 3.1.4. *Let* $A = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$ *and* $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. *A simple computation shows that* A *and* B *are similar. However another computation shows that*

$$Lat(A) = \{\{0\}, span \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \mathbb{R}^2\}$$

$$Lat(B) = \{\{0\}, span \begin{bmatrix} 1 \\ 0 \end{bmatrix}, span \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \mathbb{R}^2\}$$

Clearly, $Lat(A)$ *and* $Lat(B)$ *are not isomorphic.*

Theorem 3.1.5. *([11]) Hyper-quasi-similarity preserves nontrivial hyperinvariant invariant subspaces.*

Proof. This follows easily from the fact that hyperquasi-similarity is stronger than quasisimilarity and the fact that quasisimilarity preserves non-trivial hyperinvariant subspaces. □

Remark 3.1.6. *Kubrusly in[11] has shown that non-scalar normal operators have non-trivial hyperinvariant subspaces.*
Thus an operator quasisimilar to a non-scalar normal operator has a non-trivial hyperinvariant subspace.

Remark 3.1.7. *Hyperquasi-similarity is an equivalence relation.*

3.2 Equality Of Lattices

In this section we are going to discuss and give conditions when the invariant subspace lattices for any two or more operators on a Hilbert space $\mathscr{H}$ are equal. The following result due to [18]is well known.

Theorem 3.2.1. *[18] Metrically equivalent self-adjoint operators on a finite dimensional Hilbert space have a common non-trivial invariant subspace.*

Proof. Let $A, B, \in B(\mathscr{H})$ be self-adjoint and $\mathscr{M}, \mathscr{N}$ be nontrivial $A-invariant$ and $B-invariant$ subspaces, respectively.
Then $A^*A\mathscr{M} \subseteq \mathscr{M} \subseteq B^*BN \subseteq \mathscr{N}$ and $B^*B\mathscr{N} \subseteq \mathscr{N} \subseteq A^*AM \subseteq \mathscr{M}$ which implies that $\mathscr{M} = \mathscr{N}$. □

Theorem 3.2.2. *Two orthogonal projections on a Hilbert space $\mathscr{H}$ are metrically equivalent if and only if they are equal.*

Proof. Let P and Q be orthogonal projections on a Hilbert space $\mathscr{H}$. If $P^*P = Q^*Q$, then $P^2 = Q^2$, which implies that $P = Q$. The converse is trivial. $\square$

Corollary 3.2.3. *If P and Q are metrically equivalent projections, then $Lat(P) = Lat(Q)$.*

Theorem 3.2.4. *If A and B are metrically equivalent positive operators, then $Lat(A) = Lat(B)$.*

Proof. In this case $A = B$. The claim follows trivially. $\square$

Proposition 3.2.5. *For any operators A and B on finite-dimensional Hilbert spaces $\mathscr{H}$ and $\mathscr{K}$ over a field $\mathbb{F}$, the following are equivalent:*
(a) $Hyperlat(A \bigoplus B) = Hyperlat(A) \bigoplus Hyperlat(B)$.
(b) The minimum polynomials of A and B are relatively prime.

Theorem 3.2.6. *If $T \in B(\mathscr{H})$ is quasisimilar to a unitary operator $U \in B(\mathscr{K})$ then*

$$Hyperlat(T) \subseteq Red(U).$$

Proof. Follows from the fact that for a unitary operator U,

$$Hyperlat(U) = Red(U).$$

$\square$

Theorem 3.2.7. *(von Neumann Double Commutant Theorem) Let $\mathscr{H}$ be a Hilbert space and $a \subseteq B(\mathscr{H})$ be a unital self-adjoint *-subalgebra of $B(\mathscr{H})$. Then the following conditions are equivalent.*
(i) $a = \{a\}''$.
(ii) a is closed with respect to the weak operator topology on $B(\mathscr{H})$.
(iii) a is closed with respect to the strong operator topology (SOT) on $B(\mathscr{H})$.

If a unital (self-adjoint) *-subalgebra a of $B(\mathscr{H})$ satisfies either of the three equivalent conditions in above, we say that it is a von Neumann algebra. The Double Commutant Theorem simply asserts that the double commutant $a = \{a\}''$ of a unital self-adjoint subalgebra a of $B(\mathscr{H})$ is always strongly closed (and hence weakly closed). That is, a is strongly (and hence weakly) dense in $a = \{a\}''$. Equivalently, it says that the strongly closed unital self-adjoint subalgebras of $B(\mathscr{H})$ are always their own double commutant.

For convenience, we take a von Neumann algebra as a *- subalgebra a of $B(\mathscr{H})$ satisfying $a = \{a\}^{''}$. A von Neumann algebra is a unital, weakly closed and contains an abundance of projections. If a is a von Neumann algebra, then a is generated by the projections in a. Let $T \in B(\mathscr{H})$. We define $W^*(T)$ to be the von Neumann algebra generated by $\{I, T\}$. Note that

$$W^*(T) = \{T\}^{''} \cup \{\alpha I : \alpha \in \mathbb{C}\}.$$

From the Double Commutant Theorem, if $T = T^*$, then $\{T\}^{''} = W^*(T)$ and$\{T\}^{'}$ is a von Neumann algebra and is therefore generated by its projections. Since the projections in $\{T\}^{'}$ are also in $\{T^*\}^{'}$ it follows that the Double Commutant Theorem has the following reformulation.

$$W^*(T) = \{T : PT = TP, \forall\ projection\ P \in \{T\}^{'}\}.$$

Corollary 3.2.8. *Let $T \in B(\mathscr{H})$. Then $Lat(T) = Lat(W^*(T))$.*

Proof. Since $T \in W^*(T)$ trivially $Lat(W^*(T)) \subseteq Lat(T)$. On the other hand, $W^*(T)$ consists of polynomials in I and T, and hence $Lat(T) \subseteq Lat(W^*(T))$. Combining these two inclusions, equality follows. This proves the claim. □

Theorem 3.2.9. *Let $T, S \in B(\mathscr{H})$. If $Lat(T) = Lat(S)$ then $Hyperlat(T) = Hyperlat(S)$.*

Proof. This follows easily from the definition. □

Corollary 3.2.10. *Let $T \in B(\mathscr{H})$ then $Hyperlat(T) = Lat(\{T\}^{'})$.*

Theorem 3.2.11. *Let $A, B \in B(\mathscr{H})$. If $A \in W^*(B)$, then $Lat(B) \subseteq Lat(A)$.*

Proof. We know that $Hyperlat(T) \subseteq Lat(T)$ for any $T \in B(\mathscr{H})$ since T commutes with itself [2] that is, $T \in \{T\}^{'}$. Since $A \in W^*(B)$ we have that, $QP_{\mathscr{M}} = P_{\mathscr{M}}Q$ where $Q \in \{W^*(B)\}^{'} = \{B\}^{'} \cap \{B^*\}^{'}$ is an orthogonal projection in $\{B\}^{'}$ and $\mathscr{M} \in Hyperlat(B)$, and hence $P_{\mathscr{M}}AP_{\mathscr{M}} = P_{\mathscr{M}}A$, where $P_{\mathscr{M}} \in W^*(A)$ is an orthogonal projection of H onto $\mathscr{M}$. This means that

$$M \in Hyperlat(B) \subseteq Lat(B) \Rightarrow M \in Lat(A).$$

Thus, $M \in Lat(B) \Rightarrow \mathscr{M} \in Lat(A)$. □

Remark 3.2.12. *The converse of Theorem* 3.2.12 *is not true in general. However, if in addition, $AB = BA$ then the converse is true.*

Corollary 3.2.13. *Let $A, B \in B(\mathscr{H})$. If $A \in W^*(B)$ then $Hyperlat(B) \subseteq Hyperlat(A)$.*

Proof. This follows from the proof of Theorem 3.2.12 and the fact that $Hyperlat(T) \subseteq Lat(T)$ for any $T \in B(\mathscr{H})$. $\square$

Lemma 3.2.14. *[19] Let A be a nilpotent operator, p a polynomial. Then*

$$Lat(A) = Lat(p(A)) \; if \; and \; only \; if \; p^{'}(0) \neq 0.$$

Proof. Assume that $Lat(A) = Lat(p(A))$. If $p^{'}(0) = 0$, we may assume also that $p(0) = 0$, as removal of a scalar multiple of the identity will not change the invariant subspaces. Since A is nilpotent, there are an integer k (index of nilpotence) and a vector $x \neq 0$ such that $A^k x = 0$ and $A^{k-1}x \neq 0$. Then $A^2(A^{k-2}x + A^{k-1}x) = 0$, and hence $\mathscr{M} = [A^{k-2}x + A^{k-1}x]$ (the subspace generated by the enclosed vector) is contained in the kernel of A^2. Thus $\mathscr{M}$ is in the kernel of$p(A)$ and hence $p(A) - invariant$ [as $p(A)$ is a linear combination of powers of A of order 2 or higher]. It is easy to see that $\mathscr{M}$ is not $A - invariant$, contradicting the equality assumption.
Conversely, if $p^{'}(0) \neq 0$, we show that A and $p(A)$ have the same lattice of invariant subspaces. The inclusion $Lat(A) \subseteq Lat(p(A))$ is always true; to show the opposite inclusion, we let $\mathscr{M} \in Lat(p(A))$. We may assume with no loss of generality that $p(A) = A + c_2A^2 + \ldots..$ Since $\mathscr{M}$ is a sum of $p(A) - cyclic$ subspaces, and invariance is preserved under algebraic addition of subspaces, we may assume that $\mathscr{M}$ itself is p(A)-cyclic, i.e. $\mathscr{M} = [x, p(A)x, (P(A))^2x^2, \ldots..]$, for some vector x. It is now easy to see, by the nilpotence of A, that Ax is a linear combination of $[p(A)]^k x,\; k = 1, 2, 3, ..$ So $\mathscr{M}$ is $A - invariant$, and hence $Lat(A) = Lat(p(A))$. $\square$

Corollary 3.2.15. *[19]Let A be an arbitrary operator on a finite dimensional complex vector space, p a polynomial. Then $Lat(A) = Lat(p(A))$ if $p^{'}(\lambda) \neq 0 \; \forall \; \lambda \in \sigma(A)$.*

Proof. Every member of $Lat(A)$ is a direct sum of invariant subspaces whose restrictions of A are nilpotent operators plus scalar operators (primary summands). On each such primary subspace A is $\lambda I + N$ for some $\lambda \in \sigma(A)$and some nilpotent N. A short computation shows that the restriction of $p(A)$ to the primary subspace is

$$p(\lambda)I + p^{'}N + \frac{p^{''}(\lambda)}{2!}N^2 + \frac{p^{'''}(\lambda)}{3!}N^3 + \ldots\ldots\ldots.$$

Clearly, a subspace contained in this primary subspace is $A - invariant$ if and only if it is $N - invariant$. Therefore A and $p(A)$ leave the same subspaces invariant $p^{'}(\lambda) \neq 0$. $\square$

Definition 3.2.16. *Let $\mathscr{H}$ be a Hilbert space over a field $\mathbb{F}$. A Hilbert space homomorphism that maps $\mathscr{H}$ to itself is called an endomorphism of $\mathscr{H}$. A Hilbert space isomorphism that maps $\mathscr{H}$ to itself is called an automorphism of $\mathscr{H}$. The set of all automorphisms of $\mathscr{H}$ will be denoted $Aut(\mathscr{H})$.*

Theorem 3.2.17. *[19]Let A and B be operators on a finite dimensional complex Hilbert space $\mathscr{H}$. Then $Lat(A) = Lat(B)$ if and only if there exist a polynomial p and an invertible operator S such that (i) $p'(\lambda) \neq 0 \; \forall \; \lambda \in \sigma(A)$, (ii) $B = S^{-1}p(A)S$ and (iii) $S\mathscr{M} \in Lat(A)$ when and only when $\mathscr{M} \in Lat(A)$.*

`Proof.` Let $A = A_1 \bigoplus A_2 \bigoplus ....\bigoplus A_k$, be the Jordan decomposition of A, and $\mathscr{H} = \mathscr{H}_1 \bigoplus \mathscr{H}_2 \bigoplus ....\bigoplus \mathscr{H}_k$, the corresponding decomposition of the space $\mathscr{H}$. Then by assumption on the invariant subspace lattices, B decomposes with respect to this decomposition into a direct sum $B_1 \bigoplus B_2 \bigoplus ....\bigoplus B_k$. On each $\mathscr{H}_i$,A_i is the (algebraic) sum of a cyclic nilpotent operator and a scalar (multiple of the identity on $\mathscr{H}_i$). Since B_i leaves all $A_i - invariant$ subspaces invariant, there exist an invertible operator S_i and a scalar α_i such that $S_iA_iS_i + \alpha_i I = B_i$, where I_i is the identity operator on the space $\mathscr{H}_i$. Moreover S_i can be chosen to leave invariant all $A_i - invariant$, subspaces. For each $i \neq j$, if A_i and A_j have the same eigenvalue, then B_i and B_j have the same eigenvalue. For we may assume the eigenvalue for A_i and A_j is 0, and so there are nonzero vectors x_i and x_j such that $A_ix_i = 0$ and $A_jx_j = 0$. Thus the subspace $\mathscr{M} = [x_i \bigoplus x_j]$(generated by the vector $x_i \bigoplus x_j$) is $A-$invariant, and hence is $B-$invariant. If λ_i and λ_j are eigenvalues of B_i and B_j respectively, then

$$(B_i \bigoplus B_j)(x_i \bigoplus x_j) = \lambda_i x_i \bigoplus \lambda_j x_j$$

is a vector in $\mathscr{M}$, by invariance. Therefore $\lambda_i = \lambda_j$, and hence $\alpha_i = \alpha_j$. Thus our question reduces to the primary case, and we may assume that the primary summand A_p of A under consideration is nilpotent. Let $\mathscr{H}_p$be the primary subspace on which A_p acts, S_p the direct sum of all the S_i (obtained above) acting on subspaces of $\mathscr{H}_p$. Let B_p be the restriction of B to $\mathscr{H}_p$, and I_p, the identity operator on (or the idempotent onto)$\mathscr{H}_p$. Then the above argument shows that $B_p - \lambda_p I_p = S_p^{-1}A_pS_p$, for some scalar λ_p. Since each I_p, is a polynomial in A, we have $S^{-1}AS = B - q(A)$, where SS is the direct sum of S_p acting on primary subspaces $\mathscr{H}_p$, and q is a polynomial such that $q(A) \setminus \mathscr{H}_p$ coincides with $\lambda_p I_p$. Since S commutes with each $\lambda_p I_p$, $B = S^{-1}(q(A)+A)S = S^{-1}p(A)S$.
To see that $p'(0)$, we assume the contrary. We may also assume that A is nilpotent. For if not, we can restrict to each primary summand, which is a sum of a scalar and a nilpotent. Choose an upper triangularizing Jordan basis for A. Then the matrix of B with respect to the Jordan basis would have all entries on the first superdiagonal (the diagonal above the main diagonal) zero. There is a basis vector x annihilated by A, which is in the range of A. Then the basis vector y preceding x is the preimage of x under $A.Ay = x$. Note that B leaves the subspace $[y]$ invariant, yet A does not, contradicting our equality assumption on the lattices.
To see that (iii) holds, let $\mathscr{M} \in Lat(A)$. Then $S^{-1}p(A)S\mathscr{M} = B\mathscr{M} \subseteq \mathscr{M}$ by the lattice assumption. Thus $S\mathscr{M}$ is $p(A) - invariant$, and hence $A - invariant$ by $Corollary$3.2.16. If on the other hand $S\mathscr{M} \in Lat(A)$, then $p(A)S\mathscr{M} \subseteq S\mathscr{M}$ thus $B\mathscr{M} = S^{-1}p(A)S\mathscr{M} \subseteq \mathscr{M}$, and $\mathscr{M}$ is $B - invariant$. So $\mathscr{M} \in Lat(B) = Lat(A)$. Conversely, if the conditions are satisfied, we show that A and B have the same lattice of invariant subspaces. Let be $\mathscr{M}$

an $A - invariant$ subspace. Then $S\mathscr{M}$ is $A - invariant$, and hence is $p(A) - invariant$. Thus $B\mathscr{M} = S^{-1}p(A)S\mathscr{M} \subseteq \mathscr{M}$: So $\mathscr{M}$ is $B - invariant$. If $\mathscr{M}$ is B-invariant, then $S^{-1}p(A)S\mathscr{M} = B\mathscr{M} \subseteq \mathscr{M}$, and hence $S\mathscr{M}$ is $p(A) - invariant$. Therefore $\mathscr{M}$ is $A - invariant$ by $Corollary$3.2.16. This completes the proof. □

Corollary 3.2.18. *Let A and B be operators on a finite dimensional vector space with the same lattice of invariant subspaces. Then $S^{-1}p(A)S$ for some invertible S and some polynomial p; furthermore (i) The operator S can be chosen to be upper triangular with respect to a Jordan basis for A;*
(ii) S decomposes with respect to the Jordan decomposition of A;
(iii) S induces an automorphism on $Lat(A)$.

4 REDUCIBILITY AND SUBSPACE LATTICES

4.1 Introduction

It is clear that reducing subspaces are generally easier to treat than arbitrary invariant subspaces. Reducing subspaces find applications in wavelet expansion, multiresolution analysis(MRA) in image processing and automorphic graph theory. We now discuss some of the properties of reducing subspaces.

Theorem 4.1.1. *A subspace $\mathscr{M}$ reduces an operator T if and only if $\mathscr{M} \in Lat(T) \cap Lat(T^*)$.*

Proof. Follows easily from the definition. □

Corollary 4.1.2. *Let $T \in B(\mathscr{H})$ and $\mathscr{M}$ be a subspace of $\mathscr{H}$. Then the following statements are equivalent;*
(i) $\mathscr{M}$ reduces T.
(ii) $\mathscr{M} \in Lat(T) \cap Lat(T^*)$.
(iii) $P_{\mathscr{M}} \in \{T\}'$, where $P_{\mathscr{M}}$ is the orthogonal projection of $\mathscr{H}$ onto $\mathscr{M}$.

Theorem 4.1.3. *If $T_1 \in B(\mathscr{H}_1$ and $T_2 \in B(\mathscr{H})$ are irreducible, then every operator $A \in B(\mathscr{H}_\infty, \mathscr{H})$ that intertwines them is either zero or identity.*

Theorem 4.1.4. *If an operator A commutes with an irreducible operator T, then A is similar to a scalar operator.*

Theorem 4.1.5. *If $T \in B(\mathscr{H})$ is nilpotent of nil-index n, then*

$$Red(T) = \{\{0\}, \mathscr{H}\}.$$

Corollary 4.1.6. *Let $T \in B(\mathscr{H})$. If $Red(T) = \{\{0\}, \mathscr{H}\}$, then $T = \alpha I + S$, where S is a nilpotent operator.*

Bercovici etal [1] have proved that for a nilpotent operator $T \in B(H)$ such that $T^n = 0$, for some integer $n \geq 1, Hyperlat(T)$ is generated by the spaces $Ker(T^m)$ and $Ran(T^m), m = 0, 1, 2, ..., n$. They have also shown that $Ran(T^{n-1})$ is the smallest nontrivial hyperinvariant subspace and the $Ker(T^{n-1})$ is the largest nontrivial hyperinvariant subspace.

4.2 Relationship Between Hyperinvariant Subspaces and Reducing Subspaces Of an Operator

Theorem 4.2.1. *Let $T \in B(\mathscr{H})$ be a unitary operator. A subspace $\mathscr{M} \subseteq \mathscr{H}$ is hyperinvariant for T if and only if $\mathscr{M}$ reduces T.*

`Proof.` Suppose that $\mathscr{M} \in Hyperlat(T)$ and let $P_{\mathscr{M}}$ be the orthogonal projection of $\mathscr{H}$ onto $\mathscr{M}$. Then $AP_{\mathscr{M}} = P_{\mathscr{M}}AP_{\mathscr{M}}$ for every $A \in \{T\}^{'}$. Since T is unitary and hence normal, by Fuglede's theorem, $A^* \in \{T\}^{'}$. Thus $A^*P_{\mathscr{M}} = P_{\mathscr{M}}A^*$ and hence $AP_{\mathscr{M}} = P_{\mathscr{M}}AP_{\mathscr{M}} = P_{\mathscr{M}}A$ By Corollary 4.1.2, we have that $\mathscr{M}$ reduces T. Conversely, suppose M reduces T. Without loss of generality, suppose $AP_{\mathscr{M}} = P_{\mathscr{M}}A$. Then

$$A\mathscr{M} = AP_{\mathscr{M}}\mathscr{H} = P_{\mathscr{M}}A\mathscr{H} \subseteq P_{\mathscr{M}}\mathscr{H} = \mathscr{M}.$$

This shows that $\mathscr{M}$ is invariant under A. So, if $AP_{\mathscr{M}} = P_{\mathscr{M}}A$ for all $A \in \{T\}^{'}$, then $\mathscr{M}$ is hyperinvariant for T. □

Remark 4.2.2. *Theorem 4.2.1 says that for a unitary operator T we have that $Hyperlat(T) = Red(T)$.*

Remark 4.2.3. *Theorem 4.2.1 can be relaxed as follows.*

Theorem 4.2.4. *Let $T \in B(\mathscr{H})$ be an isometry. If $\mathscr{M} \subseteq \mathscr{H}$ is such that $T\mathscr{M} = \mathscr{M}$ then $\mathscr{M}$ reduces T.*

`Proof.` If $T\mathscr{M} = \mathscr{M}$ then $T^*\mathscr{M} = T^*T\mathscr{M} = \mathscr{M}$. This proves the claim. □

Corollary 4.2.5. *Let $T \in B(\mathscr{H})$ be an isometry. If $\mathscr{M} \subseteq \mathscr{H}$ is such that $T\mathscr{M} = \mathscr{M}$ then $Red(T) = Lat(T)$.*

`Proof.` This follows from Theorem 4.2.3 and the fact that $Red(T) \subseteq Lat(T)$, for any operator T. □

4.3 Reductive Operators

Definition 4.3.1. *An operator $T \in B(\mathscr{H})$ is reductive if all its invariant subspaces reduce it.*

Remark 4.3.2. *Note that $Red(T) = Red(T^*)$ for any operator $T \in B(\mathscr{H})$.*

We now characterize reductive operators. That is, we give a necessary and sufficient condition when an operator T is reductive.

Corollary 4.3.3. *An operator $T \in B(\mathscr{H})$ is reductive if and only if $Lat(T) = Red(T)$.*

Proof. By definition, if T is reductive, then $Lat(T) \subseteq Red(T) = Red(T^*)$. But the inclusion $Red(T) \subseteq Lat(T)$ is obvious. Combining these statements, we have equality. Conversely, suppose that

$$Lat(T) = Red(T^*).$$

Then

$$Lat(T) = Red(T^*) = Lat(T^*) \cap Lat(T) \subseteq Lat(T^*).$$

Thus $Lat(T) \subseteq Lat(T^*)$.

Corollary 4.3.4. *Let $T \in B(\mathscr{H})$. If $Lat(T) \subseteq Lat(T^*)$, then T is reductive.*

The class of reducible operators contains the class of reductive operators. However, an operator may be reducible but fail to be reductive. Thus,

$$Reductive \subseteq Reducible.$$

Note that every self-adjoint (and by extension, normal operator on a finite dimensional Hilbert space) is reductive. It is also known that every compact normal operator is reductive. It is a known fact that every operator that commutes with a non-scalar normal operator is reducible. In fact for a normal operator $T \in B(\mathscr{H})$, we have that

$$Lat(T) \cong Lat(T^*).$$

The class of reductive operators contains the class of normal operators.Thus

$$Normal \subset Reductive \subset Reducible.$$

The above inclusion is strict. For instance it has been shown in[12] that not every reductive operator in normal. Moore[15] went further and gave some conditions under which a reductive operator is normal: that such a reductive operator T must commute with an injective compact operator or T is polynomially compact or T is expressible as a sum of a normal operator and a commuting compact operator.

Example 4.3.5. *The bilateral shift B on $\ell^2(\mathbb{Z})$ defined by $B(\ldots,x_{-2},x_{-1},[x_0],x_1,x_2,\ldots) = (\ldots,x_{-2},x_{-1},x_0,x_1,x_2,\ldots)$ where $x = (\ldots,x_{-2},x_{-1},[x_0],x_1,x_2,\ldots) \in \ell^2(\mathbb{Z})$ and $[x_0]$ denotes the $0-th$ coordinate of x, is not reductive. Indeed, $\mathscr{M} = \{x \in \ell^2(\mathbb{Z}) : x_n = 0, if\ n < 0\} \in Lat(B)$ but $\mathscr{M} \nsubseteq Lat(B^*)$.*

Theorem 3.3.4 *A reductive operator is normal if and only if it has a nontrivial invariant*

subspace.
Theorem 3.3.5 *Let $T \in B(\mathscr{H})$. If a subspace $\mathscr{M} \subseteq \mathscr{H}$ is hyper-reducing then $\mathscr{M} \in Lat(\{T\}^{'}) \cap Lat(\{T^*\}^{'})$.*

Example 4.3.6. *Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ acting in $\mathbb{R}^2$*
Clearly these two operators are not similar. A simple computation shows that;

$$Lat(A) = \{\{0\}, span \begin{bmatrix} 1 \\ 0 \end{bmatrix}, span \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \mathbb{R}^2\} = Red(A)$$

and

$$Lat(B) = \{\{0\}, span \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \mathbb{R}^2\} \neq \{\{0\}, \mathbb{R}^2\} = Red(B).$$

Thus A is reductive while B is not since not every invariant subspace of B reduces B. Another computation shows that

$$\{B\}^{'} = \{X : X = \begin{bmatrix} \alpha & \beta \\ 0 & \alpha \end{bmatrix}, \alpha, \beta \in \mathbb{R}\}$$

and

$$\{A\}^{'} = \{Y : Y = \begin{bmatrix} \alpha & \beta \\ \gamma & \lambda \end{bmatrix}, \alpha, \beta, \gamma, \lambda \in \mathbb{R}\}$$

hence

$$Hyperlat(A) = \{\{0\}, \mathbb{R}^2\}$$

and

$$Hyperlat(B) = Lat(B).$$

Theorem 4.3.7. *[13] If A is a reductive operator then A can be written as a direct sum $A = A_1 \bigoplus A_2$ where A_1 is normal, A_2 is reductive and all the invariant subspaces of A_2 are hyperinvariant.*

Corollary 4.3.8. *[7] Suppose A is reductive operator such that $A = A_1 \bigoplus A_2$ then*

$$Hyperlat(A_1) \bigoplus Hyperlat(A_2)$$

and

$$Lat(A) = Hyperlat(A).$$

From theorem 4.3.7 and corollary 4.3.8 we conclude that if A is reductive and completely non-normal (that is, A has no normal direct summand) then $Lat(A) = Lat(\{A\}^{'})$.

Theorem 4.3.9. *([13]) If A is a reductive operator, then every hyperinvariant subspace of A is hyper-reducing.*

Corollary 4.3.10. *If A is a reductive operator, then $Hyperlat(A) \subseteq HyperRed(A)$.*

Corollary 4.3.10 says that if A is reductive then $Lat(\{A\}^{'}) = Lat(\{A^*\}^{'})$.

Theorem 4.3.11. *Let $T \in B(\mathscr{H})$. Then $HyperRed(T) = Lat(\{T\}^{'}) \cap Lat(\{T^*\}^{'})$.*

`Proof.`

$$\begin{aligned} HyperRed(T) &= \{M \subseteq \mathscr{H} : M \in Red(\{T\}^{'})\} \\ &= \{M \subseteq \mathscr{H} : SM \subseteq M, S^*M \subseteq M, S \in \{T\}^{'}\} \\ &= \{_{M \subseteq \mathscr{H} : M \in Lat(S) \cap Lat(S^*), S \in \{T\}^{'}}\} \\ &= \{M \subseteq \mathscr{H} : M \in Lat(\{T\}^{'}) \cap Lat(\{T^*\}^{'})\} \\ &= Lat(\{T\}^{'}) \cap Lat(\{T^*\}^{'}) \end{aligned}$$

□

Theorem 4.3.12. *Let $T \in B(\mathscr{H})$. Then*

$$HyperRed(T) = Hyperlat(T) \cap Hyperlat(T^*)$$

`Proof.` The proof follows from Theorem 4.3.11 and the fact that $Lat(\{T\}^{'}) = Hyperlat(T)$ and $Lat(\{T^*\}^{'}) = Hyperlat(T^*)$, for any $T \in B(\mathscr{H})$ □

Corollary 4.3.13. *Let $T \in B(\mathscr{H})$ be self-adjoint. Then*

$$HyperRed(T) = Hyperlat(T).$$

`Proof.` The proof follows easily from Theorem 4.3.12 and the fact that self-adjointness of T. The proof also follows from Theorem 4.3.11, the self-adjointness of T and the fact that $Lat(\{T\}^{'}) = Hyperlat(T)$. □

We know characterize the hyperlattice of a normal operator in a Hilbert space.

Theorem 4.3.14. *[1]Let $T \in B(\mathscr{H})$ be normal. Then $Hyperlat(T) = \{\mathscr{M} \subseteq \mathscr{H} : P_{\mathscr{M}} \in W^*(T)\}$.*

Corollary 4.3.15. *: Let $T \in B(\mathscr{H})$ be normal. Then every hyperinvariant subspace of T is hyperinvariant for T^*.*

Corollary 4.3.15 says $Hyperlat(T) \subseteq Hyperlat(T^*)$, for any normal operator $T \in B(\mathscr{H})$. The converse is also true. This leads to the following result.

Corollary 4.3.16. *Let $T \in B(\mathscr{H})$ be normal. Then $Hyperlat(T) = Hyperlat(T^*)$.*

Proof. Since T is normal if and only if T^* is normal, the result follows from the fact that $T^* \in \{T\}'$ if and only if $T \in \{T^*\}'$. □

Remark 4.3.17. *For a normal operator T, the hyperlattices of T and that of its adjoint coincide.*

Theorem 4.3.18. *If $T \in B(\mathscr{H})$ is an invertible reductive operator, then T^{-1} is also reducible.*

Proof. Since $T \in B(\mathscr{H})$ is reducible, by Theorem 4.3.8 it can be expressed as

$$T = \begin{bmatrix} T_1 & 0 \\ 0 & T_2 \end{bmatrix} = T_1 \bigoplus T_2,$$

with respect to the direct sum decomposition $\mathscr{H} = \mathscr{M} \bigoplus \mathscr{M}^{\perp}$, where $\mathscr{M}$ is a subspace that reduces T. Invertibility of T implies that of T_1 and T_2. Thus

$$T^{-1} = \begin{pmatrix} T_1^{-1} & 0 \\ 0 & T_2^{-1} \end{pmatrix} = T_1^{-1} \bigoplus T_2^{-1}$$

with respect to the direct sum decomposition $\mathscr{H} = \mathscr{M} \bigoplus \mathscr{M}^{\perp}$. □

Corollary 4.3.19. *Let $T \in B(\mathscr{H})$ be invertible. If a subspace $\mathscr{M} \subseteq \mathscr{H}$ reduces T, then $\mathscr{M}$ reduces T^{-1}.*

Proof. Let $P_{\mathscr{M}}$ be the orthogonal projection of $\mathscr{H}$ onto $\mathscr{M}$. Since M reduces T, we have $TP_{\mathscr{M}} = P_{\mathscr{M}}T$ By the proof of Theorem 4.3.18, $T^{-1}P_{\mathscr{M}} = P_{\mathscr{M}}T^{-1}$. This proves the claim. □

Remark 4.3.20. *From Theorem 4.3.18 and Corollary 4.3.19, we conclude that if $T \in B(\mathscr{H})$ is invertible, then $Red(T) = Red(T^{-1})$.*

Example 4.3.21. *Consider the bilateral weighted shift T_ω on $\ell^2(\mathbb{Z})$ defined by $T_\omega e_n = \omega_n e_{n+1}$, where $n \in \mathbb{Z}$ and $\{e_n\}$ the canonical orthonormal basis for $\ell^2(\mathbb{Z})$. A simple calculation shows that $T_\omega^{-1} e_n = \dfrac{1}{\omega_n} e_n$ where $n \in \mathbb{Z}$. If $\mathscr{M} = span\{e_1, e_2, ...\}$ then $\mathscr{M}$ is invariant for T_ω but is not invariant for T_ω^{-1}.*

The following result shows that taking powers of an operator $T \in B(\mathscr{H})$ preserves invariance and reduction.

Theorem 4.3.22. *Let $T \in B(\mathscr{H})$ and $\mathscr{M} \subseteq \mathscr{H}$. The following statements are true for any integer $n > 1$.*

$$(i)\ If \mathscr{M} \in Lat(T)\ then\ \mathscr{M} \in Lat(T^n).$$

$$(ii)\ If \mathscr{M} \in Red(T)\ thenmathcalM \in Red(T^n).$$

Proof. The proofs of (i) and (ii) follow easily by mathematical induction on $n \in \mathbb{N}$. In the proof of (ii), we use the fact that $\mathscr{M} \in Red(T)$ implies that $T\mathscr{M} \subseteq \mathscr{M}$ and $T^*\mathscr{M} \subseteq \mathscr{M}$. □

Theorem 4.3.23. *Let $T \in B(\mathscr{H})$ and $\mathscr{M} \subseteq \mathscr{H}$. If $\mathscr{M} \in Hyperlat(T)$ then $\mathscr{M} \in Hyperlat(T^n)$ for any integer $n > 1$.*

Proof. We need to prove that $\mathscr{M} \in Lat(S)$, where $S \in \{T\}^{'}$implies that $\mathscr{M} \in Lat(X)$ where $X = \{T^n\}^{'}$.
By Theorem 4.3.22(i), if $\mathscr{M} \in Lat(S)$ then $\mathscr{M} \in Lat(S^n)$, where $S \in \{T\}^{'}$.
By mathematical induction on $n \in \mathscr{N}$, if $S \in \{T\}^{'}$ then $S^n \in \{T\}^{'}, T^n \in \{S\}^{'}$ and $S^n \in \{\{T^n\}\}^{'}$. By letting $X = S^n$, and using Theorem 4.3.22(i) once more, the result follows. □

5 INVARIANT SUBSPACE LATTICE OPERATIONS

5.1 Introduction

The main objective in this chapter is to analyse $Lat(A)$, maps of $Lat(A)$, $for\ any\ A \in B(\mathscr{H})$ as far as possible assuming nothing about A beyond linearity and boundedness. All lattices considered in the chapter will have a zero element $\{0\}$ and a unit element $\mathscr{H}$ such that $\{0\} \subset M \subset \mathscr{H}$ for all lattice elements $\mathscr{M}$. Such a lattice is complemented if for any element $\mathscr{M}$ there exists at least one element $\mathscr{N}$ with $\mathscr{M} \cap \mathscr{N} = \{0\}$ and $\mathscr{M} + \mathscr{N} = \mathscr{H}$. If $\mathscr{M}$ and $\mathscr{N}$ are any lattice elements, we denote by $[\mathscr{M}, \mathscr{N}]$ the set of all lattice elements P with $\mathscr{M} \subset P \subset \mathscr{N}$. If each interval sublattice $[\mathscr{M}, \mathscr{N}]$ is complemented, then the lattice is said to be relatively complemented. A lattice is distributive if $(\mathscr{M} + \mathscr{N}) \cap P = (\mathscr{M} \cap P) + (\mathscr{N} \cap \mathscr{P})$ for all elements $\mathscr{M}, \mathscr{N}, \mathscr{P}$, and modular if this identity holds whenever $\mathscr{M} \subset \mathscr{P}$. A lattice L is said to be the direct sum of sublattices L_1 and L_2 if each

5.2 Implementation of the criterion

The algorithm provided in [8] resolves the problem of existence of a common invariant subspace when one of the matrices has distinct eigenvalues (requiring in the process a finite number of rational computations). Our results lead to a plausible strategy for all matrices, including the opportunity of providing a basis for the common invariant subspace. As in [8] use will be made of the following criterion for the existence of a common eigenvector.

Theorem 5.2.1. *[21]: Let $X, Y \in C^{p \times p}$ and*

$$K = \sum_{m,l=1}^{p-1} [X^m, Y^l]^* [X^m, Y^l],$$

where $[X^m, Y^l]$ denotes the commutator $X^m Y^l - Y^l X^m$. Then X and Y have a common eigenvector if and only if K is not invertible.

Consider now the following plan for discovering a $k-$dimensional $(1 < k < n)$ common invariant subspace of given operators $A, B \in C^{n \times n}$:

1. *Find s such that $A + sI$ and $B + sI$ are invertible.*
2. *Compute $X = (A + sI)^k, Y = (B + sI)^k$ and K as in the theorem above.*
3. *If K is invertible, A and B do not have a common invariant subspace of dimension k*
4. *Otherwise, compute bases for the intersections of eigenspaces of X and Y.*
5. *If the intersecting eigenspaces of X and Y contain a non-zero decomposable vector, then A and B have a common invariant subspace of dimension k. Otherwise, no such subspace exists.*

6. *Find a decomposable vector belonging to an eigenspace intersection of* $Step4$ *and factor it; its factors form a basis for a common invariant subspace of* A *and* B. *Steps* $1to3$ *of the above plan are straightforward.* $Steps2$ *and*3 *can be computationally expensive, depending on* n *and* k. $Step4$ *can be performed using* $[8, Algorithm 12.4.3]$. $Steps 5 and 6$ *can be theoretically and practically challenging; we will return to them after the following illustrative example.*

Example 5.2.2. *Let us consider whether*

$$A = \begin{bmatrix} 3 & -3 & 1 \\ 0 & 4 & 0 \\ -1 & -3 & 5 \end{bmatrix}$$

and

$$B = \begin{bmatrix} -1 & -3 & 5 \\ -2 & 6 & 2 \\ -7 & -1 & 11 \end{bmatrix}$$

have a common invariant subspace of dimension $k = 2$ *or not. The eigenvalues of* A *are all equal to* 4 *and of* B *are* $4(double)$ *and* 8. *We compute the second compounds of* A *and* B *to be:*

$$X = A^2 = \begin{bmatrix} 12 & 0 & -4 \\ -12 & 16 & -12 \\ 4 & 0 & 20 \end{bmatrix}$$

and

$$Y = B^2 = \begin{bmatrix} -12 & 8 & -36 \\ -20 & 24 & -28 \\ 44 & -8 & 68 \end{bmatrix}$$

The matrix K *of Theorem 5.3.1 is a scalar multiple of*

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

and thus it is not invertible. Using Matlab's null routine (which computes orthogonal bases, either rationally via the echelon form or via the singular value decomposition), we find that

$$Null(X - 16I) = span\{\alpha, \beta\}$$

where $\alpha = \begin{bmatrix} -1 \, 0 \, 1 \end{bmatrix}^t, \beta = \begin{bmatrix} 0 \, 1 \, 0 \end{bmatrix}^t$. *We also find that*

$$Null(Y - 16I) = span\{\mu\}$$

where $\mu = \begin{bmatrix} 1 & -1 & -1 \end{bmatrix}^t$. *Also, we find that*

$$Null(Y - 32I) = span\{\delta\}$$

where $\delta = \begin{bmatrix} 1 \, 1 & -1 \end{bmatrix}^t$.
Since $\mu, \delta \in span\{\alpha, \beta\}$ *we have that* μ, δ *are common eigenvectors of* X *and* Y. *Moreover,* μ *and* δ *are decomposable as* $\mu = x_1 \wedge x_2$ *and* $\delta = x_1 \wedge x_3$ *where*

$$x_1 = \begin{bmatrix} 1 \, 0 \, 1 \end{bmatrix}^t, \; x_2 = \begin{bmatrix} 0 \, 1 & -1 \end{bmatrix}^t, \; x_3 = \begin{bmatrix} 0 \, 1 \, 1 \end{bmatrix}^t.$$

It follows that $span\{x_1, x_2\}$ *and* $span\{x_1, x_3\}$ *are common invariant subspaces of* A *and* B. *Notice also that, when* $k = 1$, A *and* B *have a common invariant subspace of dimension* 1 *namely, the span of* x_1 *which is a common eigenvector of* A *and* B.

6 CONCLUSION AND RECOMMENDATIONS

6.1 Conclusion

The concept of the lattices of bounded linear operators on Hilbert Spaces plays a central role in the study of the structure and behavior of these operators. It tends to exploit the gaps that the study of the properties of an operator has always failed to address. For example, during our study, we have been able to realize that for a unitary operator T, $Hyperlat(T) = Red(T)$. It has also been proved that for a normal operator T, $Hyperlat(T) \subseteq Hyperlat(T^*)$, and the following inclusions hold:

$$Normal \subset Reductive \subset Reducible.$$

It have also been shown that:

1. If P and Q are metrically equivalent projections, then $Lat(P) = Lat(Q)$.
2. If A and B are metrically equivalent positive operators, then $Lat(A) = Lat(B)$.
3. An operator $T \in B(\mathscr{H})$ is reductive if and only if $Lat(T) = Red(T)$.

$$Reductive \subseteq Reducible.$$

4. It has also been shown that for $T \in B(\mathscr{H}) HyperRed(T) = Hyperlat(T) \cap Hyperlat(T^*)$
5. Also for $T \in B(\mathscr{H})$ be self-adjoint. Then

$$HyperRed(T) = Hyperlat(T).$$

6.2 Recommendations

In our study we have shown that similarity and hyper-quasisimilarity preserves non-trivial hyperinvariant subspaces. We would like to recommend research that would determine on whether similarity, almost-similarity and metric equivalence preserve hyper-reducibility or non-trivial hyper-reducing subspaces.

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