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Research Report in Mathematics, Number 05, 2018

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Submitted to the School of Mathematics in partial fulfilment for a degree in Master of Science in Applied Mathematics

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Master of Science Project
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Abstract

In this research project paper, our main aim is to solve linear and non-linear differential equations by perturbation methods. This study include the following aspects:

(1) Give some basic concepts of the regular, singular and homotopy perturbation method.

(2) Use the regular, singular and homotopy perturbation method to solve ordinary differential equations and partial differential equations.

(3) Describe the interesting concept of shallow water waves and derive the Korteweg-de Vries equation.

(4) Use homotopy perturbation method to solve the Korteweg-de Vries equation.

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Declaration and Approval

I the undersigned declare that this project report is my original work and to the best of my knowledge, it has not been submitted in support of an award of a degree in any other university or institution of learning.

Signature Date

GLADYS NGINA MUNYAO
Reg No. I56/87881/2016

In my capacity as a supervisor of the candidate, I certify that this report has my approval for submission.

Signature Date

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Dedication

This project is dedicated to my children Shirlene, Samuel, Collins and my mother Mary Mukonyo.

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Acknowledgments

First and foremost, my gratitude goes to the almighty God for granting me peace and for providing me this opportunity to proceed successfully with this work.

Many thanks to my supervisor Dr. Charles Nyandwi for the hard work he did guiding me on all aspects of this work to ensure my work was of good quality. Without his support I could not manage to come this far.

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To my family members, I take this opportunity to express the profound gratitude from my deep heart to my beloved children; Shirlene, Samuel and Collins, my mom; Mary Mukonyo and my siblings for their continuous support both spiritually and materially. I thank you for persevering the hardship I had to take you through during the accomplishment of this work.

To all whom I have not mentioned here by name and contributed to my success, feel appreciated.

Gladys Ngina Munyao

Nairobi, 2018.

1 Introduction

Many linear and nonlinear differential equations are of fundamental importance in science and technology. Actually most scientific phenomena occur nonlinearly and the equations do not have exact systematic arrangements. Therefore, these nonlinear equations cannot be solved analytically. The study of exact or approximate solutions of linear and nonlinear differential equations modelling physical problems arising in physics and engineering has been one of the challenges of mathematicians. In recent years, many researchers have paid attention to study the solutions of linear and nonlinear differential equations by using various methods. Some valuable contributions have been made to solving these differential equations using numerical techniques such as Finite Difference Method, Finite Element Method, Adomian Decomposition Method and others. These methods have their disadvantages, for example in Finite difference method discretization of the variable leads to computational complexities while Adomian Method narrow down its application due to calculation of complicated Adomian polynomials. Also, integral transform such as Laplace and Fourier transform are commonly used to solve differential equations and usefulness of these integral transforms lie in their ability to transform differential equations into algebraic equations, which allows simple and systematic solution procedures. However, using integral transform in nonlinear problems may increase its complexity.

Recently, some intermediary of analytical and numerical techniques for solving nonlinear differential equations have been dominated by the perturbation method. To overcome disadvantages of Finite Difference Method, Adomian Decomposition Method,integral transform and other methods, perturbation methods comes in handy. Perturbation method is one of the most-known methods used to solve nonlinear equations studied by a large number of researchers. Many scientists had paid more attention to the mathematical aspects of the subject while forgetting the physical verification. This loss in the physical verification of the subject was recovered by Nayfeh 1973 and Van Dyke 1975 who researched on perturbation methods in fluid mechanics. But, like other nonlinear analytical methods, perturbation methods have their own particular limitations. Firstly, almost all perturbation methods are based on an assumption that a small parameter must exist in the equation. This so called small parameter assumption greatly restricts utilization of perturbation techniques. As well known, an overwhelming majority of nonlinear problems have no small parameter at all. Secondly, the determination of small parameters seems to be a special art requiring special techniques. A suitable choice of small parameter leads to ideal results. However, an unsustainable choice of small parameters results in bad effects. Thirdly, even if there exist suitable parameters, the approximate solutions obtained by the perturbation methods are valid in the most cases, only for the small values of the parameter. So it was necessary to develop a kind of new perturbation method which does not require small parameter at all, thus the homotopy perturbation method.

The homotopy perturbation method does not depend upon a small parameter in the equation. This method, which is a combination of homotopy in topology and perturbation techniques provides us with a convenient way to obtain analytic or approximate solution to a wide variety of differential equations modelling different physical problems. Homotopy is an important part of differential topology. Homotopy techniques are generally connected to discover all bases of nonlinear algebraic equations. The homotopy techniques embeds a parameter p that typically ranges from zero to one. When the embedding parameter is zero, the equation is one of the direct framework, when it is one; the equation is the same as the first one. So the embedded parameter $p[0, 1]$ can be considered as a small parameter. In this method the solution is considered as the summation of an infinite series which usually converges rapidly to the exact solution.

The purpose of this thesis is to give information about perturbation theory in general with examples of applications to algebraic equations, ordinary differential equations and partial differential equations.The basic concepts of regular, singular and homotopy perturbation methods are explained. The concept of shallow water waves is described and the derivation of Korteweg-de Vries equation done. Moreover, the homotopy perturbation method is applied to solve the Korteweg-de Vries equation.

This thesis consist of six chapters; In chapter one we describe the background of the problem, problem statement, the objectives of the project and also present the literature review. In chapter two we discuss briefly the history of perturbation theory and in detail some basic concepts of Regular, Singular and Homotopy perturbation theory. In each case we give examples of the method as applied to algebraic and differential equations.

In chapter three, we present the homotopy perturbation method in particular by first looking at its history. Then, we present an analysis of homotopy perturbation method for solving second order partial differential equations and its implementation. In chapter four, we present the shallow water waves and Korteweg-de Vries equation by first giving the historical background of the Korteweg-de Vries equation. The derivation of the Korteweg-de Vries equation is presented in detail as well as definition of some important terms.

In chapter five, the application of homotopy perturbation method to Korteweg-de Vries equation is described using the stated initial conditions.We also consider the convergence of homotopy perturbation method in general for nonlinear functional equation. A brief presentation of an analytical exact solution by elementary operation is done by considering cases where the Korteweg-de Vries equation could have exact solution. In chapter six, we present the conclusion and give recommendation for further research.

1.1 Background of the Problem

The perturbation method is a very powerful method which has been used to solve many difficult problems in mathematics. Generally the perturbation method is not taught in our local universities either at undergraduate or post graduate level. The methods taught, for example, for solving partial differential equations are related to finding analytical solutions. As many problems in mathematical modeling using partial differential equations implies a good knowledge of the classical methods of numerical analysis applied to partial differential equations such as Finite difference method (F.D.M), Finite element method (F.E.M) and other special methods, we tried to search for other methods which are somehow intermediary between analytical methods and the more advanced numerical methods for partial differential equations to solve non-linear partial differential equations. We found it very useful to learn more about perturbation method.

The problem with the perturbation method is that it goes with the theory of simulation of infinite series and the theory of asymptotic approximation. We did not have enough time to try to learn about all those interesting and useful theories; instead we limited ourself on the perturbation theory and its application to differential equations in general and in particular to its application to partial differential equations.

By going through different perturbation methods to solve partial differential equations, we found that the homotopy perturbation method has been applied in many different cases. However, we thought that it has not been explicitly done in solving the Korteweg-de Vries equation. We therefore endeavored to learn about the origin of this equation and found that it is well derived in the context of shallow water waves.

In this project we use the homotopy perturbation method to solve the Korteweg-de Vries equation in the context of shallow water waves.

1.2 Problem Statement

Our main task in this project is to study and understand the perturbation methods of solving differential equations and specifically apply homotopy perturbation method to solve the Korteweg-de Vries equation;

$$\frac{\partial u}{\partial t} - 6u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0$$

subject to the conditions,

$$u(x,0) = \sin(\pi x) \quad u(0,t) = 0$$

1.2.1 Main Objective

To understand perturbation methods and use the homotopy perturbation method to solve Korteweg-de Vries equation.

1.2.2 Specific Objectives

1. To study and understand perturbation methods for solving ordinary differential equations and partial differential equations.

2. To study the shallow water waves and derive the Korteweg-de Vries equation.

3. To apply homotopy pertubation method to solve the Korteweg-de Vries equation.

1.3 Literature Review

Recently, many mathematicians seek new techniques to find exact or approximate solutions to nonlinear partial differential equations modelling physical problems which describe different fields of science, physics, engineering and others.One of the modern methods which has been used is the homotopy perturbation method. The first mathematician who proposed the homotopy perturbation method was Ji-Huan in 1999. This method which is a combination of homotopy in topology and classical perturbation techniques, provides us with a convenient way to obtain analytic or approximate solution of these equations. In this method the solution is considered as the summation of infinite series which usually converge rapidly to the exact solution.

Many researchers have since used homotopy perturbation method to approximate the solution of differential equations and integral equations. For example, Chang, H.K, and Liou, J. C 2006 solved wave dispersion equation for dissipative media using the homotopy perturbation technique. They were able to develop a third-order explicit approximation to find the roots of the dispersion relation for water waves that propagate over dissipative media. In 2007, Rafei M., Ganji D. and Danialli H., used homotopy perturbation method to get solution of the epidemic model. Chun,C. and Sakthivel R. 2010, solved linear and nonlinear second-order two-point boundary value problems while in the same year Gülkac V. solved the black-scholes equation for simple European option in this method to obtain a new efficient-recurrent relation to solve black-scholes equation.

Moreover, numerous researchers have used homotopy perturbation method for solving nonlinear differential equations. Vahidi A., Babolian E. and Azimzadeh Z. in 2011 solved the nonlinear Duffing's equations which yields the Maclaurin's series of the exact solution. In 2012, Zhou S. and Wu H. come up with analytical solutions of nonlinear Poisson-Boltzmann equation for colloidal particles immersed in a general electrolyte solution by homotopy perturbation technique. Yazdi A.A. 2013 solved nonlinear vibration analysis of functionally graded plate. In 2004, the homotopy perturbation method was used to solve nonlinear oscillators with discontinuous nonlinear Duffing equation and some nonlinear ordinary differential equations. For class of linear partial differential equations, Al-salf A. and Abood D.A. 2011, solved the Korteweg-de Vries equation and convergence study of homotopy perturbation method. Babolian E., Azizi A. and Saeidian J. 2009 used homotopy perturbation method to solve time-dependent differential equations.

Many more reseachers have used homotopy perturbation method for solving the class of nonlinear differential equations. Liao S. 2004, solved nonlinear partial differential equations while Yildirim A. 2009, solved the nonlinear Korteweg-de Vries equation by the homotopy perturbation method. M.D. Nausrudin, F.S. Mahadi, Salah F. et al 2014, combined homotopy perturbation method-pade approximant to acquire the approximate analytical solution of the Korteweg de Vries equation. Taghipour R. 2010, solved parabolic equations and periodic equations linear and nonlinear partial differential equations. Momani S. and Odihat Z. 2007, applied the homotopy perturbation method to solve for nonlinear partial differential equations of fractional order. However, for the system of differential equations Bataineh A.S, Noorani M.S.M and Hashim I. in 2009 solved system of second-order Boundary value problems, while Javidi M. in the same year solved system of linear fredholm integral equations. In 2007, Wang X. and Song X. studied global stability and periodic solution of a model for HIV infection of CD4+T cells, where the solution of the model was by the homotopy perturbation method.

Recently, in 2016 Aqeel Falih Jaddoa, presented homotopy perturbation method for solving inhomogeneous heat problem and vibration beam problem of the fourth order as a linear example, inhomogeneous advection problem as nonlinear example and one system of nonlinear partial differential equations. In 2017, Mohammed S. Mechee, Adil M. Al-Rammahi and Ghassan A. Al-Juaifri studied the homotopy perturbation method for solving generalized linear second-order partial differential equations. Solutions of various second-order partial differential equations were done by the method, for example the Helmholtz equation, Heat equation in 3-dimension, 3-dimmesion parabolic-like equation and so on. The approximated solution of this class of partial differential equations shows the efficiency and accuracy of the proposed method.

In this work, we present homotopy perturbation method for the solving the Korteweg-de Vries equation, a nonlinear third-order partial differential equation. We wish to improve on the presentation by Dinkar Shama and Sheo Kumar, "homotopy perturbation method for Korteweg and de Vries equation" of 2013 who worked on the infinite series upto u_3 by working up to u_4 and study the convergence of the homotopy perturbation method in general for nonlinear functional equation.

2 Perturbation Methods

2.1 History

Perturbation theory was first proposed for the solution of problems in celestial mechanics in the context of the motions of planets in the solar systems. Since the planets are very remote from each other and since their mass is small compared to the mass of the sun, the gravitational forces between the planets can be neglected and the planetary motion considered to a first approximation as taking places along Kepler's orbits which are defined by the equation of the two body problem, the two bodies being the planet and the sun.

Since astronomic data come to be known with much greater accuracy, it became necessary to consider how the motion of a planet around the sun is affected by other planets. This was the origin of the three body problem; thus, in studying the system Moon-Earth-Sun the mass ratio between the moon and the Earth was chosen as the small parameter.

J.L Lagrange and P. Laplace were the first to advance the view that the constants which describes the motion of a planet around the sun are "perturbed", as it were, by the motion of the other planets and vary as a function of time; hence the name "Perturbation theory". Perturbation theory was investigated by the classical scholars - Laplace, S. Poisson, C.F. Gauss- as a results of which the computations could be performed with a very high accuracy. The discovery of the planet Neptune in 1848 by J. Adams and U.le Venier, based on the deviations in motion of the planet Uranus, represented a triumph of perturbation theory.

2.2 Perturbation Theory

Perturbation theory comprises mathematical methods for finding an approximate solution to a problem, by starting from the exact solution of a related simpler problem. A critical feature of the technique is a middle step that breaks the problem into "solvable" and "perturbation" parts. Perturbation theory is applicable if the problem at hand cannot be solved exactly, but can be formulated by adding a "small term" to the mathematical description of the exactly solvable problem. Perturbation theory lead to an expression for the desired solution in terms of a formal power series in some "small" parameter known as a perturbation series- that quantifies the deviation from the exactly solvable problem.

The leading term in this power series is the solution of the exactly solvable problem while further terms describe the deviation in the solution due to the deviation from the initial problem. Formally; we have for the approximation to the full solution A, a series in the small parameter (here called ε), like the following:

$$A = A_0 + \varepsilon^1 A_1 + \varepsilon^2 A_2 \cdots$$

In this case, A_0 would be the known solution to the exactly solvable initial problem and A_1, A_2, $\cdots$ represents the higher-order terms which may be found iteratively by some systematic procedure. For small ε these higher-order terms in the series become successively smaller. An approximate "perturbation solution" is obtained by truncating the series, usually by keeping only the first three terms, the initial solution, the "first order" and the "second order" perturbation correction;

$$A = A_0 + \varepsilon^1 A_1 + \varepsilon^2 A_2$$

Perturbation methods start with a simplified form of the original problem, which is simple enough to be solved exactly. The basic principle and practice of perturbation theory is;

1. Find an easy problem that is close to the difficult problem. Like set $\varepsilon = 0$ and solve the resulting system (solution f_0 for definiteness).
2. Perturb the system by allowing ε to be nonzero (but small in some sense).
3. Formulate the solution to the new, perturbed system as a series $f_0 + \varepsilon f_1 + \varepsilon^2 f_2 + \cdots$
4. Expand the governing equation as a series in ε, collecting terms with equal powers of ε. Solve them in turn as far as the solution is required.
5. Solve the difficult problem by summing the series with the appropriate value of ε.

Perturbation method has three classifications namely;

(i) Regular perturbation method

(ii) Singular perturbation method

(iii) Homotopy perturbation method

2.3 Regular and Singular Perturbation Methods

Regular perturbation method is a case where a mathematical problem cannot be solved exactly or, if the exact solution is available it exhibits such an intricate dependency in the parameters that it is hard to use as such. It may be the case, however that a parameter can be identified say ε, such that the solution is available and reasonably simple for $\varepsilon = 0$. Then the solution is altered for nonzero but small ε to give a systematic answer to the problem.

Singular perturbation concern the study of problems featuring a parameter for which the solution of the problem at a limiting value of the parameter are different in character from the limit of the solution of the general problem. Modeling problems nearly always contain parameter, which are connected to the physicochemical dynamics of the system. These parameter may take a range of values. The solution obtained when the parameter is zero is called the base case. If one of the parameter is small, the behavior of the system can take trajectory that is remote from the base case. The analysis of the system having such behavior is called singular perturbation.

2.3.1 Examples on application of Regular and Singular perturbation methods

1. **Algebraic Equations**

a) Consider a quadratic equation

$$x^2 - (3 + 2\varepsilon)x + 2 + \varepsilon = 0 \tag{1}$$

when $\varepsilon = 0$ then (1) reduce to

$$x^2 - 3x + 2 = 0$$

which implies

$$(x-2)(x-1) = 0 \tag{2}$$

Whose roots are $x = 1$ and $x = 2$.

Equation (1) is called perturbed equation where as equation (2) is called un-perturbed or reduced equation.

Step 1: In determining an approximate solution is to assume the form of the expansion. Let us assume that the roots have expansion in the form

$$x = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \cdots \tag{3}$$

Here the first term x_0 is the zeroth-order term, the second term εx_1, is the first order term and the third term $\varepsilon^2 x_2$ as the second order term.
Step 2: Substitute equation (3) in the equation (1).

$$(x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \cdots)^2 - (3 + 2\varepsilon)(x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \cdots) + 2 + \varepsilon = 0 \tag{4}$$

Step 3: Using binomial theorem to expand the first

$$\begin{aligned}(x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \cdots)^2 &= x_0^2 + 2x_0(\varepsilon x_1 + \varepsilon^2 x_2 + \cdots) + \cdots + (\varepsilon x_1 + \varepsilon^2 x_2 + \cdots)^2 \\ &= x_0^2 + 2\varepsilon x_0 x_1 + 2\varepsilon^2 x_0 x_2 + \varepsilon^2 x_1^2 + \varepsilon^3 x_1 x_2 + \varepsilon^4 x_2^2 + \cdots \\ &= x_0^2 + 2\varepsilon x_0 x_1 + \varepsilon^2(2x_0 x_2 + x_1^2) + \cdots \end{aligned} \tag{5}$$

Similarly

$$\begin{aligned}(3 + 2\varepsilon)(x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \cdots) &= 3x_0 + 3\varepsilon x_1 + 3\varepsilon^2 x_2 + \cdots + 2\varepsilon x_0 + \varepsilon^2 x_1 + \varepsilon^3 x_2 + \cdots \\ &= 3x_0 + \varepsilon(3x_1 + 2x_0) + \varepsilon^2(3x_2 + 2x_1) + \cdots \end{aligned} \tag{6}$$

Substitute equation (5) and (6) in equation (4)

$$x_0^2 + 2\varepsilon x_0 x_1 + \varepsilon^2(2x_0 x_1 + x_1^2) - (3x_0 + \varepsilon(3x_1 + 2x_0) + \varepsilon^2(3x_2 + 2x_1)) + 2 + \varepsilon = 0$$

Collect the coefficients of like powers of ε yields

$$(x_0^2 - 3x_0 + 2) + \varepsilon(2x_0 x_1 - 3x_1 - 2x_0 + 1) + \varepsilon^2(2x_0 x_2 + x_1^2 - 3x_2 - 2x_1) + \cdots = 0 \tag{7}$$

Step 4: Equating the coefficient of each power of ε to zero

$$x_0^2 - 3x_0 + 2 = 0 \tag{8}$$

$$2x_0 x_1 - 3x_1 - 2x_0 + 1 = 0 \tag{9}$$

$$2x_0 x_2 + x_1^2 - 3x_2 - 2x_1 = 0 \tag{10}$$

From equation (8) $x_0 = 1$, b= 2. When $x_0 = 1$ equation (9) becomes

$$x_1 + 1 = 0 \Rightarrow x_1 = -1$$

When $x_0 = 1$ and $x_1 = -1$ equation (10) becomes;

$$2x_2 + 1 - 3x_2 + 2 = 0 \qquad \Longrightarrow x_2 - 3 = 0 \qquad \Longrightarrow x_2 = 3$$

When $x_0 = 2$, equation (9) becomes

$$x_1 - 3 = 0 \qquad \Longrightarrow x_1 = 3$$

Equation (10) becomes,

$$x_2 + 3 = 0 \qquad \Longrightarrow x_2 = -2$$

Step 5: When $x_0 = 1$, $x_1 = -1$ and $x_2 = 3$ equation (3) becomes

$$x = 1 - \varepsilon + 3\varepsilon^2 + \cdots \tag{11}$$

When $x_0 = 2$, $x_1 = 3$ and $x_2 = -3$ equation (3) becomes

$$x = 2 + 3\varepsilon - 3\varepsilon^2 + \cdots \tag{12}$$

Equations (11) and (12) are the approximations for the two roots of equation (1).

Now to verify the approximation are correct, we compare with the exact solution

$$\begin{aligned} x^2 - (3+2\varepsilon)x + 2 + \varepsilon = 0 \\ \Rightarrow x &= \frac{1}{2}[3 + 2\varepsilon \pm \sqrt{(3+2\varepsilon)^2 - 4(2+\varepsilon)}] \\ x &= \frac{1}{2}[3 + 2\varepsilon \pm \sqrt{1 + 8\varepsilon + 4\varepsilon^2}] \end{aligned} \tag{13}$$

Using binomial theorem, we have

$$\begin{aligned} (1 + 8\varepsilon + 4\varepsilon^2)^{\frac{1}{2}} &= 1 + \frac{1}{2}(8\varepsilon + 4\varepsilon^2) + \frac{(\frac{1}{2})(-\frac{1}{2})(8\varepsilon + 4\varepsilon^2)^2}{2!} + \cdots \\ &= 1 + 4\varepsilon + 2\varepsilon^2 - \frac{1}{8}(64\varepsilon^2 + \cdots) \\ &= 1 + 4\varepsilon + 6\varepsilon^2 + \cdots \end{aligned}$$

Substitute this value in equation (13) we have;

$$\begin{aligned} x &= \frac{1}{2}(3+2\varepsilon+1+4\varepsilon-6\varepsilon^2+\cdots \\ &= 2+3\varepsilon-3\varepsilon^2+\cdots \end{aligned}$$

or

$$\begin{aligned} x &= \frac{1}{2}(3+2\varepsilon-1-4\varepsilon+6\varepsilon^2+\cdots) \\ &= 1-\varepsilon+3\varepsilon^2+\cdots \end{aligned}$$

which are the same as the solution for perturbation method equation (11) and (12).This is an example of Regular perturbation Consider;

$$\varepsilon x^2+x+1=0 \tag{14}$$

Since the equation is a quadratic equation, it has two roots. For $\varepsilon \to 0$ equation (14) reduces to,

$$x+1=0 \tag{15}$$

which is of first order. Thus x is discontinuous at $\varepsilon=0$. Such perturbation are called singular perturbation problem.

$$x = x_0+\varepsilon x_1+\varepsilon^2 x_2+\cdots \tag{16}$$

Substituting equation (16) into equation (14)

$$\begin{aligned} &\varepsilon(x_0+\varepsilon x_1+\varepsilon^2 x_2+\cdots)^2+x_0+\varepsilon x_1+\varepsilon^2 x_2+\cdots+1=0 \\ &\implies \varepsilon(x_0^2+2\varepsilon x_0 x_1+\cdots)+x_0+\varepsilon x_1+\varepsilon^2 x_2+\cdots+1=0 \\ &\implies \varepsilon x_0^2+2\varepsilon^2 x_0 x_1+\cdots+x_0+\varepsilon x_1+\varepsilon^2 x_2+\cdots+1=0 \end{aligned}$$

Equating coefficient of like power of ε gives,

$$\begin{aligned} \varepsilon^0 &: x_0+1=0 \\ \varepsilon^1 &: x_1+x_0^2=0 \end{aligned}$$

When $x_0=-1, x_1=-1$, so one of the root is

$$x=-1-\varepsilon+\cdots \tag{17}$$

Thus as expected the above procedure yielded only one root. We investigate the exact solution that is,

$$x=\frac{1}{2\varepsilon}(-1\pm\sqrt{1-4\varepsilon}) \tag{18}$$

Using binomial theorem we have

$$\sqrt{1-4\varepsilon} = 1-2\varepsilon+\frac{(\frac{1}{2})(-\frac{1}{2})}{2!}(-4\varepsilon)^2+\cdots$$
$$= 1-2\varepsilon-2\varepsilon^2+\cdots \tag{19}$$

Substituting (19) in (18)

$$x = \frac{-1+1-2\varepsilon-2\varepsilon^2+\cdots}{2\varepsilon} = -1-\varepsilon+\cdots \tag{20}$$

or

$$x = \frac{-1-1+2\varepsilon+2\varepsilon^2+\cdots}{2\varepsilon} = \frac{-1}{\varepsilon}+1+\varepsilon+\cdots \tag{21}$$

Therefore, both of the roots go in powers of ε but one starts with ε^{-1}. Hence it is not surprising that the assumed expansion in (16) it failed to produce the root (21). Consequently one cannot determine the second root by perturbation technique unless its form is known. In those case, we recognize that, if the order of the equation is not to be reduced, the other tends to ∞ as $\varepsilon \to 0$ and hence, assume that the leading term has the form,

$$x = \frac{y}{\varepsilon^v} \tag{22}$$

where v must be greater than zero and needs to be determined in the course of analysis. Substitute (22)in (14)

$$\varepsilon^{1-2v}y^2+\varepsilon^v y+1+\cdots = 0$$

Since $v > 0$, the 2^{nd} term is much bigger than '1'. Hence the dominant part of (22) is,

$$\varepsilon^{1-2v}y^2+\varepsilon^v y = 0 \tag{23}$$

which demands that power of ε be the same.

$$1-2v = -v \qquad \Longrightarrow \qquad v = 1$$
$$\text{for} \quad v = 1 \qquad \Longrightarrow y = 0 \quad \text{or} \quad -1$$

First value $y = 0$, correspond to the first root $x = -1-\varepsilon$
For $y = -1$, it corresponds to second root. Thus it follows from (22)

$$x = \frac{-1}{\varepsilon}+\cdots$$

To determine more terms in the expansion of second root, we try

$$x = \frac{-1}{\varepsilon} + x_0 + \cdots \tag{24}$$

substitute it in equation (14)

$$\begin{aligned}
\implies \quad & \varepsilon(\frac{-1}{\varepsilon} + x_0 + \cdots)^2 - \frac{-1}{\varepsilon} + x_0 + \cdots + 1 = 0 \\
\implies \quad & \varepsilon((\frac{-1}{\varepsilon})^2 + \frac{2}{\varepsilon}x_0 + x_0^2 + \cdots) - \frac{-1}{\varepsilon} + x_0 + 1 + \cdots = 0 \\
\implies \quad & -2x_0 + x_)^2 + 1 + O(\varepsilon) = 0
\end{aligned}$$

$\implies \quad x_0 = 1$ and equation (24) becomes

$$x = \frac{-1}{\varepsilon} + 1 + \cdots$$

Alternatively, once v has been determined we view (22) as a transformation from x to y. Then putting $x = \frac{y}{\varepsilon}$ in (14) yields

$$y^2 + y + \varepsilon = 0$$

Which can be solved to determine both the roots because ε does not multiply the highest order.

Ordinary differential equations

(i) The basic principles underlying perturbation methods can be explained using elementary first order equation

$$\frac{dy}{dx} + \varepsilon y = 0; \qquad y(0) = 1 \tag{25}$$

Step 1: Let us assume that the solution has expansion of the form

$$y(x) = y_0(x) + \varepsilon y_1(x) + \varepsilon^2 y_2(x) + \cdots \tag{26}$$

Step 2: Inserting (26) in to defining equation (25) yields

$$[\frac{dy_0}{dx} + \varepsilon \frac{dy_1}{dx} + \varepsilon^2 \frac{dy_2}{dx} + \cdots] + \varepsilon[y_0(x) + \varepsilon y_1(x) + \varepsilon^2 y_2(x) + \cdots] = 0 \tag{27}$$

Step 3: We stipulate the following identities by matching like multiples of ε^0 ε^1, ε^2 and so on.

$$\begin{aligned}
&\varepsilon^0: \quad \frac{dy_0}{dx} = 0 \qquad \Longrightarrow y_0 = k_0 \\
&\varepsilon^1: \quad \frac{dy_1}{dx} + y_0 = 0 \qquad \Longrightarrow \frac{dy_1}{dx} = -y_0 \\
&\quad y_1 = -k_0 x + k_1 \\
&\varepsilon^2: \quad \frac{dy_2}{dx} = -y_1 \qquad \Longrightarrow \frac{dy_2}{dx} = 0 = k_0 x - k_1 \\
&\quad y_2 = \frac{k_0 x^2}{2} - k_1 x + k_2
\end{aligned}$$

where k_0 k_1 and k_2 are constants. Obviosly, the solution $y_0(x)$ corresponds to the base (when $\varepsilon = 0$); so we shall stipulate that

$$\begin{aligned}
y(0) = y_0(0) &= 1 \\
\therefore k_0 &= 1
\end{aligned}$$

This is a critical component of regular perturbation methods. Since only the base case comes with the primary boundary conditions, hence by manipulation, we must have for the other solutions.

$$y_1(0) = y_2(0) = \cdots = 0$$

This allows the determination of k_1, k_2, k_3, $\cdots$ in sequence

$$\begin{aligned}
y_1 &= -k_0 x + k_1; \qquad k_0 = 1 \\
y_1(0) &= 0 = 0 + k_1 \quad \Longrightarrow k_1 = 0 \\
y_1 &= -x \\
y_2 &= \frac{k_0 x^2}{2} - k_1 x + k_2; \quad y_0 = 1; \quad k_1 = 0 \\
y_2(0) &= 0 \qquad \Longrightarrow k_2 = 0 \\
y_2 &= \frac{x^2}{2}
\end{aligned}$$

Therefore,

$$y(x) = 1 - \varepsilon x + \frac{x^2}{2} + \cdots$$

This is identical to the first three terms of the analytical solution.

(ii) Let's consider the following differential equation;

$$\frac{dy}{dx} + y - \varepsilon y^2 = 0; \qquad y(0) = 2 \tag{28}$$

We assume that the solution has expansion of the form

$$y(x) = y_0(x) + \varepsilon y_1(x) + \varepsilon^2 y_2(x) + \cdots \tag{29}$$

we insert (29) into the defining equation (28) which yields

$$[\frac{dy_0}{dx} + \varepsilon\frac{dy_1}{dx} + \varepsilon^2\frac{dy_2}{dx} + \cdots] + \varepsilon[y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \cdots] - \varepsilon[y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \cdots]^2 = 0$$

This simplifies to

$$[\frac{dy_0}{dx} + y_0] + \varepsilon[\frac{dy_1}{dx} + y_1 - y_0^2] + \varepsilon^2[\frac{dy_2}{dx} + y_2 - 2y_0y_1] + \cdots = 0$$

In order to satisfy the initial condition $y(0) = 2$, we will have

$$y_1(0) = y_2(0) = y_3(0) = \cdots = 0$$

We collect powers of ε;

$$\begin{aligned}
\varepsilon^0: &\quad \frac{dy_0}{dx} + y_0 = 0\\
\varepsilon^1: &\quad \frac{dy_1}{dx} + y_1 - y_0^2 = 0\\
\varepsilon^2: &\quad \frac{dy_2}{dx} + y_2 - 2y_0y_1 = 0
\end{aligned}$$

Now, we solve at each order, applying the boundary conditions as we move along;

$$\frac{dy_0}{dx} + y_0 = 0; \qquad y(0) = 2$$

Integrating factor $\mu = e^{\int dx} = e^x$.

$$\begin{aligned}
\frac{dy_0}{dx}e^x + e^x y_0 &= 0\\
\frac{d}{dx}[e^x y_0] &= 0\\
\int d[e^x y_0] &= \int o dx\\
e^x y_0 &= k_0\\
y_0 &= k_0 e^{-x}; \qquad y(0) = 2\\
k_0 &= 2\\
y_0(x) &= 2e^{-x}
\end{aligned}$$

$$
\begin{aligned}
\frac{dy_1}{dx} + y_1 - y_0^2 = 0 \qquad &\implies \frac{dy_1}{dx} + y_1 - 4e^{-2x} = 0 \\
\mu = e^{\int dx} = e^x \\
\frac{dy_1}{dx}e^x + e^x y_1 = 4e^{-x} \\
\frac{d}{dx}(e^x y_1) = 4e^{-x} \\
\int d(e^x y_1) = \int 4e^{-1} dx \\
e^x y_1 = -4e^{-x} + k_1 \\
y_1 = \frac{-4e^{-x} + k_1}{e^x} \\
y_1 = -4e^{-2x} + e^{-x} k_1
\end{aligned}
$$

applying the boundary condition $y_1(0) = 0$ yields

$$
\begin{aligned}
y_1(0) = 0 = -4 + k_1 \qquad &\implies k_1 = 4 \\
y_1 = -4(e^{-2x} + 4e^{-x}) \\
y_1 = 4(e^{-x} - 4e^{-2x})
\end{aligned}
$$

Order ε^2 terms

$$
\begin{aligned}
\frac{dy_2}{dx} + y_2 = 2y_0 y_1 \\
\frac{dy_2}{dx} + y_2 = 2 \cdot 2e^{-x} \cdot 4(e^{-x} - e^{-2x}) \\
\frac{dy_2}{dx} + y_2 = 16e^{-x}(e^{-x} - e^{-2x}) \\
\mu = e^{\int dx} = e^x \\
\frac{dy_2}{dx}e^x + e^x y_2 = (16e^{-2x} - 16e^{-3x})e^x \\
\frac{d}{dx}(e^x y_2) = (16e^{-x} - 16e^{-2x}) \\
\int d(e^x y_2) = \int (16e^{-x} - 16e^{-2x}) dx \\
e^x y_2 = -16e^{-x} + 8e^{-2x} + k_2 \\
y_2 = \frac{-16e^{-x} + 8e^{-2x}}{e^x} + k_2 e^{-x} \\
y_2 = 8(e^{-3x} - 2e^{-2x}) + k_2 e^{-x}
\end{aligned}
$$

and the boundary condition $f_2(0) = 0$

$$
\begin{aligned}
y_0(0) = 0 = -8 + k_2 \qquad &\implies k_2 = 8 \\
y_2 = 8(e^{-x} - 2e^{-2x} - e^{-3x})
\end{aligned}
$$

Order ε^3 terms: The equation is;

$$\begin{aligned}
&\frac{dy_3}{dx} + y_3 - y_0^2 - 2y_0y_2 = 0 \\
&\frac{dy_3}{dx} + y_3 = 16(e^{-2x} - 2e^{-3x} + e^{-4x}) + 2\cdot 2e^{-x}\cdot 8(e^{-x} - 2e^{-2x} + e^{-3x}) \\
&\frac{dy_3}{dx} + y_3 = 16e^{-2x} - 32e^{-3x} + 16e^{-4x} + 32e^{-2x} - 64e^{-3x} + 32e^{-4x} \\
&\frac{dy_3}{dx} + y_3 = 48e^{-2x} - 96e^{-3x} + 48e^{-4x} \\
&\frac{dy_3}{dx} + y_3 = 48(e^{-2x} - 2e^{-3x} + e^{-4x})
\end{aligned}$$

$\mu = e^{\int dx} = e^x$.

$$\begin{aligned}
\frac{dy_3}{dx}e^x + e^x y_3 &= 48e^x(e^{-2x} - 2e^{-3x} + e^{-4x}) \\
\frac{d}{dx}(e^x y_3) &= 48(e^{-x} - 2e^{-2x} + e^{-3x}) \\
\int d(e^x y_3) &= \int 48(e^{-x} - 2e^{-2x} + e^{-3x})dx \\
e^x y_3 &= -48e^{-x} + 48e^{-2x} - 16e^{-3x} + k_3 \\
y_3 &= -48e^{-2x} + 48e^{-3x} - 16e^{-4x} + k_3e^{-x} \\
y_3 &= 16(-3e^{-2x} + 3e^{-3x} - e^{-4x}) + k_3e^{-x}
\end{aligned}$$

Applying the boundary condition $f_3(0) = 0$

$$\begin{aligned}
y_3(0) = 0 = 16(-1) + k_3 \qquad &\Longrightarrow k_3 = 16 \\
y_3(x) = 16(e^{-x} - 3e^{-2x} + 3e^{-3x} - e^{-4x})&
\end{aligned}$$

The solution we have found is;

$$\begin{aligned}
y(x) =& 2e^{-x} + 4\varepsilon(e^{-x} - e^{-2x}) + 8\varepsilon^2(e^{-x} - 2e^{-2x} + e^{-3x}) \\
&+ 16\varepsilon^3(e^{-x} - 3e^{-2x} + 3e^{-3x} - e^{-4x}) + \cdots
\end{aligned}$$

This is an example of a perturbation expansion which give us an insight into the full solution. Notice that, for the terms we have calculated,

$$y(x) = 2^{n+1}e^{-x}(1 - e^{-x})^n$$

which suggest a full solution,

$$\begin{aligned}
y(x) &= \sum_{n=0}^{\infty} \varepsilon^n 2^{n+1} e^{-x}(1-e^{-x}) \\
&= 2e^{-x} \sum_{n=0}^{\infty} [2\varepsilon(1-e^{-x})]^n \\
&= \frac{2e^{-x}}{1-2\varepsilon(1-e^{-x})}
\end{aligned}$$

Which is indeed the correct solution to the ordinary differential equation (28).

(iii) Let us consider a second order ordinary differential equation

$$\frac{d^2y}{d\tau^2} = -\varepsilon\frac{dy}{d\tau} - 1; \quad y(0) = 0; \quad \frac{dy}{d\tau} = 1 \tag{30}$$

we assume the expansion

$$y(\tau) = y_0(\tau) + \varepsilon y_1(\tau) + \varepsilon^2 y_2(\tau) + O(\varepsilon^3) \tag{31}$$

Then,

$$\begin{aligned}
\frac{dy}{d\tau} &= \frac{dy_0}{d\tau} + \varepsilon\frac{dy_1}{d\tau} + \varepsilon^2\frac{dy_2}{d\tau} + O(\varepsilon^3) \\
\frac{d^2y}{d\tau^2} &= \frac{d^2y_0}{d\tau^2} + \varepsilon\frac{d^2y_1}{d\tau^2} + \varepsilon^2\frac{d^2y_2}{d\tau^2} + O(\varepsilon^3)
\end{aligned}$$

Substituting equation (31) into (30) we have

$$\begin{aligned}
\frac{d^2y_0}{d\tau^2} + \varepsilon\frac{d^2y_1}{d\tau^2} + \varepsilon^2\frac{d^2y_2}{d\tau^2} + O(\varepsilon^3) + \varepsilon(\frac{dy_0}{d\tau} + \varepsilon\frac{dy_1}{d\tau} + \varepsilon^2\frac{dy_2}{d\tau} + O(\varepsilon^3)) + 1 = 0 \\
\frac{d^2y_0}{d\tau^2} + 1 + \varepsilon(\frac{d^2y_1}{d\tau^2} + \frac{dy_0}{d\tau}) + \varepsilon^2(\frac{d^2y_2}{d\tau^2} + \frac{dy_1}{d\tau}) + O(\varepsilon^3) = 0
\end{aligned}$$

Equating the coefficients of ε, it becomes;

$$\begin{aligned}
\frac{d^2y_0}{d\tau^2} + 1 = 0, \quad y(0) = 0; \quad \frac{dy}{d\tau} = 1 \\
\frac{d^2y_1}{d\tau^2} + \frac{dy_0}{d\tau} = 0, \quad y(0) = 0; \quad \frac{dy}{d\tau} = 1 \\
\frac{d^2y_2}{d\tau^2} + \frac{dy_1}{d\tau} = 0, \quad y(0) = 0; \quad \frac{dy}{d\tau} = 1
\end{aligned} \tag{32}$$

By solving the above equation we will get;

$$\frac{d^2y_0}{d\tau^2}+1=0. \qquad \implies \frac{d^2y_0}{d\tau^2}=-1$$
$$\int \frac{d}{d\tau}(\frac{dy_0}{d\tau})=\int -1 \qquad \implies \frac{dy_0}{d\tau}=-\tau+k_0$$
$$\frac{dy_0}{d\tau}(0)=1 \qquad \therefore k_o=1$$
$$\frac{dy_0}{d\tau}=-\tau+k_0 \qquad \implies \int dy_0=\int(-\tau+1)d\tau$$
$$y_0(0)=\frac{-\tau^2}{2}+\tau+k_1 \qquad \implies k_1=0 \qquad \text{since} \qquad y_0(0)=0$$

$$\therefore \quad y_0(\tau)=\tau-\frac{\tau^2}{2} \tag{33}$$

$$\frac{d^2y_1}{d\tau^2}+\frac{dy_0}{d\tau}=0$$
$$\frac{d^2y_1}{d\tau^2}-\tau+1=0 \qquad \implies \frac{d}{d\tau}(\frac{dy_1}{d\tau})=\tau-1$$
$$\int d(\frac{dy_1}{d\tau})=\int(\tau-1)d\tau \qquad \implies \frac{dy_1}{d\tau}=\frac{\tau^2}{2}-\tau+k_1$$

but

$$\frac{dy_1}{d\tau}(0)=0 \qquad \therefore k_1=0$$
$$\frac{dy_1}{d\tau}=\frac{\tau^2}{2}-\tau \qquad \implies \int dy_1=\int(\frac{\tau^2}{2}-\tau)d\tau$$
$$y_1(\tau)=\frac{\tau^3}{6}-\frac{\tau}{2}+k_2. \qquad \text{but} \qquad k_2=0 \qquad \text{since} \quad y_(0)=0$$

$$\implies y_1(\tau)=\frac{\tau^3}{6}-\frac{\tau}{2} \tag{34}$$

$$\frac{d^2y_2}{d\tau^2} + \frac{dy_1}{d\tau} = 0$$
$$\frac{d^2y_2}{d\tau^2} + \frac{\tau^2}{2} - \tau = 0 \qquad \Longrightarrow \frac{d^2y_2}{d\tau^2} = \tau - \frac{\tau^2}{2}$$
$$\int d(\frac{sy_2}{d\tau}) = \int (\tau - \frac{\tau^2}{2})d\tau \qquad \Longrightarrow \frac{dy_2}{d\tau} = \frac{\tau}{2} - \frac{\tau^3}{6} + k_2$$
$$\text{but} \quad \frac{dy_2}{d\tau} = 0 \qquad \therefore k_2 = 0$$
$$\frac{dy_2}{d\tau} = \frac{\tau}{2} - \frac{\tau^3}{6} \qquad \Longrightarrow \int dy_2 = \int (\frac{\tau}{2} - \frac{\tau^3}{6})d\tau$$
$$y_2(\tau) = \frac{\tau^3}{6} - \frac{\tau^4}{24} + k_3 \qquad k_3 = 0 \qquad \text{since} \quad y_2(0) = 0$$

$$y_2(\tau) = \frac{\tau^3}{6} - \frac{\tau^4}{24} \tag{35}$$

Putting these values in equation (31) we have the solution,

$$y(\tau) = \tau - \frac{\tau^2}{2} + \varepsilon(-\frac{\tau^2}{2} + \frac{\tau^3}{6}) + \varepsilon^2(\frac{\tau^3}{6} - \frac{\tau^4}{24}) + O(\varepsilon^3)$$

2.4 Homotopy Perturbation Method

2.4.1 Some Basic Concepts of the Homotopy Perturbation Method

Here we give some basic concepts of homotopy perturbation method. To do this, we recall the following definition:

Definition 2.4.1. *Let X and Y be two topological spaces. Two continuous function $f : X \to Y$ and $g : X \to Y$ are said to be homotopic, denoted by $f \cong g$, if there exist a continuous function $H : X \times [0,1] \to Y$ such that $H(x,0) = f(x), \forall x \in X; \quad H(x,1) = g(x), \forall x \in X$. In this case, H is said to be a homotopy.*

We illustrate this definition by considering the following example.

Example 2.4.2. *Let X and Y be any topological spaces, f be the identity function and g be the zero function, then define $H : X \times [0,1] \to Y$ by $H(x,p) = x(1-p), \forall x \in X, \forall p \in [0,1]$ Then H is a continuous function and $H(x,0) = x = f(x), \forall x \in X; \quad H(x,1) = 0 = g(x), \forall x \in X$ Therefore $f \cong g$.*

Proposition 2.4.3. *On the continuous functions $\cong$ is an equivalence relation*

Proof. Let $f : X \to Y$ be a continuous function, then define $H : X \times [0,1] \to Y$ by $H(x,p) = f(x), \forall x \in X, \forall p \in [0,1]$. Therefore

$$H(x,0) = f(x), \forall x \in X; \quad H(x,1) = f(x), \forall x \in X$$

and this implies that $f \cong f$. Therefore $\cong$ is a reflexive relation.
To prove that $\cong$ is symmetric relation, let $f \cong g$, then there exist a continuous function $H : X \times [0,1] \to Y$ such that

$$H(x,0) = f(x), \forall x \in X; \quad H(x,1) = g(x), \forall x \in X.$$

Define $K : X \times [0,1] \to Y$ by : $K(x,p) = H(x,1-p), \forall x \in X, \forall p \in [0,1]$. Then

$$K(x,0) = H(x,1) = g(x), \forall x \in X; \quad K(x,1) = H(x,0) = f(x), \forall x \in X$$

. Hence $g \cong f$.
To prove $\cong$ is a transitive relation, let $f \cong g$ and $g \cong w$, then there exist continuous functions $H : X \times [0,1] \to Y$ and $K : X \times [0,1] \to Y$ such that:

$$H(x,0) = f(x), \forall x \in X; \quad H(x,1) = g(x), \forall x \in X$$
$$K(x,0) = g(x), \forall x \in X; \quad K(x,1) = w(x), \forall x \in X$$

Define $L : X \times [0,1] \to Y$ by:

$$L(x,p) = \begin{cases} f(x), & p = 0 \\ H(x,p) + K(x,p) - g(x), & 0 < p < 1 \\ w(x), & p = 1 \end{cases}$$

. Therefore

$$L(x,0) = f(x), \forall x \in X; \quad L(x,1) = w(x), \forall x \in X.$$

$$\begin{aligned} \lim_{p \to 0^+} L(x,p) &= \lim_{p \to 0^+} [H(x,p) + K(x,p) - g(x)] \\ &= H(x,0) + K(x,0) - g(x) \\ &= f(x) + g(x) - g(x) \\ &= f(x) \\ &= L(x,0), \forall x \in X \end{aligned}$$

and

$$\begin{aligned}\lim_{p\to 1^-} L(x,p) &= \lim_{p\to 1^-} [H(x,p)+K(x,p)-g(x)] \\ &= H(x,1)+K(x,1)-g(x) \\ &= g(x)+w(x)-g(x) \\ &= w(x) \\ &= L(x,1), \forall x\in X\end{aligned}$$

Hence L is a continuous function. Therefore $f \cong w$. Hence $\cong$ is an equivalence relation on the set of all continuous function. □

Remark 2.4.4. *Let X and Y be two topological spaces,let $f : R \to R$ and $g : R \to R$ be continuous functions. Define $H : R\times[0,1] \to R$ by*

$$H(x,p) = (1-p)f(x)+pg(x), \quad \forall x\in X, \quad \forall p\in[0,1]$$

then

$$H(x,0) = f(x), \quad \forall x\in X; \quad H(x,1) = g(x), \quad \forall x\in X$$

Therefore $f \cong g$

Definition 2.4.5. *Let X and Y be two topological spaces,let $f : X \to Y$ be continuous function. The equivalence class of f_i denoted by $[f]$ is defined by:*

$$[f] = \{g| \quad g : X \to Y \text{ be continuous function and} f \cong g\}$$

and it is said to be a homotopy class of functions of f.

Definition 2.4.6. *Let X and Y be two topological spaces. Two continuous functions f and g are said to be homotopic relative to $A \subseteq X$ if there exists a continuous function $H : X\times[0,1] \to Y$ such that*

$$\begin{aligned}H(x,0) &= f(x), \quad \forall x\in X \\ H(x,1) &= g(x), \quad \forall x\in X \\ H(a,p) &= f(a)+g(a), \quad \forall p\in[0,1], \quad \forall a\in A\end{aligned}$$

Now, we demonstrate the basic idea of the homotopy perturbation method. Let us consider the following differential equation.

$$A(u) = f(x), \qquad x\in\Omega \tag{36}$$

Considering the boundary conditions of:

$$B(u, \frac{\partial u}{\partial x}) = 0, \qquad x \in \Gamma \tag{37}$$

where A is a general differential operator, B a boundary operator, $f(x)$ a known analytic function and Γ is the boundary of the domain Ω. The operator A can be generally divided into two parts of L and N where L is linear part, while N is the non-linear part in the differential equation. Therefore equation (36) can be rewritten as follows:

$$L(u) + N(u) - f(x) = 0 \tag{38}$$

By using homotopy technique, we construct a homotopy as: $v(x, p) : \Omega \times [0, 1] \to \mathbb{R}$ which satisfies the homotopy equation:

$$\begin{aligned} H(v, p) &= (1 - p)[L(v) - L(u_0)] + p[L(v) + N(v) - f(x)] = 0 \\ &\text{or} \\ H(v, p) &= L(v) - L(u_0) + pL(u_0) + p[N(v) - f(x)] = 0 \end{aligned} \tag{39}$$

where $p \in [0, 1]$ is an embedding parameter and u_0 is an initial approximation of the differential equation which satisfies boundary conditions. Obviously considering equation (38) and (39) we will have :

$$H(v, 0) = L(v) - L(u_0) = 0 \tag{40}$$

$$H(v, 1) = A(v) - f(x) = 0 \tag{41}$$

The changing process of p from zero to unity is just that of $v(x, p)$ from $u_0(x)$ to $u(x)$. Therefore;

$$\begin{aligned} L(v) - L(u_0) &\approxeq A(v) - f(x), \qquad x \in \Omega \\ \text{and} \qquad u(x_0) &\approxeq u(x), \qquad x \in \Omega \end{aligned}$$

In topology, this is called deformation and $L(v) - L(u_0)$, $L(v) + N(v) - f(x)$ are homotopy. The basic assumption is that the solution of the equations (38) and (39) can be expressed as a power series in p.

$$v(x, p) = v_0 + pv_1 + p^2 v_2 + \cdots \tag{42}$$

or

$$v(x, p) = \sum_{i=0}^{\infty} p^i v_i(x)$$

Setting $p = 1$ we get

$$u(x) = \lim_{p \to 1} v(x,p) = v_0 + v_1 + v_2 + \cdots \tag{43}$$

or

$$u(x) = \lim_{p \to 1} v(x,p) = \sum_{i=0}^{\infty} v_i(x)$$

which is the solution of equation (36).

2.4.2 Examples on application of Homotopy Perturbation Method

1. **Solving ordinary differential equations**

   (a) Consider the first order non-linear ordinary differential equation

   $$\frac{dy}{dx} + y^2(x) = 0, \qquad |x| < 1$$

   Here $A(y) = \frac{dy}{dx} + y^2$ and $f(x) = 0$. The operator A can be divided into $L(y) = \frac{dy}{dx}$ and $N(y) = y^2$. From the constructed homotopy $u : \Omega \times [0,1] \to \mathbb{R}$ which satisfies:

   $$H(u,p) = L(u) - L(y_0) + pL(y_0) + p[N(u) - f(x)] = 0 \tag{i}$$

   In this case, equation (i) becomes

   $$u'(x) - y_0'(x) + py_0'(x) + p[u^2(x)] = 0, \quad p \in [0,1]$$

   Assume the solution of the above equation can be written as a power series

   $$u(x,p) = \sum_{i=0}^{\infty} p^i u_i(x)$$

   By substituting this solution into the above equation one can have:

   $$\sum_{i=0}^{\infty} p^i u_i'(x) - y_0'(x) + py_0'(x) + p[\sum_{i=0}^{\infty} p^i u_i(x)]^2 = 0$$

   By equating the terms with identical power of p one can have

   $$p^0 : \quad u_0'(x) - y_0'(x) = 0 \tag{1.10a}$$

$$p^1: \quad u_1'(x) + y_0'(x) + u_0^2 = 0 \tag{1.10b}$$

$$p^2: \quad u_2'(x) + 2u_0(x)u_1(x) = 0 \tag{1.10c}$$

$$p^3: \quad u_3'(x) + 2u_0(x)u_2(x) + u_1^2(x) = 0 \tag{1.10e}$$

$$\vdots$$

Let $u_0(x) = y_0(x)$, then equation (1.10a) is automatically satisfied. let $y_0(x) = 1$ be the initial approximation of the differential equation, then

$$u_0(x) = 1$$

By substituting u_0 and y_0 into equation (1.10b) one can have:

$$u_1'(x) = 1$$

and this implies that:

$$u_1(x) = -x$$

By substituting u_0 and u_1 into equation (1.10c) one can have:

$$u_2'(x) = 2x$$

and this implies that:

$$u_2(x) = x^2$$

By substituting u_0 , u_1 and u_2 into equation (1.10e) one can have:

$$u_3'(x) = -3x^2$$

and this implies that

$$u_3(x) = -x^3$$

Continuing in this manner we have

$$u_i(x) = (-1)^i x^i, \quad i = 0, 1, \cdots$$

Substituting these functions into equation,

$$y(x) = \lim_{p \to 1} u(x, p) = \sum_{i=0}^{\infty} u_i(x)$$

we get

$$y(x) = \sum_{i=0}^{\infty} (-1)^i x^i = \frac{1}{1+x}$$

which is the exact solution of the ordinary differential equation.

(b) We will consider the lightwill equation

$$(x+\varepsilon y)\frac{dy}{dx}+y=0; \quad y(1)=1 \tag{44}$$

By the method, we can construct a homotopy which satisfies

$$(1-p)[\varepsilon Y\frac{dY}{dx}-\varepsilon y_0\frac{dy_0}{dx}]+p[(x+\varepsilon y)\frac{dY}{dx}+Y]=0, \quad p\in[0,1] \tag{45}$$

we can obtain a solution of (45) in the form

$$Y(x)=Y_0(x)+pY_1(x)+p^2Y_2(x)+\cdots \tag{46}$$

where $y_i(x)$; $i=1,2,\cdots$ are functions yet to be determined. By considering only first two terms of the above equation, substitute equation (46) into (45)

$$\Rightarrow(1-p)[\varepsilon(Y_0+pY_1)(\frac{dY_0}{dx}+\frac{dY_1}{dx})-\varepsilon y_0\frac{dy_0}{dx}]+$$
$$p[(x+\varepsilon Y_0+\varepsilon pY_1)(\frac{dY_0}{dx}+p\frac{dY_1}{dx})+(Y_0+pY_1)]=0$$

$$\Rightarrow(1-p)[\varepsilon Y_0(\frac{dY_0}{dx}+\frac{dY_1}{dx})+\varepsilon pY_1(\frac{dY_0}{dx}+\frac{dY_1}{dx})-\varepsilon y_0\frac{dy_0}{dx}]+$$
$$p[(x+\varepsilon Y_0+\varepsilon pY_1)(\frac{dY_0}{dx}+p\frac{dY_1}{dx})+(Y_0+pY_1)]=0$$

$$\Rightarrow\varepsilon pY_1\frac{dY_1}{dx}+(1-p)[\varepsilon Y_0\frac{dY_0}{dx}-\varepsilon y_0\frac{dy_0}{dx}]+p[(x+\varepsilon Y_0)\frac{Y_0}{dx}+Y_0]+$$
$$\varepsilon p^2Y_1(\frac{dY_0}{dx}+p\frac{dY_1}{dx})+p^2Y_1=0$$

Now we get

$$\varepsilon Y_0\frac{dY_0}{dx}-\varepsilon y_0\frac{dy_0}{dx}=0 \tag{47}$$

$$\varepsilon Y_1\frac{dY_1}{dx}+[(x+\varepsilon Y_0)\frac{dY_0}{dx}+Y_0]=0 \tag{48}$$

The initial approximation $Y_0(x)$ or $y_0(x)$ can freely chosen. Here I set

$$Y_0(x)=y_0(x)=\frac{-x}{\varepsilon}, \quad Y_0(1)=\frac{-1}{\varepsilon} \tag{49}$$

So that, the residual of equation (44) at $x = 0$ vanishes. Then substitute equation (49) into (48)

$$\varepsilon Y_1 \frac{dY_1}{dx} + [(x - \varepsilon \frac{x}{\varepsilon}) \frac{dY_0}{dx} - \frac{x}{\varepsilon}] = 0$$
$$\Rightarrow \varepsilon Y_1 \frac{dY_1}{dx} - \frac{x}{\varepsilon} = 0$$
$$\Rightarrow \varepsilon Y_1 \frac{dY_1}{dx} = \frac{x}{\varepsilon}$$
$$\Rightarrow \varepsilon^2 Y_1 dY_1 = x dx$$

Integrating both sides, we get

$$\Rightarrow \varepsilon^2 \frac{Y_1^2}{2} = \frac{x^2}{2} + c$$
$$\Rightarrow \varepsilon^2 Y_1^2 = x^2 + 2c$$
$$\Rightarrow Y_1 = \sqrt{\frac{x^2 + 2c}{\varepsilon^2}}$$

$$\varepsilon Y_1 = \sqrt{x^2 + 2c} \tag{50}$$

Putting the initial conditions;

$$Y_1(1) = 1 - Y_0 = 1 + \frac{1}{\varepsilon}$$
$$\Rightarrow \varepsilon(1 + \frac{1}{\varepsilon}) = \sqrt{1 + 2c}$$
$$1 + \varepsilon = \sqrt{1 + 2c}$$
$$1 + 2\varepsilon + \varepsilon^2 = 1 + 2c$$
$$c = \frac{\varepsilon^2 + 2\varepsilon}{2}$$

Now, putting this value in the equation (50) we get

$$Y_1 = \frac{1}{\varepsilon} \sqrt{x^2 + \varepsilon + \varepsilon^2}$$

Substitute this value in equation (46)

$$Y(X) = Y_0(x) + Y_1(x) = \frac{1}{\varepsilon} \sqrt{x^2 + \varepsilon + \varepsilon^2}$$

which is the exact solution of the equation.

(c) In this example, we demonstrate the main algorithm of homotopy perturbation method on non-linear parabolic equations with initial conditions.We consider;

$$\frac{du}{dt} = \frac{d^2u}{dx^2} + \Phi(u) + g(x,t), \quad (x,t) \in [a,b] \times (0,T)$$

with the initial condition $u(x,0) = f(x)$ where Φ is a function of u.

This problem was used by Hopkins and Wait to provide an example of a problem with a nonlinear source term:

$$\frac{du}{dt} = \frac{d^2u}{dx^2} + e^{-u} + e^{-2u}, \quad (x,t) \in [a,b] \times (0,T) \tag{51}$$

with the initial condition $u(x,0) = \ln(x+2)$.
In this example we have

$$\Phi(u) = e^{-u} + e^{-2u}, \quad g(x,t) = 0, \quad f(x) = \ln(x+2)$$

We construct the following homotopy:

$$\frac{du}{dt} - \frac{du_0}{dt} = p(\frac{d^2u}{dx^2} + e^{-u} + e^{-2u} - \frac{du_0}{dt}) \tag{52}$$

Assume the solution of equation (52) to be in the form

$$u = u_0 + pu_1 + p^2u_2 + p^3u_3 + \cdots \tag{53}$$

Substituting (53) into (52) and equating the coefficients of like powers of p, we get the following set of differential equations

$$p^0: \quad \frac{du}{dt} - \frac{du_0}{dt} = 0$$

$$p^1: \quad \frac{du_1}{dt} = \frac{d^2u_0}{dx^2} + e^{-u_0} + e^{-2u_0} - \frac{du_0}{dt}$$

$$p^2: \quad \frac{du_2}{dt} = \frac{d^2u_1}{dx^2} + u_1(-e^{-u_0} - 2e^{-2u_0})$$

$$p^3: \quad \frac{d_3}{dt} = \frac{d^2u_2}{dt} = +(-u_2 + \frac{1}{2}u_1^2)e^{-u_0} + (-2u_2 + 2u_1^2 - \frac{1}{48}u_2u_1^2)e^{-2u_0}$$

Solving the above equations, we obtain

$$u_0 = \ln(x+2)$$

$$u_1 = \frac{t}{x+2}$$

$$u_2 = \frac{-t^2}{2(x+2)^2}$$
$$u_3 = \frac{t^3}{3(x+2)^3}$$
$$\vdots$$
$$u_n = \frac{(-1)^{n+1}t^n}{n(x+2)^n}$$

therefore from the results we can obtain

$$u(x,t) = \ln(x+2) + \frac{t}{x+2} - \frac{t^2}{2(x+2)^2} + \frac{t^3}{3(x+2)^3} + \cdots + \frac{(-1)^{n+1}t^n}{n(x+2)^n}$$
$$= \ln(x+2) + \ln\left(\frac{t}{x+2} + 1\right)$$
$$= \ln(x+t+2)$$

Which is the exact solution of the problem.

(d) Consider the following system of differential equations, with initial values $y_1(0) = 1$, $y_2(0) = 0$ and $y_3(0) = 2$

$$\begin{aligned} y_1' &= y_3 - \cos(x), \\ y_2' &= y_3 - e^x, \\ y_3' &= y_1 - y_2. \end{aligned} \tag{54}$$

Using homotopy perturbation method, from the solution equation;

$$v = v_0 + pv_1 + p^2 v_2 + \cdots$$

we will obtain

$$\begin{aligned} v_1 &= v_{1,0} + pv_{1,1} p^2 v_{1,2} + \cdots, \\ v_2 &= v_{2,0} + pv_{2,1} p^2 v_{2,2} + \cdots, \\ v_3 &= v_{3,0} + pv_{3,1} p^2 v_{3,2} + \cdots. \end{aligned} \tag{55}$$

where $v_{i,j}$; $\quad i, j = 1, 2, 3, \cdots$ are functions to be determined. Setting $p = 1$ results in the approximate solutions as;

$$\begin{aligned} y_1(x) &= \lim_{p\to 1} v_1(x) &= \textstyle\sum_{k=0}^{k=2} v_{1,k}(x), \\ y_2(x) &= \lim_{p\to 1} v_2(x) &= \textstyle\sum_{k=0}^{k=2} v_{2,k}(x), \\ y_3(x) &= \lim_{p\to 1} v_3(x) &= \textstyle\sum_{k=0}^{k=2} v_{3,k}(x). \end{aligned} \tag{56}$$

According to the homotopy perturbation method, we can construct a homotopy ay system (54) as follows;

$$\begin{aligned}
(1-p)(\dot{v}_1 - v_3 - \dot{u}_{1,0}) + p(\dot{v}_1 - v_3 + \cos(x)) &= 0,\\
(1-p)(\dot{v}_2 - v_3 - \dot{u}_{2,0}) + p(\dot{v}_2 - v_3 + e^x) &= 0,\\
(1-p)(\dot{v}_3 - \dot{u}_{3,0}) + p(\dot{v}_3 - v_1 + v_2) &= 0
\end{aligned} \tag{57}$$

where dot denotes differentiation with respect to x and the initial approximation are as follows:

$$\begin{aligned}
v_{1,0}(0) &= y_1(0) = 1,\\
v_{2,0}(0) &= y_2(0) = 0,\\
v_{3,0}(0) &= y_3(0) = 2.
\end{aligned} \tag{58}$$

Substituting equation (55) and (58) into equation (57) and rearraging based on powers of p terms, we have:

$$\begin{aligned}
(\dot{v}_{1,0} - v_{3,0}) + (\dot{v}_{1,1} + \cos(x) - v_{3,1})p + (\dot{v}_{1,2} - v_{3,2})p^2 + \cdots &= 0\\
(\dot{v}_{2,0} - v_{3,0}) + (\dot{v}_{2,1} + e^x - v_{3,1})p + (\dot{v}_{2,2} - v_{3,2})p^2 + \cdots &= 0\\
(\dot{v}_{3,0}) + (v_{3,1} + v_{2,0} - v_{1,0})p + (v_{3,2} + v_{2,1} - v_{1,1})p^2 + \cdots &= 0
\end{aligned} \tag{59}$$

In order to obtain the unknowns $v_{i,j}; i,j = 1,2,3,\cdots$ we must construct and solve the following system which includes nine equations with nine unknowns:

$$\begin{aligned}
\dot{v}_{1,0} - v_{3,0} &= 0\\
\dot{v}_{1,1} + \cos(x) - v_{3,1} &= 0\\
\dot{v}_{1,2} - v_{3,2} &= 0\\
\dot{v}_{2,0} - v_{3,0} &= 0\\
\dot{v}_{2,1} + e^x - v_{3,1} &= 0\\
\dot{v}_{2,2} - v_{3,2} &= 0\\
\dot{v}_{3,0} &= 0\\
v_{3,1} + v_{2,0} - v_{1,0} &= 0\\
v_{3,2} + v_{2,1} - v_{1,1} &= 0
\end{aligned} \tag{60}$$

Therefore,

$$\begin{aligned}
v_{1,0}(x) &= 2x+1 \\
v_{1,1}(x) &= -\sin(x)+\tfrac{1}{2}x^2 \\
v_{1,2}(x) &= \sin(x)+e^x-\tfrac{1}{2}x^2-2x-1 \\
v_{2,0}(x) &= 2x \\
v_{2,1}(x) &= -e^x+\tfrac{1}{2}x^2+1 \\
v_{2,2}(x) &= \sin(x)+e^x-\tfrac{1}{2}x^2-2x-1 \\
v_{3,0}(x) &= 2 \\
v_{3,1}(x) &= x \\
v_{3,2}(x) &= \cos(x)+e^x-x-2
\end{aligned} \tag{61}$$

Therefore from equation (56)

$$y_1(x) = e^x \tag{62}$$

$$y_2(x) = \sin(x) \tag{63}$$

$$y_3(x) = \cos(x)+e^x \tag{64}$$

which is the exact solution of the system (54).

(e) In this example we solve the following nonlinear system of differential equations with exact solutions $y_1 = e^{2x}$, $y_2 = e^x$ and $y_3 = xe^x$.

$$\begin{aligned}
y_1' &= 2y_2^2 \\
y_2' &= e^x y_1 \\
y_3' &= y_2+y_3
\end{aligned} \tag{65}$$

Using our method from the solution equation (series) we obtain:

$$\begin{aligned}
v_1 &= v_{1,0}+pv_{1,1}+p^2v_{1,2}+p^3v_{1,3}+\cdots, \\
v_2 &= v_{2,0}+pv_{2,1}+p^2v_{2,2}+p^3v_{2,3}+\cdots, \\
v_3 &= v_{3,0}+pv_{3,1}+p^2v_{3,2}+p^3v_{3,3}+\cdots
\end{aligned} \tag{66}$$

where $v_{i,j}$; $\quad i,j = 1,2,3,\cdots$ are functions to be determined. Setting $p = 1$ results in the approximate solutions as;

$$\begin{aligned} y_1(x) &= \lim_{p\to 1} v_1(x) &= \textstyle\sum_{k=0}^{k=3} v_{1,k}(x), \\ y_2(x) &= \lim_{p\to 1} v_2(x) &= \textstyle\sum_{k=0}^{k=3} v_{2,k}(x), \\ y_3(x) &= \lim_{p\to 1} v_3(x) &= \textstyle\sum_{k=0}^{k=3} v_{3,k}(x). \end{aligned} \tag{67}$$

According to the HPM, we can construct a homotopy ay system (65) as follows;

$$\begin{aligned} (1-p)(\dot{v}_1 - \dot{u}_{1,0}) + p(\dot{v}_1 - v_2^2) &= 0, \\ (1-p)(\dot{v}_2 - \dot{u}_{2,0}) + p(\dot{v}_2 - e^x v_1) &= 0, \\ (1-p)(\dot{v}_3 - \dot{u}_{3,0}) + p(\dot{v}_3 - v_2 - v_3) &= 0. \end{aligned} \tag{68}$$

where dot denotes differentiation with respect to x and the initial approximation are as follows:

$$\begin{aligned} v_{1,0}(0) &= y_1(0) &= 1, \\ v_{2,0}(0) &= y_2(0) &= 1, \\ v_{3,0}(0) &= y_3(0) &= 0. \end{aligned} \tag{69}$$

Substituting equation (66) and (69) into equation (68) and rearranging based on powers of p terms, we have:

$$\begin{aligned} (\dot{v}_{1,0}) + (\dot{v}_{1,1} - 2v_{2,0}^2)p + (\dot{v}_{1,2} - 4v_{2,0}v_{2,1})p^2 + (\dot{v}_{1,3} - 4v_{2,0}v_{2,2} - 2v_{2,1}^2)p^3 + \cdots &= 0 \\ (\dot{v}_{2,0}) + (\dot{v}_{2,1} - v_{1,1}e^{-x})p + (\dot{v}_{2,2} - v_{1,2}e^{-x})p^2 + (\dot{v}_{2,3} - v_{1,3}e^{-x})p^3 \cdots &= 0 \\ (\dot{v}_{3,0}) + (\dot{v}_{3,1} + v_{2,0} - v_{3,0})p + (\dot{v}_{3,2} - v_{2,1} - v_{3,1})p^2 + (\dot{v}_{3,3} - v_{2,2} - v_{3,2})p^3 + \cdots &= 0 \end{aligned} \tag{70}$$

In order to obtain the unknowns $v_{i,j}; i,j = 1,2,3,\cdots$ we must construct and solve the following system which includes twelve equations with twelve unknowns:

$$\begin{aligned}
\dot{v}_{1,0} &= 0\\
\dot{v}_{1,1} - 2v_{2,0}^2 &= 0\\
\dot{v}_{1,2} - 4v_{2,0}v_{2,1} &= 0\\
\dot{v}_{2,0} - v_{3,0} &= 0\\
\dot{v}_{1,3} - 4v_{2,0}v_{2,2} - 2v_{2,1}^2 &= 0\\
\dot{v}_{2,0} &= 0\\
\dot{v}_{2,1} - v_{1,1}e^{-x} &= 0\\
\dot{v}_{2,2} - v_{1,2}e^{-x} &= 0\\
\dot{v}_{2,3} - v_{1,3}e^{-x} &= 0\\
\dot{v}_{3,0} &= 0\\
\dot{v}_{3,1} + v_{2,0} - v_{3,0} &= 0\\
\dot{v}_{3,2} - v_{2,1} - v_{3,1} &= 0\\
\dot{v}_{3,3} - v_{2,2} - v_{3,2} &= 0
\end{aligned} \tag{71}$$

Therefore,

$$
\begin{aligned}
v_{1,0}(x) &= 1\\
v_{1,1}(x) &= -e^{-2}+8e^{-x}+8x-7\\
v_{1,2}(x) &= \tfrac{1}{3}e^{-4x}-\tfrac{56}{9}e^{-3x}-16xe^{-2x}+6e^{-2x}+\tfrac{272}{3}e^{-x}+\\
&\quad 64xe^{-x}+\tfrac{112}{3}x-\tfrac{817}{9}\\
v_{1,3}(x) &= -9e^{-4x}+\tfrac{1448}{15}x+\tfrac{328}{5}e^{-x}-\tfrac{896}{9}xe^{-3x}-\tfrac{1096}{9}e^{-3x}-\\
&\quad \tfrac{11}{135}e^{-6x}+\tfrac{544}{225}e^{-5x}-64e^{-2x}x^2-\tfrac{80}{3}e^{-2x}x+\tfrac{3377}{9}e^{-2x}+\\
&\quad 8xe^{-4x}+448e^{x}-\tfrac{210857}{675},\\
v_{2,0}(x) &= -e^{-x}+2\\
v_{2,1}(x) &= \tfrac{1}{3}e^{-3x}-4e^{-2x}-8xe^{-x}-e^{-x}+\tfrac{14}{3},\\
v_{2,2}(x) &= -\tfrac{1}{15}e^{-5x}+\tfrac{14}{9}e^{-4x}+\tfrac{16}{3}xe^{-3x}-\tfrac{2}{9}e^{-3x}-\tfrac{184}{3}e^{-2x}-32xe^{-2x}-\tfrac{112}{3}xe^{-x}+\\
&\quad \tfrac{481}{9}e^{-x}+\tfrac{298}{45},\\
v_{2,3}(x) &= \tfrac{37}{25}e^{-5x}-\tfrac{1448}{15}xe^{-x}+\tfrac{145697}{675}e^{-x}-\tfrac{724}{5}e^{-2x}-\\
&\quad \tfrac{224}{9}xe^{4x}+\tfrac{110}{3}e^{-4x}-\tfrac{11}{945}e^{-7x}-\tfrac{272}{675}e^{6x}+\tfrac{64}{3}x^2e^{-3x}+\\
&\quad \tfrac{208}{9}xe^{-3x}-\tfrac{3169}{27}e^{-3x}-\tfrac{8}{5}xe^{-5x}-224xe^{-2x}+\tfrac{4498}{525},\\
v_{3,0}(x) &= 0\\
v_{3,1}(x) &= e^{-x}+2x-1\\
v_{3,2}(x) &= -\tfrac{1}{9}e^{-3x}+2e^{-2x}+8xe^{-x}+8e^{-x}+x^2+\tfrac{11}{3}x-\tfrac{89}{9},\\
v_{3,3}(x) &= \tfrac{1}{75}e^{-5x}-\tfrac{7}{18}e^{-4x}-\tfrac{16}{9}xe^{-3x}-\tfrac{13}{27}e^{-3x}+\tfrac{113}{3}e^{-2x}+16xe^{-2x}+\tfrac{88}{3}xe^{-x}-\\
&\quad \tfrac{289}{9}e^{-x}+\tfrac{1}{3}x^3+\tfrac{11}{6}x^2-\tfrac{49}{15}x-\tfrac{6343}{1350}
\end{aligned}
\tag{72}
$$

Therefore from equation (67)

$$y_1(x)=-3+6x+4e^{x} \tag{73}$$

$$y_2(x)=4-e^{-x}-2(1+x)e^{-x} \tag{74}$$

$$y_3(x)=-1+2x+e^{-x}+\frac{1}{2}x^2 \tag{75}$$

2. **Solving partial differential equations**

(a) Consider the inhomogeneous heat problem

$$
\begin{aligned}
u_t &= u_{xx}+\sin(x)\\
u(x,0) &= \cos(x)
\end{aligned}
\tag{73}
$$

A corresponding to the homotopy perturbation method, we will have

$$\frac{\partial u}{\partial t}-\frac{\partial u_0}{\partial t}=p[\frac{\partial^2 u}{\partial x^2}+\sin(x)-\frac{\partial u_0}{\partial t}] \tag{74}$$

We suppose that the solution of the problem (73) is in the form:

$$u=u_0+pu_1+p^2u_2+p^3u_3+\cdots \tag{75}$$

We substitute (75) into (74) and we equate the coefficients of like power p, we will have the set of differential equations:

$$\begin{array}{lcccl} p^0 & : & \frac{\partial u_0}{\partial t}-\frac{\partial u_0}{\partial t} & = & 0 \\ p^1 & : & \frac{\partial u_1}{\partial t} & = & [\frac{\partial^2 u_0}{\partial x^2}+\sin(x)-\frac{\partial u_0}{\partial t}] \\ p^2 & : & \frac{\partial u_2}{\partial t} & = & \frac{\partial^2 u_1}{\partial x^2} \\ p^3 & : & \frac{\partial u_3}{\partial t} & = & \frac{\partial^2 u_2}{\partial x^2} \\ p^4 & : & \frac{\partial u_4}{\partial t} & = & \frac{\partial^2 u_3}{\partial x^2} \end{array} \tag{76}$$

Solve equations (76) to get the solution

$$\begin{array}{lcl} u_1 & = & (\sin(x)-\cos(x))t \\ u_2 & = & (\sin(x)-\cos(x))\frac{t^2}{2!} \\ u_3 & = & (\sin(x)-\cos(x))\frac{t^3}{4!} \\ u_4 & = & (\sin(x)-\cos(x))\frac{t^4}{4!} \end{array} \tag{77}$$

So the solution will be

$$\begin{array}{lcl} u(x,t) & = & u_0+u_1+u_2+u_3+u_4+\cdots \\ & = & \cos(x)+(\sin(x)-\cos(x))t+(\sin(x)-\cos(x))\frac{t^2}{2!} \\ & & +(\sin(x)-\cos(x))\frac{t^3}{4!}+(\sin(x)-\cos(x))\frac{t^4}{4!}+\cdots \\ & = & \cos(x)(1-t+\frac{t^2}{2!}-\frac{t^3}{3!}+\frac{t^4}{4!}+\cdots) \\ & & +\sin(x)(1-t+\frac{t^2}{2!}-\frac{t^3}{3!}+\frac{t^4}{4!}+\cdots) \\ & = & \cos(x)e^{-t}+\sin(x)(1-e^{-t}) \end{array} \tag{78}$$

(b) Consider the vibrating beam problem of the fourth order:

$$\begin{array}{lcl} u_{tt} & = & -u_{xxxx} \\ u(x,0) & = & \sin(\pi x)+0.5\sin(3\pi x) \end{array} \tag{79}$$

A corresponding to the homotopy perturbation method, we will have

$$\frac{\partial^2 u}{\partial t^2}-\frac{\partial^2 u_0}{\partial t^2}=p[-\frac{\partial^4 u}{\partial x^4}-\frac{\partial^2 u_0}{\partial t^2}] \quad (80)$$

We suppose that the solution of the problem (79) is in the form:

$$u=u_0+pu_1+p^2u_2+p^3u_3+\cdots \quad (81)$$

We substitute (81) into (80) and we equate the coefficients of like power p, we will have the set of differential equations:

$$\begin{array}{rcccl} p^0 & : & \frac{\partial^0 u_0}{\partial t^0}-\frac{\partial^2 u_0}{\partial t^2} & = & 0 \\ p^1 & : & \frac{\partial^2 u_1}{\partial t^2} & = & -\frac{\partial^4 u_0}{\partial t^4}] \\ p^2 & : & \frac{\partial^2 u_2}{\partial t^2} & = & -\frac{\partial^4 u_1}{\partial x^4} \\ p^3 & : & \frac{\partial^2 u_3}{\partial t^2} & = & -\frac{\partial^4 u_2}{\partial x^4} \\ p^4 & : & \frac{\partial^2 u_4}{\partial t^2} & = & -\frac{\partial^4 u_3}{\partial x^4} \end{array} \quad (82)$$

Solve equations (82) to get the solution

$$\begin{array}{rcl} u_1 & = & -\pi^4(\sin(\pi x)+(3^4)(0.5)\sin(3\pi x))\frac{t^2}{2} \\ u_2 & = & \pi^8(\sin(\pi x)+(3^8)(0.5)\sin(3\pi x))\frac{t^4}{4!} \\ u_3 & = & -\pi^{12}(\sin(\pi x)+(3^{12})(0.5)\sin(3\pi x))\frac{t^6}{6!} \\ u_4 & = & \pi^{16}(\sin(\pi x)+(3^{16})(0.5)\sin(3\pi x))\frac{t^8}{8!} \end{array} \quad (83)$$

So the solution will be

$$\begin{array}{rcl} u(x,t) & = & u_0+u_1+u_2+u_3+u_4+\cdots \\ & = & (\sin(\pi x)+0.5\sin(3\pi x))-\pi^4(\sin(\pi x)+(3^4)(0.5)\sin(3\pi x))\frac{t^2}{2} \\ & & +\pi^8(\sin(\pi x)+(3^8)(0.5)\sin(3\pi x))\frac{t^4}{4!} \\ & & -\pi^{12}(\sin(\pi x)+(3^{12})(0.5)\sin(3\pi x))\frac{t^6}{6!} \\ & & +\pi^{16}(\sin(\pi x)+(3^{16})(0.5)\sin(3\pi x))\frac{t^8}{8!} \\ & = & \sin(\pi x)(1-\frac{(\pi^2 t)^2}{2!}+\frac{(\pi^2 t)^4}{4!}-\frac{(\pi^2 t)^6}{6!}+\frac{(\pi^2 t)^8}{8!}\cdots) \\ & & +(0.5)\sin(3\pi x)(1-\frac{(3^2\pi^2 t)^2}{2!}+\frac{(3^2\pi^2 t)^4}{4!}-\frac{(3^2\pi^2 t)^6}{6!}+\frac{(3^2\pi^2 t)^8}{8!}\cdots) \\ & = & \sin(\pi x)\cos\left(\pi^2 t\right)+(0.5)(\sin(3\pi)(\cos\left(9\pi^2 t\right)) \end{array} \quad (84)$$

(c) Consider the following partial differential equations in second order:

$$u_t \;=\; -[u_{xx}+\tfrac{2}{x}u_x+2e^{-t}\sin(x)] \quad (85)$$

Subject to the initial conditions

$$u(0,x) = \frac{\cos(x)}{x} + \sin(x) + \cdots \tag{86}$$

and a given solution $u(t,x) = e^{-t}[\frac{\cos(x)}{x} + \sin(x)]$. To solve equation (85) by homotopy perturbation method, we will have;

$$\frac{\partial u}{\partial t} - \frac{\partial u_0}{\partial t} = -p[\frac{\partial^2 u}{\partial x^2} + \frac{2}{x}\frac{\partial u}{\partial x} + 2e^{-t}\sin(x)] \tag{87}$$

Suppose that the solution of the problem (85) is in the form:

$$u = u_0 + pu_1 + p^2 u_2 + p^3 u_3 + \cdots \tag{88}$$

Substituting (88) into equation (85) and we equating the coefficients of like power p, we will have the set of differential equations:

$$\begin{array}{lcccl} p^0 & : & \frac{\partial u_0}{\partial t} - \frac{\partial u_0}{\partial t} & = & 0 \\ p^1 & : & \frac{\partial u_1}{\partial t} & = & -[\frac{\partial^2 u_0}{\partial x^2} + \frac{2}{x}\frac{\partial u}{\partial x} + 2e^{-t}\sin(x) + \frac{\partial u_0}{\partial t}] \\ p^2 & : & \frac{\partial u_2}{\partial t} & = & -[\frac{\partial^2 u_1}{\partial x^2} + \frac{2}{x}\frac{\partial u_1}{\partial x}] \\ p^3 & : & \frac{\partial u_3}{\partial t} & = & -[\frac{\partial^2 u_2}{\partial x^2} + \frac{2}{x}\frac{\partial u_2}{\partial x}] \\ \vdots & & & & \end{array} \tag{89}$$

Solve the system of equations (89) to get the solutions

$$\begin{array}{lcl} u_1 & = & -\frac{\cos x}{x}t + t\sin x + 2e^{-t}\sin x \\ u_2 & = & -3\frac{\cos x}{x}\frac{t^2}{2!} + \frac{t^2}{2!}\sin x - 2e^{-t}\sin x + 4\frac{\cos x}{x}e^{-t} \\ u_3 & = & -5\frac{\cos x}{x}\frac{t^3}{3!} + \frac{t^3}{3!}\sin x - 2e^{-t}\sin x + 8\frac{\cos x}{x}e^{-t} \\ u_4 & = & -7\frac{\cos x}{x}\frac{t^4}{4!} + \frac{t^4}{4!}\sin x - 2e^{-t}\sin x + 12\frac{\cos x}{x}e^{-t} \end{array} \tag{90}$$

Solution of equation (85) will be derived by adding these terms, so

$$\begin{array}{lcl} u(x,t) & = & u_0 + u_1 + u_2 + u_3 + \cdots \\ & = & \frac{\cos x}{x}[(1 - \frac{t}{1!} - 3\frac{t^2}{2!} - 5\frac{t^3}{3!} - 7\frac{t^4}{4!} - \cdots) \\ & & +(4 - 8 + 12 - 16 + \cdots)e^{-t}] \\ & & + \sin(x)[(1 + \frac{t}{1!} + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \cdots) \\ & & +2(1 - 1 + 1 - 1 + \cdots)e^{-t}] \\ & = & \frac{\cos(x)}{x}e^{-t} + \sin(x)e^{-t} \\ u(t,x) & = & e^{-t}[\frac{\cos(x)}{x} + \sin(x)] \end{array} \tag{91}$$

3 Homotopy Perturbation Method

3.1 History

The homotopy perturbation method was developed due to the limitations of the singular and regular perturbation methods. Firstly almost all perturbation methods are based on an assumption that a small parameter must exist in the equation. This so called small parameter assumption greatly restrict utilization of perturbation techniques. As well known, an overwhelming majority of nonlinear problems have no small parameter at all. Secondly, the determination of small parameters seems to be a special art requiring special technique. A suitable choice of small parameter leads to ideal results. However, an unsuitable choice of small parameter results in bad effects. Thirdly, even if there exist suitable parameters, the approximate solution obtained by the perturbations are valid in most cases only for the small values of the parameters. So it was necessary to develop a kind of new perturbation method which does not require small parameters at all.

Since there are some impediments with the common perturbation method, furthermore basis of the common perturbation method was upon the existence of a small parameter, developing the method for different applications is very difficult. Therefore, many different methods have recently introduced some ways to eliminate the small parameter, such as artificial parameter method, the homotopy perturbation method and the variational iteration method. Homotopy techniques are generally connected to discover all bases of non-linear algebraic equations. The homotopy technique also called the continuous mapping procedure, embeds a parameter p that typically ranges from zero to one. When the embedding parameter is zero, the equation is one of a direct framework, when it is one, the equation is the same as the original. So the embedded parameter $p[0, 1]$ can be considered as a small parameter.

The homotopy perturbation method was presented by Ji-Huan He[He, 1999] of Shanghai university in 1998 which is the coupling method of the homotopy techniques and the perturbation technique. The homotopy perturbation method is a special case of the homotopy analysis method developed by Liao Suijun in 1992. The homotopy analysis method uses a so-called convergence control parameter to guarantee the convergence of approximations series over a given interval of the physical parameters. The homotopy perturbation method is used for solving ordinary differential and partial differential equations both linear and non-linear. In this method, the solution is considered as the summation of an infinite series which usually converges rapidly to the solutions.

3.2 Analysis of homotopy perturbation method for solving second-order partial differential equation

The general second-order partial differential equation in $\mathbb{R}^n$ domain has the following form:

$$\sum_{i,j=1}^{n} a_{ij}(X_n)\frac{\partial^2 u(X_n)}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(X_n)\frac{\partial u(X_n)}{\partial x_i} + c(X_n)u(X_n) = g(X_n) \tag{92}$$

subject to the initial conditions

$$\begin{aligned} u(X_{n-1},0) &= f(X_{n-1}), \\ \frac{\partial u(X_{n-1},0)}{\partial x_n} &= h(X_{n-1}) \end{aligned}$$

such that

$$X_n = (x_1, x_2, x_3, \cdots, x_n), \quad X_{n-1} = (x_1, x_2, x_3, \cdots, x_{n-1}) \quad \text{and} \quad a_{ij}(X_n) = a_{ij}(X_n)$$

where $a_{ij}(X_n)$, $b_i(X_n)$, $c(X_n)$, $g(X_n)$ and $f(X_{n-1})$ are the given functions of n independent variables.

Now, we describe a general technique for solving second order partial differential equation in which the solution $u : \mathbb{R}^n \to \mathbb{R}$ is a function of n variables according to the following algorithms

3.2.1 The Proposed Method

Consider equation (92) and the following cases:

1. Case 1: $a_{n,n}(X_n) \neq 0$
   Firstly, we start with the initial approximation

   $$u(X_{n-1},0) = f(X_n) + xu(X_n)$$

   Secondly, we can construct a homotopy for differential equation (92) as follows

$$\begin{aligned} H(u,p) = (1-p)(\frac{\partial^2 u(X_n)}{\partial x_n^2} - \frac{\partial^2 u_0(X_n)}{\partial x_n^2}) + p(\sum_{i,j=1}^{n} a_{ij}(X_n)\frac{\partial^2 u(X_n)}{\partial x_i \partial x_j} \\ + \sum_{i=1}^{n} b_i(X_n)\frac{\partial u(X_n)}{\partial x_i} + c(X_n)u(X_n) - g(X_n)) = 0 \end{aligned} \tag{93}$$

Thirdly, suppose that the solution of the equation (92) is in the form

$$y(t) = y_0 + py_1 + p^2y_2 + p^3y_3 + \cdots \tag{94}$$

Therefore,

$$H(u,p) = (1-p)(\sum_{k=0}^{\infty} p^k \frac{\partial^2 u_k(X_n)}{\partial x_n^2} - \frac{\partial^2 u_0(X_n)}{\partial x_n^2}) + p(\sum_{k=0}^{\infty} \sum_{i,j=1}^{n} a_{ij} p^k(X_n) \frac{\partial^2 u_k(X_n)}{\partial x_i \partial x_j} + \sum_{k=0}^{\infty}\sum_{i=1}^{n} b_i p^k(X_n) \frac{\partial u_k(X_n)}{\partial x_i} + c(X_n) \sum_{k=0}^{\infty} p^k u_k(X_n) - g(X_n)) = 0 \tag{95}$$

Fourth, collecting terms of the same power of p gives, as shown in the following equations:

$$\begin{aligned}
p^0 &: \quad \tfrac{\partial^2 u_0(X_n)}{\partial x_n^2} - \tfrac{\partial^2 u_0(X_n)}{\partial x_n^2} &&= 0\\
p^1 &: \quad \tfrac{\partial^2 u_1(X_n)}{\partial x_n^2} + \sum_{i,j=1}^{n} a_{ij}(X_n)\tfrac{\partial^2 u_0(X_n)}{\partial x_i \partial x_j} \\
&\quad + \sum_{i=1}^{n} b_i(X_n)\tfrac{\partial u_0(X_n)}{\partial x_i} + c(X_n)u_0(X_n) - g(X_n) &&= 0\\
p^2 &: \quad \tfrac{\partial^2 u_2(X_n)}{\partial x_n^2} - \tfrac{\partial^2 u_1(X_n)}{\partial x_n^2} + \sum_{i,j=1}^{n} a_{ij}(X_n)\tfrac{\partial^2 u_1(X_n)}{\partial x_i \partial x_j} \\
&\quad + \sum_{i=1}^{n} b_i(X_n)\tfrac{\partial u_1(X_n)}{\partial x_i} + c(X_n)u_1(X_n) &&= 0\\
p^3 &: \quad \tfrac{\partial^2 u_3(X_n)}{\partial x_n^2} - \tfrac{\partial^2 u_2(X_n)}{\partial x_n^2} + \sum_{i,j=1}^{n} a_{ij}(X_n)\tfrac{\partial^2 u_2(X_n)}{\partial x_i \partial x_j} \\
&\quad + \sum_{i=1}^{n} b_i(X_n)\tfrac{\partial u_2(X_n)}{\partial x_i} + c(X_n)u_2(X_n) &&= 0\\
p^4 &: \quad \tfrac{\partial^2 u_4(X_n)}{\partial x_n^2} - \tfrac{\partial^2 u_3(X_n)}{\partial x_n^2} + \sum_{i,j=1}^{n} a_{ij}(X_n)\tfrac{\partial^2 u_3(X_n)}{\partial x_i \partial x_j} \\
&\quad + \sum_{i=1}^{n} b_i(X_n)\tfrac{\partial u_3(X_n)}{\partial x_i} + c(X_n)u_3(X_n) &&= 0\\
p^5 &: \quad \tfrac{\partial^2 u_5(X_n)}{\partial x_n^2} - \tfrac{\partial^2 u_4(X_n)}{\partial x_n^2} + \sum_{i,j=1}^{n} a_{ij}(X_n)\tfrac{\partial^2 u_4(X_n)}{\partial x_i \partial x_j} \\
&\quad + \sum_{i=1}^{n} b_i(X_n)\tfrac{\partial u_4(X_n)}{\partial x_i} + c(X_n)u_4(X_n) &&= 0\\
&\vdots
\end{aligned} \tag{96}$$

Hence, for $n = 2,3,4,5,\cdots$ we have

$$\begin{aligned}
p^m &: \quad \tfrac{\partial^2 u_m(X_n)}{\partial x_n^2} - \tfrac{\partial^2 u_{m-1}(X_n)}{\partial x_n^2} + \sum_{i,j=1}^{n} a_{ij}(X_n)\tfrac{\partial^2 u_{m-1}(X_n)}{\partial x_i \partial x_j} \\
&\quad + \sum_{i=1}^{n} b_i(X_n)\tfrac{\partial u_{m-1}(X_n)}{\partial x_i} + c(X_n)u_{m-1}(X_n) &&= 0
\end{aligned} \tag{97}$$

Finally, using the equation (97) with some simplification, then we get the following sequence of solutions

$$
\begin{aligned}
u_0(X_n) &= f(X_n)+xh(X_n)\\
u_1(X_n) &= -\iint_{x_n}[\tfrac{\partial^2 u_1(S_n)}{\partial s_n^2}+\sum_{i,j=1}^{n} a_{i,j}(s_n)\tfrac{\partial^2 u_0(X_n)}{\partial x_i \partial x_j}+\sum_{i=1}^{n} b_i(X_n)\tfrac{\partial u_0(X_n)}{\partial x_i}\\
&\quad +c(S_n)u_o(S_n)-g(X_n)]\,ds_n\,dx_n\\
u_2(X_n) &= -\iint_{x_n}[\tfrac{\partial^2 u_2(S_n)}{\partial s_n^2}-\tfrac{\partial^2 u_1(S_n)}{\partial s_n^2}+\sum_{i,j=1}^{n} a_{i,j}(S_n)\tfrac{\partial^2 u_1(S_n)}{\partial s_i \partial s_j}+\\
&\quad \sum_{i=1}^{n} b_i(s_n)\tfrac{\partial u_1(s_n)}{\partial s_i}+c(S_n)u_1(S_n)]\,ds_n\,dx_n\\
u_3(X_n) &= -\iint_{x_n}[\tfrac{\partial^2 u_3(S_n)}{\partial s_n^2}-\tfrac{\partial^2 u_2(S_n)}{\partial s_n^2}+\sum_{i,j=1}^{n} a_{i,j}(S_n)\tfrac{\partial^2 u_2(S_n)}{\partial s_i \partial s_j}+\\
&\quad \sum_{i=1}^{n} b_i(S_n)\tfrac{\partial u_2(S_n)}{\partial s_i}+c(S_n)u_2(S_n)]\,ds_n\,dx_n\\
u_4(X_n) &= -\iint_{x_n}[\tfrac{\partial^2 u_4(S_n)}{\partial s_n^2}-\tfrac{\partial^2 u_3(S_n)}{\partial s_n^2}+\sum_{i,j=1}^{n} a_{i,j}(S_n)\tfrac{\partial^2 u_3(S_n)}{\partial s_i \partial s_j}+\\
&\quad \sum_{i=1}^{n} b_i(S_n)\tfrac{\partial u_3(S_n)}{\partial s_i}+c(S_n)u_3(S_n)]\,ds_n\,dx_n\\
u_5(X_n) &= -\iint_{x_n}[\tfrac{\partial^2 u_5(S_n)}{\partial s_n^2}-\tfrac{\partial^2 u_4(S_n)}{\partial s_n^2}+\sum_{i,j=1}^{n} a_{i,j}(S_n)\tfrac{\partial^2 u_4(S_n)}{\partial s_i \partial s_j}+\\
&\quad \sum_{i=1}^{n} b_i(S_n)\tfrac{\partial u_4(S_n)}{\partial s_i}+c(S_n)u_4(S_n)]\,ds_n\,dx_n\\
&\vdots
\end{aligned} \tag{98}
$$

Hence the general term has the following form

$$
\begin{aligned}
u_m(X_n) &= -\iint_{x_n}[\tfrac{\partial^2 u_m(S_n)}{\partial s_n^2}-\tfrac{\partial^2 u_{m-1}(S_n)}{\partial s_n^2}+\sum_{i,j=1}^{n} a_{i,j}(S_n)\tfrac{\partial^2 u_{m-1}(S_n)}{\partial s_i \partial s_j}+\\
&\quad \sum_{i=1}^{n} b_i(S_n)\tfrac{\partial u_{m-1}(S_n)}{\partial s_i}+c(S_n)u_{m-1}(S_n)]\,ds_n\,dx_n
\end{aligned} \tag{99}
$$

$n=2,3,4,\cdots$ where $S_n=(s_1,s_2,\cdots,s_n)$.
Then the solution of the equation (92) is

$$u(X_n)=u_0(X_n)+u_1(X_n)+u_2(X_n)+u_3(X_n)+u_4(X_n)+u_5(X_n)+\cdots \tag{100}$$

2. Case 2: $a_{n,n}(X_n)=0$ and $b_n(X_n)\neq 0$
Firstly, we start with the initial approximation $u(X_{n-1},0)=f(X_{n-1})$, we start with the initial approximation $u(X_{n-1},0)=f(X_n)$.
Secondly, we can construct a homotopy for the partial differential equation (92) as

follows

$$H(u,p)=(1-p)(\frac{\partial^2 u(X_n)}{\partial x_n^2}-\frac{\partial^2 u_0(X_n)}{\partial x_n^2})+p(\sum_{i,j=1}^{n} a_{ij}(X_n)\frac{\partial^2 u(X_n)}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(X_n)\frac{\partial u(X_n)}{\partial X_i}+c(X_n)u(X_n)-g(X_n))=0 \quad (101)$$

Thirdly, suppose that the solution of the equation (92) is in the form

$$y(t)=y_0+py_1+p^2y_2+p^3y_3+\cdots \quad (102)$$

Therefore

$$H(u,p)=(1-p)(\sum_{k=0}^{\infty} p^k\frac{\partial^2 u_k(X_n)}{\partial x_n^2}-\frac{\partial^2 u_0(X_n)}{\partial x_n^2})+p(\sum_{k=0}^{\infty}\sum_{i,j=1}^{n} p^k a_{ij}(X_n)\frac{\partial^2 u_k(X_n)}{\partial x_i \partial x_j} + \sum_{k=0}^{\infty}\sum_{i=1}^{n} p^k b_i(X_n)\frac{\partial u_k(X_n)}{\partial X_i}+c(X_n)\sum_{k=0}^{\infty} p^k u_k(X_n)-g(X_n))=0 \quad (103)$$

Fourth, collecting terms of the same power of p we obtain the following equations:

$$\begin{aligned}
p^0 &: & \frac{\partial u_0(X_n)}{\partial x_n}-\frac{\partial u_0(X_n)}{\partial x_n} &= 0\\
p^1 &: & \frac{\partial u_1(X_n)}{\partial x_n}+\sum_{i,j=1}^{n} a_{ij}(X_n)\frac{\partial^2 u_0(X_n)}{\partial x_i \partial x_j}&\\
& & +\sum_{i=1}^{n-1} b_i(X_n)\frac{\partial u_0(X_n)}{\partial x_i}+c(X_n)u_0(X_n)-g(X_n) &= 0\\
p^2 &: & \frac{\partial u_2(X_n)}{\partial x_n}+\sum_{i,j=1}^{n} a_{ij}(X_n)\frac{\partial^2 u_1(X_n)}{\partial x_i \partial x_j}&\\
& & +\sum_{i=1}^{n-1} b_i(X_n)\frac{\partial u_1(X_n)}{\partial x_i}+c(X_n)u_1(X_n) &= 0\\
p^3 &: & \frac{\partial u_3(X_n)}{\partial x_n}+\sum_{i,j=1}^{n} a_{ij}(X_n)\frac{\partial^2 u_2(X_n)}{\partial x_i \partial x_j}&\\
& & +\sum_{i=1}^{n-1} b_i(X_n)\frac{\partial u_2(X_n)}{\partial x_i}+c(X_n)u_2(X_n) &= 0\\
p^4 &: & \frac{\partial u_4(X_n)}{\partial x_n}+\sum_{i,j=1}^{n} a_{ij}(X_n)\frac{\partial^2 u_3(X_n)}{\partial x_i \partial x_j}&\\
& & +\sum_{i=1}^{n-1} b_i(X_n)\frac{\partial u_3(X_n)}{\partial x_i}+c(X_n)u_3(X_n) &= 0\\
p^5 &: & \frac{\partial u_5(X_n)}{\partial x_n}+\sum_{i,j=1}^{n} a_{ij}(X_n)\frac{\partial^2 u_4(X_n)}{\partial x_i \partial x_j}&\\
& & +\sum_{i=1}^{n-1} b_i(X_n)\frac{\partial u_4(X_n)}{\partial x_i}+c(X_n)u_4(X_n) &= 0\\
&\vdots
\end{aligned} \quad (104)$$

Hence, for $n=2,3,4,5,\cdots$ we have

$$\begin{aligned} p^m \quad : \qquad & \frac{\partial u_m(X_n)}{\partial x_n} + \sum_{i,j=1}^{n} a_{ij}(X_n)\frac{\partial^2 u_{m-1}(X_n)}{\partial x_i \partial x_j} \\ & + \sum_{i=1}^{n-1} b_i(X_n)\frac{\partial u_{m-1}(X_n)}{\partial x_i} + c(X_n)u_{m-1}(X_n) \quad = \quad 0 \end{aligned} \tag{105}$$

Finally, using the equation (105) with some simplification, then we get the following sequence of solutions

$$\begin{aligned} u_0(X_n) &= f(X_n) \\ u_1(X_n) &= -\int_{x_n}[\sum_{i,j=1}^{n} a_{i,j}(X_n)\frac{\partial^2 u_0(X_n)}{\partial x_i \partial x_j} + \sum_{i=1}^{n-1} b_i(X_n)\frac{\partial u_0(X_n)}{\partial x_i} \\ &\quad + c(X_n)u_o(X_n) - g(X_n)]\, dx_n \\ u_2(X_n) &= -\int_{x_n}[\sum_{i,j=1}^{n} a_{i,j}(X_n)\frac{\partial^2 u_1(X_n)}{\partial x_i \partial x_j} + \\ &\quad \sum_{i=1}^{n-1} b_i(X_n)\frac{\partial u_1(X_n)}{\partial x_i} + c(X_n)u_1(X_n)]\, dx_n \\ u_3(X_n) &= -\int_{x_n}[\sum_{i,j=1}^{n} a_{i,j}(X_n)\frac{\partial^2 u_2(X_n)}{\partial x_i \partial x_j} + \\ &\quad \sum_{i=1}^{n-1} b_i(X_n)\frac{\partial u_2(X_n)}{\partial x_i} + c(X_n)u_2(X_n)]\, dx_n \\ u_4(X_n) &= -\int_{x_n}[\sum_{i,j=1}^{n} a_{i,j}(X_n)\frac{\partial^2 u_3(X_n)}{\partial x_i \partial x_j} + \\ &\quad \sum_{i=1}^{n-1} b_i(X_n)\frac{\partial u_3(X_n)}{\partial x_i} + c(X_n)u_3(X_n)]\, dx_n \\ u_5(X_n) &= -\int_{x_n}[\sum_{i,j=1}^{n} a_{i,j}(X_n)\frac{\partial^2 u_4(X_n)}{\partial x_i \partial x_j} + \\ &\quad \sum_{i=1}^{n-1} b_i(X_n)\frac{\partial u_4(X_n)}{\partial x_i} + c(X_n)u_4(X_n)]\, dx_n \\ &\vdots \end{aligned} \tag{106}$$

Hence the general term has the following form

$$\begin{aligned} u_m(X_n) &= -\int_{x_n}[\sum_{i,j=1}^{n} a_{i,j}(X_n)\frac{\partial^2 u_{m-1}(X_n)}{\partial x_i \partial x_j} + \\ &\quad \sum_{i=1}^{n-1} b_i(X_n)\frac{\partial u_{m-1}(X_n)}{\partial x_i} + c(X_n)u_{m-1}(X_n)]\, dx_n \end{aligned} \tag{107}$$

$n=2,3,4,\cdots$.
Then the solution of the equation (92) is

$$u(X_n) = u_0(X_n) + u_1(X_n) + u_2(X_n) + u_3(X_n) + u_4(X_n) + u_5(X_n) + \cdots \tag{108}$$

3.2.2 Implementations

In order to assess the accuracy of solving second order partial differential equation using homotopy perturbation method of some problems, we solve some example as follows,
Problem 1.
Consider the following partial differential equation:

$$u_{xx}(x,y)+u_{yy}(x,y)=0, \qquad x,y\in\mathbb{R} \tag{109}$$

Subject to the initial conditions

$$\begin{aligned} u(x,0)= &= 0 \\ u_y(x,0) &= e^{-x} \end{aligned} \tag{110}$$

Comparing equation (109) with general equation (92) we have

$$n=2, \quad a_{1,1}=1, \quad a_{2,2}=1, \quad b_1=b_2=0, \quad g=0 \quad \text{and} \quad c=0$$

The initial approximation has the form $u_0(x,y)=ye^{-x}$, substituting (97) into (109) we have

$$\begin{aligned} u_1(x,y) &= -\iint_y\left(\frac{\partial^2 u_0(x,s)}{\partial x^2}+\frac{\partial^2 u_0(x,s)}{\partial s^2}\right)ds\,dy &= -\frac{y^3}{3!}e^{-x} \\ u_2(x,y) &= -\iint_y\left(\left(\frac{\partial^2 u_1(x,s)}{\partial x^2}\right)ds\,dy\right. &= \frac{y^5}{5!}e^{-x} \\ u_3(x,y) &= -\iint_y\left(\left(\frac{\partial^2 u_2(x,s)}{\partial x^2}\right)ds\,dy\right. &= -\frac{y^7}{7!}e^{-x} \\ u_4(x,y) &= -\iint_y\left(\left(\frac{\partial^2 u_3(x,s)}{\partial x^2}\right)ds\,dy\right. &= \frac{y^9}{9!}e^{-x} \\ u_5(x,y) &= -\iint_y\left(\left(\frac{\partial^2 u_4(x,s)}{\partial x^2}\right)ds\,dy\right. &= -\frac{y^{11}}{11!}e^{-x} \end{aligned} \tag{111}$$

and

$$u_m(x,y) = -\iint_y\left(\left(\frac{\partial^2 u_{m-1}(x,s)}{\partial x^2}\right)ds\,dy\right. = (-1)^m\frac{y^{2m+1}}{(2m+1)!}e^{-x} \tag{112}$$

Then the general solution of equation (109) is written as follows

$$\begin{aligned} u(x,y) &= u_0(x,y)+u_1(x,y)+u_2(x,y)+u_3(x,y)+u_4(x,y)+u_5(x,y)+\cdots \\ &= \sin(y)e^{-x} \end{aligned}$$

Problem 2.
Consider the following Helmholtz equation:

$$u_{xx}(x,y)+u_{yy}(x,y)+8u(x,y)=0, \quad x,y\in\mathbb{R} \tag{113}$$

Subject to the initial conditions

$$\begin{aligned} u(0,y) &= \sin(2y) \\ u_y(0,y) &= 0 \end{aligned} \tag{114}$$

Comparing equation (113) with the general equation (92)we have

$$n=2, \quad a_{1,1}=1, \quad a_{2,2}=1, \quad g=0 \quad \text{and} \quad c=8$$

The initial approximation has the form $u_0(x,y)=\sin(2y)$, substituting (97) into (113) we have

$$\begin{aligned}
u_1(x,y) &= -\iint_y\left(\frac{\partial^2 u_0(x,s)}{\partial x^2}+\frac{\partial^2 u_0(x,s)}{\partial s^2}+8u_0(x,s)\right)ds\,dy &&= -\frac{(2x^2)}{2!}\sin(2y) \\
u_2(x,y) &= -\iint_y\left(\left(\frac{\partial^2 u_1(x,s)}{\partial x^2}+8u_1(x,s)\right)\right)ds\,dy &&= \frac{(2x)^4}{4!}\sin(2y) \\
u_3(x,y) &= -\iint_y\left(\left(\frac{\partial^2 u_2(x,s)}{\partial x^2}+8u_2(x,s)\right)\right)ds\,dy &&= -\frac{(2x)^6}{6!}\sin(2y) \\
u_4(x,y) &= -\iint_y\left(\left(\frac{\partial^2 u_3(x,s)}{\partial x^2}+8u_3(x,s)\right)\right)ds\,dy &&= \frac{(2x)^8}{8!}\sin(2y) \\
u_5(x,y) &= -\iint_y\left(\left(\frac{\partial^2 u_4(x,s)}{\partial x^2}+8u_4(x,s)\right)\right)ds\,dy &&= -\frac{(2x)^{10}}{10!}\sin(2y)
\end{aligned} \tag{115}$$

and

$$u_m(x,y) = -\iint_y\left(\left(\frac{\partial^2 u_{m-1}(x,s)}{\partial x^2}+8u_{m-1}(x,s)\right)\right)ds\,dy = (-1)^m\frac{y^{2m}}{(2m)!}\sin(2y) \tag{116}$$

Then the general solution of equation (113) is written as follows

$$\begin{aligned}
u(x,y) &= u_0(x,y)+u_1(x,y)+u_2(x,y)+u_3(x,y)+u_4(x,y)+u_5(x,y)+\cdots \\
&= \sin(2y)\left[1-\frac{(2x)^2}{2!}+\frac{(2x)^4}{4!}-\frac{(2x)^6}{6!}+\frac{(2x)^8}{8!}-\frac{(2x)^{10}}{10!}+\cdots\right] \\
&= \sin(2y)\cos(2y)
\end{aligned}$$

4 Shallow water waves and Korteweg-de Vries Equation

Waves are one of the most fundamental motions; waves on the water's surface and of the earthquakes, waves along springs, light waves, radio waves, sound waves, waves of cloud, waves of crowds, brain waves and many others (Toda, 1989.) This article addresses waves on the water surface. The most familiar water waves are waves at the beach caused by wind or tides, waves created by throwing a stone in a pond, by the wake of a ship in the sea, or by raindrops in a river. Despite their familiarity, these are all different types of water waves. Our concern will be only on shallow water waves where the depth of the water is much smaller than the wavelength of the disturbance of the surface. This article further discusses shallow water waves equation commonly used in oceanography and atmospheric science and in particular shallow water wave model with wave dispersion, the Korteweg-de Vries equation.

In mathematics the Korteweg-de Vries equation is a mathematical model of waves on shallow water surfaces. The Korteweg-de Vries equation was first formulated as part of an analysis of shallow water waves in canals. In 19th century the study of water waves was of much interest for applications in naval architecture and for the knowledge of tides and floods. However, it has subsequently been found to be involved in a wide range of physics phenomena especially those exhibiting shock waves, travelling waves and solitons. Certain theoretical physics phenomena in the quantum mechanics domain are explained by the means of Kortewg-de Vries model. It is used in fluid dynamics, aerodynamics and continuum mechanics as a model for shock waves formation, solitons, turbulence, boundary layer behavior and mass transport.

The Korteweg-de Vries equation has several connections to physical in addition to being the governing equation of the string in the Fermi-Pasta-Ulam-Tsingou problem in the continuum limit. It approximately describes the evolution of long, one-dimensional waves in the many physical settings including:

- Shallow water waves with weakly non-linear restoring forces.
- Long internal waves in a density stratified ocean.
- Ion acoustic waves in a plasma
- Acoustic waves on a crystal lattice

4.1 Definition of terms

Deep water: A surface wave is said to be in deep water if its wave length is much shorter than the local water depth.
Internal wave: A internal wave travel within the interior of a fluid. The maximum velocity and maximum amplitude occur within the fluid or at an internal boundary (interface). Internal waves depend on the density-stratification of the fluid.
Shallow water: A surface wave is said to be in shallow water if its wave length is much larger than the local water depth.
Shallow water waves: Shallow water waves correspond to the flow at the free surface of a body of shallow water under the force of gravity, or to the flow below a horizontal pressure surface in a fluid.
Shallow water waves equations Shallow water waves equations are set of partial differential equations that describes shallow water waves.
Solitary wave: is a localized gravity wave that maintains its coherence and hence, its visibility through properties of nonlinear hydrodynamics. Solitary waves have finite amplitude and propagate with constant speed and constant shape.
Solitons: Solitons are solitary waves that have an elastic scattering property; they retain their shape and speed after colliding with each other.
Surface wave: A surface wave travel at the free surface of a fluid.b The maximum velocity of the wave and the maximum displacement of the fluid particles occur at the free surface of the fluid.
Tsunami: A tsunami is a very long ocean wave caused by underwater earthquake, submarine volcanic eruption or by a landslide.
Wave dispersion: Wave dispersion in water waves refer to the property that long waves lower frequencies and travel faster.

4.2 Historical Background of Korteweg-de Vries Equation

The initial observation of a solitary wave in shallow water was made by John Scott Russell. Russell was a Scottish engineer and naval architect who was conducting experiments for the Union Canal company to design a more efficient canal boat. In Russell's (1844) own words:

"I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses,when the boat is suddenly stopped- no so the mass of water in the channel which it had put in motion; it accumulated around the prow of the vessel in a state of violent agitation, the suddenly having it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a round, smooth and well-defined heap of water, which continued its course along the channel apparently without speed. I followed it on horseback, and overtook it still rolling on at a an hour, preserving its original figure some thirty feet long and a foot to foot and a half in height. Its height gradually diminished, and after a chase of one or two miles i lost it in the windings of the channel, such, in the month of August 1834, was my first chance interview with that singular and beautiful phenomenon which i have called the wave of translation."

Russell built a water tank to repucate the phenomenon and research the properties of the solitary wave he had observed. In 1895, the Dutch Professor Diederik Gustav de Vries (1895) derived a partial differential equation which models the solitary waves that Russell had observed. Parenthetically the equation which now bears their names had already appeared in the seminar work on water waves published by Boussinesq (1872, 1877) and Rayleigh (1876).

In 1965, Zabusky and Kruskal realized that the Korteweg- de Vries equation arises as the continuum limit of a one dimensional an harmonic lattice used by Fermi, Pasta and Ulam (1955) to investigate "thermalization"- or how energy is distributed among the many possible oscillations in the lattice and observed that they retain their shapes and speed after collision. Interacting solitary waves merely experience a phase shift, advancing the faster and retarding the slower. In analogy with colliding particles, they coined the word "solitons" to describe these elastically colliding waves. Since the 1970's, the Korteweg-de Vries equation,Boussinesq equations and other equations that admit solitary waves and solitons solutions have been the subject of intense study for example, Remoissenet 1999, Filippov 2000, and Dauxois and Peyrard 2006. Indeed, scientist remain intrigued by the physical properties and elegant mathematical theory of the shallow water waves models.

4.3 Korteweg-de Vries Equation

The Korteweg-de Vries equation is a nonlinear partial differential equation of third order, it was originally derived to describe shallow water waves of long wavelength and small amplitude.In the derivation, Korteweg and de Vries assumed that all motion is uniform in the y-direction, along the crest of the wave. In that case, the surface elevation (above the equilibrium level h) of the wave, propagating in the x-direction is a function only of the horizontal position x (along the canal) and of time t, that is, $z=\eta(x,t)$. In terms of the physical parameter, the Korteweg-de Vries equation is given by;

$$\frac{\partial\eta}{\partial t}+\sqrt{gh}\frac{\partial\eta}{\partial x}+\frac{3}{2}\frac{\sqrt{gh}}{h}\eta\frac{\partial\eta}{\partial x}+\frac{1}{2}h^2\sqrt{gh}(\frac{1}{3}-\frac{\tau}{\rho gh^2})\frac{\partial^3\eta}{\partial x^3}=0 \tag{117}$$

where h is the uniform water depth, g is the gravitational acceleration (about $9.81m/sec^2$ at sea level), ρ is the density and τ stands for the surface tension. The dimensionless parameter $\frac{\tau}{\rho gh^2}$ is called the bond number which measures the relative strength of surface tension and the gravitational forces keeping only the first two terms in (117), the speed of the associated linear (long) wave is $c=\sqrt{gh}$. This is indeed the maximum attainable speed of propagation of gravity-induced water waves of infinitesimal amplitude. the speed of propagation of the small- amplitude solitary waves described by (117) is slightly higher.

The Korteweg-de Vries equation can be recast in dimensionless variable as

$$u_t + \alpha u u_x + u_{xxx} = 0 \tag{118}$$

where subscripts denotes partial derivatives. The parameter α can be scaled to any real number. Commonly used values are $\alpha = \pm 1$ or $\alpha = \pm 6$. The term u_t describes the time evolution of the wave propagating in one direction. Therefore, (118) is called an evolution equation. The nonlinear term $\alpha u u_x$ accounts for steepening of the waves, and the linear dispersive term u_{xxx} describes the spreading of the wave. The linear first order term $\sqrt{gh}\frac{\partial \eta}{\partial x}$ in (117) can be removed by an elementary transformation. Conversely, a linear term in u_x can be added to (118). The nonlinear steeping of the water wave can be balanced by dispersion. If so the result of these counteracting effects is a stable solitary wave with particle-like properties. A solitary wave has a finite amplitude and propagates at constant speed and without change in shape over a fairly long distance. This is in contrast to the concentric group of small-amplitude capillary waves, which disperse as they propagate.

The closed-form expression of a solitary wave solution is given by

$$u(x,t) = \frac{w - 4k^3}{\alpha k} + \frac{12k^2}{\alpha}\operatorname{sech}^2(kx - wt + \delta) \tag{119}$$

$$= \frac{w + 8k^3}{\alpha k} - \frac{12k^2}{\alpha}\tanh^2(kx - wt + \delta) \tag{120}$$

where the wave number k,the angular frequency w and δ are arbitrary constants. Requiring that $\lim_{x \to \pm\infty} u(x,t) = 0$ for all time leads to $w = 4k^3$. Then (119) and (120) reduces to

$$\begin{aligned} u(x,t) &= \tfrac{12k^2}{\alpha k}\operatorname{sech}^2\left(kx - 4k^3 t + \delta\right) \\ &= \tfrac{12k^2}{\alpha}[1 - \tanh^2\left(kx - 4k^3 t + \delta\right) \end{aligned} \tag{121}$$

The position of the hump-type wave at $t = 0$ is depicted in figure 1 for $\alpha = 6$, $k = 2$ and $\delta = 0$. As time changes, the solitary waves with amplitude $2k^2 = 8$ travels to the right at speed $v = \frac{w}{k} = 4k^2 = 16$. The speed is exactly twice the peak amplitude. So, the taller the wave the faster it travels, but it does so without change in shape. The reciprocal of the wavenumber k is a measure of the width of the sech-squared pulse.

As shown by Korteweg and de Vries (1895), equation (118) also has a simple periodic solution.

$$u(x,t) = \frac{w - 4k^3(2m-1)}{\alpha k} + \frac{12k^2 m}{\alpha} cn^2(kx + wt + \delta_i m) \tag{122}$$

which they called the cnoidal wave solution for it involves the jacobi elliptic cosine function, cn, with modulus $m; (0 < m < 1)$. The wave number k gives the characteristic width of each oscillation in the "cnoid".

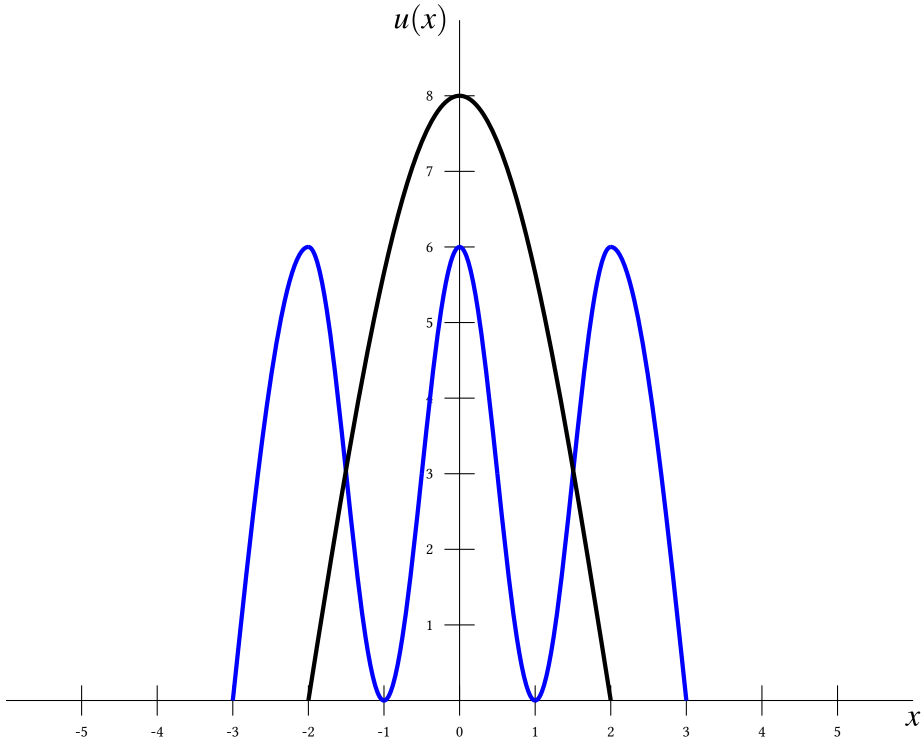

Figure 1. Solitary wave (black) and Periodic cnoidal (blue) wave profile

Three cycles of the cnoid wave are depicted in figure 1 at $t = 0$, the graph corresponding to $\alpha = 6$, $k = 2$, $m = \frac{3}{4}$, $w = 16$ and $\delta = 0$ using the property $\lim_{m \to 1} cn(\zeta; m) = \text{sech}(\zeta)$, one readily verifies that (122) reduces to (119) as m tends to 1. Pictorially, the individual oscillations then stretch infinitely for apart leaving a single-pulse solitary wave.

4.4 Derivation of the Korteweg-de Vries Equation

To begin we consider the conservation of mass equation and conservation of momentum equation.

$$\begin{aligned} \partial_t \rho + \nabla \cdot (\rho \vec{v}) &= 0 \\ \rho(\partial_t + \vec{v} \cdot \nabla)\vec{v} &= -\nabla p + \vec{f} \end{aligned} \tag{123}$$

where ρ is the density and $\vec{v}$ the velocity of the fluid, while p is the internal pressure and $\vec{f}$ is the external force density. Next assuming that our fluid is incompressible and irrotational,

$$\nabla \rho = 0, \qquad \partial_t \rho = 0, \qquad \nabla \times \vec{v} = 0 \tag{124}$$

The last equation allows us to consider the velocity in terms of some potential, and insertion of that form into the first of our hydrodynamics equation requires that this potential satisfy Laplace's equation:

$$\vec{v} = \nabla\phi, \qquad \nabla^2\phi = 0 \tag{125}$$

with the momentum density equation still to be satisfied as well. In this case the intent to consider the case when the external force is that caused by gravity, so that we will set

$$\vec{f} = -\rho g\hat{y} \tag{126}$$

Next we note the following 3-vector identity which is quite useful for irrotational vectors.

$$\vec{v} \times (\nabla \times \vec{v}) = -(\vec{v}\cdot\nabla)\vec{v} + \frac{1}{2}\nabla(\vec{v}^2) \tag{127}$$

As our velocity field has zero curl, i.e is irrotational, this allow us the easy substitution generated by equation (127), which the allows the entirety of the momentum equation to be written as a gradient

$$\nabla(\partial_t\phi + \frac{1}{2}\vec{v}^2 + \frac{p}{\rho} + gy) = 0 \tag{128}$$

we note then that the quantity under the ∇ must then depend only on the time, which is a term that may be absorbed into our potential function, since adding a function only of the time will not affect its role, namely to determine the velocities. This gives us the two equations to determine our potential, namely Laplace's equation and the following one;

$$\partial_t\phi + \frac{1}{2}(\nabla\phi)^2 + \frac{p}{\rho} + gy = \partial_t\phi + \frac{1}{2}(u^2+v^2) + \frac{p}{\rho} + gy = 0 \tag{129}$$
$$\vec{v} = \nabla\phi \equiv u\hat{x} + v\hat{y}$$

We now describe the problem, set up the geometry, and, in particular determine the boundary conditions. We are interested in the (irrotational) flow of this (incompressible) water down a long channel which is so narrow as to allow us consider it one dimensional. We therefore set up an $\hat{x}$-axis along the lentgh of the channel, and a $\hat{y}$-axis vertically, ignoring totally the $\hat{z}$-direction.

The meaning of shallow is that the waves that we want to study should be much longer than the depth of the water; a very different way of saying this is that we are making a long wavelength approximation. We describe this in some detail by saying that the depth of the water at rest is given by h, a characteristic length of the waves to be searched for is given by l and we assume that $l >> h$. On the other hand we also want, surely, to have this irrotational flow, i.e we want to avoid any turbulence in the motion; therefore, we propose as well to restrain these searched-for waves so that their amplitude is characterized by some length a, and we require that $a << h$.

It will be useful to define two small quatities, created by these requirement

$$\varepsilon \equiv \frac{a}{h}, \qquad \delta \equiv (\frac{h}{l})^2 \tag{130}$$

and we propose to treat both of these quantities as perturbations to the simple, laminar flow, and to suppose that they are both of the same (small) size. Although they appear different, since they are different powers of their ratios, we will see that, modulo overall factors, only this quadratic power of $\frac{h}{l}$ actually appears in the final equations.

As well we suppose that the surface of the water at rest is the zero for the vertical direction, so that the bottom of the channel correspond to $y = -h$, and, also we take the pressure to vanish at, and near, the surface. [This gives us what is often refereed to as Bernoulli's equation.] As the bottom is rigidly fixed, the water cannot move it, so that a boundary condition is surely that

$$|\vec{v}|^y|_{bottom} = v|_{bottom} = \frac{\partial\phi}{\partial y} = 0 \tag{131}$$

we will take some amplitude for the traveling waves, $\eta = \eta(x,t)$, so that the surface of the liquid that has waves traveling on it will be given by

$$\begin{aligned} y|_{surface} &= h + \eta(x,t) \qquad (132)\\ \Rightarrow v|_{surface} &= \frac{dy}{dt}|_{surface} = \partial_t\eta + \partial_x\eta\frac{dx}{dt}|_{surface}\\ \Rightarrow \phi_y|_{surface} &= \partial_t\eta + \phi_x|_{surface}\partial_x\eta \end{aligned}$$

We may also re-write equation (129) at the surface, remembering that the pressure vanishes there

$$\phi_t|_{surface} + \frac{1}{2}(u^2 + v^2)|_{surface} + g\eta = 0 \tag{133}$$

where we have ignored the constant term gh. We first resolve these equations for gravity forced, linear waves before obtaining the approximation that gives the Korteweg-de Vries equation. We consider first the linear approximation to all the above equations, which amounts to considering them for very small amplitudes;

$$\nabla^2\phi = 0, \qquad \phi_y(x, y = -h) = 0 \tag{134}$$

$$\text{and at the surface:} \quad \partial_t\eta - \partial_y\phi = 0 = \partial_t\phi + g\eta$$

We first eliminate the wave amplitude function, η from the three equation other than the lost one, by differentiating it and substituting, which gives

$$\partial_{tt}\phi + g\partial_y\phi = 0 \tag{135}$$

and allows us to to use last equation to determine η when we have found the potential ϕ. To do that since we know we are interested in wavelike solutions, we first propose an answer

$$\phi = Y(y)\sin(kx - wt) \tag{136}$$

The Laplace equation requires that $Y = Ae^{+ky} + be^{(}-ky)$ when this is inserted into the boundary condition at the bottom, we find that $\frac{B}{A} = e^{-2kh}$ which then gives us a form for ϕ;

$$\phi = 2Ae^{-kh}\cosh(k(y+h))\sin(kx - wt) \tag{137}$$

inserting this into the requirement we had that eliminated η_t, that is, equation (135) gives us an equation for the frequency:

$$w^2 = gk\tanh(k(y+h))|_{surface} = gk\tanh(kh) \tag{138}$$

lastly, we may use that "extra" equation to determine the wave amplitude:

$$\eta = -\frac{1}{g}\phi_t = A\sqrt{\frac{2k}{g}\sinh(2kh)\sin(kx - wt)} \tag{139}$$

we see that the wave has an interesting wavelength-dependent amplitude and a dispersion relation for its frequency that says that w is just proportional to k, with factor between them, that is, the speed equal to $c_0 = \sqrt{gh}$, for very long wavelength, that is, k near 0, but which goes more like $\sqrt{\frac{g}{k}} = \frac{c_0}{\sqrt{kh}} = 2\pi c_0\sqrt{\frac{\lambda}{h}}$ as the wavelength becomes shorter. This tell us that the important factor, defining what it means for the wavelength to be longer, or shorter is the ratio of the wavelength to the depth, h, of the channel; as well; of course, since this last relationship is the limit for very short wavelength waves travel faster and with less dispersion.

Now let us go forward with the insight created by this simpler example, and look in some detail at the next higher levels of perturbation to this problem. Returning to equation (129), and differentiating it along the channel that is, with respect to x, and evaluating at the surface of the water, we obtain

$$\phi_{xt} + \phi_x\phi_{xx} + \phi_y\phi_{xy} + g\eta_x = u_t + uu_x + vv_x + g\eta_x = 0 \tag{140}$$

we then take a standard sort of an approach for small amplitude derivations and expand ϕ in a power series in y:

$$\phi = \sum_0^\infty y^n\phi_n(x,t) \Rightarrow \phi_y = \sum_1^\infty \phi_n \tag{141}$$

Evaluating our constraints at the bottom, (131), gives us an important requirement that

$$\phi_1 = 0 \tag{142}$$

On the other hand, insertion of this sum into Laplace's equation gives us the following, generating a recursion relationship among the coefficients:

$$\sum_0^\infty y^n\{\phi_{n,xx} + (n+2)(n+1)\phi_{n+2}\} = 0 \tag{143}$$
$$\Rightarrow \phi_{n,xx} + (n+2)(n+1)\phi_{n+2} = 0$$

As we already know that $\phi_1 = 0$, this tells us that all the odd terms vanish, and we have the straightforward form for *phi* given by

$$\phi = \sum_{m=0}^{\infty} \frac{(-1)^m y^{2m}}{(2m)!} f^{(2m)}, \qquad f = \phi_0(x,t), \tag{144}$$
$$\Rightarrow \begin{cases} u = \phi_x = f_x - \frac{1}{2}y^2 f_{xxx} + \cdots \\ v = \phi_y = -y f_{xxx} + \frac{1}{6}y^3 f_{xxxx} + \cdots \end{cases}$$

where the notation $f^{(2m)}$ refers to the $2m$-derivative of f with respect to x. At this point, when dealing perturbatively with nonlinear problem, it is quite useful to first create some new dimensionless variable, which will allow us to introduce the quantities of small order.

We scale the horizontal distance in terms of its characteristic length l. We scale the time in terms of the simplest, linear speed, $c_0 \equiv \sqrt{gh}$, from equation (138) above and the amplitude in terms its maximums a, so that, eventually we define the following dimensionless variables via the following scalings, where the recall that our potential has such dimensions that its derivative, with respect to either x or y, should be a velocity:

$$\bar{x} \equiv \frac{x}{l}, \qquad \bar{y} \equiv \frac{y}{h}, \qquad \bar{t} = \frac{t}{\frac{l}{c_0}}, \qquad \eta \equiv \frac{\eta}{a}, \qquad \bar{\phi} \equiv \frac{h\phi}{alc_0} = \frac{\phi}{\varepsilon l c_0} \tag{145}$$
$$\text{the surface is now at} \quad \bar{y} = 1 + \varepsilon\bar{\eta}$$
$$\Rightarrow \begin{cases} \bar{u} = \frac{u}{\varepsilon c_0} \\ v = (\frac{\delta}{\varepsilon c_0})v, \\ \bar{f} = \frac{f}{\varepsilon l c_0} \end{cases}$$

The dimensionless versions of the equations of interest may now be worked out. It is useful to write out the equations for the (dimensionless) velocity components in this approach, to lowest order in small quantities, where we recall that ε and *delta* are of the same order of smallness, so that the symbol O^2 means any tern of second-order in these quantities, such as $\varepsilon\delta, \varepsilon^2$, of δ^2:

$$\bar{\phi} = \bar{f} - \frac{1}{2}(1+\varepsilon\eta)^2\delta\bar{f}_{\overline{xx}} + O^2 \tag{146}$$
$$\bar{u} = \bar{f}_{\bar{x}} - \frac{1}{2}\delta\bar{f}_{\overline{xxx}} + O^2$$
$$\bar{v} = -\delta[(1+\varepsilon\bar{\eta})\bar{f}_{\overline{xx}} - \frac{1}{6}\delta\bar{f}_{\overline{xxxxxx}} + O^2]$$

with that available, we now re-write our constraint on the vertical liquid velocity, as given in equation (132) in terms of these quantities, and the divide out by the common factor $\varepsilon c_0|\delta^{\frac{1}{2}}$, which gives

$$\bar{\phi y} = \delta(\bar{\eta}_{\bar{t}} + \varepsilon\bar{\eta}_{barx}\bar{\phi}_{\bar{x}}) \tag{147}$$

when our expansion for these quantities, in equation (144) are inserted, a common factor of δ emerges which divide out, and the result is then

$$\bar{\eta}_t + \varepsilon\bar{\eta}_{\bar{x}}\bar{f}_{\bar{x}} + (1+\varepsilon\bar{\eta})\bar{f}_{\overline{xx}} - \frac{1}{6}\delta\bar{f}_{\overline{xxxx}} + O^2 = 0 \tag{148}$$

In the same way our other boundary condition at the surface, equation (145), when these re-definition are inserted, and when it is divided out by the common factor ag, gives us

$$\bar{\phi}_{\bar{t}} + \frac{1}{2}\varepsilon[(\bar{\phi}_{\bar{x}})^2 + \frac{(\bar{\phi}_{\bar{y}})^2}{\delta}] + \bar{\eta} = 0 \tag{149}$$

Again when the expansion of these quantities are inserted, we obtain

$$\bar{f}_{\bar{t}} - \frac{1}{2}\delta\bar{f}_{\overline{xxt}} + \frac{1}{2}\varepsilon(\bar{f}_{\bar{x}})^2 + \bar{\eta} + O^2 = 0 \tag{150}$$

It is useful to go ahead and find how this equation changes along the channel. Therefore we differentiate it with respect to x, and the re-write both that equation and the previous equation (148) using $w \equiv \bar{f}_{\bar{x}}$. All variables are now in dimensionless form, so we don't need to write the over-bars, and remembering that all our variables are in fact dimensionless and, were we to want to go backwards we would use equations(145)

$$w_t - \frac{1}{2}\delta w_{xxt} + \varepsilon w w_x + \eta_x = 0 \tag{151}$$
$$\eta_t + \varepsilon w\eta_x + (1+\varepsilon\eta)w_x - \frac{1}{6}\delta w_{xxx} = 0$$

where, lastly, we have also stopped writing down the reminder that all these equations are correct only to second-order in ε and δ, separately. We now need to resolve this complicated pair of equations into something more manageable. The clear beginning is to make sure that the equations are satisfied to zero-th order. Therefore, ignoring both the ε-term and δ-terms, the correct pair of equation reduces simply to

$$w_t + \eta_x = 0 = w_x + \eta_t \tag{152}$$

we may separate the unknown functions in this pair of equations, which then implies that

$$w_{tt} = w_{xx} \quad \text{and} \quad \eta_{xx} = \eta_{tt} \tag{153}$$

That is to mean that at the lowest order, it is necessary that both w and η satisfies wave equations that have (dimensionless) speed l. The next requirement, then, is an answer based on the thought that w and η seems fairly similar, and that their difference are probably somewhere in, at least, first-order terms. Therefore, we now suppose that there may exist a solution for w in terms of η plus small terms

$$w \equiv \eta + \varepsilon F + \delta G + O^2 \tag{154}$$

where F and G depend on x and t and it is important to recall that the physical meanings of ε and δ are quit different, so that, in principle even though they are the same approximate size they are quite different and therefore must be studied separately. It is for this reason that we write an addition, first order term for the difference between u and η for each of them. Since we must still satisfy equations(152)and (153), we insert this answer into them, which implies that $\eta_t + \eta_x$ is of first oder in ε and δ. On the other hand, we have already seen that to lowest order we must have $w_t + w_x$ to vanish; therefore we must impose our "extra terms" that is, F and G, which are already multiplied by a small parameter, that they also have this property

$$F_x + F_t = O^1 \quad \text{and} \quad G_x + G_t = O^1 \tag{155}$$

We now insert this answer into our two equations, equations (151) and note that the purpose is to determine consistently the functions F and G in such a way that both our equations are satisfied at least to the order at which they have been written. These equations then take the following form

$$\begin{aligned} \eta_t + \eta_x + \varepsilon(F_t + \eta\eta_x) + \delta(G_t - \frac{1}{2}\eta_{xxt}) = 0 \\ \eta_x + \eta_t + \varepsilon(F_x + 2\eta\eta_x) + \delta(G_x - \frac{1}{6}\eta_{xxx}) = 0 \end{aligned} \tag{156}$$

where of course we have now dropped any explicit statement that there are higher-order terms which are being ignored. Now, subtracting these two equations we get

$$\varepsilon(F_x - F_t + \eta\eta_x) + \delta(G_x - G_t - \frac{1}{6}\eta_{xxx} + \frac{1}{2}\eta_{xxt}) = 0 \tag{157}$$

Now remember that ε and δ should be treated independently. Therefore we now end up with two rather simple equations that determine F and G,

$$2F_x = -\eta\eta_x = -\frac{1}{2}(\eta^2)_x, \quad 2G_x = \frac{2}{3}\eta_{xxx}; \qquad (158)$$
$$\Rightarrow F = -\frac{1}{4}\eta^2, \quad G = \frac{1}{3}\eta_{xx},$$
$$\Rightarrow w = \eta - \frac{1}{4}\varepsilon\eta^2 + \frac{1}{3}\delta\eta_{xx} + O^2$$

Lastly we insert this form for w into either one of the two original equations in the set equations (151) we choose the second one, which gives us the form:

$$\eta_t + \eta_x + \frac{3}{2}\varepsilon\eta\eta_x + \frac{1}{6}\delta\eta_{xxx} \qquad (159)$$

This is basically the Kortewg-de Vries equation; however we need to eliminate the first order small coefficients. We choose to first eliminate the (linear) η_x term by translating η by a constant:

$$\eta_x + \frac{3}{2}\varepsilon\eta\eta_x = \eta_x(1 + \frac{3}{2}\varepsilon\eta) = \frac{3}{2}\varepsilon\eta_x(\eta + \frac{\frac{2}{3}}{\varepsilon}) \qquad (160)$$
$$= \frac{3}{2}\varepsilon(\eta + \frac{\frac{2}{3}}{\varepsilon})_x(\eta + \frac{\frac{2}{3}}{\varepsilon}) \equiv \frac{3}{2}\varepsilon\sigma_x\sigma$$

where $\sigma \equiv \eta + \frac{\frac{2}{3}}{\varepsilon}$ is simply the original wave amplitude translated by this "large" constant value. The equation then becomes

$$\sigma_t + \frac{3}{2}\varepsilon\sigma\sigma_x = \frac{1}{6}\delta\sigma_{xxx} \qquad (161)$$

Next, we chose to re-scale σ and t as follows, and then dividing the overall equation by δ:

$$\left.\begin{array}{ll} \sigma = & (\frac{\delta}{\varepsilon})\rho, \\ t = & \frac{\tau}{\delta}, \end{array}\right\} \Rightarrow \rho_\tau + \frac{3}{2}\rho\rho_x + \frac{1}{6}\rho_{xxx} = 0 \qquad (162)$$

This is a perfectly reasonable form for our equation. On the other hand, there are still many things which are done to make it even "cleaner"; unfortunately at this point there is no consensus at all concerning what should be done. One of the options is to change, or not to change, the sign of x (or τ) so that the wave motion is toward increasing values of x instead of the other way around. Another option is to rescale the wave amplitude ρ so as to arrange the numerical constants in the equation according to one's liking. It is true that if we arrange the nonlinear term to have the coefficient G then other things becomes simpler; on the other hand, the coefficient 1 is also quite good. We choose to re-scale one last time with

$$\left.\begin{aligned} \tau &\equiv -6t \\ \rho &\equiv \tfrac{u}{9} \end{aligned}\right\} \Rightarrow u_t = uu_x + u_{xxx} \tag{163}$$

which is the general form of the Korteweg-de Vries equation.

5 Application of Homotopy Perturbation Method to Korteweg-de Vries Equation

The general form of the Korteweg-de Vries equation is presented as

$$\frac{\partial u}{\partial t} - 6u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0 \tag{164}$$

subject to the conditions

$$\begin{aligned} u(x,0) = &\quad \sin(\pi x) \\ u(0,t) = &\quad 0 \end{aligned} \tag{165}$$

Now we demonstrate the application of homotopy perturbation method to Korteweg-de Vries equation. By using homotopy technique we construct a homotopy of this problem as follows

$$H(u,p) = L(u) - L(u_0) + pL(u_0) + p[N(u) - f(x)] = 0$$

where L is the linear part and N the non-linear part.

$$L(u) = \frac{\partial u}{\partial t}, \quad N(u) = \frac{\partial^3 u}{\partial x^3} - 6u\frac{\partial u}{\partial x}, \quad f(x) = 0$$

Hence the homotopy is

$$\frac{\partial u}{\partial t} - \frac{\partial u_0}{\partial t} + p(\frac{\partial u_0}{\partial t} + \frac{\partial^3 u}{\partial x^3} - 6u\frac{\partial u}{\partial x}) = 0 \tag{166}$$

with initial approximation, $u_0 = \sin(\pi x)$ which clearly satisfies

$$\frac{\partial u}{\partial t} - \frac{\partial u_0}{\partial t} = p[6u\frac{\partial u}{\partial x} - \frac{\partial u_0}{\partial t} - \frac{\partial^3 u}{\partial x^3}]$$

the boundary conditions(165). We suppose that the solution of the problem (164) is in the form

$$u = u_0 + pu_1 + p^2 u_2 + p^3 u_3 + p^4 u_4 + \cdots \tag{167}$$

Now we substitute equation (167) in equation (166)

$$\frac{\partial}{\partial t}(u_0+pu_1+p^2u_2+p^3u_3+p^4u_4+\cdots)-\frac{\partial u_0}{\partial t}=p[6(u_0+pu_1+p^2u_2+p^3u_3+p^4u_4 \\ +\cdots)\frac{\partial}{\partial x}(u_0+pu_1+p^2u_2+p^3u_3+p^4u_4+\cdots)-\frac{\partial u_0}{\partial t}-\frac{\partial^3}{\partial x^3}(u_0+pu_1+p^2u_2+p^3u_3 \\ +p^4u_4+\cdots)]$$

L.H.S;

$$\frac{\partial u_0}{\partial t}+p\frac{\partial u_1}{\partial t}+p^2\frac{\partial u_2}{\partial t}+p^3\frac{\partial u_3}{\partial t}+p^4\frac{\partial u_4}{\partial t}+\cdots-\frac{\partial u_0}{\partial t}$$

R.H.S

$$(6pu_0+6p^2u_1+6p^2u_3+6p^4u_3+6p^5u_4+\cdots) \\ (\frac{\partial u_0}{\partial x}+p\frac{\partial u_1}{\partial x}+p^2\frac{\partial u_2}{\partial x}+p^3\frac{\partial u_3}{\partial x}+p^4\frac{\partial u_4}{\partial x}+\cdots) \\ -p\frac{\partial u_0}{\partial t}-p\frac{\partial^3 u_0}{\partial x^3}-p^2\frac{\partial^3 u_1}{\partial x^3}-p^3\frac{\partial^3 u_2}{\partial x^3}-p^4\frac{\partial^3 u_3}{\partial x^3}-p^5\frac{\partial^3 u_4}{\partial x^3}-\cdots$$

R.H.S

$$p6u_0(\frac{\partial u_0}{\partial x}+p\frac{\partial u_1}{\partial x}+p^2\frac{\partial u_2}{\partial x}+p^3\frac{\partial u_3}{\partial x}+p^4\frac{\partial u_4}{\partial x}+\cdots)+ \\ p^26u_1(\frac{\partial u_0}{\partial x}+p\frac{\partial u_1}{\partial x}+p^2\frac{\partial u_2}{\partial x}+p^3\frac{\partial u_3}{\partial x}+p^4\frac{\partial u_4}{\partial x}+\cdots)+ \\ p^36u_2(\frac{\partial u_0}{\partial x}+p\frac{\partial u_1}{\partial x}+p^2\frac{\partial u_2}{\partial x}+p^3\frac{\partial u_3}{\partial x}+p^4\frac{\partial u_4}{\partial x}+\cdots)+ \\ p^46u_3(\frac{\partial u_0}{\partial x}+p\frac{\partial u_1}{\partial x}+p^2\frac{\partial u_2}{\partial x}+p^3\frac{\partial u_3}{\partial x}+p^4\frac{\partial u_4}{\partial x}+\cdots)+ \\ p^56u_4(\frac{\partial u_0}{\partial x}+p\frac{\partial u_1}{\partial x}+p^2\frac{\partial u_2}{\partial x}+p^3\frac{\partial u_3}{\partial x}+p^4\frac{\partial u_4}{\partial x}+\cdots)+\cdots \\ -p\frac{\partial u_0}{\partial t}-p\frac{\partial^3 u_0}{\partial x^3}-p^2\frac{\partial^3 u_1}{\partial x^3}-p^3\frac{\partial^3 u_2}{\partial x^3}-p^4\frac{\partial^3 u_3}{\partial x^3}-p^5\frac{\partial^3 u_4}{\partial x^3}-\cdots$$

we compare the coefficients of like powers of p, we get

$$
\begin{aligned}
p^0 : \tfrac{\partial u_0}{\partial t} &= \tfrac{\partial u_0}{\partial t},\\
p^1 : \tfrac{\partial u_1}{\partial t} &= 6u_0\tfrac{\partial u_0}{\partial x} - \tfrac{\partial u_0}{\partial t} - \tfrac{\partial^3 u_0}{\partial x^3}\\
p^2 : \tfrac{\partial u_2}{\partial t} &= 6u_0\tfrac{\partial u_1}{\partial x} + 6u_1\tfrac{\partial u_0}{\partial x} - \tfrac{\partial^3 u_1}{\partial x^3}\\
p^3 : \tfrac{\partial u_3}{\partial t} &= 6u_0\tfrac{\partial u_2}{\partial x} + 6u_1\tfrac{\partial u_1}{\partial x} + 6u_2\tfrac{\partial u_0}{\partial x} - \tfrac{\partial^3 u_2}{\partial x^3}\\
p^4 : \tfrac{\partial u_4}{\partial t} &= 6u_0\tfrac{\partial u_3}{\partial x} + 6u_1\tfrac{\partial u_2}{\partial x} + 6u_2\tfrac{\partial u_1}{\partial x} + 6u_3\tfrac{\partial u_0}{\partial x} - \tfrac{\partial^3 u_3}{\partial x^3}\\
p^5 : \tfrac{\partial u_5}{\partial t} &= 6u_0\tfrac{\partial u_4}{\partial x} + 6u_1\tfrac{\partial u_3}{\partial x} + 6u_2\tfrac{\partial u_2}{\partial x} + 6u_3\tfrac{\partial u_1}{\partial x} + 6u_4\tfrac{\partial u_0}{\partial x} - \tfrac{\partial^3 u_4}{\partial x^3}\\
p^6 : \tfrac{\partial u_6}{\partial t} &= 6u_0\tfrac{\partial u_5}{\partial x} + 6u_1\tfrac{\partial u_4}{\partial x} + 6u_2\tfrac{\partial u_3}{\partial x} + 6u_3\tfrac{\partial u_2}{\partial x} + 6u_4\tfrac{\partial u_1}{\partial x} + 6u_5\tfrac{\partial u_0}{\partial x} - \tfrac{\partial^3 u_5}{\partial x^3}\\
&\vdots\\
p^n : \tfrac{\partial u_n}{\partial t} &= 6u_0\tfrac{\partial u_{n-1}}{\partial x} + 6u_1\tfrac{\partial u_{n-2}}{\partial x} + \cdots + 6u_{n-1}\tfrac{\partial u_0}{\partial x} - \tfrac{\partial^3 u_{n-1}}{\partial x^3}
\end{aligned}
$$

We solve the above equations using the boundary conditions given

$$u_0 = u(x,0) = \sin(\pi x); \qquad u(0,t) = 0$$

$$\frac{\partial u_0}{\partial x} = \pi\cos(\pi x); \qquad \frac{\partial u_0}{\partial t} = 0$$

$$\frac{\partial^2 u_0}{\partial x^2} = -\pi^2\sin(\pi x)$$

$$\frac{\partial^3 u_0}{\partial x^3} = -\pi^3\cos(\pi x)$$

$$\frac{\partial u_1}{\partial t} = 6[\sin(\pi x)][\pi\cos(\pi x)] + \pi^3\cos(\pi x)$$

$$\int \partial u_1 = \int [6\pi\sin(\pi x)\cos(\pi x) + \pi^3\cos(\pi x)]\partial t$$

$$u_1(x,t) = t(6\pi\sin(\pi x)\cos(\pi x) + \pi^3\cos(\pi x))$$

$$u_1(x,t) = t(3\pi\sin(2\pi x) + \pi^3\cos(\pi x))$$

$$\frac{\partial u_2}{\partial t} = 6u_0\frac{\partial u_1}{\partial x} + 6u_1\frac{\partial u_0}{\partial x} - \frac{\partial^3 u_1}{\partial x^3}$$

$$u_1 = (3\pi\sin(2\pi x) + \pi^3\cos(\pi x))t$$

$$\frac{\partial u_1}{\partial x} = (6\pi^2\cos(2\pi x) - \pi^4\sin(\pi x))t$$

$$\frac{\partial^2 u_1}{\partial x^2} = (-12\pi^3\sin(2\pi x) - \pi^5\cos(\pi x))t$$

$$\frac{\partial^3 u_1}{\partial x^3} = (-24\pi^4\cos(2\pi x) + \pi^6\sin(\pi x))t$$

so that

$$\begin{aligned}\frac{\partial u_2}{\partial t} = \; & (6\sin(\pi x))(6\pi^2\cos(2\pi x) - \pi^4\sin(\pi x))t + \\ & 6(3\pi\sin(2\pi x) + \pi^3\cos(\pi x))(\pi\cos(\pi x))t + \\ & 24\pi^4\cos(2\pi x) + \pi^6\sin(\pi x)t\end{aligned}$$

$$\begin{aligned}\frac{\partial u_2}{\partial t} = \; & (36\pi^2\sin(\pi x)\cos(2\pi x) - 6\pi^4\sin^2(\pi x))t + \\ & (18\pi^2\sin(2\pi x) + 6\pi^4\cos^2(\pi x))t + \\ & 24\pi^4\cos(2\pi x) - \pi^6\sin(\pi x)t\end{aligned}$$

Using,

$$\sin(2A) = 2\sin(A)\cos(A)$$
$$\cos(2A) = \cos^2(A) - \sin^2(A) = 2\cosh^2(A) - 1 = 1 - 2\sin^2(A)$$

we get;

$$\begin{aligned}\frac{\partial u_2}{\partial t} = \; & 36\pi^2\sin(\pi x)\cos(\pi x) + 36\pi^2\cos(2\pi x)\sin(\pi x) + \\ & 24\pi^4\cos(2\pi x) + 6\pi^4 t(\cos^2(\pi x) - \sin^2(\pi x)) - \pi^6 t\sin(\pi x)\end{aligned}$$

$$\frac{\partial u_2}{\partial t} = 36t\pi^2(3\cos^2(\pi x) - 1) + 30t\pi^4\cos(2\pi x) - \pi^6 t\sin(\pi x)$$

$$\partial u_2 = (30t\pi^4\cos(2\pi x) - \pi^6 t\sin(\pi x) + 36t\pi^2(3\cos^2(\pi x) - 1))\partial t$$

$$u_2 = 15t^2\pi^4\cos(2\pi x) - \frac{\pi^6}{2}t^2\sin(\pi x) + 18t^2\pi^2(3\cos^2(\pi x) - 1)$$

$$\begin{aligned}\frac{\partial u_2}{\partial x} & = -30t^2\pi^5\sin(2\pi x) - \tfrac{\pi^7}{2}t^2\cos(\pi x) - 108t^2\pi^3\cos(\pi x)\sin(\pi x) \\ & = -30t^2\pi^5\sin(2\pi x) - \tfrac{\pi^7}{2}t^2\cos(\pi x) - 54t^2\pi^3\sin(2\pi x)\end{aligned}$$

$$\frac{\partial^2 u_2}{\partial x^2} = -60t^2\pi^6\cos(2\pi x) + \frac{\pi^8}{2}t^2\sin(\pi x) - 108t^2\pi^4\cos(2\pi x)$$

$$\frac{\partial^3 u_2}{\partial x^3} = 120t^2\pi^7\sin(2\pi x) + \frac{\pi^9}{2}t^2\cos(\pi x) + 216t^2\pi^5\sin(2\pi x)$$

Now;

$$\begin{aligned}\frac{\partial u_3}{\partial t} = \; & (6\sin\pi x)(-30t^2\pi^5\sin(2\pi x) - \tfrac{\pi^7}{2}t^2\cos(\pi x) - 54t^2\pi^3\sin(2\pi x)) \\ & +6(3\pi\sin(2\pi x) + \pi^3\cos(\pi x))t(6\pi^2\cos(2\pi x) - \pi^4\sin(\pi x)) \\ & +6(15t^2\pi^4\cos(2\pi x) - \tfrac{\pi^6}{2}t^2\sin(\pi x) + 18t^2\pi^2(3\cos^2(\pi x) - 1)(\pi\cos(\pi x)) \\ & -20t^2\pi^7\sin(2\pi x) + \tfrac{\pi^9}{2}t^2\cos(\pi x) + 216t^2\pi^5\sin(2\pi x)\end{aligned}$$

$$\begin{aligned}\frac{\partial u_3}{\partial t} = \; & -180t^2\pi^5\sin(\pi x)\sin(2\pi x) - 3t^2\pi^3\sin(\pi x)\cos(\pi x) - 324t^2\pi^3\sin(\pi x)\sin(2\pi x) \\ & +108\pi^3t^2\sin(2\pi x)\cos(2\pi x) - 18t^2\pi^5\sin(\pi x)\sin(2\pi x) + 36t^2\pi^5\cos(\pi x)\cos(2\pi x) \\ & -6t^2\pi^7\cos(\pi x)\sin(\pi x) + 90t^2\pi^5\cos(\pi x)\cos(2\pi x) - 3t^2\pi^7\cos(\pi x)\sin(\pi x) \\ & +324t^2\pi^3\cos^3(\pi x) - 108t^2\pi^3\cos(\pi x) - 120t^2\pi^7\sin(2\pi x) - \tfrac{\pi^9}{2}t^2\cos(\pi x) \\ & -216t^2\pi^5\cos(2\pi x)\end{aligned}$$

$$\begin{aligned}\int \partial u_3 = \; & \int[-198t^2\pi^5\sin(\pi x)\sin(2\pi x) - 12t^2\pi^7\sin(\pi x)\cos(\pi x) + 126t^2\pi^5\cos(\pi x)\cos(2\pi x) \\ & -324t^2\pi^3\sin(\pi x)\sin(2\pi x) + 108\pi^3t^2\sin(2\pi x)\cos(2\pi x) + 324t^2\pi^3\cos^3(\pi x) \\ & -108t^2\pi^3\cos(\pi x) - 120t^2\pi^7\sin(2\pi x) - \tfrac{\pi^9}{2}t^2\cos(\pi x) - 218t^2\pi^5\sin(2\pi x)]\partial t\end{aligned}$$

$$\begin{aligned}u_3(x,t) = \; & -66t^3\pi^5\sin(\pi x)\sin(2\pi x) - 4t^3\pi^7\sin(\pi x)\cos(\pi x) + 42t^3\pi^5\cos(\pi x)\cos(2\pi x) \\ & -108t^3\pi^3\sin(\pi x)\sin(2\pi x) + 36\pi^3t^3\sin(2\pi x)\cos(2\pi x) + 108t^3\pi^3\cos^3(\pi x) \\ & -36t^3\pi^3\cos(\pi x) - 40t^3\pi^7\sin(2\pi x) - \tfrac{\pi^9}{6}t^3\cos(\pi x) - 72t^3\pi^5\sin(2\pi x)\end{aligned}$$

$$\begin{aligned}\frac{\partial u_3}{\partial x} = \; & -66t^3\pi^6\cos(\pi x)\sin(2\pi x) - 132t^3\pi^6\sin(\pi x)\cos(2\pi x) - 4t^3\pi^8\cos^2(\pi x) \\ & +4t^3\pi^8\sin^2(\pi x) - 42t^3\pi^6\sin(\pi x)\cos(2\pi x) - 84t^3\pi^6\cos(\pi x)\sin(2\pi x) \\ & -108t^3\pi^4\cos(\pi x)\sin(2\pi x) - 216t^3\pi^4\cos(2\pi x)\sin(\pi x) + 72t^3\pi^4\cos^2(2\pi x) \\ & -72t^3\pi^4\sin^2(2\pi x) - 324t^3\pi^4\cos^2(\pi x)\sin(\pi x) + 36t^3\pi^4\sin(\pi x) - 80t^3\pi^8\cos(2\pi x) \\ & -\tfrac{\pi^{10}}{6}t^3\sin(\pi x) - 144t^3\pi^6\cos(2\pi x)\end{aligned}$$

$$
\begin{aligned}
\frac{\partial^2 u_3}{\partial x^2} = \quad & -66t^3\pi^6(-\pi\sin(\pi x)\sin(2\pi x)+2\pi\cos(2\pi x)\cos(\pi x)) \\
& -132t^3\pi^6(-2\pi\sin(\pi x)\sin(2\pi x)+\pi\cos(\pi x)\cos(2\pi x)) \\
& -4t^3\pi^8(-2\pi\cos(\pi x)\sin(\pi x))+4t^3\pi^8(2\pi\sin(\pi x)\cos(\pi x)) \\
& -42t^3\pi^6(\pi\cos(\pi x)\cos(2\pi x)+2\pi\sin(2\pi x)\sin(\pi x)) \\
& -84t^3\pi^6(2\pi\cos(2\pi x)\cos(\pi x)-\pi\sin(\pi x)\sin(2\pi x)) \\
& -108t^3\pi^4(2\pi\cos(2\pi x)\cos(\pi x)-\pi\sin(\pi x)\sin(2\pi x)) \\
& -216t^3\pi^4(\pi\cos(\pi x)\cos(2\pi x)-2\pi\sin(2\pi x)\sin(\pi x)) \\
& +72t^3\pi^4(-2.2\pi\cos(2\pi x)\sin(2\pi x))-72t^3\pi^4(2.2\pi\sin(2\pi x)\cos(2\pi x)) \\
& -324t^3\pi^4(-2\pi\cos(\pi x)\sin^2(\pi x)+\pi\cos^3(\pi x) \\
& +36t^3\pi^5\cos(\pi x)+160t^3\pi^9\sin(2\pi x) \\
& +\frac{\pi^{11}}{6}t^3\cos(\pi x)+288t^3\pi^7\sin(2\pi x)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2 u_3}{\partial x^2} = \quad & 330t^3\pi^7\sin(\pi x)\sin(2\pi x)-262t^3\pi^7\cos(2\pi x)\cos(\pi x) \\
& +16t^3\pi^9\sin(\pi x)\cos(\pi x)-210t^3\pi^7\cos(\pi x)\cos(2\pi x) \\
& +168t^3\pi^7\sin(\pi x)\sin(2\pi x)-432t^3\pi^5\cos(\pi x)\cos(2\pi x) \\
& +540t^3\pi^5\sin(\pi x)\sin(2\pi x)-576\pi^5\sin(2\pi x)\cos(2\pi x) \\
& +628t^3\pi^5\cos(\pi x)\sin^2(\pi x)-324t^3\pi^5\cos^2(\pi x) \\
& +36t^3\pi^5\cos(\pi x)+160t^3\pi^9\sin(2\pi x)+\frac{\pi^{11}}{6}t^3\cos(\pi x)+288t^3\pi^7\sin(2\pi x)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2 u_3}{\partial x^2} = \quad & 498t^3\pi^7\sin(\pi x)\sin(2\pi x)-472t^3\pi^7\cos(2\pi x)\cos(\pi x) \\
& +16t^3\pi^9\sin(\pi x)\cos(\pi x)-432t^3\pi^5\cos(\pi x)\cos(2\pi x) \\
& +540t^3\pi^5\sin(\pi x)\sin(2\pi x)-576\pi^5\sin(2\pi x)\cos(2\pi x) \\
& +628t^3\pi^5\cos(\pi x)\sin^2(\pi x)-324t^3\pi^5\cos^2(\pi x) \\
& +36t^3\pi^5\cos(\pi x)+160t^3\pi^9\sin(2\pi x)+\frac{\pi^{11}}{6}t^3\cos(\pi x)+288t^3\pi^7\sin(2\pi x)
\end{aligned}
$$

$$\begin{aligned}\frac{\partial^3 u_3}{\partial x^3} = \;& 498t^3\pi^7(-\pi\sin(\pi x)\sin(2\pi x) + 2\pi\cos(2\pi x)\cos(\pi x))\\ & -472t^3\pi^7(-pi\sin(\pi x)\cos(2\pi x) - 2\pi\cos(\pi x)\sin(2\pi x))\\ & +16t^3\pi^9(\pi\cos^2(\pi x) - \pi\sin^2(\pi x))\\ & -432t^3\pi^5(-\pi\sin(\pi x)\cos(2\pi x) - 2\pi\cos(\pi x)\sin(2\pi x))\\ & +540t^3\pi^5(-\pi\sin(\pi x)\sin(2\pi x) + 2\pi\cos(2\pi x)\cos(\pi x))\\ & -576\pi^5(2\pi\cos^2(2\pi x) - 2\pi\sin^2(2\pi x))\\ & +628t^3\pi^5(-\pi\sin^3(\pi x) + 2\pi\sin(\pi x)\cos^2(\pi x))\\ & -324t^3\pi^5(-3\pi\cos^2(\pi x)\sin(\pi x))\\ & -36t^3\pi^6\sin(\pi x) + 320t^3\pi^9\cos(2\pi x) - \tfrac{\pi^{12}}{6}t^3\sin(\pi x) + 576t^3\pi^8\sin(2\pi x)\end{aligned}$$

$$\begin{aligned}\frac{\partial^3 u_3}{\partial x^3} = \;& -498t^3\pi^8\sin(\pi x)\sin(2\pi x) + 996t^3\pi^8\cos(2\pi x)\cos(\pi x)\\ & +472t^3\pi^8\sin(\pi x)\cos(2\pi x) + 944t^3\pi^8\cos(\pi x)\sin(2\pi x)\\ & +16t^3\pi^{10}\cos^2(\pi x) - 16t^3\pi^{10}\sin^2(\pi x)\\ & +432t^3\pi^6\sin(\pi x)\cos(2\pi x) + 864t^3\pi^6\cos(\pi x)\sin(2\pi x)\\ & -540t^3\pi^6\sin(\pi x)\sin(2\pi x) + 1080t^3\pi^6\cos(2\pi x)\cos(\pi x)\\ & -1152t^3\pi^6\cos^2(2\pi x) + 1152t^3\pi^6\sin^2(2\pi x)\\ & -628t^3\pi^6\sin^3(\pi x) + 1256t^3\pi^6\sin(\pi x)\cos^2(\pi x)\\ & +972t^3\pi^6\cos^2(\pi x)\sin(\pi x)) - 36t^3\pi^6\sin(\pi x)\\ & +320t^3\pi^9\cos(2\pi x) - \tfrac{\pi^{12}}{6}t^3\sin(\pi x) + 576t^3\pi^8\sin(2\pi x)\end{aligned}$$

Now,

$$\frac{\partial u_4}{\partial t} = 6u_0\frac{\partial u_3}{\partial x} + 6u_1\frac{\partial u_2}{\partial x} + 6u_2\frac{\partial u_1}{\partial x} + 6u_3\frac{\partial u_0}{\partial x} - \frac{\partial^3 u_3}{\partial x^3}$$

$$\begin{aligned}6u_0\frac{\partial u_3}{\partial x} = \;& -396t^3\pi^6\sin(\pi x)\cos(\pi x)\sin(2\pi x) - 792t^3\pi^6\cos(2\pi x)\sin^2(\pi x)\\ & -24t^3\pi^8\sin(\pi x)\cos^2(\pi x) + 24t^3\pi^8\sin^3(\pi x) - 252t^3\pi^6\sin^2(\pi x)\cos(2\pi x)\\ & -504t^3\pi^6\cos(\pi x)\sin(\pi x)\sin(2\pi x) - 648t^3\pi^4\sin(\pi x)\cos(\pi x)\sin(2\pi x)\\ & -1296t^3\pi^4\cos(2\pi x)\sin^2(\pi x) + 432t^3\pi^4\sin(\pi x)\cos^2(2\pi x)\\ & -432t^3\pi^4\sin(\pi x)\sin^2(2\pi x) - 1944t^3\pi^4\sin^2(\pi x)\cos^2(2\pi x)\\ & +216t^3\pi^4\sin^2(\pi x) - 480t^3\pi^8\sin(\pi x)\cos(2\pi x) + t^3\pi^6\sin^2(\pi x)\\ & -864t^3\pi^6\sin(\pi x)\cos(2\pi x)\\ 6u_1\frac{\partial u_2}{\partial x} = \;& -540t^3\pi^6\sin^2(2\pi x) - 189t^3\pi^8\sin(2\pi x)\cos(\pi x) - 972t^3\pi^4\sin^2(2\pi x)\\ & -3t^3\pi^4\cos^2(\pi x) - 324t^3\pi^6\sin(2\pi x)\cos(\pi x)\end{aligned}$$

$$
\begin{aligned}
6u_2\frac{\partial u_1}{\partial x} = \;& 540t^3\pi^6\cos^2(2\pi x) - 108t^3\pi^8\sin(\pi x)\cos(2\pi x) + 1944t^3\pi^4\cos(2\pi x) \\
& -648t^3\pi^4\cos(2\pi x) - 3t^3\pi^{10}\sin^2(\pi x) - 324t^3\pi^6\sin(\pi x)\cos^2(\pi x) \\
& +108t^3\pi^6\sin(\pi x) \\
6u_3\frac{\partial u_0}{\partial x} = \;& -396t^3\pi^6\cos(\pi x)\sin(\pi x)\sin(2\pi x) - 24t^3\pi^8\sin(\pi x)\cos^2(\pi x) \\
& +252t^3\pi^6\cos^2(\pi x)\cos(2\pi x) - 648t^3\pi^4\cos(\pi x)\sin(\pi x)\sin(2\pi x) \\
& +216t^3\pi^4\cos(\pi x)\sin(2\pi x)\cos(2\pi x) + 648t^3\pi^4\cos^4(\pi x) - 216t^3\pi^4\cos^2(\pi x) \\
& -240t^3\pi^8\cos(\pi x)\sin(2\pi x) - t^3\pi^{10}\cos^2(\pi x) - 432t^3\pi^6\cos(\pi x)\sin(2\pi x)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial u_4}{\partial t} = \;& -396t^3\pi^6\sin(\pi x)\cos(\pi x)\sin(2\pi x) - 792t^3\pi^6\cos(2\pi x)\sin^2(\pi x) \\
& -24t^3\pi^6\sin(\pi x)\cos^2(\pi x) + 24t^3\pi^8\sin^3(\pi x) - 252t^3\pi^6\sin^2(\pi x)\cos(2\pi x) \\
& -504t^3\pi^6\cos(\pi x)\sin(\pi x)\sin(2\pi x) - 648t^3\pi^4\sin(\pi x)\cos(\pi x)\sin(2\pi x) \\
& -1296t^3\pi^4\cos(2\pi x)\sin^2(\pi x) + 432t^3\pi^4\sin(\pi x)\cos^2(2\pi x) \\
& -432t^3\pi^4\sin(\pi x)\sin^2(2\pi x) - 1944t^3\pi^4\sin^2(\pi x)\cos^2(2\pi x) \\
& +216t^3\pi^4\sin^2(\pi x) - 480t^3\pi^8\sin(\pi x)\cos(2\pi x) + t^3\pi^6\sin^2(\pi x) \\
& -864t^3\pi^6\sin(\pi x)\cos(2\pi x) - 540t^3\pi^6\sin^2(2\pi x) \\
& -189t^3\pi^8\sin(2\pi x)\cos(\pi x) - 972t^3\pi^4\sin^2(2\pi x) \\
& -3t^3\pi^4\cos^2(\pi x) - 324t^3\pi^6\sin(2\pi x)\cos(\pi x) + 540t^3\pi^6\cos^2(2\pi x) \\
& -108t^3\pi^8\sin(\pi x)\cos(2\pi x) + 1944t^3\pi^4\cos(2\pi x) \\
& -648t^3\pi^4\cos(2\pi x) - 3t^3\pi^{10}\sin^2(\pi x) - 324t^3\pi^6\sin(\pi x)\cos^2(\pi x) \\
& +108t^3\pi^6\sin(\pi x) - 396t^3\pi^6\cos(\pi x)\sin(\pi x)\sin(2\pi x) - 24t^3\pi^8\sin(\pi x)\cos^2(\pi x) \\
& +252t^3\pi^6\cos^2(\pi x)\cos(2\pi x) - 648t^3\pi^4\cos(\pi x)\sin(\pi x)\sin(2\pi x) \\
& +216t^3\pi^4\cos(\pi x)\sin(2\pi x)\cos(2\pi x) + 648t^3\pi^4\cos^4(\pi x) - 216t^3\pi^4\cos^2(\pi x) \\
& -240t^3\pi^8\cos(\pi x)\sin(2\pi x) - t^3\pi^{10}\cos^2(\pi x) - 432t^3\pi^6\cos(\pi x)\sin(2\pi x) \\
& +498t^3\pi^8\sin(\pi x)\sin(2\pi x) - 996t^3\pi^8\cos(2\pi x)\cos(\pi x) \\
& -472t^3\pi^8\sin(\pi x)\cos(2\pi x) - 944t^3\pi^8\cos(\pi x)\sin(2\pi x) \\
& -16t^3\pi^{10}\cos^2(\pi x) + 16t^3\pi^{10}\sin^2(\pi x) \\
& -432t^3\pi^6\sin(\pi x)\cos(2\pi x) - 864t^3\pi^6\cos(\pi x)\sin(2\pi x) \\
& +540t^3\pi^6\sin(\pi x)\sin(2\pi x) - 1080t^3\pi^6\cos(2\pi x)\cos(\pi x) \\
& +1152t^3\pi^6\cos^2(2\pi x) - 1152t^3\pi^6\sin^2(2\pi x) \\
& +628t^3\pi^6\sin^3(\pi x) - 1256t^3\pi^6\sin(\pi x)\cos^2(\pi x) \\
& -972t^3\pi^6\cos^2(\pi x)\sin(\pi x)) + 36t^3\pi^6\sin(\pi x) \\
& -320t^3\pi^9\cos(2\pi x) + \frac{\pi^{12}}{6}t^3\sin(\pi x) - 576t^3\pi^8\sin(2\pi x)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial u_4}{\partial t} = \; & -1296t^3\pi^6\sin(\pi x)\cos(\pi x)\sin(2\pi x) - 1044t^3\pi^6\cos(2\pi x)\sin^2(\pi x) \\
& -2576t^3\pi^6\sin(\pi x)\cos^2(\pi x) + 24t^3\pi^8\sin^3(\pi x) - 1296t^3\pi^4\sin(\pi x)\cos(\pi x)\sin(2\pi x) \\
& -1296t^3\pi^4\cos(2\pi x)\sin^2(\pi x) + 432t^3\pi^4\sin(\pi x)\cos^2(2\pi x) - 432t^3\pi^4\sin(\pi x)\sin^2(\pi x) \\
& -1944t^3\pi^4\sin^2(\pi x)\cos^2(2\pi x) + 216t^3\pi^4\sin^2(\pi x) \\
& -1060t^3\pi^8\sin(\pi x)\cos(2\pi x) + t^3\pi^6\sin^2(\pi x) - 1296t^3\pi^6\sin(\pi x)\cos(2\pi x) \\
& -1692t^3\pi^6\sin^2(2\pi x) - 429t^3\pi^8\sin(2\pi x)\cos(\pi x) - 972t^3\pi^4\sin^2(2\pi x) \\
& -4t^3\pi^{10}\cos^2(\pi x) - 756t^3\pi^6\sin(2\pi x)\cos(\pi x) + 1692t^3\pi^6\cos^2(2\pi x) \\
& +1944t^3\pi^4\cos(2\pi x)\cos^2(\pi x) - 648t^3\pi^4\cos(2\pi x) - 3t^3\pi^{10}\sin^2(\pi x) \\
& +144t^3\pi^6\sin(\pi x) - 24t^3\pi^8\sin(\pi x)\cos^2(\pi x) + 252t^3\pi^6\cos^2(\pi x)\cos(2\pi x) \\
& +216t^3\pi^4\cos(\pi x)\sin(2\pi x)\cos(2\pi x) + 648t^3\pi^4\cos^4(\pi x) \\
& -216t^3\pi^4\cos(\pi x)\sin(2\pi x) + 498t^3\pi^8\sin(\pi x)\sin(2\pi x) - 996t^3\pi^8\cos(2\pi x)\cos(\pi x) \\
& -944t^3\pi^8\cos(\pi x)\sin(2\pi x) - 16t^3\pi^{10}(\cos^2(\pi x) - \sin^2(\pi x)) \\
& -864t^3\pi^6\cos(\pi x)\sin(2\pi x) + 540t^3\pi^6\sin(\pi x)\sin(2\pi x) - 1080t^3\pi^6\cos(2\pi x)\cos(\pi x) \\
& +628t^3\pi^6\sin^3(\pi x) - 320t^3\pi^9\cos(2\pi x) + t^3\frac{\pi^{12}}{6}\sin(\pi x) - 576t^3\pi^8\sin(2\pi x)
\end{aligned}
$$

.

$$
\begin{aligned}
u_4(x,t) = \; & -324t^4\pi^6\sin(\pi x)\cos(\pi x)\sin(2\pi x) - 216t^4\pi^6\cos(2\pi x)\sin^2(\pi x) \\
& -644t^4\pi^6\sin(\pi x)\cos^2(\pi x) + 6t^4\pi^8\sin^3(\pi x) - 324t^4\pi^4\sin(\pi x)\cos(\pi x)\sin(2\pi x) \\
& -324t^4\pi^4\cos(2\pi x)\sin^2(\pi x) + 108t^4\pi^4\sin(\pi x)\cos^2(2\pi x) - 108t^4\pi^4\sin(\pi x)\sin^2(\pi x) \\
& -486t^4\pi^4\sin^2(\pi x)\cos^2(2\pi x) + 54t^4\pi^4\sin^2(\pi x) \\
& -265.4t^4\pi^8\sin(\pi x)\cos(2\pi x) + \frac{1}{4}t^4\pi^6\sin^2(\pi x) - 324t^4\pi^6\sin(\pi x)\cos(2\pi x) \\
& -423t^4\pi^6\sin^2(2\pi x) - \frac{429}{4}t^4\pi^8\sin(2\pi x)\cos(\pi x) - 243t^4\pi^4\sin^2(2\pi x) \\
& -t^4\pi^{10}\cos^2(\pi x) - 189t^4\pi^6\sin(2\pi x)\cos(\pi x) + 423t^4\pi^6\cos^2(2\pi x) \\
& +486t^4\pi^4\cos(2\pi x)\cos^2(\pi x) - 162t^4\pi^4\cos(2\pi x) - \frac{3}{4}t^4\pi^{10}\sin^2(\pi x) \\
& +36t^4\pi^6\sin(\pi x) - 6t^4\pi^8\sin(\pi x)\cos^2(\pi x) + 63t^4\pi^6\cos^2(\pi x)\cos(2\pi x) \\
& +54t^4\pi^4\cos(\pi x)\sin(2\pi x)\cos(2\pi x) + 162t^4\pi^4\cos^4(\pi x) \\
& -54t^4\pi^4\cos(\pi x)\sin(2\pi x) + \frac{249}{2}t^4\pi^8\sin(\pi x)\sin(2\pi x) - 249t^4\pi^8\cos(2\pi x)\cos(\pi x) \\
& -236t^4\pi^8\cos(\pi x)\sin(2\pi x) - 4t^4\pi^{10}(\cos^2(\pi x) - \sin^2(\pi x)) \\
& -216t^4\pi^6\cos(\pi x)\sin(2\pi x) + 135t^4\pi^6\sin(\pi x)\sin(2\pi x) - 270t^4\pi^6\cos(2\pi x)\cos(\pi x) \\
& +157t^4\pi^6\sin^3(\pi x) - 80t^4\pi^9\cos(2\pi x) + t^4\frac{\pi^{12}}{24}\sin(\pi x) - 144t^4\pi^8\sin(2\pi x)
\end{aligned}
$$

.

Similarly we can solve u_5, u_6, u_7 and so on. The approximate solution of equation (164) can be obtained by setting $P = 1$ in equation (167) so that

$$u = \lim_{p\to 1} u = u_0 + u_1 + u_2 + u_4 + \cdots$$

that is

$$\begin{aligned} u = \ & \sin(\pi x) + t(6\pi \sin(\pi x)\cos(\pi x) + \pi^3\cos(\pi x)) \\ & 15t^2\pi^4\cos(2\pi x) - \tfrac{\pi^6}{2}t^2\sin(\pi x) + 18t^2\pi^2(3\cos^2(\pi x) - 1) \\ & -66t^3\pi^5\sin(\pi x)\sin(2\pi x) - 4t^3\pi^7\sin(\pi x)\cos(\pi x) + 42t^3\pi^5\cos(\pi x)\cos(2\pi x) \\ & -108t^3\pi^3\sin(\pi x)\sin(2\pi x) + 36\pi^3t^3\sin(2\pi x)\cos(2\pi x) + 108t^3\pi^3\cos^3(\pi x) \\ & -36t^3\pi^3\cos(\pi x) - 40t^3\pi^7\sin(2\pi x) - \tfrac{\pi^9}{6}t^3\cos(\pi x) - 72t^3\pi^5\sin(2\pi x) \\ & -324t^4\pi^6\sin(\pi x)\cos(\pi x)\sin(2\pi x) - 216t^4\pi^6\cos(2\pi x)\sin^2(\pi x) \\ & -644t^4\pi^6\sin(\pi x)\cos^2(\pi x) + 6t^4\pi^8\sin^3(\pi x) - 324t^4\pi^4\sin(\pi x)\cos(\pi x)\sin(2\pi x) \\ & -324t^4\pi^4\cos(2\pi x)\sin^2(\pi x) + 108t^4\pi^4\sin(\pi x)\cos^2(2\pi x) - 108t^4\pi^4\sin(\pi x)\sin^2(\pi x) \\ & -486t^4\pi^4\sin^2(\pi x)\cos^2(2\pi x) + 54t^4\pi^4\sin^2(\pi x) \\ & -265.4t^4\pi^8\sin(\pi x)\cos(2\pi x) + \tfrac{1}{4}t^4\pi^6\sin^2(\pi x) - 324t^4\pi^6\sin(\pi x)\cos(2\pi x) \\ & -423t^4\pi^6\sin^2(2\pi x) - \tfrac{429}{4}t^4\pi^8\sin(2\pi x)\cos(\pi x) - 243t^4\pi^4\sin^2(2\pi x) \\ & -t^4\pi^{10}\cos^2(\pi x) - 189t^4\pi^6\sin(2\pi x)\cos(\pi x) + 423t^4\pi^6\cos^2(2\pi x) \\ & +486t^4\pi^4\cos(2\pi x)\cos^2(\pi x) - 162t^4\pi^4\cos(2\pi x) - \tfrac{3}{4}t^4\pi^{10}\sin^2(\pi x) \\ & +36t^4\pi^6\sin(\pi x) - 6t^4\pi^8\sin(\pi x)\cos^2(\pi x) + 63t^4\pi^6\cos^2(\pi x)\cos(2\pi x) \\ & +54t^4\pi^4\cos(\pi x)\sin(2\pi x)\cos(2\pi x) + 162t^4\pi^4\cos^4(\pi x) \\ & -54t^4\pi^4\cos(\pi x)\sin(2\pi x) + \tfrac{249}{2}t^4\pi^8\sin(\pi x)\sin(2\pi x) - 249t^4\pi^8\cos(2\pi x)\cos(\pi x) \\ & -236t^4\pi^8\cos(\pi x)\sin(2\pi x) - 4t^4\pi^{10}(\cos^2(\pi x) - \sin^2(\pi x)) \\ & -216t^4\pi^6\cos(\pi x)\sin(2\pi x) + 135t^4\pi^6\sin(\pi x)\sin(2\pi x) - 270t^4\pi^6\cos(2\pi x)\cos(\pi x) \\ & +157t^4\pi^6\sin^3(\pi x) - 80t^4\pi^9\cos(2\pi x) + t^4\tfrac{\pi^{12}}{24}\sin(\pi x) - 144t^4\pi^8\sin(2\pi x) \end{aligned}$$

.

5.1 Convergence of Homotopy Perturbation Method

Now, we recall the basic concept of homotopy perturbation method. We consider the following nonlinear functional equation.

$$A(u) - f(r) = 0, \quad r \in \Omega \quad \text{,with boundary conditions,} \tag{168}$$

$$B(u, \frac{\partial u}{\partial n}) = 0, \qquad x \in \Gamma \tag{169}$$

Where A is a general functional operator, B is a boundary operator, $f(r)$ is a known analytic function, and Γ is the boundary of the domain Ω. The operator A can be divided into two parts L and N, where L is a linear operator and N a nonlinear operator.Therefore equation (168) can be rewritten as follows;

$$L(u)+N(u)-f(r)=0 \tag{170}$$

We construct a homotopy

$$\mathbb{V}(r,p):\Omega\times[0,1]\to R$$

which satisfies

$$H(v,p)=(1-p)[L(v)-L(u_0]+p[A(v)-f(r)]=0 \tag{171}$$

$$H(v,p)=L(v)-L(u_0)+pL(u_0)+p[N(v)-f(r)]=0 \tag{172}$$

Where $p\in[0,1]$ is an embedding parameter, and u_0 is an initial approximation for the solution of the equation (168) which satisfies the boundary conditions. According to homotopy perturbation method, we can first use the embedding parameter p as a small parameter, and assume that the solution of equation (172) can be written as a power series in p :

$$v=v_0+v_1p+v_2p^2+\cdots=\sum_{i=0}^{\infty}v_ip^i \tag{173}$$

Considering $p=1$, the approximate solution of equation (169) will be obtained as follows;

$$u=\lim_{p\to 1}v=v_0+v_1+v_2+\cdots \tag{174}$$

Now,to show convergence of the method, lets rewrite the equation (172) as the following

$$L(v)-L(u_0)=p[f(r)-L(u_0)-N(v)] \tag{175}$$

substituting (174) into (175) leads to;

$$L(\sum_{i=0}^{\infty}v_ip^i)-L(u_0)=p[f(r)-L(u_0)-N(\sum_{i=0}^{\infty}v_ip^i)] \tag{176}$$

$$\text{So}\quad \sum_{i=0}^{\infty}L(v_ip^i)-L(u_0)=p[f(r)-L(u_0)-N(\sum_{i=0}^{\infty}v_ip^i)] \tag{177}$$

According to Maclaurin expansion of $N(\sum_{i=0}^{\infty} v_i p^i)$ with respect to p, we have;

$$N(\sum_{i=0}^{\infty} v_i p^i) = \sum_{n=0}^{\infty} (\frac{1}{n!} \frac{\partial^n}{\partial p^n} N(\sum_{i=0}^{\infty} v_i p^i))_{p=0} p^i \tag{178}$$

$$\text{We get} \quad (\frac{\partial^n}{\partial p^n} N(\sum_{i=0}^{\infty} v_i p^i))_{p=0} = (\frac{\partial^n}{\partial p^n} N(\sum_{i=0}^{n} v_i p^i))_{p=0} \tag{179}$$

$$\text{Then} \quad N(\sum_{i=0}^{\infty} v_i p^i) = \sum_{n=0}^{\infty} (\frac{1}{n!} \frac{\partial^n}{\partial p^n} N(\sum_{i=0}^{\infty} v_i p^i))_{p=0} p^i \tag{180}$$

$$\text{We set} \quad H_n(v_0, v_1, \cdots, v_n) = (\frac{1}{n!} \frac{\partial^n}{\partial p^n} N(\sum_{i=0}^{n} v_i p^i))_{p=0}, \quad n = 0, 1, 2, 3, \cdots \tag{181}$$

Where $H_n' s$ are the so called He's polynomials, Then;

$$N(\sum_{i=0}^{n} v_i p_i) = \sum_{n=0}^{\infty} H_i p^i \tag{182}$$

Substituting (182) into (177), we get

$$\sum_{i=0}^{\infty} L(v_i) - L(u_0) = p[f(r) - L(u_0) - \sum_{n=0}^{\infty} H_i p^i] \tag{183}$$

By equating the terms with the identical powers of p:

$$\begin{aligned} p^0 &: L(v_0) - L(u_0) = 0 \\ p^1 &: L(v_1) = f(r) - L(u_0) - H_0, \\ p^2 &: L(v_2) = -H_1, \\ &\vdots \\ p^{n+1} &: L(v_{n+1}) = -H_n \\ &\vdots \end{aligned} \tag{184}$$

So we derive,

$$\begin{cases} v_0 = & u_0 \\ v_1 = & L^{-1}[f(r)] - u_0 - L^{-1}(H_0) \\ v_2 = & -L^{-1}(H_1), \\ \vdots & \\ v_{n+1} = & -L^{-1}(H_n) \\ \vdots & \end{cases} \tag{185}$$

Theorem 5.1.1. *Homotopy perturbation method used to find the solution of equation (168) is equivalent to determining the following sequence;*

$$s_n = v_1 + \cdots + v_n, \tag{186}$$
$$s_0 = 0$$

By using the iterative scheme:

$$s_{n+1} = -L^{-1}N_n(s_n + v_0) - u_0 + L^{-1}(f(r)), \quad \textit{where} \tag{187}$$

$$N_n(\sum_{i=0}^{n} v_i) = \sum_{n=0}^{n} H_i, \quad n = 0, 1, 2, \cdots \tag{188}$$

`Proof.` For $n = 0$, from (187), we have;

$$s_1 = -L^{-1}N_0(s_0 + v_0) - u_0 + L^{-1}(f(r)), \tag{189}$$
$$= -L^{-1}(H_0) - u_0 + L^{-1}(.f(r)) \quad \text{then}$$
$$v_1 = -L^{-1}(H_0) - u_0 + L^{-1}(.f(r)), \tag{190}$$
$$\text{for n=1} \quad S_2 = -L^{-1}N_1(s_1 + v_0) - u_0 + L^{-1}(f(r))$$
$$= -L^{-1}(H_0 + H_1) - u_0 + L^{-1}(f(r))$$
$$= -L^{-1}(H_1) + v_1 \tag{191}$$

According to $s_2 = v_1 + v_2$, we get;

$$v_2 = -L^{-1}(H_1) \tag{192}$$

Now we prove this theorem by induction. Let's assume that

$$v_{k+1} = -L^{-1}(H_k), \quad \text{for k=1,2, ..., n-1, So}$$
$$s_{n+1} = -L^{-1}N_n(s_n + v_0) - u_0 + L^{-1}(f(r))$$
$$= -L^{-1}(\sum_{n=0}^{n} H_i) - u_0 + L^{-1}(f(r))$$
$$= -\sum_{n=0}^{n} L^{-1}(H_i) - u_0 + L^{-1}(f(r))$$
$$= v_1 + v_2 + \cdots + v_n$$
$$= -L^{-1}(H_n) \tag{193}$$

Then, from (186) it can be shown;

$$v_{n+1} = -L^{-1}(H_n) \tag{194}$$

which is the same as the result of (185) from the homotopy perturbation method, and the theorem is proved □

Theorem 5.1.2. *Let B be a Banach space.*

(a) $\sum_{i=0}^{\infty} v_i$ *obtained by equation (185), converge to* $s \in B$, *if* $\exists (0 \leq \lambda \leq 1)$ *such that*

$$(\forall n \in N \Rightarrow ||v_n|| \leq \lambda ||v_{n-1}||) \tag{195}$$

(b) $s = \sum_{n=1}^{\infty} v_n$, *satisfies in*

$$s = -L^{-1}N(s+v_0) - u_0 + L^{-1}(f(r)) \tag{196}$$

Proof. (a) we have

$$||s_{n+1} - s_n|| = ||v_{n+1}|| \leq \lambda ||v_n|| \leq \lambda^2 ||v_{n-1}|| \leq \cdots \leq \lambda^{n+1} ||v_0|| \tag{197}$$

for any $n, m \in N, \quad n \geq m$, we derive;

$$\begin{aligned} ||s_n - s_m|| &= ||(s_n - s_{n-1}) + (s_{n-1} - s_{n-2}) + \cdots + (s_{m+1} - s_m)||, \\ &\leq ||s_n - s_{n-1}|| + ||s_{n-1} - s_{n-2}|| + \cdots + ||s_{m+1} - s_m||, \\ &\leq \lambda^n ||v_0|| + \lambda^{n-1} ||v_0|| + \cdots + \lambda^{m+1} ||v_0|| \leq (\lambda^n + \lambda^{n-1} + \cdots + \lambda^{m+1}) ||v_0||, \\ &\leq (\lambda^{m+1} + \cdots + \lambda^n + \cdots) ||v_0|| \leq \lambda^{m+1} (1 + \lambda + \cdots + \lambda^n + \cdots) ||v_0|| \\ &\leq \frac{\lambda^{m+1}}{1-\lambda} ||v_0|| \end{aligned} \tag{198}$$

So

$$\lim_{n,m \to \infty} ||s_n - s_m|| = 0 \tag{199}$$

Then $\{s_n\}$ is Cauchy sequence in Banach space, and it is convergent, that is $\exists s \in B$, such that $\lim_{n \to \infty} s_n = \sum_{n=1}^{\infty} v_n = s$

(b) ,From equation (187), we have

$$\begin{aligned} \lim_{n \to \infty} s_{n+1} &= -L^{-1} \lim_{n \to \infty} N_n(s_n + v_0) - u_0 + L^{-1}(f(r)) \\ &= -L^{-1} \lim_{n \to \infty} N_n(\sum_{i=0}^{n} v_i) - u_0 + L^{-1}(f(r)) \end{aligned} \tag{200}$$

$$
\begin{aligned}
s &= -L^{-1}\lim_{n\to\infty}\sum_{i=0}^{n} H_i - u_0 + L^{-1}(f(r)) \\
&= -L^{-1}\sum_{i=0}^{\infty} H_i - u_0 + L^{-1}(f(r))
\end{aligned}
$$

But by equation (188) and (182) for $p=1$, we derive

$$\sum_{i=0}^{\infty} H_i = N(\sum_{i=0}^{\infty} v_i), \quad \text{so} \tag{201}$$

$$s = -L^{-1}N(\sum_{i=0}^{n} v_i) - u_0 + L^{-1}(f(r)) \tag{202}$$

$$s = -L^{-1}N(s+v_0) - u_0 + L^{-1}(f(r)) \tag{203}$$

Lemma 1. Equation (196) is equivalent to;

$$L(u) + N(u) - f(r) = 0 \tag{204}$$

Proof. We rewrite equation (196) as follows

$$s + u_0 = -L^{-1}N(s+v_0) + L^{-1}(f(r)) \tag{205}$$

By applying the operator L to equation (204), we derive;

$$L(s+u_0) = -N(s+v_0) + f(r), \quad \text{but} \quad u_0 = v_0, \quad \text{then;}$$

$$L(s+v_0) + N(s+v_0) = f(r) \tag{206}$$

By considering $u = s + v_0 = \sum_{n=0}^{\infty} v_i$, equation (204)has been derived which is the original equation. Then the solution of equation (196) is the same as solution of $A(u) - f(r) = 0$. □

Definition 1. For every $i \in N$, we define;

$$\lambda_i = \begin{cases} \frac{||v_{i+1}||}{||v_i||}, & ||v_i|| \neq 0, \\ 0, & ||v_i|| = 0 \end{cases}$$

In theorem 5.1.2, $\sum_{n=0}^{\infty} v_i$ converges to exact solution, when $0 \leq \lambda_i < 1$. If v_i and v_i' are obtained by two different homotopy, and $\lambda_i < \lambda_i'$ for each $i \in N$, the rate of convergence of $\sum_{n=0}^{\infty} v_i$ is higher than $\sum_{n=0}^{\infty} v_i'$.

5.2 Exact Solution of the Korteweg-de Vries Equation

This exact solution was presented by Klaus Brauer 2000. The author's aim was to present an analytical exact result to the Korteweg-de Vries equation by means of elementary operation. Using the typical short denotations the problem can be formulated as:

$$u_t(x,t) + 6u(x,t)u_x(x,t) + u_{xxx}(x,t) = 0 \tag{207}$$

The aim here is to show how a general solution to (207),can be obtained without considering initial conditions and boundary conditions. The solutions to (207) are called solitons or solitary waves. Now, we know that the simplest mathematical wave is a function of the form $u(x,t) = f(x-ct)$ which for example is a solution to the simple partial differential equation$u_t + cu_x = 0$ where c denotes the speed of the wave. The well known wave equation $u_{tt} - c^2 u_{xx} = 0$ leads to two wave fronts represented by terms $f(x-ct)$ and $f(x+ct)$. Having this in mind,we start with a trial solution for (207) of the form,

$$u(x,t) = z(x - \beta t) = z(\phi) \tag{208}$$

denoting the parameter c above by β and the function f by z. Substituting the trial solution (208) into (207) we are led to the ordinary differential equation

$$-\beta \frac{dz}{d\phi} + 6z\frac{dz}{d\phi} + \frac{d^3 z}{d\phi^3} = 0 \tag{209}$$

integrating (209) we get;

$$-\beta z + 3z^2 + \frac{d^2 z}{d\phi^2} = c_1 \tag{210}$$

where c_1 is the costant of integration. In order to obtain a first order equation for z a multiplication with $\frac{dz}{d\phi}$ is done, that is

$$-\beta z\frac{dz}{d\phi} + 3z^2\frac{dz}{d\phi} + \frac{d^2 z}{d\phi^2}\cdot\frac{dz}{d\phi} = c_1\frac{dz}{d\phi}$$
$$\Rightarrow -\beta z dz + 3z^2 dz + \frac{d^2 z}{d\phi^2} dz = c_1 dz$$

integrating on both sides gives;

$$\frac{-\beta}{2} z^2 + z^3 + \frac{1}{2}(\frac{dz}{d\phi})^2 = c_1 z + c_2 \tag{211}$$

Now it is required that in case $x \to \pm\infty$ we should have $z \to 0$, $\frac{dz}{d\phi} \to 0$, $\frac{d^2 z}{d\phi^2} \to 0$. From these requirements it follows $c_1 = c_2 = 0$.

Remark: More general solution can be found for other choices of c_1 and c_2. These solutions can be represented in terms of elliptic integrals.

Now, with $c_1 = c_2 = 0$. equation (211) can be written as

$$(\frac{dz}{d\phi})^2 = z^2(\beta - 2z) \tag{212}$$

By separation of variables we write

$$\int_0^z \frac{d\psi}{\psi\sqrt{\beta - 2\psi}} = \int_0^\phi d\eta \tag{213}$$

The choice of 0 for the lower integration limits does not bring any loss of generality since the starting point can be transformed linearly. The integration of the left hand side of (213) can be done by using a transform'

$$s = \frac{1}{2}\beta\,\mathrm{sech}^2(w) \tag{214}$$

The role of s here is played by the variable ψ and we obtain

$$\beta - 2\psi = \beta(1 - \mathrm{sech}^2(w)) = \beta\tanh^2(w) \tag{215}$$

Since $\cosh^2(w) - \sinh[2]w = 1$ holds. Also we have;

$$\frac{d\psi}{dw} = -\beta\frac{\sinh(w)}{\cosh^3(w)} \tag{216}$$

The upper integration limit of the left hand integral in (213) due to (214) is transformed to

$$w = \mathrm{sech}^{-1}\left(\sqrt{\frac{2z}{\beta}}\right) \tag{217}$$

Substituting (215), (216) and (217) into (213) we get

$$\begin{aligned}
\psi &= -\frac{2}{\sqrt{\beta}}\int_0^w \frac{1}{\mathrm{sech}^2(w)\cdot\tanh(w)}\cdot\frac{\sinh(w)}{\cosh^3(w)}dw \\
&= -\frac{2}{\sqrt{\beta}}\int_0^w \frac{\cosh^2(w)\cdot\cosh(w)}{\sinh(w)}\cdot\frac{\sinh(w)}{\cosh^3(w)}dw \\
&\Rightarrow \psi = -\frac{2}{\sqrt{\beta}}\int_0^w dw = -\frac{2}{\sqrt{\beta}}w
\end{aligned}$$

with (214) the transform back to ϕ is done and we obtain

$$\phi = -\frac{2}{\sqrt{\beta}} \operatorname{sech}^{-1}\left(\sqrt{\frac{2z}{\beta}}\right) \Rightarrow z(\phi) = \frac{\beta}{2} \operatorname{sech}^2\left(\frac{\sqrt{\beta}}{2}\phi\right)$$

Now we use (208) and we finally get

$$u(x,t) = \frac{\beta}{2} \operatorname{sech}^2\left([\frac{\sqrt{\beta}}{2}(x-\beta t)]\right) \tag{218}$$

Remarks: In order to have a real solution the quantity β must be a positive number. As noticed from (218) for $\beta > 0$ the solitary wave moves to the right. The amplitude is proportional to the speed which is indicated by the value of β. Thus larger amplitude solitary waves move with a higher speed than smaller amplitude waves.

If instead of (214) we select the transformation

$$s = -\frac{1}{2}\beta \operatorname{csch}^2(w) \tag{219}$$

then similarly we will obtain another solution

$$u(x,t) = -\frac{\beta}{2} \operatorname{csch}^2\left([\frac{\sqrt{\beta}}{2}(x-\beta t)]\right) \tag{220}$$

Remark: The solution (220) is an irregular solution to the Korteweg-de Vries equation. It has a singularity for vanishing argument of the cosech-function, that is, for the line in the $(x-t)$-phase with $x-\beta t = 0 \Leftrightarrow t = \frac{1}{\beta}x$.

Note: There is a limitation to getting reliable solution analytically as boundary conditions were not used in obtaining the analytical solution in contrast with the separation of variable method used for linear partial differential equations where analytical solutions are constrained to both the boundary and initial conditions. Also the issue of assumptions like $c_1 = c_2 = 0$ (the constants of integration) limits the aspect of getting reliable solutions analytically. Other choices of c_1 and c_2 further complicates the solution which can be represented in terms of elliptic integrals. The analytical solution give reliable values for time close to the initial condition $t = 0$. Therefore we get the analytical solution only for the interval $0 \leq t \leq 1$.

6 Conclusion

In this research project paper,we have studied the perturbation theory and its application to differential equations.We have also studied shallow water waves and derived the Korteweg-de Vries equation.Then we have applied the homotopy perturbation method to solve this nonlinear partial differential equation of third order,the Kortewg-de Vries equation.

The solution we have got for the Korteweg-de Vries equation contain few terms of a series which may converge or diverge.Naturally what we know from the convergence theory is that even to calculate the coefficients of a perturbation series is not that easy.Actually,it has been for sometime now a big concern of many mathematicians to formulate formal series related to the solution of given problems.This is due to precisely the difficulties of calculating the coefficients as illustrated in our case.

The big problem is to decide whether or not those formal series converge or atleast are summable. However the advantage of perturbation method is that it gives a relatively very good results as long as the parameter used is very small. In our work we have shown that in some particular cases the convergence can be obtained but need of convergence proofs is still an on going problem(Giovanni Gallavotti).Some cases of convergence lead to the singularities of the function and then require analytical continuation process to determine area of validity of those solutions.In our case we couldnot go that way. To avoid the approach of possible analytic continuation it is also observed that convergence proofs in most interesting cases require a multiscale analysis.

As already said,the Korteweg-de Vries equation was first formulated as part of an analysis of shallow water waves in canals.In the 19th century the study of water waves was of much interest for applications in naval architecture and for the knowledge of tides and floods.However it has subsequently been found to be involved in a wide range of physics phenomena especially those exhibiting shock waves,travelling waves and solitons. The Korteweg-de Vries equation is applicable in condensed matter and semiconductor physics through nonlinear optics and laser physics,hydrodynamics,meteorology and plasma physics,protein systems and neurophysiology,areas of geophysics especially in magma flow and conduit waves.Perhaps the boldest application of Korteweg-de Vries equation to date has been to the understanding of the Great Red Spot(GRS) and other features in the Jovian atmosphere,seen in cloud patterns,such as the South Equatorial Disturbance,the Dark South Tropical Streak,the Hollow and the White Ovals, and in particular also,the South Tropical Disturbance.The Korteweg-de Vries equation is used in fluid dynamics,aerodynamics and continuum mechanics as a model for shock waves formation,solitons,turbulence,boundary layer behavior and mass transport.

In our work we considered the convergence of homotopy perturbation method in general for nonlinear functional equation and we didnot have time to apply to this particular case.We also considered cases where the Korteweg-de Vries equation could have exact solution,but the problem of its interpretation remain in the relationship of shallow water waves.With this in mind a possible future work can be done as we link the mathematical treatment to the physical meaning in real situations.

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