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Master Project in Mathematical Statistics

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Research Report in Mathematics, Number 27, 2018

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November 2018

Submitted to the School of Mathematics in partial fulfilment for a degree in Master of Science in Mathematical Statistics

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Master of Science Project

Submitted to the School of Mathematics in partial fulfilment for a degree in Master of Science in Mathematical Statistics

Prepared for The Director
Graduate School
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ABSTRACT

The goal of this project is to construct the Generalized Inverse gaussian distribution under different parameterizations; using the special function called the modified Bessel function of the third kind. This is the only special function that has been used all through in this research. Under Modified Bessel function of the third kind, various definitions, properties and alternative forms have been studied. With the parameters of the Generalized inverse gaussian distribution reguating both the concetration and scaling of densities, other parameters are introduced leading to GIG distributions but of different forms. These parameterizations are Sichel, Jorgensen's, Willmot's, Barndorff-Nielsen and Allenis Paramaterrizations.

Special cases under each parameterizations has led to various distributons which have been treated as sub-models of the GIG distributions. These distributions are inverse gaussian, receiprocal inverse gaussian, gamma, Inverse gaussian, exponential, positive hyperbolic and Levy distributions. Their statistical properties such as the r th moment, the Laplace transform and modality of the special submodels have also been studied. Depending on the sign of v, it has been established that the Generalized inverse gaussian distribution is viewed as either the first or the last hitting times for a certain diffusion process where the Inverse and the reciprocal inverse gaussian distributions were among the sub-models of the Generalized inverse gaussian distributions.

It has been established that the Generalized inverse gaussian distributions is seen to belong to the family of generalized gamma convolution. By introducing other parameters, we have seen that the resultant distribution has four parameters. This is the Power Generalized Inverse gaussian distribution. We have also established that the inverse of a Generalized inverse gaussian distribution is a special case of the power Generalized inverse gaussian distribution where the power is one.

Under Sichel and Barndorff-Nielsen parameterizations, convolution properties have been proved using the Laplace technique. It has been shown that multiplying the two laplaces of the specific special cases, gives back the laplace distribution with the GIG distribution parameters. Under Sichel parameterization, we have come up with the Sichel distribution wich is as a resut of mixing the Poisson and the Generalized inverse gaussian distribution. The Sichel distribution has been expressed recussively, then arriving at its properties. Moreover, it has been established that special cases under Allen's and Willmot's parameterizations has led to the same statical properties as the other cases established earlier.

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DECLARATION AND APPROVAL

I the undersigned declare that this project report is my original work and to the best of my knowledge, it has not been submitted in support of an award of a degree in any other university or institution of learning.

Signature Date

KEVIN ODHIAMBO NYAWADE
Reg No. I56/87382/2016

In my capacity as a supervisor of the candidate, I certify that this report has my approval for submission.

Signature Date

Prof. J.A.M Ottieno
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DEDICATION

To my parents, brothers and sisters for their steadfast support throughout my studies.

ABBREVIATIONS AND NOTATIONS

Abbreviations and notations of various chapetes are in those specific chapters. Generally, abbreviations and notations used are given as below.

$K_v(\omega)$- Modofied Bessel function of the third kind
$E(X^r)$- The r th moment of a distribution
$L_X(s)$- The laplace transform of distribution
$f(x)$ - The probality mass function of a Generalized inverse gaussian distribution
pdf - probablity density function
cdf - cummulative distribution function
L_{GIG}- The Laplace of the Generalized inverse gaussian distribution.
$g(\lambda)$- mixing distribution.
$f(x|\lambda)$-The conditional properly.
pgf - Probability Generating Function.

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Kevin Odhiambo Nyawade

Nairobi, 2018.

1 GENERAL INTRODUCTION

1.1 Background Information

Probablity distributions are key in statistics. A probability distribution is a mathematical function which provides the probability of occurences which have different possible outcomes in any experiment. In other words, its a detailed description of any random phenomenon in terms of the probabilities of various events. For this case, an experiment or a survey is a good example of a random phenomenon We have two major subdivisions of Probability distributions. **Discrete probability distribution:** This is only applcable under scenarios when set of possible outcomes are discrete, example is tossing a coin. It is mostly encoded by a list of the probabilites which are descrete known as **probability mass function. On the other hand, Continous probability distriution** is where possible scenarios can only take set that are contionous range. Example is real numbers. This type of distribution is described by probability distribution functions. The key areas of any distribution are the constructions using various method, estimation of various parameters depending on the type of distribution and their applications.

Some methods of constructing various distributions include: Power series, transformation, special functions, generator approach, Langararian expansion, Geomerty and trigonometry, mixtures, recussive relations in probablities, differential equetions, stochastic process, hazard functions and sum of independent random ariables.

In this work, we used modified Bessel function of the third kind which is one of the special functions to construct the Generalized inverse gaussian distribution. Moreover, we used the change of variable technique and the cummulative distribution technique to come up with other forms of the GIG distributions.

Historically, the first appearence of the GIG distribution was discovered by Halphen, E (1941). This is the reason as to why it is also known as Halphen type A distribution. During this time, there was a study of extreme hydrological events. Halphen family of distributions was used to model the maximum annual flood series. The following were taken into account; the dseign of hydrolaulic structures, forecasting of flood involving estimation of flood flows achieved by fitting a probability distribution to observed data, with specified return period and optimal operation of reservoirs. In his study of population frequencies, Good(1953) proposed the GIG distribution. Population frequencies of species and estimation were studied in the Biometrica. A random sample drawn from a population of species of animals and if a population is represented r times in the sample of the population size N, then r/N is not a good estimate of the entire population of frequency f, when r is small.

This work consideres the five parameters which are used to come up with the GIG distribution by using the modified Bessel functon of the third kind. Using the change of variable technique and the cummulative distribution technique, the resultantant new distribution is called the Power-GIG distribution which has an extra parameter compaired to the GIG distribution. The inverse of a GIG distribution is a special case of the Power-GIG distribution. Some unique propositions of the Laplaces of various GIG distributions has been proved. Taking various Laplaces of the GIG distributions, their product results to the same Laplaces of the GIG distribuutions with the same paramters.

1.2 Definitions and Terminologies

A random variable $(\bullet)$ with function $f(\bullet)$ has the following conditions

$$f(\bullet) \geq 0 \quad and \quad \int_{-\infty}^{\infty} f(\bullet) \quad d(\bullet) = 1$$

Where $f(\bullet)$ is the pdf in a continous form
In a descrete form, we define;

$$\sum_{-\infty}^{\infty} f(\bullet) = 1 \quad and \quad f(\bullet) \geq 0$$

$f(\bullet)$ is the pmf

Generalized Distribution

A distribution is said to be generalized if it nests various distributions. To generalize a distribution, means adding a another parameter to an existing parameters of a given distribution. This is done by formulating general concepts from specific instances by basing on common statistical properties of a given pdf. GIG is therefore advategious over other distributions since it nests various distributions.

Parameterization

This is a mathematical process that consist of expressing a model, system or process as a function of some specific independent quantities which are called **Parameters.** By definition,

$$2K_{\nu}(\omega) = \int_{0}^{\infty} y^{\nu-1} \exp -\frac{\omega}{2}(y + \frac{1}{y}) \quad dy \qquad \mathbf{1.0}$$

Due to the various parameterizations; the Sichel's, Jorgensen's, Willmot's, Barndorff-Nielsen and Allen's parameterizations, we have the Generalized inverse gaussian distributions. This has been done by letting ω eqaul to various parameters, we substitute ω with the following,

1. $\omega = \sqrt{\psi\phi}$ this is the Sichel's parameterization
2. $\omega = \frac{\mu}{\beta}$, where $\phi = \mu^2$ and $\psi = \frac{1}{\beta^2}$ Willmot's parameterization
3. $\omega = \rho\sigma$, where $\psi = \rho^2$ and $\phi = \sigma^2$ Barndorff-Nielsen parameterization
4. $\omega = 2\sigma\theta^{\frac{1}{2}}$, where $\psi = 2\theta$ and $\phi = 2\sigma^2$ Allen's parameterization
5. $\omega = \sqrt{\psi\phi}$ and $\eta = \sqrt{\frac{\phi}{\psi}}$ implying $\phi = \omega\eta$ and $\psi = \frac{\omega}{\eta}$this is the Jorgensen's parameterization

All the above parameterizations are as a result of Sichel's Parameterization since the original parametrs ψ and ϕ have been equeted to other parameters.

1.3 Statement of the Problem

In construction of the Generalized Inverse gaussian distribution, we choose on the modified Bessel function of the third kind which is a very significant special function used in the construction. Choosing on other special functions or the use of mixture method in the costruction has not yet proved to hold waters in the GIG distribution construction, hence giving room for us to explore further the use of modified Bessel function of the third kind as a special function. Generalized Inverse gaussian distribution which is as a result of various parameterizations, hence by use of change of variable technique, we obtain it. GIGD has various sub-models due to special and limiting cases. In this case, by letting the parameters to be equal to certain consistent values, we obtain the sub-models which are inverse gaussian, reciprocal inverse gaussian, gamma, inverse gamma, exponential, positive hyperbolic and the Levy distribution. GIGD and its submodels are quite flexible to work with despite their complexity nature.
The problem is to work with all the five parameterizations; Sichel's, Barndorff-Nielsen, Allen's, Willmot's and Jorgensen's parameterizations. The aim being to construct the GIGD based on the all the above parameterizations and study their statistical propeties (r-th moment, Laplace transform and modality). Due to the special and limiting cases, we various sub-models and their statical properties mentioned above.

1.4 Objectives

1.4.1 General Objectives

The main objective is to construct the Generalized Inverse Gaussian distributions, under different parameterizations.

1.4.2 Specific Objectives

1. To explore modified Bessel function of the third kind.
2. To construct Generalized Inverse Gaussian distribution under different parameterizations.
3. To study of the Generalized Inverse Gaussian distribution properties under different parameterizations.

1.5 Literature Review

Generalized Inverse Gaussian distributions can be constructed using the special function; modified Bessel function of the third kind. Several ways of extending GIG distribution can be used to extend the GIG distribution. These are the change of variable technique and Cummulative distribution technique. Taking some sub models of the GIG and the GIG distributions in different forms give rise to the GIG distibution.

The first appearence of GIG distribution was discovered by Halphen, E(1941). This is the reason behind the GIG distribution called Halphen type A distribution. Good (1953), proposed the GIG distribution during his study of population frequences. As seen earlier, GIG distribution nests the modified Bessel function of the third kind, whereby the parameters regulate both the concetration and scaling of densities.

Sichel (1974) used the GIG distribution to construct the mixture of poisson distribution hence calling it the Sichel distribution. Barndorff-Nielsen and Halgreen (1977) studied the convolution properties. They studied the GIG distributions infinite divisibility and the equvalence properties. The result was that the inverse Gaussian and receiprocal Inverse gaussian are respectively the first and the last hitting times for a Brownian mortion.

Halgreen(1979) found out that the GIG distribution belong to the family of generalized gamma convolution. Jorgensen(1982), completed the study of Halphen distribution. They introduced the two parameters leading to a GIG density of another form. Due to the restrictions of the parameters, the GIG distribution leads to the Inverse Gaussian, Receiprocal

Inverse Gaussian, gamma and the invese gamma distributions. Eberline and Hmmerstein (2004) showed the detailed proof of the GIG belonging to the family of generalized gamma convolution. Eberline and Hammerstein (2004) reviwed the fact that GIG distribution belong to the family of generalized gamma convolution. Madan, Roynette and Yor (2008) have shown that the Black-Scholes formular in finance can be expressed interms of the distribution function of the GIG distribution function of GIG variables.

Lemonte and Cardeiro (2010) proposed the Exponentiated GIG distribution. The exponentiated standard gamma distribution was extended then structural proprties of the resultant distribution by expanding its moments were studied, mgf and the order statistics. They further discussed the MLE of the model parameters and importance of the new model showed by use of real data.

Devorge (2012) showed random variate generation for the GIG distribution. Uniformly efficient and simple random variate generator for every parameter range of the GIG was provided. In this work, general algorithm and that which works for all densities that are propotional to a long concave function even if the normalization constant is not known has been provided.

Kondou and Ley (2014) studied characterization of the GIG laws. They reviewed several characterization theorems of the GIG on the positive real line. These characterization theorems have been surveyed then two new characterizations based on the Maximum Likelihood estimation and Stein method established.

Hasebe and Szpojankowski (2017) studied properties of free GIG. They showed that, free GIG distributions have same properties with the classical one. They provided that free GIG distributions is unimodal, divisible and free regular freely infinitely . The distributions in this class that are freely self decomposable were determined.

1.6 Methods

The methods used in the construction are:

1. Special functions. Modified Bessel function of the third kind is the only type of special function that has been used.
2. Direct intergration and substitution.
3. Cummulative distribution and change of variable techniques.

1.7 Significance of the Study

Generalized Inverse Gaussian distributions has been applied in many fields i.e used to model a lifetime data. **In actual Data.** It has been used in diverse real phenomenon such as waiting time. Jorgensen(1982) applied the GIG distribution in modelling of waiting time. **In Neural activity.** Iyengar and Liao (1977), used Generlaized Inverse Gaussian distribution in modelling the neural activity. **Extreme hydrological events.** Cheban et al(2010), applied GIG distribution in his research work of of mixed estimation methods for Halphen distributions with application in extreme hydrologic events. **GIG as a mixing distribution:** the distribution can be used as a mixing distribution. Using it as a mixing distribution can be helpful since taking other statistical distribution, we can get back to a distribution that can be expressed interms of a Bessel function or puerly new distribution or purely new distribution. Therefore, in this study, Modified Bessel function of the third kind has been used to obtain the GIG distribution under different parameterizations. It is one of the most crucial special type of distribution in constructiong the GIG distribution. This study has also made use of the generalized distributions nesting other distributions. Generalized distributions include, GIG distribution, 3 parameter generalized lindly and transmited exponential. Though, the main focuss has been the GIG distributions mostly. The main reason for working with GIG distribution throughout is because of its flexiblity to work with it. Various identities have been deduced as a result of various paramaterizations and also under generalization of GIG.

1.8 Outline of the Project

The rest of the project is outlined as follows. Chapter 1 introduces the topic. It has executive summary of the work. In Chapter 2, Modified Bessel function of the third kind, its properties and various theorems have been deeply investigated. The relationship of Modified Bessel function of the third kind and the first kind also studied.

In chapter 3, Generalized inverse gaussian distribution based on the Sichel's parameterization has been constructed using the modified Bessel function of the third kind as the only special function used in the construction. Sichel parameterization has led to various sub-models of the Generalized inverse gaussian distribution. Under this section, we have derived some statisical properties of the special cases of the Generalized inverse Gaussian distribution based on this parameterization. Later in this chapter, we have the convolution properties which have been proved using the Laplace technique. This has given rise to the Sichel Distribution which is as a result of miture of poisson and the GIG distributions. We have expressed the Sichel distribution in a recussive form.

In chapter 4, we have generalized the GIG based on the Barndorff Nielsen and Allen's parameterizations. Under each paramaterization, we have the proved the propositions based on the Laplace transform. We have derived various special cases wich are the sub-models of the GIG distributions and their properties. We constructed the GIG distribution based on the Willmot's and Jorgensen's parameterizations. Special cases under the two parameterizations have been covered with their properties. These are the last parameterizations that have been covered in detailed.

In chapter 5, we have conluded by giving the general conclusion and the recomendation. This chapter outlines in summary what we have not covered but is relevant as far as this topic is concerned. In otherwords, we have given the possible areas of research in Generalized Inverse gaussian distribution.

2 MODIFIED BESSEL FUNCTION OF THE THIRD KIND: DEFINITIONS AND PROPERTIES

2.1 Introduction

In this chapter, we have presented various equivalent definitions of modified Bessel function of the third kind, properties and alternative forms. Some of the properties in a recussive relation and summation forms have been studied. There is a section on the special cases of the modified Bessel function of the third kind which has been done by varying or assigning the parameters various values then studying various relationships. We have concluded this chapter by expressing the modified Bessel function of the third kind interms of the first kind. Some relations have been established.

2.2 Definition and Its Properties

A modified Bessel function of the third kind with index v and of order ω denoted by $K_v\omega$:

$$2K_v(\omega) = \int_0^\infty x^{v-1} \exp -\frac{\omega}{2}(x+\frac{1}{x}) \quad dx \tag{2.1}$$

Property 2.1 (Symmetry)

$$K_v(\omega) = K_{-v}(\omega) \tag{2.2}$$

Proof:

Let

$$x = \frac{1}{y} \Rightarrow dx = -\frac{dy}{y^2}$$

$$\begin{aligned}
\therefore K_v(\omega) &= \frac{1}{2}\int_\infty^0 \left(\frac{1}{y}\right)^{v-1} e^{-\frac{\omega}{2}(\frac{1}{y}+y)} \left(-\frac{dy}{y^2}\right) \\
&= \frac{1}{2}\int_0^\infty \frac{1}{y^{v+1}} e^{-\frac{\omega}{2}(y+\frac{1}{y})} \quad dy \\
&= \frac{1}{2}\int_0^\infty y^{-v-1} e^{-\frac{\omega}{2}(y+\frac{1}{y})} \quad dy \\
&= K_{-v}(\omega)
\end{aligned}$$

Property 2.2 (Derivative 1)

$$\frac{\partial}{\partial w}K(\omega) = K_v'(\omega) = -\frac{1}{2}[K_{v+1}(\omega) + K_{v-1}(\omega)] \tag{2.3}$$

Proof

$$\begin{aligned}
\frac{d}{d\omega}K_v(\omega) &= \frac{1}{2}\frac{d}{d\omega}\int_0^\infty x^{v-1}\exp -\frac{\omega}{2}(y+\frac{1}{x}) \quad dx \\
&= \frac{1}{2}\int_0^\infty x^{v-1}[-\frac{1}{2}(x+\frac{1}{x})]\exp -\frac{\omega}{2}\left(x+\frac{1}{x}\right) \quad dx \\
&= \frac{1}{2}(-\frac{1}{2})\{\int_0^\infty x^{(v+1)-1}\exp -\frac{\omega}{2}(x+\frac{1}{x})dx + \int_0^\infty x^{(v-1)-1}\exp -\frac{\omega}{2}(x+\frac{1}{x}) \quad dx\} \\
&= -\frac{1}{2}\{\frac{1}{2}\int_0^\infty x^{(v+1)-1}\exp -\frac{\omega}{2}(x+\frac{1}{2}) \quad dx + \int_0^\infty x^{(v-1)-1}\exp -\frac{\omega}{2}(x+\frac{1}{x}) \quad dx\}
\end{aligned}$$

$\therefore$

$$K_v'(\omega) = -\frac{1}{2}[K_{v+1}(\omega) + K_{v-1}(\omega)]$$

An alternative form of definition 1 is given in the following

Proposition (2.1)

$$K_v(\omega) = \frac{1}{2}\left(\frac{\omega}{2}\right)^v \int_0^\infty t^{-v-1}\exp\{-t - \frac{\omega^2}{4t}\} \quad dt \tag{2.4}$$

Proof

Let

$$x = \frac{\omega}{2t} \Rightarrow dx = -\frac{\omega}{2t^2} \quad dt$$

$$\begin{aligned}
K_v(\omega) &= \frac{1}{2}\int_\infty^0 (\frac{\omega}{2t})^{v-1}\exp -\frac{\omega}{2}(\frac{\omega}{2t} + \frac{2t}{\omega})(-\frac{\omega}{2t^2}) \quad dt \\
&= \frac{1}{2}(\frac{\omega}{2})^v \int_0^\infty \frac{1}{t^{v+1}}\exp\{-\frac{\omega}{2}(\frac{\omega}{2t} + \frac{2t}{\omega})\} \quad dt \\
&= \frac{1}{2}(\frac{\omega}{2})^v \int_0^\infty \frac{1}{t^{v+1}}\exp -(\frac{\omega^2}{4t} + t) \quad dt \\
&= \frac{1}{2}(\frac{\omega}{2})^v \int_0^\infty t^{-v-1}\exp\{-t - \frac{\omega^2}{4t}\} \quad dt
\end{aligned}$$

Alternatively,

$$K_v(\omega) = \frac{(\frac{\omega}{2})^v}{2}\int_0^\infty t^{-v-1}\exp -t - \frac{\omega^2}{4t} \quad dt$$

Let

$$x = \frac{2t}{\omega} \Rightarrow dx = \frac{2}{\omega} \quad dt$$

$$\begin{aligned}
\therefore K_{-v}(\omega) &= \frac{1}{2}\int_0^\infty x^{-v-1}\exp -\frac{\omega}{2}(x+\frac{1}{x}) \quad dx \\
&= \frac{1}{2}\int_0^\infty (\frac{2t}{\omega})^{-v-1}\frac{2}{\omega}\exp -\frac{\omega}{2}(\frac{2t}{\omega}+\frac{\omega}{2t}) \quad dt \\
&= \frac{1}{2}\int_0^\infty \frac{2^{v-1}}{\omega^{v-1}}t^{-v-1}\exp\left[-t-\frac{\omega^2}{4t}\right] \quad dt \\
&= \frac{1}{2}\left(\frac{\omega}{2}\right)^v\int_0^\infty t^{-v-1}\exp\left[-(t+\frac{\omega^2}{4t})\right] \quad dt
\end{aligned}$$

Property 2.3 (Derivative II)

$$\frac{\partial}{\partial\omega}K_v(\omega) = \frac{v}{\omega}K_v(\omega) - K_{v+1}(\omega) \tag{2.5}$$

Proof

Differentiating equation (2.4),

$$\begin{aligned}
\frac{\partial}{\partial\omega}K_v(\omega) &= \frac{1}{2}\frac{\partial}{\partial\omega}\left(\frac{\omega}{2}\right)^v\int_0^\infty t^{-v-1}\exp\left[-t-\frac{\omega^2}{4t}\right] \quad dt \\
&= \frac{1}{2}\left(\frac{v}{2}(\frac{\omega}{2})^{v-1}\int_0^\infty t^{-v-1}\exp(-t-\frac{\omega^2}{4t}) \quad dt + (\frac{\omega}{2})^v\frac{\partial}{\partial\omega}\int_0^\infty t^{-v-1}\exp(-t-\frac{\omega^2}{4t}) \quad dt\right) \\
&= \frac{1}{2}\left(\frac{1}{2}\left(\frac{\omega}{2}\right)^{-1}\left(\frac{\omega}{2}\right)^v\int_0^\infty t^{-v-1}\exp\left[-t-\frac{\omega^2}{4t}\right] \quad dt + \left(\frac{\omega}{2}\right)^v\int_0^\infty t^{-v-1}\left(-\frac{\omega}{2t}\right)\exp\left(-t-\frac{\omega^2}{4t}\right)dt\right) \\
&= \frac{v}{\omega}K_v(\omega) - \left(\frac{\omega}{2}^{v+1}\int_0^\infty t^{-(v+1)-1}\exp\left(-t-\frac{\omega^2}{4t}\right) \quad dt\right) \\
&= \frac{v}{\omega}K_v(\omega) - K_{v+1}(\omega)
\end{aligned}$$

By letting v=0 we have

Corollary 2.3.1

$$\frac{\partial}{\partial\omega}K_0(\omega) = -K_1(\omega) \tag{2.6}$$

Property 2.4 (Recursive Relation)

$$\frac{2v}{\omega}K_v(\omega) = K_{v+1}(\omega) - K_{v-1}(\omega) \tag{2.7}$$

Proof

Equating (2.3) and (2.5): i.e properties 2.2 and 2.3, we have

$$\begin{aligned} -\frac{1}{2}\left[K_{v+1}(\omega)+K_{v-1}(\omega)\right] &= \frac{v}{\omega}K_v(\omega) - K_{v+1}(\omega) \\ \therefore K_{v+1}(\omega)+K_{v-1}(\omega) &= -\frac{2v}{\omega}K_v(\omega)+2K_{v+1}(\omega) \\ \therefore K_{v+1}(\omega) &= \frac{2v}{\omega}K_v + K_{v-1}(\omega) \end{aligned}$$

Alternatively, consider definition 1, i.e,

$$K_v(\omega) = \frac{1}{2}\int_0^\infty x^{v-1}\exp\left[-\frac{\omega}{2}(x+\frac{1}{x})\right] \quad dx$$

Using intergration by parts, let

$$u = \exp\left[-\frac{\omega}{2}(x+\frac{1}{x})\right] \Rightarrow du = -\frac{\omega}{2}(1-\frac{1}{x})\exp\left[-\frac{\omega}{2}(x+\frac{1}{x})\right] \quad dx$$

and

$$dv = x^{v-1} \quad dx \Rightarrow v = \frac{x^v}{\alpha}$$

$$\begin{aligned} K_v(\omega) &= \frac{x^v}{2v}\exp\left[-\frac{\omega}{2}(x+\frac{1}{x})\right]\Bigg|_0^\infty + \frac{1}{2}\int_0^\infty x^v\frac{\omega}{2}\left(1-\frac{1}{x^2}\right)e^{-\frac{\omega}{2}(x+\frac{1}{x})} \quad dx \\ &= 0+\frac{1}{2}\frac{\omega}{2v}\int_0^\infty \left(x^v - x^{v-2}\right)e^{-\frac{\omega}{2}(x+\frac{1}{x})} \quad dx \\ &= \frac{\omega}{2v}\frac{1}{2}\int_0^\infty x^v e^{-\frac{\omega}{2}(x+\frac{1}{x})} \quad dx - \frac{\omega}{2v}\frac{1}{2}\int_0^\infty x^{v-2}e^{-\frac{\omega}{2}(x+\frac{1}{x})} \quad dx \\ \therefore K_v(\omega) &= \frac{\omega}{2v}K_{v+1}(\omega) - \frac{\omega}{2v}K_{v-1} \\ \therefore 2vK_v(\omega) &= \omega K_{v+1}(\omega) - \omega K_{v-1}(\omega) \\ \therefore K_{v+1}(\omega) &= \frac{2v}{\omega}K_v(\omega) + K_{v-1}(\omega) \end{aligned}$$

Corollary 2.4.1

$$K_{\frac{3}{2}}(\omega) = \left(1+\frac{1}{\omega}\right)K_{\frac{1}{2}}(\omega) \tag{2.8}$$

and

$$K_{\frac{5}{2}}(\omega) = \left(1+\frac{3}{\omega}+\frac{3}{\omega^2}\right)K_{\frac{1}{2}}(\omega) \tag{2.9}$$

Proof

Put $v = \frac{1}{2}$ in (2.7)

$$\begin{aligned} K_{\frac{3}{2}}(\omega) &= \frac{1}{\omega} K_{\frac{1}{2}} + K_{-\frac{1}{2}}(\omega) \\ &= \frac{1}{\omega} K_{\frac{1}{2}}(\omega) + K_{\frac{1}{2}}(\omega) \\ &= \left(1 + \frac{1}{\omega}\right) K_{\frac{1}{2}}(\omega) \end{aligned}$$

Also if

$$v = -\frac{1}{2} \quad in \quad (2.7), \quad we \quad obtain,$$

$$\begin{aligned} K_{\frac{1}{2}}(\omega) &= -\frac{1}{\omega} K_{-\frac{1}{2}}(\omega) + K_{-\frac{3}{2}}(\omega) \\ (1 + \frac{1}{\omega}) K_{\frac{1}{2}}(\omega) &= K_{\frac{3}{2}}(\omega) \quad as \quad in \quad (2.8) \end{aligned}$$

Next put $v = \frac{3}{2}$ in (2.7)

$$\begin{aligned} K_{\frac{5}{2}}(\omega) &= \frac{3}{\omega} K_{\frac{3}{2}}(\omega) + K_{\frac{1}{2}}(\omega) \\ &= \frac{3}{\omega}\left(1 + \frac{1}{\omega}\right) K_{\frac{1}{2}}(\omega) + K_{\frac{1}{2}}(\omega) + K_{\frac{1}{2}}(\omega) \\ &= \left[\frac{3}{\omega}(1 + \frac{1}{\omega}) + 1\right] K_{\frac{1}{2}}(\omega) \\ &= \left(1 + \frac{3}{\omega} + \frac{3}{\omega^2}\right) K_{\frac{1}{2}}(\omega) \end{aligned}$$

Property 2.5 (Derivative of log)

$$\frac{\partial}{\partial \omega} \log K_v(\omega) = \frac{v}{\omega} - \frac{K_{v+1}(\omega)}{K_v(\omega)} \tag{2.10}$$

Proof

$$\begin{aligned} \frac{\partial}{\partial \omega} \log K_v(\omega) &= \frac{1}{K_v(\omega)} \frac{\partial}{\partial \omega} K_v(\omega) \\ &= \frac{1}{K_v(\omega)} \left[\frac{v}{\omega} K_v(\omega) - K_{v+1}(\omega)\right] \quad by \quad property \quad 2.3 \\ \therefore \frac{\partial}{\partial \omega} \log K_v(\omega) &= \frac{v}{\omega} - \frac{K_{v+1}(\omega)}{K_v(\omega)} \quad by \quad property \quad (2.2) \end{aligned}$$

$$
\begin{aligned}
\frac{\partial}{\partial\omega}\log K_v(\omega) &= \frac{1}{K_v(\omega)}\frac{\partial}{\partial\omega}K_v(\omega)\\
&= \frac{1}{K_v(\omega)}\left(-\frac{1}{2}\left[K_{v+1}(\omega)+K_{v-1}(\omega)\right]\right)\\
\therefore \frac{\partial}{\partial\omega}\log K_v(\omega) &= \frac{1}{K_v(\omega)}\left(-\frac{1}{2}\left[2K_{v+1}(\omega)-\frac{2v}{\omega}K_v(\omega)\right]\right)\\
&= \frac{1}{K_v(\omega)}\left(-K_{v+1}(\omega)+\frac{v}{\omega}K_v(\omega)\right)\\
&= \frac{v}{\omega}-\frac{K_{v+1}(\omega)}{K_v(\omega)}
\end{aligned}
$$

By property (2.4)

$$K_{v+1}(\omega) = \frac{2v}{\omega}K_v(\omega)+K_{v-1}(\omega)$$

$$K_{v-1}(\omega) = K_{v+1}(\omega)-\frac{2v}{\omega}K_v(\omega)$$

The modified Bessel function of the third kind can be expressed interms of the hyperbolic functions as

Proposition (2.2)

$$K_v(\omega) = \frac{1}{2}\int_{-\infty}^{\infty} e^{-\omega\cosh t}\cosh vt \quad dt \tag{2.11}$$

Proof

By property 2.1

$K_v(\omega) = K_{-v}(\omega)$

Adding $K_v(\omega)$ on both sides we have,

$$
\begin{aligned}
K_v(\omega) &= K_{-v}(\omega)\\
2K_v(\omega) &= K_v(\omega)+K_{-v}(\omega)\\
&= \int_0^{\infty} x^{v-1}e^{-\frac{\omega}{2}(x+\frac{1}{x})} \quad dx+\frac{1}{2}\int_0^{\infty} x^{-v-1}e^{-\frac{\omega}{2}(x+\frac{1}{x})} \quad dx\\
&= \frac{1}{2}\int_0^{\infty}\left(x^{v-1}+x^{-v-1}\right)e^{-\frac{\omega}{2}(x+\frac{1}{x})} \quad dx
\end{aligned}
$$

Let

$$x = e^t \Rightarrow dx = e^t dt$$

$$
\begin{aligned}
\therefore 2K_v(\omega) &= \frac{1}{2}\left[\int_{-\infty}^{\infty}\left(e^{t(v-1)}+e^{t(-v-1)}\right)e^{-\frac{\omega}{2}(e^t+e^{-t})}e^t \quad dt\right] \\
&= \frac{1}{2}\left[\int_{-\infty}^{\infty}\left(e^{tv}+e^{-tv}\right)e^{-\frac{\omega}{2}(e^t+e^{-t})} \quad dt\right] \\
&= \int_{-\infty}^{\infty}\left(\frac{e^{vt}+e^{-vt}}{2}\right)e^{-\omega(\frac{e^t+e^{-t}}{2})} \quad dt \\
&= \frac{1}{2}\int_{-\infty}^{\infty}\cosh vte^{\omega\cosh t} \quad dt
\end{aligned}
$$

Where

$$\cosh x = \frac{e^x+e^{-x}}{2}$$

2.3 Definition 2 and its Properties

$$K_v(\omega) = \left(\frac{\omega}{2}\right)^v \frac{\Gamma(\frac{1}{2})}{\Gamma(v+\frac{1}{2})}\int_1^{\infty}(t^2-1)^{v-\frac{1}{2}}e^{-\omega t} \quad dt \tag{2.12}$$

which satisfies the differential equation

$$\omega^2\frac{d^2y}{d\omega^2}+\omega\frac{dy}{d\omega}-(\omega^2+v^2)y=0$$

In terms of hyperbolic functions we have

Proposition (2.3)

$$K_v(\omega) = \left(\frac{\omega}{2}\right)^v \frac{\Gamma(\frac{1}{2})}{\Gamma(v+\frac{1}{2})}\int_0^{\infty}(\sinh\theta)^{2v}e^{-\omega\cosh\theta} \quad d\theta \tag{2.13}$$

Proof

Let

$$t=\cosh\theta \Rightarrow dt=\sinh\theta \quad d\theta$$

$$
\begin{aligned}
\therefore K_v(\omega) &= \left(\frac{\omega}{2}\right)^v \frac{\Gamma(\frac{1}{2})}{\Gamma(v+\frac{1}{2})}\int_0^\infty \left(\cosh^2\theta - 1\right)^{v-\frac{1}{2}} e^{-\omega\cosh\theta}\sinh\theta \quad d\theta \\
&= \left(\frac{\omega}{2}\right)^v \frac{\Gamma(\frac{1}{2})}{\Gamma(v+\frac{1}{2})}\int_0^\infty \left[\left(\frac{e^\theta+e^{-\theta}}{2}\right)^2 - 1\right]^{v-\frac{1}{2}} e^{-\omega\cosh\theta}\sinh\theta \quad d\theta \\
&= \left(\frac{\omega}{2}\right)^v \frac{\Gamma(\frac{1}{2})}{\Gamma(v+\frac{1}{2})}\int_0^\infty \left(\frac{e^{2\theta}+2+e^{-2\theta}-4}{4}\right)^{v-\frac{1}{2}} e^{-\omega\cosh\theta}\sinh\theta \quad d\theta \\
&= \left(\frac{\omega}{2}\right)^v \frac{\Gamma(\frac{1}{2})}{\Gamma(v+\frac{1}{2})}\int_0^\infty \left(\frac{e^{2\theta}-2+e^{-2\theta}}{4}\right)^{v-\frac{1}{2}} e^{-\omega\cosh\theta}\sinh\theta \quad d\theta \\
&= \left(\frac{\omega}{2}\right)^v \frac{\Gamma(\frac{1}{2})}{\Gamma(v+\frac{1}{2})}\int_0^\infty \left(\frac{e^{\theta}-e^{-\theta}}{2}\right)^{2(v-\frac{1}{2})} e^{-\omega\cosh\theta}\sinh\theta \quad d\theta \\
&= \left(\frac{\omega}{2}\right)^v \frac{\Gamma(\frac{1}{2})}{\Gamma(v+\frac{1}{2})}\int_0^\infty (\sinh\theta)^{2v-1} e^{-\omega\cosh\theta}\sinh\theta \quad d\theta \\
&= \left(\frac{\omega}{2}\right)^v \frac{\Gamma(\frac{1}{2})}{\Gamma(v+\frac{1}{2})}\int_0^\infty (\sinh\theta)^{2v} e^{-\omega\cosh\theta} \quad d\theta
\end{aligned}
$$

Definition 2 can also be expressed in summation form as given in the following

Proposition (2.4)

$$
\begin{aligned}
(a) \quad K_v(\omega) &= \sqrt{\frac{\pi}{2\omega}}e^{-\omega}\sum_{i=0}^{\infty}\frac{\Gamma(v+\frac{1}{2})}{i!\Gamma(v+\frac{1}{2}-i)}\frac{\Gamma(v+\frac{1}{2}+i)}{\Gamma(v+\frac{1}{2})}(2\omega)^{-i} && \textbf{(i)} \\
&= \sqrt{\frac{\pi}{2\omega}}e^{-\omega}\left(1+\sum_{i=1}^{\infty}\frac{\Gamma(v+\frac{1}{2})}{i!\Gamma(v+\frac{1}{2}-i)}\frac{\Gamma(v+\frac{1}{2}+i)}{\Gamma(v+\frac{1}{2})}(2\omega)^{-i}\right) && \textbf{(ii)}
\end{aligned}
$$

Which can be further expressed as

$$
(b) \quad K_v(\omega) = \sqrt{\frac{\pi}{2\omega}}e^{-\omega}\left(1+\sum_{n=1}^{\infty}\prod_{i=1}^{n}\frac{\left[4v^2-(2i-1)^2\right]}{n!(8\omega)^n}\right)
$$

Proofs
Part (a)

$$\begin{aligned}
K_v(\omega) &= \left(\frac{\omega}{2}\right)^v \frac{\Gamma(\frac{1}{2})}{\Gamma(v+\frac{1}{2})} \int_1^\infty \left(t^2-1\right)^{v-\frac{1}{2}} e^{-\omega t} \quad dt \\
&= \left(\frac{\omega}{2}\right)^v \frac{\Gamma(\frac{1}{2})}{\Gamma(v+\frac{1}{2})} e^{-\omega+\omega} \int_1^\infty \left(t^2-1\right)^{v-\frac{1}{2}} e^{-\omega t} \quad dt \\
&= \left(\frac{\omega}{2}\right)^v \frac{\Gamma(\frac{1}{2})}{\Gamma(v+\frac{1}{2})} e^{-\omega} \int_1^\infty \left(t^2-1\right)^{v-\frac{1}{2}} e^{-\omega t+\omega} \quad dt \\
&= \left(\frac{\omega}{2}\right)^v \frac{\Gamma(\frac{1}{2})}{\Gamma(v+\frac{1}{2})} e^{-\omega} \int_1^\infty \left(t^2-1\right)^{v-\frac{1}{2}} e^{-\omega(t-1)} \quad dt
\end{aligned}$$

Now let

$$y = \omega(t-1) \Rightarrow t = 1 + \frac{y}{\omega} \Rightarrow dt = \frac{dy}{\omega}$$

$$\begin{aligned}
\therefore K_v(\omega) &= \left(\frac{\omega}{2}\right)^v \frac{\Gamma\frac{1}{2}}{\Gamma(v+\frac{1}{2})} \frac{e^{-\omega}}{\omega} \int_0^\infty \left[2\left(1+\frac{y}{2\omega}\right)\right]^{v-\frac{1}{2}} e^{-y} \quad dy \\
&= \left(\frac{\omega}{2}\right)^v \frac{\Gamma(\frac{1}{2})}{\Gamma(v+\frac{1}{2})} \frac{e^{-\omega}}{\omega} \int_0^\infty \left[4(1+\frac{y}{2\omega})\frac{y}{\omega}\right]^{v-\frac{1}{2}} e^{-y} \quad dy \\
&= \left(\frac{\omega}{2}\right)^v \frac{\Gamma(\frac{1}{2})}{\Gamma(v+\frac{1}{2})} \frac{e^{-\omega}}{\omega} \int_0^\infty \left[2^2(1+\frac{y}{2\omega})\frac{y}{\omega}\right]^{v-\frac{1}{2}} e^{-y} \quad dy \\
&= 2^{2v-1} \left(\frac{\omega}{2}\right)^v \frac{\Gamma(\frac{1}{2})}{\Gamma(v+\frac{1}{2})} \frac{e^{-\omega}}{\omega} \int_0^\infty \left(\frac{y}{2\omega}\right)^{v-\frac{1}{2}} \left(1+\frac{y}{2\omega}\right)^{v-\frac{1}{2}} e^{-y} \quad dy \\
&= 2^{2v-1} \left(\frac{\omega}{2}\right)^v \frac{\Gamma(\frac{1}{2})}{\Gamma(v+\frac{1}{2})} \frac{e^{-\omega}}{\omega} \int_0^\infty \left(\frac{y}{2\omega}\right)^{v-\frac{1}{2}} \sum_{i=0}^\infty \binom{v-\frac{1}{2}}{i} \left(\frac{y}{2\omega}\right)^i e^{-y} \quad dy \\
&= 2^{2v-1} \left(\frac{\omega}{2}\right)^v \frac{\Gamma(\frac{1}{2})}{\Gamma(v+\frac{1}{2})} \frac{e^{-\omega}}{\omega} \int_0^\infty \sum_{i=0}^\infty \binom{v-\frac{1}{2}}{i} \left(\frac{y}{2\omega}\right)^{v-\frac{1}{2}+i} e^{-y} \quad dy \\
&= \frac{2^{2v-1}\left(\frac{\omega}{2}\right)^v \Gamma\frac{1}{2} e^{-\omega}}{\Gamma(v+\frac{1}{2})\omega} \sum_{i=0}^\infty \left(\left(\frac{1}{2\omega}\right)^{v-\frac{1}{2}+i} \int_0^\infty \binom{v-\frac{1}{2}}{i} y^{v-\frac{1}{2}+i} e^{-y} \right) \quad dy
\end{aligned}$$

$$
\begin{aligned}
\therefore K_v(\omega) &= \frac{2^{2v-1}\left(\frac{\omega}{2}\right)^v \Gamma(\frac{1}{2})e^{-\omega}}{\Gamma(v+\frac{1}{2})\,(\omega)^{v-\frac{1}{2}}}\sum_{i=0}^{\infty}(2\omega)^{-i}\binom{v-\frac{1}{2}}{i}\int_0^{\infty} y^{v-\frac{1}{2}+i+1-1}e^{-y} \quad dy \\
&= \frac{2^{2v}}{2\omega}\frac{\left(\frac{\omega}{2}\right)^v \Gamma\frac{1}{2}e^{-\omega}}{\Gamma(v+\frac{1}{2})\,(\omega)^{v-\frac{1}{2}}}\sum_{i=0}^{\infty}(2\omega)^{-i}\binom{v-\frac{1}{2}}{i}\Gamma(v+\frac{1}{2}+i) \\
&= \frac{\left(\frac{4\omega}{2}\right)^v}{2\omega}\frac{\Gamma(\frac{1}{2})e^{-\omega}}{\Gamma(v+\frac{1}{2})\,(\omega)^{v-\frac{1}{2}}}\sum_{i=0}^{\infty}(2\omega)^{-i}\binom{v-\frac{1}{2}}{i}\Gamma(v+\frac{1}{2}+i) \\
K_v(\omega) &= \frac{\Gamma(\frac{1}{2})}{(2\omega)^{\frac{1}{2}}}\frac{e^{-\omega}}{\Gamma(v+\frac{1}{2})}\sum_{i=0}^{\infty}(2\omega)^{-i}\binom{v-\frac{1}{2}}{i}\Gamma(v+\frac{1}{2}+i) \\
&= \sqrt{\frac{\pi}{2\omega}}e^{-\omega}\sum_{i=0}^{\infty}\binom{v-\frac{1}{2}}{i}\frac{\Gamma(v+\frac{1}{2}+i)}{\Gamma(v+\frac{1}{2})}(2\omega)^{-i} \\
&= \sqrt{\frac{\pi}{2\omega}}e^{-\omega}\sum_{i=0}^{\infty}\frac{\Gamma(v+\frac{1}{2})}{i!\Gamma(v+\frac{1}{2}-i)}\frac{\Gamma(v+\frac{1}{2}+i)}{\Gamma(v+\frac{1}{2})}(2\omega)^{-i} \\
&= \sqrt{\frac{\pi}{2\omega}}e^{-\omega}\sum_{i=0}^{\infty}\frac{\Gamma(v+\frac{1}{2})}{i!\Gamma(v+\frac{1}{2}-i)}(2\omega)^{-i} \\
&= \sqrt{\frac{\pi}{2\omega}}e^{-\omega}\left[1+\sum_{i=1}^{\infty}\frac{\Gamma(v+\frac{1}{2})}{i!\Gamma(v+\frac{1}{2}-i)}(2\omega)^{-i}\right]
\end{aligned}
$$

Part (b)

$K_v(\omega)$ can be expressed as follows;

$$
\begin{aligned}
K_v(\omega) &= \sqrt{\frac{\pi}{2\omega}}e^{-\omega}\left[1+\frac{\Gamma(v+\frac{1}{2}+1)}{1!\Gamma(v+\frac{1}{2}-1)}(2\omega)^{-i}+\sum_{i=2}^{\infty}\frac{\Gamma(v+\frac{1}{2}+i)}{i!\Gamma(v+\frac{1}{2}-1)}(2\omega)^{-i}\right] \\
&= \sqrt{\frac{\pi}{2\omega}}e^{-\omega}\left[1+\frac{(v+\frac{1}{2})(v-\frac{1}{2})}{1!2\omega}+\sum_{i=2}^{\infty}\frac{\Gamma(v+\frac{1}{2}+i)}{i!\Gamma(v+\frac{1}{2}-i)}(2\omega)^{-i}\right] \\
&= \sqrt{\frac{\pi}{2\omega}}e^{-\omega}\left[1+\frac{v^2-\frac{1}{4}}{1!2\omega}+\sum_{i=2}^{\infty}\frac{\Gamma(v+\frac{1}{2}+i)}{i!\Gamma(v+\frac{1}{2}-i)}(2\omega)^{-i}\right] \\
&= \sqrt{\frac{\pi}{2\omega}}e^{-\omega}\left[1+\frac{4v^2-1}{1!8\omega}+\frac{\Gamma(v+\frac{1}{2}+1)}{2!\Gamma(v+\frac{1}{2}-2)}(2\omega)^{-2}+\sum_{i=3}^{\infty}\frac{\Gamma(v+\frac{1}{2}+i)}{i!\Gamma(v+\frac{1}{2}-i)}(2\omega)^{-i}\right] \\
&= \sqrt{\frac{\pi}{2\omega}}e^{-\omega}\left[1+\frac{4v^2-1}{1!8\omega}+\frac{(v+\frac{1}{2}+1)(v+\frac{1}{2})(v+\frac{1}{2}-1)(v+\frac{1}{2}-2)}{2!(2\omega)^2}+\sum_{i=3}^{\infty}\frac{\Gamma(v+\frac{1}{2}+i)}{i!\Gamma(v+\frac{1}{2}-i)}(2\omega)^{-i}\right] \\
&= \sqrt{\frac{\pi}{2\omega}}e^{-\omega}\left[1+\frac{4v^2-1}{1!8\omega}+\frac{(v+\frac{3}{2})(v+\frac{1}{2})(v-\frac{1}{2})(v-\frac{3}{2})}{2!(2\omega)^2}+\sum_{i=3}^{\infty}\frac{\Gamma(v+\frac{1}{2}+i)}{i!\Gamma(v+\frac{1}{2}-i)}(2\omega)^{-i}\right] \\
&= \sqrt{\frac{\pi}{2\omega}}e^{-\omega}\left[1+\frac{4v^2-1}{1!8\omega}+\frac{(v+\frac{3}{2})(v+\frac{1}{2})(v-\frac{1}{2})(v-\frac{3}{2})}{2!(2\omega)^2}+\sum_{i=3}^{\infty}\frac{\Gamma(v+\frac{1}{2}+i)}{i!\Gamma(v+\frac{1}{2}-i)}(2\omega)^{-i}\right] \\
&= \sqrt{\frac{\pi}{2\omega}}e^{-\omega}\left[1+\frac{4v^2-1}{1!8\omega}+\frac{(v^2-\frac{9}{4})(v^2-\frac{1}{4})}{2!4^2(2\omega)^2}+\sum_{i=3}^{\infty}\frac{\Gamma(v+\frac{1}{2}+i)}{i!\Gamma(v+\frac{1}{2}-i)}(2\omega)^{-i}\right] \\
&= \sqrt{\frac{\pi}{2\omega}}e^{-\omega}\left[1+\frac{4v^2-1}{1!8\omega}+\frac{(4v^2-1)(4v^2-9)}{2!(8\omega)^2}+\sum_{i=3}^{\infty}\frac{\Gamma(v+\frac{1}{2}+i)}{i!\Gamma(v+\frac{1}{2}-i)}(2\omega)^{-i}\right] \\
&= \sqrt{\frac{\pi}{2\omega}}e^{-\omega}\left[1+\frac{4v^2-1}{1!8\omega}+\prod_{i=1}^{2}\frac{\left[4v^2-(2i-1)^2\right]}{2!(8\omega)^2}+\sum_{i=3}^{\infty}\frac{\Gamma(v+\frac{1}{2}+i)}{i!\Gamma(v+\frac{1}{2}-i)}(2\omega)^{-i}\right] \\
&= \sqrt{\frac{\pi}{2\omega}}e^{-\omega}\left[1+\frac{4v^2-1}{1!8\omega}+\prod_{i=1}^{2}\frac{\left[4v^2-(2i-1)^2\right]}{2!(8\omega)^2}+\frac{\Gamma(v+\frac{7}{2})}{3!\Gamma(v-\frac{5}{2})}(2\omega)^{-3}+\sum_{i=3}^{\infty}\frac{\Gamma(v+\frac{1}{2}+i)}{i!\Gamma(v+\frac{1}{2}-i)}(2\omega)^{-i}\right] \\
&= \sqrt{\frac{\pi}{2\omega}}e^{-\omega}\left[1+\frac{4v^2-1}{1!8\omega}+\prod_{i=1}^{2}\frac{\left[4v^2-(2i-1)^2\right]}{2!(8\omega)^2}+\frac{(v^2-\frac{25}{4})(v^2-\frac{9}{4})(v^4-\frac{1}{4})}{3!(2\omega)^3}+\sum_{i=3}^{\infty}\frac{\Gamma(v+\frac{1}{2}+i)}{i!\Gamma(v+\frac{1}{2}-i)}(2\omega)^{-i}\right] \\
&= \sqrt{\frac{\pi}{2\omega}}e^{-\omega}\left[1+\frac{4v^2-1}{1!8\omega}+\prod_{i=1}^{2}\frac{\left[4v^2-(2i-1)^2\right]}{2!(8\omega)^2}+\prod_{i=1}^{3}\frac{\left[4v^2-(2i-1)^2\right]}{3!(8\omega)^3}+\sum_{i=3}^{\infty}\frac{\Gamma(v+\frac{1}{2}+i)}{i!\Gamma(v+\frac{1}{2}-i)}(2\omega)^{-i}\right] \\
\therefore K_v(\omega) &= \sqrt{\frac{\pi}{2\omega}}e^{-\omega}\left[1+\sum_{n=1}^{\infty}\prod_{i=1}^{n}\frac{\left[4v^2-(2i-1)^2\right]}{n!(8\omega)^n}\right]
\end{aligned}
$$

Corollary 2.4.1

(a) $K_{n+\frac{1}{2}}(\omega)=\sqrt{\frac{\pi}{2\omega}}e^{-\omega}\left[1+\sum_{i=1}^{n}\frac{(n+i)!}{i!(n-i)!}(2\omega)^{-i}\right]$

(b) $K_{n-\frac{1}{2}}(\omega)=\sqrt{\frac{\pi}{2\omega}}e^{-\omega}\left[1+\sum_{i=1}^{n-1}\frac{(n+i-1)!}{i!(n-i-1)!}(2\omega)^{-i}\right]$

where n is a positive interger.

Proof

From proposition 2.4 put
$v=n+\frac{1}{2}$

$$\begin{aligned}K_{n+\frac{1}{2}}(\omega)&=\sqrt{\frac{\pi}{2\omega}}e^{-\omega}\left[1+\sum_{i=1}^{\infty}\frac{\Gamma(n+1+i)}{i!\Gamma(n+1-i)}(2\omega)^{-i}\right]\\&=\sqrt{\frac{\pi}{2\omega}}e^{-\omega}\left[1+\sum_{i=1}^{\infty}\frac{(n+i)!}{i!(n-i)!}(2\omega)^{-i}\right]\\&=\sqrt{\frac{\pi}{2\omega}}e^{-\omega}\left[1+\sum_{i=1}^{n}\frac{(n+i)!}{i!(n-i)!}(2\omega)^{-i}\right]\end{aligned}$$

Putting $v=n-\frac{1}{2}$ we have

$$\begin{aligned}K_{n-\frac{1}{2}}(\omega)&=\sqrt{\frac{\pi}{2\omega}}e^{-\omega}\left[1+\sum_{i=1}^{\infty}\frac{\Gamma(n+i)!}{i!\Gamma(n-i)}(2\omega)^{-i}\right]\\&=\sqrt{\frac{\pi}{2\omega}}e^{-\omega}\left[1+\sum_{i=1}^{\infty}\frac{(n+i-1)!}{i!(n-i-1)!}(2\omega)^{-i}\right]\\&=\sqrt{\frac{\pi}{2\omega}}e^{-\omega}\left[1+\sum_{i=1}^{n-1}\frac{(n+i-1)!}{i!(n-i-1)!}(2\omega)^{-i}\right]\end{aligned}$$

Corollary 2.4.2

(a) $K_{\frac{1}{2}}(\omega) = K_{-\frac{1}{2}}(\omega) = \sqrt{\frac{\pi}{2\omega}}e^{-\omega}$

(b) $K_{\frac{3}{2}}(\omega) = \sqrt{\frac{\pi}{2\omega}}e^{-\omega}(1+\frac{1}{\omega})$

(c) $K_{\frac{5}{2}}(\omega) = \sqrt{\frac{\pi}{2\omega}}e^{-\omega}\left(1+\frac{3}{\omega}+\frac{3}{\omega^2}\right)$

Proof

From corollary 2.4.1 put $n=0$

$$\therefore K_{\frac{1}{2}}(\omega) = K_{-\frac{1}{2}}(\omega) = \sqrt{\frac{\pi}{2\omega}}e^{-\omega}$$

putting $n=1$,

$$\begin{aligned}\therefore K_{\frac{3}{2}}(\omega) &= \sqrt{\frac{\pi}{2\omega}}e^{-\omega}\left[1+\frac{2!}{i!0!}(2\omega)^{-i}\right]\\ &= \sqrt{\frac{\pi}{2\omega}}e^{-\omega}\left[1+\frac{1}{\omega}\right]\end{aligned}$$

putting $n=2$, we have

$$\begin{aligned}K_{\frac{5}{2}}(\omega) &= \sqrt{\frac{\pi}{2\omega}}e^{-\omega}\left[1+\sum_{1=1}^{2}\frac{(2+i)!}{i!(2-i)}(2\omega)^{-i}\right]\\ \therefore K_{\frac{5}{2}}(\omega) &= \sqrt{\frac{\pi}{2\omega}}e^{-\omega}\left[1+\frac{3!}{1!1!}(2\omega)^{-1}+\frac{4!}{2!0!}(2\omega)^{-2}\right]\\ \therefore K_{\frac{5}{2}}(\omega) &= \sqrt{\frac{\pi}{2\omega}}e^{-\omega}\left[1+\frac{6}{1!2\omega}+\frac{24}{2!4\omega^2}\right]\\ &= \sqrt{\frac{\pi}{2\omega}}e^{-\omega}\left[1+\frac{3}{\omega}+\frac{3}{\omega^2}\right]\end{aligned}$$

2.4 Modified Bessel function of the third kind in terms of modified Bessel function of the first kind

By expanding the generating function $e^{\frac{\omega}{2}(t-\frac{1}{t})}$ we have

$$\begin{aligned}
e^{\frac{\omega}{2}(t-\frac{1}{t})} &= e^{\frac{\omega t}{2}} e^{-\frac{\omega}{2t}} \\
&= e^{\frac{\omega}{2}t} e^{-\frac{-\omega}{2}t^{-1}} \\
&= \sum_{m=0}^{\infty} \frac{\left(+\frac{\omega}{2}t\right)^m}{m!} \sum_{k=0}^{\infty} \left(-\frac{\omega}{2}t^{-1}\right)^k \\
&= \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{\omega}{2}\right)^{m+k} t^{m-k}}{m!k!}
\end{aligned}$$

Let

$$m-k = n \Rightarrow m = n+k$$

$$\begin{aligned}
\therefore e^{\frac{\omega}{2}(t-\frac{1}{t})} &= \sum_{n+k=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^k}{(n+k)!} \left(\frac{\omega}{2}\right)^{n+2k} t^n \\
&= \sum_{n=-k}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^k}{(n+k)!} \frac{1}{k!} \left(\frac{\omega}{2}\right)^{2k} t^n \\
&= \sum_{n=-\infty}^{\infty} \left[\sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(n+k+1)k!} \left(\frac{\omega}{2}\right)^{2k+n} \right] t^n
\end{aligned} \tag{2.20}$$

The coefficient of t^n is

$$J_n(\omega) \sum_{K=0}^{\infty} \frac{(-1)^k}{\Gamma(n+k+1)k!} \left(\frac{\omega}{2}\right)^{n+2k} \tag{2.21}$$

Let

$$I_n(\omega) \sum_{K=0}^{\infty} \frac{1}{\Gamma(n+k+1)k!} \left(\frac{\omega}{2}\right)^{n+2k}$$

Replace n by v>0

$$I_v(\omega) = \sum_{k=0}^{\infty} \frac{1}{k!\Gamma(v+k+1)} \left(\frac{\omega}{2}\right)^{v+2k}$$

which is called a modified Bessel function of the first kind. The relationship between modified Bessel functions of the third and first kind is given by

$$K_v(\omega) = \frac{\pi}{2\sin v\pi} \left[I_{-v}(\omega) - I_v(\omega)\right] \tag{2.22}$$

3 GENERALIZED INVERSE GAUSSIAN DISTRIBUTION BASED ON SICHEL'S PARAMETERIZATION ($\omega = \sqrt{\psi\phi}$)

3.1 Introduction

In this chapter, we constructed GIG distribution under Sichel's parameterization. Special cases of GIG also discussed. Under this parameterization, we have a vast area on the special cases wich gave rise to various distributions. These distributions are known as the GIG distribution sub-models. Throughout this chapeter, we have various statistical properties which we have studied in detailed. Properties cosidered are r-th moment, Laplace transform and modality; all the properties have been discussed under the GIG distribution and its sub-models as well.

3.2 Construction and properties

From (2.1) we let

$$\omega = \sqrt{\psi\phi} \tag{3.1}$$

$$K_v(\sqrt{\psi\phi}) = \frac{1}{2}\int_0^\infty z^{v-1} e^{-\frac{\sqrt{\psi\phi}}{2}(z+\frac{1}{z})} \quad dz \tag{3.2}$$

We shall refer to parameterization (3.1) as Sichel's parameterization. Now let us consider the following transformation

$$z = \sqrt{\frac{\psi}{\phi}} x \Rightarrow dz = \sqrt{\frac{\psi}{\phi}} dx \tag{3.3}$$

Therefore, (3.2) becomes

$$K_v(\sqrt{\psi\phi}) = \frac{1}{2}\int_0^\infty \left(\sqrt{\frac{\psi}{\phi}}\right)^v x^{v-1}e^{-\frac{1}{2}(\psi x+\frac{\phi}{x})} \quad dx$$

$$K_v(\sqrt{\psi\phi}) = \frac{1}{2}\int_0^\infty \left(\sqrt{\frac{\psi}{\phi}}\right)^v e^{-\frac{1}{2}(\psi x+\frac{\phi}{x})} \quad dx$$

$$K_v(\sqrt{\psi\phi}) = \frac{1}{2}\left(\sqrt{\frac{\psi}{\phi}}\right)^v \int_0^\infty x^{v-1}e^{-\frac{1}{2}(\psi x+\frac{\phi}{x})} \quad dx$$

$$\therefore 1 = \int_0^\infty \left(\sqrt{\frac{\psi}{\phi}}\right)^v \frac{x^{v-1}e^{-\frac{1}{2}(\psi x+\frac{\phi}{x})}}{2K_v(\sqrt{\psi\phi})} \quad dx \tag{3.4}$$

$$\therefore f(x) = \left(\sqrt{\frac{\psi}{\phi}}\right)^v \frac{x^{v-1}e^{-\frac{1}{2}(\psi x+\frac{\phi}{x})}}{2K_v(\sqrt{\psi\phi})} \quad dx \quad x>0; -\infty<v<\infty, \quad \phi,\psi \geq 0 \tag{3.5}$$

Is a Generalized Inverse Gaussian (GIG) pdf variable X with parameters v, ϕ, ψ

$$f(x) = \frac{\left(\sqrt{\frac{\psi}{\phi}}\right)^v x^{v-1}e^{-\frac{1}{2}(\psi x+\frac{\phi}{x})}}{2\left[\frac{1}{2}\left(\sqrt{\frac{\psi}{\phi}}\right)^v \int_0^\infty x^{v-1}e^{-\frac{1}{2}(\psi x+\frac{\phi}{x})} \quad dx\right]}$$

$$= \frac{x^{v-1}e^{-\frac{1}{2}(\psi x+\frac{\phi}{x})}}{\int_0^\infty x^{v-1}e^{-\frac{1}{2}(\psi x+\frac{\phi}{x})}dx}, \quad x>0; \quad -\infty<v<\infty, \quad \phi \geq 0, \quad \psi \geq 0 \tag{3.6}$$

The random Variable X from the GIG pdf will be denoted as

$$X \sim GIG(v, \phi, \psi) \tag{3.7}$$

3.3 Properties

3.3.1 Property 3.1: Moments

$$E(X^r) = \left(\sqrt{\frac{\phi}{\psi}}\right)^r \frac{K_{v+r}(\sqrt{\psi}\phi)}{K_v(\sqrt{\psi\phi})} \tag{3.8}$$

$$E(X) = \left(\sqrt{\frac{\phi}{\psi}}\right) \frac{K_{v+1}(\sqrt{\psi}\phi)}{K_v(\sqrt{\psi\phi})} \tag{3.9}$$

$$Var(X) = \left(\sqrt{\frac{\phi}{\psi}}\right)^2 \left(\frac{K_{v+2}(\sqrt{\psi}\phi)}{K_v(\sqrt{\psi\phi})} - \left[\frac{K_{v+1}(\sqrt{\psi}\phi)}{K_v(\sqrt{\psi\phi})}\right]^2\right) \tag{3.10}$$

Proof

$$\begin{aligned}
\therefore E(X^r) &= \int_0^\infty x^r \frac{\left(\sqrt{\frac{\psi}{\phi}}\right)^v}{2K_v(\sqrt{\psi\phi})} x^{v-1} \exp -\{\frac{1}{2}(\psi x + \frac{\phi}{x})\} d(x) \\
&= \frac{1}{K_v\sqrt{\psi\phi}} \frac{1}{2} \int_0^\infty x^{v+r-1} \exp -\{\frac{1}{2}(\psi x + \frac{\phi}{x})\} dx \\
&= \frac{\left(\sqrt{\frac{\psi}{\phi}}\right)^v}{K_v(\psi\phi)} \frac{1}{2} \int_0^\infty x^{v+r-1} \exp\left(-\frac{1}{2}(\psi x + \frac{\phi}{x})\right) \quad dx \\
&= \frac{\left(\sqrt{\frac{\psi}{\phi}}\right)^v}{K_v(\psi\phi)} \frac{1}{2} \int_0^\infty \frac{\left(\sqrt{\frac{\psi}{\phi}}\right)^{v+r} \exp\left(-\frac{1}{2}(\psi x + \frac{\phi}{x})\right) \quad dx}{\left(\sqrt{\frac{\psi}{\phi}}\right)^{v+r}} \\
&= \left(\sqrt{\frac{\phi}{\psi}}\right)^r \frac{K_{v+r}(\sqrt{\psi\phi})}{K_v(\sqrt{\psi\phi})}
\end{aligned}$$

Therefore, to get the mean and variance,

$$\begin{aligned}
E(X^r) &= \left(\sqrt{\frac{\phi}{\psi}}\right)^r \frac{K_{v+r}(\sqrt{\psi}\phi)}{K_v(\sqrt{\psi\phi})} \\
E(X) &= \left(\sqrt{\frac{\phi}{\psi}}\right) \frac{K_{v+1}(\sqrt{\psi}\phi)}{K_v(\sqrt{\psi\phi})} \\
E(X^2) &= \left(\sqrt{\frac{\phi}{\psi}}\right)^2 \frac{K_{v+2}(\sqrt{\psi}\phi)}{K_v(\sqrt{\psi\phi})} \\
Var(X) &= \left(\sqrt{\frac{\phi}{\psi}}\right)^2 \left(\frac{K_{v+2}(\sqrt{\psi}\phi)}{K_v(\sqrt{\psi\phi})} - \left[\frac{K_{v+1}(\sqrt{\psi}\phi)}{K_v(\sqrt{\psi\phi})}\right]^2\right)
\end{aligned}$$

3.3.2 Property 3.2: Laplace transform

The Laplace transform of a GIG distributed random variable X with parameters $v, \phi, \quad and \quad \psi$ denoted by $L_{GIG(v,\phi,\psi)}$

$$L_{GIG(v,\phi,\psi)} = \left(\sqrt{\frac{\psi}{2s+\psi}}\right)^{v} \frac{K_v(\sqrt{(2s+\psi)\phi})}{K_v(\sqrt{\psi\phi})} \tag{3.11}$$

Proof

$$\begin{aligned}
\therefore L_X(s) &= \int_0^\infty e^{-sX} \frac{\left(\sqrt{\frac{\psi}{\phi}}\right)^v}{2K_v(\sqrt{\psi\phi})} x^{v-1} \exp -\frac{1}{2}(\psi x + \frac{\phi}{x}) dx \\
&= \frac{\left(\sqrt{\frac{\psi}{\phi}}\right)^v}{2K_v(\sqrt{\psi\phi})} \int_0^\infty x^{v-1} \exp\{\frac{-2sx}{2} - \frac{\psi x}{2} - \frac{\phi}{2x}\} dx \\
&= \frac{\left(\sqrt{\frac{\psi}{\phi}}\right)^v}{2K_v(\sqrt{\psi\phi})} \int_0^\infty x^{v-1} \exp\{-\frac{1}{2}[(2s+\psi)x + \frac{\phi}{x}]\} dx
\end{aligned}$$

$$\begin{aligned}
\therefore L_X(s) &= \frac{\left(\sqrt{\frac{\psi}{\phi}}\right)^v}{2K_v(\sqrt{\psi\phi})} \int_0^\infty x^{v-1} \exp\{\frac{-(2s+\psi)}{2}[x + \frac{\phi}{2s+\psi} * \frac{1}{x}]\} dx \\
&= \frac{\left(\sqrt{\frac{\psi}{\phi}}\right)^v}{K_v(\sqrt{\psi\phi})} \left(\sqrt{\frac{\psi}{2s+\psi}}\right)^v K_v(\sqrt{(2s+\psi)\phi}) \\
&= \left(\sqrt{\frac{\psi}{2s+\psi}}\right)^v \frac{K_v(\sqrt{(2s+\psi)\phi})}{K_v(\sqrt{\psi\phi})}
\end{aligned}$$

3.3.3 Property 3.3: Modality

The mode of GIG distribution is which maximizes the pdf f(x). This peak is obtained by solving the equation

$$\frac{d}{dx} f(x) = 0$$

Therefore,

$$\begin{aligned}
X_{mode} &= \frac{(v-1) + \sqrt{(v-1)^2 + \psi\phi}}{\psi}, \quad \psi > 0 \\
and \quad X_{mode} &= \frac{\phi}{2(1-v)}, \quad \psi = 0
\end{aligned} \tag{3.12}$$

Proof

$$\frac{df}{dx} = 0$$

$$\frac{d}{dx}\left(x^{v-1}e^{-\frac{1}{2}(\psi x+\frac{\phi}{x})}\right) = 0$$

$$(v-1)x^{v-2}e^{-\frac{1}{2}(\psi x+\frac{\phi}{x})} + x^{v-1}e^{-\frac{1}{2}(\phi x+\frac{\phi}{x})} = 0$$

$$\frac{d}{dx}\left(x^{v-1}e^{-\frac{1}{2}(\psi x+\frac{\phi}{x})}\right) = 0$$

$$(v-1) + x\left[-\frac{\psi}{2} + \frac{\phi}{2x^2}\right] = 0$$

$$(v-1) - x\frac{\psi}{2} + \frac{\phi}{2x} = 0$$

$$2(v-1)x - \psi x^2 + \phi = 0$$

$$\psi x^2 - 2(v-1)x - \phi = 0$$

$$\psi = 0 \Rightarrow -2(v-1)x - \phi = 0$$

$$x = \frac{\phi}{2(1-v)}$$

The mode is

$$X_{mode} = \frac{(v-1) + \sqrt{(v-1)^2 + \psi\phi}}{\psi}, \quad \psi > 0$$

$$and \quad X_{mode} = \frac{\phi}{2(1-v)}, \quad \psi = 0$$

3.4 Special cases of GIG Distribution

3.4.1 Inverse Gaussian Distribution

When

$$v = -\frac{1}{2} \tag{3.13}$$

$$f(x) = \left(\sqrt{\frac{\psi}{\phi}}\right)^{-\frac{1}{2}} \frac{x^{-\frac{3}{2}}e^{-\frac{1}{2}(\psi x+\frac{\phi}{x})}}{2K_{-\frac{1}{2}(\sqrt{\psi\phi})}}$$

$$= \sqrt{\frac{\phi}{2\pi x^3}}e^{\sqrt{\psi\phi}}e^{-\frac{1}{2}(\psi x+\frac{\phi}{x})} \tag{3.14}$$

$$E(X^r) = \left(\sqrt{\frac{\phi}{\psi}}\right)^r \frac{K_{-\frac{1}{2}+r}\left(\sqrt{\psi\phi}\right)}{K_{-\frac{1}{2}(\sqrt{\psi\phi})}} \tag{3.15}$$

$$E(X) = \left(\sqrt{\frac{\phi}{\psi}}\right) \frac{\sqrt{\frac{\pi}{2\sqrt{\psi\phi}}}e^{-\sqrt{\psi\phi}}}{\sqrt{\frac{\pi}{2\sqrt{\psi\phi}}}e^{-\sqrt{\psi\phi}}} = \left(\sqrt{\frac{\phi}{\psi}}\right)$$

$$\begin{aligned} E(X^2) &= \left(\sqrt{\frac{\phi}{\psi}}\right)^2 \frac{K_{\frac{3}{2}}(\sqrt{\psi\phi})}{K_{-\frac{1}{2}}(\sqrt{\psi\phi})} \\ &= \left(\sqrt{\frac{\phi}{\psi}}\right)^2 \left(1+\sqrt{\psi\phi}\right) \end{aligned} \tag{3.16}$$

$$\begin{aligned} VarX &= \left(\sqrt{\frac{\phi}{\psi}}\right)^2 (1+\sqrt{\psi\phi}) - \left(\sqrt{\frac{\phi}{\psi}}\right)^2 \\ &= \left(\sqrt{\frac{\phi}{\psi}}\right)^2 \left((1+\sqrt{\psi\phi})-1\right) \\ &= \left(\sqrt{\frac{\phi^3}{\psi}}\right) \end{aligned} \tag{3.17}$$

$$\begin{aligned} L_X(s) &= \left(\sqrt{\frac{\psi}{2s+\psi}}\right)^{-\frac{1}{2}} \frac{K_{-\frac{1}{2}}(\sqrt{(2s+\psi)\phi})}{K_{-\frac{1}{2}(\sqrt{\psi\phi})}} \\ &= e^{\sqrt{\psi\phi}-\sqrt{(2s+\psi)\phi}} \end{aligned} \tag{3.18}$$

$$\begin{aligned} X_{mode} &= \frac{-3+\sqrt{9+4(\phi\psi)}}{2(\sqrt{\psi\phi})} \\ &= \frac{3}{2}\left[\sqrt{1+\frac{4}{9}\psi\phi}-1\right] \end{aligned} \tag{3.19}$$

3.4.2 Reciprocal Inverse Gaussian

This is the case when

$$v = \frac{1}{2}, \quad \phi > 0, \quad \psi > 0 \tag{3.20}$$

Then the pdf of RIG distribution is

$$f(x) = \left(\sqrt{\frac{\psi}{\phi}}\right)^{\frac{1}{2}} \frac{x^{-\frac{1}{2}} e^{-\frac{1}{2}(\psi x + \frac{\phi}{x})}}{2K_{\frac{1}{2}(\sqrt{\psi\phi})}} = \left(\frac{2\psi}{\pi x}\right)^{\frac{1}{2}} e^{-\frac{1}{2}(\psi x + \frac{\phi}{x})} \quad for \quad x > 0; \quad \phi > 0, \quad \psi > 0 \tag{3.21}$$

The moments are:

$$E(X^r) = \left(\sqrt{\frac{\phi}{\psi}}\right)^r \frac{K_{\frac{1}{2}+r}(\sqrt{\psi\phi})}{K_{\frac{1}{2}}(\sqrt{\phi\psi})} \tag{3.22}$$

$$\begin{aligned} E(X) &= \sqrt{\frac{\phi}{\psi}} \frac{K_{\frac{3}{2}}(\sqrt{\psi\phi})}{K_{\frac{1}{2}}(\sqrt{\psi\phi})} \\ &= \sqrt{\frac{\phi}{\psi}} \left(1 + \sqrt{\phi\psi}\right) \frac{K_{\frac{1}{2}}(\sqrt{\psi\phi})}{K_{\frac{1}{2}}(\sqrt{\psi\phi})} \\ &= \sqrt{\frac{\phi}{\psi}} \left(1 + \sqrt{\phi\psi}\right) \end{aligned} \tag{3.23}$$

$$\begin{aligned} Var(X) &= \left(\sqrt{\frac{\phi}{\psi}}\right)^2 \left[3\sqrt{\psi\phi} + \psi\phi - \psi^2\phi^2\right] \\ &= \left(\sqrt{\frac{\phi}{\psi}}\right)^2 \left(3\sqrt{\psi\phi} + \psi\phi(1 - \psi\phi)\right) \end{aligned} \tag{3.24}$$

The Laplace of RIG,

$$\begin{aligned} L_{GIG(\frac{1}{2},\phi,\psi)} &= \left(\sqrt{\frac{\psi}{2s+\psi}}\right)^{\frac{1}{2}} \frac{K_{\frac{1}{2}}(\sqrt{(2s+\psi)\phi})}{K_{\frac{1}{2}}(\sqrt{\psi\phi})} \\ &= \left(\sqrt{\frac{\psi}{2s+\psi}}\right)^{\frac{1}{2}} \frac{\left[\frac{\pi}{2\sqrt{(2s+\psi)\phi}}\right]^{\frac{1}{2}} e^{-\sqrt{(2s+\psi)}}}{\left[\frac{\pi}{2\sqrt{\psi\phi}}\right]^{\frac{1}{2}} e^{-\sqrt{\psi\phi}}} \\ &= \left(\sqrt{\frac{\psi}{2s+\psi}}\right)^{\frac{1}{2}} \left(\sqrt{\frac{\psi\phi}{(2s+\phi)\phi}}\right)^{\frac{1}{2}} e^{\sqrt{\psi\phi} - \sqrt{(2s+\psi)\phi}} \end{aligned}$$

$$
\begin{aligned}
L_{GIG(\frac{1}{2},\phi,\psi)} &= \sqrt{\frac{\psi}{2s+\psi}}e^{\sqrt{\psi\phi}-\sqrt{(2s+\psi)\phi}} \\
&= \sqrt{\frac{\psi}{2s+\psi}}L_{GIG(-\frac{1}{2},\phi,\psi)} \\
&= \left(\frac{\psi}{2s+\psi}\right)^{\frac{1}{2}} L_{GIG(-\frac{1}{2},\phi,\psi)} \\
&= L_{GIG(\frac{1}{2},0,\psi)}L_{GIG(-\frac{1}{2},\phi,\psi)}
\end{aligned} \tag{3.25}
$$

where

$$
L_{GIG(\frac{1}{2},0,\psi)} = \left(\frac{\psi}{2s+\psi}\right)^{\frac{1}{2}}
$$

is the Laplace of gamma distribution with parameters $\frac{1}{2}$, $\frac{\psi}{2}$ as will be shown in section 3.4.3

Thus the Laplace transform of a reciprocal inverse Gaussian is the product of the Laplace transform of a Gamma $(\frac{1}{2}, \frac{\psi}{2})$ and the Laplace of an inverse Gaussian distribution. This further implies that a convolution of a Gamma $(\frac{1}{2}, \frac{\psi}{2})$ and an $IG(\phi, \psi)$ as will be shown later.

The mode of RIG is

$$
\begin{aligned}
\therefore X_{mode} &= \frac{(-\frac{1}{2}) + \sqrt{(\frac{1}{4}) + \psi\phi}}{\sqrt{\psi\phi}} \\
&= \frac{-\frac{1}{2} + \sqrt{\frac{1}{4}(1 + \frac{\psi\phi}{4})}}{\sqrt{\psi\phi}} \\
&= \frac{\sqrt{1 + \frac{\psi\phi}{4}} - 1}{2\sqrt{\psi\phi}}
\end{aligned} \tag{3.26}
$$

3.4.3 Gamma Distribution

When

$$
v > 0, \quad \phi = 0, \quad \psi > 0 \tag{3.27}
$$

Using formula (3.6) we have

$$f(x) = \frac{x^{v-1}e^{-\frac{\psi}{2}}}{\int_0^\infty x^{v-1}e^{-\frac{\psi}{2}x}dx}$$
$$= \frac{x^{v-1}e^{-\frac{\psi}{2}x}}{\frac{\Gamma(v)}{\left(\frac{\psi}{2}\right)^v}}$$
$$= \frac{\left(\frac{\psi}{2}\right)^v}{\Gamma(v)}e^{-\frac{\psi}{2}x}x^{v-1} \quad x > 0; \quad v > 0 \tag{3.28}$$

This is called a gamma pdf with parameters v and $\frac{\psi}{2}$

Thus

$$GIG(v, 0, \psi) = Gamma(v, \frac{\psi}{2})$$

$$E(X^r) = \frac{x^{r+v-1}e^{-\frac{1}{2}(\psi x + \frac{\phi}{x})}}{\int_0^\infty x^{r+v-1}e^{-\frac{1}{2}(\psi x + \frac{\phi}{x})} \quad dx}$$

when $\phi = 0$

$$E(X^r) = \frac{x^{r+v-1}e^{-\frac{\psi}{2}x}}{\int_0^\infty x^{r+v-1}e^{-\frac{\psi}{2}x} \quad dx}$$
$$= \int_0^\infty x^r \frac{\left(\frac{\psi}{2}\right)^v}{\Gamma(v)}x^{v-1}e^{-\frac{\psi}{2}x} \quad dx$$
$$= \frac{\left(\frac{\psi}{2}\right)}{\Gamma(v)}\int_0^\infty x^{v+r-1}e^{-\frac{\psi}{2}x}$$
$$= \left(\frac{2}{\psi}\right)^r \frac{\Gamma(v+r)}{\Gamma(v)} \tag{3.29}$$

$$E(X) = \left(\frac{2}{\psi}\right)\frac{\Gamma(v+1)}{\Gamma(v)} = \frac{2}{\psi}v \tag{3.30}$$
$$E(X^2) = \left(\frac{2}{\psi}\right)^2 (v+1)v$$
$$\therefore Var(X) = \left(\frac{2}{\psi}\right)^2 v \tag{3.31}$$

$$L_X(s) = \frac{\left(\frac{\psi}{2}\right)^v}{\left(\frac{\psi}{2}+s\right)^v}$$
$$= \left(\frac{\frac{\psi}{2}}{\frac{\psi}{2}+s}\right)^v$$

i.e

$$L_{GIG(v,0,\psi)} = \left(\frac{\psi}{\psi+2s}\right)^v \tag{3.32}$$

Modality

$$\frac{d}{dx}f(x) = 0$$
$$\frac{d}{dx}\left[\frac{\left(\frac{\psi}{2}\right)^v}{\Gamma v}e^{-\frac{\psi}{2}x}x^{v-1}\right] = 0$$
$$(v-1)x^{v-2}e^{-\frac{\psi}{2}x} + x^{v-1}\left[-\frac{\psi}{2}e^{-\frac{\psi}{2}x}\right] = 0$$
$$(v-1)x^{v-2} - x^{v-1}\frac{\psi}{2} = 0$$
$$2(v-1)x^{v-2} - x^{v-1}\psi = 0$$
$$2(v-1)x = \psi$$
$$\therefore X_{mode} = \frac{\psi}{2(v-1)} \quad if \quad v \neq 1$$

and

$$X_{mode} = \infty \quad if \quad v = 1 \tag{3.33}$$

3.4.4 Exponential Distribution

This is a special case of a gamma distribution when

$$v = 1, \quad \phi = 0, \quad \psi > 0 \tag{3.34}$$

$$f(x) = \frac{e^{\frac{\psi}{2}x}}{\int_0^\infty e^{-\frac{\psi}{2}dx}}$$
$$= \frac{e^{\frac{\psi}{2}x}}{\int_0^\infty x^{1-1}e^{-\frac{\psi}{2}dx}}$$
$$\therefore f(x) = \frac{e^{-\frac{\psi}{2}x}}{\frac{1}{\frac{\psi}{2}}}$$
$$= \frac{\psi}{2}e^{-\frac{\psi}{2}x} \quad for \quad x > 0 \quad and \quad \psi > 0$$

$$E(X^r) = \left(\frac{2}{\psi}\right)^r \Gamma(r+1) = \left(\frac{2}{\psi}\right)^r r! \tag{3.35}$$

This is an exponential distribution with parameters $\frac{\psi}{2} > 0$

$$\begin{aligned}
E(X) &= \frac{1}{\left(\frac{\psi}{2}\right)} = \frac{2}{\psi} \\
\therefore E(X^2) &= \frac{2}{\left(\frac{\psi}{2}\right)^2} \\
Var(X) &= \frac{2}{\left(\frac{\psi}{2}\right)^2} - \frac{1}{\left(\frac{\psi}{2}\right)^2} \\
&= \frac{1}{\left(\frac{\psi}{2}\right)^2}
\end{aligned} \tag{3.36}$$

$$L_X(s) = \left(\frac{\psi}{2}\right) \int_0^\infty e^{-(\frac{\psi}{2}+s)x} \quad dx$$

Let

$$\frac{\psi}{2} + s = \lambda$$

$$\begin{aligned}
L_X(s) &= \frac{\psi}{2} \int_0^\infty e^{-\lambda x} \quad dx \\
&= \frac{\psi}{2} \left[-\frac{e^{-\lambda x}}{\lambda} \right]_0^\infty \\
&= \frac{\psi}{2} \frac{1}{\lambda} \\
\therefore &= \frac{\frac{\psi}{2}}{\frac{\psi}{2} + s} = \frac{\psi}{2s + \psi}
\end{aligned} \tag{3.37}$$

To get the modality

$$\begin{aligned}
\frac{d}{dx} f(x) &= 0 \\
\frac{d}{dx} \left[\frac{\psi}{2} e^{-\frac{\psi}{2}x} \right] &= 0 \\
\frac{\psi}{2} \frac{d}{dx} \left[e^{-\frac{\psi}{2}x} \right] &= 0 \\
\frac{\psi}{2} \left(-\frac{\psi}{2} \right) \left(e^{-\frac{\psi}{2}x} \right) &= 0 \\
e^{-\frac{\psi}{2}x} &= 0 \\
X_{mode} &= \infty
\end{aligned} \tag{3.38}$$

3.4.5 Inverse Gamma Distribution

This is the case when

$$v < 0, \quad \phi > 0, \quad \psi = 0 \tag{3.39}$$

Using formula (3.6) we have

$$f(x) = \frac{x^{v-1} e^{-\frac{\phi}{2}\frac{1}{x}}}{\int_0^\infty x^{v-1} e^{-\frac{\phi}{2}\frac{1}{x}}}, x > 0; \quad v < 0, \quad \phi > 0 \tag{3.40}$$

$$I = \int_0^\infty x^{v-1} e^{-\frac{\phi}{2}\frac{1}{x}} \quad dx$$

$$Put \quad x = \frac{1}{y} \Rightarrow dx = -\frac{dy}{y^2}$$

$$\begin{aligned} I &= \int_0^\infty \left(\frac{1}{y}\right)^{v-1} e^{-\frac{\phi}{2}y} \left(-\frac{dy}{y^2}\right) \\ &= \int_0^\infty \frac{1}{y^{v-1+2}} e^{-\frac{\phi}{2}y} \quad dy \\ &= \frac{\Gamma(-v)}{\left(\frac{\phi}{2}\right)^{-v}} \end{aligned}$$

$$\therefore f(x) = \frac{\left(\frac{\phi}{2}\right)^{-v}}{\Gamma - v} x^{v-1} e^{-\frac{\phi}{2}\frac{1}{x}}, \quad x > 0 \tag{3.41}$$

Let

$$v = -\lambda \quad where \quad \lambda > 0$$

$$\begin{aligned} f(x) &= \frac{\left(\frac{\phi}{2}\right)^{\lambda}}{\Gamma\lambda} x^{-\lambda-1} e^{-\frac{\phi}{2}\frac{1}{x}} \\ &= \frac{\left(\frac{\phi}{2}\right)^{\lambda}}{\Gamma\lambda} e^{-\frac{\phi}{2}\frac{1}{x}} x^{-\lambda-1} \end{aligned} \tag{3.42}$$

Which is an inverse Gamma pdf with v<0

Thus

$$X \sim GIG(v, \phi, 0) = Inverse \quad Gamma(-v, \frac{\phi}{2}), \quad where v < 0 \quad and \quad \phi > 0$$

$$E(X^r) = \int_0^\infty x^r \frac{\left(\frac{\phi}{2}\right)^\lambda}{\Gamma\lambda} e^{-\frac{\phi}{2}\frac{1}{x}} x^{-\lambda-1} \quad dx$$
$$= \frac{\left(\frac{\phi}{2}\right)^\lambda}{\Gamma\lambda} \int_0^\infty x^{-\lambda+r-1} e^{-\frac{\phi}{2}\frac{1}{x}} \quad dx$$

$$Let \quad x = \frac{1}{y} \Rightarrow dx = -\frac{dy}{y^2}$$

$$E(X^r) = \frac{\left(\frac{\phi}{2}\right)^\lambda}{\Gamma\lambda} \int_0^\infty \frac{1}{y^{-\lambda+r-1}} e^{-\frac{\phi}{2}y} - \frac{dy}{y^2}$$
$$= \frac{\left(\frac{\phi}{2}\right)^\lambda}{\Gamma\lambda} \int_0^\infty \frac{1}{y^{-\lambda+r+1}} e^{-\frac{\phi}{2}y} \quad dy$$
$$= \frac{\left(\frac{\phi}{2}\right)^\lambda}{\Gamma\lambda} \int_0^\infty y^{\lambda-r-1} e^{-\frac{\phi}{2}} \quad dy$$
$$= \frac{\left(\frac{\phi}{2}\right)^\lambda}{\Gamma\lambda} \frac{\Gamma(\lambda-r)}{\left(\frac{\phi}{2}\right)^{\lambda-r}} \tag{3.43}$$

when $\lambda > 1$

$$\therefore E(X) = \frac{\left(\frac{\phi}{2}\right)^\lambda}{\Gamma\lambda} \frac{\Gamma(\lambda-1)}{\left(\frac{\phi}{2}\right)^{\lambda-1}}$$
$$= \frac{\phi}{2} \frac{1}{\lambda-1}$$
$$\therefore E(X^2) = \frac{\left(\frac{\phi}{2}\right)^\lambda}{\Gamma\lambda} \frac{\Gamma(\lambda-2)}{\left(\frac{\phi}{2}\right)^{\lambda-2}}$$
$$= \left(\frac{\phi}{2}\right)^2 \frac{1}{(\lambda-1)(\lambda-1)} \tag{3.44}$$

$$Var(X) = \left(\frac{\phi}{2}\right)^2 \frac{1}{(\lambda-1)(\lambda-2)} - \left(\frac{\phi}{2}\right)^2 \frac{1}{(\lambda-1)(\lambda-1)}$$
$$= \left(\frac{\phi}{2}\right)^2 \frac{1}{\lambda-1}\left[\frac{\lambda-1-\lambda+2}{(\lambda-1)(\lambda-2)}\right]$$
$$= \left(\frac{\phi}{2}\right)^2 \frac{1}{(\lambda-1)^2(\lambda-2)} \quad for \quad \lambda > 2 \tag{3.45}$$

The Laplace Transform is

$$L_X(s) = \frac{\left(\frac{\phi}{2}\right)^\lambda}{\Gamma\lambda}\int_0^\infty e^{-\frac{\phi}{2}\frac{1}{x}-sx}x^{-\lambda-1} \quad dx$$
$$= \frac{\left(\frac{\phi}{2}\right)^\lambda}{\Gamma\lambda}\int_0^\infty x^{-\lambda-1}e^{-\frac{1}{2}(2sx+\frac{\phi}{x})} \quad dx$$

Let

$$x = \sqrt{\frac{\phi}{2s}}z \Rightarrow dx = \sqrt{\frac{\phi}{2s}}dz$$

$$L_x(s) = \frac{\left(\frac{\phi}{2}\right)^\lambda}{\Gamma\lambda}\int_0^\infty \left(\sqrt{\frac{\phi}{2s}}z\right)^{-\lambda-1} \exp\left(-\frac{2s}{2}\left[\sqrt{\frac{\phi}{2s}}z + \frac{\phi}{2s}\frac{1}{\frac{1}{\sqrt{\frac{\phi}{2s}}}}\right]\right)\sqrt{\frac{\phi}{2s}} \quad dz$$
$$= 2\frac{\left(\frac{\phi}{2}\right)^\lambda}{\Gamma(\lambda)}\left(\sqrt{\frac{\phi}{2s}}\right)^{-\lambda-1} K_{-\lambda}(\sqrt{(2s)\phi})$$
$$= 2\left(\frac{\phi}{2}\right)^\lambda\left(\sqrt{\frac{\phi}{2s}}\right)^{-\lambda-1}\frac{K_{-\lambda}\sqrt{(2s)\phi}}{\Gamma(\lambda)}$$
$$= 2\left[\sqrt{\frac{\phi}{2}}\right]^\lambda \frac{K_{-\lambda}(\sqrt{2\psi s})}{\Gamma(\lambda)} \tag{3.46}$$

For modality

$$\frac{d}{dx}\left[x^{-\lambda-1}e^{-\frac{\phi}{2}(\frac{1}{x})}\right] = 0$$
$$(-\lambda+1)x^{-\lambda-2} - \left[-\frac{\phi}{2}\frac{1}{x^2}\right] = 0$$
$$\therefore -(\lambda+1)x^{-(\lambda+2)} + \frac{\phi}{2x^2} = 0$$
$$\frac{-(\lambda+1)}{x^\lambda} + \frac{\phi}{2} = 0$$

$$\therefore \frac{x^{\lambda}}{\lambda+1} = \frac{2}{\phi}$$
$$\therefore x = \left[\frac{2\lambda+2}{\phi}\right]^{\frac{1}{\lambda}} \tag{3.47}$$

3.4.6 Levy Distribution

This is a special case of inverse gamma distribution when

$$v = -\frac{1}{2}, \quad \phi > 0, \quad \psi = 0 \tag{3.48}$$

Therefore $\lambda = \frac{1}{2}$

$$f(x) = \frac{\left(\frac{\phi}{2}\right)^{\frac{1}{2}}}{\Gamma(\frac{1}{2})} e^{-\frac{\phi}{2}\frac{1}{x}} x^{-\frac{1}{2}-1}$$

i.e

$$\begin{aligned} f(x) &= \sqrt{\frac{\phi}{2\pi}} x^{-\frac{\phi}{2}\frac{1}{x}} \\ &= \sqrt{\frac{\phi}{2\pi x^3}} e^{-\frac{\phi}{2}\frac{1}{x}}, \quad x > 0; \quad \phi > 0 \\ E(X^r) &= \left(\frac{\phi}{2}\right)^r \frac{\Gamma(\frac{1}{2}-r)}{\Gamma(\frac{1}{2})} \\ &= \left(\frac{\phi}{2}\right)^r \frac{\Gamma(\frac{1}{2}-r)}{\sqrt{\pi}} \end{aligned} \tag{3.49}$$

$$\begin{aligned} E(X) &= \left(\frac{\phi}{2}\right) \frac{\Gamma(\frac{1}{2}-1)}{\sqrt{\pi}} = \left(\frac{\phi}{2}\right) \frac{\Gamma(-\frac{1}{2})}{\sqrt{\pi}} \\ \therefore E(X) &= \frac{\phi}{2} \frac{\Gamma(-\frac{1}{2}+1)}{-\frac{1}{2}\Gamma(\frac{1}{2})} \\ \therefore E(X) &= -\phi \end{aligned}$$

$$E(X^2) = \left(\frac{\phi}{2}\right)^2 \frac{\Gamma(-\frac{3}{2}+1)}{-\frac{3}{2}\Gamma(\frac{1}{2})}$$

$$\begin{aligned} E(X^2) &= \frac{\phi^2}{3} \\ \therefore Var(X) &= \frac{\phi^2}{3} - (-\phi)^2 = \frac{\phi^2}{3} - \phi^2 = -\frac{2}{3}\phi^2 \end{aligned} \tag{3.50}$$

Alternatively using

$$
\begin{aligned}
Var(x) &= \left(\frac{\phi}{2}\right)^2 \frac{1}{(\lambda-1)^2(\lambda-2)} \\
&= \left(\frac{\phi}{2}\right)^2 \frac{1}{(\frac{1}{2}-1)^2(\frac{1}{2}-2)} \\
&= \frac{\phi^2}{4}\frac{1}{(-\frac{1}{2})(-\frac{3}{2})} \\
&= -\frac{2}{3}\phi^2
\end{aligned}
$$

Remark: $\lambda > 2 \Rightarrow v < -2$ for Var X to exist

Laplace transform is

$$
\begin{aligned}
L_X(s) &= 2\left(\sqrt{\frac{\phi s}{2}}\right)^{\frac{1}{2}} \frac{K_{-\frac{1}{2}}(\sqrt{2\phi s})}{\Gamma(\frac{1}{2})} \\
&= \frac{2}{\sqrt{\pi}}\left(\sqrt{\frac{\phi s}{2}}\right)^{\frac{1}{2}} \left[\frac{\pi}{2\sqrt{2\phi s}}\right]^{\frac{1}{2}} e^{-\sqrt{2\phi s}} \\
&= \left[\frac{4}{\pi}\sqrt{\frac{\phi s}{2}}\frac{\pi}{2\sqrt{2\phi s}}\right]^{\frac{1}{2}} e^{-\sqrt{2\phi s}} \\
&= \left[2\sqrt{\frac{\phi s}{2}\frac{1}{2\phi s}}\right]^{\frac{1}{2}} e^{-\sqrt{2\phi s}} \\
&= e^{-\sqrt{2\phi s}}
\end{aligned} \tag{3.51}
$$

3.4.7 Positive Hyperbolic Distribution

This is the case when

$$
v = 1, \quad \phi > 0, \quad \psi > 0 \tag{3.52}
$$

$$
X \sim GIG(1, \phi, \psi)
$$

So put v=1 in the results obtained in sections 3.2 and 3.3

$$
f(x) = \left(\sqrt{\frac{\psi}{\phi}}\right)\frac{e^{-\frac{1}{2}(\psi x + \frac{\phi}{x})}}{2K_1(\sqrt{\psi\phi})} \quad x > 0; \quad \phi > 0, \quad \psi > 0
$$

This is called a positive hyperbolic distribution.

$$
\begin{aligned}
E(X^r) &= \int_0^\infty x^r \frac{\left(\frac{\psi}{\phi}\right) e^{-\frac{1}{2}(\psi x+\frac{\phi}{x})}}{2K_1(\sqrt{\psi\phi})} \quad dx \\
&= \frac{\left(\frac{\psi}{\phi}\right)}{K_1(\sqrt{\psi\phi})} \frac{1}{2} \int_0^\infty x^r e^{-\frac{1}{2}(\psi x+\frac{\phi}{x})} \quad dx \\
&= \frac{K_{1+r}}{K_1(\sqrt{\psi\phi})} \int_0^\infty \frac{\left(\frac{\psi}{\phi}\right)^{1+r}}{\left(\sqrt{\frac{\psi}{\phi}}\right)^r} x^{1+r-1} \frac{e^{-\frac{1}{2}(\psi x+\frac{\phi}{x})}}{K_{1+r}(\sqrt{\frac{\psi}{\phi}})} \quad dx \\
&= \left(\sqrt{\frac{\phi}{\psi}}\right)^r \frac{K_{1+r}(\sqrt{\psi\phi})}{K_1(\sqrt{\psi\phi})}
\end{aligned} \tag{3.53}
$$

$$
\begin{aligned}
\therefore E(X) &= \left(\sqrt{\frac{\phi}{\psi}}\right) \frac{K_2\left(\sqrt{\psi\phi}\right)}{K_1(\sqrt{\psi\phi})} \\
\therefore E(X^2) &= \left(\sqrt{\frac{\phi}{\psi}}\right)^2 \frac{K_3\left(\sqrt{\psi\phi}\right)}{K_1(\sqrt{\psi\phi})} \\
Var(X) &= \left(\sqrt{\frac{\phi}{\psi}}\right)^2 \left[\frac{K_3\left(\sqrt{\psi\phi}\right)}{K_1(\sqrt{\psi\phi})} - \frac{K_2^2\left(\sqrt{\psi\phi}\right)}{K_1^2(\sqrt{\psi\phi})}\right]
\end{aligned} \tag{3.54}
$$

$$
L_{GIG(1,\phi,\psi)} = \sqrt{\frac{\psi}{2s+\psi}} \frac{K_1(\sqrt{(2s+\psi)\phi})}{K_1(\sqrt{\phi\psi})} \tag{3.55}
$$

Modality

$$
\begin{aligned}
\frac{d}{dx}\left[e^{-\frac{\sqrt{\psi\phi}}{2}(x+\frac{1}{x})}\right] &= 0 \\
-\frac{\sqrt{\psi\phi}}{2}(1-\frac{1}{x^2})e^{-\frac{\sqrt{\psi\phi}}{2}(x+\frac{1}{x})} &= 0 \\
-\sqrt{\psi\phi} + \frac{\sqrt{\psi\phi}}{x^2} &= 0 \\
\sqrt{\psi\phi}x^2 + \sqrt{\psi\phi} &= 0 \\
\therefore X_{mode} &= 1
\end{aligned} \tag{3.56}
$$

3.4.8 Harmonic Distribution

This is a case when

$$v = 0, \quad \phi = an, \quad \psi = \frac{a}{n} \tag{3.57}$$

$$f(x) = \frac{x^{-1}e^{-\frac{a}{2}(\frac{x}{n}+\frac{n}{x})}}{2K_0(a)} \tag{3.58}$$

$$E(X^r) = n^r \frac{K_r(\frac{1}{n})}{K_0(a)} \tag{3.59}$$

$$E(X) = \frac{nK_1(\frac{1}{n})}{K_0(a)}$$

$$VarX = n^2\left[\frac{K_2(\frac{1}{n})}{K_0(a)} - \left[\frac{K_1(\frac{1}{n})}{K_0(a)}\right]^2\right] \tag{3.60}$$

$$L_{GIG(0,an,\frac{a}{n})} = \frac{K_0(\sqrt{2ans+a^2})}{K_0(a)} \tag{3.61}$$

$$X_{mode} = \frac{-1+\sqrt{1+a^2}}{a}$$

$$= \sqrt{\frac{1}{a^2}+1} - \frac{1}{a} \tag{3.62}$$

3.4.9 Generalized Harmonic Distribution

Barndorff-Nielsen (1971) constructed a GIG distribution as follows:-
X and Y are independent gamma II distributed random variables with parametrs $(\zeta, 2\rho^{-1})$ and $(\eta, 2\rho)$ respectively. Their joint pdf is

$$f(x,y) = f(x)f(y) = \frac{1}{(2\rho^{-1})^{\zeta}\Gamma(\zeta)} e^{-\frac{x}{2\rho^{-1}}} x^{\zeta-1} \frac{e^{-\frac{y}{2\rho}} y^{\eta-1}}{(2\rho)^{\eta}\Gamma(\eta)}$$

$$\therefore f(x,y) = \frac{x^{\zeta-1}y^{\eta-1}e^{-\frac{\rho}{2}x-\frac{y}{2\rho}}}{(2\rho)^{\eta-\zeta}\Gamma(\zeta)\Gamma(\eta)} \tag{3.63}$$

Let

$$t = \sqrt{\frac{y}{x}} \quad and \quad u = \sqrt{yx}$$

Therefore

$$ut = \sqrt{\frac{y}{x}yx} = y \quad and \quad \frac{u}{t} = \sqrt{\frac{yx}{y}x} = x$$

$$x = \frac{u}{t} \quad and \quad y = ut \Rightarrow \frac{2u}{t}$$

The above is from Jacobian

The new pdf is

$$
\begin{aligned}
g(u,t) &= f(x,y)|J| \\
&= \frac{x^{\zeta-1}y^{\eta-1}e^{-\frac{\rho}{2}x-\frac{y}{2\rho}}}{(2\rho)^{\eta-\zeta}\Gamma(\zeta)\Gamma(\eta)}\frac{2u}{t} \\
\therefore g(u,t) &= \frac{x^{\zeta-1}ut^{\eta-1}e^{-\frac{\rho}{2}\frac{u}{t}-\frac{ut}{2\rho}}}{(2\rho)^{\eta-\zeta}\Gamma(\zeta)\Gamma(\eta)}\frac{2u}{t} \\
&= \frac{u^{\zeta-1+\eta-1+1}t^{-\zeta+1+\eta-1-1}}{2^{\eta-\zeta-1}\rho^{\eta-\zeta}\Gamma(\zeta)\Gamma(\eta)}\exp\left(-\frac{1}{2}\left[\frac{ut}{\rho}+\frac{\rho u}{t}\right]\right) \\
&= \frac{u^{\eta+\zeta-1}t^{\eta-\zeta-1}}{2^{\eta-\zeta-1}\rho^{\eta-\zeta}\Gamma(\zeta)\Gamma(\eta)}\exp\left(-\frac{1}{2}\left[\frac{ut}{\rho}+\frac{\rho u}{t}\right]\right) \\
\therefore g(u) &= \int_0^\infty g(u,t)dt \Rightarrow g(t/u) = \frac{g(u,t)}{g(u)} \\
g(t/u) &= \frac{t^{\eta-\zeta-1}\exp(-\frac{1}{2}(\frac{ut}{\rho}+\frac{\rho u}{t}))}{\int_0^\infty t^{\eta-\zeta-1}\exp\left(-\frac{1}{2}(\frac{ut}{\rho}+\frac{u\rho}{t})\right)}dt \\
&= \frac{t^{\lambda-1}\exp(-\frac{1}{2}(\frac{ut}{\rho}+\frac{\rho u}{t}))}{\int_0^\infty t^{\lambda-1}\exp\left(-\frac{1}{2}(\frac{ut}{\rho}+\frac{u\rho}{t})\right)}dt
\end{aligned} \tag{3.64}
$$

Considering

$$
\begin{aligned}
I &= \int_0^\infty t^{\lambda-1}\exp\left(-\frac{1}{2}(\frac{ut}{\rho}+\frac{u\rho}{t})\right)dt \\
&= \int_0^\infty t^{\lambda-1}\exp\left(-\frac{u}{2\rho}(t+\frac{\rho^2}{t})\right)dt
\end{aligned}
$$

Let

$$
t = \rho z \Rightarrow dt = \rho dz
$$

$$
\begin{aligned}
I &= \int_0^\infty \rho^\lambda z^{\lambda-1}\exp\left(-\frac{u}{2\rho}(\rho z+\frac{\rho^2}{\rho z})\right)dz \\
&= 2\rho^\lambda\frac{1}{2}\int_0^\infty z^{\lambda-1}\exp\left(-\frac{u}{2}(z+\frac{1}{z})\right)dz \\
&= 2\rho^\lambda K_\lambda(u)
\end{aligned}
$$

$$\therefore g(t|u) = \frac{t^{\lambda-1}\exp\left[-\frac{1}{2}(\frac{u}{\rho}t + \frac{u\rho}{t})\right]}{2\rho^{\lambda}K_{\lambda}(u)}$$
$$= \frac{\left(\frac{1}{\rho}\right)^{\lambda} t^{\lambda-1}\exp\left[-\frac{1}{2}(\frac{ut}{\rho} + \frac{u\rho}{t})\right]}{2K_{\lambda}(u)} \tag{3.66}$$

$$\psi = \frac{u}{\rho} \quad and \quad \phi = u\rho$$

$$\therefore \sqrt{\psi\phi} = \sqrt{\frac{u}{\rho}u\rho} = u$$
$$= \frac{\psi}{\phi}\sqrt{\frac{u}{\rho}\frac{1}{u\rho}} = \frac{1}{\rho}$$

$$T|U = u \sim GIG(\lambda, u\rho, \frac{u}{\rho})$$

when $\lambda > 0$ then

$$T|U = u \sim GIG(0, u\rho, \frac{u}{\rho})$$

which is a harmonic distribution.

$$E(T^r|U = u) = \left(\frac{\phi}{\psi}\right)^r \frac{K_{v+r}\sqrt{\psi\phi}}{K_v(\sqrt{\psi\phi})} \tag{3.67}$$
$$= \rho^r \frac{K_{\lambda+r}(u)}{K_{\lambda}(u)}$$
$$E\left[T|U = u\right] = \rho\frac{K_{\lambda+1}(u)}{K_{\lambda}(u)}$$
$$E\left[T^2|U = u\right] = \rho^2\frac{K_{\lambda+2}(u)}{K_{\lambda}(u)}$$
$$\therefore Var(T|U = u) = \rho^2\left[\frac{K_{\lambda+2}(u)}{K_{\lambda}(u)} - \left[\frac{K_{\lambda+1}(u)}{K_{\lambda}(u)}\right]^2\right] \tag{3.68}$$

3.5 Related Distributions

Using GIG distribution, other distributions can be obtained as follows;

1. Powers of a GIG distributed random variable; the inverse of a GIG random variable is a special case
2. Log of a GIG distributed random variable
3. Convolutions of random variables of GIG(Sums of distributions and its special cases)

4. Sichel distributions which is a mixture of a Poisson and a GIG mixing distribution which has been discussed.

3.5.1 Distribution based on translational transformation

(a) Let

$$Y = rX \tag{3.69}$$

where

$$X \sim GIG(v, \phi, \psi)$$

Then

$$Y \sim GIG(v, r\phi, \frac{\psi}{r})$$

Proof

$$x = \frac{y}{r} \Rightarrow \frac{dx}{dy} = \frac{1}{r}$$

$$\begin{aligned}
g(y) &= f(x)|J| = f(x)\frac{1}{r} \\
\therefore g(y) &= \left(\sqrt{\frac{\psi}{\phi}}\right)^{v}\left(\frac{y}{r}\right)^{v-1}\exp\frac{\left[-\frac{1}{2}\left[\psi(\frac{y}{r}) + \phi(\frac{r}{y})\right]\right]}{2K_v(\sqrt{\psi\phi})}\frac{1}{r} \\
&= \left(\sqrt{\frac{\psi}{\phi}}\right)^{v}\frac{y^{v-1}}{r^{v}}\exp\frac{\left[-\frac{1}{2}\left(\frac{\psi}{r}y + \frac{r\phi}{y}\right)\right]}{2K_v(\sqrt{\psi\phi})} \\
&= \left(\frac{1}{r}\sqrt{\frac{\psi}{\phi}}\right)^{v}\frac{y^{v-1}}{2K_v(\sqrt{\psi\phi})}\exp\left[-\frac{1}{2}\left(\frac{\psi}{r}y + \frac{r\phi}{y}\right)\right] \\
&= \left(\sqrt{\frac{\psi}{r^2\phi}}\right)^{v} y^{v-1}\frac{\exp\left[-\frac{1}{2}\left(\frac{\psi}{r}y + \frac{r\phi}{y}\right)\right]}{2K_v(\sqrt{\frac{\psi}{r}r\phi})} \\
&= \left(\sqrt{\frac{\psi}{r.r\phi}}\right)^{v} y^{v-1}\frac{\exp\left[-\frac{1}{2}\left(\frac{\psi}{r}y + \frac{r\phi}{y}\right)\right]}{2K_v(\sqrt{\frac{\psi}{r}r\phi})}
\end{aligned}$$

$$\therefore Y \sim GIG(v, r\phi, \frac{\psi}{r})$$

1. Let

$$Z = lnX \tag{3.70}$$

where X is a continous random variable

Then

$$E(Z) = E[lnX] = \frac{d}{dt}M_z(t)|_{t=0}$$

where

$$\begin{aligned} M_z(t) &= E(e^{tz}) \\ &= mgf \quad of \quad Z \\ &= E[X^t] \end{aligned}$$

To determine $E\left[\ln X\right]$

$$\begin{aligned} \frac{\partial}{\partial v}K_v(\sqrt{\psi\phi}) &= \frac{1}{2}\int_0^\infty \left[\frac{d}{dv}\left(\sqrt{\frac{\psi}{\phi}}\right)^v x^{v-1}\right] e^{-\frac{1}{2}(\psi x+\frac{\phi}{x})} \quad dx \\ &= \frac{1}{2}\int_0^\infty \left[\left(\sqrt{\frac{\psi}{\phi}}\right)^v \ln\sqrt{\frac{\psi}{\phi}}x^{v-1}\left(\sqrt{\frac{\psi}{\phi}}x^{v-1}\ln X\right)\right] e^{-\frac{1}{2}(\psi x+\frac{\phi}{x})} \\ &= \frac{1}{2}\left(\sqrt{\frac{\psi}{\phi}}\right)^v \left[\int_0^\infty \left(\ln\left(\frac{\psi}{\phi}\right)x^{v-1} + x^{v-1}\ln X\right) e^{-\frac{1}{2}(\psi x+\frac{\phi}{x})}\right] \quad dx \\ &= \ln\sqrt{\frac{\psi}{\phi}}\int_0^\infty \frac{1}{2}\left(\sqrt{\frac{\psi}{\phi}}\right)^v x^{v-1}e^{-\frac{1}{2}(\psi x+\frac{\phi}{x})} \quad dx + \frac{1}{2}\int_0^\infty (\ln X)\left(\sqrt{\frac{\psi}{\phi}}\right)x^{v-1}e^{-\frac{1}{2}(\psi x+ \\ &= \left(\ln\sqrt{\frac{\psi}{\phi}}\right) + K_v(\sqrt{\psi\phi})K_v(\sqrt{\psi\phi})\int_0^\infty \frac{(\ln X)\left(\sqrt{\frac{\psi}{\phi}}\right)x^{v-1}e^{-\frac{1}{2}(\psi x+\frac{\phi}{x})}dx}{2K_v(\sqrt{\psi\phi})} \\ &= \left(\ln\sqrt{\frac{\psi}{\phi}}\right)K_v(\sqrt{\psi\phi}) + K_v(\sqrt{\psi\phi})E[\ln X] \\ \therefore K_v(\sqrt{\psi\phi})E[\ln X] &= (-\ln\sqrt{\frac{\psi}{\phi}})K_v(\sqrt{\psi\phi}) + \frac{\partial}{\partial v}(\sqrt{\psi\phi}) \\ \therefore E[\ln X] &= -\ln\sqrt{\frac{\psi}{\phi}} + \frac{\partial}{\partial v}K_v(\sqrt{\psi\phi}) \\ \therefore E[\ln X] &= \ln\sqrt{\frac{\phi}{\psi}} + \frac{\partial}{\partial v}K_v(\sqrt{\psi\phi}) \end{aligned}$$

2. when

$$X \sim GIG(v,\phi,\psi) \tag{3.71}$$

then

$$E(X^r) = \left(\frac{\phi}{\psi}\right)^{\frac{r}{2}}\frac{K_{v+r}(\sqrt{\psi\phi})}{K_v(\sqrt{\psi\phi})}$$

and

$$E[lnX] = \frac{d}{dt}E[X^t]|_{t=0}$$

Alternatively, let

$$Z = \ln X \Rightarrow M_z(t) = E[e^{tz}] = E[e^{t\ln x}]$$

$$\begin{aligned}
\therefore M_z(t) &= E[X^t] \\
&= \int_0^\infty x^t \frac{\left(\sqrt{\frac{\psi}{\phi}}\right)^v x^{v-1} e^{-\frac{1}{2}(\psi x + \frac{\phi}{x})} dx}{2K_v(\sqrt{\psi\phi})} \\
&= \frac{\left(\sqrt{\frac{\psi}{\phi}}\right)^v}{2K_v(\sqrt{\psi\phi})} \int_0^\infty x^{t+v-1} e^{-\frac{1}{2}(\psi x + \frac{\phi}{x})} dx \\
&= \frac{K_{t+v}(\sqrt{\psi\phi})}{K_v\sqrt{\psi\phi}} \frac{\left(\sqrt{\frac{\psi}{\phi}}\right)^v}{\left(\sqrt{\frac{\psi}{\phi}}\right)^{t+v}} \\
\therefore M_z(t) &= \left(\sqrt{\frac{\phi}{\psi}}\right)^t \frac{K_{t+v}(\sqrt{\psi\phi})}{K_v(\sqrt{\psi\phi})} \\
\frac{d}{dt}M_z(t) &= \frac{1}{K_v(\sqrt{\psi\phi})}\left[\left(\sqrt{\frac{\phi}{\psi}}\right)^t \ln\sqrt{\frac{\phi}{\psi}} K_{t+v}(\sqrt{\psi\phi}) + \left(\sqrt{\frac{\phi}{\psi}}\right)^t \frac{d}{dt}K_{t+v}(\sqrt{\psi\phi})\right] \\
E[\ln X] &= \frac{d}{dt}M_z(t)|_{t=0} \\
&= \frac{1}{K_v(\sqrt{\psi\phi})}\left[(\ln\sqrt{\frac{\phi}{\psi}})K_v(\sqrt{\psi\phi}) + \frac{d}{dt}K_v(\sqrt{\psi\phi})\right] \\
E[\ln X] &= \ln\sqrt{\frac{\phi}{\psi}} + \frac{\frac{d}{dt}K_v(\sqrt{\psi\phi})}{K_v(\sqrt{\psi\phi})}
\end{aligned}$$

3. when

$$X \sim GIG(v, 0, \psi) = \Gamma(v, \frac{\psi}{2}) \tag{3.72}$$

then

$$E[lnX] = \psi(t) - ln\frac{\psi}{2}$$

where

$$\psi(t) = \frac{\Gamma'(t)}{\Gamma(t)} = \frac{d}{dt}\log\Gamma(t) = digamma \quad function$$

The pdf of a Gamma Distribution is

$$f(x) = \frac{\left(\frac{\psi}{2}\right)^v}{\Gamma(v)} e^{-\frac{\psi}{2}x}, \quad x > 0; \psi > 0$$

$$
\begin{aligned}
\therefore f(x)dx &= \int_0^\infty \frac{x^{v-1}\left(\frac{\psi}{2}\right)^v e^{-\frac{\psi}{2}x}}{\Gamma(v)} dx \\
\therefore 1 &= \int_0^\infty \frac{x^{v-1}\left(\frac{\psi}{2}\right)^v e^{-\frac{\psi}{2}x}}{\Gamma(v)} dx \\
\therefore \Gamma(v) &= \int_0^\infty \left(\frac{\psi}{2}\right)^v x^{v-1} e^{-\frac{\psi}{2}x} dx \\
\therefore \frac{d}{dv}\Gamma(v) &= \int_0^\infty \frac{d}{dv}\left[\left(\frac{\psi}{2}\right)^v x^{v-1}\right] e^{-\frac{\psi}{2}x} dx \\
&= \int_0^\infty \left[\left(\frac{\psi}{2}\right)^v \ln\frac{\psi}{2} x^{v-1} + x^{v-1} + x^{v-1}\ln X \left(\frac{\psi}{2}\right)^v\right] e^{-\frac{\psi}{2}x} dx \\
&= \left(\frac{\psi}{2}\right)^v \left[\int_0^\infty (\ln\frac{\psi}{2}) x^{v-1} e^{-\frac{\psi}{2}x} dx + \int_0^\infty (\ln X) x^{v-1} e^{-\frac{\psi}{2}x} dx\right] \\
&= \left(\frac{\psi}{2}\right)^v \left(\ln\frac{\psi}{2}\right) \frac{\Gamma(v)}{\left(\frac{\psi}{2}\right)^v} + \Gamma(v) E[\ln X] \\
\therefore \frac{d}{dv} &= \Gamma(v)\ln\left(\frac{\psi}{2}\right) + \Gamma(v)E[\ln X] \\
\Gamma(v)E[\ln X] &= -\Gamma(v)\ln\left(\frac{\psi}{2}\right) + \frac{d}{dv}\Gamma(v) \\
E[\ln X] &= -\ln\left(\frac{\psi}{2}\right) + \frac{\frac{d}{dv}\Gamma(v)}{\Gamma(v)} \\
&= -\ln\left(\frac{\psi}{2}\right) + \psi(v) \\
\therefore E[\ln X] &= \psi(v) - \psi(v)
\end{aligned}
$$

where

$$\psi(v) = \frac{\Gamma'(v)}{\Gamma(v)}$$

is a digamma function.

Alternatively,

$$\begin{aligned}
M_z(t) &= E[X^t] \\
&= \int_0^\infty x^t \frac{\left(\frac{\psi}{2}\right)^v}{\Gamma(v)} e^{-\frac{\psi}{2}x} x^{v-1} dx \\
&= \frac{\left(\frac{\psi}{2}\right)^v}{\Gamma(v)} \int_0^\infty t^{t+v-1} e^{-\frac{\psi}{2}x} dx \\
&= \frac{\left(\frac{\psi}{2}\right)^v}{\Gamma(v)} \frac{\Gamma(t+v)}{\left(\frac{\psi}{2}\right)^{t+v}} \\
\frac{d}{dt} M_z(t) &= \frac{1}{\Gamma(v)} \left[\left(\frac{2}{\psi}\right)^t \ln \frac{2}{\psi} \Gamma(t+v) + \left(\frac{2}{\psi}\right)^t \frac{d}{dt} \Gamma(t+v) \right] \\
&= \ln \frac{\psi}{2} + \frac{\frac{d}{dt}\Gamma(v)}{\Gamma(v)} \\
\therefore E[\ln X] &= \frac{\frac{d}{dt}\Gamma(v)}{\Gamma(v)} - \ln \frac{\psi}{2}
\end{aligned}$$

$$\begin{aligned}
\therefore E[\ln Z] &= \frac{d}{dt} M_z(t)|_{t=0} \\
i.e \quad E[\ln X] &= \frac{1}{\Gamma(v)} \left[\Gamma(v) \ln \frac{2}{\psi} + \frac{d}{dt} \Gamma(v) \right]
\end{aligned}$$

4. when

$$X \sim GIG(v, \phi, 0) = Inverse \quad Gamma(v, \frac{\phi}{2}) \tag{3.73}$$

then

$$E[lnX] = \ln \frac{\phi}{2} - \psi(v)$$

Proof

$$\begin{aligned}
f(x) &= \frac{\left(\frac{\phi}{2}\right)^{-v}}{\Gamma(-v)} e^{-\frac{\phi}{2}\frac{1}{x}} x^{v-1} \\
\therefore \Gamma(-v) &= \int_0^\infty \left(\frac{\phi}{2}\right)^{-v} x^{v-1} e^{-\frac{\phi}{2}\frac{1}{x}} dx \\
\frac{d}{dv}\Gamma(v) &= \int_0^\infty \left[-\left(\frac{\phi}{2}\right)^{-v} \left(\ln \frac{\phi}{2}\right) x^{v-1} (\ln X) \left(\frac{\phi}{2}\right)^{-1} \right] e^{-\frac{\phi}{2}\frac{1}{x}} dx \\
&= -\left(\frac{\phi}{2}\right)^{-v} \ln \frac{\phi}{2} \int_0^\infty x^{v-1} e^{-\frac{\phi}{2}\frac{1}{x}} dx + \left(\frac{\phi}{2}\right)^{-v} \int_0^\infty x^{v-1} (\ln X) e^{-\frac{\phi}{x}\frac{1}{x}} dx \\
&= -\left(\ln \frac{\phi}{2}\right) \Gamma(-v) + \Gamma(-v) \left(\frac{\psi}{2}\right)^{-v} \frac{\int_0^\infty (\ln X) x^{v-1} e^{-\frac{\phi}{2}\frac{1}{x}}}{\Gamma(-v)} \\
&= -\left(\ln \frac{\phi}{2}\right) \Gamma(-v) + \Gamma(-v) E[\ln X]
\end{aligned}$$

$$
\begin{aligned}
\therefore E[\ln X] &= \frac{d}{dv}\Gamma(-v) + \ln\frac{\phi}{2} \\
E[\ln X] &= \frac{\frac{d}{dv}\Gamma(-v)}{\Gamma(-v)} + \ln\frac{\phi}{2} \\
&= -\frac{\Gamma'(-v)}{\Gamma(-v)} + \ln\frac{\phi}{2} \\
\therefore E[\ln X] &= \ln\frac{\phi}{2} - \psi(-v)
\end{aligned}
$$

Alternatively,

$$
\begin{aligned}
M_z(t) = E[X^t] &= \int_0^\infty x^t \frac{\left(\frac{\phi}{2}\right)^{-v}}{\Gamma(-v)} e^{-\frac{\phi}{2}\frac{1}{x}} x^{v-1} dx \\
&= \frac{1}{\Gamma(-v)}\left(\frac{\phi}{2}\right)^{-v} \int_0^\infty x^{t+v-1} e^{-\frac{\phi}{2}\frac{1}{x}} dx
\end{aligned}
$$

Let

$$
y = \frac{\phi}{2}\frac{1}{x} \Rightarrow x = \frac{\phi}{2y} \quad and \quad dx = -\frac{\phi}{2y^2}dy
$$

$$
\begin{aligned}
\therefore M_z(t) = E[X^t] &= \frac{1}{\Gamma(-v)}\left(\frac{\phi}{2}\right)^{t+v-1} e^{-y}\frac{\phi}{2}\frac{dy}{y^2} \\
&= \frac{1}{\Gamma(-v)}\left(\frac{\phi}{2}\right)^{-v}\left(\frac{\phi}{2}\right)^{t+v} \int_0^\infty \frac{1}{y^{t+v+1}} e^{-y} dy \\
\therefore M_z(t) &= \frac{1}{\Gamma(-v)}\left(\frac{\phi}{2}\right)^{-v}\left(\frac{\phi}{2}\right)^{t+v} \Gamma(-t-v) \\
&= \left(\frac{\phi}{2}\right)^{t} \frac{\Gamma(-t-v)}{\Gamma(-v)} \\
\Gamma(-v)\frac{d}{dt}M_z(t) &= \left(\frac{\phi}{2}\right)^{t}\left(\ln\frac{\phi}{2}\right)\Gamma(-t-v) + \left(\frac{\phi}{2}\right)^{t}\frac{d}{dt}\Gamma(-t-v) \\
\Gamma(-v)E[\ln X] &= \frac{d}{dt}M_z(t)|_{t=0} \\
&= \left(\ln\frac{\phi}{2}\right)\Gamma(-v) - \frac{d}{dt}\Gamma(-v) \\
\therefore E[\ln X] &= \ln\frac{\phi}{2} - \frac{\frac{d}{dt}\Gamma(-v)}{\Gamma(-v)}
\end{aligned}
$$

3.5.2 The Inverse of a GIG distributed Random variable $(Y = X^{-1})$

Let $Y = \frac{1}{X}$ where $X \sim GIG(v, \psi\phi)$
then

$$
Y = \frac{1}{X} \sim GIG(-v, \psi, \phi) \tag{3.74}
$$

Proof

$X = \frac{1}{Y} \Rightarrow \frac{dx}{dy} = -\frac{1}{y^2} \Rightarrow |J| = \frac{1}{y^2}$

$$\begin{aligned}
g(y) &= f(x)|J| \\
&= \frac{\left(\frac{\psi}{\phi}\right)^{\frac{v}{2}}}{2K_v(\sqrt{\psi\phi})} x^{v-1}|J| \exp\left(-\frac{1}{2}(\psi x + \frac{\phi}{x})\right) \\
&= \frac{\left(\frac{\psi}{\phi}\right)^{\frac{v}{2}}}{2K_v(\sqrt{\psi\phi})} \left(\frac{1}{y}\right)^{v-1} |J| \exp\left(-\frac{1}{2}(\psi x + \frac{\phi}{x})\right) \\
&= \frac{\left(\frac{\psi}{\phi}\right)^{\frac{v}{2}}}{2K_v(\sqrt{\psi\phi})} \left(\frac{1}{y}\right)^{v-1} \left(\frac{1}{y^2}\right) \exp\left(-\frac{1}{2}(\psi x + \frac{\phi}{x})\right) \\
&= \frac{\left(\frac{\psi}{\phi}\right)^{\frac{v}{2}}}{2K_v(\sqrt{\psi\phi})} \left(\frac{1}{y}\right)^{v-1} \left(\frac{1}{y^2}\right) \exp\left(-\frac{1}{2}(\psi \frac{1}{y} + \phi y)\right) \\
&= \frac{\left(\frac{\psi}{\phi}\right)^{\frac{v}{2}}}{2K_v(\sqrt{\psi\phi})} y^{-v-1} \exp\left(-\frac{1}{2}(\psi \frac{1}{y} + \phi y)\right) \\
&= \left(\sqrt{\frac{\psi}{\phi}}\right)^{v} \frac{y^{-v-1} \exp\left(-\frac{1}{2}(\phi y) + \frac{\psi}{y}\right)}{2K_{-v}(\sqrt{\psi\phi})}
\end{aligned} \tag{3.75}$$

Therefore
$Y = \frac{1}{X} \sim GIG(-v, \psi, \phi)$

$$
\begin{aligned}
E[Y^r] &= \int_0^\infty \frac{y^r\left(\sqrt{\frac{\phi}{\psi}}\right)^{-v} y^{-v-1}}{2K_{-v}(\sqrt{\phi\psi})} \exp\left(-\frac{1}{2}(\phi y + \frac{\psi}{y})\right) \quad dy \\
&= \frac{\left(\sqrt{\frac{\phi}{\psi}}\right)^{-v}}{2K_{-v}(\sqrt{\phi\psi})} \int_0^\infty \frac{\left(\sqrt{\frac{\phi}{\psi}}\right)^{r-v} y^{r-v-1}}{\left(\sqrt{\frac{\phi}{\psi}}\right)^{r-v}} \exp\left(-\frac{1}{2}(\phi y + \frac{\psi}{y})\right) \quad dy \\
&= \left(\sqrt{\frac{\phi}{\psi}}\right)^{-r} \frac{K_{r-v}(\sqrt{\phi\psi})}{K_{-v}(\sqrt{\phi\psi})} \\
&= \left(\sqrt{\frac{\phi}{\psi}}\right)^{-r} \frac{K_{-v+r}(\sqrt{\phi\psi})}{K_{-v}(\sqrt{\phi\psi})} \\
&= \left(\sqrt{\frac{\psi}{\phi}}\right)^{r} \frac{K_{-v+r}(\sqrt{\phi\psi})}{K_{-v}(\sqrt{\phi\psi})} \\
\therefore E(Y) &= \left(\sqrt{\frac{\psi}{\phi}}\right)^{1} \frac{K_{-v+1}(\sqrt{\phi\psi})}{K_{-v}(\sqrt{\phi\psi})} = \left(\sqrt{\frac{\phi}{\psi}}\right)^{-1} \frac{K_{-v+1}(\sqrt{\phi\psi})}{K_{-v}(\sqrt{\phi\psi})} \\
E(Y^2) &= \left(\sqrt{\frac{\phi}{\psi}}\right)^{-2} \frac{K_{-v+2}(\sqrt{\phi\psi})}{K_{-v}(\sqrt{\phi\psi})} \\
\therefore VarY &= \left(\sqrt{\frac{\phi}{\psi}}\right) \left[\frac{K_{-v+2}(\sqrt{\phi\psi})}{K_{-v}(\sqrt{\phi\psi})} - \left[\frac{K_{-v+1}(\sqrt{\phi\psi})}{K_{-v}(\sqrt{\phi\psi})}\right]^2\right]
\end{aligned}
\tag{3.76}
$$

The Laplace transform:

$$
\begin{aligned}
L_Y(s) &= E[e^{-sY}] \\
&= \int_0^\infty e^{-sY} \left(\sqrt{\frac{\phi}{\psi}}\right)^{-v} \frac{y^{-v-1}}{2K_v(\sqrt{\phi\psi})} \exp\left(-\frac{1}{2}(\phi y + \frac{\psi}{y})\right) \quad dy \\
&= \frac{\left(\sqrt{\frac{\psi}{\phi}}\right)^{v}}{2K_v(\sqrt{\phi\psi})} \int_0^\infty \frac{\left(\frac{\psi}{2s+\phi}\right)^{v}}{\left(\frac{\psi}{\phi+2s}\right)^{v}} y^{-v-1} \exp\left[-\frac{1}{2}\left[(2s+\phi)y + \frac{\psi}{y}\right]\right] \quad dy \\
L_Y(s) &= \left(\sqrt{\frac{2s+\phi}{\phi}}\right)^{v} \frac{K_v(\sqrt{(2s+\phi)\psi})}{K_v(\sqrt{\phi\psi})}
\end{aligned}
\tag{3.77}
$$

Modality

$$\frac{df}{dy} = 0$$

$$\frac{d}{dy}\left(y^{-v-1} e^{-\frac{1}{2}(\phi y + \frac{\psi}{y})}\right) = 0$$

$$(-v-1)y^{-v-2}e^{-\frac{1}{2}(\phi y+\frac{\phi}{y})}+y^{-v-1}e^{-\frac{1}{2}(\phi y+\frac{\psi}{y})}=0$$
$$\frac{d}{dy}\left(y^{-v-1}e^{-\frac{1}{2}(\phi y+\frac{\psi}{y})}\right)=0$$
$$(-v-1)-y\frac{\phi}{2}+\frac{\psi}{2y}=0$$
$$2(v+1)y-\phi y^2+\psi=0$$
$$\phi y^2+2(v+1)y-\psi=0$$

$$y=\frac{-2(v+1)\pm\sqrt{4(v+1)^2+4\psi\phi}}{2\phi}$$
$$y=\frac{-(v+1)\pm\sqrt{(v+1)^2+\psi\phi}}{\phi}$$
$$since \quad y>0 \quad y=\frac{-(v+1)+\sqrt{(v+1)^2+\psi\phi}}{\phi}$$

The mode is

$$y=\frac{-(v+1)+\sqrt{(v+1)^2+\psi\phi}}{\phi}, \quad \phi>0$$
$$and \quad y=\frac{\psi}{2(v+1)}, \quad when \quad \phi=0 \tag{3.78}$$

Remark: The results of Y can be obtained from the results of X and by replacing v with -v and interchanging ϕ and ψ

3.5.3 The Power of a GIG-Distributed Random Variable $(Y=X^\theta)$

If X is a GIG distribution with parameters (v,ψ,ϕ), then $Y=X^\theta$, has an Extended-GIG distribution with parameters $v,\psi,\phi,and\theta$

1. Using the change of variable technique.

$$Y=X^\theta$$

where

$$X\sim GIG(v,\phi,\psi) \quad and \quad \infty<\theta<\infty$$

$$\theta X^{\theta-1}dx=dy$$
$$\frac{dy}{dx}=\theta X^{\theta-1}$$
$$\frac{dx}{dy}=\frac{1}{\theta}y^{\frac{1}{\theta}-1}$$

$$
\begin{aligned}
f(x) &= \frac{\left(\sqrt{\frac{\psi}{\phi}}\right)^{v}}{2K_v(\sqrt{\psi\phi})} x^{v-1} e^{-\frac{1}{2}(\psi x + \frac{\phi}{x})}, \quad x > 0 \\
g(y) &= f(x)|J| \\
&= \frac{\left(\sqrt{\frac{\psi}{\phi}}\right)^{v}}{2K_v(\sqrt{\psi\phi})} x^{v-1} e^{-\frac{1}{2}(\psi x + \frac{\phi}{x})} |\frac{1}{\theta} y^{\frac{1}{\theta}-1}| \\
&= \frac{1}{\theta} \frac{\left(\sqrt{\frac{\psi}{\phi}}\right)^{v}}{2K_v(\sqrt{\psi\phi})} y^{\frac{1}{\theta}(v-1)} y^{\frac{1}{\theta}-1} e^{-\frac{1}{2}(\psi y^{\frac{1}{\theta}+\frac{\phi}{y^{\frac{1}{\theta}}}})} \\
&= \frac{1}{\theta} \frac{\left(\sqrt{\frac{\psi}{\phi}}\right)^{v}}{2K_v(\sqrt{\psi\phi})} y^{\frac{v}{\theta}-\frac{1}{\theta}+\frac{1}{\theta}-1} e^{-\frac{1}{2}(\psi y^{\frac{1}{\theta}+\frac{\phi}{y^{\frac{1}{\theta}}}})} \\
g(y) &= \frac{1}{\theta} \frac{\left(\sqrt{\frac{\psi}{\phi}}\right)^{v}}{2K_v(\sqrt{\psi\phi})} y^{\frac{v}{\theta}-1} e^{-\frac{1}{2}(\psi y^{\frac{1}{\theta}} + \frac{\phi}{y^{\frac{1}{\theta}}})} \qquad \textbf{(3.79)}
\end{aligned}
$$

This is a Power-GIG distribution. It has four parameters contrary to the GIG distribution itself. The parameters are $(\lambda, \frac{1}{\theta}, \psi, \phi)$

Proof 2: Using Cummulative Distribution Technique.

Using the cmmulative distribution technique, we let

$$Y = X^{\theta}$$

$$
\begin{aligned}
F(x) &= Prob(X \le x) \\
G(y) &= Prob(Y \le y) \\
G(y) &= Prob(Y \le y) \\
&= Prob(X^{\theta} \le y) \\
&= Prob(X \le y^{\frac{1}{\theta}}) \\
g(y) &= \frac{1}{\theta} f(y^{\frac{1}{\theta}-1})
\end{aligned}
$$

But

$$f(x) = \frac{\left(\sqrt{\frac{\psi}{\phi}}\right)^{v}}{2K_v(\sqrt{\psi\phi})} x^{v-1} e^{-\frac{1}{2}(\psi x + \frac{\phi}{x})}, \quad x > 0$$

$$\therefore g(y) = \left(\sqrt{\frac{\psi}{\phi}}\right)^{v} \frac{1}{2K_v(\sqrt{\psi\phi})} \frac{1}{\theta} y^{\frac{1}{\theta}(v-1)} y^{\frac{1}{\theta}-1} e^{-\frac{1}{2}(\psi y^{\frac{1}{\theta} + \frac{\phi}{y^{\frac{1}{\theta}}}})}$$

$$= \frac{1}{\theta} \left(\sqrt{\frac{\psi}{\phi}}\right)^{v} \frac{1}{2K_v(\sqrt{\psi\phi})} y^{\frac{v}{\theta}-1} e^{-\frac{1}{2}(\psi y^{\frac{1}{\theta}} + \frac{\phi}{y^{\frac{1}{\theta}}})}$$

This is the power GIG distribution with the four parametrs.

$$g(y) = \left(\sqrt{\frac{\psi}{\phi}}\right)^{v} \frac{\frac{1}{\theta}}{2K_v(\sqrt{\phi\psi})} y^{(\frac{1}{\theta})v-1} \exp\left(-\frac{1}{2}(\psi y^{\frac{1}{\theta}} + \frac{\phi}{y^{\frac{1}{\theta}}})\right)$$

$$\therefore E(Y^r) = \int_0^{\infty} y^r f(y) dy$$

$$= \int_0^{\infty} y^r \left(\sqrt{\frac{\psi}{\phi}}\right)^{v} \frac{\frac{1}{\theta}}{2K_v(\psi\phi)} y^{(\frac{1}{\theta})v-1} \exp\left(-\frac{1}{2}(\psi y^{\frac{1}{\theta}} + \frac{\phi}{y^{\frac{1}{\theta}}})\right) \quad dy$$

$$= \int_0^{\infty} \left(\sqrt{\frac{\psi}{\phi}}\right)^{v} \frac{\frac{1}{\theta}}{2K_v(\psi\phi)} y^{\frac{1}{\theta}v+r-1} \exp\left(-\frac{1}{2}(\psi y^{\frac{1}{\theta}} + \frac{\phi}{y^{\frac{1}{\theta}}})\right) \quad dy$$

$$= \left(\sqrt{\frac{\phi}{\psi}}\right)^{r\theta} \frac{K_{v+r\theta}(\sqrt{\psi\phi})}{K_v(\sqrt{\psi\phi})}$$

$$\therefore E(Y^r) = \left(\sqrt{\frac{\psi}{\phi}}\right)^{r\theta} \frac{K_{v+r\theta}(\sqrt{\psi\phi})}{K_v(\sqrt{\psi\phi})} \tag{3.80}$$

The Modality

$$\frac{d}{dy} f(y) = 0$$

$$\frac{d}{dy}\left[\left(\sqrt{\frac{\psi}{\phi}}\right)^{v} \frac{\frac{1}{\theta}}{2K_v} y^{\frac{1}{\theta}v-1} \exp\left(-\frac{1}{2}(\psi y^{\frac{1}{\theta}} + \frac{\phi}{y^{\frac{1}{\theta}}})\right)\right] = 0$$

$$\frac{d}{dy}\left[\frac{1}{\theta} y^{\frac{1}{\theta}v} \exp\left(-\frac{\sqrt{\psi\phi}}{2}(y^{\frac{1}{\theta}} + \frac{1}{y^{\frac{1}{\theta}}})\right)\right] = 0$$

$$\frac{1}{\theta}\left((v\frac{1}{\theta} - 1) y^{v\frac{1}{\theta}-2} - y^{\frac{1}{\theta}v-1} \frac{\sqrt{\psi\phi}}{2}(\frac{1}{\theta} y^{\frac{1}{\theta}-1} + \frac{1}{\theta} y^{-\frac{1}{\theta}-1})\right) = 0$$

$$Y_{mode} = \frac{2(v\frac{1}{\theta^2} - \frac{1}{\theta}) \pm \sqrt{4(v\frac{1}{\theta^2} - \frac{1}{\theta})^2 + 4\sqrt{\psi\phi}\frac{1}{\theta^2}(\sqrt{\psi\phi}\frac{1}{\theta^2})}}{2(\sqrt{\psi\phi}\frac{1}{\theta^2})} \tag{3.81}$$

3.6 Convolutions of a class of a GIG Distribution

Convolution arises as the operation interms of pdf that corresponds to the addition of independent random variables to forming linear combinations of random variables.
Let $Z = X + Y$ where X and Y are independent continous random variable. The problem is to find the distribution of Z.

Method 1: Convolution Approach

$$g(z) = \frac{d}{dz}G(z) = \int_0^z f_1(z-y)f_2(y)dy \tag{3.82}$$

Where $f_1(x)$ and $f_2(y)$ are the pdfs of X and Y.

Proof

$$\begin{aligned}
Z &= X + Y \\
\therefore G(z) &= Prob\left[Z \le z\right] \\
&= Prob\left[X + Y \le z\right] \\
&= Prob\left[x \le z - y\right] \\
&= Prob\left[x \le z - y; 0 < y < z\right] \\
&= \int_0^z \int_0^{z-y} f_1(x)f_2(y)dxdy \\
&= \int_0^z \left[\int_0^{z-y} f_1(x)dx\right] f_2(y)dy \\
&= \int_0^z F_1(z-y)f_2(y)dy
\end{aligned}$$

$$g(z) = \frac{d}{dz}G(z) = \int_0^z f_1(z-y)f_2(y)dy \tag{3.83}$$

Method 2: Laplace Transform Technique

$$L_Z(s) = L_X(s)L_Y(s) \tag{3.84}$$

Proof

$$
\begin{aligned}
L_Z(s) &= E\left[e^{sZ}\right] \\
&= E\left[e^{s(X+Y)}\right] \\
&= E\left[e^{sX} + e^{sY}\right] \\
&= E(e^{sX})E(e^{sY}) \\
&= L_X(s)L_Y(s)
\end{aligned}
$$

Using the Laplace Technique, we shall prove the following proposition;

Proposition (3.3)

Let X_1 and X_2 be two independent gamma random variables with parameters (v_1, ψ) and (v_2, ψ), the sum is also a Gamma with parameters $v_1 + v_2$ and ψ

$$L_{GIG(v_1,0,\psi)} * L_{GIG(v_2,0,\psi)} = L_{GIG(v_1+v_2,0,\psi)} \tag{3.85}$$

Proof

$$Z = X + Y$$

Where

$$X \sim GIG(v_1, 0, \psi) \quad and \quad Y \sim GIG(v_2, 0, \psi)$$

$$
\begin{aligned}
L_{GIG(v_1,0,\psi)} * L_{GIG(v_2,0,\psi)} &= \left(\frac{\frac{\psi}{2}}{\frac{\psi}{2}+s}\right)^{v_1} * \left(\frac{\frac{\psi}{2}}{\frac{\psi}{2}+s}\right)^{v_2} \\
&= \left(\frac{\frac{\psi}{2}}{\frac{\psi}{2}+s}\right)^{v_1+v_2} \quad as \quad from \quad (3.32) \\
\therefore &= L_{GIG(v_1+v_2,0,\psi)}
\end{aligned}
$$

Therefore, the product of the Laplaces of two gamma distributions is the Laplace of two gamma distributions.

Proposition (3.4)

$$L_{GIG(-v,\phi,\psi)} * L_{GIG(v,0,\psi)} = L_{GIG(v,\phi,\psi)} \tag{3.86}$$

$$X \sim GIG(-v, \phi, \psi) \quad and \quad Y \sim GIG(v, 0, \psi)$$

From (3.32) and (3.77)

Proof

$$\begin{aligned}
L_{GIG(-v,\phi,\psi)} * L_{GIG(v,0,\psi)} &= \left(\sqrt{\frac{\psi}{\psi+2s}}\right)^{-v} \frac{K_{-v}(\sqrt{\phi(\psi+2s)})}{K_{-v}(\sqrt{\psi\phi})} * \left(\frac{\frac{\psi}{2}}{\frac{\psi}{2}+s}\right)^{v} \\
&= \left(\sqrt{\frac{\psi}{\psi+2s}}\right)^{-v} \frac{K_{-v}\sqrt{\phi(2s+\psi)}}{K_{-v}(\sqrt{\psi\phi})} \left(\frac{\psi}{\psi+2s}\right)^{v} \\
&= \left(\sqrt{\left(\frac{\psi+2s}{\psi}\right)\left(\frac{\psi}{2s+\psi}\right)^{2}}\right)^{v} \frac{K_{-v}(\sqrt{\phi(2s+\psi)})}{K_{-v}(\sqrt{\psi\phi})} \\
&= \left(\sqrt{\frac{\psi}{2s+\psi}}\right)^{v} \frac{K_{v}(\sqrt{\phi(2s+\psi)})}{K_{v}(\sqrt{\psi\phi})} \qquad \textbf{(3.87)} \\
&= L_{GIG(v,\phi,\psi)}
\end{aligned}$$

Therefore, the product of the Laplace transforms of the GIG distribution with parameter -v and the Laplace of a gamma, is the Laplace transform of a GIG distribution with parameters (v,ϕ,ψ)

Note: $GIG(-v,\psi,\phi) \neq GIG(-v,\phi,\psi)$

Proposition (3.5)

$$L_{GIG(-\frac{1}{2},\phi_1,\psi)} * L_{GIG(-\frac{1}{2},\phi_2,\psi)} = L_{GIG(-\frac{1}{2},\phi_1+\phi_2,\psi)} \qquad \textbf{(3.88)}$$

Proof

$$\begin{aligned}
L_X(s) &= e^{\sqrt{\psi\phi}-\sqrt{\phi(\psi+2s)}} \quad from \quad (3.18) \\
&= e^{\sqrt{\phi}(\sqrt{\psi}-\sqrt{\psi+2s})} \\
\therefore L_z(s) &= L_x(s)L_y(s) \\
&= e^{\sqrt{\phi_1}(\sqrt{\psi}-\sqrt{\psi+2s})} e^{\sqrt{\phi_2}(\sqrt{\psi}-\sqrt{\psi+2s})} \\
&= e^{(\sqrt{\phi_1}+\sqrt{\phi_2})(\sqrt{\psi}-\sqrt{\psi+2s})} \qquad \textbf{3.89}
\end{aligned}$$

This is the Laplace Transform of $Z \sim GIG(-\frac{1}{2},\phi_1+\phi_2,\psi)$

Proposition (3.6)

$$L_{GIG(-\frac{1}{2},\phi_1,\psi)} * L_{GIG(\frac{1}{2},\phi_2,\psi)} = L_{GIG(\frac{1}{2},\phi_1+\phi_2,\psi)} \qquad \textbf{(3.90)}$$

Proof

$$
\begin{aligned}
L_X(s) &= L_{GIG}(-\frac{1}{2},\phi,\psi) \quad from \quad (3.18)\\
&= e^{\sqrt{\phi_1}(\sqrt{\psi}-\sqrt{\psi+2s})}\\
L_Y(s) &= \left(\frac{\psi}{\psi+2s}\right)^{\frac{1}{2}} e^{\sqrt{\phi_2}(\sqrt{\psi}-\sqrt{\psi+2s})} \quad from \quad (3.25)\\
\therefore L_Z(s) &= L_X(s)L_Y(s)\\
&= e^{\sqrt{\phi_1}(\sqrt{\psi}-\sqrt{\psi+2s})}\left(\frac{\psi}{\psi+2s}\right)^{\frac{1}{2}} e^{\sqrt{\phi_2}(\sqrt{\psi}-\sqrt{\psi+2s})}\\
&= \left(\frac{\psi}{\psi+2s}\right)^{\frac{1}{2}} e^{\sqrt{\phi_1}(\sqrt{\psi}-\sqrt{\psi+2s})}e^{\sqrt{\phi_2}(\sqrt{\psi}-\sqrt{\psi+2s})}\\
&= \left(\frac{\psi}{\psi+2s}\right)^{\frac{1}{2}} e^{(\sqrt{\phi_1}+\sqrt{\phi_2})(\sqrt{\psi}-\sqrt{\psi+2s})}
\end{aligned}
$$

Which is a Laplace transform of $Z \sim GIG(\frac{1}{2},\phi_1+\phi_2,\psi)$

3.6.1 A GIG distribution as a mixing distribution

In this case, GIG is used as a mixing distribution. Poisson and the GIG distribution are mixed giving rise to a distribution called Sichel' Distribution.

$$f(x/\lambda) = \frac{(\exp(-\lambda t))(\lambda t)^x}{x!}, x = 0,1,2,...,\lambda > 0$$

$$g(\lambda) = \frac{\left(\sqrt{\frac{\psi}{\phi}}\right)^{v}\lambda^{v-1}\exp-\frac{1}{2}(\psi\lambda+\frac{\phi}{\lambda})}{2K_v(\sqrt{\psi\phi})}, \quad \lambda > 0$$

$$
\begin{aligned}
f_x(t) &= \int_0^\infty \frac{(\exp - \lambda t)\lambda t^x}{x!} \frac{\left(\sqrt{\frac{\psi}{\phi}}\right)^v \lambda^{v-1} \exp - \frac{1}{2}(\psi\lambda + \frac{\phi}{\lambda})}{2K_v(\sqrt{\psi\phi})} \quad d(\lambda) \\
&= \frac{t^x(\sqrt{\frac{\psi}{\phi}})^v}{2x!K_v\sqrt{\psi\phi}} \int_0^\infty \lambda^{v+x-1} \exp\left(-\lambda t - \frac{1}{2}\psi\lambda - \frac{\phi}{2\lambda}\right) \quad d\lambda \\
&= c\frac{1}{2}\int_0^\infty \lambda^{v+x-1} \exp\left(-\frac{2\lambda t}{2} - \frac{1}{2}\psi\lambda - \frac{\phi}{2\lambda}\right) \quad d\lambda \\
&\quad where \quad c = \frac{t^x(\sqrt{\frac{\psi}{\phi}})^v}{x!K_v\sqrt{\psi\phi}} \\
f_x(t) &= c * \frac{1}{2}\int_0^\infty \lambda^{v+x-1} \exp\{-\frac{1}{2}[(2t+\psi)\lambda + \frac{\phi}{\lambda}]\} \quad d\lambda \\
&= c * \frac{1}{2}\int_0^\infty \lambda^{v+x-1} \exp\{-\frac{1}{2}(2t+\psi)[\lambda + \frac{\phi}{2t+\psi} * \frac{1}{\lambda}]\} \quad d\lambda \\
&\quad Let \lambda = \sqrt{\frac{\phi}{2t+\psi}z} \\
&\quad \therefore d\lambda = \sqrt{\frac{\phi}{2t+\psi}} \quad dz \\
f_x(t) &= c * \frac{1}{2}\int_0^\infty \left(\sqrt{\frac{\phi}{2t+\psi}z}\right)^{v+x-1} \exp\left(-\frac{1}{2}(2t+\psi)\left(\sqrt{\frac{\phi}{2t+\psi}}z + \frac{\phi}{2t+\psi} * \frac{1}{\sqrt{\frac{\phi}{2t+\psi}}z}\right)\right) \sqrt{\frac{\phi}{2t+\psi}}
\end{aligned}
$$

$\therefore$

$$
\begin{aligned}
f_x(t) &= \left(\sqrt{\frac{\phi}{2t+\psi}}\right)^{v+x} \int_0^\infty z^{v+x-1} \exp\left(-\frac{1}{2}(2t+\psi)\frac{\phi}{2t+\psi}\left(z+\frac{1}{z}\right)\right) \quad dz \\
&= c * \left(\sqrt{\frac{\phi}{2t+\psi}}\right)^{v+x} \frac{1}{2}\int_0^\infty z^{v+x-1} \exp\left(-\frac{\sqrt{\phi(2t+\psi)}}{2}\left(z+\frac{1}{z}\right)\right) \quad dz \\
&= c * \left(\sqrt{\frac{\phi}{2t+\psi}}\right)^{v+x} K_{v+x}(\sqrt{\phi(2t+\psi)}) \\
&= \frac{t^x\left(\sqrt{\frac{\psi}{\phi}}\right)^v}{x!K_v\sqrt{\psi\phi}}(\sqrt{\frac{\phi}{2t+\psi}})^{v+x} K_{v+x}(\sqrt{\phi(2t+\psi)}) \\
&= \frac{t^x}{x!}\left(\sqrt{\frac{\psi}{\phi}\frac{\phi}{2t+\psi}}\right)^v \frac{K_{v+x}(\sqrt{\phi(2t+\psi)})}{K_v\sqrt{\psi\phi}} \left(\sqrt{\frac{\phi}{2t+\psi}}\right)^x
\end{aligned} \tag{3.91}
$$

3.6.2 Poisson-GIG in a Recursive Form

$K_{v+1}(\omega) = \frac{2v}{\omega}K_v(\omega) + K_{v-1}(\omega)$

$$f(x) = \frac{t^x}{x!}\left(\sqrt{\frac{\psi}{2t+\psi}}\right)^v\left(\sqrt{\frac{\phi}{2t+\psi}}\right)^x\frac{K_{v+x}(\sqrt{\phi(2t+\psi)})}{K_v\sqrt{\psi\phi}}$$

Therefore

$$f(x-1) = \frac{t^{x-1}}{(x-1)!}\left(\sqrt{\frac{\psi}{2t+\psi}}\right)^v\left(\sqrt{\frac{\phi}{2t+\psi}}\right)^{x-1}\frac{K_{v+x-1}(\sqrt{\phi(2t+\psi)})}{K_v\sqrt{\psi\phi}}$$

$$f(x+1) = \frac{t^{x+1}}{(x+1)!}\left(\sqrt{\frac{\psi}{2t+\psi}}\right)^v\left(\sqrt{\frac{\phi}{2t+\psi}}\right)^{x+1}\frac{K_{v+x+1}(\sqrt{\phi(2t+\psi)})}{K_v\sqrt{\psi\phi}}$$

$$f(x+1) = \frac{\frac{t^{x+1}}{(x+1)!}\left(\sqrt{\frac{\psi}{2t+\psi}}\right)^v\left(\sqrt{\frac{\phi}{2t+\psi}}\right)^{x+1}}{K_v\sqrt{\psi\phi}}\left(\frac{2(v+x)}{\sqrt{\phi(2t+\psi)}}K_{v+x}(\sqrt{\phi(2t+\psi)}) + K_{v+x-1}(\sqrt{\phi(2t+\psi)})\right)$$

$$f(x+1) = \frac{\frac{t^{x+1}}{(x+1)!}\left(\sqrt{\frac{\psi}{2t+\psi}}\right)^v\left(\sqrt{\frac{\phi}{2t+\psi}}\right)^{x+1}}{K_v\sqrt{\psi\phi}}\frac{2(v+x)}{\sqrt{\phi(2t+\psi)}}\frac{K_{v+x}(\sqrt{\phi(2t+\psi)})}{K_v\sqrt{\psi\phi}}$$
$$+\frac{t^{x+1}}{(x+1)!}\left(\sqrt{\frac{\psi}{2t+\psi}}\right)^v\left(\sqrt{\frac{\phi}{2t+\psi}}\right)^{x+1}\frac{K_{v+x-1}\left(\sqrt{\phi(2t+\psi)}\right)}{K_v\sqrt{\psi\phi}}$$
$$= \frac{t}{x+1}\left(\sqrt{\frac{\phi}{2t+\psi}}\right)\left(\frac{t^x}{x!}\left(\sqrt{\frac{\psi}{2t+\psi}}\right)^v\right)\left(\sqrt{\frac{\phi}{2t+\psi}}\right)^x\frac{K_{v+x}\left(\sqrt{\phi(2t+\psi)}\right)}{K_v\sqrt{\psi\phi}}\}\frac{2(v+x)}{\sqrt{\phi(2t+\psi)}}$$
$$+\frac{t^2}{(x+1)x}\left(\sqrt{\frac{\phi}{2t+\psi}}\right)^2\left(\frac{t^{x-1}}{(x-1)!}(\sqrt{\frac{\psi}{2t+\psi}})^v\right)\left(\sqrt{\frac{\phi}{2t+\psi}}\right)^{x-1}\frac{K_{v+x-1}\left(\sqrt{\phi(2t+\psi)}\right)}{K_v\sqrt{\psi\phi}}\}$$

$$f(x+1) = \frac{t}{x+1}\left(\sqrt{\frac{\phi}{2t+\psi}}\right)\frac{2(v+x)}{\sqrt{\phi(2t+\psi)}}f(x) + \frac{t^2}{(x+1)x}\frac{\phi}{2t+\psi}f(x-1)$$

$$\therefore f(x+1) = \frac{t}{x+1} * \frac{2(v+x)}{2t+\psi}f(x) + \frac{\phi t^2}{x(x+1)(2t+\psi)}f(x-1)$$
$$\Rightarrow x(x+1)(2t+\psi)f(x+1) = 2tx(x+v)f(x) + \phi t^2 f(x-1)$$
$$for \quad x = 0,1,2,3,..., where \quad f(-1) = 0$$

$$x(x+1)(2t+\psi)f(x+1) = 2tx(x+v)f(x) + \phi t^2 f(x-1) \tag{3.92}$$

3.6.3 PGF of Poisson-GIG Distribution

$$
\begin{aligned}
G(s) &= \sum_{x=0}^{\infty} f(x)s^x \\
&= \sum_{x=0}^{\infty} \frac{t^x}{x!}\left(\sqrt{\frac{\psi}{2t+\psi}}\right)^v \left(\sqrt{\frac{\phi}{2t+\psi}}\right)^x \frac{K_{v+x}(\sqrt{\phi(2t+\psi)})}{K_v\sqrt{\psi\phi}} s^x \\
&= \sum_{x=0}^{\infty} \frac{\frac{t^x}{x!}\left(\sqrt{\frac{\psi}{2t+\psi}}\right)^v \left(\sqrt{\frac{\phi}{2t+\psi}}s\right)^x}{K_v\sqrt{\psi\phi}} \frac{1}{2}\int_0^{\infty} y^{v+x-1}\exp -\frac{1}{2}\sqrt{\phi(2t+\psi)}\left(y+\frac{1}{y}\right) \quad dy \\
&= \int_0^{\infty}\{\frac{\left(\sqrt{\frac{\psi}{2t+\psi}}\right)^v}{2K_v\sqrt{\psi\phi}}\left(\sum_{x=0}^{\infty}\frac{t^x}{x!}\left(\sqrt{\frac{\phi}{2t+\psi}}\right)^x s^x y^x\right) y^{v-1}\exp -\frac{1}{2}\sqrt{\phi(2t+\psi)}(y+\frac{1}{y})\} \quad dy \\
G(s) &= \int_0^{\infty}\left(\frac{\left(\sqrt{\frac{\psi}{2t+\psi}}\right)^v}{2K_v\sqrt{\psi\phi}}\sum_{x=0}^{\infty}\frac{\left(tsy\sqrt{\frac{\phi}{2t+\psi}}\right)^x}{x!} y^{v-1}\exp -\frac{1}{2}\sqrt{\phi(2t+\psi)}\left(y+\frac{1}{y}\right)\right) \quad dy \\
&= \int_0^{\infty}\frac{\left(\sqrt{\frac{\psi}{2t+\psi}}\right)^v}{2K_v\sqrt{\psi\phi}} y^{v-1}\exp\left(\left(tsy\sqrt{\frac{\phi}{2t+\psi}}-\frac{1}{2}\sqrt{\phi(2t+\psi)}y-\frac{1}{2}\sqrt{\phi(2t+\psi)}\frac{1}{y}\right)\right) \quad dy \\
&= \frac{\left(\sqrt{\frac{\psi}{2t+\psi}}\right)^v}{2K_v\sqrt{\psi\phi}}\int_0^{\infty} y^{v-1}\exp\left(\frac{2ts}{2}\sqrt{\frac{\phi}{2t+\psi}}y-\frac{1}{2}\sqrt{\phi(2t+\psi)}y-\frac{1}{2}\sqrt{\phi(2t+\psi)}\frac{1}{y}\right) \quad dy \\
&= \frac{(\sqrt{\frac{\psi}{2t+\psi}})^v}{2K_v\sqrt{\psi\phi}}\int_0^{\infty} y^{v-1}\exp\{-\frac{1}{2}[(\sqrt{\phi(2t+\psi)}-2ts\sqrt{\frac{\phi}{2t+\psi}})y+\sqrt{\phi(2t+\psi)}\frac{1}{y}]\} \quad dy \\
&= \frac{\left(\sqrt{\frac{\psi}{2t+\psi}}\right)^v}{2K_v\sqrt{\psi\phi}}\int_0^{\infty} y^{v-1}\exp\left(-\frac{1}{2}\left((\frac{(2t+\psi)\sqrt{\phi}}{\sqrt{2t+\psi}}-\frac{2ts\sqrt{\phi}}{\sqrt{2t+\psi}})y+\sqrt{\phi(2t+\psi)}\frac{1}{y}\right)\right) \quad dy
\end{aligned}
$$

$$
\begin{aligned}
\therefore G(s) &= \frac{\left(\sqrt{\frac{\psi}{2t+\psi}}\right)^v}{2K_v\sqrt{\psi\phi}}\left(\sqrt{\frac{\psi+2t}{\psi+2t-2ts}}\right)^v 2K_v\left(\sqrt{\phi(2s+\psi)}\right) \\
&= \left(\sqrt{\frac{\psi}{\psi+2t-2ts}}\right)^v \frac{K_v(\sqrt{\phi(2s+\psi)})}{K_v\sqrt{\psi\phi}} \qquad (3.93)
\end{aligned}
$$

Alternatively,

$$
\begin{aligned}
G(s) &= \sum_{x=0}^{\infty} f(x)s^x \\
&= \sum_{x=0}^{\infty}[\int_0^{\infty} \frac{\exp-(\lambda t)(\lambda t)^x}{x!} g(\lambda)d\lambda]s^x \\
&= \int_0^{\infty} \exp-(\lambda t)[\sum_{x=0}^{\infty} \frac{(\lambda ts)^x}{x!}]g\lambda d\lambda \\
&= \int_0^{\infty} \exp-(\lambda t)\exp-(\lambda ts)g\lambda d\lambda \\
\therefore G(s) &= \int_0^{\infty} \exp-\lambda t(1-s))g\lambda d\lambda \\
&= E[\exp -t(1-s)\Lambda] \\
&= L_{\Lambda}(t(1-s))
\end{aligned}
\tag{3.94}
$$

is the Laplace of a mixing distribution at $t(1-s)$.

$$
\begin{aligned}
L_{\Lambda}(s) &= \left(\sqrt{\frac{\psi}{2s+\psi}}\right)^{v} \frac{K_v(\sqrt{\phi(2s+\psi)})}{K_v\sqrt{\psi\phi}} \\
G(s) = L_{\Lambda}(t(1-s)) &= \left(\sqrt{\frac{\psi}{2t(1-s)+\psi}}\right)^{v} \frac{K_v\sqrt{\phi(2(t(1-s))+\psi)}}{K_v\sqrt{\psi\phi}}
\end{aligned}
\tag{3.95}
$$

Table 1. GIG and Related Distributions using Sichel's Parameterization

Dist	f(x)	$E(X^r)$	E(X)
GIG	$\left(\sqrt{\frac{\psi}{\phi}}\right)^{v}\frac{x^{v}e^{\frac{1}{2}(\psi x+\frac{\phi}{x})}}{2K_{v}(\sqrt{\psi\phi})}$	$\left(\sqrt{\frac{\phi}{\psi}}\right)^{r}\frac{K_{v+r}(\sqrt{\psi\phi})}{K_{v}(\sqrt{\psi\phi})}$	$\left(\sqrt{\frac{\phi}{\psi}}\right)\frac{K_{v+1}(\sqrt{\psi\phi})}{K_{v}(\sqrt{\psi\phi})}$
IG	$\sqrt{\frac{\phi}{2\pi x^{3}}}e^{\sqrt{\psi\phi}}e^{-\frac{1}{2}(\psi x+\frac{\phi}{x})}$	$\left(\sqrt{\frac{\phi}{\psi}}\right)^{r}\frac{K_{-\frac{1}{2}+r}(\sqrt{\psi\phi})}{K_{-\frac{1}{2}}(\sqrt{\psi\phi})}$	$\sqrt{\frac{\phi}{\psi}}$
RIG	$\left(\frac{2\psi}{\pi x}\right)^{\frac{1}{2}}e^{-\frac{1}{2}(\psi x+\frac{\phi}{x})}$	$\left(\sqrt{\frac{\phi}{\psi}}\right)^{r}\frac{K_{\frac{1}{2}}(\sqrt{\psi\phi})}{K_{\frac{1}{2}}(\sqrt{\psi\phi})}$	$\sqrt{\frac{\phi}{\psi}}(1+\sqrt{\psi\phi})$
Gamma	$\frac{(\frac{\psi}{2})^{v}}{\Gamma(v)}e^{-\frac{\psi}{2}x}x^{v-1}$	$\left(\frac{2}{\psi}\right)^{r}\frac{\Gamma(v+r)}{\Gamma(v)}$	$\frac{2}{\psi}v$
Exponential	$\frac{\psi}{2}e^{-\frac{\psi}{2}x}$	$r!\left(\frac{2}{\psi}\right)^{r}$	$\frac{2}{\psi}$
Inverse gamma	$\frac{(\frac{\phi}{2})^{\lambda}}{\Gamma(\lambda)}e^{-\frac{\phi}{2}\frac{1}{x}}x^{-\lambda-1}$	$\frac{(\frac{\phi}{2})^{\lambda}}{\Gamma(\lambda)}\frac{\Gamma(\lambda-r)}{(\frac{\phi}{2})^{\lambda-r}}$	$\frac{\phi}{2(\lambda-1)}$
Levy	$\frac{(\frac{\phi}{2})}{\Gamma(\frac{1}{2})}e^{-\frac{\phi}{2}\frac{1}{x}}x^{-\frac{1}{2}-1}$	$\left(\frac{\phi}{2}\right)^{r}\frac{\Gamma(\frac{1}{2}-r)}{\sqrt{\pi}}$	$\left(\frac{\phi}{2}\right)\frac{\Gamma(-\frac{1}{2})}{\sqrt{(\pi)}}$
Hyperbolic	$\left(\sqrt{\frac{\psi}{\phi}}\right)\frac{e^{\frac{1}{2}(\psi x+\frac{\phi}{x})}}{2K_{1}(\sqrt{\psi\phi})}$	$\left(\sqrt{\frac{\phi}{\psi}}\right)^{r}\frac{K_{1+r}(\sqrt{\psi\phi})}{K_{1}(\sqrt{\psi\phi})}$	$\left(\sqrt{\frac{\phi}{\psi}}\right)\frac{K_{2}(\sqrt{\psi\phi})}{K_{1}(\sqrt{\psi\phi})}$
Harmonic	$\frac{x^{v-1}e^{-\frac{a}{2}(\frac{x}{n}+\frac{n}{x})}}{2K_{0}(a)}$	$\frac{n^{r}K_{r}(\frac{1}{n})}{K_{0}(a)}$	$\frac{nK_{1}(\frac{1}{n})}{K_{0}(a)}$

Table 2. Summary of the Variance, Laplace and modes of GIG and related Distributions

Dist	Var(X)	Laplace	Mode
GIG	$\left(\sqrt{\frac{\phi}{\psi}}\right)^{2}\left(\frac{K_{v+2}(\sqrt{\psi\phi})}{K_{v}(\sqrt{\psi\phi})}-\left[\frac{K_{v}(\sqrt{\psi\phi})}{K_{v}(\sqrt{\psi\phi})}\right]\right)$	$\left(\frac{\psi}{2s+\psi}\right)^{v}\frac{K_{v}(\sqrt{(2s+\psi)\phi})}{K_{v}(\sqrt{\psi\phi})}$	$\frac{(v-1)+\sqrt{(v-1)^{2}+\psi\phi}}{\psi}$
IG	$\sqrt{\frac{\phi^{3}}{\psi}}$	$e^{\sqrt{\psi\phi}-\sqrt{(2s+\psi)\phi}}$	$\frac{3}{2}\left[\sqrt{1+\frac{4}{9}\psi\phi}-1\right]$
RIG	$\left(\sqrt{\frac{\phi}{\psi}}\right)^{2}(3\sqrt{\phi\psi}+\psi\phi(1-\psi\phi))$	$\sqrt{\frac{\psi}{\psi+2s}}e^{\sqrt{\psi\phi}-\sqrt{(2s+\psi)\phi}}$	$\frac{\sqrt{1+\frac{\psi\phi}{4}}-1}{2\sqrt{\psi\phi}}$
Gamma	$\left(\frac{2}{\psi}\right)^{2}v$	$\left(\frac{\psi}{\psi+2s}\right)^{v}$	$\frac{\psi}{2(v-1)}$
Exponential	$\left(\frac{2}{\psi}\right)^{2}$	$\frac{\psi}{2s+\psi}$	∞
Inverse gamma	$\left(\frac{\phi}{2}\right)^{2}\frac{1}{(\lambda-1)^{2}(\lambda-2)}$	$2\left[\sqrt{\frac{\phi}{2}}\right]^{\lambda}\frac{K_{\lambda}(\sqrt{2s\psi s})}{\Gamma(\lambda)}$	$\left[\frac{2(\lambda+1)}{\phi}\right]$
Levy	$-\frac{2}{3}\phi^{2}$	$e^{-\sqrt{2\phi s}}$	$\frac{\phi}{3}$
Hyperbolic	$\left(\sqrt{\frac{\phi}{\psi}}\right)^{2}\left(\frac{K_{3}(\sqrt{\psi\phi})}{K_{1}(\sqrt{\psi\phi})}-\left[\frac{K_{2}(\sqrt{\psi\phi})}{K_{1}(\sqrt{\psi\phi})}\right]\right)$	$\sqrt{\frac{\psi}{2s+\psi}}\frac{K_{1}(\sqrt{(2s+\psi)\phi})}{K_{1}(\sqrt{\psi\phi})}$	1
Harmonic	$n^{2}\left[\frac{K_{2}(\frac{1}{n})}{K_{0}(a)}-\left(\frac{K_{1}(\frac{1}{n})}{K_{0}(a)}\right)\right]$	$\frac{K_{0}(\sqrt{2ams+a^{2}})}{K_{0}(a)}$	$\sqrt{\frac{1}{a^{2}}+1}-\frac{1}{a}$

4 GENERALIZED INVERSE GAUSSIAN DISTRIBUTION BASED ON OTHER PARAMETERIZATIONS

4.1 Introduction

In this chapter, we have constructed the Generalized inverse gaussian distribution based on the Barndorff-Nielsen, Allen, Willmot and Jorgensen Parameterizations. Using Sichel's Paramterization, we substitute various parameters to come up with GIG distributions based on all the above mentioned paramterizations.

Properties of the distributions based on the two above parameterizations have been studied. These properties are r-th moment, the Laplace transform and the modality.

Going by every property mentioned above, we have the special cases of the GIG distribution in which we have studied their properties. These resulting distributions are the sub-models of the GIG distributions.

4.2 Barndorff-Nielsen Parameterization: $\omega = \rho\sigma$

Using Sichel's parameterizations, let $\psi = \rho^2$ and $\phi = \sigma^2$ we obtain GIG distribution based on Barndorff-Nielsen parameterizations.

4.2.1 Construction

Let $\omega = \rho\sigma$

$$\therefore K_v(\rho\sigma) = \frac{1}{2}\int_0^\infty z^{v-1} e^{-\frac{\rho\sigma}{2}(z+\frac{1}{z})} \tag{4.0}$$

$$\therefore K_v(\rho\sigma) = \frac{1}{2}\left(\frac{\rho}{\sigma}\right)^v \int_0^\infty x^{v-1} e^{-\frac{1}{2}(\rho^2 x+\frac{\sigma^2}{x})} \quad dx$$

$$1 = \int_0^\infty \left(\frac{\rho}{\sigma}\right)^v \frac{x^{v-1} e^{-\frac{1}{2}(\rho^2 x+\frac{\sigma^2}{x})}}{2K_v(\rho\sigma)} \quad dx$$

$$\therefore f(x) = \left(\frac{\rho}{\sigma}\right)^v \frac{x^{v-1} e^{-\frac{1}{2}(\rho^2 x+\frac{\sigma^2}{x})}}{2K_v(\rho\sigma)} \quad dx \quad x > 0; -\infty < v < \infty, \quad \rho, \quad \sigma \tag{4.1}$$

Therefore

$$f(x) = \frac{\left(\frac{\rho}{\sigma}\right)^{v} x^{v-1} e^{-\frac{1}{2}(\rho^2 x + \frac{\sigma^2}{x})}}{2\left[\frac{1}{2}\left(\frac{\rho}{\sigma}\right)^{v} \int_0^\infty x^{v-1} e^{-\frac{1}{2}(\rho^2 x + \frac{\sigma^2}{x})} \quad dx\right]}$$

$$= \frac{x^{v-1} e^{-\frac{1}{2}(\rho^2 x + \frac{\sigma^2}{x})}}{\int_0^\infty x^{v-1} e^{-\frac{1}{2}(\rho^2 x + \frac{\sigma^2}{x})} \quad dx}, \quad x > 0; \quad -\infty < v < \infty, \quad \sigma \geq 0, \quad \rho \geq 0 \qquad \textbf{(4.2)}$$

This is a GIG distribution with parameters, (v, ρ, σ)

$$E(X^r) = \left(\frac{\sigma}{\rho}\right)^r \frac{K_{v+r}(\rho\sigma)}{K_v(\rho\sigma)}$$

$$Therefore,$$

$$E(X) = \left(\frac{\sigma}{\rho}\right) \frac{K_{v+1}(\rho\sigma)}{K_v(\rho\sigma)}$$

$$E(X^2) = \left(\frac{\sigma}{\rho}\right)^2 \frac{K_{v+2}(\rho\sigma)}{K_v(\rho\sigma)}$$

$$Var(X) = \left(\frac{\sigma}{\rho}\right)^2 \frac{K_{v+2}(\rho\sigma)}{K_v(\rho\sigma)} - \left(\frac{\sigma}{\rho}\right)^2 \left[\frac{K_{v+1}(\rho\sigma)}{K_v(\rho\sigma)}\right]^2 \qquad \textbf{(4.3)}$$

The Laplace Transform

$$L_X(s) = \left(\sqrt{\frac{\rho^2}{2s+\rho^2}}\right)^v \frac{K_v\sqrt{(2s+\rho^2)\sigma^2}}{K_v \rho\sigma}$$

$$= \left(\frac{\rho}{\sqrt{2s+\rho^2}}\right)^v \frac{K_v\sqrt{(2s+\rho^2)}\sigma}{K_v \sigma\rho} \qquad \textbf{(4.4)}$$

To get the modality The mode is

$$X_{mode} = \frac{(v-1) + \sqrt{(v-1)^2 + \rho^2\sigma^2}}{\rho^2}, \quad \rho > 0 \qquad \textbf{(4.5)}$$

$$and \quad X_{mode} = \frac{\sigma^2}{2(1-v)}, \quad \rho = 0$$

4.3 Special cases of GIG Distribution based on the Barndorff-Nielsen Paramterization

4.3.1 Iverse Gaussian Distribution

When

$$v = -\frac{1}{2} \tag{4.6}$$

$$\begin{aligned} f(x) &= \left(\frac{\rho}{\sigma}\right)^{-\frac{1}{2}} \frac{x^{-\frac{3}{2}} e^{-\frac{1}{2}(\rho^2 x + \frac{\sigma^2}{x})}}{2K_{-\frac{1}{2}(\rho\sigma)}} \\ &= \sqrt{\frac{\sigma^2}{2\pi x^3}} \frac{e^{-\frac{1}{2}(\rho^2 x + \frac{\phi^2}{x})}}{e^{-\rho\sigma}} \\ \therefore f(x) &= \sqrt{\frac{\sigma^2}{2\pi x^3}} e^{\rho\sigma} e^{-\frac{1}{2}(\rho^2 x + \frac{\sigma^2}{x})} \quad x > 0 \quad and \quad \phi \geq 0 \end{aligned}$$

$$\begin{aligned} \therefore E(X) &= \left(\frac{\sigma}{\rho}\right) \frac{K_{\frac{1}{2}}(\rho\sigma)}{K_{-\frac{1}{2}(\rho\sigma)}} \\ E(X^2) &= \left(\frac{\sigma}{\rho}\right)^2 \frac{K_{\frac{3}{2}}(\sigma\rho)}{K_{-\frac{1}{2}(\rho\sigma)}} \\ \therefore Var(X) &= \left(\frac{\sigma}{\rho}\right)^2 \left[\frac{K_{\frac{3}{2}}(\rho\sigma)}{K_{-\frac{1}{2}(\rho\sigma)}} - \frac{K^2_{\frac{1}{2}}(\rho\sigma)}{K^2_{-\frac{1}{2}(\rho\sigma)}}\right] \end{aligned}$$

$$E(X) = \left(\sqrt{\frac{\sigma^2}{\rho^2}}\right) \frac{\sqrt{\frac{\pi}{2\sqrt{\rho^2\sigma^2}}} e^{-\sqrt{\rho^2\sigma^2}}}{\sqrt{\frac{\pi}{2\sqrt{\rho^2\sigma^2}}} e^{-\sqrt{\rho^2\sigma^2}}} = \left(\sqrt{\frac{\sigma^2}{\rho^2}}\right) = \frac{\sigma}{\rho} \tag{4.7}$$

$$\begin{aligned} E(X^2) &= \left(\sqrt{\frac{\sigma^2}{\rho^2}}\right)^2 \left(1 + \sqrt{\rho^2\sigma^2}\right) \\ &= \left(\frac{\sigma}{\rho}\right)^2 (1 + \rho\sigma) \end{aligned} \tag{4.8}$$

$$VarX = \left(\sqrt{\frac{\sigma^2}{\rho^2}}\right)^2 \left(\sqrt{\rho^2\sigma^2}\right)$$
$$= \left(\frac{\sigma^3}{\rho}\right) \tag{4.9}$$

$$L_x(s) = E(e^{sX})$$
$$= \int_0^\infty e^{sX} f(x) \quad dx$$
$$= \left(\frac{\rho}{\sqrt{2s+\rho}}\right)^{-\frac{1}{2}} \frac{K_{-\frac{1}{2}}(\sqrt{(2s+\rho)\sigma})}{K_{-\frac{1}{2}(\rho\sigma)}}$$

$$L_X(s) = e^{\sqrt{\rho^2\sigma^2}-\sqrt{(2s+\rho^2)\sigma^2}}$$
$$= e^{\rho\sigma-\sqrt{(2s+\rho^2)}\sigma} \tag{4.10}$$
$$X_{mode} = \frac{-3+\sqrt{9+4(\sigma^2\rho^2)}}{2(\sqrt{\rho^2\sigma^2})}$$
$$= \frac{-3+\sqrt{9+4(\sigma^2\rho^2)}}{2(\rho\sigma)} \tag{4.11}$$

4.3.2 Reciprocal Inverse Gaussian

This is the case when

$$v = \frac{1}{2}, \quad \sigma > 0, \quad \rho > 0 \tag{4.12}$$

Then the pdf of RIG distribution is

$$f(x) = \left(\sqrt{\frac{\rho^2}{\sigma^2}}\right)^{\frac{1}{2}} \frac{x^{-\frac{1}{2}} e^{-\frac{1}{2}(\rho^2 x + \frac{\sigma^2}{x})}}{2K_{\frac{1}{2}(\sqrt{\rho^2\sigma^2})}} = \left(\frac{2\rho^2}{\pi x}\right)^{\frac{1}{2}} e^{-\frac{1}{2}(\rho^2 x + \frac{\sigma^2}{x})} \quad for \quad x > 0; \quad \sigma > 0, \quad \rho > 0 \tag{4.13}$$

The moments are:

$$E(X^r) = \left(\sqrt{\frac{\sigma^2}{\rho^2}}\right)^r \frac{K_{\frac{1}{2}+r}(\sqrt{\rho^2\sigma^2})}{K_{\frac{1}{2}}(\sqrt{\sigma^2\rho^2})} = \left(\frac{\sigma}{\rho}\right)^r \frac{K_{\frac{1}{2}+r}(\rho\sigma)}{K_{\frac{1}{2}}(\sigma\rho)} \tag{4.14}$$

$$\begin{aligned} E(X) &= \left(\frac{\sigma}{\rho}\right) \frac{K_{\frac{1}{2}+1}(\rho\sigma)}{K_{\frac{1}{2}}(\sigma\rho)} \\ &= \left(\frac{\sigma}{\rho}\right)(1+\rho\sigma)\frac{K_{\frac{1}{2}}(\rho\sigma)}{K_{\frac{1}{2}}(\rho\sigma)} \\ &= \left(\frac{\sigma}{\rho}\right)(1+\rho\sigma) \end{aligned} \tag{4.15}$$

$$\begin{aligned} Var(X) &= \left(\frac{\sigma}{\rho}\right)^2 \left[3\sigma\rho + \rho^2\sigma^2 - (\sigma^2\rho^2)^2\right] \\ &= \left(\frac{\sigma}{\rho}\right)^2 \left(3\sigma\rho + \rho^2\sigma^2(1-\sigma^2\rho^2)\right) \end{aligned} \tag{4.16}$$

The Laplace of RIG,

$$\begin{aligned} L_{GIG(\frac{1}{2},\sigma,\rho)} &= \sqrt{\frac{\rho^2}{2s+\rho^2}} e^{\rho\sigma - \sqrt{(2s+\rho^2)\sigma^2}} \\ &= \sqrt{\frac{\rho^2}{2s+\rho^2}} L_{GIG(-\frac{1}{2},\sigma,\rho)} \\ &= L_{GIG(\frac{1}{2},0,\rho)} L_{GIG(-\frac{1}{2},\sigma,\rho)} \end{aligned} \tag{4.17}$$

where

$$L_{GIG(\frac{1}{2},0,\rho)} = \left(\frac{\rho^2}{2s+\rho^2}\right)^{\frac{1}{2}}$$

is the Laplace of a gamma distribution with parameters $\frac{1}{2}$ and $\frac{\rho^2}{2}$

The mode of RIG is

$$\begin{aligned} X_{mode} &= \frac{(\frac{1}{2}-1) + \sqrt{(\frac{1}{2})^2 + \rho^2\sigma^2}}{\rho\sigma} \\ \therefore X_{mode} &= \frac{(-\frac{1}{2}) + \sqrt{(\frac{1}{4}) + \sigma^2\rho^2}}{\rho\sigma} \\ &= \frac{\sqrt{1+\frac{\rho^2\sigma^2}{4}} - 1}{2\rho\sigma} \end{aligned} \tag{4.18}$$

4.3.3 Gamma Distribution

When

$$v > 0, \quad \sigma = 0, \quad \rho > 0 \tag{4.19}$$

Using formula (3.6) we have

$$f(x) = \frac{\left(\frac{\rho^2}{2}\right)^v}{\Gamma(v)} e^{-\frac{\rho^2}{2}x} x^{v-1} \quad x > 0; \quad v > 0 \tag{4.20}$$

This is called a gamma pdf with parameters v and $\frac{\rho^2}{2}$

$$E(X^r) = \frac{x^{r+v-1} e^{-\frac{1}{2}(\psi x + \frac{\phi}{x})}}{\int_0^\infty x^{r+v-1} e^{-\frac{1}{2}(\psi x + \frac{\phi}{x})} \quad dx}$$

when $\phi = 0$

$$E(X^r) = \left(\frac{2}{\rho^2}\right)^r \frac{\Gamma(v+r)}{\Gamma(v)} \tag{4.21}$$

$$E(X) = \left(\frac{2}{\rho^2}\right) \frac{\Gamma(v+1)}{\Gamma(v)} = \frac{2}{\rho^2} v \tag{4.22}$$

$$E(X^2) = \left(\frac{2}{\rho^2}\right)^2 \frac{\Gamma(v+2)}{\Gamma(v)} = \left(\frac{2}{\rho^2}\right)^2 (v+1)v$$

$$\therefore Var(X) = \left(\frac{2}{\rho^2}\right)^2 (v+1)v - \left(\frac{2}{\rho^2}\right)^2 v^2 = \left(\frac{2}{\rho^2}\right)^2 v \tag{4.23}$$

$$L_X(s) = \left(\frac{\frac{\rho^2}{2}}{\frac{\rho^2}{2} + s}\right)^v \tag{4.24}$$

i.e

$$L_{GIG(v,0,\rho)} = \left(\frac{\rho^2}{\rho^2 + 2s}\right)^v \tag{4.25}$$

To get the modality

$$\frac{d}{dx}f(x) = 0$$

$$\frac{d}{dx}\left[\frac{\left(\frac{\rho^2}{2}\right)^v}{\Gamma v}e^{-\frac{\rho^2}{2}x}x^{v-1}\right] = 0$$

$$\therefore X_{mode} = \frac{\rho^2}{2(v-1)}, v \neq 1$$

and

$$X_{mode} = \infty \quad if \quad v = 1 \tag{4.26}$$

4.3.4 Exponential Distribution

This is a special case of a gamma distribution with

$$v = 1, \quad \sigma = 0, \quad \rho > 0 \tag{4.27}$$

$$\begin{aligned} f(x) &= \frac{e^{-\frac{\rho^2}{2}x}}{\frac{1}{\frac{\rho^2}{2}}} \\ &= \frac{\rho^2}{2}e^{-\frac{\rho^2}{2}x} \quad for \quad x > 0 \quad and \quad \rho > 0 \end{aligned}$$

$$E(X^r) = \left(\frac{2}{\rho^2}\right)^r \Gamma(r+1) = r!\left(\frac{2}{\rho^2}\right)^r \tag{4.28}$$

This is an exponential distribution with parameters $\frac{\psi}{2} > 0$

$$\begin{aligned} E(X) &= \frac{1}{\left(\frac{\rho^2}{2}\right)} = \frac{2}{\rho^2} \\ \therefore E(X^2) &= \frac{2}{\left(\frac{\rho^2}{2}\right)^2} \\ Var(X) &= \frac{2}{\left(\frac{\rho^2}{2}\right)^2} - \frac{1}{\left(\frac{\rho^2}{2}\right)^2} \\ &= \frac{1}{\left(\frac{\rho^2}{2}\right)^2} \end{aligned} \tag{4.29}$$

$$
\begin{aligned}
L_X(s) &= \frac{\rho^2}{2}\frac{1}{\lambda} \\
\therefore &= \frac{\frac{\rho^2}{2}}{\frac{\rho^2}{2}+s} = \frac{\rho^2}{2s+\rho^2}
\end{aligned}
\tag{4.30}
$$

To get the modality

$$
\begin{aligned}
\frac{d}{dx}\left[\frac{\rho^2}{2}e^{-\frac{\rho^2}{2}x}\right] &= 0 \\
\frac{\rho^2}{2}\frac{d}{dx}\left[e^{-\frac{\rho^2}{2}x}\right] &= 0 \\
\frac{\rho^2}{2}\left(-\frac{\rho^2}{2}\right)\left(e^{-\frac{\rho^2}{2}x}\right) &= 0 \\
e^{-\frac{\rho^2}{2}x} &= 0 \\
X_{mode} &= \infty
\end{aligned}
\tag{4.31}
$$

4.3.5 Inverse Gamma Distribution

This is the case when

$$
v<0, \quad \sigma>0, \quad \rho=0
\tag{4.32}
$$

$$
f(x) = \frac{\left(\frac{\sigma^2}{2}\right)^{-v}}{\Gamma-v}x^{v-1}e^{-\frac{\sigma^2}{2}\frac{1}{x}}, \quad x>0
\tag{4.33}
$$

$$
\begin{aligned}
f(x) &= \frac{\left(\frac{\sigma^2}{2}\right)^{\lambda}}{\Gamma\lambda}x^{-\lambda-1}e^{-\frac{\sigma^2}{2}\frac{1}{x}} \\
&= \frac{\left(\frac{\sigma^2}{2}\right)^{\lambda}}{\Gamma\lambda}e^{-\frac{\sigma^2}{2}\frac{1}{x}}x^{-\lambda-1}
\end{aligned}
\tag{4.34}
$$

Which is an inverse Gamma pdf with v<0

Thus

$$
X \sim GIG(v,\sigma,0) = Inverse \quad Gamma(-v,\frac{\sigma^2}{2}), \quad wherev<0 \quad and \quad \sigma>0
$$

$$E(X^r) = \frac{\left(\frac{\sigma^2}{2}\right)^{\lambda}}{\Gamma\lambda} \frac{\Gamma(\lambda - r)}{\left(\frac{\sigma^2}{2}\right)^{\lambda - r}} \tag{4.35}$$

when $\lambda > 1 \Rightarrow -v > 1 \Rightarrow v < -1$

$$\begin{aligned}
\therefore E(X) &= \frac{\left(\frac{\sigma^2}{2}\right)^{\lambda}}{\left(\frac{\sigma^2}{2}\right)^{\lambda-1}} \frac{\Gamma(\lambda - 1)}{\Gamma(\lambda)} \\
&= \frac{\sigma^2}{2} \frac{1}{\lambda - 1} \qquad (4.36) \\
\therefore E(X^2) &= \frac{\left(\frac{\sigma^2}{2}\right)^{\lambda}}{\Gamma\lambda} \frac{\Gamma(\lambda - 2)}{\left(\frac{\sigma^2}{2}\right)^{\lambda-2}} \\
&= \left(\frac{\sigma^2}{2}\right)^2 \frac{1}{(\lambda - 1)(\lambda - 1)} \qquad (4.37)
\end{aligned}$$

$$\begin{aligned}
Var(X) &= \left(\frac{\sigma^2}{2}\right)^2 \frac{1}{\lambda - 1} \left[\frac{\lambda - 1 - \lambda + 2}{(\lambda - 1)(\lambda - 2)}\right] \\
&= \left(\frac{\sigma^2}{2}\right)^2 \frac{1}{(\lambda - 1)^2(\lambda - 2)} \quad for \quad \lambda > 2 \qquad (4.38)
\end{aligned}$$

The Laplace Transform is

$$\begin{aligned}
L_x(s) &= 2\left(\frac{\sigma^2}{2}\right)^{\lambda} \left(\sqrt{\frac{\sigma^2}{2s}}\right)^{-\lambda-1} \frac{K_{-\lambda}\sqrt{(2s)\sigma^2}}{\Gamma(\lambda)} \\
&= 2\left[\sqrt{\frac{\sigma^2}{2}}\right]^{\lambda} \frac{K_{-\lambda}(\sqrt{2\sigma^2 s})}{\Gamma(\lambda)} \qquad (4.39)
\end{aligned}$$

For modality

$$\begin{aligned}
\frac{-(\lambda + 1)}{x^{\lambda}} + \frac{\sigma^2}{2} &= 0 \\
\therefore \frac{x^{\lambda}}{\lambda + 1} &= \frac{2}{\sigma^2} \\
\therefore x &= \left[\frac{2(\lambda + 1)}{\sigma^2}\right]^{\frac{1}{\lambda}} \qquad (4.40)
\end{aligned}$$

4.3.6 Levy Distribution

This is a special case of inverse gamma distribution when When

$$v = -\frac{1}{2}, \quad \sigma > 0, \quad \rho = 0 \tag{4.41}$$

Therefore $\lambda = \frac{1}{2}$

$$f(x) = \frac{\left(\frac{\sigma^2}{2}\right)^{\frac{1}{2}}}{\Gamma(\frac{1}{2})} e^{-\frac{\sigma^2}{2}\frac{1}{x}} x^{-\frac{1}{2}-1}$$

i.e

$$\begin{aligned}
f(x) &= \sqrt{\frac{\sigma^2}{2\pi}} x^{-\frac{\sigma^2}{2}\frac{1}{x}} \\
&= \sqrt{\frac{\sigma^2}{2\pi x^3}} e^{-\frac{\sigma^2}{2}\frac{1}{x}}, \quad x > 0; \quad \sigma > 0 \\
E(X^r) &= \left(\frac{\sigma^2}{2}\right)^r \frac{\Gamma(\frac{1}{2}-r)}{\Gamma(\frac{1}{2})} \\
&= \left(\frac{\sigma^2}{2}\right)^r \frac{\Gamma(\frac{1}{2}-r)}{\sqrt{\pi}}
\end{aligned} \tag{4.42}$$

$$\begin{aligned}
E(X) &= \left(\frac{\sigma^2}{2}\right) \frac{\Gamma(\frac{1}{2}-1)}{\sqrt{\pi}} = \left(\frac{\sigma^2}{2}\right) \frac{\Gamma(-\frac{1}{2})}{\sqrt{\pi}} \\
\therefore E(X) &= \frac{\sigma^2}{2} \frac{\Gamma(-\frac{1}{2}+1)}{-\frac{1}{2}\Gamma(\frac{1}{2})} \\
&= \frac{\sigma^2}{2} \frac{\Gamma(\frac{1}{2})}{-\frac{1}{2}\Gamma(\frac{1}{2})} \\
\therefore E(X) &= \sigma^2
\end{aligned}$$

$$E(X^2) = \left(\frac{\sigma^2}{2}\right)^2 \frac{\Gamma(-\frac{3}{2}+1)}{-\frac{3}{2}\Gamma(\frac{1}{2})}$$

$$\begin{aligned}
E(X^2) &= \left(\frac{\sigma^2}{2}\right)^2 \frac{\Gamma(-\frac{1}{2})}{-\frac{3}{2}\Gamma(\frac{1}{2})} \\
&= \left(\frac{\sigma^2}{2}\right)^2 \frac{\Gamma(-\frac{1}{2}+1)}{-\frac{3}{2}(-\frac{1}{2})\Gamma(\frac{1}{2})} \\
&= \left(\frac{\sigma^2}{2}\right)^2 \frac{1}{\frac{3}{4}} = \frac{\sigma^4}{3} \\
\therefore Var(X) &= \frac{\sigma^4}{3} - (\sigma^2)^2 = \frac{\sigma^4}{3} - \sigma^4 = -\frac{2}{3}\sigma^4
\end{aligned} \tag{4.43}$$

Alternatively using

$$\begin{aligned}
Var(x) &= \left(\frac{\sigma^2}{2}\right)^2 \frac{1}{(\lambda-1)^2(\lambda-2)} \\
&= \left(\frac{\sigma^2}{2}\right)^2 \frac{1}{(\frac{1}{2}-1)^2(\frac{1}{2}-2)} \\
&= \frac{\sigma^4}{4}\frac{1}{(-\frac{1}{2})(-\frac{3}{2})} \\
&= -\frac{2}{3}\sigma^4
\end{aligned}$$

Remark: $\lambda > 2 \Rightarrow v < -2$ for Var X to exist

Laplace transform is

$$\begin{aligned}
L_X(s) &= 2\left(\sqrt{\frac{\sigma^2 s}{2}}\right)^{\frac{1}{2}} \frac{K_{-\frac{1}{2}}(\sqrt{2\sigma^2 s})}{\Gamma(\frac{1}{2})} \\
&= \left[2\sqrt{\frac{\sigma^2 s}{2}\frac{1}{2\sigma^2 s}}\right]^{\frac{1}{2}} e^{-\sqrt{2\sigma^2 s}} \\
&= e^{-\sqrt{2\sigma^2 s}} = e^{-\sqrt{2s}\sigma} \qquad (4.44)
\end{aligned}$$

4.3.7 Positive Hyperbolic Distribution

This is the case when

$$v = 1, \quad \sigma > 0, \quad \rho > 0 \qquad (4.45)$$

$$X \sim GIG(1, \sigma, \rho)$$

So put v=1 in the results obtained in sections 3.2 and 3.3

$$f(x) = \left(\sqrt{\frac{\rho^2}{\sigma^2}}\right) \frac{e^{-\frac{1}{2}(\rho^2 x + \frac{\sigma^2}{x})}}{2K_1(\rho\sigma)} = \left(\frac{\rho}{\sigma}\right) \frac{e^{-\frac{1}{2}(\rho^2 x + \frac{\sigma^2}{x})}}{2K_1(\rho\sigma)} \quad x > 0; \quad \sigma > 0, \quad \rho > 0$$

$$E(X^r) = \left(\frac{\sigma}{\rho}\right)^r \frac{K_{1+r}(\rho\sigma)}{K_1(\sigma\rho)} \qquad (4.46)$$

$$\therefore E(X) = \left(\frac{\sigma}{\rho}\right)\frac{K_2(\rho\sigma)}{K_1(\sigma\rho)}$$
$$\therefore E(X^2) = \left(\frac{\sigma}{\rho}\right)^2\frac{K_3(\rho\sigma)}{K_1(\rho\sigma)}$$
$$Var(X) = \left(\frac{\sigma}{\rho}\right)^2\frac{K_3(\rho\sigma)}{K_1(\sigma\rho)} - \left(\frac{\sigma}{\rho}\right)^2\frac{K_2^2(\rho\sigma)}{K_1^2(\rho\sigma)}$$
$$= \left(\frac{\sigma}{\rho}\right)^2\left[\frac{K_3(\rho\sigma)}{K_1(\rho\sigma)} - \frac{K_2^2(\rho\sigma)}{K_1^2(\rho\sigma)}\right] \quad \textbf{(4.47)}$$

$$L_{GIG(1,\sigma,\rho)} = \sqrt{\frac{\rho^2}{2s+\rho^2}}\frac{K_1(\sqrt{(2s+\rho^2)\sigma^2})}{K_1(\rho\sigma)} \quad \textbf{(4.48)}$$

Modality

$$\sigma\rho x^2 + \sigma\rho = 0$$
$$\therefore X_{mode} = 1 \quad \textbf{(4.49)}$$

4.3.8 Convolution of a class of GIG Distribution

As proved earlier in chapter 3, using the Laplace transform technique, where $L_z(s) = L_x(s)L_y(s)$. The following proposition hold.

Proposition (4.1)

$$L_{GIG}(v_1,0,\rho) * L_{GIG}(v_2,0,\rho) = L_{GIG}(v_1+v_2,0,\rho) \quad \textbf{(4.50)}$$

Proof

$$L_z(s) = L_x(s) * L_y(s)$$
$$L_z(s) = \left(\frac{\rho^2}{\rho^2+2s}\right)^{v_1+v_2}$$
$$= L_{GIG}(v_1+v_2,0,\rho)$$

This is the Laplace transform wich is as a result of the product of the two laplaces of gamma distribution.
$Z \sim L_{GIG}(v_1+v_2,0,\rho)$

Proposition (4.2)

$$L_{GIG}(-v,\sigma,\rho)*L_{GIG}(v,0,\rho)=L_{GIG}(v,\sigma,\rho) \tag{4.51}$$

Proof

$$\begin{aligned} L_z(s) &= \left(\frac{\rho}{\sqrt{\rho^2+2s}}\right)^v \frac{K_v(\sigma\sqrt{\rho^2+2s})}{K_v(\rho\sigma)} \\ &= L_{GIG}(v,\sigma,\rho) \end{aligned}$$

Proposition (4.3)

$$L_{GIG}(-\frac{1}{2},\sigma_1,\rho)*L_{GIG}(-\frac{1}{2},\sigma_2,\rho)=L_{GIG}(-\frac{1}{2},\sigma_1+\sigma_2,\rho) \tag{4.52}$$

Proof

$$\begin{aligned} L_{GIG}(-\frac{1}{2},\sigma,\rho) &= L_x(s) \\ &= e^{\rho\sigma_1-\sigma_1\sqrt{\rho^2+2s}} \\ \therefore L_z(s) &= L_x(s)L_y(s) \\ &= e^{\rho\sigma_1-\sigma_1\sqrt{\rho^2+2s}}e^{\rho\phi_2-\sigma_2\sqrt{\rho^2+2s}} \\ &= e^{(\sigma_1+\sigma_2)(\rho-\sqrt{\rho^2+2s})} \\ \therefore &= L_{GIG}(-\frac{1}{2},\sigma_1+\sigma_2,\rho) \end{aligned}$$

Proposition (4.4)

$$L_{GIG}(-\frac{1}{2},\sigma_1,\rho)*L_{GIG}(\frac{1}{2},\sigma_2,\rho)=L_{GIG}(\frac{1}{2},\sigma_1+\sigma_2,\rho) \tag{4.53}$$

Proof

$$\begin{aligned} L_y(s) &= L_{GIG}(\frac{1}{2},\sigma_2,\rho) \\ &= \left(\frac{\frac{\rho}{\rho^2+2s}}{\frac{1}{\rho}}\right)^{\frac{1}{2}} e^{\rho\sigma_2-\sigma_2\sqrt{\rho^2+2s}} \\ &= \frac{\rho}{\sqrt{\rho^2+2s}}e^{\rho\sigma_2-\sigma_2\sqrt{\rho^2+2s}} \\ \therefore L_z(s) &= L_x(s)L_y(s) \\ &= e^{\rho\sigma_1-\sigma_1\sqrt{\rho^2+2s}}\frac{\rho}{\sqrt{\rho^2+2s}}e^{\rho\sigma_2-\sigma_2\sqrt{\rho^2+2s}} \\ &= \frac{\rho}{\sqrt{\rho^2+2s}}e^{(\sigma_1+\sigma_2)(\rho-\sqrt{\rho^2+2s})} \end{aligned}$$

4.4 Allen's Parameterization: $\omega = 2\sigma\theta^{\frac{1}{2}}$

In Allen's Parameterization, we substitute the following in Sichel's Parameterization,

$$\psi = 2\theta \quad and \quad \phi = 2\sigma^2 \tag{4.54}$$

$$f(x) = \frac{\left(\frac{\theta^{\frac{1}{2}}}{\sigma}\right)^{v}}{2K_v(2\sigma\theta^{\frac{1}{2}})} x^{v-1} e^{\theta x + \frac{\sigma^2}{x}}, x > 0 \tag{4.55}$$

$$\begin{aligned} f(x) &= \frac{\left(\frac{\theta^{\frac{1}{2}}}{\sigma}\right)^{v} x^{v-1} e^{-(\theta x + \frac{\sigma^2}{x})}}{2K_v(2\sigma\theta^{\frac{1}{2}})} \\ &= \frac{\left(\frac{\theta^{\frac{1}{2}}}{\sigma}\right)^{v} x^{v-1} e^{-(\theta x + \frac{\sigma^2}{x})}}{2\left[\frac{1}{2}\int_0^\infty \left(\frac{\theta^{\frac{1}{2}}}{\sigma}\right)^{v} x^{v-1} e^{-(\theta x + \frac{\sigma^2}{x})} dx\right]} \\ &= \frac{x^{v-1} e^{-(\theta x + \frac{\sigma^2}{x})}}{\left[\int_0^\infty x^{v-1} e^{-(\theta x + \frac{\sigma^2}{x})} dx\right]} \end{aligned} \tag{4.56}$$

For $x > 0, \quad -\infty < x < \infty, \quad \theta \geq 0, \quad \sigma > 0$

$$E(X^r) = \left(\frac{\theta^{\frac{1}{2}}}{\sigma}\right)^{-r} \frac{K_{r+v}(2\sigma\theta^{\frac{1}{2}})}{K_v(2\sigma\theta^{\frac{1}{2}})} \tag{4.57}$$

$$= \left(\frac{\sigma}{\theta^{\frac{1}{2}}}\right)^{r} \frac{K_{r+v}(2\sigma\theta^{\frac{1}{2}})}{K_v(2\sigma\theta^{\frac{1}{2}})}$$

$$\therefore E(X) = \left(\frac{\sigma}{\theta^{\frac{1}{2}}}\right) \frac{K_{1+v}(2\sigma\theta^{\frac{1}{2}})}{K_v(2\sigma\theta^{\frac{1}{2}})}$$

$$\begin{aligned} \therefore Var(X) &= \left(\frac{\sigma}{\theta^{\frac{1}{2}}}\right)^{2} \frac{K_{v+2}(2\sigma\theta^{\frac{1}{2}})}{K_v(2\sigma\theta^{\frac{1}{2}})} - \left(\frac{\sigma}{\theta^{\frac{1}{2}}}\right)^{2} \frac{K_{v+1}^2(2\sigma\theta^{\frac{1}{2}})}{K_v^2(2\sigma\theta^{\frac{1}{2}})} \\ &= \left(\frac{\sigma}{\theta^{\frac{1}{2}}}\right)^{2} \left[\frac{K_{v+2}(2\sigma\theta^{\frac{1}{2}})}{K_v(2\sigma\theta^{\frac{1}{2}})} - \frac{K_{v+1}^2(2\sigma\theta^{\frac{1}{2}})}{K_v^2(2\sigma\theta^{\frac{1}{2}})}\right] \end{aligned} \tag{4.58}$$

The Laplace Transform

$$L_X(s) = \left(\frac{2\theta}{2s+2\theta}\right)^{\frac{v}{2}} \frac{K_v(2\sqrt{\sigma^2(s+\theta)})}{K_v(2\sigma\theta^{\frac{1}{2}})} \tag{4.59}$$

Modality

$$(v-1)-\theta x+\frac{\sigma^2}{x}=0$$

$$\theta x^2-(v-1)x+\sigma^2=0$$

$$X_{mode}=\frac{(v-1)\pm\sqrt{(v-1)^2+4\sigma^2\theta}}{2\theta}$$

$$\therefore when \quad \theta>0, \quad X_{mode}=\frac{(v-1)+\sqrt{(v-1)^2+4\sigma^2\theta}}{2\theta} \tag{4.60}$$

when $\theta=0$

$$X_{mode}=\frac{\sigma^2}{(v-1)}$$

4.5 Special cases and their Properties

4.5.1 Inverse Gaussian Distribution

When

$$v=-\frac{1}{2} \tag{4.61}$$

$$\begin{aligned} f(x) &= \frac{\left(\frac{\theta^{\frac{1}{2}}}{\sigma}\right)x^{-\frac{1}{2}-1}e^{-(\theta x+\frac{\sigma^2}{x})}}{\left[2\frac{\pi}{(2\sigma\theta^{\frac{1}{2}})}\right]^{\frac{1}{2}}e^{-(2\sigma\theta^{\frac{1}{2}})}} \\ &= \frac{x^{-\frac{3}{2}}e^{2\sigma\theta^{\frac{1}{2}}}e^{-(\theta x+\frac{\sigma^2}{x})}}{\frac{\sqrt{\pi x^3}}{\sigma}} \\ &= \frac{\sigma x^{-\frac{3}{2}}e^{2\sigma\theta^{\frac{1}{2}}}e^{-(\theta x+\frac{\sigma^2}{x})}}{\sqrt{\pi x^3}} \end{aligned}$$

$$E(X^r)=\left(\frac{\sigma}{\theta^{\frac{1}{2}}}\right)^r$$

$$E(X)=\left(\frac{\sigma}{\theta^{\frac{1}{2}}}\right)$$

$$\begin{aligned} E(X^2) &= \left(\frac{\sigma}{\theta^{\frac{1}{2}}}\right)^2\frac{K_{\frac{3}{2}}(2\sigma\theta^{\frac{1}{2}})}{K_{-\frac{1}{2}}(2\sigma\theta^{\frac{1}{2}})} \\ &= \left(\frac{\sigma}{\theta^{\frac{1}{2}}}\right)^2\left(1+2\sigma\theta^{\frac{1}{2}}\right) \end{aligned} \tag{4.62}$$

$$
\begin{aligned}
VarX &= \left(\frac{\sigma}{\theta^{\frac{1}{2}}}\right)^2 \left(2\sigma\theta^{\frac{1}{2}}\right) \\
&= \frac{2\sigma^3}{\theta^{\frac{1}{2}}}
\end{aligned}
\tag{4.63}
$$

$$
L_X(s) = \left(\sqrt{\frac{2\theta}{2s+2\theta}}\right)^{-\frac{1}{2}} \frac{K_{-\frac{1}{2}}(\sqrt{(2s+2\theta)2\sigma^2})}{K_{-\frac{1}{2}(2\sigma\theta^{\frac{1}{2}})}}
$$

$$
= e^{2\sigma\theta^{\frac{1}{2}} - \sqrt{(2s+2\theta)2\sigma^2}} \tag{4.64}
$$

$$
= e^{2\sigma\theta^{\frac{1}{2}} - 2\sqrt{(s+\theta)2\sigma^2}}
$$

$$
X_{mode} = \frac{-3+\sqrt{9+4(4\sigma^2\theta)}}{4(\sigma\theta^{\frac{1}{2}})}
$$

$$
= \frac{3}{2}\left[\sqrt{1+\frac{16}{9}\theta\sigma^2} - 1\right] \tag{4.65}
$$

4.5.2 Reciprocal Inverse Gaussian

When

$$
v = \frac{1}{2}, \quad \sigma > 0, \quad \theta > 0 \tag{4.66}
$$

Then the pdf of RIG distribution is

$$
f(x) = \left(\frac{\theta^{\frac{1}{2}}}{\sigma}\right)^{\frac{1}{2}} \frac{x^{-\frac{1}{2}} e^{-(\theta x + \frac{\sigma^2}{x})}}{2K_{\frac{1}{2}(2\sigma\theta^{\frac{1}{2}})}} = \left(\frac{4\theta}{\pi x}\right)^{\frac{1}{2}} e^{-(\theta x + \frac{\sigma^2}{x})} \quad for \quad x > 0; \quad \sigma > 0, \quad \theta > 0
\tag{4.67}
$$

The moments are:

$$
E(X^r) = \left(\frac{\sigma^2}{\theta^{\frac{1}{2}}}\right)^r \frac{K_{\frac{1}{2}+r}(2\sigma\theta^{\frac{1}{2}})}{K_{\frac{1}{2}}(2\sigma\theta^{\frac{1}{2}})} \tag{4.68}
$$

$$
E(X) = \left(\frac{\sigma^2}{\theta^{\frac{1}{2}}}\right) \frac{K_{\frac{1}{2}+1}(2\sigma\theta^{\frac{1}{2}})}{K_{\frac{1}{2}}(2\sigma\theta^{\frac{1}{2}})}
$$

$$
= \left(\frac{\sigma^2}{\theta^{\frac{1}{2}}}\right)(1+2\sigma\theta^{\frac{1}{2}}) \frac{K_{\frac{1}{2}}(2\sigma\theta^{\frac{1}{2}})}{K_{\frac{1}{2}}(2\sigma\theta^{\frac{1}{2}})} = \left(\frac{\sigma^2}{\theta^{\frac{1}{2}}}\right)(1+2\sigma\theta^{\frac{1}{2}})
$$

$$Var(X) = \left(\frac{\sigma}{\theta}\right)^2 \left(6\sigma\theta^{\frac{1}{2}} + 4\sigma^2\theta(1 - 4\sigma^2\theta)\right) \tag{4.69}$$

The Laplace of RIG,

$$\begin{aligned}
L_{GIG(\frac{1}{2},\sigma,\theta)} &= \sqrt{\frac{2\theta}{2s+2\theta}} e^{2\sigma\theta^{\frac{1}{2}} - \sqrt{(2s+2\theta)2\sigma}} \\
&= \sqrt{\frac{2\theta}{2s+2\theta}} L_{GIG(-\frac{1}{2},\theta,\sigma)} \\
&= \left(\frac{2\theta}{2s+2\theta}\right)^{\frac{1}{2}} L_{GIG(-\frac{1}{2},\sigma,\theta)} \\
&= \left(\frac{\theta}{s+\theta}\right)^{\frac{1}{2}} L_{GIG(-\frac{1}{2},\sigma,\theta)} \qquad = L_{GIG(\frac{1}{2},0,\theta)} L_{GIG(-\frac{1}{2},\sigma,\theta)}
\end{aligned} \tag{4.70}$$

where

$$L_{GIG(\frac{1}{2},0,\theta)} = \left(\frac{\theta}{s+\theta}\right)^{\frac{1}{2}}$$

The mode of RIG is

$$\begin{aligned}
X_{mode} &= \frac{-\frac{1}{2} + \sqrt{\frac{1}{4}(1 + \frac{4\theta\sigma^2}{4})}}{2\sigma\theta^{\frac{1}{2}}} \\
&= \frac{-\frac{1}{2} + \sqrt{\frac{1}{4}(1 + \theta\sigma^2)}}{2\sigma\theta^{\frac{1}{2}}} \\
X_{mode} &= \frac{\sqrt{1+\theta\sigma^2} - 1}{4\sigma\theta^{\frac{1}{2}}}
\end{aligned} \tag{4.71}$$

4.5.3 Gamma Distribution

When

$$v > 0, \quad \sigma = 0, \quad \theta > 0 \tag{4.72}$$

$$f(x) = \frac{(\theta)^v}{\Gamma(v)} e^{-\theta x} x^{v-1} \quad x > 0; \quad v > 0 \tag{4.73}$$

$$E(X^r) = \frac{x^{r+v-1} e^{-(\theta x + \frac{\sigma^2}{x})}}{\int_0^\infty x^{r+v-1} e^{-(\theta x + \frac{\sigma^2}{x})} \quad dx}$$

when $\sigma = 0$

$$E(X^r) = \frac{\Gamma(v+r)}{\Gamma(v)\left(\frac{2\theta}{2}\right)^r}$$
$$= \left(\frac{1}{\theta}\right)^r \frac{\Gamma(v+r)}{\Gamma(v)} \tag{4.74}$$

$$E(X) = \left(\frac{1}{\theta}\right)\frac{\Gamma(v+1)}{\Gamma(v)} = \frac{1}{\theta}v \tag{4.75}$$
$$E(X^2) = \left(\frac{1}{\theta}\right)^2 \frac{\Gamma(v+2)}{\Gamma(v)} = \left(\frac{1}{\theta}\right)^2 (v+1)v$$
$$\therefore Var(X) = \left(\frac{1}{\theta}\right)^2 (v+1)v - \left(\frac{1}{\theta}\right)^2 v^2 = \left(\frac{1}{\theta}\right)^2 v \tag{4.76}$$

$$L_X(s) = \left(\frac{\theta}{\theta+s}\right)^v \tag{4.77}$$

i.e

$$L_{GIG(v,0,\theta)} = \left(\frac{\theta}{\theta+s}\right)^v \tag{4.78}$$

To get the modality

$$\therefore X_{mode} = \frac{\theta}{(v-1)}, v \neq 1$$

and

$$X_{mode} = \infty \quad if \quad v = 1 \tag{4.79}$$

4.5.4 Exponential Distribution

When

$$v = 1, \quad \sigma = 0, \quad \theta > 0 \tag{4.80}$$

$$f(x) = \frac{e^{-\theta x}}{\frac{1}{\theta}}$$
$$= \theta e^{-\theta x} \quad for \quad x > 0 \quad and \quad \theta > 0$$

$$E(X^r) = \left(\frac{1}{\theta}\right)^r \Gamma(r+1) = r!\left(\frac{1}{\theta}\right)^r \quad (4.81)$$

This is an exponential distribution with parameters $\theta > 0$

$$\begin{aligned} E(X) &= \frac{1}{\theta} \\ \therefore E(X^2) &= \frac{2}{(\theta)^2} \\ Var(X) &= \frac{2}{(\theta)^2} - \frac{1}{(\theta)^2} \\ &= \frac{1}{\left(\frac{\psi}{2}\right)^2} \end{aligned} \quad (4.82)$$

$$\begin{aligned} L_X(s) &= \theta\frac{1}{\lambda} \\ \therefore &= \frac{\theta}{\theta+s} \end{aligned}$$

To get the modality

$$\begin{aligned} e^{-\theta x} &= 0 \\ X_{mode} &= \infty \end{aligned} \quad (4.83)$$

4.5.5 Inverse Gamma Distribution

$$v < 0, \quad \sigma > 0, \quad \theta = 0 \quad (4.84)$$

$$\begin{aligned} f(x) &= \frac{(\sigma^2)^\lambda}{\Gamma\lambda} x^{-\lambda-1} e^{-\sigma^2\frac{1}{x}} \\ &= \frac{(\sigma^2)^\lambda}{\Gamma\lambda} e^{-\sigma^2\frac{1}{x}} x^{-\lambda-1} \end{aligned} \quad (4.85)$$

Which is an inverse Gamma pdf with v<0

$$E(X^r) = \frac{(\sigma^2)^\lambda}{\Gamma\lambda}\frac{\Gamma(\lambda-r)}{(\sigma^2)^{\lambda-r}} \quad (4.86)$$

when $\lambda > 1 \Rightarrow -v > 1 \Rightarrow v < -1$

$$\therefore E(X) = \frac{\left(\sigma^2\right)^{\lambda}}{\left(\sigma^2\right)^{\lambda-1}} \frac{\Gamma(\lambda-1)}{\Gamma(\lambda)}$$

$$= \sigma^2 \frac{1}{\lambda - 1} \tag{4.87}$$

$$\therefore E(X^2) = \frac{\left(\sigma^2\right)^{\lambda}}{\Gamma\lambda} \frac{\Gamma(\lambda-2)}{\left(\sigma^2\right)^{\lambda-2}}$$

$$= \left(\sigma^2\right)^2 \frac{1}{(\lambda-1)(\lambda-1)} \tag{4.88}$$

$$Var(X) = \left(\sigma^2\right)^2 \frac{1}{\lambda-1} \left[\frac{\lambda - 1 - \lambda + 2}{(\lambda-1)(\lambda-2)}\right]$$

$$= \left(\sigma^4\right) \frac{1}{(\lambda-1)^2(\lambda-2)} \quad for \quad \lambda > 2 \tag{4.89}$$

The Laplace Transform is

$$L_x(s) = 2\left(\sigma^2\right)^{\lambda} \left(\sqrt{\frac{\sigma^2}{s}}\right)^{-\lambda-1} \frac{K_{-\lambda}\sqrt{(2s)2\sigma^2}}{\Gamma(\lambda)}$$

$$= 2\left[\sigma^2\right]^{\lambda} \frac{K_{-\lambda}(2\sqrt{\sigma^2 s})}{\Gamma(\lambda)} \tag{4.90}$$

For modality

$$\frac{-(\lambda+1)}{x^{\lambda}} + \sigma^2 = 0$$

$$\therefore \frac{x^{\lambda}}{\lambda+1} = \frac{1}{\sigma^2}$$

$$\therefore x = \left[\frac{(\lambda+1)}{\sigma^2}\right]^{\frac{1}{\lambda}} \tag{4.91}$$

4.5.6 Levy Distribution

$$v = -\frac{1}{2}, \quad \sigma > 0, \quad \theta = 0 \tag{4.92}$$

$$f(x) = \frac{\left(\sigma^2\right)^{\frac{1}{2}}}{\Gamma(\frac{1}{2})} e^{-\sigma^2 \frac{1}{x}} x^{-\frac{1}{2}-1}$$

i.e

$$\begin{aligned}
f(x) &= \sqrt{\frac{\sigma^2}{\pi}} x^{-\sigma^2 \frac{1}{x}} \\
&= \sqrt{\frac{\sigma^2}{\pi x^3}} e^{-\sigma^2 \frac{1}{x}}, \quad x > 0; \quad \sigma > 0 \\
E(X^r) &= \left(\sigma^2\right)^r \frac{\Gamma(\frac{1}{2} - r)}{\Gamma(\frac{1}{2})} \\
&= \left(\sigma^2\right)^r \frac{\Gamma(\frac{1}{2} - r)}{\sqrt{\pi}} \qquad \textbf{(4.93)}
\end{aligned}$$

$$\begin{aligned}
E(X) &= \left(\sigma^2\right) \frac{\Gamma(\frac{1}{2} - 1)}{\sqrt{\pi}} = \left(\sigma^2\right) \frac{\Gamma(-\frac{1}{2})}{\sqrt{\pi}} \\
\therefore E(X) &= \sigma^2 \frac{\Gamma(-\frac{1}{2} + 1)}{-\frac{1}{2}\Gamma(\frac{1}{2})} \\
&= \sigma^2 \frac{\Gamma(\frac{1}{2})}{-\frac{1}{2}\Gamma(\frac{1}{2})} \\
\therefore E(X) &= -(2\sigma^2) \qquad \textbf{(4.94)}
\end{aligned}$$

$$\begin{aligned}
E(X^2) &= \left(\sigma^2\right)^2 \frac{\Gamma(\frac{1}{2} - 2)}{\Gamma(\frac{1}{2})} \\
&= \left(\sigma^2\right)^2 \frac{\Gamma(-\frac{3}{2})}{\Gamma(\frac{1}{2})} \\
&= \left(\sigma^2\right)^2 \frac{\Gamma(-\frac{3}{2} + 1)}{-\frac{3}{2}\Gamma(\frac{1}{2})}
\end{aligned}$$

$$\begin{aligned}
E(X^2) &= \left(\sigma^2\right)^2 \frac{\Gamma(-\frac{1}{2})}{-\frac{3}{2}\Gamma(\frac{1}{2})} \\
&= \left(\sigma^2\right)^2 \frac{\Gamma(-\frac{1}{2} + 1)}{-\frac{3}{2}(-\frac{1}{2})\Gamma(\frac{1}{2})} \\
&= \left(\sigma^2\right)^2 \frac{1}{\frac{3}{4}} = \frac{\phi^2}{3} \\
\therefore Var(X) &= \frac{4\sigma^4}{3} - 4\sigma^4 = -\frac{8}{3}\sigma^4 \qquad \textbf{(4.95)}
\end{aligned}$$

Alternatively using

$$Var(x) = \sigma^4 \frac{1}{(-\frac{1}{2})(-\frac{3}{2})}$$
$$= -\frac{8}{3}\sigma^4$$

Remark: $\lambda > 2 \Rightarrow v < -2$ for Var X to exist

Laplace transform is

$$L_X(s) = \left[2\sqrt{\frac{2\sigma^2 s}{2}\frac{1}{2\phi s}}\right]^{\frac{1}{2}} e^{-\sqrt{2(2\sigma^2)s}}$$
$$= e^{-2\sqrt{2\sigma^2 s}} \tag{4.96}$$

4.5.7 Positive Hyperbolic Distribution

When

$$v = 1, \quad \sigma > 0, \quad \theta > 0 \tag{4.97}$$

So put v=1 in the results obtained in sections 3.2 and 3.3

$$f(x) = \left(\frac{\theta^{\frac{1}{2}}}{\sigma}\right)\frac{e^{-(\theta x + \frac{\sigma}{x})}}{2K_1(2\sigma\theta^{\frac{1}{2}})} \quad x > 0; \quad \sigma > 0, \quad \theta > 0$$

$$E(X^r) = \left(\frac{\theta^{\frac{1}{2}}}{\sigma}\right)^r \frac{K_{1+r}(2\sigma\theta^{\frac{1}{2}})}{K_1(2\sigma\theta^{\frac{1}{2}})} \tag{4.98}$$

$$\therefore E(X) = \left(\frac{\theta^{\frac{1}{2}}}{\sigma}\right)\frac{K_2(2\sigma\theta^{\frac{1}{2}})}{K_1(2\sigma\theta^{\frac{1}{2}})}$$
$$\therefore E(X^2) = \left(\frac{\theta^{\frac{1}{2}}}{\sigma}\right)^v \frac{K_3(2\sigma\theta^{\frac{1}{2}})}{K_1(2\sigma\theta^{\frac{1}{2}})}$$
$$Var(X) = \left(\frac{\theta^{\frac{1}{2}}}{\sigma}\right)\left[\frac{K_3\left(2\sigma\theta^{\frac{1}{2}}\right)}{K_1(2\sigma\theta^{\frac{1}{2}})} - \frac{K_2^2\left(2\sigma\theta^{\frac{1}{2}}\right)}{K_1^2(2\sigma\theta^{\frac{1}{2}})}\right] \tag{4.99}$$

$$L_{GIG(1,\sigma,\theta)} = \sqrt{\frac{2\theta}{2s+2\theta}}\frac{K_1(\sqrt{(2s+2\theta)2\sigma^2})}{K_1(2\sigma\theta^{\frac{1}{2}})}$$
$$= \sqrt{\frac{\theta}{s+\theta}}\frac{K_1(2\sqrt{(s+\theta)\sigma^2})}{K_1(2\sigma\theta^{\frac{1}{2}})} \tag{4.100}$$

Modality

$$-(2\sigma\theta^{\frac{1}{2}})+\frac{2\sigma\theta^{\frac{1}{2}}}{x^2}=0$$

$$(2\sigma\theta^{\frac{1}{2}})x^2+(2\sigma\theta^{\frac{1}{2}})=0$$

$$\therefore X_{mode}=1 \tag{4.101}$$

4.6 Willmot and Jorgensen Parameterizations

4.6.1 Introduction

In this section, covered the GIG distribution under two different parameterizations; Willmot and Jorgensen. Under each, we constructed the GIG distributions. We studied properties such as the r th moment, the Laplace transforms and the modality of the distributions. Going by each special cases, we determined the same properties such as the r th moment, the Laplace transforms and the modalities of the special cases of the GIG distributions.

4.6.2 Willmot's Parameterization: $\omega=\frac{\mu}{\beta}$

Given the Sichel's paramterization, $\omega=\sqrt{\phi\psi}$ we therefore substitute the following,

$$When \quad \phi=\mu^2 \quad and \quad \psi=\frac{1}{\beta^2} \tag{4.102}$$

$$\therefore \omega=\sqrt{\frac{\mu^2}{\beta^2}}=\frac{\mu}{\beta}$$

$$K_v\left(\sqrt{\frac{\mu^2}{\beta^2}}\right)=\frac{1}{2}\int_0^\infty x^{v-1}e^{-\frac{1}{2}\sqrt{\frac{\mu^2}{\beta^2}}(x+\frac{1}{x})} \quad dx$$

$$K_v\left(\frac{\mu}{\beta}\right)=\frac{1}{2}\int_0^\infty x^{v-1}\exp\left(-\frac{\mu}{2\beta}\left(x+\frac{1}{x}\right)\right) \quad dx$$

Therefore the GIG is a pdf given by

$$f(x)=\frac{\mu^{-v}x^{v-1}}{2K_v(\frac{\mu}{\beta})}\exp\{-\frac{(x^2+\mu^2)}{2\beta x}\} \tag{4.103}$$

Where
$\mu,\beta>0, for \quad x>0$

$$E(X^r) = \mu^r \frac{K_{v+r}(\frac{\mu}{\beta})}{K_v(\frac{\mu}{\beta})} \tag{4.104}$$

$$\therefore E(X) = \mu \frac{K_{v+1}(\frac{\mu}{\beta})}{K_v(\frac{\mu}{\beta})} \tag{4.105}$$

$$E(X^2) = \mu^2 \frac{K_{v+2}(\frac{\mu}{\beta})}{K_v(\frac{\mu}{\beta})}$$

$$Var(X) = \mu^2 \frac{K_{v+2}(\frac{\mu}{\beta})}{K_v(\frac{\mu}{\beta})} - \left[\mu \frac{K_{v+1}(\frac{\mu}{\beta})}{K_v(\frac{\mu}{\beta})}\right]^2$$

$$\therefore = \mu^2 \left[\frac{K_{v+2}(\frac{\mu}{\beta})}{K_v(\frac{\mu}{\beta})} - \frac{K^2_{v+1}(\frac{\mu}{\beta})}{K^2_v(\frac{\mu}{\beta})}\right] \tag{4.106}$$

$$L_X(s) = \frac{\mu^{-v}}{2K_v\frac{\mu}{\beta}} 2 \left(\frac{\mu^2}{2\beta\left(\frac{1}{2\beta+s}\right)^{\frac{v}{2}}}\right) K_v 2\sqrt{\frac{\mu^2}{2\beta}(\frac{1}{2\beta}+s))}$$

$$= \frac{1}{K_v\frac{\mu}{\beta}} \frac{1}{(1+2\beta s)^{\frac{v}{2}}} K_v(2\sqrt{\frac{\mu^2}{2\beta(\frac{1}{2\beta}+s)}})$$

$$= (1+2\beta s)^{-\frac{v}{2}} \frac{K_v(\frac{\mu}{\beta}\sqrt{1+2\beta s})}{K_{(\frac{\mu}{\beta})}} \tag{4.107}$$

According to Willmot's notations,

$$L(s) = (1+2\beta s)^{\frac{-\alpha}{2}} \frac{K_\alpha\{\mu\beta^{-1}(1+2\beta s)^{\frac{1}{2}}\}}{K_\alpha(\mu\beta^{-1})} \tag{4.108}$$

To get the modality

$$X_{mode} = \frac{(v-1)+\sqrt{(v-1)^2+\frac{\mu^2}{\beta^2}}}{\frac{1}{\beta^2}} \quad \beta > 0$$

$$X_{mode} = \frac{\mu^2}{2(v-1)} \quad for \quad \beta = 0$$

4.7 Special Cases

4.7.1 Inverse Gaussian Distribution

When

$$v = -\frac{1}{2} \tag{4.109}$$

$$\begin{aligned} f(x) &= \mu(\frac{1}{2\pi\beta x^3})^{\frac{1}{2}} \exp\left(-\frac{(x^2+\mu^2-2\beta x)}{2\beta x}\right) \\ &= \left(\frac{\frac{\mu^2}{\beta}}{2\pi x^3}\right)^{\frac{1}{2}} \exp\{-\frac{\mu^2}{\beta}(\frac{(x-\mu)^2}{2\mu^2 x})\} \end{aligned}$$

$$E(X^r) = \beta^r \frac{K_{r-\frac{1}{2}}(\frac{\mu}{\beta})}{K_{\frac{1}{2}}(\frac{\mu}{\beta})}$$

$$But \quad K_{r-\frac{1}{2}} = \sqrt{\frac{\pi}{\frac{2\mu}{\beta}}} \exp -\frac{\mu}{\beta}\{1+\sum_{i=1}^{r-1}\frac{(r+i-1)!}{(r-i-1)!}\frac{(\frac{2\mu}{\beta})^{-i}}{i!}\}$$

$$\begin{aligned} K_{-\frac{1}{2}}(\frac{\mu}{\beta}) &= K_{\frac{1}{2}}(\frac{\mu}{\beta}) \\ &= \sqrt{\frac{\pi}{\frac{2\mu}{\beta}}} \exp -\frac{\mu}{\beta} \\ E(x^r) &= \mu^r\{1+\sum_{i=1}^{r-1}\frac{(r+i-1)!}{(r-i-1)!}\frac{(\frac{2\mu}{\beta})^{-i}}{i!}\} \\ E(X^r) &= \mu^r \end{aligned} \tag{4.110}$$

Therefore
$E(X) = \mu$

$$\begin{aligned} E(X^2) &= \mu^2\{1+\sum_{i=1}^{r-1}\frac{(2+i-1)!}{(2-i-1)!}\frac{(\frac{2\mu}{\beta})^{-i}}{i!}\} \\ &= \mu^2\{1+\frac{2!}{0!}\frac{(\frac{2\mu}{\beta})^{-i}}{1!}\} \\ &= \mu^2+(1+\frac{\mu}{\beta}) \end{aligned}$$

$$
\begin{aligned}
Var(X) &= E(X^2) - [E(X)]^2 \\
&= \mu^2 + (1 + \frac{\mu}{\beta}) - \mu^2 \\
&= 1 + \frac{\mu}{\beta} \qquad (4.111)
\end{aligned}
$$

The Laplace Transform

$$
\begin{aligned}
L_X(s) &= (\sqrt{1+2\beta s})^{\frac{1}{2}} \frac{K_{-\frac{1}{2}}\{1+2\beta s\}^{-\frac{1}{2}}}{K_{-\frac{1}{2}}\mu\beta^{-1}} \\
&= (\sqrt{1+2\beta s})^{\frac{1}{2}} \frac{(\frac{\pi}{2\mu\beta^{-1}\sqrt{1+2\beta s}})^{\frac{1}{2}} \exp(-\mu\beta^{-1})\sqrt{1+2\beta s}}{(\frac{\pi}{2\mu\beta^{-1}})^{\frac{1}{2}} \exp(-\mu\beta^{-1})} \\
&= (\frac{\pi\sqrt{1+2\beta s}}{2\mu\beta^{-1}\sqrt{1+2\beta s}} \frac{2\mu\beta^{-1}}{\pi})^{\frac{1}{2}} \exp\{\mu\beta^{-1} - \mu\beta^{-1}\sqrt{1+2\beta s}\} \\
&= \exp\{-\frac{\mu}{\beta}[(1+2\beta s)^{\frac{1}{2}} - 1]\} \qquad (4.112)
\end{aligned}
$$

To get the modality

$$
\frac{d}{dx}\left[x^{-\frac{3}{2}} \exp\left(-\frac{\mu}{2\beta}(x + \frac{1}{x})\right)\right] = 0
$$

$$
-3x - \frac{\mu}{\beta}(x^2 - 1) = 0
$$

$$
\therefore X_{mode} = \frac{-3 + \sqrt{9 + 4\frac{\mu^2}{\beta^2}}}{\frac{2\mu}{\beta}}
$$

4.7.2 Case 2: $v = \frac{1}{2}$

When

$$
v = \frac{1}{2} \qquad (4.113)
$$

$$
f(x) = \left(\frac{\beta^{-1}}{2\pi x}\right)^{\frac{1}{2}} \exp\{-\frac{\mu^2}{2\beta x}(1 - \frac{x}{\mu})\}
$$

$$
\begin{aligned}
E(X^r) &= \frac{\mu^r K_{r+\frac{1}{2}}(\mu\beta^{-1})}{K_{\frac{1}{2}}(\mu\beta^{-1})} \\
&= \mu^r\{1 + \sum_{i=1}^{r} \frac{(r+i)!(2\omega)^{-i}}{(r-i)!i!}\} \qquad (4.114)
\end{aligned}
$$

$$\therefore E(X) = \mu\{1 + \frac{1}{\omega}\} \quad where \quad \omega = \mu\beta^{-1} \tag{4.115}$$

$$\begin{aligned} E(X^2) &= \mu^2\{1 + \sum_{i=1}^{2} \frac{(2+i)!(2\omega)^{-i}}{(2-i)!i!}\} \\ &= \mu^2\{1 + \frac{3!(2\mu\beta^{-1})^{-1}}{1!1!} + \frac{4!(2\mu\beta^{-1})^{-2}}{0!2!}\} \\ &= \mu^2\{1 + \frac{3}{2\mu\beta^{-1}} + \frac{3}{(\mu\beta^{-1})^2}\} \end{aligned}$$

$$\begin{aligned} Var(X) &= \mu^2\{\frac{\beta}{\mu} + \frac{2\beta^2}{\mu^2}\} \\ &= \mu + 2\beta^2 \end{aligned} \tag{4.116}$$

$$\begin{aligned} L_X(s) &= (\sqrt{1+2\beta s})^{\frac{1}{2}}(\frac{\pi 2\mu\beta^{-1}}{2\mu\beta^{-1}\sqrt{(1+2\beta s)\pi}})^{\frac{1}{2}} \frac{\exp(-\mu\beta^{-1}\sqrt{1+2\beta s})}{\exp(-\mu\beta^{-1})} \\ &= (\frac{1}{\sqrt{1+2\beta s}})\exp\{-\mu\beta^{-1}\sqrt{1+2\beta s} + \mu\beta^{-1}\} \end{aligned} \tag{4.117}$$

Modality

$$\frac{d}{dx}f(x) = 0$$

$$-1 - x^{-1}\frac{\mu}{\beta}(1 - \frac{1}{x^2}) = 0$$

$$X_{mode} = \frac{-1 + \sqrt{1 + \frac{\mu^2}{4\beta^2}}}{2\frac{\mu}{\beta}} \tag{4.118}$$

4.7.3 Gamma Distribution

When

$$\mu^2 = 0, \quad \frac{1}{\beta^2} > 0 \tag{4.119}$$

$$f(x) = \frac{\left(\frac{1}{2\beta^2}\right)^{\nu}}{\Gamma(\nu)} e^{-\frac{1}{2\beta^2}x} x^{\nu-1}, \quad x > 0, \quad \frac{1}{\beta^2} > 0 \tag{4.120}$$

This is a gamma distribution with parameters v and $\frac{1}{2\beta^2}$

$$E(X^r) = \frac{\Gamma(v+r)}{\Gamma(v)\left(\frac{1}{2\beta^2}\right)^r} \tag{4.121}$$

$$\begin{aligned}
E(X) &= \frac{\Gamma v}{\Gamma v\left(\frac{1}{2\beta^2}\right)} \\
&= \frac{1}{\frac{1}{2\beta^2}} \\
and \quad E(X^2) &= \frac{\Gamma(v+2)}{\Gamma v\left(\frac{1}{2\beta^2}\right)^2} \\
\therefore Var(X) &= \frac{1}{\left(\frac{1}{2\beta^2}\right)^2}\left[\frac{\Gamma(v+2)}{\Gamma(v)} - 1\right]
\end{aligned} \tag{4.122}$$

The Laplace of gamma distribution is

$$\begin{aligned}
L_x(s) &= E\left(e^{-sX}\right) \\
&= \left(\frac{\frac{1}{2\beta^2}}{\frac{1}{2\beta^2}+s}\right)^v
\end{aligned} \tag{4.123}$$

Modality

$$\frac{d}{dx}f(x) = 0$$

$$\therefore X_{mode} = \frac{1}{2\beta^2(v-1)}$$

When

$$v = -\frac{1}{2}, \quad \mu^2 = 0, \quad \frac{1}{\beta^2} > 0 \tag{4.124}$$

$$f(x) = \frac{\left(\frac{1}{2\beta^2}\right)^{-\frac{1}{2}}}{\Gamma(-\frac{1}{2})} e^{-\frac{1}{2\beta^2}x} x^{-\frac{1}{2}-1}, \quad x > 0, \quad \frac{1}{\beta^2} > 0 \tag{4.125}$$

$$E(X^r) = \frac{\Gamma(-\frac{1}{2}+r)}{\Gamma(-\frac{1}{2})\left(\frac{1}{2\beta^2}\right)^r} \tag{4.126}$$

$$E(X) = \frac{1}{\frac{1}{2\beta^2}} \tag{4.127}$$

$$and \quad E(X^2) = \frac{\Gamma(-\frac{1}{2}+2)}{\Gamma(-\frac{1}{2})\left(\frac{1}{2\beta^2}\right)^2}$$

$$\therefore Var(X) = \frac{1}{\left(\frac{1}{2\beta^2}\right)^2}\left[\frac{\Gamma(-\frac{1}{2}+2)}{\Gamma(-\frac{1}{2})} - 1\right] \tag{4.128}$$

$$L_x(s) = \frac{\left(\frac{1}{2\beta^2}\right)^{-\frac{1}{2}}}{\left(\frac{1}{2\beta^2}+s\right)^{-\frac{1}{2}}}$$

$$= \left(\frac{\frac{1}{2\beta^2}}{\frac{1}{2\beta^2}+s}\right)^{-\frac{1}{2}} \tag{4.129}$$

Modality

$$\frac{d}{dx}f(x) = 0$$

$$\frac{d}{dx}\left[\frac{\left(\frac{1}{2\beta^2}\right)^{-\frac{1}{2}}}{\Gamma-\frac{1}{2}} e^{-\frac{1}{2\beta^2}x} x^{-\frac{1}{2}-1}\right] = 0$$

$$\therefore X_{mode} = -\frac{1}{3\beta^2}$$

When

$$v = \frac{1}{2}, \quad \mu^2 = 0, \frac{1}{\beta^2} > 0 \tag{4.130}$$

$$f(x) = \frac{\left(\frac{1}{2\beta^2}\right)^{\frac{1}{2}}}{\Gamma(\frac{1}{2})} e^{-\frac{1}{2\beta^2}x} x^{\frac{1}{2}-1}, \quad x > 0, \quad \frac{1}{\beta^2} > 0$$

$$E(X^r) = \frac{\Gamma(\frac{1}{2}+r)}{\Gamma(\frac{1}{2})\left(\frac{1}{2\beta^2}\right)^r}$$

$$E(X) = \frac{1}{\frac{1}{2\beta^2}}$$

$$and \quad E(X^2) = \frac{\Gamma(\frac{1}{2}+2)}{\Gamma(\frac{1}{2})\left(\frac{1}{2\beta^2}\right)^2}$$

$$\therefore Var(X) = \frac{1}{\left(\frac{1}{2\beta^2}\right)^2}\left[\frac{\Gamma(\frac{1}{2}+2)}{\Gamma(\frac{1}{2})} - 1\right]$$

$$\begin{aligned} L_X(s) &= E\left(e^{-sX}\right) \\ &= \frac{\left(\frac{1}{2\beta^2}\right)^{\frac{1}{2}}}{\left(\frac{1}{2\beta^2}+s\right)^{\frac{1}{2}}} \\ &= \left(\frac{\frac{1}{2\beta^2}}{\frac{1}{2\beta^2}+s}\right)^{\frac{1}{2}} \end{aligned}$$

Modality

$$(\frac{1}{2}-1)x^{\frac{1}{2}-2} - x^{\frac{1}{2}-1}\frac{1}{2\beta^2} = 0$$

$$x = \frac{\frac{1}{2\beta^2}}{(\frac{1}{2}-1)}$$

$$x = \frac{1}{2\beta^2(\frac{1}{2}-1)}$$

$$\therefore X_{mode} = -\frac{1}{\beta^2}$$

4.7.4 Exponential Distribution

$$v = 1, \quad \mu^2 = 0, \quad \frac{1}{\beta^2} > 0 \tag{4.131}$$

$$f(x) = \frac{1}{2\beta^2}e^{-\frac{1}{2\beta^2}x} \quad for \quad x > 0 \quad and \quad \frac{1}{\beta^2} > 0$$

$$E(X^r) = \left(2\beta^2\right)^r\Gamma(r+1) = r!\left(2\beta^2\right)^r \tag{4.132}$$

This is an exponential distribution with parameters $\frac{1}{2\beta^2} > 0$

$$
\begin{aligned}
E(X) &= 2\beta^2 \\
\therefore E(X^2) &= \frac{2}{\left(\frac{1}{2\beta^2}\right)^2} \\
Var(X) &= \frac{1}{\left(\frac{1}{2\beta^2}\right)^2}
\end{aligned} \tag{4.133}
$$

$$X_{mode} = \infty \tag{4.134}$$

4.7.5 Inverse Gamma Distribution

This is the case when

$$v < 0, \quad \mu^2 > 0, \quad \frac{1}{\beta^2} = 0 \tag{4.135}$$

$$f(x) = \frac{\left(\frac{\mu^2}{2}\right)^\lambda}{\Gamma\lambda} e^{-\frac{\mu^2}{2}\frac{1}{x}} x^{-\lambda-1} \tag{4.136}$$

Which is an inverse Gamma pdf with v<0

$$E(X^r) = \frac{\left(\frac{\mu^2}{2}\right)^\lambda}{\Gamma\lambda} \frac{\Gamma(\lambda - r)}{\left(\frac{\mu^2}{2}\right)^{\lambda-r}} \tag{4.137}$$

when $\lambda > 1$

$$\therefore E(X) = \frac{\mu^2}{2}\frac{1}{\lambda - 1} \tag{4.138}$$

$$
\begin{aligned}
\therefore E(X^2) &= \frac{\left(\frac{\mu^2}{2}\right)^\lambda}{\Gamma\lambda} \frac{\Gamma(\lambda - 2)}{\left(\frac{\mu^2}{2}\right)^{\lambda-2}} \\
&= \left(\frac{\mu^2}{2}\right)^2 \frac{1}{(\lambda-1)(\lambda-1)}
\end{aligned} \tag{4.139}
$$

$$
\begin{aligned}
Var(X) &= \left(\frac{\mu^2}{2}\right)^2 \frac{1}{\lambda-1}\left[\frac{\lambda-1-\lambda+2}{(\lambda-1)(\lambda-2)}\right] \\
&= \left(\frac{\mu^2}{2}\right)^2 \frac{1}{(\lambda-1)^2(\lambda-2)} \quad for \quad \lambda > 2
\end{aligned} \tag{4.140}
$$

$$
\begin{aligned}
L_X(s) &= 2\left(\frac{\mu^2}{2}\right)^{\lambda}\left(\sqrt{\frac{\mu^2}{2s}}\right)^{-\lambda-1}\frac{K_{-\lambda}\sqrt{(2s)\mu^2}}{\Gamma(\lambda)} \\
&= 2\left[\sqrt{\frac{\mu^2}{2}}\right]^{\lambda}\frac{K_{-\lambda}(\sqrt{2\mu^2 s})}{\Gamma(\lambda)} \qquad (4.141)
\end{aligned}
$$

For modality

$$
\begin{aligned}
\therefore \frac{x^{\lambda}}{\lambda+1} &= \frac{2}{\mu^2} \\
\therefore X_{mode} &= \left[\frac{2(\lambda+1)}{\mu^2}\right]^{\frac{1}{\lambda}} \qquad (4.142)
\end{aligned}
$$

4.7.6 Levy Distribution

This is a special case of inverse gamma when

$$
v = -\frac{1}{2}, \quad \mu^2 > 0, \quad \frac{1}{\beta^2} = 0 \qquad (4.143)
$$

Therefore $\lambda = \frac{1}{2}$

$$
f(x) = \frac{\left(\frac{\mu^2}{2}\right)^{\frac{1}{2}}}{\Gamma(\frac{1}{2})} e^{-\frac{\mu^2}{2}\frac{1}{x}} x^{-\frac{1}{2}-1}
$$

i.e

$$
\begin{aligned}
f(x) &= \sqrt{\frac{\mu^2}{2\pi}} x^{-\frac{\mu^2}{2}\frac{1}{x}} \\
&= \sqrt{\frac{\mu^2}{2\pi x^3}} e^{-\frac{\mu^2}{2}\frac{1}{x}}, \quad x > 0; \quad \mu^2 > 0 \\
E(X^r) &= \left(\frac{\mu^2}{2}\right)^{r}\frac{\Gamma(\frac{1}{2}-r)}{\Gamma(\frac{1}{2})} \\
&= \left(\frac{\mu^2}{2}\right)^{r}\frac{\Gamma(\frac{1}{2}-r)}{\sqrt{\pi}} \qquad (4.144)
\end{aligned}
$$

$$
\begin{aligned}
E(X) &= \frac{\mu^2}{2}\frac{\Gamma(\frac{1}{2})}{-\frac{1}{2}\Gamma(\frac{1}{2})} \\
\therefore E(X) &= -(\mu^2) \qquad (4.145)
\end{aligned}
$$

$$
E(X^2) = \left(\frac{\mu^2}{2}\right)^{2}\frac{\Gamma(-\frac{3}{2}+1)}{-\frac{3}{2}\Gamma(\frac{1}{2})}
$$

$$
\begin{aligned}
E(X^2) &= \left(\frac{\mu^2}{2}\right)^2 \frac{1}{\frac{3}{4}} = \frac{\mu^4}{3} \\
\therefore Var(X) &= \frac{\mu^4}{3} - (\mu^4) = \frac{\mu^4}{3} - \mu^4 = -\frac{2}{3}\mu^4
\end{aligned} \tag{4.146}
$$

Alternatively using

$$
\begin{aligned}
Var(x) &= \frac{\mu^4}{4}\frac{1}{(-\frac{1}{2})(-\frac{3}{2})} \\
&= -\frac{2}{3}\mu^4
\end{aligned}
$$

Remark: $\lambda > 2 \Rightarrow v < -2$ for Var X to exist

Laplace transform is

$$
\begin{aligned}
L_X(s) &= \left[2\sqrt{\frac{\mu^2 s}{2}\frac{1}{2\mu^2 s}}\right]^{\frac{1}{2}} e^{-\sqrt{2\mu^2 s}} \\
&= e^{-\sqrt{2\mu^2 s}}
\end{aligned} \tag{4.147}
$$

4.7.7 Positive Hyperbolic Distribution

When

$$
v = 1, \quad \mu > 0, \quad \beta > 0 \tag{4.148}
$$

$$
\begin{aligned}
f(x) &= \frac{\mu^{-1}x^{1-1}\exp\left(-\frac{(x^2+\mu^2)}{2\beta x}\right)}{2K_1(\frac{\mu}{\beta})} \\
&= \mu^{-1}\exp\left(-\frac{(x^2+\mu^2)}{2\beta x}\right), \quad x > 0, \quad \beta > 0, \quad \mu > 0
\end{aligned}
$$

This is a positive hyperbolic distribution.

4.8 Jorgensen's Parameterization

4.8.1 Introduction

In this section, we shall construct the GIG distribution under Jorgensen parameterization and determine various properties. Most of the properties have been discussed in the previous sections under the special cases.

4.8.2 Construction

Using Sichel's Parameterization, $\omega = \sqrt{\phi\psi}$ and $\eta = \sqrt{\frac{\psi}{\phi}}$

Therefore

$$\psi = \omega\eta \quad and \quad \phi = \frac{\omega}{\eta} \tag{4.149}$$

Given that;

$$K_v(\omega) = \frac{1}{2}\int_0^{\infty} x^{v-1}\exp\left[-\frac{1}{2}(\psi x + \frac{\phi}{x})\right] \quad dx$$

and $\phi, \quad \psi > 0$ we have the notation $\eta = \sqrt{\frac{\pi}{\phi}}$ and $\omega = \sqrt{\phi\psi}$

$$\begin{aligned} K_v(\omega) &= \int_0^{\infty} x^{v-1}\exp\left[-\frac{1}{2}(\psi x + \frac{\phi}{x})\right] \quad dx \\ &= \int_0^{\infty} x^{v-1}\left[-\frac{1}{2}(\omega\eta x^{-1} + \frac{\omega}{\eta}x)\right] \quad dx \end{aligned}$$

We let $x = \eta z \Rightarrow dx = \eta dz$

$$\begin{aligned} K_v(\psi,\phi) &= \int_0^{\infty} (\eta z)^{v-1}\exp\left[-\frac{\omega}{2}(\frac{\eta}{\eta z} + \frac{\eta z}{\eta})\right]\eta dz \\ &= \eta^v\int_0^{\infty} z^{v-1}\exp\left[-\frac{\omega}{2}(\frac{1}{z}+z)\right] \quad dz \\ &= 2\eta^v\frac{1}{2}\int_0^{\infty}\exp\left[-\frac{\omega}{2}(\frac{1}{z}+z)\right] \quad dz \\ \therefore 2\left(\sqrt{\frac{\phi}{\psi}}\right)^v K_v(\sqrt{\psi\phi}) &= \int_0^{\infty} x^{v-1}\exp\left[-\frac{1}{2}(\psi x + \frac{\phi}{x})\right] \quad dx \\ 1 &= \frac{x^{v-1}\exp\left[-\frac{1}{2}(\psi x + \frac{\phi}{x})\right]}{2\left(\sqrt{\frac{\phi}{\psi}}\right)^v K_v(\sqrt{\psi\phi})} \\ \therefore f(x) &= \frac{x^{v-1}\exp\left[-\frac{1}{2}(\psi x + \frac{\phi}{x})\right]}{2\left(\sqrt{\frac{\phi}{\psi}}\right)^v K_v(\sqrt{\psi\phi})} \\ &= \frac{x^{v-1}e^{-\frac{1}{2}\left(\omega\eta x + \frac{\omega}{\eta x}\right)}}{2\eta^v K_v(\omega)} \end{aligned}$$

$$f(x) = \frac{x^{v-1}e^{-\frac{\omega}{2}\left(\eta x + \frac{1}{\eta x}\right)}}{2\eta^v K_v(\omega)} \qquad -\infty < v < \infty, \quad \eta > 0, \quad \omega \geq 0 \tag{4.150}$$

$$f(x) = \frac{x^{v-1} e^{-\frac{\omega}{2}(\eta x + \frac{1}{\eta x})}}{\int_0^\infty x^{v-1} e^{-\frac{\omega}{2}(\eta x + \frac{1}{\eta x})} dx}, \quad x > 0; \quad -\infty < v < \infty, \quad \eta \geq 0, \quad \omega \geq 0 \tag{4.151}$$

4.8.3 Property 3.1: Moments

$$E(X^r) = \left(\frac{1}{\eta}\right)^r \frac{K_{v+r}(\omega)}{K_v(\omega)} \tag{4.152}$$

$$E(X) = \left(\frac{1}{\eta}\right) \frac{K_{v+1}(\omega)}{K_v(\omega)} \tag{4.153}$$

$$Var(X) = \left(\frac{1}{\eta}\right)^2 \left(\frac{K_{v+2}(\omega)}{K_v(\omega)} - \left[\frac{K_{v+1}(\omega)}{K_v(\omega)}\right]^2\right) \tag{4.154}$$

4.8.4 Property 3.2: Laplace transform

The Laplace is

$$L_X(s) = \left(\sqrt{\frac{\omega\eta}{2s + \omega\eta}}\right)^v \frac{K_v(\sqrt{(2s + \omega\eta)\frac{\omega}{\eta}})}{K_v(\omega)} \tag{4.155}$$

4.8.5 Property 3.3: Modality

The mode of GIG distribution is which maximizes the pdf f(x). This peak is obtained by solving the equation

$$\frac{d}{dx} f(x) = 0$$

Therefore,

$$X_{mode} = \frac{(v-1) + \sqrt{(v-1)^2 + \omega^2}}{\omega\eta}, \quad \omega > 0$$

$$and \quad X_{mode} = \frac{\frac{\omega}{\eta}}{2(1-v)}, \quad \omega = 0 \tag{4.156}$$

4.9 Special cases of GIG Distribution

4.9.1 Inverse Gaussian Distribution

When

$$v = -\frac{1}{2} \tag{4.157}$$

$$f(x) = \sqrt{\frac{\frac{\omega}{\eta}}{2\pi x^3}} e^{\omega} e^{-\frac{\omega}{2}(\eta x + \frac{1}{\eta x})}$$

$$E(X^r) = \left(\frac{1}{\eta}\right)^r$$

$$L_X(s) = e^{\omega - \sqrt{(2s+\omega\eta)\frac{\omega}{\eta}}}$$
$$X_{mode} = \frac{3}{2}\left[\sqrt{1 + \frac{4}{9}\omega^2} - 1\right] \quad \textbf{(4.158)}$$

4.9.2 Reciprocal Inverse Gaussian

This is the case when

$$v = \frac{1}{2}, \quad \omega > 0, \quad \eta > 0 \quad \textbf{(4.159)}$$

Then the pdf of RIG distribution is

$$f(x) = \left(\frac{2\omega\eta}{\pi x}\right)^{\frac{1}{2}} e^{-\frac{\omega}{2}(\eta x + \frac{1}{\eta x})} \quad for \quad x > 0; \quad \omega > 0, \quad \eta > 0 \quad \textbf{(4.160)}$$

The moments are:

$$E(X^r) = \left(\frac{1}{\eta}\right)^r \frac{K_{\frac{1}{2}+r}(\omega)}{K_{\frac{1}{2}}(\omega)} \quad \textbf{(4.161)}$$
$$E(X) = \frac{1}{\eta}(1+\omega) \quad \textbf{(4.162)}$$

$$Var(X) = \left(\frac{1}{\eta}\right)^2 \left(3\omega + \omega^2(1-\omega^2)\right) \quad \textbf{(4.163)}$$

The Laplace of RIG,

$$\begin{aligned} L_X(s) &= \sqrt{\frac{\omega\eta}{2s+\omega\eta}} e^{\omega - \sqrt{(2s+\omega\eta)\frac{\omega}{\eta}}} \\ &= \sqrt{\frac{\omega\eta}{2s+\omega\eta}} L_{GIG(-\frac{1}{2},\frac{\omega}{\eta},\omega\eta)} \\ &= L_{GIG(\frac{1}{2},0,\omega\eta)} L_{GIG(-\frac{1}{2},\frac{\omega}{\eta},\omega\eta)} \end{aligned} \quad \textbf{(4.164)}$$

where

$$L_{GIG(\frac{1}{2},0,\omega\eta)} = \left(\frac{\omega\eta}{2s+\omega\eta}\right)^{\frac{1}{2}}$$

is the Laplace of a gamma distribution with parameters $\frac{1}{2}$ and $\frac{\omega\eta}{2}$

The mode of RIG is

$$X_{mode} = \frac{\sqrt{1+\frac{\omega^2}{4}}-1}{2\omega} \tag{4.165}$$

4.9.3 Gamma Distribution

When

$$v > 0, \quad \frac{\omega}{\eta} = 0, \quad \omega\eta > 0 \tag{4.170}$$

$$f(x) = \frac{\left(\frac{\omega\eta}{2}\right)^v}{\Gamma(v)} e^{-\frac{\omega\eta}{2}x} x^{v-1} \quad x > 0; \quad v > 0 \tag{1.171}$$

This is called a gamma pdf with parameters v and $\frac{\psi}{2}$

Thus

$$GIG(v,0,\omega\eta) = Gamma(v, \frac{\omega\eta}{2})$$

$$E(X^r) = \left(\frac{2}{\omega\eta}\right)^r \frac{\Gamma(v+r)}{\Gamma(v)} \tag{4.172}$$

$$\begin{aligned}
E(X) &= \left(\frac{2}{\omega\eta}\right)\frac{\Gamma(v+1)}{\Gamma(v)} = \frac{2}{\omega\eta}v \\
E(X^2) &= \left(\frac{2}{\omega\eta}\right)^2\frac{\Gamma(v+2)}{\Gamma(v)} = \left(\frac{2}{\omega\eta}\right)^2(v+1)v \\
\therefore Var(X) &= \left(\frac{2}{\omega\eta}\right)^2(v+1)v - \left(\frac{2}{\omega\eta}\right)^2 v^2 = \left(\frac{2}{\omega\eta}\right)^2 v
\end{aligned} \tag{4.173}$$

$$L_X(s) = \left(\frac{\frac{\omega\eta}{2}}{\frac{\omega\eta}{2}+s}\right)^v$$

The mode is

$$\therefore X_{mode} = \frac{\omega\eta}{2(v-1)}, v \neq 1$$

and

$$X_{mode} = \infty \quad if \quad v = 1 \tag{4.174}$$

4.9.4 Exponential Distribution

This is a special case of a gamma distribution with

$$v = 1, \quad \omega\eta > 0 \tag{4.175}$$

$$\begin{aligned} f(x) &= \frac{e^{-\frac{\omega\eta}{2}x}}{\frac{1}{\frac{\omega\eta}{2}}} \\ &= \frac{\omega\eta}{2} e^{-\frac{\omega\eta}{2}x} \quad for \quad x > 0 \quad and \quad \omega\eta > 0 \end{aligned}$$

$$E(X^r) = \left(\frac{2}{\omega\eta}\right)^r \Gamma(r+1) = r! \left(\frac{2}{\omega\eta}\right)^r \tag{4.176}$$

This is an exponential distribution with parameters $\frac{\psi}{2} > 0$

$$\begin{aligned} E(X) &= \frac{2}{\omega\eta} \\ \therefore E(X^2) &= \frac{2}{\left(\frac{\omega\eta}{2}\right)^2} \\ Var(X) &= \frac{2}{\left(\frac{\omega\eta}{2}\right)^2} - \frac{1}{\left(\frac{\omega\eta}{2}\right)^2} \\ &= \frac{1}{\left(\frac{\omega\eta}{2}\right)^2} \end{aligned} \tag{4.177}$$

$$L_X(s) = \frac{\frac{\omega\eta}{2}}{\frac{\omega\eta}{2} + s} = \frac{\omega\eta}{2s + \omega\eta} \tag{4.178}$$

$$\begin{aligned} e^{-\frac{\omega\eta}{2}x} &= 0 \\ X_{mode} &= \infty \end{aligned} \tag{4.179}$$

4.9.5 Inverse Gamma Distribution

This is the case when

$$v < 0, \quad \omega\eta = 0 \tag{4.180}$$

$$f(x) = \frac{\left(\frac{\frac{\omega}{\eta}}{2}\right)^{\lambda}}{\Gamma\lambda} e^{-\frac{\frac{\omega}{\eta}}{2}\frac{1}{x}} x^{-\lambda-1} \tag{4.181}$$

Which is an inverse Gamma pdf with v<0

Thus

$$X \sim GIG(v, \frac{\omega}{\eta}, 0) = Inverse \quad Gamma(-v, \frac{\omega}{2\eta}), \quad wherev < 0 \quad and \quad \frac{\omega}{\eta} > 0$$

$$E(X^r) = \frac{\left(\frac{\omega}{2\eta}\right)^{\lambda}}{\Gamma\lambda} \frac{\Gamma(\lambda - r)}{\left(\frac{\omega}{2\eta}\right)^{\lambda - r}} \tag{4.182}$$

when $\lambda > 1$

$$\therefore E(X) = \frac{\omega}{2\eta} \frac{1}{\lambda - 1}$$
$$\therefore E(X^2) = \left(\frac{\omega}{2\eta}\right)^2 \frac{1}{(\lambda - 1)(\lambda - 1)} \tag{4.183}$$

$$Var(X) = \left(\frac{\omega}{2\eta}\right)^2 \frac{1}{(\lambda - 1)^2(\lambda - 2)} \quad for \quad \lambda > 2 \tag{4.184}$$

The Laplace Transform is

$$L_x(s) = 2\left[\sqrt{\frac{\omega}{2\eta}}\right]^{\lambda} \frac{K_{-\lambda}(\sqrt{2\omega\eta s})}{\Gamma(\lambda)} \tag{4.185}$$

For modality

$$X_{mode} = \left[\frac{2(\lambda + 1)}{\frac{\omega}{\eta}}\right]^{\frac{1}{\lambda}} \tag{4.186}$$

4.9.6 Levy Distribution

Is a special case of inverse gamma distribution when

$$v = -\frac{1}{2}, \quad \frac{\omega}{\eta} > 0$$

Therefore $\lambda = \frac{1}{2}$

$$f(x) = \frac{\left(\frac{\omega}{2\eta}\right)^{\frac{1}{2}}}{\Gamma(\frac{1}{2})} e^{-\frac{\omega}{2\eta}\frac{1}{x}} x^{-\frac{1}{2}-1}$$

i.e

$$f(x) = \sqrt{\frac{\omega}{2\eta\pi x^3}} e^{-\frac{\omega}{2\eta}\frac{1}{x}}, \quad x > 0; \quad \frac{\omega}{2\eta} > 0$$

$$E(X^r) = \left(\frac{\omega}{2\eta}\right)^r \frac{\Gamma(\frac{1}{2} - r)}{\Gamma(\frac{1}{2})}$$

$$= \left(\frac{\omega}{2\eta}\right)^r \frac{\Gamma(\frac{1}{2} - r)}{\sqrt{\pi}} \tag{4.187}$$

$$E(X) = \left(\frac{\omega}{2\eta}\right) \frac{\Gamma(-\frac{1}{2})}{\sqrt{\pi}}$$

$$\therefore E(X) = \frac{\omega}{2\eta} \frac{\Gamma(-\frac{1}{2} + 1)}{-\frac{1}{2}\Gamma(\frac{1}{2})}$$

$$= \frac{\omega}{2\eta} \frac{\Gamma(\frac{1}{2})}{-\frac{1}{2}\Gamma(\frac{1}{2})}$$

$$\therefore E(X) = -\frac{\omega}{\eta} \tag{4.188}$$

$$E(X^2) = \left(\frac{\omega}{2\eta}\right)^2 \frac{\Gamma(-\frac{3}{2} + 1)}{-\frac{3}{2}\Gamma(\frac{1}{2})}$$

$$E(X^2) = \left(\frac{\omega}{2\eta}\right)^2 \frac{1}{\frac{3}{4}} = \frac{(\frac{\omega}{\eta})^2}{3}$$

$$\therefore Var(X) = \frac{\left(\frac{\omega}{\eta}\right)^2}{3} - (-\frac{\omega}{\eta})^2 = \frac{(\frac{\omega}{\eta})^2}{3} - (\frac{\omega}{\eta})^2 = -\frac{2}{3}\left(\frac{\omega}{\eta}\right)^2 \tag{4.189}$$

Alternatively using

$$\begin{aligned} Var(x) &= \frac{(\frac{\omega}{\eta})^2}{4}\frac{1}{(-\frac{1}{2})(-\frac{3}{2})} \\ &= -\frac{2}{3}\left(\frac{\omega}{\eta}\right)^2 \end{aligned}$$

Remark: $\lambda > 2 \Rightarrow v < -2$ for Var X to exist

Laplace transform is

$$\begin{aligned} L_X(s) &= \left[2\sqrt{\frac{\omega s}{2\eta}\frac{1}{2\frac{\omega}{\eta}s}}\right]^{\frac{1}{2}} e^{-\sqrt{2\frac{\omega}{\eta}s}} \\ &= e^{-\sqrt{2\frac{\omega}{\eta}s}} \end{aligned} \tag{4.190}$$

4.9.7 Positive Hyperbolic Distribution

This is the case when

$$v = 1, \quad \frac{\omega}{\eta} > 0, \quad \omega\eta > 0$$

$$X \sim GIG(1, \frac{\omega}{\eta}, \omega\eta)$$

So put v=1 in the results obtained in sections 3.2 and 3.3

$$f(x) = (\eta)\frac{e^{-\frac{\omega}{2}(\eta x + \frac{1}{\eta x})}}{2K_1(\omega)} \quad x > 0; \quad \frac{\omega}{\eta} > 0, \quad \omega\eta > 0$$

This is called a positive hyperbolic distribution.

$$E(X^r) = \left(\frac{1}{\eta}\right)^r \frac{K_{1+r}(\omega)}{K_1(\omega)} \tag{4.191}$$

$$\therefore E(X) = \left(\frac{1}{\eta}\right)\frac{K_2(\omega)}{K_1(\omega)} \tag{4.192}$$

$$\begin{aligned} \therefore E(X^2) &= \left(\frac{1}{\eta}\right)^2 \frac{K_3(\omega)}{K_1(\omega)} \\ Var(X) &= \left(\frac{1}{\eta}\right)^2 \frac{K_3(\omega)}{K_1(\omega)} - \left(\frac{1}{\eta}\right)^2 \frac{K_2^2(\omega)}{K_1^2(\omega)} \\ &= \left(\frac{1}{\eta}\right)^2 \left[\frac{K_3(\omega)}{K_1(\omega)} - \frac{K_2^2(\omega)}{K_1^2(\omega)}\right] \end{aligned} \tag{4.193}$$

$$L_{GIG(1,\frac{\omega}{\eta},\omega\eta)} = \sqrt{\frac{\omega\eta}{2s+\omega\eta}} \frac{K_1(\sqrt{(2s+\omega\eta)\frac{\omega}{\eta}})}{K_1(\omega)} \tag{4.194}$$

Modality is

$$-\omega + \frac{\omega}{x^2} = 0$$

$$\omega x^2 + \omega = 0$$

$$\therefore X_{mode} = 1 \tag{4.195}$$

5 CONCLUSSIONS AND RECOMMENDATIONS

5.1 Summary of Results and challanges

The objective of this study was to construct the Generalized inverse gaussian distributions under different parameterizations using modified Bessel function of the third kind wich is a special function.

Modified Bessel function of the third kind

Modified Bessel function of the third kind is one of the special functions that have been used throughout in construction of GIG distribution. Eventhough, there are quiet a number of special functions, we have only used it as the only special function in this work. Due to different parameterizations, we have derived Generalized inverse Gaussian distributions.

GIG distributions

We have seen that various forms of GIG distributions have been constructed. This was only by use of modified Bessel function of the third kind. We therefore worked with the five parameterizations to come up with various forms of GIG distributions. These are; The Sichel's, Jorgensen's, Allen's, Willmot's, and Barndorff-Nielsen Parameterizations. From Sichel's, we came up with all the other four parameterizations. In otherwords, we treated the other four parameterizations as special cases of the Sichel's parametrization. Under each, we have covered the following thematic areas; construction and properties, special cases which have given rise to GIG sub-models distributions, the convolution properties, power and the inverse of the GIG distribution which was only covered under Sichel's and Barndorff-Nielsen parameterizations. Using the change of variable technique and the cummulative distribution function technique, we came up with the power of the GIG distribution.

5.2 Recommendations

Modified Bessel function of the third kind

The only special function used is the modified Bessel function of the third kind. Other special functions such as Gamma, Beta, Modified Bessel functions of first and second kind and many other special distributions could be used as well.

Other Methods

Futher work could be to identify other methods in constructing the Generalized inverse Gaussian distributions, such as the use of mixture method and use of special functions other than the one used in this research work. GIG distribution could be used as a mixing distribution to come up with other distributions. Other special functions that could be used are the generalized Lindly, Paretto and Trasmuted exponential which are distributions of the finite mixtures. Other special functions that could be used are gamma, betta, of first and second kind Bessel functions .

Properties

In this work, we have only costructed the Generalized Inverse Gaussian distributions using the based on five parametrizations which have led to various forms of the GIG. We have obtained the general formular for various properties such as the moments, the laplace transforms and the modalities under all four parameterizations.

We have generalized the Generalized inverse Gaussian distributions and their properties. There were some special cases which led to sub-models of the GIG distributions. These are; the Inverse gaussian, the reciprocal inverse gaussian, the gamma, inverse gamma, exponential, positive hyperbolic and the Levy distributions. We have therefore worked on their statistical properties like the r th moment, the Laplace trasform and their modalities.

However, a lot has not been done on other properties and estimating parameters of GIG, the power GIG distribution and the GIG submodels as well. Extensive work in these areas would be very important.

Inference on Parameters and Applications

The main forcus of this work is on the construction of Generalized inverse Gaussian distribution based on the five parameterizations. Under theory of estimation, estimations, testing of hypotheses and extensive applications in real data situation of the Generalized inverse Gaussian distributions and other related distributions and its sub-models such as the inverse gaussian, the reciprocal inverse gaussian, gamma, the inverse gamma, exponential, positive hyperbolic and the Levy distributions are major areas for further research.

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