PROPERTIES OF FUNCTION SPACES DEFINED ON SETS OF CONTINUOUS FUNCTIONS ON BITOPOLOGICAL SPACES

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DECLARATION

This thesis is a presentation of my original research work. Wherever contributions of others are involved, every effort is made to indicate this clearly, with due reference to the literature, and acknowledgement. The matter embodied in this thesis has not been submitted in any other institution for the award of any other degree.

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DEDICATION

To my loving wife Lucy Wamuyu and my sons Dylan Muturi and Dwaine Munene.

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May the almighty God bless you all.

ABSTRACT

In function spaces, the set $C(Y,Z)$ of continuous functions from the topological space Y to the topological space Z is considered. Topologies are defined on this set to form the function space $C_\tau(Y,Z)$. Topological properties such as compactness, and separation axioms, as well as splitting and admissibility properties of topologies defined on the set $C(Y,Z)$ have been studied in this space. In bitopological spaces, topological concepts such as compactness, disconnectedness, separation axioms among others, have been generalized to the space (Y,τ_1,τ_2) and relationship between these generalized topological properties and the corresponding properties on the spaces (Y,τ_1) and (Y,τ_2) studied.

By combining these two fields and generalizing concepts in function spaces to topologized sets of continuous functions defined on bitopological spaces, the following function spaces are obtained; $s-C_\tau(Y,Z)$, $p-C_\omega(Y,Z)$, $(1,2)-C_\varphi(Y,Z)$ and $(2,1)-C_\xi(Y,Z)$. In the spaces $p-C_\omega(Y,Z)$ and $s-C_\tau(Y,Z)$, pairwise splitting (p-splitting), pairwise admissible (p-admissible), supremum splitting (s-splitting) and supremum admissible (s-admissible) topologies are defined, and relationship with splitting and admissible topologies on $1-C_\varsigma(Y,Z)$ and $2-C_\zeta(Y,Z)$ established. It is proved that if both the spaces $1-C_\varsigma(Y,Z)$ and $2-C_\zeta(Y,Z)$ have splitting (admissible) topology, then the spaces $s-C_\tau(Y,Z)$ and $p-C_\omega(Y,Z)$ have p-splitting (p-admissible) and s-splitting (s-admissible) topologies respectively. Separation axioms are generalized to the space $p-C_\omega(Y,Z)$ and compared with separation axioms defined on the spaces (Z,δ_i) for $i=1,2$, $1-C_\varsigma(Y,Z)$ and $2-C_\zeta(Y,Z)$, (Z,δ_1,δ_2), as well as on the space $s-C_\tau(Y,Z)$. It is shown that the space $p-C_\omega(Y,Z)$ is a ${}_pT_\circ$, ${}_pT_1$, ${}_pT_2$ and ${}_p$regular, if the spaces (Z,δ_1) and (Z,δ_2) both are $T_\circ$, T_1, T_2 and regular. The space $p-C_\omega(Y,Z)$ is also shown to be ${}_pT_\circ$, ${}_pT_1$, ${}_pT_2$ and ${}_p$regular, if the space (Z,δ_1,δ_2) is pairwise-$T_\circ$, pairwise-T_1, pairwise-T_2 and pairwise regular. The space $p-C_\omega(Y,Z)$ is also proved to be ${}_pT_0$, ${}_pT_1$, ${}_pT_2$ and ${}_p$regular, if the spaces $1-C_\varsigma(Y,Z)$ and $2-C_\zeta(Y,Z)$ are both T_0, T_1, T_2 and regular. Separation axioms defined on the space $s-C_\tau(Y,Z)$ are also compared with those defined on $p-C_\omega(Y,Z)$. It is proved that the space $s-C_\tau(Y,Z)$ is T_0, T_1 and T_2, if the space $p-C_\omega(Y,Z)$ is ${}_pT_0$, ${}_pT_1$ and ${}_pT_2$. The concept of compactness is also extended to closed subsets of the spaces $1-C_\varsigma(Y,Z)$ and $2-C_\zeta(Y,Z)$ in the space $s-C_\tau(Y,Z)$, this culminates into a proof of a variant of Arzela-Ascoli theorem.

TABLE OF CONTENTS

Contents **Page**

NOTATIONS TERMINOLOGIES AND DEFINITIONS

Most of these notations and terminologies can be found in general topology books, others are product of this work.

(Y, τ_1, τ_2)	A bitopological space in which the set $Y \neq \phi$ is assigned unique topologies τ_1 and τ_2.
$(Y, \tau_1 \vee \tau_2)$	A topological space in which the set $Y \neq \phi$ is assigned the topology $\tau_1 \vee \tau_2$ generated by basis $\tau_1 \cup \tau_2$.
Z^Y	Collection of all functions mapping set Y to the set Z.
$C(Y,Z)$	Collection of all continuous functions mapping topological space Y to the topological space Z.
$s-C(Y,Z)$	Collection of all continuous functions mapping $(Y, \tau_1 \vee \tau_2)$ to $(Z, \delta_1 \vee \delta_2)$.
$p-C(Y,Z)$	Collection of all continuous functions mapping (Y, τ_1, τ_2) to (Z, δ_1, δ_2).
$d-C(Y,Z)$	Collection of all continuous functions mapping $(Y, \tau_1 \wedge \tau_2)$ to $(Z, \delta_1 \wedge \delta_2)$.
$1-C(Y,Z)$	Collection of all continuous functions mapping (X, τ_1) to (Y, δ_1).
$2-C(Y,Z)$	Collection of all continuous functions mapping (X, τ_2) to (Y, δ_2).
$(1,2)-C(Y,Z)$	Collection of all continuous functions mapping (X, τ_1) to (Y, δ_2).
$(2,1)-C(Y,Z)$	Collection of all continuous functions mapping (X, τ_2) to (Y, δ_1).
τ_{co}	Compact-open topology or k-topology.
τ_p	Point-open topology or topology of pointwise convergence.
$C_\tau(Y,Z)$	A function space in which a topology τ is defined on the set $C(Y,Z)$.
$\tau(E)$	The class of all topologies that can be generated from the non empty set E.
$\bigvee_\alpha \tau_\alpha$	The "join" of topologies τ_α; it is a topology whose basis is

	the set $\tau_1 \cup \tau_2$.
$\bigwedge_\alpha \tau_\alpha$	The "meet" of topologies τ_α; it is the topology generated from the intersection of all τ_α topologies.
$\mathcal{O}_Z(Y)$	The collection of open sets in the space Y for every open set in the space Z.
Directed system Δ	A partially ordered system with the property that for any $\mu, \mu^{'} \in \Delta$, there exist a $\mu^{"} \in \Delta$ with $\mu^{"} \geq \mu$ and $\mu^{"} \geq \mu^{'}$ (Birkhoff, 1948).
Even continuity	A family F of mappings of X to Y is evenly continuous if for every $x \in X$, every $y \in Y$ and any neighbourhood V of y, there exists a neighbourhood U of x and a neighbourhood W of y such that $e[(F \cap M(\{x\}, W)) \times U] \subset V$ (Engelking, 1989).
Δ-directed set	A function on a directed system Δ with values in the space Y and it is also denoted by $\{y_\mu\}_{\mu \in \Delta}$ (Birkhoff, 1948).
Generalized topology	Let Y be a non empty set. A collection τ of subsets of Y is called a generalized topology on Y, if $\phi \in \tau$ and τ is closed under arbitrary union (Csaszar, 2002).
Jointly continuous	A topology τ on the set $C(Y,Z)$ is jointly continuous if the function $e : C_\tau(Y,Z) \times Y \to Z$ is continuous (Kelley, 1955).
Lattice	A partly ordered set P any two of whose elements for example, x and y, have a g.l.b. or 'meet' $(x \wedge y)$, and l.u.b. or 'join' $(x \vee y)$. Let E consist of the subgroups of any group, and let inclusion mean set-inclusion, then the terms 'join' and 'meet' have their usual meaning of union and intersection (Birkhoff, 1948).
N-open set	A set A of the space (Y, τ_1, τ_2) is called an "N-open set", if and only if it is open in the space $(Y, \tau_1 \vee \tau_2)$, where $\tau_1 \vee \tau_2$ is the supremum topology on Y containing τ_1 and τ_2 (Jabbar and Nasir, 2010).

N-compactness	A space (Y, τ_1, τ_2) is said to be an "N-compact space", if and only if every N-open cover of Y has a finite subcover (Jabbar and Nasir, 2010).
Open function	A function $f : X \to Y$ is called an open function, if the image of every open set in X is open in Y.
Tube lemma	Consider the product space $X \times Y$, where Y is compact. If N is an open set of $X \times Y$ containing the slice $x_\circ \times Y$ of $X \times Y$, then N contains some tube $W \times Y$ about $x_\circ \times Y$, where W is a neighbourhood of $x_\circ \in X$ (Munkres, 2000).
Totally disconnected	A bitopological space (Y, τ_1, τ_2) is said to be totally disconnected if for every two distint points x and y, there exist a disconnection $Y = A\|B$ with $x \in A$ and $y \in B$
Submap	If $G \subset Y$ and $H \subset Z$ and $f : Y \to Z$, then the function $f\|_{G,H} : G \to H$ is called a submap provided $f(G) \subset H$.

CHAPTER ONE

INTRODUCTION

1.1 Background information

The study of function spaces dates back to the nineteenth century. It came about from the need to study convergence of sequences of functions. Function spaces have since been incorporated in various areas of mathematics. In set theory, the set Z^Y of all mappings from the set Y to the set Z is considered. In general topology, a topology such as compact-open topology or point-open topology can be assigned to a collection of continuous functions defined on a topological space Y to a topological space Z. In algebraic topology, the study of homotopy theory is essentially that of discrete invariants of function spaces. In linear algebra, the set of all linear transformations mapping a vector space U to a vector space V over the same field, forms a vector space. These vector spaces when defined over a topological field form topological vector spaces, which are studied in functional analysis as Hilbert spaces and Banach spaces.

Let Z^Y denote the set of all the functions from a topological space Y to a topological space Z. Suppose Y is a finite set consisting of the elements $y_1, y_2, y_3, y_4 ...., y_n$ and has a discrete topology, then Z^Y is the collection of all functions mapping each element of the space Y to the space Z. Each of such functions can be considered as an n-tuple of points in Z. That is $(f(y_1), f(y_2), f(y_3)......, f(y_n))$ can be viewed as a set of points in Z (Belk, 2015). The set Z^Y can therefore be associated with the Cartesian power $Z^n = Z \times Z \times Z..... \times Z$ $(n\ times)$. Generally, Z^Y can be expressed as $\prod_{y \in Y} Z_y$. If the collection of all the functions mapping Y to Z are continuous, then such a collection is represented by the notation $C(Y,Z)$, which forms the function space $C_\tau(Y,Z)$ when a topology τ is assigned to it (Georgiou et al., 1996).

Different topologies are defined on the set $C(Y,Z)$ to form the function space $C_\tau(Y,Z)$. One of the common topology defined on the set $C(Y,Z)$ is the topology (τ_p) of pointwise convergence, this topology is defined by the subbasis $S(y,V) = \{g \in C(Y,Z) : g(y) \in V\}$, where $y \in Y$ and V is open in the space Z (Kelley, 1955). It borrows its name from pointwise convergence of a sequence of points, only that in this case, a sequence of functions is considered. In product spaces, topologies are defined either by using product of open sets (box topology) or by using inverses of projection mappings (product topology). Suppose U and V are any two open sets in Y and Z respectively, then $U \times V$ form the basis for box topology on $Y \times Z$. If $\pi_1 : Y \times Z \to Y$ and $\pi_2 : Y \times Z \to Z$ are projection mappings, then $\pi_1^{-1}(U) = (U \times Z)$ and $\pi_2^{-1}(V) = (Y \times V)$

are open sets in $Y \times Z$. The subbasis for the space $Y \times Z$ generated by the sets of the form $S = \pi_1^{-1}(U) \cup \pi_2^{-1}(V)$ is called product topology (Munkres, 2000). Suppose $Y = \{a, b\}$, then $Z^Y = Z \times Z$. If U is open in Z, then $S(a, U) = U \times Z = \pi_1^{-1}(U)$ and $S(b, U) = Z \times U = \pi_1^{-1}(U)$. The pre-image $\pi_1^{-1}(U)$ is an open set in the subbasis for $Z \times Z$, hence, topology of pointwise convergence is equivalent to product topology, but this only holds when Y is a finite set (Belk, 2015). If $\mathcal{B}$ is the subbasis for the topological space Z, then the subbasis for the topology of pointwise convergence on the set $C(Y, Z)$, constitutes sets of the form $S(y, B) = \{y \in Y, B \in \mathcal{B}\}$ (Dugundji, 1978). If the class $\{U_y\}_{y \in Y}$ of open sets is contained in the space Z, then the product $\prod_{y \in Y} U_y = \{f \in C(Y, Z) : f(y) \in U_y, \forall\ y \in Y\}$ is an open box contained in the set $C(Y, Z)$. The collection of all open boxes forms a basis for the box topology on the set $C(Y, Z)$. The product topology and box topology coincide when Y is a finite set. If $\mathcal{B}$ is the basis for a topological space Z, then the collection $\{\prod B_y : B_y \in \mathcal{B}, \forall\ y \in Y\}$ forms the basis for the box topology on Z^Y (Munkres, 2000).

The subbases for the set-open topologies defined on the set $C(Y, Z)$, consist of sets from both the space Y and the space Z. A good example of a set-open topology is the compact-open topology (τ_{co}), whose subbasis consist of sets of the form $S(U, V) = \{f \in C(Y, Z) : f(U) \subset V\}$, for U compact in the space Y and V open in the space Z (Fox, 1945). For the family $\mathcal{B}$ of sets forming the subbasis for the space Z, the collection $S(U, V) = \{U \subset Y, V \in \mathcal{B}\}$ of sets, forms the subbasis for the set-open topology on the set $C(Y, Z)$ (Dugundji, 1978). If U is open, then $S(U, V)$ forms the subbasis for open-open topology, if U is closed or bounded, then $S(U, V)$ forms a subbasis for closed-open topology or bounded-open topology respectively. Some of these topologies are equivalent under given conditions. For example, the topology of pointwise convergence is equivalent to the closed-open topology, whenever Y is a T_2 compact space. The topology of pointwise convergence is equivalent to compact-open topology, provided all compact subsets of the space Y are finite sets, or Y is a T_1 space. The compact-open topology is equivalent to the topology of pointwise convergence, provided that Y is a discrete space (Porter, 1993).

Topologies defined on function spaces are classified as either admissible or splitting, this is done by use of the notion of continuous functions defined on product spaces (Fox, 1945; Arens, 1946), or by use of "continuous convergence" of directed sets (generalized sequences) (Arens and Dugundji, 1951), or even by use of exponential functions as was done by Engelking (1989). A topology τ defined on the set $C(Y, Z)$ is said to be admissible, if whenever an open set W containing $f(y)$ in the space Z is given, there is an open set V containing y in the space Y and an open set U containing f in the space $C_\tau(Y,Z)$, such that $f \in U$ and $y \in V$ implies $f(y) \in W$. Explicitly, the definition above by Arens (1946) states that a topology τ is admissible, if the evaluation function $e : Y \times C(Y,Z) \to Z$ is continuous.

Let X be another topological space such that h maps the space $X \times Y$ at $y \in Y$ for each $x \in X$, to the space Z, and h^* maps the space X to the space $C_\tau(Y,Z)$. By defining the function ${h^*}_x = h(x)$, where ${h^*}_x(y) = h(x,y)$ for every $y \in Y$, one is bound to note that the functions h and h^* have a one-to-one correspondence either; when Y is regular and locally compact, or when the spaces X and Y satisfy the first countability axiom and the set $C(Y,Z)$ has compact-open topology (Fox 1945), or when $X \times Y$ is a k-space (Dugundji, 1978). Actually, the spaces $Z^{(X\times Y)}$ and $(Z^Y)^X$ have been shown by Engelking (1989) to be homeomorphic to one another. The continuity of the function h basically depends on the topologies defined on the spaces X, Y and Z, this is unlike the continuity of h^*, where only the topology defined on the set $C(Y,Z)$ come into play. Since the continuity of the evaluation mapping (e) can be used to show that the continuity of h^* implies the continuity of h, it follow that τ is also an admissible topology on the set $C(Y,Z)$ if and only if the continuity of h^* implies the continuity of h (Arens and Dugundji, 1951). A topology τ is also termed splitting, if the continuity of the function h implies the continuity of the function h^*. These definitions of admissible and splitting topologies coincides with those defined using also generalized sequences by Arens and Dugundji (1951), or those defined using exponential functions by Engelking (1989).

At any one time, there is at most one proper admissible topology. Such topology is both the greatest splitting topology and least admissible topology, and it is called splitting-admissible topology (Arens and Dugundji, 1951). This topology is the intersection of all topologies defined on the set $C(Y,Z)$ that satisfy the admissibility property, or the union of all the topologies satisfying the splitting property (Georgiou et al., 2007). The k-topology will always satisfy admissibility property, provided that the space Y is regular and locally compact. If no restrictions are attached to the spaces Y and Z, then the k-topology will satisfy the splitting property (Fox, 1945). An admissible topology will always be finer than a splitting topology, any topology finer than admissible topology will always be an admissible topology, and any topology courser than a splitting topology will always be a splitting topology (Engelking, 1989). A good example of admissible topology is the discrete topology, while a good example of splitting topology is the indiscrete topology.

Birkhoff (1948) defines a lattice to be a partially ordered set P for which any two elements belonging to it have a greatest lower bound (g.l.b) or ”meet” denoted by $\bigwedge$, and a least upper bound (l.u.b) or ”join” denoted by $\bigvee$. If E is any non empty set, then the class $\tau(E)$ of all topologies of E forms a lattice. The lattice $\tau(E)$ of all splitting topologies will always satisfy the splitting property, while the topology $\bigvee_\alpha \tau_\alpha$ is the greatest splitting topology (Arens and Dugundji, 1951). Any k-topology is the greatest splitting topology either when Y is a completely regular space and Z is any arbitrary space, or when Z is a metric space containing non-degenerate arc. The fact that $\bigwedge_\alpha \tau_\alpha$ is courser than any τ_α, makes $\tau(E)$ fail to form a lattice of admissible topologies as shown by Arens and Dugundji (1951).

For arbitrary families $\mathcal{A}$ of spaces X and $\mathcal{A}_\circ$ of spaces $X_\circ$ subspaces of X, splitting and admissible topologies have been defined on the set $C(Y,Z)$ with respect to these arbitrary families. These topologies are $(\mathcal{A},\mathcal{A}_\circ)$-splitting and $(\mathcal{A},\mathcal{A}_\circ)$-admissible respectively. Splitting and admissible topologies have further been shown to coincide with $(\mathcal{A},\mathcal{A}_\circ)$-splitting and $(\mathcal{A},\mathcal{A}_\circ)$-admissible topologies respectively (Georgiou, 2009). If τ_1 and τ_2 are any two topologies defined on the set $C(Y,Z)$ with respect to $(\mathcal{A},\mathcal{A}_\circ)$, such that $\tau_1 \subset \tau_2$ and τ_2 is $(\mathcal{A},\mathcal{A}_\circ)$-splitting, then τ_1 is also $(\mathcal{A},\mathcal{A}_\circ)$-splitting, and if τ_1 is $(\mathcal{A},\mathcal{A}_\circ)$-admissible, then τ_2 is $(\mathcal{A},\mathcal{A}_\circ)$-admissible (Georgiou, 2009). Generalization of splitting and admissible topologies has also been extended to function spaces defined on generalized topological spaces by Gupta and Sarma (2015). In particular, it has been shown that a topology τ defined on the set $C(Y,Z)$, whose subbasis consist of sets from the generalized topology of Y and Z, is typically admissible.

The concepts of jointly continuous and evenly continuous as defined by Kelley (1955), have a relation with the evaluation function defined on the space $C_\tau(Y,Z)$. A topology τ defined on the set $C(Y,Z)$ is termed jointly continuous by Kelley (1955), if the evaluation function $e: C_\tau(Y,Z) \times Y \to Z$ is continuous. Thus, any admissible topology is jointly continuous and so is any other topology finer than the admissible topology. The set F subset of $C(Y,Z)$ is termed evenly continuous, if $e[(F \cap S(\{y\},W)) \times U] \subset V$ (Engelking, 1989). Therefore, a topology τ defined on F is jointly continuous or admissible, if F is evenly continuous. The concept of jointly continuous and evenly continuous are important in proving compactness for a closed subset F of $C_\tau(Y,Z)$ in Ascoli theorem (Kelley, 1955).

Generally, for any set-open topology defined on the set $C(Y,Z)$, the topological properties of the space Y and Z interact with those of the space $C_\tau(Y,Z)$. These interactions are what makes the study of function spaces interesting. In particular, emphasis is given on topological properties of the set $C(Y,Z)$ that can be deduced from those of the spaces Y and Z. For example, the separation axioms T_0, T_1, T_2, regular and completely regular, defined on the space $C_\tau(Y,Z)$, where τ is a compact-open topology, solely depends on the space Z having the same separation properties (Arens, 1946).

Let Y be a non empty set, two topologies τ_1 and τ_2 can be defined on Y to form a bitopological space denoted by (Y,τ_1,τ_2) (Kelly, 1963). An investigation of a set with two different topologies defined on it, makes it possible on some occasions to obtain a combined effect, that is, to get more information than one would acquire if one considered the same set with each topology separately. Good examples are the generalized separation axioms defined by Kelly (1963), which comprise of; pairwise Hausdorff, pairwise regular, pairwise completely regular and pairwise normal spaces. In general the pattern of these axioms involves mixing of the two topologies of the bitopological space in a certain way. For example, (Y,τ_1,τ_2) is said to be pairwise Hausdorff, if two unique points can be separated by disjoint τ_1-open and τ_2-open set. The space (Y,τ_1,τ_2) is pairwise regular, if a τ_1-closed (τ_2-closed) set and a point not belonging to this set, can be separated by disjoint τ_2-open (τ_1-open) and τ_1-open (τ_2-open) sets, and finally pairwise normal, if disjoint τ_1-closed and τ_2-closed sets can be separated by disjoint τ_2-open and τ_1-open sets respectively. Other combined effects include; pairwise open cover by Fletcher et al. (1969), pairwise compactness by Swart (1971) and Kim (1968) and pairwise connectedness by Pervin (1967), among others.

The concept of continuity of functions in topological spaces has also been extended to bitopological spaces. The function $f:(X,\tau_1,\tau_2)\to(Y,\mu_1,\mu_2)$ is defined to be pairwise continuous (resp. pairwise open, pairwise closed), if the functions $f:(X,\tau_1)\to(Y,\mu_1)$ and $f:(X,\tau_2)\to(Y,\mu_2)$ are continuous (resp. pairwise open, pairwise closed) (Pervin, 1970). The function f is also defined by Dvalishvili (2005) to be (i,j)-continuous, if the function $f:(Y,\tau_i)\to(Z,\delta_j)$ is continuous for $i,j\in\{1,2\}$, $i\neq j$. The function $f:(Y,\tau_1,\tau_2)\longrightarrow(Z,\delta_1,\delta_2)$ is defined to be pairwise$_1$-continuous (p_1-continuous) by Tallafha et al. (1999), if it is both (i,j)-continuous and (j,i)-continuous. The function f is also defined to be a p-homeomorphism, if f is a bijection, p-continuous and f^{-1} is also p-continuous (Dvalishvili, 2005). The concept of p-continuous and p-homeomorphism has been used to generalize topological properties to bitopological spaces.

It is a general observation that the study of function spaces in particular the space $C_\tau(Y,Z)$, has generally concentrated on topologies defined on it and properties of these topologies, as well as conditions under which such properties of topologies hold. Separation axioms as well as compactness have also been studied on the function space $C_\tau(Y,Z)$. While studies of topological concepts such as connectedness, separation axioms and compactness have also been done on bitopological spaces, it would be interesting to observe how properties studied on function spaces would vary when generalized to a topologized set of continuous functions between two bitopological spaces.

1.2 Research problem

The study of function spaces on topological spaces has generally concentrated on topologies defined on set of continuous functions between topological spaces, properties of those topologies, as well as conditions under which such properties hold. Separation axioms and compactness have also been studied in function spaces. In bitopological spaces, generalized topological properties and how such properties interact with the topological properties in individual spaces are explored. There is need to check whether properties of function spaces as well as bitopological spaces can be generalized to sets of continuous functions between two bitopological spaces. It would also be interesting to observe how these generalized properties compare with known results both in function spaces and bitopological spaces.

1.3 Objectives

1.3.1 General objective

This work investigates properties of function spaces and bitopological spaces on sets of s and p continuous functions defined on bitopological spaces, and how they relate to properties of sets of continuous functions defined on topological spaces, as well as to properties of bitopological spaces.

1.3.2 Specific objectives

The specific objectives of this study were to;

(i) Define function spaces $p-C_{\omega}(Y,Z)$, $s-C_{\tau}(Y,Z)$, $(1,2)-C_{\varphi}(Y,Z)$ and $(2,1)-C_{\xi}(Y,Z)$, and establish the relationship among them, as well as with the spaces $1-C_{\varsigma}(Y,Z)$ and $2-C_{\zeta}(Y,Z)$.

(ii) Establish the relationship between p-splitting (p-admissible) and s-splitting (s-admissible) topologies defined on the spaces $p-C_{\omega}(Y,Z)$ and $s-C_{\tau}(Y,Z)$ respectively, and splitting and admissible topologies defined on $1-C_{\varsigma}(Y,Z)$ and $2-C_{\zeta}(Y,Z)$.

(iii) Investigate separation axioms on the spaces $p-C_\omega(Y,Z)$ and $s-C_\tau(Y,Z)$, and how they relate to the separation axioms defined on the spaces (Z,δ_i) for $i=1,2$, (Z,δ_1,δ_2), $1-C_\varsigma(Y,Z)$ and $2-C_\zeta(Y,Z)$ and also on $s-C_\tau(Y,Z)$.

(iv) Investigate compactness of closed subsets of $1-C_\varsigma(Y,Z)$ and $2-C_\zeta(Y,Z)$ in the space $s-C_\tau(Y,Z)$.

1.4 Significance of the study

The theory of function spaces cuts across different areas of mathematics. In set theory, the set Z^Y consisting of functions from the set Y to the set Z is studied. In algebraic topology, homotopy theory is studied and deals with invariants in function spaces. In category theory, function spaces are simply exponential objects. In topology, function spaces give a frame work in which convergence of sequences of functions can be studied. All the studies above converge in functional analysis, which basically deals with formulation of properties of transformations of functions defined on some set to $\mathbb{R}$ or $\mathbb{C}$. While continuous functions have applications in computer aided design systems, geo information systems and building information models through establishing links between structured collections of "topological primitives" like edges, vertices, faces, and volumes, other use of functions especially when studied in algebraic topology is in modelling. One such modelling has been done on human connectome by Sizemore et al. (2018). Some Hilbert spaces and Banach spaces are function spaces and have been used in mathematical formulation of quantum mechanics. This study like many others, enriches the field of function spaces with new concepts that could be applied in the field of functional analysis, as well as algebraic topology.

CHAPTER TWO

LITERATURE REVIEW

2.1 Function spaces

Geometry beyond the Euclidean spaces $\mathbb{R}^n$ was advanced by Italian geometers such as Ascoli, Arzel'a and Hadamard, part of their work entailed the study of topology of pointwise convergence and the topology of uniform convergence on function spaces. It was Frechet (1906) who expounded further the concept of function spaces and distinguished it from the calculus of variations. Two important types of generalized spaces were isolated, the L-spaces, where the notion of limit was based on an axiomatization of convergent sequences, and the L-spaces on which a distance function could be defined. Frechet (1906) doctoral dissertation considered a distance function $d : X \times X \to \mathbb{R}^+$ between any two general objects x and y of a given set X such that they satisfied four axioms namely; $d(x,y) \geq 0$, $d(x,y) = 0$ if and only if $x = y$, $d(x,y) = d(y,x)$, and $d(x,y) \leq d(x,z) + d(z,y)$, $\forall$ x, y and z in the set X. These postulates culminated into the concept of supremum metric topology. The concept of function spaces on topological spaces was picked up and expounded further by Fox (1945) who used compact-open topology to compare continuity of the functions $h : X \times Y \to Z$, for each fixed $x \in X$, and the continuity of the function $h^* : X \to C_\tau(Y,Z)$. Fox (1945) was more concerned on other properties of the space Y other than local compactness, for which the function h^* would be continuous and correspond to the continuous function h. Eventually, Fox (1945) was able to show that by restricting the range of the space X, replacing the condition of local compactness on the space Y with the first countability axiom and introducing compact-open topology on the set $C(Y,Z)$, continuity of the function h^* implied continuity of the function h. This essentially meant that there existed an onto function ϕ from the set $C((X \times Y),Z)$ to the set $C(X, C_\tau(Y,Z))$ as was shown by Morita (1956). The function ϕ was also proved by Jackson (1951) and Brown (1964), to be a homeomorphism, provided that the spaces X and Y were both locally separable, or that the space Y was regular and locally compact.

One common property of topology τ defined on the set $C(Y,Z)$ was admissibility. Arens (1946) proved the property of admissibility using convergence of sequence of functions contained in the set $C(Y,Z)$. In particular, it was shown that when the space Y was locally compact and τ was a k-topology defined on the set $C(Y,Z)$, the sequence f_n converged to the function f, if and only if the sequence $f_n(y_n)$ converged to the image $f(y)$, whenever y_n converged to y. This convergence implied that the k-topology τ was admissible and the strongest admissible topology for which the evaluation function $e : C_\tau(Y,Z) \times Y \to Z$ was continuous. Evaluation function was also formulated by Engelking (1989) using composition of continuous functions. Engelking (1989) showed that if $id_{Y^X} \times i_X : Y^X \times X \to Y^X \times X^{\{p\}}$, $\Sigma : Y^X \times X^{\{p\}} \to Y^{\{p\}}$ and $i_Y : Y \to Y^{\{p\}}$ were continuous functions, then the evaluation function $e : Y^X \times X \to Y$ was equivalent to the composite function ${i_Y}^{-1} \circ \Sigma \circ (id_{Y^X} \times i_X)$. It is easily seen that there is a correlation between the definition of admissible topology using evaluation function and using functions defined on Cartesian product. For if X is replaced with $C_\tau(Y,Z)$, the function h^* mapping X to $C_\tau(Y,Z)$, becomes an identity function whose associated function h, is the evaluation function defined on the Cartesian product $C_\tau(Y,Z) \times Y$. Topology τ defined on the set $C(Y,Z)$ was also characterized as splitting topology conversely to the characterization of admissible topology done by Arens and Dugundji (1951).

The concepts of splitting and admissibility on topologies defined on the set $C(Y,Z)$ have also been proven using convergence and continuous convergence of directed sets. Arens and Dugundji (1951) observed that a topology τ was admissible on the set $C(Y,Z)$, if the sequence f_n viewed as a directed set of functions in $C(Y,Z)$ converging continuously to f in $C(Y,Z)$, the sequence $f_n(y_n)$ converged to $f(y)$, whenever y_n converged to y. It was also observed that a topology τ satisfied splitting property, if for every directed system Δ and Δ-directed set $\{f_\mu\}$ in $C(Y,Z)$, the continuous convergence of the sequence f_μ to f implied convergence of f_μ to f with respect to τ. The property of splitting topology was also shown by Arens and Dugundji (1951) to hold when continuous convergence of the sequence f_μ to f, implied convergence of the sequence f_μ to f in the space $C_\tau(Y,Z)$, and for the continuous function $h : X \times Y \to Z$. Arens and Dugundji (1951) also proved that the property of admissibility was satisfied, if the convergence of f_μ to f in $C(Y,Z)$, implied continuous convergence of f_μ to f. One interesting observation noted by the duo was that the k-topology defined on the set $C(Y,Z)$, where Y was a regular locally compact space and Z was an arbitrary space, was an acceptable topology (both splitting and admissible). A comparative analysis of topologies given to $C(Y,Z)$ carried out by Arens and Dugundji (1951) showed that; a topology coarser than splitting topology is splitting, a topology finer than admissible topology is admissible and a splitting topology is coarser than an admissible topology.

By considering arbitrary collection $\{t_\alpha\}$ of splitting topologies as a lattice as defined by Birkhoff (1948), Arens and Dugundji (1951) were able to show that $\bigwedge_\alpha t_\alpha$ and $\bigvee_\alpha t_\alpha$ were also splitting topologies. The case was different for arbitrary collection of admissible topologies, even though admissible topologies under the operation of "join" resulted into another admissible topology. This vaguely implied that the union and intersection of splitting topologies was also splitting but only the union of admissible topologies was an admissible topology.

The relation between continuous functions h and h^* extensively studied by Fox (1945) and Arens and Dugundji (1951), can best be explained using exponential function ϕ studied by Morita (1956) and Jackson (1951). If the continuity of h^* is implied by that of h, then $\phi : C((X \times Y), Z)) \to C(X, C(Y,Z))$ is easily seen to be a continuous function (Engelking, 1989). Therefore, a topology τ satisfies splitting property, if for the mapping $\phi : C((X \times Y), Z)) \to C(X, C(Y,Z))$, every space X, and for an $f \in C((X \times Y), Z)$, the mapping $\phi(f)$ belongs to $C(X, C(Y,Z))$. That is, if $f \in C((X \times Y), Z)$, then $\phi(f)(x) \in C(Y,Z)$ implying that $\phi((C(X \times Y, Z))) \subset C(X, C_\tau(Y,Z))$. Conversely, τ is said to be admissible topology, if for every space X and any $g \in C(X, C(Y,Z))$, the mapping $\phi^{-1}(g)$ belongs to $C((X \times Y), Z))$, that is,
$\phi(C(X, C_\tau(Y,Z))) \subset (C(X \times Y, Z))$ (Engelking, 1989).

Nets have also been used to characterize the concept of splitting and admissible topologies in function spaces. McCoy and Ntantu (1988) observed that the net $\{f_i\}$ in the set $C(X, \mathbb{R})$ converged to f in $C(X, \mathbb{R})$, if and only if for each $x \in X$, $\{f_i(x)\}$ converged to $f(x) \in \mathbb{R}$. A topology τ on $C(X, \mathbb{R})$ was then shown to be splitting, if and only if whenever $\{f_i\}$ was a net in $C(X, \mathbb{R})$ converging to $f \in C(X, \mathbb{R})$ continuously, then $\{f_i\}$ converged to $f \in C_\tau(X, \mathbb{R})$. A topology τ was also shown to be admissible if whenever $\{f_i\}$ was a net in $C(X, \mathbb{R})$ converging to $f \in C(X, \mathbb{R})$, then $\{f_i\}$ converged continuously to $f \in C_\tau(X, \mathbb{R})$. By viewing a directed system as a net, the concepts of splitting and admissibility as captured by Arens and Dugundji (1951) relates to the above concepts by McCoy and Ntantu (1988).

For the space Z satisfying the separation axioms T_0, T_1, T_2, regular and completely regular, Arens (1946) showed that the space $C_\tau(Y,Z)$ where τ was a compact-open topology, inherited the same separation axioms. The above case failed to hold when Z was a normal space, this was because the set $C(Y,Z)$ was homeomorphic to the product $\prod_{y \in Y} Z_y$ where each Z_y was a copy of Z, and the product of normal spaces need not be normal.

The topology of pointwise convergence is equivalent to the subspace topology defined on the set $C(Y,Z)$ generated by Tychonoff topology on Z^Y, and solely depended on the topology of the space Z (Willard, 1970). This topology was shown by Kelley (1955) to be the least topology for which the evaluation function was continuous. It is worth noting that if $p_y : Z^Y \to Z$ is a continuous mapping, then $Z^Y \bigcap_{i=1}^{n} {p_{y_i}}^{-1}(U) = M(y_i, U)$ for U open in Z, is the subbasis for the topology of pointwise convergence on Z^Y (Engelking, 1989). One importance of topology of pointwise convergence, is that it facilitates compactness for subsets of function spaces to be defined. Kelley (1955) showed that for a set $\mathcal{F} \subset C(Y,Z)$ to be compact relative to topology of pointwise convergence, it was sufficient that the set $\mathcal{F}$ be pointwise closed in the space $C_\tau(Y,Z)$, and for the set $\mathcal{F}(y)$ to have a compact closure for each point y in Y. Kelley (1955) showed that the concept of even continuity and joint continuity were related by the statement; $\mathcal{F}$ is evenly continuous if and only if for each $y \in Y$ and $z \in Z$ and for each open set U containing z, there are open sets V containing y and W containing z such that $\{f : f \in \mathcal{F} \text{ and } f(y) \in W\} \times V$ is carried into U by a natural map. The natural map here is simply the evaluation function. Using the above statement relating even continuity and jointly continuity, Kelley (1955) was able to show that, for a pointwise topology τ defined on the set $\mathcal{F}$ of evenly continuous functions, mapping a normal space X to a regular space Y, the set $\overline{\mathcal{F}}$ was also evenly continuous, and the topology τ was jointly continuous on $\overline{\mathcal{F}}$.

The concept of jointly continuous topology and even continuity plays a critical role in the definition of compactness on subsets of function spaces, which forms the basis for Arzela-Ascoli theorem. Kelley (1955) showed that a subset $\mathcal{F}$ of $C_\tau(Y,Z)$, for a regular locally compact space Y, a regular space Z and compact-open topology τ, was compact if and only if; $\mathcal{F}$ was closed in $C_\tau(Y,Z)$, the closure of $\mathcal{F}(y)$ was compact for each $y \in Y$, and $\mathcal{F}$ was evenly continuous. The subset $\mathcal{F}$ of $C(Y,Z)$ was also shown to be evenly continuous, if it was compact relative to the point-open topology.

For a singleton set X, Dugundji (1978) showed that the continuous mapping $T : C(X,Y) \times C(Y,Z) \to C(X,Z)$ was simply the evaluation function $\omega : C(Y,Z) \times Y \to Z$ defined by $\omega(f,y) = f(y)$. Dugundji (1978) also proved that for the spaces X, Y and Z, where $A \subset X$, $B \subset Y$ were compact and V open subset of the space Z, the class of sets $(A \times B, V)$ formed the subbasis for k-topology defined on the space $C((X \times Y), Z)$. Using this subbasis, it was possible for Dugundji (1978) to provide an alternative proof of homeomorphism between the spaces $C((X \times Y), Z)$ and $C(X, C(Y,Z))$, different from that of Jackson (1951) and Brown (1964).

Recent studies in function spaces defined on topological spaces have mainly concentrated on generalization of properties of topologies as well as characterization of compactness on function space $C_\tau(Y,Z)$. Georgiou (2009) has extended the concept of splitting and admissible properties of topologies defined on the set $C(Y,Z)$ to the family $(\mathcal{A}, \mathcal{A}_\circ)$ of the spaces $(X, X_\circ)$, where $X_\circ$ is a subspace of X. In particular a topology τ on $C(Y,Z)$ is defined to be $(\mathcal{A},\mathcal{A}_\circ)$-splitting, if for every $(X,X_\circ) \in (\mathcal{A},\mathcal{A}_\circ)$, the continuity of the map $g : X \times Y \to Z$ implied the continuity of the map $g^*|_{X_\circ} : X_\circ \to C_\tau(Y,Z)$, where $g^* : X \to C_\tau(Y,Z)$. Georgiou (2009) also defined a topology τ on $C(Y,Z)$ to be $(\mathcal{A},\mathcal{A}_\circ)$-admissible, if for every $(X,X_\circ) \in (\mathcal{A},\mathcal{A}_\circ)$, the continuity of the map $h : X \to C_\tau(Y,Z)$ implied the continuity of the map $h^*|_{X_\circ \times Y} : X_\circ \times Y \to Z$, where $h^* : X \times Y \to Z$. With these definitions, Georgiou (2009) was able to show that, if a topology τ defined on $C(Y,Z)$ was splitting (admissible), then it was also $(\mathcal{A},\mathcal{A}_\circ)$-splitting (admissible). Gupta and Sarma (2015, 2017), also generalized splitting and admissibility properties of topologies, to topologies on function spaces defined on generalized topological spaces, as well as to topologies on sets of continuous multifunctions from the space Y to the space Z, using continuous convergence of generalized nets and nets respectively. Bartch and Pope (2011) have characterized compactness on $\mathcal{H}$ subset of $C(Y,Z)$ using subsets of the space Y and set-open topologies defined on the set $C(Y,Z)$, this was made possible by considering the set $C(A,Z)$ where A was a subset of the space Y and ensuring that $\mathcal{H}$ was evenly continuous for every A in the set-open topology. The proof of compactness obtained was similar to the one earlier obtained by Bartch (2004) using the concept of hyperspaces. To help expound further on the theory of function spaces, there is need to consider function spaces defined on more richer spaces than those considered by Georgiou (2009) and Gupta and Sarma (2015). One such space is the bitopological space introduced by Kelly (1963).

2.2 Bitopological spaces

Concepts in topological spaces such as separation axioms, compactness, connectedness and continuity among other, have been generalized to bitopological spaces and new results obtained. Such results include pairwise separation axioms, pairwise compactness and pairwise continuity.

In the work of Reilly (1972), it was shown that pairwise T_2 implied pairwise T_1 which implied pairwise $T_\circ$. Pairwise T_1 was shown to imply T_1 in both the spaces (Y,τ_1) and (Y,τ_2). In Reilly's (1972) work, it was also shown that if either the space (Y,τ_1) or (Y,τ_2) was $T_\circ$, then the space (Y,τ_1,τ_2) was pairwise $T_\circ$, and if either the space (Y,τ_1) or (Y,τ_2) was T_1, then the space (Y,τ_1,τ_2) was pairwise $T_\circ$. Pairwise $T_\circ$ in the space (Y,τ_1,τ_2) did not necessarily imply that (Y,τ_1) or (Y,τ_2) was a $T_\circ$ space.

The 'topologicalness' of various separation axioms in bitopological spaces has been extensively covered by Lal (1978). In the work, it was shown that the space (Y,τ_1,τ_2) was pairwise $T_\circ$, if and only if the space $(Y, \tau_1 \vee \tau_2)$ was $T_\circ$, and also if the spaces (Y, τ_1) and (Y, τ_2) were both $T_\circ$. The space (Y, τ_1, τ_2) was shown to be pairwise T_1, if the spaces (Y, τ_1) and (Y, τ_2) were both T_1, and also if the space $(Y, \tau_1 \vee \tau_2)$ was T_1. Lal (1978) also proved that the space $(Y, \tau_1 \vee \tau_2)$ was T_2, if the space (Y, τ_1, τ_2) was pairwise T_2, and regular if the space (Y, τ_1, τ_2) was pairwise regular.

The concept of compactness in bitopological spaces was first introduced by Kim (1968), Kim (1968) defined the space (Y, τ_1, τ_2) to be $(1,2)$-compact, if $\tau_1(V) = \{\phi, X\} \cup \{U \cup V : U \in \tau_1\}$ was compact for every non empty set V of τ_2. If the space Y was both $(1,2)$-compact and $(2,1)$-compact, then Y was said to be K-compact. Reilly (1972) noted that the space (Y, τ_1, τ_2) was pairwise locally compact, if τ_1 was locally compact with respect to τ_2 and τ_2 was locally compact with respect to τ_1. Swart (1971) observed that pairwise compactness in the space (Y, τ_1, τ_2) implied compactness in the space $(Y, \tau_1 \vee \tau_2)$. Lal (1978) noted that the space (Y, τ_1, τ_2) was pairwise compact, if and only if the space $(Y, \tau_1 \vee \tau_2)$ was compact, and only if the spaces (Y, τ_1) and (Y, τ_2) were compact. This was an extension of Mrsevic and Reilly (1996) work, which compared compactness as defined by Swart (1971) and compactness as defined by Fletcher et al. (1969). Jabbar and Nasir (2010) introduced the concept of N-open sets in space (Y, τ_1, τ_2) and using this concept, defined N-compactness.

CHAPTER THREE

CONTINUITY OF FUNCTIONS ON FUNCTION SPACES DEFINED ON BITOPOLOGICAL SPACES

3.1 Introduction

In this chapter, the spaces $(Y, \tau_1 \vee \tau_2)$ and (Y, τ_1, τ_2) are considered, and s-continuous and d-continuous function defined on them. Relationships among p-continuity, s-continuity, p_1-continuity and d-continuity are also established. In subsequent section, the spaces $s - C_\tau(Y,Z)$, $1 - C_\varsigma(Y,Z)$, $2 - C_\zeta(Y,Z)$, $(1,2) - C_\varphi(Y,Z)$ and $(2,1) - C_\xi(Y,Z)$ are defined, and continuous functions between any two of them established. A homeomorphism is also established between the spaces $1 - C_\varsigma(Y,Z)$ and $(2,1) - C_\xi(Y,Z)$.

For the bitopological space (Y, τ_1, τ_2), topologies τ_1 and τ_2 when combined can form a base or a subbasis for a topology defined on set Y. The notation $\tau_1 \vee \tau_2$ in this work will imply topology defined on the set Y generated by the basis $\tau_1 \cup \tau_2$. Such topology will be called supremum topology. The set Y together with its supremum topology $\tau_1 \vee \tau_2$ will be denoted by $(Y, \tau_1 \vee \tau_2)$. The notation $\tau_1 \wedge \tau_2$ will imply topology generated by the basis $\tau_1 \cap \tau_2$, thus the basis $\tau_1 \cap \tau_2$ for the topology $\tau_1 \wedge \tau_2$ is simply the topology $\tau_1 \wedge \tau_2$.

3.2 Continuous functions defined on bitopological spaces

In this section, s-continuous and d-continuous functions are introduced, relationships among s-continuous functions, p-continuous functions, p_1-continuous functions and d-continuous functions are studied.

The following definitions will be important in proving subsequent theorems.

Definition 3.2.1. *(Muturi N. E. et al., 2017). Subset A of a bitopological space $(Y, \tau_1 \vee \tau_2)$ is a supremum-open set or simply s-open set, if $A = V_1 \cup V_2$, where $V_1 \in \tau_1$ and $V_2 \in \tau_2$.*

Definition 3.2.2. *(Muturi N. E. et al., 2017). A function $f : (Y, \tau_1 \vee \tau_2) \longrightarrow (Z, \delta_1 \vee \delta_2)$ is s-continuous, if the inverse image of each s-open subset of Z is s-open in Y.*

Definition 3.2.3. *(Muturi N. E. et al., 2017). A function $f : (Y, \tau_1 \wedge \tau_2) \longrightarrow (Z, \delta_1 \wedge \delta_2)$ is double-continuous (d-continuous), if for every $V \in \delta_1 \cap \delta_2$, $f^{-1}(V) \in \tau_1 \cap \tau_2$.*

Theorem 3.2.4. *(Muturi E. N. et al., 2017). The function $f : (Y, \tau_1 \vee \tau_2) \longrightarrow (Z, \delta_1 \vee \delta_2)$ is s-continuous, if the function $f : (Y, \tau_1, \tau_2) \longrightarrow (Z, \delta_1, \delta_2)$ is p-continuous.*

Proof. Let the function f be p-continuous, and let $U_1 \in \delta_1$ and $U_2 \in \delta_2$ such that $U_1 \cup U_2 \in \delta_1 \vee \delta_2$. Since f is p-continuous, then $f^{-1}(U_1) \in \tau_1$ and $f^{-1}(U_2) \in \tau_2$, implying that $f^{-1}(U_1) \cup f^{-1}(U_2) \in \tau_1 \vee \tau_2$. But $f^{-1}(U_1) \cup f^{-1}(U_2) = f^{-1}(U_1 \cup U_2)$. Hence, the function $f : (Y, \tau_1 \vee \tau_2) \longrightarrow (Z, \delta_1 \vee \delta_2)$ is s-continuous. ∎

The converse of Theorem 3.2.4 is not always true as illustrated by the example below.

Example 3.2.5. *Let $\tau_1 = \{Y, \phi, \{a\}, \{b\}, \{a,b\}\}$ and $\tau_2 = \{Y, \phi, \{c\}\}$ be topologies on $Y = \{a,b,c\}$, and $\delta_1 = \{Z, \phi, \{3\}\}$ and $\delta_2 = \{Z, \phi, \{2\}\}$ be topologies defined on $Z = \{1,2,3\}$. Let $f : Y \to Z$ be defined by $f(a) = 1$, $f(b) = 2$, and $f(c) = 3$, then the function $f : (Y, \tau_1 \vee \tau_2) \longrightarrow (Z, \delta_1 \vee \delta_2)$ is s-continuous but not p-continuous on (Y, τ_1, τ_2). Observe that $(\tau_1 \vee \tau_2) = \{Y, \phi, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}\}$ and $(\delta_1 \vee \delta_2) = \{Z, \phi, \{2\}, \{3\}, \{2,3\}\}$, and for every U is open $(\delta_1 \vee \delta_2)$, $f^{-1}(U)$ is open in $\tau_1 \vee \tau_2)$, therefore f is s-continuous. But f is not p-continuous since for the open set $\{2\} \in \delta_2$, $f^{-1}(\{2\})$ is not open in τ_2.*

Theorem 3.2.6. *(Muturi E. N. et al., 2017). The function $f : (Y, \tau_1 \vee \tau_2) \longrightarrow (Z, \delta_1 \vee \delta_2)$ is s-continuous, if the function $f : (Y, \tau_1, \tau_2) \longrightarrow (Z, \delta_1, \delta_2)$ is p_1-continuous.*

Proof. Let the function f be p_1-continuous and let $U_1 \in \delta_1$ and $U_2 \in \delta_2$ such that $U_1 \cup U_2 \in \delta_1 \vee \delta_2$. Since f is p_1-continuous, then $f^{-1}(U_1) \in \tau_2$ and $f^{-1}(U_2) \in \tau_1$. Thus, $f^{-1}(U_1) \cup f^{-1}(U_2) \in \tau_1 \vee \tau_2$. It remains to put $f^{-1}(U_1) \cup f^{-1}(U_2) = f^{-1}(U_1 \cup U_2)$. Hence, the function $f : (Y, \tau_1 \vee \tau_2) \longrightarrow (Z, \delta_1 \vee \delta_2)$ is s-continuous. ∎

The converse of Theorem 3.2.6 is not always true as illustrated by the example below.

Example 3.2.7. *Let $\tau_1 = \{Y, \phi, \{a\}, \{b\}, \{a,b\}\}$ and $\tau_2 = \{Y, \phi, \{c\}\}$ be topologies on $Y = \{a,b,c\}$, and $\delta_1 = \{Z, \phi, \{2\}\}$ and $\delta_2 = \{Z, \phi, \{3\}\}$ be topologies defined on $Z = \{1,2,3\}$. Let $f : Y \to Z$ be defined by $f(a) = 1$, $f(b) = 2$, and $f(c) = 3$, then the function $f : (Y, \tau_1 \vee \tau_2) \longrightarrow (Z, \delta_1 \vee \delta_2)$ is s-continuous but not p_1-continuous. Observe that $(\tau_1 \vee \tau_2) = \{Y, \phi, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}\}$ and $(\delta_1 \vee \delta_2) = \{Z, \phi, \{2\}, \{3\}, \{2,3\}\}$, and for every U is open $(\delta_1 \vee \delta_2)$, $f^{-1}(U)$ is open in $\tau_1 \vee \tau_2)$, therefore f is s-continuous. But f is not p_1-continuous since for the open set $\{3\} \in \delta_2$, $f^{-1}(\{3\})$ is not open in τ_1.*

Remark 3.2.8. *p-continuity and p_1-continuity imply d-continuity as illustrated in the following propositions.*

Proposition 3.2.9. *(Muturi E. N. et al., 2017). Let the function $f : (Y, \tau_1, \tau_2) \longrightarrow (Z, \delta_1, \delta_2)$ be p-continuous, then the function $f : (Y, \tau_1 \wedge \tau_2) \longrightarrow (Z, \delta_1 \wedge \delta_2)$ is d-continuous.*

Proof. Let U be open in $\delta_1 \wedge \delta_2$, then $U \in \delta_1 \cap \delta_2$, implying that $U \in \delta_1$ and $U \in \delta_2$, this implies further that $f^{-1}(U) \in \tau_1$ and $f^{-1}(U) \in \tau_2$. Therefore, $f^{-1}(U) \in \tau_1 \cap \tau_2$ implying that $f^{-1}(U)$ is open in $\tau_1 \wedge \tau_2$. Thus f is d-continuous. ■

Proposition 3.2.10. *(Muturi E. N. et al., 2017). Let the function $f : (Y, \tau_1, \tau_2) \longrightarrow (Z, \delta_1, \delta_2)$ be p_1-continuous, then the function $g : (Y, \tau_1 \wedge \tau_2) \longrightarrow (Z, \delta_1 \wedge \delta_2)$ is d-continuous.*

Proof. The proof is similar to that of Proposition 3.2.9. ■

Remark 3.2.11. *The relationships between the spaces $(Y, \tau_1 \vee \tau_2)$, (Z, δ_i) for $i = 1, 2$ and $(Z, \delta_1 \wedge \delta_2)$ are established in the following propositions.*

Proposition 3.2.12. *(Muturi E. N. et al., 2017). The function $h : (Y, \tau_1 \vee \tau_2) \longrightarrow (Z, \delta_i)$ is continuous for $i = 1, 2$, if the function $f : (Y, \tau_1 \vee \tau_2) \longrightarrow (Z, \delta_1 \vee \delta_2)$ is s-continuous.*

Proof. The function $g : (Z, \delta_1 \vee \delta_2) \rightarrow (Z, \delta_i)$ is continuous since $\delta_i \subset \delta_1 \vee \delta_2$ for $i = 1, 2$. Therefore $h = g \circ f$ is continuous as a composition of continuous functions. ■

Proposition 3.2.13. *(Muturi E. N. et al., 2017). The function $\rho : (Y, \tau_1 \vee \tau_2) \longrightarrow (Z, \delta_1 \wedge \delta_2)$ is continuous for $i = 1, 2$, if the function $f : (Y, \tau_1 \vee \tau_2) \longrightarrow (Z, \delta_1 \vee \delta_2)$ is s-continuous.*

Proof. Let $h : (Z, \delta_i) \rightarrow (Z, \delta_1 \wedge \delta_2)$for $i = 1, 2$ and $g : (Z, \delta_1 \vee \delta_2) \rightarrow (Z, \delta_i)$ for $i = 1, 2$, then h and g are continuous functions since $\delta_1 \wedge \delta_2 \subset \delta_i$ and $\delta_i \subset \delta_1 \vee \delta_2$ for $i = 1, 2$. Let $\rho = h \circ g \circ f$, then ρ is a continuous function since it is a composition of continuous functions. ■

3.3 Continuous functions on function spaces defined on bitopological spaces

For bitopological spaces (Y, τ_1, τ_2) and (Z, δ_1, δ_2), and the open continuous functions $f : (Y, \tau_2) \longrightarrow (Y, \tau_1)$, $g : (Z, \delta_1) \longrightarrow (Z, \delta_2)$ and $h : (Y, \tau_1) \longrightarrow (Z, \delta_1)$, the following sets of continuous functions can be defined. The set $i - C(Y, Z)$ for $i = 1, 2$, the set $(i, j) - C(Y, Z)$ for $i, j = 1, 2$ and $i \neq j$, the set $s - C(Y, Z)$, the set $p - C(Y, Z)$ and the set $d - C(Y, Z)$.

Different set-open topologies can be defined on sets of continuous functions. Such topologies include; point-open topology, compact-open topology, bounded-open topology, closed-open topology and open-open topology, among others. In this section, subbasis for open-open topologies on the sets $1 - C(Y, Z)$, $2 - C(Y, Z)$, $(1, 2) - C(Y, Z)$, $(2, 1) - C(Y, Z)$ and $s - C(Y, Z)$

are defined, giving rise to the function spaces $1-C_{\varsigma}(Y,Z)$, $2-C_{\zeta}(Y,Z)$, $(1,2)-C_{\varphi}(Y,Z)$, $(2,1)-C_{\xi}(Y,Z)$ and $s-C_{\tau}(Y,Z)$. Functional relationships between these function spaces are studied.

Definition 3.3.1. *(Muturi E. N. et al., 2017). The collection $S(U,V)_i=\{f\in i-C(Y,Z):f(U)\subset V\}$ of sets, for U open in Y and V open in Z, forms the subbasis for the open-open topology defined on $i-C(Y,Z)$, for $i=1,2$.*

Definition 3.3.2. *(Muturi E. N. et al., 2017). The collection $S(U,V)_{(i,j)}=\{f\in(i,j)-C(Y,Z):f(U)\subset V\}$ of sets, where U is an open set in Y and V is an open set in Z, forms the subbasis for the open-open topology defined on $(i,j)-C(Y,Z)$, for $i,j=1,2$ and $i\neq j$.*

Definition 3.3.3. *(Muturi E. N. et al., 2017). The collection $S(U,V)_s=\{f\in s-C(Y,Z):f(U)\subset V\}$ of sets, for $U\in\tau_1\vee\tau_2$ and $V\in\delta_1\vee\delta_2$, forms the subbasis for the supremum open-open topology defined on $s-C(Y,Z)$.*

Remark 3.3.4. *Let $S(U_1,V_1)$ be open in $1-C_{\varsigma}(Y,Z)$ and $S(U_2,V_2)$ be open in $2-C_{\zeta}(Y,Z)$. From Definition 3.2.1, $U_1\cup U_2=U$ is open in $\tau_1\vee\tau_2$ and $V_1\cup V_2=V$ is open in $\delta_1\vee\delta_2$. Therefore $S(U,V)_s$ is open in $s-C_{\tau}(Y,Z)$.*

Continuous functions are established between the spaces $(1,2)-C_{\varphi}(Y,Z)$, $(2,1)-C_{\xi}(Y,Z)$, $1-C_{\varsigma}(Y,Z)$ and $2-C_{\zeta}(Y,Z)$. All topologies are assumed to be open-open topologies unless specified.

Proposition 3.3.5. *(Muturi E. N. et al., 2017). Let $f:(Y,\tau_2)\longrightarrow(Y,\tau_1)$, $g:(Z,\delta_1)\longrightarrow(Z,\delta_2)$ and $h:(Y,\tau_1)\longrightarrow(Z,\delta_1)$ be open and continuous functions. Then the function $\mu: 1-C_{\varsigma}(Y,Z)\longrightarrow 2-C_{\zeta}(Y,Z)$ defined by $\mu_{(g,f)}(h)=g\circ h\circ f$ is continuous.*

Proof. Let $S(U,V)_2$ be open in $2-C_{\zeta}(Y,Z)$, then $g\circ h\circ f\in 2-C_{\zeta}(Y,Z)$. Now $\mu^{-1}(S(U,V)_2)$= $\{h\in 1-C(Y,Z):h(f(U))\subset g^{-1}(V)\ for\ U\in\tau_2\ and\ V\in\delta_2\}$=$S(f(U),g^{-1}(V))_1$, which is open in $1-C_{\varsigma}(Y,Z)$. ■

Proposition 3.3.6. *(Muturi E. N. et al., 2017). Let $f:(Y,\tau_2)\longrightarrow(Y,\tau_1)$ and $h:(Y,\tau_1)\longrightarrow(Z,\delta_1)$ be open and continuous functions. Then the function $\rho: 1-C_{\varsigma}(Y,Z)\longrightarrow(2,1)-C_{\xi}(Y,Z)$ defined by $\rho_f(h)=h\circ f$ is continuous.*

Proof. Let $S(U,V)_{2,1}$ be open in $(2,1)-C_{\xi}(Y,Z)$, then $h\circ f\in(2,1)-C_{\xi}(Y,Z)$. Now $\rho^{-1}(S(U,V)_{2,1})$=$\{h\in 1-C(Y,Z):h(f(U))\subset V$ for $U\in\tau_2$ and $V\in\delta_1\}$=$S(f(U),V)_1$, which is open in $1-C_{\varsigma}(Y,Z)$. ■

Proposition 3.3.7. *(Muturi E. N. et al., 2017). Let $g:(Z,\delta_1)\longrightarrow(Z,\delta_2)$ and $h:(Y,\tau_1)\longrightarrow(Z,\delta_1)$ be open and continuous functions. Then the function $\omega:1-C_\varsigma(Y,Z)\longrightarrow(1,2)-C_\varphi(Y,Z)$ defined by $\omega_g(h)=g\circ h$ is continuous.*

Proof. Let $S(U,V)_{1,2}$ be open in $(1,2)-C_\varphi(Y,Z)$, then $g\circ h\in(1,2)-C_\varphi(Y,Z)$. The set $\omega^{-1}(S(U,V)_{1,2})$=$\{h\in 1-C(Y,Z):h(U)\subset g^{-1}(V)$ for $U\in\tau_1$ and $V\in\delta_2\}$=$S(U,g^{-1}(V))_1$ is open in $1-C_\varsigma(Y,Z)$. ■

Proposition 3.3.8. *(Muturi E. N. et al., 2017). Let $f:(Y,\tau_2)\longrightarrow(Y,\tau_1)$, $g:(Z,\delta_1\longrightarrow(Z,\delta_2)$ and $h:(Y,\tau_1)\rightarrow(Z,\delta_1)$ be open and continuous functions. Then the following functions are continuous;*

(i) $\alpha:(2,1)-C_\xi(Y,Z)\longrightarrow 1-C_\varsigma(Y,Z)$ *defined by* $\alpha(h\circ f)=h_f$

(ii) $\beta:2-C_\zeta(Y,Z)\longrightarrow(1,2)-C_\varphi(Y,Z)$ *defined by* $\beta(g\circ h\circ f)=(g\circ h)_f$

Proof. *(i)* Let U be open in τ_2 and V be open in δ_1, then $S(f(U),V)_1$ is open in $1-C_\varsigma(Y,Z)$. Now $\alpha^{-1}S(f(U),V)_1$=$\{(h\circ f)\in(2,1)-C(Y,Z):h(f(U))\subset V,U\in\tau_2$ and $V\in\delta_1\}$=$\{(h\circ f)\in(2,1)-C(Y,Z):(h\circ f)(U)\subset V,U\in\tau_2$ and $V\in\delta_1\}$=$S(U,V)_{2,1}$, which is open in $(2,1)-C_\xi(Y,Z)$.
(ii) Let U be open in τ_2 and V be open in δ_2, then $S(f(U),V)_{1,2}$ is open in $(1,2)-C_\varphi(Y,Z)$. Now $\beta^{-1}S(f(U),V)_{1,2}$=$\{(g\circ h\circ f)\in 2-C(Y,Z):g(h(f(U)))\subset V,U\in\tau_2$ and $V\in\delta_2\}$=$\{(g\circ h\circ f)\in 2-C(Y,Z):(g\circ h\circ f)(U)\subset V,U\in\tau_2$ and $V\in\delta_2\}$=$S(U,V)_2$, which is open in $2-C_\zeta(Y,Z)$. ■

Proposition 3.3.9. *(Muturi E. N. et al., 2017). Let $f:(Y,\tau_2)\longrightarrow(Y,\tau_1)$, $g:(Z,\delta_1\longrightarrow(Z,\delta_2)$ and $h:(Y,\tau_1)\rightarrow(Z,\delta_1)$ be open and continuous functions, then the function $\beta\circ\mu\circ\alpha:(2,1)-C_\xi(Y,Z)\longrightarrow(1,2)-C_\varphi(Y,Z)$ defined by $(\beta\circ\mu\circ\alpha)_g(h\circ f)=(g\circ h)_f$ is continuous.*

Proof. The function $\beta\circ\mu\circ\alpha$ is a composite function of continuous functions defined in Proposition 3.3.5 and Proposition 3.3.8. ■

The lemma that follows help to prove the subsequent theorem.

Lemma 3.3.10. *(Willard, 1970). Let Y be a regular space, if F is a compact subset of Y, U open is open in Y and $F\subset U$, then for some open set V in Y, $F\subset V\subset\overline{V}\subset U$.*

Theorem 3.3.11. *(Muturi E. N. et al., 2017). Let ς be a k-topology on $1-C(Y,Z)$, Y be a regular and locally compact space, Z a Hausdorff space and $S(U,V)_1$ be compact subset of $1-C_\varsigma(Y,Z)$, then for the continuous functions $\mu:1-C_\varsigma(Y,Z)\longrightarrow 2-C_\zeta(Y,Z)$ and*

$\beta : 2 - C_{\zeta}(Y,Z) \longrightarrow (1,2) - C_{\varphi}(Y,Z)$, *the function* $T : C(1 - C_{\varsigma}(Y,Z), 2 - C_{\zeta}(Y,Z)) \times C(2 - C_{\zeta}(Y,Z), (1,2) - C_{\varphi}(Y,Z)) \to C(1 - C_{\varsigma}(Y,Z), (1,2) - C_{\varphi}(Y,Z))$ *is continuous with respect to closed-open topology.*

Proof. Let $(S(U,V)_1, S(U,V)_{1,2})$ be neighbourhood of ω in $C(1 - C_{\varsigma}(Y,Z), (1,2) - C_{\varphi}(Y,Z))$, then $\beta^{-1}(S(U,V)_{1,2}$ is open in $2 - C_{\zeta}(Y,Z)$. Now, $\mu(S(U,V)_1) \subset \beta^{-1}(S(U,V)_{1,2}$. Since $\mu(S(U,V)_1)$ is compact, then by the lemma above, there exist an open set $S(A,B)_2$ such that $\mu(S(U,V)_1) \subset S(A,B)_2 \subset \overline{S(A,B)}_2 \subset \beta^{-1}(S(U,V)_{1,2})$. This implies that $\mu \in (S(U,V)_1, S(A,B)_2)$ and $\beta \in (\overline{S(A,B)}_2, S(U,V)_{1,2})$. Therefore, $T((S(U,V)_1, S(A,B)_2), (\overline{S(A,B)}_2, S(U,V)_{1,2})) \subset (S(U,V)_1, S(U,V)_{1,2})$, implying that the function T is continuous. ■

Theorem 3.3.12. *(Muturi E. N. et al., 2017). Let* $f : (Y, \tau_2) \longrightarrow (Y, \tau_1)$ *and* $h : (Y, \tau_1) \to (Z, \delta_1)$ *be open and continuous functions, then the function* $\rho : 1 - C_{\varsigma}(Y,Z) \longrightarrow (2,1) - C_{\xi}(Y,Z)$ *defined by* $\rho_f(h) = h \circ f$ *is a homeomorphism.*

Proof. Let h_1 and h_2 be functions in $1 - C_{\varsigma}(Y,Z)$, then $\rho_f(h_1) = h_1 \circ f$ and $\rho_f(h_2) = h_2 \circ f$. Suppose $\rho_f(h_1) = \rho_f(h_2)$, then $h_1 \circ f = h_2 \circ f$, implying that $h_1 = h_2$. Hence, ρ_f is a 1-1 function. The function ρ_f is an onto function since for any $h \circ f \in (2,1) - C_{\xi}(Y,Z)$ their exist the function $h \in 1 - C_{\varsigma}(Y,Z)$ and from Proposition 3.3.6, $\rho_f(h) = h \circ f$ is open and continuous. Continuity of ρ_f^{-1} follows from Proposition 3.3.8 part *(i)*. ■

Proposition 3.3.13. *(Muturi E. N. et al., 2017). The function* $j : 1 - C_{\varsigma}(Y,Z) \to s - C_{\tau}(Y,Z)$ *is continuous.*

Proof. Let $S(A,B)_s$ be open in $s - C_{\tau}(Y,Z)$, then $j^{-1}(S(A,B)_s) = \{f \in s - C(Y,Z) : f(A) \subset B\} = \{f \in s - C(Y,Z) : f|_{1-C(Y,Z)}(A) \subset B\} = S(A,B)_1$, which is open in $1 - C_{\varsigma}(Y,Z)$. ■

Proposition 3.3.14. *(Muturi E. N. et al., 2017). The function* $j : 2 - C_{\zeta}(Y,Z) \to s - C_{\tau}(Y,Z)$ *is continuous.*

Proof. Let $S(A,B)_s$ be open in $s - C_{\tau}(Y,Z)$, then $j^{-1}(S(A,B)_s) = \{f \in s - C(Y,Z) : f(A) \subset B\} = \{f \in s - C(Y,Z) : f|_{2-C(Y,Z)}(A) \subset B\} = S(A,B)_2$, which is open in $2 - C_{\varsigma}(Y,Z)$. ■

Remark 3.3.15. *From the results above, both* $1 - C_{\varsigma}(Y,Z)$ *and* $2 - C_{\zeta}(Y,Z)$ *can be considered as subspaces of* $s - C_{\tau}(Y,Z)$*. The spaces* $1 - C_{\varsigma}(Y,Z)$ *and* $2 - C_{\zeta}(Y,Z)$ *will be important in proving splitting and admissibility properties of topologies in* $p - C_{\omega}(Y,Z)$ *and* $s - C_{\tau}(Y,Z)$ *in the next chapter, as well as proving separation axioms and compactness on function spaces defined on bitopological spaces in subsequent chapters.*

CHAPTER FOUR

SPLITTING AND ADMISSIBLE TOPOLOGIES DEFINED ON THE SET OF CONTINUOUS FUNCTIONS BETWEEN BITOPOLOGICAL SPACES

4.1 Introduction

For the bitopological spaces (Y,τ_1,τ_2) and (Z,δ_1,δ_2), the following sets of continuous functions are considered. The set $i-C(Y,Z)$ for $i=1,2$, the set $s-C(Y,Z)$ and the set $p-C(Y,Z)$. In this section, p-splitting, p-admissible, s-splitting and s-admissible topologies on $p-C(Y,Z)$ and $s-C(Y,Z)$ respectively, are defined and explored. The relationship between splitting, p-splitting and s-splitting, as well as admissible, p-admissible and s-admissible are established. Exponential functions are defined on the space $s-C_\tau(Y,Z)$ and a simpler proof for comparing s-splitting topology and s-admissible topology provided.

4.2 Pairwise splitting and pairwise admissible topologies defined on the set $p-C(Y,Z)$

Definition 4.2.1. *(Muturi E. N. et al., 2018(a)). The collection $S((U,V),(A,B))_p=\{f\in p-C(Y,Z): f(U)\subset V$ and $f(A)\subset B\}$ of sets, for U open in τ_1, V open in δ_1, A open in τ_2 and B open in δ_2, forms the subbasis for the open-open topology on $p-C(Y,Z)$. If U and A are compact subsets of Y, then $S((U,V),(A,B))_p$ forms the subbasis for compact open topology.*

The results that follow, help to define p-splitting and p-admissible topologies on $p-C(Y,Z)$. The concept of pairwise continuity as well as splitting and admissible topologies defined on both $1-C(Y,Z)$ and $2-C(Y,Z)$ are employed. These results culminates in the proof of pairwise splitting and pairwise admissible topologies defined on $p-C(Y,Z)$. The functions h and h^* as used here, are defined as follows; $h^*(x)=h_x$, where $h_x:(Y,\tau_i)\to(Z,\delta_i)$, for $i=1,2$, is defined by $h_x(y)=h(x,y)$.

Proposition 4.2.2. *(Muturi E. N. et al., 2018(a)). The function $h^*:(X,\sigma)\to p-C_\omega(Y,Z)$ is pairwise continuous if the functions $h^*:(X,\sigma)\to 1-C_\varsigma(Y,Z)$ and $h^*:(X,\sigma)\to 2-C_\zeta(Y,Z)$ are continuous, where $h:(X,\sigma)\times(Y,\tau_i)\to(Z,\delta_i)$ for $i=1,2$.*

Proof. Let $h^*:(X,\sigma)\to 1-C_\varsigma(Y,Z)$ and $h^*:(X,\sigma)\to 2-C_\zeta(Y,Z)$ be continuous functions. Then for each fixed $x\in X$, the functions $h_x:(Y,\tau_1)\to(Z,\delta_1)$ and $h_x:(Y,\tau_2)\to(Z,\delta_2)$ are continuous. By definition of pairwise continuity, the function $h_x : (Y,\tau_1,\tau_2) \to (Z,\delta_1,\delta_2)$ is continuous for each fixed $x \in X$. Since $h_x = h^*(x)$, then the function $h^* : (X,\sigma) \to p - C_\omega(Y,Z)$ is continuous. ■

Proposition 4.2.3. (Muturi E. N. et al., 2018(a)). *The function $h : (X,\sigma) \times (Y,\tau_1,\tau_2) \to (Z,\delta_1,\delta_2)$ is pairwise continuous for every fixed $x \in X$, if the induced functions $h : (X,\sigma) \times (Y,\tau_1) \to (Z,\delta_1)$ and $h : (X,\sigma) \times (Y,\tau_2) \to (Z,\delta_2)$ are continuous for every fixed $x \in X$.*

Proof. Let $h : (X,\sigma) \times (Y,\tau_1) \to (Z,\delta_1)$ and $h : (X,\sigma) \times (Y,\tau_2) \to (Z,\delta_2)$ be continuous functions for every fixed $x \in X$, then the functions $h_x : (Y,\tau_1) \to (Z,\delta_1)$ and $h_x : (Y,\tau_2) \to (Z,\delta_2)$ are continuous. By definition of pairwise continuity, the function $h_x : (Y,\tau_1,\tau_2) \to (Z,\delta_1,\delta_2)$ defined by $h_x(y) = h(x,y)$ is continuous for every fixed $x \in X$. Since $h(x)(y) = h(x,y)$, then $h_x(y) = h(x)(y)$ implying that the function $h : (X,\sigma) \times (Y,\tau_1,\tau_2) \to (Z,\delta_1,\delta_2)$ is continuous for every fixed $x \in X$. ■

The Propositions above motivates the following Definitions.

Definition 4.2.4. (Muturi E. N. et al., 2018(a)). *A topology ω on $p - C(Y,Z)$ is said to be pairwise splitting (p-splitting), if the continuity of the functions $h : (X,\sigma) \times (Y,\tau_1) \to (Z,\delta_1)$ and $h : (X,\sigma) \times (Y,\tau_2) \to (Z,\delta_2)$ for every fixed $x \in X$, implies that of $h^* : (X,\sigma) \to p - C_\omega(Y,Z)$ for every $x \in X$.*

Definition 4.2.5. (Muturi E. N. et al., 2018(a)). *A topology ω on $p - C(Y,Z)$ is said to be pairwise admissible (p-admissible), if the continuity of the functions $h^* : (X,\sigma) \to 1 - C_\varsigma(Y,Z)$ and $h^* : (X,\sigma) \to 2 - C_\zeta(Y,Z)$ implies that of $h : (X,\sigma) \times (Y,\tau_1,\tau_2) \to (Z,\delta_1,\delta_2)$.*

Remark 4.2.6. *Using the Propositions and Definitions above, the proof of pairwise splitting and pairwise admissibility of topologies defined on $p - C(Y,Z)$ are established.*

Theorem 4.2.7. (Muturi E. N. et al., 2018(a)). *Let $h : (X,\sigma) \times (Y,\tau_1) \to (Z,\delta_1)$ and $h : (X,\sigma) \times (Y,\tau_2) \to (Z,\delta_2)$ be continuous functions, then the compact-open topology ω defined on $p - C(Y,Z)$ is pairwise splitting.*

Proof. Let $h : (X,\sigma) \times (Y,\tau_1) \to (Z,\delta_1)$ and $h : (X,\sigma) \times (Y,\tau_2) \to (Z,\delta_2)$ be continuous functions and let $x_\circ \in X$ such that $h^*(x_\circ) \in S((U,V)(A,B))_p$, where $S((U,V)(A,B))_p$ is open in $p - C_\omega(Y,Z)$. Therefore, $h^*(x_\circ) \in S(U,V)_1$ and $h^*(x_\circ) \in S(A,B)_2$, implying that $x_\circ \times U \subset h^{-1}(V)$ and $x_\circ \times A \subset h^{-1}(B)$. Since U and A are compact, then by Munkres's (2000) tube lemma, there exist an open set W neighbourhood of $x_\circ$, such that $W \times U \subset h^{-1}(V)$ and $W \times A \subset$

$h^{-1}(B)$. This implies that $h^*(W) \subset S(U,V)_1$ and $h^*(W) \subset S(A,B)_2$, implying further that $h^* : (X,\sigma) \to 1-C_\varsigma(Y,Z)$ and $h^* : (X,\sigma) \to 2-C_\zeta(Y,Z)$ are continuous functions. By Proposition 4.2.2, the function $h^* : (X,\sigma) \to p-C_\omega(Y,Z)$ is continuous and by Definition 4.2.4, topology ω is pairwise splitting on $p-C(Y,Z)$. ■

Theorem 4.2.8. *(Muturi E. N. et al., 2018(a)). Let $h^* : (X,\sigma) \to 1-C_\varsigma(Y,Z)$ and $h^* : (X,\sigma) \to 2-C_\zeta(Y,Z)$ be continuous functions, then the compact-open topology ω defined on $p-C(Y,Z)$ is pairwise admissible for locally compact spaces (Y,τ_1) and (Y,τ_2).*

Proof. Let ς and ζ be compact-open topologies on $1-C(Y,Z)$ and $2-C(Y,Z)$ respectively, such that the evaluation functions $e : 1-C_\varsigma(Y,Z)\times Y \to Z$ and $e : 2-C_\zeta(Y,Z)\times Y \to Z$ are continuous. Let $h^* : (X,\sigma) \to 1-C_\varsigma(Y,Z)$ and $h^* : (X,\sigma) \to 2-C_\zeta(Y,Z)$ be continuous functions and $i : (Y,\tau_1) \to (Y,\tau_1)$ and $i : (Y,\tau_2) \to (Y,\tau_2)$ be identity functions. Then $e\circ(g\times i) : (X,\sigma)\times(Y,\tau_1) \to (Z,\delta_1)$ and $e\circ(g\times i) : (X,\sigma)\times(Y,\tau_2) \to (Z,\delta_2)$ are continuous functions. By Proposition 4.2.3, the function $e\circ(g\times i) : (X,\sigma)\times(Y,\tau_1,\tau_2) \to (Z,\delta_1,\delta_2)$ is continuous for every fixed $x \in X$, and by Definition 4.2.5, topology ω defined on $p-C(Y,Z)$ is pairwise admissible. ■

Remark 4.2.9. *From Theorem 4.2.7 and Theorem 4.2.8, it is clear that if topologies ς and ζ are splitting or admissible topologies on $1-C(Y,Z)$ and $2-C(Y,Z)$, then topology ω on $p-C(Y,Z)$ is p-splitting or p-admissible topology.*

4.3 Supremum splitting and supremum admissible topologies defined on the set $s-C(Y,Z)$

From Chapter Three, it was noted that if $f \in p-C(Y,Z)$, then $f \in s-C(Y,Z)$, but the converse was not true. This result together with Proposition 4.2.2 and Proposition 4.2.3 motivates the following two propositions. The function f and f^* as used here are defined as follows; $f^*(x) = f_x$, where $f_x : (Y,\tau_i) \to (Z,\delta_i)$, for $i = 1,2$, is defined by $f_x(y) = f(x,y)$.

Proposition 4.3.1. *(Muturi E. N. et al., 2018(a)). The function $f : (X,\sigma)\times(Y,\tau_1\vee\tau_2) \to (Z,\delta_1\vee\delta_2)$ is continuous, if the functions $f : (X,\sigma)\times(Y,\tau_1) \to (Z,\delta_1)$ and $f : (X,\sigma)\times(Y,\tau_2) \to (Z,\delta_2)$ are both continuous.*

Proof. Let the functions $f : (X,\sigma)\times(Y,\tau_1) \to (Z,\delta_1)$ and $f : (X,\sigma)\times(Y,\tau_2) \to (Z,\delta_2)$ defined by $f(x)(y) = f(x,y)$ be both continuous. Define the associated function $f_x : (Y,\tau_i) \to (Z,\delta_i)$ for

$i = 1,2$, by $f_x(y) = f(x,y)$, $\forall x \in X$. Then, the functions f and f_x have a one to one correspondence, and hence $f_x : (Y,\tau_i) \to (Z,\delta_i)$ for $\forall x \in X$ and $i = 1,2$ is continuous. From Theorem 3.2.4, it follows that the function $f_x : (Y,\tau_1 \vee \tau_2) \to (Z,\delta_1 \vee \delta_2)$ is s-continuous $\forall x \in X$. Hence, $f : (X,\sigma) \times (Y,\tau_1 \vee \tau_2) \to (Z,\delta_1 \vee \delta_2)$ is continuous for every $x \in X$. ■

Proposition 4.3.2. *(Muturi E. N. et al., 2018(a)). The function $f^* : (X,\sigma) \to s - C_\tau(Y,Z)$ is continuous, if the functions $f^* : (X,\sigma) \to 1 - C_\varsigma(Y,Z)$ and $f^* : (X,\sigma) \to 2 - C_\zeta(Y,Z)$ are continuous.*

Proof. Let $f^* : (X,\sigma) \to 1 - C_\varsigma(Y,Z)$ and $f^* : (X,\sigma) \to 2 - C_\zeta(Y,Z)$ be continuous functions. Then, for the functions $f : (X,\sigma) \times (Y,\tau_1) \to (Z,\delta_1)$ and $f : (X,\sigma) \times (Y,\tau_2) \to (Z,\delta_2)$, the associated functions $f_x : (Y,\tau_1) \to (Z,\delta_1)$ and $f_x : (Y,\tau_2) \to (Z,\delta_2)$ defined by $f_x = f^*(x)$ $\forall x \in X$ are continuous. From Theorem 3.2.4, it follows that the function $g_x : (Y,\tau_1 \vee \tau_2) \to (Z,\delta_1 \vee \delta_2)$ is s-continuous $\forall x \in X$. Since $f_x = f^*(x)$, then the function $f^* : (X,\sigma) \to s - C(Y,Z)$ is continuous. ■

Definition 4.3.3. *(Muturi E. N. et al., 2018(a)). A topology τ on $s - C(Y,Z)$ is said to be supremum splitting (s-splitting), if the continuity of the functions $f : (X,\sigma) \times (Y,\tau_1) \to (Z,\delta_1)$ and $f : (X,\sigma) \times (Y,\tau_2) \to (Z,\delta_2)$, for every fixed $x \in X$, implies that of $g : (X,\sigma) \to s - C_\tau(Y,Z)$.*

Definition 4.3.4. *(Muturi E. N. et al., 2018(a)). A topology τ on $s - C(Y,Z)$ is said to be supremum admissible (s-admissible), if the continuity of the functions $f^* : (X,\sigma) \to 1 - C_\varsigma(Y,Z)$ and $f^* : (X,\sigma) \to 2 - C_\zeta(Y,Z)$, implies that of $f : (X,\sigma) \times (Y,\tau_1 \vee \tau_2) \to (Z,\delta_1 \vee \delta_2)$, for each $x \in X$.*

Remark 4.3.5. *Using the Propositions and Definitions above, the proof of supremum splitting and supremum admissibility of topologies defined on $s - C(Y,Z)$ are established.*

Theorem 4.3.6. *(Muturi E. N. et al., 2018(a)). A compact open topology τ is s-splitting, if the continuity of the functions $f : (X,\sigma) \times (Y,\tau_1) \to (Z,\delta_1)$ and $f : (X,\sigma) \times (Y,\tau_2) \to (Z,\delta_2)$, implies continuity of the function $f^* : (X,\delta) \to s - C_\tau(Y,Z)$.*

Proof. Let $f : (X,\sigma) \times (Y,\tau_1) \to (Z,\delta_1)$ and $f : (X,\sigma) \times (Y,\tau_2) \to (Z,\delta_2)$ be continuous functions for every fixed $x \in X$. Then from Proposition 4.3.1, the function $f : (X,\sigma) \times (Y,\tau_1 \vee \tau_2) \to (Z,\delta_1 \vee \delta_2)$ is continuous. Let $x_\circ \in X$ where $S(U,V)_s$ is open in $s - C_\tau(Y,Z)$, then $f^*(x_\circ) \in S(U,V)_s$, implying that $x_\circ \times U \subset f^{-1}(V)$. Since U is compact, then by Munkres's (2000) tube lemma, there exist an open set W neighbourhood of $x_\circ$ such that $W \times U \subset f^{-1}(V)$. This implies that $f^*(W) \subset S(U,V)_s$, implying further that $f^* : (X,\sigma) \to s - C_\tau(Y,Z)$ is continuous functions. By Definition 4.3.3, topology τ is pairwise splitting on $s - C(Y,Z)$. ■

Theorem 4.3.7. *(Muturi E. N. et al., 2018(a)). Let $f^* : (X,\sigma) \to 1 - C_\varsigma(Y,Z)$ and $f^* : (X,\sigma) \to 2 - C_\zeta(Y,Z)$ be continuous functions, then the compact-open topology τ defined on $s - C(Y,Z)$ is s-admissible for locally compact spaces (Y,τ_1) and (Y,τ_2).*

Proof. Let ς and ζ be compact-open topologies on $1 - C(Y,Z)$ and $2 - C(Y,Z)$ respectively and let $f^* : (X,\sigma) \to 1 - C_\varsigma(Y,Z)$ and $f^* : (X,\sigma) \to 2 - C_\zeta(Y,Z)$ be continuous functions, then by Proposition 4.3.2, the function $f^* : (X,\sigma) \to s - C_\tau(Y,Z)$ is continuous. Let $i : (Y,\tau_1 \vee \tau_2) \to (Y,\tau_1 \vee \tau_2)$ be an identity function and let $e : s - C_\tau(Y,Z) \times (Y,\tau_1 \vee \tau_2) \to (Z,\delta_1 \vee \delta_2)$ be an evaluation mapping. Since τ is compact-open topology, then the evaluation mapping e is continuous and the composite mapping $e \circ (f^* \times i) : (X,\sigma) \times (Y,\tau_1 \vee \tau_2) \to (Z,\delta_1 \vee \delta_2)$ is also continuous in Y for each $x \in X$. By Definition 4.3.4, topology τ is s-admissible. ■

Remark 4.3.8. *From Theorem 4.3.6 and Theorem 4.3.7, it is clear that if topologies ς and ζ are splitting or admissible topologies on $1 - C(Y,Z)$ and $2 - C(Y,Z)$, then topology τ on $s - C(Y,Z)$ is s-splitting or s-admissible topology.*

4.4 Exponential mappings defined on the space $s - C_\tau(Y,Z)$

Consider the topological spaces (X,σ), $(Z,\delta_1 \vee \delta_2)$ and $(Y,\tau_1 \vee \tau_2)$, and where $(Y,\tau_1 \vee \tau_2)$ is a locally compact Hausdorff space.

Definition 4.4.1. *(Muturi E. N. et al., 2018(a)). Consider the exponential mapping $\Lambda : C(X \times Y,Z) \to C(X, s - C_\varphi(Y,Z))$, defined by $\Lambda(f)(x)(y) = f(x,y)$ for each $f \in C(X \times Y,Z)$, $x \in X$ and $y \in Y$. A topology φ on $s - C(Y,Z)$ is called s-splitting topology, if Λ is a continuous function with respect to φ.*

Definition 4.4.2. *(Muturi E. N. et al., 2018(a)). Consider the exponential mapping Λ^{-1} : $C(X, s - C_\varphi(Y,Z)) \to C(X \times Y,Z)$, defined by $\Lambda^{-1}((g)(x,y)) = g(x)(y)$ where $g \in C(X, s - C_\varphi(Y,Z))$ for each $(x,y) \in X \times Y$. A topology φ on $s - C(Y,Z)$ is called s-admissible topology, if the function Λ^{-1} is continuous with respect to φ.*

Proposition 4.4.3. *McCoy and Ntantu (1988) The function $\Lambda^{-1} \circ \Lambda : C(X \times Y,Z) \to C(X \times Y,Z)$ is continuous.*

Proof. Observe that $(\Lambda^{-1} \circ \Lambda(f))(x,y) = \Lambda^{-1}(\Lambda(f))(x,y) = \Lambda(f)(x)(y) = f(x,y)$. Implying that $\Lambda^{-1} \circ \Lambda(f) = f$. Hence, $\Lambda^{-1} \circ \Lambda$ is an identity function and therefore continuous. ■

Proposition 4.4.4. McCoy and Ntantu (1988) *The function* $\Lambda \circ \Lambda^{-1} : C(X, s - C_\varphi(Y,Z)) \to C(X, s - C_\varphi(Y,Z))$ *is continuous.*

Proof. Observe that $(\Lambda \circ \Lambda^{-1}(f))(x)(y) = \Lambda(\Lambda^{-1}(f))(x)(y) = \Lambda^{-1}(f)(x,y) = f(x)(y)$. Implying that $\Lambda \circ \Lambda^{-1}(f) = f$. Hence, $\Lambda \circ \Lambda^{-1}$ is an identity function and therefore continuous. ■

Proposition 4.4.5. (Muturi E. N. et al., 2018(a)). *the exponential mapping* $\Lambda : C(X \times Y, Z) \to C(X, s - C_\varphi(Y,Z))$*, defined by* $\Lambda(f)(x)(y) = f(x,y)$ *for each* $f \in C(X \times Y, Z)$*,* $x \in X$ *and* $y \in Y$*, is a homeomorphism.*

Proof. From Proposition 4.4.3 and Proposition 4.4.4, it follows that Λ is a homeomorphism. ■

Proposition 4.4.6. (Muturi E. N. et al., 2018(a)). *The function* $i : C(X, s - C_{\varphi_1}(Y,Z)) \to C(X, s - C_{\varphi_2}(Y,Z))$ *is continuous if and only if* $\varphi_2 \subset \varphi_1$.

Proof. The function i is continuous if and only if $S(W, S(U,V)) \in \varphi_2$ implies that $i^{-1}(S(W, S(U,V))) \in \varphi_1$, but i is an identity function, therefore, $i^{-1}(S(W, S(U,V))) = S(W, S(U,V))$. Hence, i is continuous if and only if $S(W, S(U,V)) \in \varphi_2$ implies $S(W, S(U,V)) \in \varphi_1$. ■

Theorem 4.4.7. (Muturi E. N. et al., 2018(a)). *Let* φ_1 *and* φ_2 *be topologies defined on the function space* $s - C(Y,Z)$.

(i) If φ_1 *is an s-splitting topology on* $s - C(Y,Z)$ *and* $\varphi_2 \subset \varphi_1$*, then* φ_2 *is also an s-splitting topology on* $s - C(Y,Z)$.

(ii) If φ_1 *is an s-admissible topology on* $s - C(Y,Z)$ *and* $\varphi_1 \subset \varphi_2$*, then* φ_2 *is also an s-admissible topology on* $s - C(Y,Z)$.

(iii) If φ_1 *is an s-splitting topology on* $s - C(Y,Z)$ *and* φ_2 *an admissible topology on* $s - C(Y,Z)$*, then* $\varphi_1 \subset \varphi_2$.

Proof.

(i) Let φ_1 be s-splitting topology, then by Definition 4.4.1 the function $\Lambda : C(X \times Y, Z) \to C(X, s - C_{\varphi_1}(Y,Z))$, defined by $\Lambda(f)(x)(y) = f(x,y)$ for each $f \in C(X \times Y, Z)$, $x \in X$ and

$y \in Y$, is continuous with respect to φ_1. Let φ_2 be any other topology such that $\varphi_2 \subset \varphi_1$, then by Proposition 4.4.6, the function $i : C(X, s - C_{\varphi_1}(Y,Z)) \to C(X, s - C_{\varphi_2}(Y,Z))$ is continuous. Now the composite function $i \circ \Lambda : C(X \times Y, Z) \to C(X, s - C_{\varphi_2}(Y,Z))$ is continuous with respect to φ_2, implying that φ_2 is also s-splitting topology.

(ii) Let φ_1 be s-admissible topology, then by Definition 4.4.2 the function $\Lambda^{-1} : C(X, s - C_{\varphi_1}(Y,Z)) \to C(X \times Y, Z)$ defined by $\Lambda^{-1}((g)(x,y)) = g(x)(y)$ where $g \in C(X, s - C_{\varphi}(Y,Z))$ for each $(x,y) \in X \times Y$, is continuous with respect to φ_1. Let $\varphi_1 \subset \varphi_2$, then by Proposition 4.4.6, the function $i : C(X, s - C_{\varphi_2}(Y,Z)) \to C(X, s - C_{\varphi_1}(Y,Z))$ is continuous. Now the composite function $\Lambda^{-1} \circ i : C(X, s - C_{\varphi_2}(Y,Z)) \to C(X \times Y, Z)$ is continuous with respect to φ_2. Hence, φ_2 is also s-admissible topology.

(iii) Let φ_1 be a s-splitting topology, then by Definition 4.4.1 the function $\Lambda : C(X \times Y, Z) \to C(X, s - C_{\varphi_1}(Y,Z))$, defined by $\Lambda(f)(x)(y) = f(x,y)$ for each $f \in C(X \times Y, Z)$, $x \in X$ and $y \in Y$, is continuous with respect to φ_1. Let φ_2 be s-admissible topology, then by Definition 4.4.2 the function $\Lambda^{-1} : C(X, s - C_{\varphi_1}(Y,Z)) \to C(X \times Y, Z)$ defined by $\Lambda^{-1}((g)(x,y)) = g(x)(y)$ where $g \in C(X, s - C_{\varphi}(Y,Z))$ for each $(x,y) \in X \times Y$, is continuous with respect to φ_1. Now the composite function $\Lambda \circ \Lambda^{-1} : C((X,\sigma), s - C_{\varphi_2}(Y,Z)) \to C((X,\sigma), s - C_{\varphi_1}(Y,Z))$ is continuous by Proposition 4.4.6, implying that $\varphi_1 \subset \varphi_2$. ■

CHAPTER FIVE

SEPARATION AXIOMS ON FUNCTION SPACES DEFINED ON BITOPOLOGICAL SPACES

5.1 Introduction

In this chapter, separation axioms are introduced on the space $p-C_{\omega}(Y,Z)$. The relationship between these separation axioms and those defined on the spaces (Z,δ_i) for $i=1,2$, (Z,δ_1,δ_2), $1-C_{\varsigma}(Y,Z)$, $2-C_{\zeta}(Y,Z)$ and $s-C_{\tau}(Y,Z)$ are established. A number of the results obtained concurs with the results of Reilly (1972) on bitopological spaces, those of Lal (1978) on pairwise concepts in bitopological spaces, as well as those of Arens (1946) on function spaces.

The role of separation axioms on the function spaces $p-C_{\omega}(Y,Z)$ and $s-C_{\tau}(Y,Z)$ is that it allows restrictive conditions that give rise to more stronger properties defined on this spaces, such properties include but not limited to continuity of functions, splitting and admissibility properties and compactness.

5.2 Separation axioms on the space $p-C_{\omega}(Y,Z)$

Separation axioms defined on the function space $C_{\tau}(Y,Z)$ are generalized to the space $p-C_{\omega}(Y,Z)$ of pairwise continuous functions between bitopological spaces. The notation $_pT_i$ for $i=0,1,2$ and $_p$regular, denotes separation axioms defined on $p-C_{\omega}(Y,Z)$, to differentiate them from pairwise separation axioms defined on bitopological space (Y,τ_1,τ_2) and normally denoted by $p-T_i$ for $i=0,1,2$ and p-regular.

The definition of separation axioms are generalized to function space $p-C_{\omega}(Y,Z)$ as follows.

Definition 5.2.1. *(Muturi E. N. et al., 2018(b)). A function space $p-C_{\omega}(Y,Z)$ is said to be a $_pT_{\circ}$-space, if for any two distinct functions f and g in $p-C_{\omega}(Y,Z)$, there exist an open set $S((U_1,V_1)(A_1,B_1))_p=\{f\in p-C_{\omega}(Y,Z): f(U_1)\subset V_1$ and $f(A_1)\subset B_1\}$ neighborhood of f not containing g, or $S((U_2,V_2)(A_2,B_2))_p=\{g\in p-C(Y,Z): g(U_2)\subset V_2$ and $g(A_2)\subset B_2\}$ neighborhood of g not containing f.*

Definition 5.2.2. (Muturi E. N. et al., 2018(b)). *A function space $p-C_\omega(Y,Z)$ is said to be a ${}_pT_1$-space, if for any two distinct functions f and g in $p-C_\omega(Y,Z)$, there exist open sets $S((U_1,V_1)(A_1,B_1))_p = \{f \in p-C(Y,Z) : f(U_1) \subset V_1$ and $f(A_1) \subset B_1\}$ neighborhood of f not containing g, and $S((U_2,V_2)(A_2,B_2))_p = \{g \in p-C_\omega(Y,Z) : g(U_2) \subset V_2$ and $g(A_2) \subset B_2\}$ neighborhood of g not containing f.*

Definition 5.2.3. (Muturi E. N. et al., 2018(b)). *A function space $p-C_\omega(Y,Z)$ is said to be a ${}_pT_2$-space, if for any two distinct functions f and g in $p-C(Y,Z)$, there exist disjoint open sets $S((U_1,V_1)(A_1,B_1))_p = \{f \in p-C(Y,Z) : f(U_1) \subset V_1$ and $f(A_1) \subset B_1\}$ and $S((U_2,V_2)(A_2,B_2))_p = \{g \in p-C(Y,Z) : g(U_2) \subset V_2$ and $g(A_2) \subset B_2\}$ neighborhoods of f and g respectively.*

Definition 5.2.4. (Muturi E. N. et al., 2018(b)). *A function space $p-C_\omega(Y,Z)$ is said to be a ${}_p$regular space, if for any two distinct functions f and g in $p-C(Y,Z)$ and a closed set $\overline{S((\mathcal{U},\mathcal{V})(\mathcal{A},\mathcal{B})}$ in $p-C(Y,Z)$ such that $g \notin \overline{S(\mathcal{U},\mathcal{V})(\mathcal{A},\mathcal{B})}$, there exist disjoint open sets $S((U_1,V_1)(A_1,B_1))_p = \{f \in p-C(Y,Z) : f(U_1) \subset V_1$ and $f(A_1) \subset B_1\}$ containing $\overline{S((\mathcal{U},\mathcal{V})(\mathcal{A},\mathcal{B})}$ and $S((U_2,V_2)(A_2,B_2))_p = \{g \in p-C(Y,Z) : g(U_2) \subset V_2$ and $g(A_2) \subset B_2\}$ neighborhood of g.*

Remark 5.2.5. *For convenience, the following notations will also be used. If both the spaces (Z,δ_1) and (Z,δ_2) and both the function spaces $1-C_\varsigma(Y,Z)$ and $2-C_\zeta(Y,Z)$ have a topological property P, then it will be denoted by $b-P$, and if the bitopological space (Z,δ_1,δ_2) has the property P, then it will be denoted by $p-P$.*

5.3 Comparison of separation axioms defined on the spaces $p-C_\omega(Y,Z)$, (Z,δ_1), (Z,δ_2) and (Z,δ_1,δ_2)

In this section, relationships between separation axioms defined on the spaces $p-C_\omega(Y,Z)$, (Z,δ_1) and (Z,δ_2) are established.

Theorem 5.3.1. (Muturi E. N. et al., 2018(b)). *Let (Z,δ_1) and (Z,δ_2) be $b-T_\circ$ spaces, then $p-C_\omega(Y,Z)$ is a ${}_pT_\circ$ space.*

Proof. Let f and g be unique functions in $p-C(Y,Z)$ such that for every $y \in Y$, $f(y) \neq g(y)$ and let (Z,δ_1) and (Z,δ_2) be $b-T_\circ$ spaces. Then there exist an open set $U_1 \in \delta_1$ such that $f(y) \in U_1$, $g(y) \notin U_1$ and $U_2 \in \delta_2$ such that $f(y) \in U_2$ and $g(y) \notin U_2$ or $V_1 \in \delta_1$ such that $g(y) \in V_1$, $f(y) \notin V_1$ and $V_2 \in \delta_2$ such that $g(y) \in V_2$, $f(y) \notin V_2$. Hence, there exist an open set $S((\{y\},U_1)(\{y\},U_2))_p$ in $p-C_\omega(Y,Z)$ neighbourhood of f not containing g or an open set $S((\{y\},V_1)(\{y\},V_2))_p$ in $p-C_\omega(Y,Z)$ neighborhood of g not containing f. Therefore, the space $p-C_\omega(Y,Z)$ is a ${}_pT_\circ$ space. ■

Theorem 5.3.2. (Muturi E. N. et al., 2018(b)). *Let (Z,δ_1) and (Z,δ_2) be $b-T_1$ spaces, then $p-C_\omega(Y,Z)$ is a ${}_pT_1$ space.*

Proof. Let f and g be unique functions in $p-C(Y,Z)$ such that for every $y \in Y$, $f(y) \neq g(y)$ and let (Z,δ_1) and (Z,δ_2) be $b-T_1$ spaces. Then there exist an open set U_1 and V_1 in δ_1 such that $f(y) \in U_1$, $g(y) \notin U_1$ and $g(y) \in V_1$, $f(y) \notin V_1$, and also U_2 and V_2 in δ_2 such that $f(y) \in U_2$, $g(y) \notin U_2$ and $g(y) \in V_2$, $f(y) \notin V_2$. Hence, there exist an open set $S((\{y\},U_1)(\{y\},U_2))_p$ in $p-C_\omega(Y,Z)$ neighbourhood of f not containing g and open set $S((\{y\},V_1)(\{y\},V_2))_p$ in $p-C_\omega(Y,Z)$ neighborhood of g not containing f. Therefore, the space $p-C_\omega(Y,Z)$ is a ${}_pT_1$ space. ■

Theorem 5.3.3. (Muturi E. N. et al., 2018(b)). *Let (Z,δ_1) and (Z,δ_2) be $b-T_2$ spaces, then $p-C_\omega(Y,Z)$ is a ${}_pT_2$ space.*

Proof. Let f and g be unique functions in $p-C(Y,Z)$ such that for every $y \in Y$, $f(y) \neq g(y)$ and let (Z,δ_1) and (Z,δ_2) be $b-T_2$ spaces. Then there exist disjoint open sets $U_1 \in \delta_1$ and $V_1 \in \delta_1$ and also $U_2 \in \delta_2$ and $V_2 \in \delta_2$ such that $f(y) \in U_1$ and $g(y) \in V_1$, and also $f(y) \in U_2$ and $g(y) \in V_2$ respectively. Hence, there exist disjoint open sets $S((\{y\},U_1)(\{y\},U_2))_p$ in $p-C_\omega(Y,Z)$ neighbourhood of f and $S((\{y\},V_1)(\{y\},V_2))_p$ in $p-C_\omega(Y,Z)$ neighborhood of g. Therefore, the space $p-C_\omega(Y,Z)$ is a ${}_pT_2$ space. ■

Theorem 5.3.4. (Muturi E. N. et al., 2018(b)). *Let the spaces (Z,δ_1) and (Z,δ_2) be b-regular, then $p-C_\omega(Y,Z)$ with compact open topology ω is a ${}_p$regular space.*

Proof. Let f and g be unique functions in $p-C(Y,Z)$ such that $\forall y \in Y$ $f(y) \neq g(y)$ and let $S((U_i,V_i)(U_j,(V_j))$
$= \{f \in p-C(Y,Z) : f(U_i) \subset V_i \text{ and } f(U_j) \subset V_j)\}$ for $U_i \in \tau_1$, $V_i \in \delta_1$, $U_j \in \tau_2$ and $V_j \in \delta_2$ for $i,j = 1,2,3,4....n$ be the neighbourhood system for f. Since U_i and U_j are compact, then both $f(U_i)$ and $f(U_j)$ are also compact, and since (Z,δ_1) and (Z,δ_2) are b-regular spaces, then there exist open sets A_i and B_j in δ_1 and δ_2 respectively, for $i,j = 1,2,3,4....n$, such that $f(U_i) \subset A_i$, $f(U_j) \subset B_j$, $\overline{A_i} \subset V_i$ and $\overline{B_j} \subset V_j$. This implies that $S((U_i,A_i)(U_j,B_j)) \subset S((U_i,\overline{A}_i)(U_j,\overline{B}_j)) \subset S((U_i,V_i)(U_j,V_j))$. Suppose that

$\overline{S((U_i,A_i)(U_j,B_j))} \subset S((U_i,\overline{A}_i)(U_j,\overline{B}_j))$, let $g \notin S((U_i,V_i)(U_j,V_j))$, then it follows that $g \notin S((U_i,\overline{A}_i)(U_j,\overline{B}_j))$, implying further that for some point $y \in Y$, $g(y) \in \overline{A_i}^{\,c}$ and $g(y) \in \overline{B_j}^{\,c}$. Thus, $S((\{y\},\overline{A_i}^{\,c})(\{y\},\overline{B_j}^{\,c}))$ is a neighbourhood system for g which does not intersect $S((U_i,\overline{A}_i)(U_j,\overline{B}_j))$. Since $\overline{S((U_i,A_i)(U_j,B_j))} \subset S((U_i,\overline{A}_i)(U_j,\overline{B}_j))$, then $\overline{S((U_i,A_i)(U_j,B_j))} \subset S((U_i,V_i)(U_j,V_j))$. Therefore the sets $\bigcap_{i=1}^{n} S((\{y\},\overline{A_i}^{\,c})(\{y\},\overline{B_j}^{\,c}))$ and $\bigcap_{i,j=1}^{n} S((U_i,V_i)(U_j,V_j))$ are disjoint open sets containing g and $\overline{\bigcap_{i=1}^{n} S((U_i,A_i)(U_j,B_j))}$ respectively, hence $p-C_\omega(Y,Z)$ is a $_p$regular space. ■

The subsequent theorems presents the relationships between separation axioms defined on the spaces $p-C_\omega(Y,Z)$ and (Z,δ_1,δ_2).

Theorem 5.3.5. *(Muturi E. N. et al., 2018(b)). Let (Z,δ_1,δ_2) be $p-T_\circ$ space, then $p-C_\omega(Y,Z)$ is a $_pT_\circ$ space.*

Proof. Let f and g be unique functions in $p-C(Y,Z)$ such that for every $y \in Y$, $f(y) \neq g(y)$, since (Z,δ_1,δ_2) is a $p-T_\circ$ space, then there exist an open set $U_1 \in \delta_1$ containing $f(y)$ but not $g(y)$ or $V_2 \in \delta_2$ containing $g(y)$ but not $f(y)$. Suppose there exist an open set $U_1 \in \delta_1$ containing $f(y)$ but not $g(y)$, then by pairwise continuity of f, there is also an open set $U_2 \in \delta_2$ also containing $f(y)$ but not $g(y)$. Suppose there exist an open set $V_2 \in \delta_2$ containing $g(y)$ but not $f(y)$, then by pairwise continuity of g, there is also an open set $V_1 \in \delta_1$ containing $g(y)$ but not $f(y)$. Either way, there exist an open set $S((\{y\},U_1)(\{y\},U_2))_p$ in $p-C_\omega(Y,Z)$, neighbourhood of f not containing g, or an open set $S((\{y\},V_1)(\{y\},V_2))_p$ in $p-C_\omega(Y,Z)$, neighborhood of g not containing f. Therefore, the space $p-C_\omega(Y,Z)$ is a $_pT_\circ$ space. ■

Theorem 5.3.6. *(Muturi E. N. et al., 2018(b)). Let (Z,δ_1,δ_2) be $p-T_1$ space, then $p-C_\omega(Y,Z)$ is a $_pT_1$ space.*

Proof. Let f and g be unique functions in $p-C(Y,Z)$ such that for every $y \in Y$, $f(y) \neq g(y)$, since (Z,δ_1,δ_2) is a $p-T_1$ space. Then there exist an open set $U_1 \in \delta_1$ neighbourhood of $f(y)$ and not $g(y)$ and an open set $V_2 \in \delta_2$ neighbourhood of $g(y)$ and not $f(y)$. But since f and g are both $\tau_1-\delta_1$ and $\tau_2-\delta_2$ continuous, it follows that there exist $U_2 \in \delta_2$ neighbourhood of $f(y)$ and not $g(y)$ and $V_2 \in \delta_2$ neighbourhood of $g(y)$ and not $f(y)$. Hence, there exist an open set $S((\{y\},U_1)(\{y\},U_2))_p$ in $p-C_\omega(Y,Z)$ neighbourhood of f not containing g and open set $S((\{y\},V_1)(\{y\},V_2))_p$ in $p-C_\omega(Y,Z)$ neighborhood of g not containing f. Therefore, the space $p-C_\omega(Y,Z)$ is a $_pT_1$ space. ■

Theorem 5.3.7. *(Muturi E. N. et al., 2018(b)). Let (Z,δ_1,δ_2) be totally disconnected $p-T_2$ space, then $p-C_\omega(Y,Z)$ is a ${}_pT_2$ space.*

Proof. Let f and g be unique functions in $p-C(Y,Z)$ such that for every $y\in Y$, $f(y)\neq g(y)$, since (Z,δ_1,δ_2) is a totally disconnected $p-T_2$ space, then there exist disjoint open sets $U_1\in\delta_1$ and $V_2\in\delta_2$ containing $f(y)$ and $g(y)$ respectively, such that $U_1\cup V_2=Y$. But since f and g are both $\tau_1-\delta_1$ and $\tau_2-\delta_2$ continuous, it follows that there exist open sets $U_2\in\delta_2$ containing $f(y)$ and $V_1\in\delta_1$ containing $g(y)$. Suppose $U_2={V_2}^c\in\delta_2$ and $V_1={U_1}^c\in\delta_1$. Now, ${V_2}^c\cup{U_1}^c=(V_2\cap U_1)^c=(\phi)^c=Y$, implying that $U_2\cup V_1=Y$, Now, $U_2\cap V_1={V_2}^c\cap{U_1}^c=(V_2\cup U_1)^c=Y^c=\phi$. Therefore the sets U_2 and V_1 are disjoint open sets, neighbourhoods of $f(y)$ and $g(y)$ respectively. Therefore the sets $S((\{y\},U_1)(\{y\},U_2))_p$ and $S((\{y\},V_1)(\{y\},V_2))_p$ in $p-C_\omega(Y,Z)$ are disjoint open sets, neighbourhoods of f and g respectively. Hence, $p-C_\omega(Y,Z)$ is a ${}_pT_2$ space. ■

Theorem 5.3.8. *(Muturi E. N. et al., 2018(b)). Let the space (Z,δ_1,δ_2) be pairwise regular, then $p-C_\omega(Y,Z)$ is a ${}_p$regular space.*

Proof. Let f and g be unique functions in $p-C(Y,Z)$ such that $\forall y\in Y$ $f(y)\neq g(y)$ and let $S((U_i,V_i)(U_j,(V_j)) = \{f\in p-C(Y,Z) : f(U_i)\subset V_i \textit{ and } f(U_j)\subset V_j)\}$ for $U_i\in\tau_1$, $V_i\in\delta_1$, $U_j\in\tau_2$ and $V_j\in\delta_2$ for $i,j=1,2,3,4....n$ be the neighbourhood system for f. Now, U_i and U_j are both compact, therefore $f(U_i)$ and $f(U_j)$ are also compact. Since (Z,δ_1,δ_2) is pairwise regular space and f is pairwise continuous, then δ_1 regularity with respect to δ_2 implies that there exist open sets B_j in δ_2 for $j=1,2,3,4....n$, such that $f(U_i)\subset B_j$ and $\overline{B_j}\subset V_j$. Since $f(U_j)\subset V_j$ and $\overline{B_j}\subset V_j$, then there exist some B_j's such that $f(U_j)\subset B_j$ and $\overline{B_j}\subset V_j$. This implies that $S(U_j,B_j)\subset S(U_j,\overline{B}_j)\subset S(U_j,V_j)$. Suppose that $\overline{S(U_j,B_j)}\subset S(U_j,\overline{B}_j)$, let $g\notin S(U_j,V_j)$, then it follows that $g\notin S(U_j,\overline{B}_j)$, implying further that for some point $y\in Y$, $g(y)\in\overline{B_j}^{\,c}$. Thus, $S(\{y\},\overline{B_j}^{\,c})$ is a neighbourhood system for g which does not intersect $S(U_j,\overline{B}_j)$. Since $\overline{S(U_j,B_j)}\subset S(U_j,\overline{B}_j)$, then $\overline{S(U_j,B_j)}\subset S(U_j,V_j)$. Therefore $\bigcap_{j=1}^{n} S(\{y\},\overline{B_j}^{\,c})$ and $\bigcap_{j=1}^{n} S(U_j,V_j)$ are $\tau_2-\delta_2$ disjoint open sets neighbourhoods of g and $\overline{\bigcap_{i=1}^{n} S(U_j,B_j)}$ respectively. Now, δ_2 regularity with respect to δ_1 implies that there exist open sets A_i in δ_1 for $i=1,2,3,4....n$, such that $f(U_j)\subset A_i$ and $\overline{A_i}\subset V_i$. But $f(U_i)\subset V_i$ and $\overline{A_i}\subset V_i$, therefore there exist A_i's such that $f(U_i)\subset A_i$ and $\overline{A_i}\subset V_i$. This implies that $S(U_i,A_i)\subset S(U_i,\overline{A}_i)\subset S(U_i,V_i)$. Suppose that $\overline{S(U_i,A_i)}\subset S(U_i,\overline{A}_i)$, let $g\notin S(U_i,V_i)$, then it follows that $g\notin S(U_i,\overline{A}_i)$, implying further that for some point $y \in Y$, $g(y) \in \overline{A_i}^{\,c}$. Thus, $S(\{y\}, \overline{A_i}^{\,c})$ is a neighbourhood system for g which does not intersect $S(U_i, \overline{A}_i)$. Since $\overline{S(U_i, A_i)} \subset S(U_i, \overline{A}_i)$, then $\overline{S(U_i, A_i)} \subset S(U_i, V_i)$. Therefore $\bigcap_{i=1}^{n} S(\{y\}, \overline{A_i}^{\,c})$ and $\bigcap_{i=1}^{n} S(U_i, V_i)$ are $\tau_1 - \delta_1$ disjoint open sets, neighbourhoods of g and $\overline{\bigcap_{i=1}^{n} S(U_i, A_i)}$ respectively. Let $f \in \overline{S(U_i, A_i)}$ and $f \in \overline{S(U_j, B_j)}$ imply that $f \in \overline{S((U_i, A_i), (U_j, B_j))}$, then $\bigcap_{i,j=1}^{n} S((\{y\}, \overline{A_i}^{\,c})(\{y\}, \overline{B_j}^{\,c}))$ and $\bigcap_{i,j=1}^{n} S((U_i, V_i)(U_j, V_j))$ are disjoint open sets neighbourhoods of g and $\overline{\bigcap_{i,j=1}^{n} S((U_i, A_i), (U_j, B_j))}$ respectively in $p - C_{\omega}(Y, Z)$. Therefore $p - C_{\omega}(Y, Z)$ is a ${}_p$regular space. ■

5.4 Comparison of separation axioms between the spaces $p - C_{\omega}(Y, Z)$, $s - C_{\tau}(Y, Z)$, $1 - C_{\varsigma}(Y, Z)$ and $2 - C_{\zeta}(Y, Z)$

The relationships between separation axioms defined on the spaces $p - C_{\omega}(Y, Z)$, $1 - C_{\varsigma}(Y, Z)$ and $2 - C_{\zeta}(Y, Z)$ are established in this section.

Theorem 5.4.1. *(Muturi E. N. et al., 2018(b)). The function space $p - C_{\omega}(Y, Z)$ is a ${}_pT_{\circ}$-space, if and only if the function spaces $1 - C_{\varsigma}(Y, Z)$ and $2 - C_{\zeta}(Y, Z)$ are $b - T_{\circ}$-spaces.*

Proof. Let f and g be unique functions in $p - C(Y, Z)$ such that for every $y \in Y$, $f(y) \neq g(y)$ and let $1 - C_{\varsigma}(Y, Z)$ and $2 - C_{\zeta}(Y, Z)$ be $b - T_{\circ}$ spaces. Since f and g are both $\tau_1 - \delta_1$ and $\tau_2 - \delta_2$ continuous, there exist open sets $S(\{y\}, U_1)$ and $S(\{y\}, U_2)$ neighbourhoods of f but not g, or $S(\{y\}, V_1)$ and $S(\{y\}, V_2)$ neighbourhoods of g but not f for U_1, $V_1 \in \delta_1$ and U_2, $V_2 \in \delta_2$. Hence, there exist an open set $S((\{y\}, U_1)(\{y\}, U_2))_p$ in $p - C_{\omega}(Y, Z)$ neighbourhood of f not containing g or open set $S((\{y\}, V_1)(\{y\}, V_2))_p$ in $p - C_{\omega}(Y, Z)$ neighborhood of g not containing f. Therefore, $p - C_{\omega}(Y, Z)$ is a ${}_pT_{\circ}$ space.

Conversely, Suppose $p - C_{\omega}(Y, Z)$ is a ${}_pT_{\circ}$ space, then for any two functions f and g in $p - C(Y, Z)$ such that $f(y) \neq g(y)$ for $\forall y \in Y$, there exist an open set $S((\{y\}, U_1)(\{y\}, U_2))_p$ in $p - C_{\omega}(Y, Z)$ neighbourhood of f not containing g or an open set $S((\{y\}, V_1)(\{y\}, V_2))_p$ in $p - C_{\omega}(Y, Z)$ neighborhood of g not containing f. If $S((\{y\}, U_1)(\{y\}, U_2))_p$ is in $p - C_{\omega}(Y, Z)$, then $S((\{y\}, U_1)(\{y\}, U_2))_p = \{f \in p - C(Y, Z) : f(y) \subset U_1 \textit{ and } f(y) \subset U_2\} = \{\{f \in p - C(Y, Z) : f(y) \subset U_1\} \textit{ and } \{f \in p - C(Y, Z) : f(y) \subset U_2\}\}$. Now, $\{f \in p - C(Y, Z) : f(y) \subset U_1\} = \{f \in 1 - C(Y, Z) : f(y) \subset U_1\} = S(\{y\}, U_1)$ and $\{f \in p - C(Y, Z) : f(y) \subset U_2\} = \{f \in 2 - C(Y, Z) : f(y) \subset U_2\} = S(\{y\}, U_2)$. These two sets are open and are both neighborhood of f in $1 - C_{\varsigma}(Y, Z)$ and $2 - C_{\zeta}(Y, Z)$ respectively.

If $S((\{y\},V_1)(\{y\},V_2))_p$ is in $p-C_\omega(Y,Z)$, then in a similar manner, two open sets $S(\{y\},V_1)$ and $S(\{y\},V_2)$ both neighborhood of g in $1-C_\varsigma(Y,Z)$ and $2-C_\zeta(Y,Z)$ are obtained. Either way, there exist open sets $S(\{y\},U_1)$ in $1-C_\varsigma(Y,Z)$ and $S(\{y\},U_2)$ in $2-C_\zeta(Y,Z)$ neighborhoods of f but not g or $S(\{y\},V_1)$ in $1-C_\varsigma(Y,Z)$ and $S(\{y\},V_2)$ in $2-C_\zeta(Y,Z)$ neighborhoods of g but not f. Hence, $1-C_\varsigma(Y,Z)$ and $2-C_\zeta(Y,Z)$ are $b-T_\circ$-spaces. ■

Theorem 5.4.2. *(Muturi E. N. et al., 2018(b)). The function space $p-C_\omega(Y,Z)$ is a ${}_pT_1$-space, if and only if the function spaces $1-C_\varsigma(Y,Z)$ and $2-C_\zeta(Y,Z)$ are $b-T_1$-spaces.*

Proof. Let f and g be unique functions in $p-C(Y,Z)$ such that $\forall y\in Y$ $f(y)\neq g(y)$, and let $1-C_\varsigma(Y,Z)$ be a T_1 space such that $S(U_1,V_1)$ is neighborhood of f and not g and $S(U_2,V_2)$ is neighborhood of g and not f. Let $2-C_\zeta(Y,Z)$ also be a T_1 space such that $S(A_1,B_1)$ is a neighborhood of f and not g and $S(A_2,B_2)$ is neighborhood of g and not f. Since f and g are both $\tau_1-\delta_1$ and $\tau_2-\delta_2$ continuous, then $S((U_1,V_1)(A_1,B_1))_p = \{f\in p-C(Y,Z) : f(U_1)\subset V_1$ and $f(A_1)\subset B_1\}$ is a neighborhood of f and not g and $S((U_2,V_2)(A_2,B_2))_p = \{g\in p-C(Y,Z) : g(U_2)\subset V_2$ and $g(A_2)\subset B_2\}$ is neighborhood of g and not f. Hence, $p-C_\omega(Y,Z)$ is a ${}_pT_1$ space.

Conversely, let $p-C_\omega(Y,Z)$ be a ${}_pT_1$ space and let f and g be unique functions in $p-C(Y,Z)$ such that $\forall y\in Y$ $f(y)\neq g(y)$, then there exist two open sets $S((U_1,V_1)(A_1,B_1))_p = \{f\in p-C(Y,Z) : f(U_1)\subset V_1$ and $f(A_1)\subset B_1\}$ for U_1 open in τ_1, V_1 open in δ_1, A_1 open in τ_2 and B_1 open in δ_2, neighborhood of f not containing g and $S((U_2,V_2)(A_2,B_2))_p = \{g\in p-C(Y,Z) : g(U_2)\subset V_2$ and $g(A_2)\subset B_2\}$ for U_2 open in τ_1, V_2 open in δ_1, A_2 open in τ_2 and B_2 open in δ_2, neighborhood of g not containing f. But $S((U_1,V_1)(A_1,B_1))_p = \{f\in p-C(Y,Z) : f(U_1)\subset V_1$ and $f(A_1)\subset B_1\} = \{f\in 1-C(Y,Z) : f(U_1)\subset V_1\} = S(U_1,V_1)$ and $S((U_1,V_1)(A_1,B_1))_p = \{f\in p-C(Y,Z) : f(U_1)\subset V_1$ and $f(A_1)\subset B_1\} = \{f\in 2-C(Y,Z) : f(A_1)\subset B_1\} = S(A_1,B_1)$ which are both neighborhoods of f not containing g in $1-C_\varsigma(Y,Z)$ and $2-C_\zeta(Y,Z)$ respectively. Similarly $S(U_2,V_2)$ and $S(A_2,B_2)$ are both neighborhoods of g not containing f in $1-C_\varsigma(Y,Z)$ and $2-C_\zeta(Y,Z)$ respectively. Hence, $S(U_1,V_1)$ in $1-C_\varsigma(Y,Z)$ and $S(A_1,B_1)$ in $2-C_\zeta(Y,Z)$ are neighborhoods of f not containing g, and $S(U_2,V_2)$ in $1-C_\varsigma(Y,Z)$ and $S(A_2,B_2)$ in $2-C_\zeta(Y,Z)$ are neighborhoods of g not containing f. Therefore, $1-C_\varsigma(Y,Z)$ and $2-C_\zeta(Y,Z)$ are $b-T_1$ spaces. ■

Theorem 5.4.3. *(Muturi E. N. et al., 2018(b)). The function space $p-C_{\omega}(Y,Z)$ is a ${}_pT_2$-space, if and only if the function spaces $1-C_{\varsigma}(Y,Z)$ and $2-C_{\zeta}(Y,Z)$ are $b-T_2$-spaces.*

Proof. Let f and g be unique functions in $p-C(Y,Z)$ such that $\forall y \in Y$ $f(y) \neq g(y)$, and let $1-C_{\varsigma}(Y,Z)$ be a T_2 space such that $S(U_1,V_1)$ and $S(U_2,V_2)$ are disjoint open sets, neighbourhoods of f and g respectively. Also, Let $2-C_{\zeta}(Y,Z)$ be a T_2 space such that $S(A_1,B_1)$ and $S(A_2,B_2)$ are disjoint open sets, neighbourhoods of f and g respectively. Now, pairwise continuity of f and g allows one to pick $S((U_1,V_1)(A_1,B_1))_p = \{f \in p-C(Y,Z) : f(U_1) \subset V_1$ and $f(A_1) \subset B_1\}$ and $S((U_2,V_2)(A_2,B_2))_p = \{g \in p-C(Y,Z) : g(U_2) \subset V_2$ and $g(A_2) \subset B_2\}$ as disjoint open sets in $p-C_{\omega}(Y,Z)$, containing f and g respectively. Hence $p-C_{\omega}(Y,Z)$ is a ${}_pT_2$ space.

Conversely, let $p-C_{\omega}(Y,Z)$ be a ${}_pT_2$-space and let f and g be unique functions in $p-C(Y,Z)$ such that $\forall y \in Y$ $f(y) \neq g(y)$, then there exist two disjoint open sets $S((U_1,V_1)(A_1,B_1))_p = \{f \in p-C(Y,Z) : f(U_1) \subset V_1$ and $f(A_1) \subset B_1\}$ for U_1 open in τ_1, V_1 open in δ_1, A_1 open in τ_2 and B_1 open in δ_2, neighborhood of f, and $S((U_2,V_2)(A_2,B_2))_p$ $= \{g \in p-C(Y,Z) : g(U_2) \subset V_2$ and $g(A_2) \subset B_2\}$ for U_2 open in τ_1, V_2 open in δ_1, A_2 open in τ_2 and B_2 open in δ_2, neighborhood of g. But $S((U_1,V_1)(A_1,B_1))_p = \{f \in p-C(Y,Z) : f(U_1) \subset V_1$ *and* $f(A_1) \subset B_1\}$=$\{\{f \in p-C(Y,Z) : f(U_1) \subset V_1\}$ and $\{f \in p-C(Y,Z) : f(A_1) \subset B_1\}\}$. Now $\{f \in p-C(Y,Z) : f(U_1) \subset V_1\} = \{f \in 1-C(Y,Z) : f(U_1) \subset V_1\} = S(U_1,V_1)$, and $\{f \in p-C(Y,Z) : f(A_1) \subset B_1\} = \{f \in 2-C(Y,Z) : f(A_1) \subset B_1\} = S(A_1,B_1)$. These two sets are open and are both neighborhood of f in $1-C_{\varsigma}(Y,Z)$ and $2-C_{\zeta}(Y,Z)$ respectively. In a similar manner, $S(U_2,V_2)$ and $S(A_2,B_2)$ are both open set, neighborhood of g in $1-C_{\varsigma}(Y,Z)$ and $2-C_{\zeta}(Y,Z)$ respectively. Now, $S(U_1,V_1)$ and $S(U_2,V_2)$ in $1-C_{\varsigma}(Y,Z)$ are disjoint open neighborhoods of f and g respectively. Also, $S(A_1,B_1)$ and $S(A_2,B_2)$ in $2-C_{\zeta}(Y,Z)$ are disjoint open neighbourhood of f and g respectively. Therefore, $1-C_{\varsigma}(Y,Z)$ and $2-C_{\zeta}(Y,Z)$ are $b-T_2$ spaces. ■

Theorem 5.4.4. *(Muturi E. N. et al., 2018(b)). The function space $p-C_{\omega}(Y,Z)$ is a ${}_p$regular space, if the function spaces $1-C_{\varsigma}(Y,Z)$ and $2-C_{\zeta}(Y,Z)$ are b-regular spaces.*

Proof. let f and g be unique functions in $p-C(Y,Z)$ such that $\forall y \in Y$ $f(y) \neq g(y)$, and let $1-C_{\varsigma}(Y,Z)$ and $2-C_{\zeta}(Y,Z)$ be b-regular. Then for a closed set $\overline{S(U_1,V_1)}$ in $1-C_{\varsigma}(Y,Z)$ such that $f \notin \overline{S(U_1,V_1)}$, there exist disjoint open sets $S(A_1,B_1)$ and $S(C_1,D_1)$ such that $f \in S(A_1,B_1)$ and $\overline{S(U_1,V_1)} \subset S(C_1,D_1)$. Similarly, for a closed set $\overline{S(U_2,V_2)}$ in

$2-C_\zeta(Y,Z)$ such that $f \notin \overline{S(U_2,V_2)}$, there exist disjoint open sets $S(A_2,B_2)$ and $S(C_2,D_2)$ such that $f \in S(A_2,B_2)$ and $\overline{S(U_2,V_2)} \subset S(C_2,D_2)$. Since f is pairwise continuous, we have that $f \in S((A_1,B_1)(A_2,B_2))$. Now, suppose $g \in \overline{S(U_1,V_1)} \subset S(C_1,D_1)$ and $g \in \overline{S(U_2,V_2)} \subset S(C_2,D_2)$ imply that $g \in \overline{S((U_1,V_1)(U_2,V_2))}$, then $g \in \overline{S((U_1,V_1)(U_2,V_2))} \subset S((C_1,D_1)(C_2,D_2))$. Now $\overline{S((U_1,V_1)(U_2,V_2))}$ is a closed subset of $p-C_\omega(Y,Z)$ not containing f, and $S((C_1,D_1)(C_2,D_2))$ and $S((A_1,B_1)(A_2,B_2))$ are disjoint open sets containing $\overline{S((U_1,V_1)(U_2,V_2))}$ and f respectively. Therefore $p-C_\omega(Y,Z)$ is a $_p$regular space. ■

Comparison of separation axiom on the spaces $s-C_\tau(Y,Z)$ and $p-C_\omega(Y,Z)$ is carried out in the theorems that follow. The normal definition of separation axiom on the space $C_\tau(Y,Z)$ hold for the space $s-C_\tau(Y,Z)$.

Theorem 5.4.5. *(Muturi E. N. et al., 2018(b)). Let $p-C_\omega(Y,Z)$ be a $_pT_\circ$-space, then the function space $s-C_\tau(Y,Z)$ is a $T_\circ$-space.*

Proof. Let $p-C_\omega(Y,Z)$ be a $_pT_\circ$, then for any distinct functions f and g in $p-C(Y,Z)$ such that $f(y) \neq g(y)\ \forall y \in Y$, there exist an open set $S((U_1,V_1)(A_1,B_1))_p = \{f \in p-C(Y,Z) : f(U_1) \subset V_1$ and $f(A_1) \subset B_1\}$ for U_1 open in τ_1, V_1 open in δ_1, A_1 open in τ_2 and B_1 open in δ_2, neighborhood of f not containing g, or $S((U_2,V_2)(A_2,B_2))_p = \{g \in p-C(Y,Z) : g(U_2) \subset V_2$ and $g(A_2) \subset B_2\}$ for U_2 open in τ_1, V_2 open in δ_1, A_2 open in τ_2 and B_2 open in δ_2, neighborhood of g not containing f. Since p-continuity imply s-continuity, the set $S((U_1,V_1)(A_1,B_1))_p$ can also be expressed as follows, $S((U_1,V_1)(A_1,B_1))_p = \{f \in s-C(Y,Z) : f|_{(U_1,V_1)}(U_1) \subset V_1 \ and \ f|_{(A_1,B_1)}(A_1) \subset B_1\} = \{f \in s-C(Y,Z) : f(U) \subset V \ for \ U = U_1 \cup A_1 \ and \ V = V_1 \cup B_1\} = S(U,V)_s$ which is a neighborhood of f and not g. Also, $S((U_2,V_2)(A_2,B_2))_p = \{g \in s-C(Y,Z) : g|_{(U_2,V_2)}(U_2) \subset V_2 \ and \ g|_{(A_2,B_2)}(A_2) \subset B_2\} = \{g \in s-C(Y,Z) : g(A) \subset B \ for \ A = U_2 \cup A_2 \ and \ B = V_2 \cup B_2\} = S(A,B)_s$ which is a neighborhood of g and not f. Hence, the function space $s-C_\tau(Y,Z)$ is a $T_\circ$-space. ■

Theorem 5.4.6. *(Muturi E. N. et al., 2018(b)). The function space $s-C_\tau(Y,Z)$ is a T_1 space, if the function space $p-C_\omega(Y,Z)$ is a $_pT_1$ space.*

Proof. Let $p-C_\omega(Y,Z)$ be a $_pT_1$ space and let f and g be unique functions in $p-C(Y,Z)$ such that $\forall y \in Y\ f(y) \neq g(y)$, then there exist two open sets $S((U_1,V_1)(A_1,B_1))_p = \{f \in p-C(Y,Z) : f(U_1) \subset V_1$ and $f(A_1) \subset B_1\}$ for U_1 open in τ_1, V_1 open in δ_1, A_1 open in τ_2 and B_1 open in δ_2, neighborhood of f and not g and $S((U_2,V_2)(A_2,B_2))_p = \{g \in p-C(Y,Z) : g(U_2) \subset V_2$ and $g(A_2) \subset B_2\}$ for U_2 open in τ_1, V_2 open in δ_1, A_2 open in τ_2 and B_2

open in δ_2, neighborhood of g and not f. Now, since p-continuity imply s-continuity, the set $S((U_1,V_1)(A_1,B_1))_p$ can also be expressed as follows, $S((U_1,V_1)(A_1,B_1))_p = \{f \in s-C(Y,Z) : f|_{(U_1,V_1)}(U_1) \subset V_1 \text{ and } f|_{(A_1,B_1)}(A_1) \subset B_1\} = \{f \in s-C(Y,Z) : f(U) \subset V \text{ for } U = U_1 \cup A_1 \text{ and } V = V_1 \cup B_1\} = S(U,V)_s$ which is a neighborhood of f and not g, and $S((U_2,V_2)(A_2,B_2))_p = \{g \in s-C(Y,Z) : g|_{(U_2,V_2)}(U_2) \subset V_2 \text{ and } g|_{(A_2,B_2)}(A_2) \subset B_2\} = \{g \in s-C(Y,Z) : g(A) \subset B \text{ for } A = U_2 \cup A_2 \text{ and } B = V_2 \cup B_2\} = S(A,B)_s$ which is a neighborhood of g and not f. Hence, the space $s-C_\tau(Y,Z)$ is a T_1 space. ■

Theorem 5.4.7. *(Muturi E. N. et al., 2018(b)). The function space $s-C_\tau(Y,Z)$ is an T_2-space, if the function space $p-C_\omega(Y,Z)$ is a ${}_pT_2$ space.*

Proof. Let $p-C_\omega(Y,Z)$ be a ${}_pT_2$ space and let f and g be unique functions in $p-C(Y,Z)$ such that $\forall y \in Y$ $f(y) \neq g(y)$, then there exist two disjoint open sets $S((U_1,V_1)(A_1,B_1))_p = \{f \in p-C(Y,Z) : f(U_1) \subset V_1 \text{ and } f(A_1) \subset B_1\}$ for U_1 open in τ_1, V_1 open in δ_1, A_1 open in τ_2 and B_1 open in δ_2, neighborhood of f and $S((U_2,V_2)(A_2,B_2))_p = \{g \in p-C(Y,Z) : g(U_2) \subset V_2 \text{ and } g(A_2) \subset B_2\}$ for U_2 open in τ_1, V_2 open in δ_1, A_2 open in τ_2 and B_2 open in δ_2, neighborhood of g. Since p-continuity imply s-continuity, the set $S((U_1,V_1)(A_1,B_1))_p$ can also be expressed as follows, $S((U_1,V_1)(A_1,B_1))_p = \{f \in s-C(Y,Z) : f|_{(U_1,V_1)}(U_1) \subset V_1 \text{ and } f|_{(A_1,B_1)}(A_1) \subset B_1\} = \{f \in s-C(Y,Z) : f(U) \subset V \text{ for } U = U_1 \cup A_1 \text{ and } V = V_1 \cup B_1\} = S(U,V)_s$. Similarly, $S((U_2,V_2)(A_2,B_2))_p = \{g \in s-C(Y,Z) : g|_{(U_2,V_2)}(U_2) \subset V_2 \text{ and } g|_{(A_2,B_2)}(A_2) \subset B_2\} = \{g \in s-C(Y,Z) : g(A) \subset B \text{ for } A = U_2 \cup A_2 \text{ and } B = V_2 \cup B_2\} = S(A,B)_s$. The sets $S(U,V)_s$ and $S(A,B)_s$ are disjoint open sets neighbourhoods of f and g respectively. Therefore, the space $s-C_\tau(Y,Z)$ is a T_2 space. ■

CHAPTER SIX

COMPACTNESS ON SUBSETS OF SETS OF CONTINUOUS FUNCTIONS DEFINED ON BITOPOLOGICAL SPACES

6.1 Introduction

Let F_s be a subset of the space $s-C_\tau(Y,Z)$, F_1 a subset of the space $1-C_\varsigma(Y,Z)$ and F_2 a subset of the space $2-C_\zeta(Y,Z)$. In this chapter, joint continuity of topologies defined on F_1 and F_2 as well as even continuity of the same sets are introduced. Compactness is also proven for the subsets F_1 of $1-C_\varsigma(Y,Z)$ and F_2 of $2-C_\zeta(Y,Z)$ in the space $s-C_\tau(Y,Z)$.

The following definitions and lemma by Kelley (1955) and Engelking (1989) are considered.

Definition 6.1.1. *(Kelley, 1955). A topology τ for the set $C(Y,Z)$ is said to be jointly continuous if and only if the function $e: C_\tau(Y,Z)\times Y \to Z$ is continuous.*

Definition 6.1.2. *(Engelking, 1989). A set F is an evenly continuous subset of $C_\tau(Y,Z)$ if $\forall y \in Y$, $\forall z \in Z$ and a neighbourhood V of z, there exist a neighbourhood U of y and a neighbourhood W of z such that for all $f \in F_s$ with $f(y)\in W$, then $f(U)\subset V$. That is, $e(F\cap S(\{y\},W)\times U)\subset V$, where e is the evaluation function on $C_\tau(Y,Z)$ and τ is the topology of pointwise convergence.*

Lemma 6.1.3. *(Engelking, 1989). For every pair Y and Z of topological spaces, any subset A of Y and any closed subset B of Z, the set $S(A,B)$ is closed in the space $C(Y,Z)$ with topology of pointwise convergence.*

6.2 Even continuity for the subsets F_1 of $1-C_\varsigma(Y,Z)$ and F_2 of $2-C_\zeta(Y,Z)$ and their jointly continuous topologies.

In this section, even continuity for the subset F_1 of $1-C_\varsigma(Y,Z)$ and joint continuity of the induced topology on F_1 is proven, the proofs of the same for F_2 subset of $2-C_\zeta(Y,Z)$ can be done in a similar manner.

Proposition 6.2.1. *Let $F_s \subset s-C_\tau(Y,Z)$ be an evenly continuous family of mappings and let τ be the topology of pointwise convergence, then the induced topology τ_{F_s} on F_s is jointly continuous.*

Proof. Let $f \in F_s$, $y \in Y$, $z = f(y)$ and let V be the neighbourhood of z in Z. It suffices to show that the restriction $e|_{F_s \times (Y,\tau_1 \vee \tau_2)}$ of the evaluation mapping e is continuous with respect to topology τ_{F_s} for F_s. Since F_s is evenly continuous, there exist an open set U neighbourhood of y and an open set W neighbourhood of z such that $e(F_s \cap S(\{y\},W) \times U) \subset V$. Now, the set $F_s \cap S(\{y\},W)$ is open in topology τ_{F_s} for F_s whenever W is open in Z. Since the evaluation mapping is continuous, we have that $e|_{F_s \times (Y,\tau_1 \vee \tau_2)}(F_s \cap S(\{y\},W) \times U) \subset V$, hence $e|_{F_s \times (Y,\tau_1 \vee \tau_2)}$ is continuous, implying that τ_{F_s} is jointly continuous. ■

Proposition 6.2.2. *Let F_s be evenly continuous subset of $s - C_\tau(Y,Z)$ and let F_1 be a subset of $1 - C_\varsigma(Y,Z)$, if τ and ς are topologies of pointwise convergence, then the induced topology ς_{F_1} on F_1 is jointly continuous.*

Proof. It suffices to show that the restriction $e|_{F_1 \times (Y,\tau_1 \vee \tau_2)} : F_1 \times (Y, \tau_1 \vee \tau_2) \to (Z, \delta_1 \vee \tau_2)$ is continuous with respect to topology ς_{F_1} for F_1. Let F_s be evenly continuous, then from Proposition 6.2.1 the mapping $e|_{F_s \times (Y,\tau_1 \vee \tau_2)} : F_s \times (Y, \tau_1 \vee \tau_2) \to (Z, \delta_1 \vee \tau_2)$ is continuous. From Proposition 3.3.13 function $h : 1 - C_\varsigma(Y,Z) \to s - C_\tau(Y,Z)$ is a continuous mapping, therefore the submap $h|_{F_1,F_s} : F_1 \to F_s$ is continuous. Let $i : (Y, \tau_1 \vee \tau_2) \to (Y, \tau_1 \vee \tau_2)$ be an identity mapping, then the composite function $e|_{F_s \times (Y,\tau_1 \vee \tau_2)} \circ (h|_{F_1,F_s} \times i) = e|_{F_1 \times (Y,\tau_1 \vee \tau_2)} :$ $F_1 \times (Y, \tau_1 \vee \tau_2) \to (Z, \delta_1 \vee \delta_2)$ is continuous, implying that ς_{F_1} is jointly continuous. ■

Proposition 6.2.3. *Let $(Y, \tau_1 \vee \tau_2)$ be an arbitrary topological space, $(Z, \delta_1 \vee \delta_2)$ a regular space, F_s a compact subspace of $s - C_\tau(Y,Z)$ and topology c_{F_s} on F_s jointly continuous, if (F_1, c_{F_1}) is a compact subspace of $1 - C_\varsigma(Y,Z)$, then F_1 is evenly continuous.*

Proof. Since $(Z, \delta_1 \vee \delta_2)$ is a regular space, it follows that $s - C_\tau(Y,Z)$ is a Hausdorff space with a topology τ of pointwise convergence. Now, the induced topology τ_{F_s} on F_s is also a topology of pointwise convergence, hence the identity function $i : (F_s, c_{F_s}) \to (F_s, \tau_{F_s})$ is continuous. Since (F_s, τ_{F_s}) is also a Hausdorff space, then c_{F_s} and τ_{F_s} coincide, implying that τ_{F_s} is also jointly continuous. If ς_{F_1} is an induced topology on F_1, then by Proposition 6.2.2, ς_{F_1} is also jointly continuous. Suppose that $y \in Y$, $z \in Z$, and U is an open neighbourhood of z. Let W be a closed neighbourhood of z such that $W \subset U$, if $\forall f \in K \subset F_1$, $f(y) \in W$, then (K, ς_{F_1}) is closed by Lemma 6.1.3. Now, (K, ς_{F_1}) is a closed subset of a compact space (F_1, c_{F_1}), hence (K, ς_{F_1}) is also compact. Since ς_{F_1} is jointly continuous, then $e|_{F_1 \times (Y,\tau_1 \vee \tau_2)}$ is continuous, hence for the compact set $K \times \{y\}$ contained in $e|_{F_1\times(Y,\tau_1\vee\tau_2)}{}^{-1}(U)$, there exist an open set V of y such that $e|_{F_1\times(Y,\tau_1\vee\tau_2)}(K\times V)\subset e(K\times V)\subset U$, hence F_1 is evenly continuous. ■

Proposition 6.2.4. *Let $(Y,\tau_1\vee\tau_2)$ be an arbitrary topological space, $(Z,\delta_1\vee\delta_2)$ a regular space, F_s an evenly continuous compact subset of $s-C_\tau(Y,Z)$ and τ a topology of pointwise convergence. If F_1 is a closed subset of $1-C_\varsigma(Y,Z)$, then F_1 is evenly continuous in the space $s-C_\tau(Y,Z)$.*

Proof. Let $y\in Y$, $z\in Z$ and V a neighbourhood of z. Since the space Z is regular, then for any open set V neighbourhood of z, there exist an open set W neighbourhood of z such that $\overline{W}\subset V$. The set $S(\{y\},\overline{W})$ is a closed subset of $s-C_\tau(Y,Z)$ by Lemma 6.1.3, since a compact subset of a regular space (which is a hausdorff space) is closed, then $F_s\cap S(\{y\},\overline{W})$ is closed. Now, $F_s\cap S(\{y\},\overline{W})$ is a closed subset of a compact space F_s, therefore it is compact. Since $F_1\cap S(\{y\},\overline{W})\subset F_s\cap S(\{y\},\overline{W})$ is closed, then $F_1\cap S(\{y\},\overline{W})$ is also compact. F_s is evenly continuous, therefore by Proposition 6.2.2, the function $e|_{F_1\times(Y,\tau_1\vee\tau_2)}$ is continuous, and for the compact set $(F_1\cap S(\{y\},\overline{W}))\times\{y\}$, there exist open set U subsets of $(Y,\tau_1\vee\tau_2)$ containing $\{y\}$, such that $e|_{F_1\times(Y,\tau_1\vee\tau_2)}(F_1\cap S(\{y\},W)\times U)\subset e(F_1\cap S(\{y\},W)\times U)\subset V$. Hence F_1 is evenly continuous. ■

The following lemma by McCoy and Ntantu (1988) is considered in proving the next proposition.

Lemma 6.2.5. *(McCoy and Ntantu, 1988). Let F_s be evenly continuous subset of $s-C_\tau(Y,Z)$, then $\overline{F_s}$ is evenly continuous in the space $s-C_\tau(Y,Z)$.*

Proposition 6.2.6. *Let $(Y,\tau_1\vee\tau_2)$ be an arbitrary topological space, $(Z,\delta_1\vee\delta_2)$ a regular space, F_s an evenly continuous subset of $s-C_\tau(Y,Z)$ and τ a topology of pointwise convergence. If F_1 is a closed subset of $1-C_\varsigma(Y,Z)$, then the closure of F_1 in the space $s-C_\tau(Y,Z)$ is evenly continuous.*

Proof. Since F_s is evenly continuous, then by Proposition 6.2.4, the set F_1 is also evenly continuous in the space $s-C_\tau(Y,Z)$, and by Lemma 6.2.5, the closure of F_1 in the space $s-C_\tau(Y,Z)$ is evenly continuous. ■

6.3 Compactness criterion for the closed subsets F_1 of $1-C_\varsigma(Y,Z)$ and F_2 of $2-C_\zeta(Y,Z)$ in the space $s-C_\tau(Y,Z)$

In this section, the proof of compactness for the closed subset F_1 of $1-C_\varsigma(Y,Z)$ in the space $s-C_\tau(Y,Z)$ is provided. The proof of compactness of the subset F_2 of $2-C_\zeta(Y,Z)$ in the space $s-C_\tau(Y,Z)$ can be done in a similar manner.

The following is an Arzela Ascoli theorem.

Theorem 6.3.1. *Kelley (1955). Let C be a family of all continuous functions from a regular locally compact space X to a regular space Y, and let C have the compact-open topology. Then the subset F of C is compact if and only if;*

(a) F is closed in C,

(b) the closure of $F(x)$ is compact for each $x \in X$, and

(c) F is evenly continuous.

The theorem that follows is a variant of the above Arzela Ascoli theorem.

Theorem 6.3.2. *Let $(Y,\tau_1 \vee \tau_2)$ be regular locally compact space, $(Z,\delta_1 \vee \delta_2)$ be regular space and F_s be evenly continuous subset of $s-C_\tau(Y,Z)$. A closed subset F_1 of $1-C_\varsigma(Y,Z)$ where ς is compact open topology, is compact in the space $s-C_\tau(Y,Z)$ if and only if $\overline{F_1(y)}$ is compact $\forall y \in Y$.*

Proof. Let $\overline{F_1(y)}$ be compact $\forall y \in Y$ where $\overline{F_1(y)} = \overline{e(F_1 \times \{y\})} \subset Z$, then by Tychonoff theorem, the product $\prod\limits_{y\in Y} \overline{F_1(y)}$ is compact, since F_1 is a closed subset of $\prod\limits_{y\in Y} F_1(y)$, then $\overline{F_1} \subset \prod\limits_{y\in Y} \overline{F_1(y)}$, implying that $\overline{F_1}$ is pointwise compact as a closed subset of a compact space. Since F_s is evenly continuous, then by Proposition 6.2.4, F_1 is evenly continuous and by Proposition 6.2.6, $\overline{F_1}$ is also evenly continuous, implying that $e_1 = e|_{\overline{F_1}\times(Y,\tau_1\vee\tau_2)}$ is continuous, that is $e_1 : \overline{F_1} \times (Y,\tau_1 \vee \tau_2) \to (Z,\delta_1 \vee \delta_2)$ is continuous with respect to the induced topology $p_{\overline{F_1}}$ of pointwise convergence on $\overline{F_1}$. Let ς_{F_1} be the induced compact open topology on F_1, then $p_{\overline{F_1}} \subset \varsigma_{\overline{F_1}}$ and the function $h : (\overline{F_1}, \varsigma_{\overline{F_1}}) \to (\overline{F_1}, p_{\overline{F_1}})$ from a compact space to a Hausdorff space is a homeomorphism, hence the induced topology of pointwise convergence $p_{\overline{F_1}}$ on $\overline{F_1}$, coincides with the induced compact open topology $\varsigma_{\overline{F_1}}$ on $\overline{F_1}$. Now, $e_1 : \overline{F_1} \times (Y,\tau_1 \vee \tau_2) \to (Z,\delta_1 \vee \delta_2)$ is continuous with respect to the induced compact open topology $\tau_{\overline{F_1}}$, this implies that $(\overline{F_1}, \varsigma_{\overline{F_1}})$ is compact. Since F_1 is a closed set, then $F_1 = \overline{F_1}$, implying that F_1 is compact.

Conversely, let F_1 be closed subset of $1-C_{\varsigma}(Y,Z)$ compact in the space $s-C_{\tau}(Y,Z)$. Since F_s is evenly continuous, it follows from Proposition 6.2.4 that F_1 is also evenly continuous, and therefore $e_1 = e|_{F_1\times(Y,\tau_1\vee\tau_2)}$ is continuous, that is, $e_1 : F_1 \times (Y,\tau_1 \vee \tau_2) \to (Z,\delta_1 \vee \delta_2)$ is pointwise continuous, since $p_{\overline{F_1}} \subset \varsigma_{\overline{F_1}}$ on F_1, then $e_1 : F_1 \times (Y,\tau_1 \vee \tau_2) \to (Z,\delta_1 \vee \delta_2)$ is also continuous with respect to the induced compact open topology $\varsigma_{\overline{F_1}}$ on F_1. Now, since F_1 is compact, then the image of a F_1 under e_1 is compact in the space $(Z,\delta_1 \vee \delta_2)$. It follows that $e_1(F_1 \times \{y\})$ is a compact subset of Z, but $e(F_1 \times \{y\}) = F_1(y)$, futhermore, a compact subset of a Hausdorff space is closed. Hence $\overline{F_1(y)}$ is compact $\forall y \in Y$. ■

CHAPTER SEVEN

CONCLUSION AND FURTHER RESEARCH

7.1 Conclusion

By considering continuous functions from bitopological space (Y,τ_1,τ_2) to the bitopological space (Z,δ_1,δ_2), the following sets of continuous functions are defined; $p-C(Y,Z)$, $s-C(Y,Z)$, $(1,2)-C(Y,Z)$, $(2,1)-C(Y,Z)$ and $d-C(Y,Z)$. A function f is s-continuous if it is p-continuous or p_1-continuous, the converse of this relation is not true. If f is p-continuous or p_1-continuous, then f is also d-continuous. The composition of the open continuous functions $f:(Y,\tau_2)\longrightarrow(Y,\tau_1)$, $g:(Z,\delta_1)\longrightarrow(Z,\delta_2)$ and $h:(Y,\tau_1)\longrightarrow(Z,\delta_1)$ give rise to the continuous functions; $\mu: 1-C_\varsigma(Y,Z)\longrightarrow 2-C_\zeta(Y,Z)$ defined by $\mu_{(g,f)}(h)=g\circ h\circ f$, $\rho: 1-C_\varsigma(Y,Z)\longrightarrow(2,1)-C_\xi(Y,Z)$ defined by $\rho_f(h)=h\circ f$, $\omega: 1-C_\varsigma(Y,Z)\longrightarrow(1,2)-C_\varphi(Y,Z)$ defined by $\omega_g(h)=g\circ h$, $\alpha:(2,1)-C_\xi(Y,Z)\longrightarrow 1-C_\varsigma(Y,Z)$ defined by $\alpha(h\circ f)=h_f$ and $\beta: 2-C_\zeta(Y,Z)\longrightarrow(1,2)-C_\varphi(Y,Z)$ defined by $\beta(g\circ h\circ f)=(g\circ h)_f$. The function $\rho: 1-C_\varsigma(Y,Z)\longrightarrow(2,1)-C_\xi(Y,Z)$ defined by $\rho_f(h)=h\circ f$ is a homeomorphism, while $1-C_\varsigma(Y,Z)$ and $2-C_\zeta(Y,Z)$ are established as subspaces of $s-C_\tau(Y,Z)$.

Pairwise splitting, pairwise admissibility, supremum splitting and supremum admissibility properties of topologies are introduced on the spaces $p-C_\omega(Y,Z)$ and $s-C_\tau(Y,Z)$ respectively. These properties of topologies depends on splitting and admissibility of topologies defined on $1-C_\varsigma(Y,Z)$ and $2-C_\zeta(Y,Z)$. Exponential mapping is defined on the space $s-C_\tau(Y,Z)$ and comparison of topologies defined on $s-C(Y,Z)$ made. A simpler proof is provided that shows that any topology finer than s-admissible topology is s-admissible, any topology courser than s-splitting topology is s-splitting, while s-splitting topology is always courser than s-admissible topology.

Separation axioms are generalized on the space $p-C_\omega(Y,Z)$ and redefined as ${}_pT_\circ$, ${}_pT_1$, ${}_pT_2$ and ${}_p$regular. It is shown that the space $p-C_\omega(Y,Z)$ is a ${}_pT_\circ$, ${}_pT_1$, ${}_pT_2$ and ${}_p$regular if the spaces (Z,δ_1) and (Z,δ_2) both are $T_\circ$, T_1, T_2 and regular. The space $p-C_\omega(Y,Z)$ is also shown to be is ${}_pT_\circ$, ${}_pT_1$, ${}_pT_2$ and ${}_p$regular if the space (Z,δ_1,δ_2) is pairwise-$T_\circ$, pairwise-T_1, pairwise-T_2 and pairwise regular. The space $p-C_\omega(Y,Z)$ is proved to be ${}_pT_0$, ${}_pT_1$, ${}_pT_2$ and ${}_p$regular if the spaces $1-C_\varsigma(Y,Z)$ and $2-C_\zeta(Y,Z)$ are both T_0, T_1, T_2 and regular. Separation axioms defined on the space $s - C_\tau(Y,Z)$ are also compared with those defined on $p - C_\omega(Y,Z)$. It is proved that the space $s - C_\tau(Y,Z)$ is T_0, T_1 and T_2, if the space $p - C_\omega(Y,Z)$ is a ${}_pT_0$, ${}_pT_1$ and ${}_pT_2$ space.

The evaluation functions are shown to be continuous when restricted to the subsets F_1 and F_2 of $1 - C_\varsigma(Y,Z)$ and $2 - C_\zeta(Y,Z)$ respectively. Furthermore, if F_1 and F_2 are closed, then they are evenly continuous in the space $s - C_\tau(Y,Z)$, provided F_s is evenly continuous compact subset of $s - C_\tau(Y,Z)$. The closures of F_1 and F_2 are also shown to be evenly continuous under specific conditions. Finally, it is proven that the closed sets F_i for $i = 1,2$, are compact in the space $s - C_\tau(Y,Z)$, if and only if $\overline{F_1(y)}$ for $i = 1,2$ is compact, $\forall y \in Y$.This is a slight variation of Arzela Ascoli theorem on compactness of subsets of function spaces.

7.2 FURTHER RESEARCH

This thesis sets the groundwork for more generalization of properties of function spaces to topologized sets of continuous functions defined on bitopological spaces. It also makes it possible to generalize more bitopologicals concepts to these new spaces. It would also be interesting to see how a pair of topologies can be defined on function spaces defined on topological spaces or even bitopological spaces, allowing generalization of concepts such as pairwise continuity, pairwise separation axioms, pairwise compactness and pairwise connectedness among other concepts to these new function spaces. Converses of some of the theorems involving splitting and admissibility of topologies, separation axioms and compactness defined on $s - C_\tau(Y,Z)$ and $p - C(Y,Z)$, remain as open problems.

APPENDIX

PUBLICATIONS

PAPER I: **Continuity of functions on function spaces defined on bitopological spaces**
Muturi N. E., Pokhariyal G. P. and Khalagai J. M.
Journal of Advanced Studies in Topology 8: 2 (2017), 130–134.

PAPER II: **Splitting and admissible topologies defined on the set of continuous functions on bitopological spaces**
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PAPER III: **Separation axioms on function spaces defined on bitopological spaces**
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Continuity of functions on function spaces defined on bitopological spaces

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Abstract

In this paper, relationships between continuous functions defined on the spaces (Y, τ_1, τ_2), $(Y, \tau_1 \vee \tau_2)$, $(Y, \tau_1 \wedge \tau_2)$ and (Y, τ_i) for $i = 1, 2$ are examined. Function spaces $s - C_\tau(Y, Z)$, $p - C_\omega(Y, Z)$, $1 - C_\varsigma(Y, Z)$, $2 - C_\zeta(Y, Z)$, $(1, 2) - C_\varphi(Y, Z)$ and $(2, 1) - C_\xi(Y, Z)$ are defined and continuous functions between them explored. A homeomorphism is also established between the spaces $1 - C_\varsigma(Y, Z)$ and $(2, 1) - C_\xi(Y, Z)$.

Keywords: s-continuity, i-continuity, p-continuity, p_1-continuity, d-continuity, open-open topology.
2010 MSC: 54A10, 54C35, 54E55.

1. Introduction

For any two topological spaces Y and Z, $C(Y, Z)$ denotes the set of continuous functions from Y to Z. The two commonly defined topologies on $C(Y, Z)$ are compact open topology denoted by τ_{co} and point open topology denoted by τ_p. Fox [1] defined compact open topology to be the topology generated by the subbasis $S(U, V) = \{f \in C(Y, Z) : f(U) \subset V\}$, for U compact subset of Y and V open subset of Z. Point open topology stems from the notion of convergence sequence of functions and is generated by the subbasis $S(y, V) = \{f \in C(Y, Z) : f(y) \in V\}$, for $y \in Y$ and V open subset of Z. Both compact open topology and point open topology have been shown to be open-open topologies by Porter [5]. The set $C(Y, Z)$ on which topology τ is defined, is called a function space and is denoted by $C_\tau(Y, Z)$.

The notion of bitopological space where two topologies τ_1 and τ_2 are defined on a non empty set Y and denoted by (Y, τ_1, τ_2), is due to Kelly [2]. The concept of pairwise continuity on bitopological spaces is due to Pervin [3].

In this paper, the spaces (Y, τ_1, τ_2) and $(Y, \tau_1 \vee \tau_2)$ are considered and s-continuous, p_1-continuous and d-continuous function defined on them. Relationships between p-continuity, s-continuity, p_1-continuity and d-continuity are also explored. Finally, the spaces $s - C_\tau(Y, Z)$, $p - C_\omega(Y, Z)$, $1 - C_\varsigma(Y, Z)$, $2 - C_\zeta(Y, Z)$, $(1, 2) - C_\varphi(Y, Z)$

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and $(2,1)-C_\xi(Y,Z)$ are defined and continuous functions between them studied. A homeomorphism is also established between the spaces $1-C_\varsigma(Y,Z)$ and $(2,1)-C_\xi(Y,Z)$.

2. Preliminaries

For the bitopological space (Y,τ_1,τ_2), topologies τ_1 and τ_2 when combined can form a base or a subbasis for a topology on the set Y. The notation $\tau_1 \vee \tau_2$ in this paper will imply topology on the set Y generated by the basis $\tau_1 \cup \tau_2$. Such topology will be called supremum topology on the set Y. The set Y together with its supremum topology $\tau_1 \vee \tau_2$ will be denoted by $(Y,\tau_1 \vee \tau_2)$.

The following definitions are used in this paper.

Definition 2.1. [3] A function $f:(Y,\tau_1,\tau_2) \longrightarrow (Z,\delta_1,\delta_2)$, is said to be pairwise continuous (p-continuous) if the induced functions $f:(Y,\tau_1) \longrightarrow (Z,\delta_1)$ and $f:(Y,\tau_2) \longrightarrow (Z,\delta_2)$ are continuous.

Definition 2.2. [6] A function $f:(Y,\tau_1,\tau_2) \longrightarrow (Z,\delta_1,\delta_2)$ is said to be (i,j)-continuous, if the function $f:(Y,\tau_i) \rightarrow (Z,\delta_j)$ is continuous for $i,j \in \{1,2\}$, $i \neq j$. The function $f:(Y,\tau_1,\tau_2) \longrightarrow (Z,\delta_1,\delta_2)$ is called pairwise$_1$-continuous (p_1-continuous) if it is both (i,j)-continuous and (j,i)-continuous.

The following definitions are introduced in this paper.

Definition 2.3. A subset A of a bitopological space $(Y,\tau_1 \vee \tau_2)$ is called a *supremum*-open set or simply s-open set if $A = U_1 \cup U_2$, where $U_1 \in \tau_1$ and $U_2 \in \tau_2$.

Definition 2.4. A function $f:(Y,\tau_1 \vee \tau_2) \longrightarrow (Z,\delta_1 \vee \delta_2)$, is said to be a s-continuous, if the inverse image of each s-open subset of Z is s-open in Y.

Definition 2.5. A function $f:(Y,\tau_1,\tau_2) \longrightarrow (Z,\delta_1,\delta_2)$ is said to be double-continuous (d-continuous), if for every $U \in \delta_1 \cap \delta_2$, $f^{-1}(U) \in \tau_1 \cap \tau_2$.

Among the commonly studied topologies on the set $C(Y,Z)$ of continuous functions, is the class of set-set topologies which comprise of compact open topology, point open topology, closed open topology and bounded open topology among others. Some of these topologies are equivalent under some given conditions. For example the point open topology is equivalent to the closed open topology whenever the Y is a T_2 compact space, the point open topology is equivalent to the compact open topology provided all compact subsets of of the space Y are finite sets, and the compact open topology on $C(Y,Z)$ is equivalent to point open topology provided Y is a discrete space [5].

For bitopological spaces (Y,τ_1,τ_2) and (Z,δ_1,δ_2), the following sets of continuous functions can be defined. The set $i-C(Y,Z)$ of all i-continuous functions for $i=1,2$, the set $i,j-C(Y,Z)$ of all (i,j)-continuous functions for $i,j=1,2$ and $i \neq j$, the set $s-C(Y,Z)$ of all s-continuous functions, the set $p-C(Y,Z)$ of all pairwise continuous functions and the set $d-C(Y,Z)$ of all double continuous functions.

Definition 2.6. The sets of the form $S(U,V)=\{f \in C(Y,Z) : f(U) \subset V\}$ for U open in Y and V open in Z, defines the subbasis for the open-open topology τ on the set $C(Y,Z)$ [5]. If U is a compact subset, then $S(U,V)$ defines the subbasis for compact open topology τ on $C(Y,Z)$ [1].

Definition 2.7. The sets of the form $S(y,V)=\{f \in C(Y,Z) : f(y) \in V\}$ for $y \in Y$ and V open in Z, defines the subbasis for the point open topology τ on the set $C(Y,Z)$.

From the above definitions, the following definitions are introduced.

Definition 2.8. The sets of the form $S(U,V)_i=\{f \in i-C(Y,Z) : f(U) \subset V\}$ for U open in Y, V open in Z, defines the subbasis for the open-open topology on the set $i-C(Y,Z)$ for $i=1,2$.

Definition 2.9. The sets of the form $S(U,V)_{(i,j)} = \{f \in (i,j) - C(Y,Z) : f(U) \subset V\}$ for U open in Y, V open in Z, defines the subbasis for the open-open topology on the set $(i,j) - C(Y,Z)$ for $i,j = 1,2, i \neq j$.

Definition 2.10. The sets of the form $S(U,V)_s = \{f \in s - C(Y,Z) : f(U) \subset V\}$ for $U \in \tau_1 \vee \tau_2$ and $V \in \delta_1 \vee \delta_2$, defines the subbasis for the supremum open-open topology on the set $s - C(Y,Z)$.

Definition 2.11. The sets of the form $S(U,V)_d = \{f \in d - C(Y,Z) : f(U) \subset V\}$ for $U \in \tau_1 \cap \tau_2$ and $V \in \delta_1 \cap \delta_2$, defines the subbasis for the minimum open-open topology τ on the set $d - C(Y,Z)$.

3. Continuous functions defined on bitopological spaces

In this section, s-continuous and d-continuous functions are introduced, relationships between s-continuous functions, p-continuous functions, p_1-continuous functions and i-continuous functions are studied.

Theorem 3.1. The function $f : (Y, \tau_1 \vee \tau_2) \longrightarrow (Z, \delta_1 \vee \delta_2)$ is s-continuous if the function $f : (Y, \tau_1, \tau_2) \longrightarrow (Z, \delta_1, \delta_2)$ is p-continuous.

Proof. Let the function f be p-continuous and let $U_1 \in \delta_1$ and $U_2 \in \delta_2$ such that $U_1 \cup U_2 \in \delta_1 \vee \delta_2$. We need to show that $f^{-1}(U_1 \cup U_2)$ is open in $\tau_1 \vee \tau_2$. Since f is p-continuous, then $f^{-1}(U_1) \in \tau_1$ and $f^{-1}(U_2) \in \tau_2$, this implies that $f^{-1}(U_1) \cup f^{-1}(U_2) \in \tau_1 \vee \tau_2$. But $f^{-1}(U_1) \cup f^{-1}(U_2) = f^{-1}(U_1 \cup U_2)$, hence the function $f : (Y, \tau_1 \vee \tau_2) \longrightarrow (Z, \delta_1 \vee \delta_2)$ is s-continuous. □

Corollary 3.2. Let function $f : (Y, \tau_1 \vee \tau_2) \longrightarrow (Z, \delta_1 \vee \delta_2)$ be s-continuous, then the function $g : (Y, \tau_i) \longrightarrow (Z, \delta_i)$ is i continuous for $i = 1, 2$.

Proof. The function g is continuous as a submap of s-continuous function f. □

The converse of theorem 3.1 is not always true as shown by the following example.

Example 3.3. Let $\tau_1 = \{Y, \phi, \{a\}, \{b\}, \{a,b\}\}$ and $\tau_2 = \{Y, \phi, \{c\}\}$ be topologies on $Y = \{a,b,c\}$, and $\delta_1 = \{Z, \phi, \{3\}\}$ and $\delta_2 = \{Z, \phi, \{2\}\}$ be topologies defined on $Z = \{1,2,3\}$. Let $f : Y \to Z$ be defined by $f(a) = 1$, $f(b) = 2$, and $f(c) = 3$, then the function $f : (Y, \tau_1 \vee \tau_2) \longrightarrow (Z, \delta_1 \vee \delta_2)$ is s-continuous but not p-continuous on (Y, τ_1, τ_2).

Theorem 3.4. The function $f : (Y, \tau_1 \vee \tau_2) \longrightarrow (Z, \delta_1 \vee \delta_2)$ is s-continuous if the function $f : (Y, \tau_1, \tau_2) \longrightarrow (Z, \delta_1, \delta_2)$ is p_1-continuous.

Proof. Let the function f be p_1-continuous and let $U_1 \in \delta_1$ and $U_2 \in \delta_2$ such that $U_1 \cup U_2 \in \delta_1 \vee \delta_2$. We need to show that $f^{-1}(U_1 \cup U_2)$ is open in $\tau_1 \vee \tau_2$. Since f is p_1-continuous, then $f^{-1}(U_1) \in \tau_2$ and $f^{-1}(U_2) \in \tau_1$, thus $f^{-1}(U_1) \cup f^{-1}(U_2) \in \tau_1 \vee \tau_2$. It remains to put $f^{-1}(U_1) \cup f^{-1}(U_2) = f^{-1}(U_1 \cup U_2)$, hence the function $f : (Y, \tau_1 \vee \tau_2) \longrightarrow (Z, \delta_1 \vee \delta_2)$ is s-continuous. □

The converse of theorem 3.4 is not always true as shown by the following example.

Example 3.5. Let $\tau_1 = \{Y, \phi, \{a\}, \{b\}, \{a,b\}\}$ and $\tau_2 = \{Y, \phi, \{c\}\}$ be topologies on $Y = \{a,b,c\}$, and $\delta_1 = \{Z, \phi, \{2\}\}$ and $\delta_2 = \{Z, \phi, \{3\}\}$ be topologies defined on $Z = \{1,2,3\}$. Let $f : Y \to Z$ be defined by $f(a) = 1$, $f(b) = 2$, and $f(c) = 3$, then the function $f : (Y, \tau_1 \vee \tau_2) \longrightarrow (Z, \delta_1 \vee \delta_2)$ is s-continuous but not p_1-continuous.

Proposition 3.6. Let the function $f : (Y, \tau_1, \tau_2) \longrightarrow (Z, \delta_1, \delta_2)$ be p-continuous, then the function $g : (Y, \tau_1 \wedge \tau_2) \longrightarrow (Z, \delta_1 \wedge \delta_2)$ is d-continuous.

Proof. The function g is continuous as a submap of p-continuous function f. □

Proposition 3.7. Let the function $f : (Y, \tau_1, \tau_2) \longrightarrow (Z, \delta_1, \delta_2)$ be p_1-continuous, then the function $g : (Y, \tau_1 \wedge \tau_2) \longrightarrow (Z, \delta_1 \wedge \delta_2)$ is d-continuous.

Proof. The function g is continuous as a submap of p_1-continuous function f. □

Proposition 3.8. Let the function $f : (Y, \tau_1 \vee \tau_2) \longrightarrow (Z, \delta_1 \vee \delta_2)$ be s-continuous, then the function $g : (Y, \tau_1 \wedge \tau_2) \longrightarrow (Z, \delta_1 \wedge \delta_2)$ is d-continuous.

Proof. The function g is continuous as a submap of s-continuous function f. □

4. Continuous functions defined on function spaces

In this section, subbasis for open-open topologies on sets $s - C(Y,Z)$, $p - C(Y,Z)$, $1 - C(Y,Z)$, $2 - C(Y,Z)$, $(1,2) - C(Y,Z)$ and $(2,1) - C(Y,Z)$ are defined, giving rise to the function spaces $s - C_\tau(Y,Z)$, $p - C_\omega(Y,Z)$, $1 - C_\varsigma(Y,Z)$, $2 - C_\zeta(Y,Z)$, $(1,2) - C_\varphi(Y,Z)$ and $(2,1) - C_\xi(Y,Z)$. Functional relationships between these function spaces are studied. All topologies here are assumed to be open-open topologies unless specified.

Proposition 4.1. Let $f : (Y,\tau_2) \longrightarrow (Y,\tau_1)$, $g : (Z,\delta_1) \longrightarrow (Z,\delta_2)$ and $h : (Y,\tau_1) \longrightarrow (Z,\delta_1)$ be open and continuous functions. Then the function $\mu : 1 - C_\varsigma(Y,Z) \longrightarrow 2 - C_\zeta(Y,Z)$ defined by $\rho_{(g,f)}(h) = g \circ h \circ f$ is continuous.

Proof. Let $S(U,V)_2$ be open in $2 - C_\zeta(Y,Z)$, then $g \circ h \circ f \in 2 - C_\zeta(Y,Z)$. Now $\mu^{-1}(S(U,V)_2)$=$\{h \in 1 - C(Y,Z) : h(f(U)) \subset g^{-1}(V) \; for \; U \in \tau_2 \; and \; V \in \delta_2\}$=$S(f(U), g^{-1}(V))_1$, which is open in $1 - C_\varsigma(Y,Z)$. □

Proposition 4.2. Let $f : (Y,\tau_2) \longrightarrow (Y,\tau_1)$ and $h : (Y,\tau_1) \longrightarrow (Z,\delta_1)$ be open and continuous functions. Then the function $\varrho : 1 - C_\varsigma(Y,Z) \longrightarrow (2,1) - C_\xi(Y,Z)$ defined by $\varrho_f(h) = h \circ f$ is continuous.

Proof. Let $S(U,V)_{2,1}$ be open in $(2,1) - C_\xi(Y,Z)$, then $h \circ f \in (2,1) - C_\xi(Y,Z)$. Now $\varrho^{-1}(S(U,V)_{2,1})$=$\{h \in 1 - C(Y,Z) : h(f(U)) \subset V$ for $U \in \tau_2$ and $V \in \delta_1\}$=$S(f(U),V)_1$, which is open in $1 - C_\varsigma(Y,Z)$. □

Proposition 4.3. Let $g : (Z,\delta_1) \longrightarrow (Z,\delta_2)$ and $h : (Y,\tau_1) \longrightarrow (Z,\delta_1)$ be open and continuous functions. Then the function $\omega : 1 - C_\varsigma(Y,Z) \longrightarrow (1,2) - C_\varphi(Y,Z)$ defined by $\omega_g(h) = g \circ h$ is continuous.

Proof. Let $S(U,V)_{1,2}$ be open in $(1,2) - C_\varphi(Y,Z)$, then $g \circ h \in (1,2) - C_\varphi(Y,Z)$. The set $\omega^{-1}(S(U,V)_{1,2})$=$\{h \in 1 - C(Y,Z) : h(U) \subset g^{-1}(V)$ for $U \in \tau_1$ and $V \in \delta_2\}$=$S(U, g^{-1}(V))_1$ is open in $1 - C_\varsigma(Y,Z)$. □

Proposition 4.4. Let $f : (Y,\tau_2) \longrightarrow (Y,\tau_1)$, $g : (Z,\delta_1 \longrightarrow (Z,\delta_2)$ and $h : (Y,\tau_1) \to (Z,\delta_1)$ be open and continuous functions. Then the following functions are continuous;

(i) $\alpha : (2,1) - C_\xi(Y,Z) \longrightarrow 1 - C_\varsigma(Y,Z)$ defined by $\alpha(h \circ f) = h_f$
(ii) $\beta : 2 - C_\zeta(Y,Z) \longrightarrow (1,2) - C_\varphi(Y,Z)$ defined by $\beta(g \circ h \circ f) = (g \circ h)_f$

Proof. (i) Let U be open in τ_2 and V be open in δ_1, then $S(f(U),V)_1$ is open in $1 - C_\varsigma(Y,Z)$. Now $\alpha^{-1}S(f(U),V)_1$=$\{(h \circ f) \in (2,1) - C(Y,Z) : h(f(U)) \subset V, U \in \tau_2$ and $V \in \delta_1\}$=$\{(h \circ f) \in (2,1) - C(Y,Z) : (h \circ f)(U) \subset V, U \in \tau_2$ and $V \in \delta_1\}$=$S(U,V)_{2,1}$, which is open in $(2,1) - C_\xi(Y,Z)$.
(ii) Let U be open in τ_2 and V be open in δ_2, then $S(f(U),V)_{1,2}$ is open in $(1,2) - C_\varphi(Y,Z)$. Now $\beta^{-1}S(f(U),V)_{1,2}$=$\{(g \circ h \circ f) \in 2 - C(Y,Z) : g(h(f(U))) \subset V, U \in \tau_2$ and $V \in \delta_2\}$=$\{(g \circ h \circ f) \in 2 - C(Y,Z) : (g \circ h \circ f)(U) \subset V, U \in \tau_2$ and $V \in \delta_2\}$=$S(U,V)_2$, which is open in $2 - C_\zeta(Y,Z)$. □

Proposition 4.5. Let $f : (Y,\tau_2) \longrightarrow (Y,\tau_1)$, $g : (Z,\delta_1 \longrightarrow (Z,\delta_2)$ and $h : (Y,\tau_1) \to (Z,\delta_1)$ be open and continuous functions, then the function $\beta \circ \mu \circ \alpha : (2,1) - C_\xi(Y,Z) \longrightarrow (1,2) - C_\varphi(Y,Z)$ defined by $(\beta \circ \mu \circ \alpha)_g(h \circ f) = (g \circ h)_f$ is continuous.

Proof. The function $\beta \circ \mu \circ \alpha$ is a composite function of continuous functions defined in proposition 4.4 (i), proposition 4.1 and proposition 4.4 (ii). □

Theorem 4.6. Let ς be compact open topology on $1 - C(Y,Z)$, Y be a regular and locally compact space, Z a Hausdorff space and $S(U,V)_1$ be compact subset of $1 - C_\varsigma(Y,Z)$, then the function $T : C(1 - C_\varsigma(Y,Z), 2 - C_\zeta(Y,Z)) \times C(2 - C_\zeta(Y,Z), (1,2) - C_\varphi(Y,Z)) \to C(1 - C_\varsigma(Y,Z), (1,2) - C_\varphi(Y,Z))$ is continuous with respect to closed open topology.

Proof. To prove this theorem, we consider the following lemma by Willard [4].
In a regular space, if F is compact, U open and $F \subset U$, then for some open set V, $F \subset V \subset \overline{V} \subset U$.
Let $(S(U,V)_1, S(U,V)_{1,2})$ be neighbourhood of ω in $C(1 - C_\varsigma(Y,Z), (1,2) - C_\varphi(Y,Z))$, then from proposition 4.4 *(ii)*, $\beta^{-1}(S(U,V)_{1,2}$ is open in $2 - C_\zeta(Y,Z)$. Now, $\mu(S(U,V)_1) \subset \beta^{-1}(S(U,V)_{1,2}$. Since $\mu(S(U,V)_1)$ is

compact, then by the above lemma, there exist an open set $S(A,B)_2$ such that $\mu(S(U,V)_1) \subset S(A,B)_2 \subset \overline{S(A,B)}_2 \subset \beta^{-1}(S(U,V)_{1,2})$. This implies that $\mu \in (S(U,V)_1, S(A,B)_2)$ and $\beta \in (\overline{S(A,B)}_2, S(U,V)_{1,2})$. Therefore $T((S(U,V)_1, S(A,B)_2), (\overline{S(A,B)}_2, S(U,V)_{1,2})) \subset (S(U,V)_1, S(U,V)_{1,2})$ implying that the function T is continuous. □

Theorem 4.7. The function $\varrho : 1 - C_\varsigma(Y,Z) \longrightarrow (2,1) - C_\xi(Y,Z)$ defined by $\varrho_f(h) = h \circ f$ is a homeomorphism.

Proof. Let h_1 and h_2 be functions in $1 - C_\varsigma(Y,Z)$, then $\varrho_f(h_1) = h_1 \circ f$ and $\varrho_f(h_2) = h_2 \circ f$. Suppose $\varrho_f(h_1) = \varrho_f(h_2)$, then $h_1 \circ f = h_2 \circ f$, implying that $h_1 = h_2$, hence ϱ_f is a 1-1 function. The function ϱ_f is an onto function since for any $h \circ f \in (2,1) - C_\xi(Y,Z)$ their exist the function $h \in 1 - C_\varsigma(Y,Z)$ and from proposition 4.2, $\varrho_f(h) = h \circ f$ is open and continuous. Continuity of ϱ_f^{-1} follows from Proposition 4.4 *(i)*. □

Proposition 4.8. The function $j : 1 - C_\varsigma(Y,Z) \to s - C_\tau(Y,Z)$ is continuous.

Proof. Let $S(U,V)_s$ be open in $s - C_\tau(Y,Z)$, then $j^{-1}(S(U,V)_s) = \{f \in s - C(Y,Z) : f(U) \subset V\} = \{f \in s - C(Y,Z) : f|_{1-C(Y,Z)}(U) \subset V\} = S(U,V)_1$, which is open in $1 - C_\varsigma(Y,Z)$. □

Proposition 4.9. The function $j : 2 - C_\zeta(Y,Z) \to s - C_\tau(Y,Z)$ is continuous.

Proof. The proof is similar to that of Proposition 4.8. □

References

[1] Fox R. H., *On topologies for function spaces*, American Mathematical Society **27** (1945), 427-432.

[2] Kelly J. C., *Bitopological Spaces*, Proc. London Math Soc. **13** (1963), 71- 89.

[3] Pervin W. J., *Connectedness in Bitopological Spaces*, Nederl. Akad. Wetensch. Proc. Ser . A70, I ndag. Math. **29** (1967), 369-372.

[4] Willard S., *General Topology*, Addison-Wesley Publishing Company, USA, 1970.

[5] Porter K. F., *The open-open topology for function spaces*, Internat. J. Math. and Math. Scl. **16** (1993), no. 1, 111-116.

[6] Tallafha A. and Al-Bsoul A. and Fora A., *Countable Dense Homogeneous Bitopological Spaces*, Tr. J. of Mathematics **23** (1999), 233-242.

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SPLITTING AND ADMISSIBLE TOPOLOGIES DEFINED ON THE SET OF CONTINUOUS FUNCTIONS BETWEEN BITOPOLOGICAL SPACES

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ABSTRACT

In this paper, p-splitting, p-admissible, s-splitting and s-admissible topologies on the sets p−C(Y, Z) and s−C(Y, Z) are defined and their properties explored. exponential functions are introduced in function spaces and s-splitting and s-admissible topologies defined on s-C(Y, Z) compared using these mappings.

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Keywords: *p-splitting, p-admissible, s-splitting and s-admissible topologies.*

1. INTRODUCTION

Let X, Y and Z be topological spaces, the set of all continuous functions from Y to Z is denoted by $C(Y, Z)$. This set when given a topology τ forms the function space $C_\tau(Y,Z)$. For any function $h : X \times Y \to Z$ which is continuous in Y for each fixed $x \in X$, there is an associated map $h^* : X \to C_\tau(Y,Z)$. The function h^* is defined as follows, $h^*(x) = h_x$, where $h_x(y) = h(x, y)$ for every $y \in Y$ (Fox [3]). Arens and Dugundji [1] defines a topology τ defined on $C(Y, Z)$ to be splitting, if the continuity of the mapping h implies the continuity of the mapping h^*. Topology τ defined on $C(Y, Z)$ is said to be admissible, if the continuity of the mapping h^* implies the continuity of the mapping h. The latter is also defined, if the evaluation mapping $e: C_\tau(Y, Z) \times Y \to Z$ defined by $e(f, y) = f(y)$ is continuous. For the bitopological spaces (Y, τ_1, τ_2) and (Z, δ_1, δ_2) introduced by Kelly [4], the following sets of continuous functions have been defined. The set $i-C(Y, Z)$ of all i-continuous functions for i=1,2, the set $p-C(Y,Z)$ of all pairwise continuous functions and the set $s-C(Y, Z)$ of all supremum continuous functions (Muturi *et.al* [6] and Dvalishvili [2]). In this paper, we generalize bitopological concepts to function spaces defined on bitopological space and introduce p-splitting, p-admissible, s-splitting and s-admissible topologies on the set $p-C(Y, Z)$ and $s-C(Y, Z)$. exponential functions are also defined on function spaces and and s-splitting and s-admissible topologies defined on the set $s-C(Y, Z)$ compared.

2. PRELIMINARIES

The following definition are important in this work.

Definition 2.1: *(Pervin* [5]*). A function* $f : (Y,\tau_1,\tau_2) \to (Z,\delta_1,\delta_2)$, *is said to be pairwise continuous (p-continuous) if the induced functions* $f : (Y,\tau_1) \to (Z,\delta_1)$ *and* $f : (Y,\tau_2) \to (Z,\delta_2)$ *are continuous.*

Definition 2.2: *(Muturi et al.* [6]*). A subset A of a bitopological space* $(Y,\tau_1\vee\tau_2)$ *is called a supremum-open set or simply s-open set if* $A = U_1\cup U_2$, *where* $U_1 \in \tau_1$ *and* $U_2 \in \tau_2$.

Definition 2.3: *(Muturi et al.* [6]*). A function* $f : (Y,\tau_1\vee\tau_2) \to (Z,\delta_1\vee \delta_2)$, *is said to be s-continuous, if the inverse image of each s-open subset of Z is s-open in Y .*

Definition 2.4: *The set of all pairwise continuous functions from the bitopological space* (Y,τ_1,τ_2) *to the bitopological space* (Z,δ_1,δ_2) *is denoted by* $p-C(Y,Z)$, *and the set of all supremum continuous function from the bitopological space* $(Y,\tau_1 \vee \tau_2)$ *to the bitopological space* $(Z,\delta_1 \vee \delta_2)$ *is denoted by* $s-C(Y,Z)$.

Definition 2.5: *The sets of the form* $S((U,V),(A,B))_p = \{f \in p-C(Y,Z) : f(U) \subset V \text{ and } f(A) \subset B\}$ *for U open in* τ_1, *V open in* δ_1, *A open in* τ_2 *and B open in* δ_2, *defines the subbasis for the open-open topology on the set* $p-C(Y,Z)$.

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3. PAIRWISE SPLITTING AND PAIRWISE ADMISSIBLE TOPOLOGIES DEFINED ON THE SET $p{-}C(Y, Z)$

In this section, we explore pairwise splitting and pairwise admissible topologies defined on the set $p{-}C(Y, Z)$.

Proposition 3.1: *The function* $h : (X, \sigma) \times (Y, \tau_1, \tau_2) \to (Z, \delta_1, \delta_2)$ *is pairwise continuous in* Y *for each fixed* $x \in X$, *if the functions* $h : (X, \sigma) \times (Y, \tau_1) \to (Z, \delta_1)$ *and* $h : (X, \sigma) \times (Y, \tau_2) \to (Z, \delta_2)$ *are continuous in* Y *for each fixed* $x \in X$.

Proof: Let $h : (X, \sigma) \times (Y, \tau_1) \to (Z, \delta_1)$ and $h : (X, \sigma) \times (Y, \tau_2) \to (Z, \delta_2)$ be continuous functions in Y for each fixed $x \in X$, then the functions $h_x : (Y, \tau_1) \to (Z, \delta_1)$ and $h_x : (Y, \tau_2) \to (Z, \delta_2)$ are continuous. By definition of pairwise continuity, the function $h_x : (Y, \tau_1, \tau_2) \to (Z, \delta_1, \delta_2)$ is continuous for each $x \in X$. Since $h_x(y) = h(x, y)$ and $h(x)(y) = h(x, y)$, then $h_x(y) = h(x)(y)$, implying that the function $h : (X, \sigma) \times (Y, \tau_1, \tau_2) \to (Z, \delta_1, \delta_2)$ is continuous in Y for each fixed $x \in X$.

Proposition 3.2: *The function* $h^* : (X, \sigma) \to p{-}C_\omega(Y, Z)$ *is pairwise continuous, if the functions* $h^* : (X, \sigma) \to 1{-}C_\varsigma(Y, Z)$ *and* $h^* : (X, \sigma) \to 2{-}C_\zeta(Y, Z)$ *are continuous, where* $h : (X, \sigma)\times(Y, \tau_i) \to (Z, \delta_i)$ *for* $i = 1,2$.

Proof: Let $h^* : (X, \sigma) \to 1{-}C_\varsigma(Y, Z)$ and $h^* : (X, \sigma) \to 2{-}C_\zeta(Y, Z)$ be continuous functions. Then for each fixed $x \in X$, the functions $h_x : (Y, \tau_1) \to (Z, \delta_1)$ and $h_x : (Y, \tau_2) \to (Z, \delta_2)$ are continuous. By definition of pairwise continuity, the function $h_x : (Y, \tau_1, \tau_2) \to (Z, \delta_1, \delta_2)$ is continuous for each $x \in X$. Since $h_x = h^*(x)$, then the function $h^* : (X, \sigma) \to p{-}C_\omega(Y, Z)$ is continuous.

From the above propositions, we introduce the following definitions.

Definition 3.3: *A topology* ω *on* $p{-}C(Y, Z)$ *is said to be pairwise splitting (p-splitting) if the continuity of the functions* $h : (X, \sigma) \times (Y, \tau_1) \to (Z, \delta_1)$ *and* $h : (X, \sigma) \times (Y, \tau_2) \to (Z, \delta_2)$ *in* Y *for each fixed* $x \in X$, *implies that of* $h^* : (X,\sigma) \to p{-}C_\omega(Y,Z)$.

Definition 3.4: *A topology* ω *on* $p{-}C(Y, Z)$ *is said to be pairwise admissible (p-admissible) if the continuity of the functions* $h^* : (X, \sigma) \to 1{-}C_\varsigma(Y, Z)$ *and* $h^* : (X, \sigma) \to 2{-}C_\zeta(Y, Z)$ *implies that of* $h : (X, \sigma) \times (Y, \tau_1, \tau_2) \to (Z, \delta_1, \delta_2)$ *in* Y *for each fixed* $x \in X$.

Theorem 3.5: *Let* $h : (X, \sigma) \times (Y, \tau_1) \to (Z, \delta_1)$ *and* $h : (X, \sigma) \times (Y, \tau_2) \to (Z, \delta_2)$ *be continuous functions, then the compact open topology* ω *defined on* $p{-}C(Y, Z)$ *is pairwise splitting.*

Proof: Let $h : (X, \sigma)\times(Y, \tau_1) \to (Z, \delta_1)$ and $h : (X, \sigma)\times(Y, \tau_2) \to (Z, \delta_2)$ be continuous functions in Y for each fixed $x \in X$, and let $x_0 \in X$ such that $h^*(x_0) \in S((U, V)(A, B))_p$, where $S((U, V)(A,B))_p$ is open in $p{-}C(Y, Z)$. Then $h^*(x_0) \in S(U, V)_1$ and $h^*(x_0) \in S(A, B)_2$, implying that $x_0 \times U \subset h^{-1}(V)$ and $x_0 \times A \subset h^{-1}(B)$. Since U and A are compact, then by tube lemma there exist an open set W neighbourhood of x_0 such that $W \times U \subset h^{-1}(V)$ and $W \times A \subset h^{-1}(B)$, this implies that $h^*(W) \subset S(U, V)_1$ and $h^*(W) \subset S(A, B)_2$, implying further that $h^* : (X, \sigma) \to 1{-}C_\varsigma(Y, Z)$ and $h^* : (X, \sigma) \to 2{-}C_\zeta(Y, Z)$ are continuous functions. By proposition 3.2, the function $h^* : (X, \sigma) \to p{-}C_\omega(Y, Z)$ is continuous and by definition 3.3, topology ω is pairwise splitting on $p{-}C(Y, Z)$.

Theorem 3.6: *Let* $h^* : (X, \sigma) \to 1{-}C_\varsigma(Y, Z)$ *and* $h^* : (X, \sigma) \to 2{-}C_\zeta(Y, Z)$ *be continuous functions, then the compact open topology* ω *defined on* $p{-}C(Y, Z)$ *is pairwise admissible for locally compact spaces* (Y, τ_1) *and* (Y, τ_2).

Proof: Let ς and ζ be compact open topologies on $1-C(Y,Z)$ and $2-C(Y,Z)$ respectively such that the evaluation functions $e: 1-C_\varsigma(Y,Z)\times Y \to Z$ and $e: 2-C_\varsigma(Y,Z)\times Y \to Z$ are continuous. Let $h^* : (X, \sigma) \to 1{-}C_\varsigma(Y, Z)$ and $h^* : (X, \sigma) \to 2{-}C_\zeta(Y, Z)$ be continuous functions and $i : (Y, \tau_1) \to (Y, \tau_1)$ and $i : (Y, \tau_2) \to (Y, \tau_2)$ be identity functions, then $e \circ (h^* \times i) : (X, \sigma) \times (Y, \tau_1) \to (Z, \delta_1)$ and $e \circ (h^* \times i) : (X, \sigma) \times (Y, \tau_2) \to (Z, \delta_2)$ are continuous functions. By proposition 3.1, the function $e\circ(h^*\times i) : (X, \sigma)\times(Y, \tau_1, \tau_2) \to (Z, \delta_1, \delta_2)$ is continuous in Y for each fixed $x \in X$ and by definition 3.4, topology ω defined on $p{-}C(Y, Z)$ is pairwise admissible.

Remark 3.7: *From theorem 3.5 and theorem 3.6, we conclude that* τ *on* $p{-}C(Y, Z)$ *is p-splitting or p-admissible topology if* ς *and* ζ *are splitting or admissible topologies on* $1{-}C(Y, Z)$ *and* $2{-}C(Y, Z)$ *respectively.*

4. SUPREMUM SPLITTING AND SUPREMUM ADMISSIBLE TOPOLOGIES DEFINED ON THE SET s-C(Y, Z)

In this section, supremum splitting and supremum admissible topologies are introduced on the set $s{-}C(Y, Z)$.

Definition 4.1: *A topology τ on $s-C(Y, Z)$ is said to be supremum splitting (s-splitting) if the continuity of the functions $f : (X, \sigma) \times (Y, \tau_1) \to (Z, \delta_1)$ and $f : (X, \sigma) \times (Y, \tau_2) \to (Z, \delta_2)$ in Y for each fixed $x \in X$, implies that of $f^* : (X, \sigma) \to s-C_\tau(Y, Z)$.*

Definition 4.2: *A topology τ on $s-C(Y, Z)$ is said to be supremum admissible (s-admissible) if the continuity of the functions $f^* : (X, \sigma) \to 1-C_\varsigma(Y, Z)$ and $f^* : (X, \sigma) \to 2-C_\zeta(Y, Z)$, implies that of $f : (X, \sigma) \times (Y, \tau_1 \vee \tau_2) \to (Z,\delta_1\vee \delta_2)$ in Y for each fixed $x \in X$.*

Proposition 4.3: *The function $f : (X, \sigma) \times (Y, \tau_1 \vee \tau_2) \to (Z, \delta_1 \vee \delta_2)$ is continuous if the functions $f : (X, \sigma) \times (Y, \tau_1) \to (Z, \delta_1)$ and $f : (X, \sigma) \times (Y, \tau_2) \to (Z, \delta_2)$ are continuous.*

Proof: Let the functions $f : (X, \sigma) \times (Y, \tau_1) \to (Z, \delta_1)$ and $f : (X, \sigma) \times (Y, \tau_2) \to (Z, \delta_2)$ be continuous in Y for each fixed $x \in X$. then the associated functions $f_x : (Y,\tau_1) \to (Z, \delta_1)$ and $f_x : (Y, \tau_2) \to (Z,\delta_2)$ defined by $f_x(y) = f(x, y)$, are continuous $\forall x \in X$. From theorem 3.1 [6], it follows that the function $f_x : (Y, \tau_1 \vee \tau_2) \to (Z, \delta_1 \vee \delta_2)$ is s-continuous $\forall x \in X$. Since $f_x(y) = f(x, y)$ and $f(x)(y) = f(x, y)$, then $f_x(y) = f(x)(y)$ and hence $f : (X, \sigma)\times(Y, \tau_1\vee\tau_2) \to (Z, \delta_1\vee\delta_2)$ is continuous in Y for each fixed $x \in X$.

Proposition 4.4: *The function $f^* : (X, \sigma) \to s-C_\tau(Y, Z)$ is continuous if the functions $f^* : (X, \sigma) \to 1-C_\varsigma(Y, Z)$ and $f^* : (X, \sigma) \to 2-C_\zeta(Y, Z)$ are continuous.*

Proof: Let $f^* : (X, \sigma) \to 1-C_\varsigma(Y, Z)$ and $f^* : (X, \sigma) \to 2-C_\zeta(Y, Z)$ be a continuous functions, then for the functions $f : (X, \sigma)\times(Y, \tau_1) \to (Z, \delta_1)$ and $f : (X, \sigma) \times (Y, \tau_2) \to (Z, \delta_2)$, the associated functions $f_x : (Y, \tau_1) \to (Z, \delta_1)$ and $f_x : (Y, \tau_2) \to (Z, \delta_2)$ defined by $f_x = f^*(x)$, $\forall x \in X$ are continuous. From theorem 3.1 [6], it follows that the function $f_x : (Y, \tau_1\vee\tau_2) \to (Z, \delta_1\vee \delta_2)$ is s-continuous $\forall x \in X$. Since $f_x = f^*(x)$, then the function $f^* :(X,\sigma) \to s-C(Y,Z)$ is continuous.

Theorem 4.5: *A compact open topology τ is s-splitting if the continuity of the functions $f : (X, \sigma)\times(Y, \tau_1) \to (Z, \delta_1)$ and $f : (X, \sigma)\times(Y, \tau_2) \to (Z, \delta_2)$ implies continuity of the function $f^* : (X, \delta) \to s-C_\tau(Y, Z)$.*

Proof: Let $f : (X, \sigma)\times(Y, \tau_1) \to (Z, \delta_1)$ and $f : (X, \sigma)\times(Y, \tau_2) \to (Z, \delta_2)$ be continuous functions in Y for each fixed $x \in X$. Then from proposition 4.3, the function $f : (X, \sigma)\times(Y, \tau_1\vee\tau_2) \to (Z, \delta_1\vee\delta_2)$ is continuous. Let $x_0 \in X$ and $S(U,V)_s$ be open in $s-C_\tau(Y, Z)$, then $f^*(x_0) \in S(U, V)_s$, implying that $x_0 \times U \subset f^{-1}(V)$. Since U is compact, then by tube lemma, there exist an open set W neighbourhood of x_0 such that $W \times U \subset f^{-1}(V)$. This implies that $f^*(W) \subset S(U,V)_s$, implying further that $f^* : (X, \sigma) \to s-C_\tau(Y, Z)$ is continuous functions. By definition 4.1, topology τ is s-splitting on $s-C(Y, Z)$.

Theorem 4.6: *Let $f^* : (X, \sigma) \to 1-C_\varsigma(Y, Z)$ and $f^* : (X, \sigma) \to 2-C_\zeta(Y, Z)$ be continuous functions, then the compact open topology τ defined on $s-C(Y, Z)$ is s-admissible for locally compact spaces (Y, τ_1) and (Y, τ_2).*

Proof: Let ς and ζ be compact open topologies on $1-C(Y, Z)$ and $2-C(Y, Z)$ respectively, and let $f^* : (X, \sigma) \to 1-C_\varsigma(Y,Z)$ and $f^* : (X, \sigma) \to 2-C_\zeta(Y, Z)$ be a continuous functions, then by proposition 4.4, the function $f^* : (X, \sigma) \to s-C_\tau(Y, Z)$ is continuous. Let $i : (Y, \tau_1 \vee \tau_2) \to (Y, \tau_1 \vee \tau_2)$ be an identity function and let $e : s-C_\tau(Y, Z) \times (Y, \tau_1\vee \tau_2) \to (Z, \delta_1\vee \delta_2)$ be an evaluation mapping. Since τ is compact open topology, then the evaluation mapping e is continuous and the composite mapping $e\circ(f^*\times i) : (X, \sigma) \times (Y, \tau_1 \vee \tau_2) \to (Z, \delta_1 \vee \delta_2)$ is also continuous in Y for each fixed $x \in X$. By definition 4.2, topology τ is s-admissible.

Remark 4.7: *From theorem 4.5 and theorem 4.6, we note that if ς and ζ are splitting or admissible topologies on $1-C(Y,Z)$ and $2-C(Y,Z)$ respectively, then τ on $s-C(Y,Z)$ is s-splitting or s-admissible topology.*

5. EXPONENTIAL MAPPINGS DEFINED ON FUNCTION SPACES

Let (X, σ), $(Z, \delta_1\vee\delta_2)$ be arbitrary spaces and let $(Y, \tau_1\vee\tau_2)$ be locally compact Hausdorff space.

Definition 5.1: *Consider the exponential mapping $\Lambda : C(X \times Y,Z) \to C(X,s - C_\phi(Y, Z))$, defined by $\Lambda(f)(x)(y) = f(x, y)$ for each $f \in C(X \times Y,Z)$, $x \in X$ and $y \in Y$. A topology ϕ on $s-C(Y, Z)$ is called s-splitting topology if Λ is a continuous function with respect to ϕ.*

Definition 5.2: *Consider the exponential mapping $\Lambda^{-1} : C(X, s-C_\phi(Y, Z)) \to C(X \times Y,Z)$, defined by $\Lambda^{-1}((g)(x, y)) = g(x)(y)$ where $g \in C(X, s-C_\phi(Y, Z))$ for each $(x, y) \in X \times Y$. A topology ϕ on $s-C(Y, Z)$ is called s-admissible topology if the function Λ^{-1} is continuous with respect to ϕ.*

Proposition 5.3: *The function* $\Lambda^{-1} \circ \Lambda : C(X \times Y, Z) \rightarrow C(X \times Y, Z)$ *is continuous.*

Proof: Observe that $(\Lambda^{-1} \circ \Lambda(f))(x, y) = \Lambda^{-1}(\Lambda(f))(x, y) = \Lambda(f)(x)(y) = f(x, y)$. Implying that $\Lambda^{-1}\circ\Lambda(f) = f$, hence $\Lambda^{-1}\circ\Lambda$ is an identity function.

Proposition 5.4: *The function* $\Lambda \circ \Lambda^{-1} : C(X, s{-}C_{\phi}(Y, Z)) \rightarrow C(X, s{-}C_{\phi}(Y, Z))$ *is continuous.*

Proof: Observe that $(\Lambda\circ\Lambda{-}1(f))(x)(y) = \Lambda(\Lambda{-}1(f))(x)(y) = \Lambda{-}1(f)(x, y) = f(x)(y)$. Implying that $\Lambda \circ \Lambda{-}1(f) = f$, hence $\Lambda \circ \Lambda^{-1}$ is an identity function.

Remark 5.5: *From proposition 5.3 and proposition 5.4, it follows that* Λ *is a homeomorphism.*

Proposition 5.6: *The function* $i : C(X, s{-}C_{\phi 1}(Y, Z)) \rightarrow C(X, s{-}C_{\phi 2}(Y, Z))$ *is continuous if and only if* $\phi_2 \subset \phi_1$.

Proof: The function i is continuous if and only if $S(W, S(U,V)) \in \phi_2$ implies that $i^{-1}(S(W, S(U,V))) \in \phi_1$, but i is an identity function, therefore $i^{-1}(S(W, S(U, V))) = S(W, S(U, V))$. Hence i is continuous if and only if $S(W, S(U, V)) \in \phi_2$ implies $S(W, S(U, V)) \in \phi_1$.

Theorem 5.7: *The following statements are true;*

(i) Let ϕ_1 *be s-splitting topology on* $s{-}C(Y, Z)$ *and let* $\phi_2 \subset \phi_1$*, then* ϕ_2 *is also s-splitting topology on* $s{-}C(Y, Z)$*.*
(ii) Let ϕ_1 *be s-admissible topology on* $s{-}C(Y, Z)$ *and let* $\phi_1 \subset \phi_2$*, then* ϕ_2 *is also s-admissible topology on* $s{-}C(Y, Z)$*.*
(iii) Let ϕ_1 *be s-splitting topology on* $s{-}C(Y, Z)$ *and let* ϕ_2 *be admissible topology on* $s{-}C(Y, Z)$*, then* $\phi_1 \subset \phi_2$*.*

Proof:

(i) Let ϕ_1 be s-splitting topology, then by definition 5.1 the function $\Lambda : C(X \times Y, Z) \rightarrow C(X, s{-}C_{\phi 1}(Y, Z))$, defined by $\Lambda(f)(x)(y) = f(x, y)$ for each $f \in C(X \times Y, Z)$, $x \in X$ and $y \in Y$, is continuous with respect to ϕ_1. Let ϕ_2 be any other topology such that $\phi_2 \subset \phi_1$, then by proposition 5.6, the function $i : C(X, s{-}C_{\phi 1}(Y, Z)) \rightarrow C(X, s{-}C_{\phi 2}(Y, Z))$ is continuous. Now the composite function $i \circ \Lambda : C(X \times Y, Z) \rightarrow C(X, s{-}C_{\phi 2}(Y, Z))$ is continuous with respect to ϕ_2, implying that ϕ_2 is also s-splitting topology.

(ii) Let ϕ_1 be s-admissible topology, then by definition 5.2 the function $\Lambda^{-1} : C(X, s{-}C_{\phi 1}(Y, Z)) \rightarrow C(X \times Y, Z)$ defined by $\Lambda^{-1}((g)(x, y)) = g(x)(y)$ where $g \in C(X, s{-}C_{\phi}(Y, Z))$ for each $(x, y) \in X\times Y$, is continuous with respect to ϕ_1. Let $\phi_1 \subset \phi_2$, then by proposition 5.6, the function $i : C(X, s{-}C_{\phi 2}(Y, Z)) \rightarrow C(X, s{-}C_{\phi 1}(Y, Z))$ is continuous. Now the composite function $\Lambda^{-1}\circ i : C(X, s{-}C_{\phi 2}(Y, Z)) \rightarrow C(X \times Y, Z)$ is continuous with respect to ϕ_2. Hence ϕ_2 is also s-admissible topology.

(iii) Let ϕ_1 be s-splitting topology, then by definition 5.1 the function $\Lambda : C(X \times Y, Z) \rightarrow C(X, s{-}C_{\phi 1}(Y, Z))$, defined by $\Lambda(f)(x)(y) = f(x, y)$ for each $f \in C(X \times Y, Z)$, $x \in X$ and $y \in Y$, is continuous with respect to ϕ_1. Let ϕ_2 be s-admissible topology, then by definition 5.2 the function $\Lambda^{-1} : C(X, s{-}C_{\phi 1}(Y, Z)) \rightarrow C(X \times Y, Z)$ defined by $\Lambda^{-1}((g)(x, y)) = g(x)(y)$ where $g \in C(X, s{-}C_{\phi}(Y, Z))$ for each $(x, y) \in X \times Y$, is continuous with respect to ϕ_1. Now the composite function $\Lambda\circ\Lambda^{-1} : C((X, \sigma), s{-}C_{\phi 2}(Y, Z)) \rightarrow C((X, \sigma), s{-}C_{\phi 1}(Y, Z))$ is continuous by proposition 5.6, implying that $\phi_1 \subset \phi_2$.

REFERENCES

[1] Arens, R. F. and Dugundji J., *Topologies for function space*, Pacific J. Math. **1** (1951), 5-31.
[2] Dvalishvili B. P., *Bitopological Spaces, Theory, Relations with Generalized Algebraic Structures, and Applications*, Elsevier Ltd, San Diego, USA, 2005.
[3] Fox R. H., *On topologies for function spaces*, American Mathematical Society **27** (1945), 427-432.
[4] Kelly J. C., *Bitopological Spaces*., Proc. London Math Soc. **13** (1963), 71- 89.
[5] Pervin W. J., *Connectedness in Bitopological Spaces*, Nederl. Akad. Wetensch. Proc. Ser. A70, Indag. Math. **29** (1967) 369-372.
[6] Muturi N. E., Pokhariyal G.P. and Khalaghai J. M., *Continuity of functions on function spaces defined on bitopological spaces.* Journal of Advanced Studies in Topology **8 (2)** (2017), 130-134.

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Separation axioms on function spaces defined on bitopological spaces

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Abstract

In this paper, we generalize separation axioms to the function space $p-C_{\omega}(Y,Z)$ and study how they relate to separation axioms defined on the spaces (Z,δ_i) for $i=1,2$, (Z,δ_1,δ_2), $1-C_{\varsigma}(Y,Z)$ and $2-C_{\zeta}(Y,Z)$. We show that the space $p-C_{\omega}(Y,Z)$ is ${}_pT_\circ$, ${}_pT_1$, ${}_pT_2$ and ${}_p$regular, if the spaces (Z,δ_1) and (Z,δ_2) are both $T_\circ$, T_1, T_2 and regular respectively. The space $p-C_{\omega}(Y,Z)$ is also shown to be ${}_pT_\circ$, ${}_pT_1$, ${}_pT_2$ and ${}_p$regular, if the space (Z,δ_1,δ_2) is $p-T_\circ$, $p-T_1$, $p-T_2$ and p-regular respectively. Finally, the space $p-C_{\omega}(Y,Z)$ is shown to be ${}_pT_\circ$, ${}_pT_1$, ${}_pT_2$ and ${}_p$regular, if and only if the spaces $1-C_{\varsigma}(Y,Z)$ and $2-C_{\zeta}(Y,Z)$ are both T_0, T_1, T_2, and only if the spaces $1-C_{\varsigma}(Y,Z)$ and $2-C_{\zeta}(Y,Z)$ are both regular respectively.

Keywords: bitopological space, function space, separation axiom.
2010 MSC: 54A10, 54C35, 54D10, 54E55.

1. Introduction

The set of all continuous functions from a topological space Y to a topological space Z is denoted by $C(Y,Z)$. Several topologies have been defined on this set as seen in [3], [1] and [2]. The non empty set Y when assigned two unique topologies τ_1 and τ_2, forms a bitopological space (Y,τ_1,τ_2) (see [5]). A number of function spaces have been defined on sets of continuous functions between two bitopological spaces (Y,τ_1,τ_2) and (Z,δ_1,δ_2), examples of such function spaces include; $s-C_{\tau}(Y,Z)$, $p-C_{\omega}(Y,Z)$, $1-C_{\varsigma}(Y,Z)$ and $2-C_{\zeta}(Y,Z)$ (see [8]).

Separation axioms allows one to separates points from points, points from closed sets and closed sets from each other using open sets. These axioms play a critical role in topology in that, apart from characterizing continuous mappings, they also provide restrictive conditions on which other topological properties and structures can be defined on a given non empty set. Studies of separation axioms on function spaces are covered in [1], [4] and [12]. Pairwise separation axioms have been introduced on bitopological spaces in [5], while in [6] and [11], comparisons have been made between separation axioms defined on the spaces

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(Y, τ_1) and (Y, τ_2), (Y, τ_1, τ_2) and $(Y, \tau_1 \vee \tau_2)$. In this paper, we generalize separation axioms to the function space $p - C_\omega(Y, Z)$, and study how they relate to separation axioms defined on topological spaces (Z, δ_i) for $i = 1, 2$, pairwise separation axioms defined on bitopological space (Z, δ_i, δ_2), as well as separation axioms defined on function spaces $1 - C_\varsigma(Y, Z)$ and $2 - C_\zeta(Y, Z)$.

2. Preliminaries

The following definitions are considered in this paper.

Definition 2.1. A function $f : (Y, \tau_1, \tau_2) \longrightarrow (Z, \delta_1, \delta_2)$, is said to be pairwise continuous (p-continuous) or $\tau_1 - \delta_1$ and $\tau_2 - \delta_2$ continuous, if the induced functions $f : (Y, \tau_1) \longrightarrow (Z, \delta_1)$ and $f : (Y, \tau_2) \longrightarrow (Z, \delta_2)$ are both continuous (see [10]).

Definition 2.2. The collection $S((U, V), (A, B))_p = \{f \in p - C(Y, Z) : f(U) \subset V$ and $f(A) \subset B\}$ of sets, for U open in τ_1, V open in δ_1, A open in τ_2 and B open in δ_2, forms the subbasis for the open-open topology ω on $p - C(Y, Z)$ (the set of all pairwise continuous functions). If U and A are compact subsets of Y, then $S((U, V), (A, B))_p$ forms the subbasis for compact open topology. The set of all pairwise continuous functions endowed with topology ω is denoted by $p - C_\omega(Y, Z)$ (see [9]) .

Definition 2.3. The space (Y, τ_1, τ_2) is said to be pairwise $T_\circ$ $(p - T_\circ)$, if for each pair of distinct points of Y, there is a τ_1 open set or τ_2 open set containing one of the points, but not the other (see [7]).

Definition 2.4. The space (Y, τ_1, τ_2) is said to be pairwise T_1 $(p - T_1)$, if for each pair of distinct points $x, y \in Y$, there is a τ_1 open set U and a τ_2 open set V, such that $x \in U$, $y \notin U$ and $x \notin V$, $y \in V$ (see [11]).

Definition 2.5. The space (Y, τ_1, τ_2) is said to be pairwise T_2 $(p - T_2)$, if for two distinct points $x, y \in Y$, there is a τ_1 open set U and τ_2 open set V, such that $x \in U$, $y \in V$ and $U \cap V = \phi$ (see [5]).

Definition 2.6. In the space (Y, τ_1, τ_2), τ_1 is said to be regular with respect to τ_2, if for each $y \in Y$ and τ_1 closed set F such that $y \notin F$, there exist τ_1 open set U and τ_2 open set V such that $x \in U$, $F \subset V$ and $U \cap V = \phi$. The space (Y, τ_1, τ_2) is said to be pairwise regular (p-regular), if it is both τ_1 regular with respect to τ_2 and τ_2 regular with respect to τ_1 (see [5]).

Let (Y, τ_1, τ_2) and (Z, δ_1, δ_2) be bitopological spaces, and let U_1 and U_2 be open sets in τ_1, V_1 and V_2 be open sets in δ_1, A_1 and A_2 be open sets in τ_2 and B_1 and B_2 be open sets in δ_2. Let ${}_pT_i$ for $i = 0, 1, 2$ and ${}_p$regular, denote separation axioms defined on $p - C_\omega(Y, Z)$, to differentiate them from pairwise separation axioms defined on bitopological space (Y, τ_1, τ_2).

The following definitions are introduced.

Definition 2.7. A function space $p - C_\omega(Y, Z)$ is said to be a ${}_pT_\circ$-space, if for any two distinct functions f and g in $p - C(Y, Z)$, there exist an open set $S((U_1, V_1)(A_1, B_1))_p = \{f \in p - C(Y, Z) : f(U_1) \subset V_1$ and $f(A_1) \subset B_1\}$ neighborhood of f not containing g, or $S((U_2, V_2)(A_2, B_2))_p = \{g \in p - C(Y, Z) : g(U_2) \subset V_2$ and $g(A_2) \subset B_2\}$ neighborhood of g not containing f.

Definition 2.8. A function space $p - C_\omega(Y, Z)$ is said to be a ${}_pT_1$-space, if for any two distinct functions f and g in $p - C(Y, Z)$, there exist open sets $S((U_1, V_1)(A_1, B_1))_p = \{f \in p - C(Y, Z) : f(U_1) \subset V_1$ and $f(A_1) \subset B_1\}$ neighborhood of f not containing g, and $S((U_2, V_2)(A_2, B_2))_p = \{g \in p - C(Y, Z) : g(U_2) \subset V_2$ and $g(A_2) \subset B_2\}$ neighborhood of g not containing f.

Definition 2.9. A function space $p - C_\omega(Y, Z)$ is said to be a ${}_pT_2$-space, if for any two distinct functions f and g in $p - C(Y, Z)$, there exist disjoint open sets $S((U_1, V_1)(A_1, B_1))_p = \{f \in p - C(Y, Z) : f(U_1) \subset V_1$ and $f(A_1) \subset B_1\}$ and $S((U_2, V_2)(A_2, B_2))_p = \{g \in p - C(Y, Z) : g(U_2) \subset V_2$ and $g(A_2) \subset B_2\}$ neighborhoods of f and g respectively.

Definition 2.10. A function space $p - C_\omega(Y, Z)$ is said to be a ${}_p$regular space, if for any two distinct functions f and g in $p - C(Y, Z)$ and a closed set $\overline{S((U, V)(A, B)}$ in $p - C(Y, Z)$ such that $g \notin \overline{S(U, V)(A, B)}$, there exist disjoint open sets $S((U_1, V_1)(A_1, B_1))_p = \{f \in p - C(Y, Z) : f(U_1) \subset V_1$ and $f(A_1) \subset B_1\}$ containing $\overline{S((U, V)(A, B)}$ and $S((U_2, V_2)(A_2, B_2))_p = \{g \in p - C(Y, Z) : g(U_2) \subset V_2$ and $g(A_2) \subset B_2\}$ neighborhood of g.

3. Comparison of separation axioms defined on the spaces $p - C_\omega(Y, Z)$, (Z, δ_1), (Z, δ_2) and (Z, δ_1, δ_2)

Let P denote a topological property, If both the topological spaces (Z, δ_1) and (Z, δ_2), and both the function spaces $1 - C_\varsigma(Y, Z)$ and $2 - C_\zeta(Y, Z)$ have the property P, then it will be denoted by $b - P$. In this section, we establish the relationship between ${}_pT_\circ$, ${}_pT_1$, ${}_pT_2$ and ${}_p$regular separation axioms defined on the function space $p - C_\omega(Y, Z)$, and $b - T_\circ$, $b - T_1$, $b - T_2$ and b-regular separation axioms defined on the topological spaces (Z, δ_1) and (Z, δ_2), as well as $p - T_\circ$, $p - T_1$, $p - T_2$ and p-regular separation axioms defined on bitopological space (Z, δ_1, δ_2). We provide proof for ${}_pT_2$ and ${}_p$regularity on $p - C_\omega(Y, Z)$ whenever (Z, δ_1) and (Z, δ_2) are $b - T_2$ and b-regular spaces, and also ${}_pT_\circ$, ${}_pT_2$ and ${}_p$regularity on $p - C_\omega(Y, Z)$, whenever (Z, δ_1, δ_2) is $p - T_0$, $p - T_2$ and p-regular space. The proofs for the other separation axioms can be done in a similar manner.

Theorem 3.1. Let (Z, δ_1) and (Z, δ_2) be $b - T_2$ spaces, then $p - C_\omega(Y, Z)$ is a ${}_pT_2$ space.

Proof. Let f and g be unique functions in $p - C(Y, Z)$ such that for every $y \in Y$, $f(y) \neq g(y)$, and let (Z, δ_1) and (Z, δ_2) be $b - T_2$ spaces. Then there exist disjoint open sets $U_1 \in \delta_1$ and $V_1 \in \delta_1$ and also $U_2 \in \delta_2$ and $V_2 \in \delta_2$ such that $f(y) \in U_1$ and $g(y) \in V_1$, and also $f(y) \in U_2$ and $g(y) \in V_2$ respectively. Now, the disjoint open sets $S((\{y\}, U_1)(\{y\}, U_2))_p$ and $S((\{y\}, V_1)(\{y\}, V_2))_p$ in $p - C_\omega(Y, Z)$, are neighbourhoods of f and g respectively in the space $p - C_\omega(Y, Z)$. Therefore, the space $p - C_\omega(Y, Z)$ is a ${}_pT_2$ space. □

Theorem 3.2. Let the spaces (Z, δ_1) and (Z, δ_2) be b-regular, then $p - C_\omega(Y, Z)$ with compact open topology ω is a ${}_p$regular space.

Proof. Let f and g be unique functions in $p - C(Y, Z)$ such that $\forall y \in Y$ $f(y) \neq g(y)$ and let $S((U_i, V_i)(U_j, (V_j))$ $= \{f \in p - C(Y, Z) : f(U_i) \subset V_i \textit{ and } f(U_j) \subset V_j)\}$ for $U_i \in \tau_1$, $V_i \in \delta_1$, $U_j \in \tau_2$ and $V_j \in \delta_2$ for $i, j = 1, 2, 3, 4....n$ be the neighbourhood system for f. Since U_i and U_j are compact, then both $f(U_i)$ and $f(U_j)$ are also compact, and since (Z, δ_1) and (Z, δ_2) are b-regular spaces, then there exist open sets A_i and B_j in δ_1 and δ_2 respectively, for $i, j = 1, 2, 3, 4....n$, such that $f(U_i) \subset A_i$, $f(U_j) \subset B_j$, $\overline{A_i} \subset V_i$ and $\overline{B_j} \subset V_j$. This implies that $S((U_i, A_i)(U_j, B_j)) \subset S((U_i, \overline{A}_i)(U_j, \overline{B}_j)) \subset S((U_i, V_i)(U_j, V_j))$. Suppose that $\overline{S((U_i, A_i)(U_j, B_j))} \subset S((U_i, \overline{A}_i)(U_j, \overline{B}_j))$, let $g \notin S((U_i, V_i)(U_j, V_j))$, then it follows that $g \notin S((U_i, \overline{A}_i)(U_j, \overline{B}_j))$, implying further that for some point $y \in Y$, $g(y) \in \overline{A_i}^{\,c}$ and $g(y) \in \overline{B_j}^{\,c}$. Thus, $S((\{y\}, \overline{A_i}^{\,c})(\{y\}, \overline{B_j}^{\,c}))$ is a neighbourhood system for g which does not intersect $S((U_i, \overline{A}_i)(U_j, \overline{B}_j))$. Since $\overline{S((U_i, A_i)(U_j, B_j))} \subset S((U_i, \overline{A}_i)(U_j, \overline{B}_j))$, then $\overline{S((U_i, A_i)(U_j, B_j))} \subset S((U_i, V_i)(U_j, V_j))$. Therefore the sets $\bigcap_{i=1}^{n} S((\{y\}, \overline{A_i}^{\,c})(\{y\}, \overline{B_j}^{\,c}))$ and $\bigcap_{i,j=1}^{n} S((U_i, V_i)(U_j, V_j))$ are disjoint open sets containing g and $\overline{\bigcap_{i=1}^{n} S((U_i, A_i)(U_j, B_j))}$ respectively, hence $p - C_\omega(Y, Z)$ is a ${}_p$regular space. □

Theorem 3.3. Let (Z, δ_1, δ_2) be $p - T_\circ$ space, then $p - C_\omega(Y, Z)$ is a ${}_pT_\circ$ space.

Proof. Let f and g be unique functions in $p - C(Y, Z)$ such that for every $y \in Y$, $f(y) \neq g(y)$, since (Z, δ_1, δ_2) is a $p - T_\circ$ space, then there exist an open set $U_1 \in \delta_1$ containing $f(y)$ but not $g(y)$ or $V_2 \in \delta_2$ containing $g(y)$ but not $f(y)$. Suppose there exist an open set $U_1 \in \delta_1$ containing $f(y)$ but not $g(y)$, then by pairwise continuity of f, we can find an open set $U_2 \in \delta_2$ also containing $f(y)$ but not $g(y)$. Suppose there exist an open set $V_2 \in \delta_2$ containing $g(y)$ but not $f(y)$, then by pairwise continuity of g, we can also find an open set $V_1 \in \delta_1$ containing $g(y)$ but not $f(y)$. Either way, there exist an open set $S((\{y\}, U_1)(\{y\}, U_2))_p$ in $p - C_\omega(Y, Z)$, neighbourhood of f not containing g, or an open set $S((\{y\}, V_1)(\{y\}, V_2))_p$ in $p - C_\omega(Y, Z)$, neighborhood of g not containing f. Therefore, the space $p - C_\omega(Y, Z)$ is a ${}_pT_\circ$ space. □

Theorem 3.4. Let (Z, δ_1, δ_2) be totally disconnected $p - T_2$ space, then $p - C_\omega(Y, Z)$ is a ${}_pT_2$ space.

Proof. Let f and g be unique functions in $p - C(Y, Z)$ such that for every $y \in Y$, $f(y) \neq g(y)$, since (Z, δ_1, δ_2) is a totally disconnected $p - T_2$ space, then there exist disjoint open sets $U_1 \in \delta_1$ and $V_2 \in \delta_2$ containing $f(y)$ and

$g(y)$ respectively, such that $U_1 \cup V_2 = Y$. But since f and g are both $\tau_1 - \delta_1$ and $\tau_2 - \delta_2$ continuous, it follows that there exist open sets $U_2 \in \delta_2$ containing $f(y)$ and $V_1 \in \delta_1$ containing $g(y)$. Suppose $U_2 = V_2{}^c \in \delta_2$ and $V_1 = U_1{}^c \in \delta_1$. Now, $V_2{}^c \cup U_1{}^c = (V_2 \cap U_1)^c = (\phi)^c = Y$, implying that $U_2 \cup V_1 = Y$, Now, $U_2 \cap V_1 = V_2{}^c \cap U_1{}^c = (V_2 \cup U_1)^c = Y^c = \phi$. Therefore the sets U_2 and V_1 are disjoint open sets, neighbourhoods of $f(y)$ and $g(y)$ respectively. Therefore the sets $S((\{y\}, U_1)(\{y\}, U_2))_p$ and $S((\{y\}, V_1)(\{y\}, V_2))_p$ in $p - C_\omega(Y, Z)$ are disjoint open sets, neighbourhoods of f and g respectively. Hence, $p - C_\omega(Y, Z)$ is a ${}_pT_2$ space. □

Theorem 3.5. Let the space (Z, δ_1, δ_2) be pairwise regular, then $p - C_\omega(Y, Z)$ is a ${}_p$regular space.

Proof. Let f and g be unique functions in $p - C(Y, Z)$ such that $\forall y \in Y\ f(y) \neq g(y)$ and let $S((U_i, V_i)(U_j, (V_j)) = \{f \in p - C(Y, Z) : f(U_i) \subset V_i \textit{ and } f(U_j) \subset V_j)\}$ for $U_i \in \tau_1$, $V_i \in \delta_1$, $U_j \in \tau_2$ and $V_j \in \delta_2$ for $i, j = 1, 2, 3, 4....n$ be the neighbourhood system for f. Now, U_i and U_j are both compact, therefore $f(U_i)$ and $f(U_j)$ are also compact. Since (Z, δ_1, δ_2) is pairwise regular space, then δ_1 regularity with respect to δ_2 implies that there exist open sets B_j in δ_2 for $j = 1, 2, 3, 4....n$, such that $f(U_j) \subset B_j$ and $\overline{B_j} \subset V_j$. This implies that $S(U_j, B_j) \subset S(U_j, \overline{B}_j) \subset S(U_j, V_j)$. Suppose that $\overline{S(U_j, B_j)} \subset S(U_j, \overline{B}_j)$, let $g \notin S(U_j, V_j)$, then it follows that $g \notin S(U_j, \overline{B}_j)$, implying further that for some point $y \in Y$, $g(y) \in \overline{B_j}^{\,c}$. Thus, $S(\{y\}, \overline{B_j}^{\,c})$ is a neighbourhood system for g which does not intersect $S(U_j, \overline{B}_j)$. Since $\overline{S(U_j, B_j)} \subset S(U_j, \overline{B}_j)$, then $\overline{S(U_j, B_j)} \subset S(U_j, (V_j)$. Therefore $\bigcap_{j=1}^{n} S(\{y\}, \overline{B_j}^{\,c})$ and $\bigcap_{j=1}^{n} S(U_j, \overline{B}_j)$ are $\tau_2 - \delta_2$ disjoint open sets neighbourhoods of g and $\overline{\bigcap_{i=1}^{n} S(U_j, B_j)}$ respectively. Now, δ_2 regularity with respect to δ_1 implies that there exist open sets A_i in δ_1 for $i = 1, 2, 3, 4....n$, such that $f(U_i) \subset A_i$ and $\overline{A_i} \subset V_i$. This implies that $S(U_i, A_i) \subset S(U_i, \overline{A}_i) \subset S(U_i, V_i)$. Suppose that $\overline{S(U_i, A_i)} \subset S(U_i, \overline{A}_i)$, let $g \notin S(U_i, V_i)$, then it follows that $g \notin S(U_i, \overline{A}_i)$, implying further that for some point $y \in Y$, $g(y) \in \overline{A_i}^{\,c}$. Thus, $S(\{y\}, \overline{A_i}^{\,c})$ is a neighbourhood system for g which does not intersect $S(U_i, \overline{A}_i)$. Since $\overline{S(U_i, A_i)} \subset S(U_i, \overline{A}_i)$, then $\overline{S(U_i, A_i)} \subset S(U_i, V_i)$. Therefore $\bigcap_{i=1}^{n} S(\{y\}, \overline{A_i}^{\,c})$ and $\bigcap_{i=1}^{n} S(U_i, V_i)$ are $\tau_1 - \delta_1$ disjoint open sets, neighbourhoods of g and $\overline{\bigcap_{i=1}^{n} S(U_i, A_i)}$ respectively. Let $f \in \overline{S(U_i, A_i)}$ and $f \in \overline{S(U_j, B_j)}$ imply that $f \in \overline{S((U_i, A_i), (U_j, B_j))}$, then $\bigcap_{i,j=1}^{n} S((\{y\}, \overline{A_i}^{\,c})(\{y\}, \overline{B_j}^{\,c}))$ and $\bigcap_{i,j=1}^{n} S((U_i, V_i)(U_j, V_j))$ are disjoint open sets neighbourhoods of g and $\overline{\bigcap_{i,j=1}^{n} S((U_i, A_i), (U_j, B_j))}$ respectively in $p - C_\omega(Y, Z)$. Therefore $p - C_\omega(Y, Z)$ is a ${}_p$regular space. □

4. Comparison of separation axioms defined on the spaces $p - C_\omega(Y, Z)$, $1 - C_\varsigma(Y, Z)$ and $2 - C_\zeta(Y, Z)$

The relationship between ${}_pT_\circ$, ${}_pT_1$, ${}_pT_2$ and ${}_p$regular separation axioms defined on the function space $p - C_\omega(Y, Z)$, and $b - T_\circ$, $b - T_1$, $b - T_2$ and b-regular separation axioms defined on function spaces $1 - C_\varsigma(Y, Z)$ and $2 - C_\zeta(Y, Z)$, are established in this section. We provide proof for ${}_pT_2$ and ${}_p$regular separation axioms on $p - C_\omega(Y, Z)$ whenever (Z, δ_1) and (Z, δ_2) are $b - T_2$ and b-regular spaces, and also $b - T_2$ property on (Z, δ_1) and (Z, δ_2) whenever $p - C_\omega(Y, Z)$ is a ${}_pT_2$ space. The proofs of the other separation axioms on the function space $p - C_\omega(Y, Z)$ can be done in a similar manner as that of ${}_pT_2$.

Theorem 4.1. The function space $p - C_\omega(Y, Z)$ is a ${}_pT_2$-space, if and only if the function spaces $1 - C_\varsigma(Y, Z)$ and $2 - C_\zeta(Y, Z)$ are $b - T_2$-spaces.

Proof. Let f and g be unique functions in $p - C_\omega(Y, Z)$ such that $\forall y \in Y\ f(y) \neq g(y)$, and let $1 - C_\varsigma(Y, Z)$ be a T_2 space such that $S(U_1, V_1)$ and $S(U_2, V_2)$ are disjoint open sets, neighbourhoods of f and g respectively. Also, let $2 - C_\zeta(Y, Z)$ be a T_2 space such that $S(A_1, B_1)$ and $S(A_2, B_2)$ are disjoint open sets, neighbourhoods f and g respectively. Now, pairwise continuity of f and g allows us to pick $S((U_1, V_1)(A_1, B_1))_p = \{f \in p - C(Y, Z) :$

$f(U_1) \subset V_1$ and $f(A_1) \subset B_1\}$ and $S((U_2,V_2)(A_2,B_2))_p = \{g \in p-C(Y,Z) : g(U_2) \subset V_2$ and $g(A_2) \subset B_2\}$ as disjoint open sets in $p-C_\omega(Y,Z)$, containing f and g respectively. Hence $p-C_\omega(Y,Z)$ is a ${}_pT_2$ space.

Conversely, let $p-C_\omega(Y,Z)$ be a ${}_pT_2$-space and let f and g be unique functions in $p-C_\omega(Y,Z)$ such that $\forall y \in Y\ f(y) \neq g(y)$, then there exist two disjoint open sets $S((U_1,V_1)(A_1,B_1))_p = \{f \in p-C(Y,Z) : f(U_1) \subset V_1$ and $f(A_1) \subset B_1\}$ for U_1 open in τ_1, V_1 open in δ_1, A_1 open in τ_2 and B_1 open in δ_2, neighborhood of f, and $S((U_2,V_2)(A_2,B_2))_p = \{g \in p-C(Y,Z) : g(U_2) \subset V_2$ and $g(A_2) \subset B_2\}$ for U_2 open in τ_1, V_2 open in δ_1, A_2 open in τ_2 and B_2 open in δ_2, neighborhood of g. But $S((U_1,V_1)(A_1,B_1))_p = \{f \in p-C(Y,Z) : f(U_1) \subset V_1\ \ and\ \ f(A_1) \subset B_1\}$=$\{\{f \in p-C(Y,Z) : f(U_1) \subset V_1\}$ and $\{f \in p-C(Y,Z) : f(A_1) \subset B_1\}\}$. Now $\{f \in p-C(Y,Z) : f(U_1) \subset V_1\} = \{f \in 1-C(Y,Z) : f(U_1) \subset V_1\} = S(U_1,V_1)$, and $\{f \in p-C(Y,Z) : f(A_1) \subset B_1\} = \{f \in 2-C(Y,Z) : f(A_1) \subset B_1\} = S(A_1,B_1)$. These two sets are open and are both neighborhood of f in $1-C_\varsigma(Y,Z)$ and $2-C_\zeta(Y,Z)$ respectively. In a similar manner, $S(U_2,V_2)$ and $S(A_2,B_2)$ are both open set, neighborhood of g in $1-C_\varsigma(Y,Z)$ and $2-C_\zeta(Y,Z)$ respectively. Now, $S(U_1,V_1)$ and $S(U_2,V_2)$ in $1-C_\varsigma(Y,Z)$ are disjoint open neighborhoods of f and g respectively. Also, $S(A_1,B_1)$ and $S(A_2,B_2)$ in $2-C_\zeta(Y,Z)$ are disjoint open neighbourhood of f and g respectively. Therefore, $1-C_\varsigma(Y,Z)$ and $2-C_\zeta(Y,Z)$ are $b-T_2$ spaces. □

Theorem 4.2. The function space $p-C_\omega(Y,Z)$ is a ${}_p$regular space, if $1-C_\varsigma(Y,Z)$ and $2-C_\zeta(Y,Z)$ are b-regular spaces.

Proof. let f and g be unique functions in $p-C_\omega(Y,Z)$ such that $\forall y \in Y\ f(y) \neq g(y)$, and let $1-C_\varsigma(Y,Z)$ and $2-C_\zeta(Y,Z)$ be b-regular. Then for a closed set $\overline{S(U_1,V_1)}$ in $1-C_\varsigma(Y,Z)$ such that $f \notin \overline{S(U_1,V_1)}$, there exist disjoint open sets $S(A_1,B_1)$ and $S(C_1,D_1)$ such that $f \in S(A_1,B_1)$ and $\overline{S(U_1,V_1)} \subset S(C_1,D_1)$. Similarly, for a closed set $\overline{S(U_2,V_2)}$ in $2-C_\zeta(Y,Z)$ such that $f \notin \overline{S(U_2,V_2)}$, there exist disjoint open sets $S(A_2,B_2)$ and $S(C_2,D_2)$ such that $f \in S(A_2,B_2)$ and $\overline{S(U_2,V_2)} \subset S(C_2,D_2)$. Since f is pairwise continuous, we have that $f \in S((A_1,B_1)(A_2,B_2))$. Now, suppose $g \in \overline{S(U_1,V_1)} \subset S(C_1,D_1)$ and $g \in \overline{S(U_2,V_2)} \subset S(C_2,D_2)$ imply that $g \in \overline{S((U_1,V_1)(U_2,V_2))}$, then $g \in \overline{S((U_1,V_1)(U_2,V_2))} \subset S((C_1,D_1)(C_2,D_2))$. Now $\overline{S((U_1,V_1)(U_2,V_2))}$ is a closed subset of $p-C_\omega(Y,Z)$ not containing f, and $S((C_1,D_1)(C_2,D_2))$ and $S((A_1,B_1)(A_2,B_2))$ are disjoint open sets containing $\overline{S((U_1,V_1)(U_2,V_2))}$ and f respectively. Therefore $p-C_\omega(Y,Z)$ is a ${}_p$regular space. □

5. Conclusion

The function space $p-C_\omega(Y,Z)$ is a ${}_pT_\circ$, ${}_pT_1$, ${}_pT_2$ and ${}_p$regular space, if the topological spaces (Z,δ_1) and (Z,δ_2) are $b-T_\circ$, $b-T_1$, $b-T_2$ and b-regular spaces, and also if the bitopological space (Z,δ_1,δ_2) is $p-T_\circ$, $p-T_1$, $p-T_2$ and p-regular space. The function space $p-C_\omega(Y,Z)$ is also ${}_pT_\circ$, ${}_pT_1$, ${}_pT_2$ and ${}_p$regular, if and only if the function spaces $1-C_\varsigma(Y,Z)$ and $2-C_\zeta(Y,Z)$ are $b-T_\circ$, $b-T_1$ and $b-T_2$, and only if the function spaces $1-C_\varsigma(Y,Z)$ and $2-C_\zeta(Y,Z)$ are b-regular spaces. The set $C(Y,Z)$ can be expressed as a cartesian product $\prod_{y\in Y} Z_y$. Since the product of normal spaces need not be normal, it follows that the space $p-C_\omega(Y,Z)$ need not be normal whenever (Z,δ_1) and (Z,δ_2) are both normal spaces, and also whenever (Z,δ_1,δ_2) is pairwise normal. The results so far obtained can be extended to the space $s-C_\tau(Y,Z)$ and be used to characterize compactness in the space $s-C_\tau(Y,Z)$.

References

[1] R. F. Arens, *A Topology for Spaces of Transformations*, The Annals of Mathematics **47** (1946), no. 2, 480-495.
[2] R. F. Arens and J. Dugundji, *Topologies for function space*, Pacific J. Math. **1** (1951), 5-31.
[3] R. H. Fox, *On topologies for function spaces*, American Mathematical Society **27** (1945), 427-432.
[4] R. Engelking, *General Topology*, Heldermann Verlag, Berlin-West, 1989.
[5] J. C. Kelly, *Bitopological Spaces*, London Math Soc. **13** (1963), 71- 89.
[6] S. Lal, *Pairwise concepts in bitopological spaces*, J. Austral. Math. Soc. Ser. **26** (1978), 241-250.

[7] M. G. Murdeshwar and S. A. Naimpally, *Quasi-uniform Topological Spaces*, Monograph Noordhoof Ltd., 1966.
[8] N. E. Muturi, G.P. Pokhariyal and J. M. Khalagai, *Continuity of functions on function spaces defined on bitopological spaces*, Journal of Advanced Studies in Topology **8** (2017), no. 2, 130-134.
[9] N. E. Muturi, J. M. Khalagai and G. P. Pokhariyal, *Splitting and admissible topologies defined on the set of continuous functions on bitopological spaces*, International Journal of Mathematical Archive **9** (2018), no. 1, 65-68.
[10] W. J. Pervin, *Connectedness in Bitopological Spaces*, Nederl. Akad. Wetensch. Proc. Ser . A70, Indag. Math. **29** (1967), 369-372.
[11] I. L. Reilly, *On bitopological separation properties*, Nanta Math. **5** (1972), no. 2, 14-25.
[12] S. Willard, *General Topology*, Addison-Wesley Publishing Company, U.S.A, 1970.

REFERENCES

[1] Arens R. F. (1946). A Topology for Spaces of Transformations, *the Annals of Mathematics*, (2)47: 480-495.

[2] Arens R. F. and Dugundji J. (1951). Topologies for function space, *Pacific J. Math.*, 1: 5-31.

[3] Bartsch R. (2004). On a nice embedding and the Ascoli-theorem, *New Zealand Journal of Mathematics*, 33(1): 25–39.

[4] Bartsch R. and Poppe H. (2011). Compactness in function spaces with splitting Topologies, *Rostock Math. Kolloq*, 66: 69–73.

[5] Belk J. M. (2015). *Function spaces*, lecture notes, Topology Math 351 Bard College, Spring 2015.

[6] Birkhoff G. (1948). Lattice theory, *Amer. Math. Soc. Colloquium Publications*, vol. 25.

[7] Brown R. (1964). Function space and product topology, *Quart. J. Math. Oxford*, 15: 238-250.

[8] Csaszar A. (2002). Generalized topology, generalized continuity, *Acta Math. Hungar.*, 96: 351 - 357.

[9] Dvalishvili B. P. (2005). *Bitopological Spaces, Theory, Relations with Generalized Algebraic Structures, and Applications*, Elsevier Ltd, San Diego, U.S.A.

[10] Dugundji J. (1978). *Topology (12th edition)*, Allyn and Bacon Inc., United States of America.

[11] Engelking R. (1989). *General Topology*, Heldermann Verlag, Berlin-West.

[12] Fletcher P., Hoyles H. B. and Patty C. W. (1969). The comparison of Topologies, *Duke Math. Journal*, 36: 325-331.

[13] Frechet M. (1906). Sur quelques points du calcul fonctionnel, *Rendiconti del Circolo Matematico di Palermo* 22:1–74.

[14] Fox R. H. (1945). On topologies for function spaces, *American Mathematical Society*, 27:427-432.

[15] Georgiou D. N. (2009). Topologies on Function Spaces and Hyperspaces, *Applied general Topology*, 10: 159-171.

[16] Georgiou D. N., Iliadis S.D. and Mynard F. (2007). *Function Space Topologies. Open Problems in Topology 2*, Elsevier, Amsterdam.

[17] Georgiou D. N., Illadis S. D. and Papadopoulos B. K. (1996). Topologies on function spaces, *Journal of Mathematical Sciences*, 81(2): 2506-2507.

[18] Gupta A. and Sarma R. D. (2015). Function space topologies for generalized topological spaces, *Journal of Advanced Research in Pure Mathematics*, 7(4): 103-112.

[19] Gupta A. and Sarma R. D. (2017). A study of function space topologies for multifunctions, *Appl. Gen. Topol.*, 18(2): 331-344.

[20] Jabbar N. A. and Nasir A. I. (2010). Some Types of Compactness in Bitopological Spaces, *IBN Al-Hiatham J. for Pure and Appl. Sci.*, 23(1).

[21] Jackson J. R. (1951). Spaces of Mapping with application to Homotopy theory, *Proc. Amer. Math. Soc.*, 3: 327-333.

[22] Lal S. (1978). Pairwise concepts in bitopological spaces, *J. Austral. Math. Soc. Ser.*, 26:241-250.

[23] Kelley J. L. (1955). *General Topology*, Springer, New-York.

[24] Kelly J. C. (1963). Bitopological Spaces, *London Math Soc.*, 13: 71- 89.

[25] Kim Y. W. (1968). Pairwise compactness, *Publ. Math*, 15: 87-90.

[26] McCoy R. A. and Ntantu I. (1988). *Topological Properties of Spaces of Continuous Functions*, Springer-Verlag Berlin Heidelberg, Germany.

[27] Morita K. (1956). Note on Mapping Spaces, *Proc. Japan Acad.*, 32(9): 671-675.

[28] Mrsevic M. and Reilly I. L. (1996). Covering and Connectedness Properties of a Topological Space and it is Associated Topology of α-subsets, *Indian J. pure apple. Math.*, 27(10): 995-1004.

[29] Munkres J. R. (2000). *Topology (2nd edition)*, Prentice Hall, United States of America.

[30] Muturi N. E., Pokhariyal G.P. and Khalagai J. M. (2017). Continuity of functions on function spaces defined on bitopological spaces, *Journal of Advanced Studies in Topology*, 8(2): 130–134.

[31] Muturi N. E., Khalagai J. M. and Pokhariyal G. P. (2018(a)). Splitting and admissible topologies defined on the set of continuous functions on bitopological spaces, *International Journal of Mathematical Archive*, 9(1): 65-68.

[32] Muturi N. E., Pokhariyal G.P. and Khalagai J. M. (2018(b)). Separation axioms on function spaces defined on bitopological spaces, *Journal of Advanced Studies in Topology*, 9(2): 113–118.

[33] Pervin W. J. (1970). *Foundations of General Topology*. Acad. Press. New-York.

[34] Pervin W. J. (1967). Connectedness in bitopological spaces, *Nederl. Akad. Wetensch. Proc. Ser. A70, Indag. Math.* 29: 369-372.

[35] Porter K. F. (1993). The Open-Open Topology for Function Spaces, *International Journal Of Mathematics And Science*, 1(16): 111-116.

[36] Reilly I. L. (1972). On bitopological separation properties, *Nanta Math.*, 5(2): 14-25.

[37] Sizemore A. E., Giusti C., Kahn A. Betzel R. F. and Bassett D. S. (2018). Cliques and Cavities in the Human Connectome, *J. Comput Neurosci*, 44: 115-145.

[38] Swart J. (1971). Total disconnectedness in bitopological spaces and product bitopological spaces, *Indag. Math.*, 33:135-145.

[39] Tallafha A. Al-Bsoul A. and Fora A. (1999). Countable Dense Homogenous Bitopological spaces. *Tr. J. of Mathematics*, 23: 233-244.

[40] Willard S. (1970). *General Topology*, Addison-Wesley Publishing Company, United States of America.