On Ricci Solitons as Quasi-Einstein metrics

Leon Fidele Ruganzu Uwimbabazi
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UNIVERSITY OF NAIROBI
COLLEGE OF BIOLOGICAL
AND PHYSICAL SCIENCES
SCHOOL OF MATHEMATICS
P.O.BOX 30197, Nairobi, Kenya.

DECLARATION AND APPROVAL

DECLARATION

I the undersigned declare that this thesis contains my own work. To the best of my knowledge, no portion of this work has been submitted in support of an application for another degree or qualification of this or any other university or other institution of learning.

SIGN············ DATE ············

Leon Fidele RUGANZU UWIMBABAZI

APPROVAL

This Thesis has been presented with our approval as University Supervisors

SIGN············ DATE ············

Prof G.P POKHARIYAL
University of Nairobi
School of Mathematics

SIGN············ DATE ············

Dr. S.K. Moindi
University of Nairobi
School of Mathematics.

SIGN············ DATE ············

Dr. J. N. Katende
University of Nairobi
School of Mathematics.

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Dedication

To my wife Chantal and my daughter Iris.

Abstract

This thesis is the key to good understanding of differential geometry with para- Kenmotsu and Lorentzian Para- Sasakian structure and it is organized as follows. In chapter one, the preliminaries and definitions are introduced,where, Manifolds, differentiable structures, Riemannian Manifolds and Ricci flows are defined. In chapter two the relevant literature is reviewed and Propositions and theorems proved in area are included. In chapter three, Ricci solitons on para- Kenmotsu Manifolds satisfying $(\xi, .)_s.W_8 = 0$ and $(\xi, .)_{W_8}.S = 0$ are discussed and we have proved that the Para- Kenmotsu manifolds satisfying $(\xi, .)_{W_8}.S = 0$. are quasi- Einstein Manifolds and those satisfying $(\xi, .)_S.W_8 = 0$, are Einstein Manifolds.Also it has been proved that the para- Kenmotsu manifolds with cyclic Ricci tensor and $\eta-$ Ricci soliton structure are quasi-Einstein manifolds . In chapter four, Ricci solitons on Lorentzian Para- Sasakian manifolds satisfying $(\xi, .)_s.W_8 = 0$ and $(\xi, .)_{W_8}.S = 0$ are treated and it has been proved that Lorentzian Para- Sasakian manifolds satisfying $(\xi, .)_s.W_8 = 0$ and having $\eta-$ Ricci soliton structure are quasi-Einstein manifolds and those satisfying $(\xi, .)_{W_8}.S = 0$ are Einstein manifolds. In chapter five, we discuss Ricci solitons on Lorentzian Para- Sasakian manifolds satisfying $(\xi, .)_s.W_2 = 0$ and $(\xi, .)_{W_2}.S = 0$ and it was found that, Lorentzian Para- Sasakian manifolds satisfying $(\xi, .)_s.W_2 = 0$ and having $\eta-$ Ricci soliton structure are Einstein or quasi-Einstein manifolds according to the value of μ and λ. In Chapter six, results are discussed and the connection between Ricci solitons and Einstein metrics on Para- Kenmotsu and Lorentzian Para Sasakian Manifolds has been established.

Contents

Chapter 1

Introduction

In this chapter, theory that is necessary through our research is introduced. As smoothness is the foundation of this research, we will start by reminding the basic theory starting by differentiability on $\mathbb{R}^n$.

1.1 Preliminaries and definitions

A topological space is said to be Hausdorff if for each pair of its distinct points, there exist neighborhoods with empty intersection. A locally Euclidean space is a topological space, such that each point has a neighborhood homeomorphic to an open subset of the Euclidean space. A manifold of dimension n is defined as a Hausdorff, locally Euclidean space of dimension n. A topological manifold is said to be differentiable (or smooth), if differentiable structure is defined on it. The manifolds are classified on the basis of their structures. Given a curve γ, on smooth manifold, its tangent vector(or simply a vector) is defined as the derivative of a differentiable function f, in direction of γ at origin. A vector field on smooth manifold M is an assignment of tangent vector at each point of M. A geodesic is a curve, such that its vector field is parallel along the given curve. The Riemannian metric is a positive definite bilinear, which is symmetrical by nature. The Riemannian metric tensor is useful in definition of metric properties, on differentiable manifolds, such as angles between vectors, curvature tensor, Riemannian curvature tensor , Ricci tensor and geodesics. A manifold M is said to be Riemannian manifold if a Riemannian metric tensor is defined on its tangent vector space. A pseudo-Riemannian manifold is a pair (M, g), where M is a smooth manifold and g is a metric tensor that is not positive- definite. A real manifold is differential manifold whose tangent vector space is real vector space. By introducing the complex structures in the manifolds, we obtain complex manifolds. Due to different structures we can introduce various manifolds. Manifolds are classified as even or odd dimensional according to the dimensions of their respective tangent vector spaces. An odd dimensional manifold is said to be Sasakian if the Sasakian structure is defined on it. An odd dimensional manifold is said to be para- Sasakian if para- Sasakian structure is given on it. A Lorentzian para Sasakian manifold is an almost - contact structure is given. If vector space and its dual have the same geodesics (are in geodesics correspondence), then the expression of Weyl curvature tensor is obtained. new curvature tensors have been defined by Pokhariyal and Mishra [1970] and Pokhariyal [1982a] on basis of Weyl curvature tensor having different combination of vector field associated to Ricci tensor and metric tensor. Ricci flows are partial differential equations whose variable is a metric tensor of a Riemannian manifold. Einstein manifolds are fixed points of Ricci flows and Ricci solitons are their generalized fixed points. Ricci solitons areAlso used in quasi-Einstein manifold. Ricci solitons On antisymmetric and semisymmetric para- Kenmonsu with respect to W_8 and on semisymmetric and anti symmetric Lorentzian para-Sasakian with respect to W_2 and W_8 have been studied.

1.2 Euclidean space

The Euclidean space $\mathbb{R}^n$, is a model of all manifolds, as every manifold looks, locally as $\mathbb{R}^n$. That is why the notion of smooth functions on $\mathbb{R}^n$, is introduced.The euclidean space $\mathbb{R}^n$ is the set of $n-$tuples $(x^1, \cdots, x^n)$, where x^i are real numbers. An element of $\mathbb{R}^n$ is called point of $\mathbb{R}^n$.In particular $\mathbb{R}$, $\mathbb{R}^2$ and $\mathbb{R}^3$ are respectively, called the line, the plane and space.The $\mathbb{R}^n$ is a vector space with operations:

$$\begin{cases} X + Y = (x^1 + Y^1, \cdots, x^n + y^n) \quad \forall X, Y \in \mathbb{R}^n \\ aX = (ax^1, \cdots, ax^n) \quad \forall a \in \mathbb{R}. \end{cases} \tag{1.2.1}$$

That is why, the elements of $\mathbb{R}^n$ are also, called vectors in $\mathbb{R}^n$.

Definition 1.2.1. *Let k be a positive integer and U an open subset of $\mathbb{R}^n$. A real valued function $f : U \to \mathbb{R}$ is said to be C^k at a point $p \in U$, if its partial derivatives*

$$\frac{\partial^j f}{\partial x^{i_1} \cdots \partial x^{i_j}},$$

of all orders $j \leq k$ exist and are continuous. A vector valued function $F : U \subset \mathbb{R}^n \to \mathbb{R}^m$, such that $F(p) = (F^1, \cdots, F^m)$ is said to be C^k if all its components $F^1, \cdots, F^m$ are C^k. A C^∞ function is called smooth.

1.3 Tangent vector in $\mathbb{R}^n$

The aim of this section is description of tangent vectors in $\mathbb{R}^n$ that will later be important to manifolds. In order to differentiate vectors from point, a vector will be presented as a column vector,

$$V = \begin{pmatrix} v^1 \\ v^2 \\ \vdots \\ v^n \end{pmatrix}. \tag{1.3.1}$$

Definition 1.3.1. *the tangent space $T_p(\mathbb{R}^n)$ is the vector space of all column vectors emanating from p.*

The line through a point $P = \begin{pmatrix} p^1 & \cdots & p^n \end{pmatrix}$ with direction $V = \begin{pmatrix} v^1 \\ \vdots \\ v^n \end{pmatrix}$ in $\mathbb{R}^n$ is given by the parameterization equation:

$$l(t) = \begin{pmatrix} p^1 + tv^1, & \cdots, & p^n + tv^n \end{pmatrix}. \tag{1.3.2}$$

If F is a smooth function in an open subset of $\mathbb{R}^n$ containing P, its directional derivative at P in the direction V is given by,

$$D_V F = \lim_{t \to 0} \frac{F(l(t)) - F(P)}{t} = \left. \frac{df}{dt} \right|_{t=0}. \tag{1.3.3}$$

Using the chain rule in(1.3.3), we get,

$$D_V F = \sum_{i=1}^{n} \frac{dl^i(0)}{dt} \frac{\partial F}{\partial x^i}(P) = \sum_{i=1}^{n} v^i \frac{\partial F}{\partial x^i}(P). \tag{1.3.4}$$

That is,

$$D_V = \sum_{i=1}^{n} v^i \frac{\partial}{\partial x^i}\bigg|_P. \tag{1.3.5}$$

The association of the directional derivative D_V to the tangent vector V allows us to characterize tangent vectors as certain operators on differentiable functions as their definition on manifolds. In this work the equivalence relations play important role as the spaces under study are considered to agree on some open subsets. In the following equivalence relation is introduced together with germ of functions, as example.

Definition 1.3.2. *A relation on a set S is a subset R of its Cartesian product $S \times S$, such that, given $x, y \in S$ we write $x \sim y$ if $(x, y) \in R$. The relation R is said to be equivalence relation if the following properties are satisfied $\forall x, y, z \in S$:*

- *reflexivity: $x \sim x$,*
- *symmetry: $x \sim y \Rightarrow y \sim x$,*
- *transitivity: $x \sim y$ and $y \sim z \Rightarrow x \sim z$.*

The function with the same values on some neighborhood of P have the same directional derivative. That is why an equivalence relation is introduced on the smooth functions defined in some neighborhood of P.

Definition 1.3.3. *Consider the set of all pairs (f, U), where U is an open subset of $\mathbb{R}^n$ containing P and $f : U \to \mathbb{R}$ is a smooth function, then (f, U) and (g, V) are said to be equivalent if there is a neighborhood $W \subset U \cap V$ such that $f = g$ on W. The equivalence class of (f, U) is called germ of f at P.*

The set of all germs of smooth functions on $\mathbb{R}^n$ at P is denoted $C_P^\infty(\mathbb{R}^n)$. Let us now introduce the notion of algebra and one of linear transformations.

Definition 1.3.4. *An algebra over a field K is a vector space A over K with a multiplication map:$\mu : A \times A \to A$, denoted by $\mu(a, b) = ab$, such that $\forall a, b, c \in A$ and $r \in K$, the following properties are satisfied.*

- *associativity:*$(ab)c = a(bc)$,
- *distributivity:* $(a + b)c = ac + bc$,
- *homogeneity:* $r(ab) = (ra)b = a(rb)$.

Definition 1.3.5. *A map $L : V \to W$ between two vector spaces over a field K is said to be linear if $\forall r \in K$ and $\forall u, v \in V$, the following properties are satisfied:*

- $L(u + v) = L(u) + L(v)$,
- $L(rv) = rL(v)$.

We are now in position of introducing the notion of derivation at a point P. For each tangent vector v at a point P of $\mathbb{R}^n$, the direction derivative at P gives a map of real vector spaces $D_v : C_p^\infty(\mathbb{R})^n \to \mathbb{R}$. The D_v is linear and satisfies the Leibniz rule as the partial derivatives $\left.\frac{\partial}{\partial x^i}\right|_P$ have these properties.Any linear map $D : C_p^\infty(\mathbb{R})^n \to \mathbb{R}$, satisfying Leibniz rule is called derivation at P on $C_P^\infty(\mathbb{R}^n)$. The set of all derivations at P is denoted by $\mathcal{D}_P(\mathbb{R}^n)$ and it is a vector space over $\mathbb{R}$. We now, know that directional derivatives are all $P-$ point derivatives. Thus, there is a map $\phi : T_P(\mathbb{R}^n) \to \mathcal{D}_P(\mathbb{R}^n)$ defined as,

$$\phi : v \to D_v = \sum_{i=1}^{n} v^i \frac{\partial}{\partial x^i}\bigg|_P .$$

In the following the notion of vector field is introduced on $\mathbb{R}^n$ and it will be useful by its generalization on manifolds.

Definition 1.3.6. *A vector field on an open subset U of $\mathbb{R}^n$ is a function $X : U \cap \mathbb{R}^n \to T_P(\mathbb{R})^n$. That is , a map that assigns a tangent to each point of U.*

Since $T_P\mathbb{R}^n$ has $\frac{\partial}{\partial x^i}\Big|_P$ as basis, the vector X_P is a linear combination

$$\sum_{i=1}^{n} a^i(P) \frac{\partial}{\partial x^i}\bigg|_P, P \in U, a^i(P) \in \mathbb{R}.$$

A vector field X $\sum_{i=1}^n a^i \frac{\partial}{\partial x^i}$, where a^i are functions on U. A vector field is said to smooth on U if the coefficients functions are all smooth on U. The vector field X on an open subset U of $\mathbb{R}^n$ can be defined as a derivation as follows. From a smooth function f, we define a new function Xf on U given by:$(Xf)(P) = X_P f \forall P \in U$. Using the fact that

$$X = \sum_{i=1}^{n} a^i \frac{\partial}{\partial x^i},$$

we get

$$(Xf)(P) = \sum_{i=1}^{n} a^i(P)\frac{\partial f}{\partial x^i}(P).$$

1.4 Manifolds and Differentiable structures

1.4.1 Manifolds

In this section, basic definitions and properties of differentiable manifolds and differentiable functions are introduced. Both topological manifolds and differentiable structures are discussed.

Definition 1.4.1. *Let M be topological space. A covering of M is a collection of open subsets of M whose union is M. A covering $(U_\alpha)\alpha \in A$ is called locally finite if each point of M has neighbourhood which intersects only finitely many of the sets U_α.*

Definition 1.4.2. *A topological space M is Hausdorff if for any $p_1, p_2 \in M$ with $p_1 \neq p_2$ there exist open sets U_1 containing p_1 and U_2containing p_2 such that $U_1 \cap U_2 = \emptyset$.*

Definition 1.4.3. *A Hausdorff space is said to be paracompact if $\forall (U_\alpha)_{\alpha \in A}$, covering of M, $\exists$ a locally finite covering $(V_\beta)_{\beta \in B}$, which is refinement of $(U_\alpha)\alpha \in A$.*

Definition 1.4.4. *A topological space M is said to be $n-$ manifold if:*

- *$\forall p \in M$ there is an open neighbourhood U of p and a function $\varphi : U \to \mathbb{R}^n$, that, is a homeomorphism onto an open subset of $\mathbb{R}^n$,*

- *M is Hausdorff,*

- *M is paracompact.*

Here U is called a coordinate neighbourhood and φ is said to be a coordinate map. The function $f^i = x^i \circ \varphi$, where x^i denotes the i^{th} canonical coordinate on $\mathbb{R}^n$ are called the coordinate functions and the pair (U, φ) is called a local chart.

1.5 Differentiable structures

Definition 1.5.1. *A function $f : U \to \mathbb{R}^n$,where U is an open subset of $\mathbb{R}^n$, is called smooth or C^∞ if all its partial derivatives exist and are continuous on U.*

Definition 1.5.2. *A smooth atlas $\mathfrak{A}$ on $n-$ manifold M is a collection of coordinate charts $(U_\alpha, \varphi_\alpha)_{\alpha \in A}$ such that the following holds:*

- $\cup_{\alpha \in A} U_\alpha = M$,
- $\varphi_\alpha \circ \varphi_\beta^{-1} : \varphi_\beta(U_\alpha \cap U_\beta) \to \mathbb{R}^n$ *exist and are smooth,* $\forall \alpha, \beta \in A$.

Definition 1.5.3. *A smooth atlas $\mathfrak{F}$ is said to be differential structure if it is maximal, that is, if (U, φ) is coordinate chart such that $\varphi \circ \varphi_\alpha^{-1}$ and $\varphi_{\alpha \circ} \varphi^{-1}$ are smooth on $\varphi_\alpha(U \cap U_\alpha)$ and $\varphi(U \cap U_\alpha)$, respectively, then $(U, \varphi) \in \mathfrak{F}$.*

Definition 1.5.4. *A differentiable manifold of class C^∞ with dimension n is a pair $(M, \mathfrak{F})$ consisting of a topological manifold of dimension n, M and a smooth differentiable structure on M.*

1.6 Computations on manifolds.

In the following section some techniques of computations on manifolds are given. Both algebraic and analysis methods are included.

1.6.1 Tangent Vectors and vector fields.

Let p be a point of differentiable manifold M, a tangent vector to M at p is a function $V : C^{\infty}(M) \to \mathbb{R}$ such that:

- $V(f+g) = V(f) + V(g)$: Linearity property,
- $V(fg) = V(f)g(p) + f(p)V(g)\ \forall f, g \in C^{\infty}(M)$: Leibniz property.

The set of all tangent vectors to M at p forms an $n-$dimensional vector space denoted by T_p. The tangent vector notion is related to velocity vector of a curve $\gamma : (-\varepsilon, \varepsilon) \to M$ as follows. If $\gamma(0) = p$, then we associate to the velocity vector of γ at p the tangent vector $X \in T_p$, such that $X(f) = \dfrac{d}{dt} f(\gamma(t))_{t=0}$. In this work we will denote the set of all tangent vectors on M by TM. That is, $TM = \cup_{p \in M} T_p M$. Here TM is called tangent bundle. Note that the tangent bundle is a differentiable manifold of dimension $2n$, where n is dimension of M.

Definition 1.6.1. *A vector field X on M is a map which assigns to each point p of M a tangent vector $V(p)$.*

The set of all vector fields on M forms a vector field denoted by$\mathfrak{X}(M)$. The dual of tangent vector space on M at p is called cotangent space and is denoted T_p^*.

1.7 Tensors and differential forms

Let us consider the tangent vector space on M at p, T_pM, a $k-$ tensor is areal multi linear function defined on $T_pM \times T_pM \times \cdots \times T_pM$ of k copies of T_pM. The set of all $k-$ tensors is a vector space denoted by $\mathcal{T}^k(T_p^*M)$. If we consider the $k-$ tensors on T_pM, we obtain the space $\mathcal{T}^k(T_pM)$, and these tensors are called contravariant tensors, while $\mathcal{T}^k(T_pM)$ are covariant tensors. The mixed tensors (k,m) on T_pM are multilinear functions k copies of T_pM and m copies of T_p^*M. The space of all (k,m) is denoted by $\mathcal{T}^{(k,m)}(T_pM^*, T_pM)$.

Definition 1.7.1. *A $(k,m)-$tensor field is a map that assigns a tensor $\mathcal{T} \in \mathcal{T}^{(k,m)}(T_pM^*, T_pM)$.*

Definition 1.7.2. *Given a $k-$tensor T and an $m-$tensor S, their tensor product is defined as $(k+m)-$tensor $T \otimes S$ given by*

$$T \otimes S(V_1, \ldots, V_k, V_k + 1, \ldots, V_{k+m}) = T(V_1, \ldots, V_k)S(V_{k+1}, \ldots, V_{k+m}). \tag{1.7.1}$$

Definition 1.7.3. *A symmetric tensor $\mathcal{T}$, with respect to i and j is a tensor such that*

$$\mathcal{T}(V_1, \cdots, V_i, \cdots, V_j, \cdots, V_k) = \mathcal{T}(V_1, \cdots, V_j, \cdots, V_i, \cdots, V_k). \tag{1.7.2}$$

Definition 1.7.4. *A alternating tensor $\mathcal{T}$, with respect to i and j is a tensor such that*

$$\mathcal{T}(V_1, \cdots, V_i, \cdots, V_j, \cdots, V_k) = -\mathcal{T}(V_1, \cdots, V_j, \cdots, V_i, \cdots, V_k). \tag{1.7.3}$$

The space of alternating $k-$ tensors is a subspace of $\mathcal{T}(T_p^*M)$ denoted by $\bigwedge^k(T_p^*M)$. Considering S_k,the group of all possible permutations of $\{1,\ldots,k\}$,an alternating tensor is defined from a $k-$tensor $T\in\mathfrak{T}^k(V^*)$, as follows: if $\sigma\in S_k$, we get $\sigma(V_1,\ldots,V_k)=(V_{\sigma(1)},\ldots,V_{\sigma(k)})$. From T, a new alternating $k-$tensor, denoted by $Alt(T)$ is defined by

$$Alt(T)=\frac{1}{k!}\sum_{\sigma\in S_k}(sgn\sigma)(T\circ\sigma),$$

where $sgn\sigma$ is $+1$ if σ even permutation and -1, when σ is odd permutation. We are now in position of defining the wedge product between alternating tensors as follows.

Definition 1.7.5. *If $T\in\bigwedge^k(V^*)$ and $S\in\bigwedge^m(V^*)$, then $T\wedge S$ is a $k+m$ alternating tensor and is given by*

$$T\wedge S=\frac{(k+m)!}{k!m!}Alt(T\otimes S).$$

Definition 1.7.6. *Let M be C^∞ manifold. A form of degree k or $k-$form on M is a field of alternating $k-$tensor defined on M. That is, a map $\omega:M\to\bigwedge^k(T_p^*M)$ such that $\forall p\in M$ we have $\omega(p)=\omega_p\in\bigwedge^k(T_p^*M)$.*

1.8 Differentiable maps on manifolds

Let M and N be differentiable manifolds of dimensions m and n, respectively. A map $\phi:M\to N$ is said to be smooth if $\forall(U,\varphi)$ on M and (V,ψ) on N, $\psi\circ\varphi^{-1}$ is smooth. Further $\phi:M\to N$ is diffeomorphism if:

- ϕ is bijective,

- its inverse exists and is also smooth.

Definition 1.8.1. *Let $C^\infty(M)$ be the algebra of differentiable functions on M. The differential map atp of $\phi : M \to N$ is the map $\phi_{*p} : M \to T_{\phi(p)}N$, $(\phi_{*p}V)(f) = V(f \circ \phi)\ \forall f \in C^\infty(M)$.*

Definition 1.8.2. *Let $\phi : M \to N$ be smooth map . A point $p \in M$ is said to be a critical point of ϕ if $\phi_* : T_pM \to T_{\phi(p)}N$ is not surjective. A point $q \in N$ is said to be critical value of ϕ if $\phi^{-1}(q)$ contains a critical point of ϕ.*

Definition 1.8.3. *Let $\phi : M \to N$ be smooth map,then:*

- *ϕ is an immersion if ϕ_* is one- to- one $\forall p \in M$,*
- *the pair (M, ϕ) is submanifold of N if ϕ is injective and immersion.*
- *M is said to be submanifold of N if it is subset of N and the inclusion map map of M in N is injective,*
- *ϕ is embedding it is one-to- one,immersion and homeomorphism into,*
- *ϕ is submersion if ϕ_* is surjective $\forall p \in M$.*

1.9 One-Parameter Groups of Transformations and Flows

Definition 1.9.1. *Let M be a smooth manifold. A one-parameter group of transformations, on M is a differentiable map $\varphi : \mathbb{R} \times M \to M$, such that*

- $\varphi(0, x) = x$,
- $\varphi(s, \varphi(t, x)) = \varphi(s + t, x)$.

Putting $\varphi(t,x) = \varphi_t(x)$*, then, we get* $\forall t \in \mathbb{R}$*, a new differentiable map* $\varphi_t : M \to M$*. From the above definition, we have,* $\varphi_{t+s} = \varphi_t \circ \varphi_s$ *and* φ_0 *is identity map of* M*. That is, each map* φ_t *has an inverse ,*φ_{-t}*, which is also differentiable.*

Each one- parameter group of transformations φ on M defines a family of curves on M, called the group orbits. The map $\varphi_x : \mathbb{R} \to M$ defined as $\varphi_x(t) = \varphi(t,x)$ is a differentiable curve in M $\forall x \in M$. Since $\varphi_x(0) = x$, the tangent vector to the curve ,φ_x at $t = 0$ is an element of T_xM. The vector field, such that $X_x = (\varphi)_x^{'}$ is called infinitesimal generator of φ. Let X be a vector field on M and I be an open subset of $\mathbb{R}$. A curve $\gamma : I \to M$ is siad to be integral curve of X if $\gamma^{'}(t) = X_{\gamma(t)}, \forall t \in I$. In some cases φ is not defined for $\mathbb{R}$. However if $\forall x \in M$, there exists a neighbourhood U of x and $\varepsilon > 0$, such that φ is defined on $U \times (-\epsilon, \varepsilon)$ and is differentiable, the map φ is called a flow or a local one group of transformations.

1.10 Lie bracket, lie derivative and affine Connection

In this section, lie bracket, lie derivative and affine connection are defined.

Definition 1.10.1. *Let M be a smooth manifold. A lie bracket is map* $\begin{bmatrix} & \end{bmatrix} : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ *such that* $\begin{bmatrix} X & Y \end{bmatrix} = XY - YX$ $\forall X, Y \in \mathfrak{X}(M)$.

Definition 1.10.2. *Let X be infinitesimal generator of φ, φ_x be the integral curve of X starting at x and f a differentiable function, then*

$$\lim_{t\to 0}\left(\frac{\varphi_t^* f - f}{t}\right)(x) = \lim_{t\to 0}\frac{f(\varphi_x(t)) - f(\varphi_x(0))}{t} = (Xf)(x),$$

is called lie derivative of f with respect to X and it is denoted $L_X f$.

Let $X, Y \in \mathfrak{X}(M)$, then the lie derivative of Y with respect to X is equal to the lie bracket of X and Y.

Definition 1.10.3. *Let M be a C^∞ manifold. An affine connection on M is a bilinear map,* $\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ *such that:*

- $\nabla_{fX} Y = f \nabla_x Y$,
- $\nabla_X (fY) = XfY + f\nabla_X Y$,

$\forall X, Y \in \mathfrak{X}(M)$ *and* $\forall f \in C^\infty(M)$,

$\nabla_X Y$ stands for $\nabla(X, Y)$ and it is also called covariant derivative of Y with respect to X.

1.11 Riemannian manifold

Definition 1.11.1. *A Riemannian metric is a symmetric and positive defined 2- tensor field $g \in \mathfrak{T}^2(M)$. That is,*

$$\begin{cases} g(X,Y) = g(Y,X) \\ g(X,X) > 0 \quad \textit{if} \quad X \neq 0 \quad \forall X, Y \in \mathfrak{X}(M). \end{cases} \tag{1.11.1}$$

Definition 1.11.2. *A pseudo-Riemannian metric is a $(0,2)$ tensor such that :*

$$\begin{cases} g(X,Y) = g(Y,X), \\ g(X,Y) = 0 \Longleftrightarrow X = 0 \quad \forall X, Y \in \mathfrak{X}(M). \end{cases} \tag{1.11.2}$$

Definition 1.11.3. *A manifold in which a Riemannian metric is defined is called Riemannian manifold and one with pseudo-Riemannian metric is called pseudo-Riemannian manifold. All the manifolds in this research are pseudo-Riemannian manifolds. Given a metric on Riemannian manifold M, we can define smooth function given by $g(X,Y)_p$ $\forall X, y \in \mathfrak{X}(M)$.*

Definition 1.11.4. *Given a metric g on C^∞ manifold M, a connection ∇, on M is said to be compatible with the metric if:*
$Xg(Y,Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X)Z$, $\forall X, Y, Z \in \mathfrak{X}(M)$.

This connection is known as the Levi- Civita connection. With the help of this connection, the curvature is defined as:
$R(X,Y) = \nabla_{[X,Y]} + \nabla_Y \nabla_X - \nabla_X \nabla_Y$, and the torsion defined as
$T(X,Y) = \nabla_X - \nabla_y - [X,Y]$. These tensors help to measure the commutativity and the flatness of a given manifold respectively. By allowing the

curvature tensor to act on the third vector field Z, we obtain a new curvature tensor given by

$$R(X,Y)Z = \nabla_{[x,y]}Z + \nabla_Y\nabla_X Z - \nabla_X\nabla_Y Z, \tag{1.11.3}$$

which is known as Riemannian curvature tensor. In terms of local coordinates it is given by :

$$R(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j})\frac{\partial}{\partial x^k} = \sum R^l_{kij}\frac{\partial}{\partial x^l}. \tag{1.11.4}$$

The second covariant derivative is defined as:

$$\nabla^2_{X,Y}Z = \nabla_X(\nabla_Y Z) - \nabla_{\nabla_X Y}Z. \tag{1.11.5}$$

Using (1.11.3) in (1.11.5), we get

$$\nabla^2_{Y,X}Z - \nabla^2_{X,Y}Z = R(X,Y)Z. \tag{1.11.6}$$

An other important notion of curvature is Ricci curvature tensor. Before introducing it, let us recall that the trace of linear transformation of Euclidean vector space of finite dimensional is given by:

$$tr(f) = \sum_{i=1}^{n} g(f(e_i), e_i), \tag{1.11.7}$$

where $(e_1, \cdots, e_n)$ is the orthonormal basis of this euclidean vector space.

Definition 1.11.5. *Let (M, g) be a Riemannian manifold. The Ricci curva-*

ture tensor of M is a $(0,2)$ tensor defined as:

$$S(X,Y) = \sum_{i=1}^{n} g(R(X,e_i)Y,e_i) = \sum_{i=1}^{n} R(X,e_i,Y,e_i), \tag{1.11.8}$$

where $(e_1,\cdots,e_n)$ is any orthonormal basis of T_PM.

Definition 1.11.6. *The scalar curvature tensor S is the trace of Ricci curvature tensor. That is,*

$$S = \sum_{i\neq J} R(e_i,e_j,e_i,e_j). \tag{1.11.9}$$

Definition 1.11.7. *On the Riemannian manifold (M,g), the Ricci operator is defined by the following equation.*

$$g(QX,Y) = S(X,Y). \tag{1.11.10}$$

The Riemannian curvature tensor has served as important tool in the Riemannian geometry study. The Weyl projective curvature tensor W, the conformal curvature tensor V, the concircular curvature tensor C and the coharmonic curvature tensor L are introduced from Riemannian curvature tensor and Ricci curvature tensor as follows.

Definition 1.11.8. *the Weyl curvature tensor is defined by*

$$W(X,Y)Z = R(X,Y)Z + \frac{1}{n-1}[S(X,Y)Z - S(Y,Z)X]. \tag{1.11.11}$$

Definition 1.11.9. *The conformal curvature tensor is defined as,*

$$V(X,Y)Z = \{R(X,Y)Z - \frac{1}{n-2}S(Y,Z)X - S(X,Z)Y - g(X,Z)QY + g(Y,Z)QX + \frac{S}{(n-1)(n-2)}[g(Y,Z)X - g(X,Z)Y]\}. \tag{1.11.12}$$

Definition 1.11.10. *The concircular curvature tensor is defined by*

$$C(X,Y)Z = R(X,Y)Z - \frac{S}{n(n-1)}[g(Y,Z)X - g(X,Z)Y] \tag{1.11.13}$$

Definition 1.11.11. *The conharmonic curvature tensor is defined by*

$$L(X,Y)Z = R(X,Y)Z - \frac{1}{n-2}[S(Y,Z)X - S(X,Z)Y + g(X,Z)QY + g(Y,Z)QX]. \tag{1.11.14}$$

W_2 curvature tensor and W_8 curvature tensor that are used in this research have been introduced from projective Weyl tensor by Pokhariyal and Mishra, and are defined as follows.

Definition 1.11.12. *W_2 curvature is defined by Pokhariyal and Mishra [1970]*

$$W_2(X,Y)Z = R(X,Y)Z + \frac{1}{n-1}[g(X,Z)QY - g(Y,Z)QX]. \tag{1.11.15}$$

Definition 1.11.13. *W_8 curvature tensor is defined by Pokhariyal [1982a]*

$$W_8(X,Y)Z = R(X,Y)Z + \frac{1}{n-1}[S(X,Y)Z - S(Y,Z)X]. \tag{1.11.16}$$

The notion of antisymmetric and that of semi-symmetric Riemannnian manifolds with respect to a tensor T are important to our research, and they are defined in the following.

Definition 1.11.14. *The fact that a Riemaian manifold (M, g) is said to be antisymmetric with respect to a tensor T is denoted by $(\xi, .)_T.S = 0$ and defined by relation:*

$$S(T(\xi, X)Y, Z) + S(Y, T(\xi, X)Z) = 0. \tag{1.11.17}$$

Definition 1.11.15. *The fact that a Riemannian manifold (M, g) is semisymmetric with respect to a tensor T is denoted as $(\xi, .)_S.T = 0$ and given by the relation:*

$$\begin{aligned} S(X, T(Y, Z)V)\xi - S(\xi, T(Y, Z)V)X+ \\ S(X, Y)T(\xi, Z)V - S(\xi, Y)T(X, Z)V \\ +S(X, Z)T(Y, \xi)V - S(\xi, Z)T(Y, X)V \\ +S(X, V)T(Y, Z)\xi - S(\xi, V)T(Y, Z)X = 0, \\ \forall X, Y, Z, V \in \mathfrak{X}(M). \end{aligned} \tag{1.11.18}$$

1.12 complex Manifolds and Almost complex structure

1.12.1 Complex structure

In the following section Complex manifolds are introduced, as all the manifolds under our study are particular cases of complex manifolds. Also Almost complex structure are introduced in order to explain how algebraic computations are extended to complex manifolds. Noting that the complex manifolds connect several mathematics areas and that their theory is almost analogous to one of differentiable manifolds, accept that all the functions considered here, have to be holomorphic instead of being smooth. We start by defining holomorphic functions. In order to make notations as comprehensible as possible, let us start by the function of one variable. Let Ω an open subset of complex numbers $\mathbb{C}$, and f be a complex valued function on Ω, such that $f(x, y) = u(x, y)+iv(x, y)$, where $u(x, y)$ and $v(x, y)$ are real-valued functions from an open subset of $\mathbb{R}^2$.Let us consider a complex number $z = x + iy$ and its conjugate $\overline{z} = x - iy$. Considering z and $\overline{z}$ as mapping defined on $\mathbb{R}^2$ to $\mathbb{R}^2$, we get,

$$dz = dx + idy \tag{1.12.1}$$

and

$$d\overline{z} = dx - idy, \tag{1.12.2}$$

combining (1.12.1) and (1.12.2), we get:

$$dx = \frac{1}{2}(dz + d\overline{z}) \tag{1.12.3}$$

and

$$dy = \frac{1}{2i}(dz - d\overline{z}). \tag{1.12.4}$$

Now considering $f : \Omega \subset \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, we get:

$$df = \frac{\partial u}{\partial x}dx + i\frac{\partial v}{\partial x}dx + \frac{\partial u}{\partial x}dy + i\frac{\partial v}{\partial y}dy. \tag{1.12.5}$$

Introducing (1.12.2) and (1.12.3), in (1.12.4), we get

$$df = \frac{1}{2}(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x})(dz + d\overline{z}) + \frac{1}{2i}(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y})(dz - d\overline{z}). \quad (1.12.6)$$

Rearranging terms in (1.12.6), we get:

$$df = \frac{1}{2}(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} + \frac{1}{i}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y})dz + \frac{1}{2}(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} - \frac{1}{i}\frac{\partial u}{\partial y} - \frac{\partial v}{\partial y})d\overline{z}. \quad (1.12.7)$$

That is,

$$df = \frac{1}{2}(\frac{\partial}{\partial x}(u+iv)+\frac{1}{i}\frac{\partial}{\partial y}(u+iv))dz+\frac{1}{2}(\frac{\partial}{\partial x}(u+iv)-\frac{1}{i}\frac{\partial}{\partial y}(u+iv))d\overline{z}. \quad (1.12.8)$$

Finally, using the fact that $f(x, y) = u + iv$, we get:

$$df = \frac{1}{2}(\frac{\partial f}{\partial x} + \frac{1}{i}\frac{\partial f}{\partial y})dz + \frac{1}{2}(\frac{\partial f}{\partial x} - \frac{1}{i}\frac{\partial f}{\partial y})d\overline{z}. \quad (1.12.9)$$

Putting

$$\frac{1}{2}(\frac{\partial f}{\partial x} + \frac{1}{i}\frac{\partial f}{\partial y}) = \frac{\partial}{\partial z},$$

and

$$\frac{1}{2}(\frac{\partial f}{\partial x} - \frac{1}{i}\frac{\partial f}{\partial y}) = \frac{\partial f}{\partial \overline{z}},$$

we get

$$df = \frac{\partial f}{\partial z}dz + \frac{\partial f}{\partial \overline{z}}d\overline{z}.$$

Note that $\frac{\partial f}{\partial z}$ and $\frac{\partial f}{\partial \overline{z}}$ are not ordinary partial differential equations, but in computations they behave as partial derivatives. That is,their definition is in accordance with the chain rule within the change of independent variables x

and y to z and $\overline{z}$. A function $f \in C^1(\Omega)$ is holomorphic if $\frac{\partial f}{\partial \overline{z}} = 0$. That is:

$$\frac{\partial f}{\partial x} = \frac{1}{i}\frac{\partial f}{\partial y}, \tag{1.12.10}$$

and from the fact that $f = u + iv$, we get

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}. \tag{1.12.11}$$

That is:

$$\begin{cases} \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \\ \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}, \end{cases} \tag{1.12.12}$$

which are known as Cauchy- Riemannian conditions. In the following we are going to extend the notion of holomorphic function to the function of several complex variables. Let $\mathbb{C}^n$ be $n-$ complex vector space, Z denote $(z_1, \cdots, Z_n)$, where $Z_j = x_j + iy_j, j = 1, \cdots, n$ and Ω be an open subset of $\mathbb{C}^n$. Assume that $f(Z)$ is a complex- valued function which is continuously differentiable as function of $2n$ variables $x_1, y_1, \cdots, x_n, y_n$. Setting :

$$\frac{\partial f}{\partial Z_j} = \frac{1}{2}(\frac{\partial f}{\partial x_j} + \frac{1}{i}\frac{\partial f}{\partial y_j}) \tag{1.12.13}$$

and

$$\frac{\partial f}{\partial \bar{Z}_j} = \frac{1}{2}(\frac{\partial f}{\partial x_j} - \frac{1}{i}\frac{\partial f}{\partial y_j}). \tag{1.12.14}$$

The definition of holomorphic function of several complex variables is given as follows:

Definition 1.12.1. *$f(Z)$ is said to be holomorphic on Ω, if $\frac{\partial f}{\partial \bar{Z}_j} = 0, j = 1, \cdots, n$.*

We are now in position of defining the complex manifold.

Definition 1.12.2. *Let M be a $2n-$ real dimensional manifold. An holomorphic atlas for M is a collection of charts $(U_\alpha, \varphi_\alpha)\alpha \in A$, where*

$$\varphi_\alpha : U_\alpha \to V_\alpha \subset \mathbb{R}^{2n} \cong \mathbb{C}^n,$$

such that:

1. $M = \cup_\alpha U_\alpha$

2. $\varphi_{\alpha\beta} = \varphi_\alpha \circ \varphi_\beta^{-1}$ *are holomorphic.*

Definition 1.12.3. *Two holomorphic atlases $(U_\alpha, \varphi_\alpha)$ and $(\tilde{U}_\alpha, \tilde{\varphi}_\alpha)$ are said to be equivalent if $\varphi_\alpha \circ \tilde{\varphi}_\beta^{-1}$ is holomorphic $\quad\forall\alpha, \beta$.*

Definition 1.12.4. *A complex manifold M is a smooth manifold in which the equivalence class of holomorphic atlases is defined.*

1.12.2 Almost complex structure

Almost complex structure on real vector space

In the following the almost complex is introduced in order justify the complexification of the vector spaces and vector bundles. On a real vector space V, a complex structure is defined as follows:

Definition 1.12.5. *An almost complex structure on V is a linear map $J : V \to V$, such that $J^2 = -I$, where I is identity transformation of V. The endomorphism $J : \mathbb{R}^2 \to \mathbb{R}^2 : (x_1, y_1, \cdots, x_n, y_n) \to (y_1, -x_1, \cdots, y_n, -x_n)$ is called the standard almost complex structure.*

The relation $J^2 = -I$ implies that its eigenvalues are $\pm i$, so that for V as real vector space, J has no eigenspace. By considering the complexification $V_{\mathbb{C}} = V_{\otimes}\mathbb{R}\mathbb{C}$ any linear transformation $T : V \to V$ can be extended to the linear map $T : V_{\mathbb{C}} \to V_{\mathbb{C}}$ and J is extended to $J : V_{\mathbb{C}} \to V_{\mathbb{C}}$ and J, such that $J^2 = -I$. In this case J has two eigenspaces denoted as $V^{1,0}$ and $V^{0,1}$ with eigenvalues i and $-i$,respectively.

Almost complex structure on vector bundles

In this section, the notion of almost complex manifold is extended to tangent bundle TM on smooth manifold M. Here $T_{\mathbb{C}}M$ is obtained by complexifying fiber by fiber.

Definition 1.12.6. *Let TM be a real vector bundle on a smooth manifold M. An almost complex structure on TM is a morphism $J : TM \to TM$, such that $J^2 = -I$.*

Given such bundle, we have a decomposition $T_{\mathbb{C}}M = T^{1,0}M \oplus T^{0,1}M$, where $J\Big|_{T^{1,0}M}(TM) = iTM$,$J\Big|_{T^{0,1}M}(TM) = -iTM$ and this decomposition exists fiber by fiber.

Definition 1.12.7. *Let M be a real smooth manifold. An almost complex structure on M is defined on its vector bundle TM. Denoting the standard almost complex structure on $\mathbb{R}^{2n}$ by J_{st} and given a holomorphic $\varphi : TU \to TU$, we get a bundle map$J : TU \to TU$ given by $J = D\varphi^{-1} \circ J_{st} \circ D\varphi$.*

Definition 1.12.8. *$T^{1,0}M$ is the holomorphic tangent bundle of M and we have the decomposition $(T^*M)_{\mathbb{C}} = (T^*M)^{1,0} \oplus (T^*M)^{0,1}$. In terms of local* *coordinates we say that $x^j + +iy^j$ are called holomorphic coordinates and*

$$J(\frac{\partial}{\partial x^j}) = \frac{\partial}{\partial y^j}, \quad J(\frac{\partial}{\partial y^j}) = -\frac{\partial}{\partial x^j}. \tag{1.12.15}$$

$$J(dx^j) = -dy^j, \quad J(dy^j) = dx^j. \tag{1.12.16}$$

1.12.3 Para- Kenmotsu Manifolds

Definition 1.12.9. *An $(2n+1)-$ dimensional manifolds is almost para-contact if*

$$\varphi\xi = 0 \tag{1.12.17}$$

$$\eta(\xi) = 1 \tag{1.12.18}$$

$$\varphi^2(X) = X - \eta(X) \otimes \xi \tag{1.12.19}$$

$$g(\varphi., \varphi.) = -g + \eta \otimes \eta, \tag{1.12.20}$$

$(M, \varphi, \xi, \eta, g)$ is called almost paracontact manifold, φ the structure endomorphism, ξ the characteristic vector field and η the Paracontact form. Examples of almost paracontact metric structure are given in Ivanov et al. [2010]. From the definition it follows that :

$$\eta(X) = g(X, \xi), \tag{1.12.21}$$

and ξ is a unit vector field.

Definition 1.12.10. *An almost paracontact metric structure $(M, \varphi, \xi, \eta, g)$ is said to be Para- Kenmotsu if the Levi- Civita connection ∇ of g satisfies* *the following equation :*

$$(\nabla_X \varphi)Y = g(\varphi X, Y)\xi - \eta(Y)\varphi X \forall X, Y \in \mathfrak{X}(M), \tag{1.12.22}$$

where $\mathfrak{X}(M)$ is the algebra of vector fields on M.

Ricci flows and Ricci solitons

Definition 1.12.11. *Let (M, g_o) be a Riemannian manifold, then the Ricci flow is evolution equation that evolves the metric tensor and it is defined as:*

$$\begin{cases} \frac{\partial}{\partial t} g(t) = -2S(g(t)) \\ g(0) = g_0, \end{cases} \tag{1.12.23}$$

where $S(g(t))$ is the Ricci curvature of the metric $g(t)$. Recall that, the flat metric has zero Ricci curvature and it does not evolve under Ricci flow. Those Manifolds with vanishing Ricci curvature are considered as fixed points of Ricci flow.The generalized fixed points of Ricci flows are manifolds which change only by diffeomorphisms and rescaling under Ricci flow.
That is, there exist $\varphi_t : M^n \to M^n$ time dependent family of diffeomorphisms such that $\varphi_0 = Id$ and a time depending scale factor $\sigma(0) = 1$, such that $g(t) = \sigma(t)\varphi_t^* g(0)$. The generalized fixed points are also called **Ricci solitons**.

Definition 1.12.12. *A manifold is called quasi- Einstein if the Ricci curvature tensor field S is linear combination of g and the tensor product of a non- zero 1-form η satisfying $\eta(x) = g(X, \xi)$, for ξ a unit vector field, and it is Einstein if S is collinear with g.*

1.12.4 Lorentzian Para-Sasakian Manifolds

If M is m- dimensional differentiable manifold, φ a $(1,1)-$type tensor field, η $1-$form and g a Lorentzian metric on M, then

Definition 1.12.13. *(φ, ξ, η, g) is said to be Lorentzian structure on M if:*

$$\varphi\xi = 0, \quad \eta(\varphi) = 0, \tag{1.12.24}$$

$$\eta(\xi) = -1, \quad \varphi^2 = I + \eta \otimes \xi, \tag{1.12.25}$$

$$g(\varphi., \varphi.) = g + \eta \otimes \eta, \tag{1.12.26}$$

$$(\nabla_X \varphi)Y = g(X,Y)\xi + 2\eta(X)\eta(Y)\xi + \eta(Y)X \quad \forall X, Y \in \mathfrak{X}(M), \tag{1.12.27}$$

where $\mathfrak{X}(M)$ is algebra of vector fields on M and ∇ is the Levi- Civita connection associated to g. and $\eta \otimes \eta(X,Y)$ stands for $\eta(X)\eta(Y)$.

1.13 Research problem and Objectives

1.13.1 Research problem

The curvature tensors W_2 and W_8 have been studied by various authors. Thus for example, for Sasakian manifold W_2 has been studied by Pokhariyal [1982b]; while for a P- Sasakian manifold Matsumuto and Milrai [1986] have studied W_2 curvature tensor. Venkatesha et al. [2011], have studied

Lorentzian para- Sasakian satisfying certain conditions on W_2- curvature tensor. Motivated by the all important role of W_2 and $W8-$ curvature tensors in the study of certain differential geometric structures we have made detailed study of these tensors on para -Kenmotsu and Lorentzian para- Sasakian manifolds with Ricci solitons. This helped in classification of manifolds as Einstein or quasi- Einstein which has been one of problems that interested different differential geometers and physicists for several years.

1.13.2 Objectives

General objective

To study the relation between Ricci solitons, Einstein and quasi- Einstein manifolds on Para- Kenmotsu and Lorentzian Para - Sasakian Manifolds with particular structures.

1.13.3 Specific objectives

These were to:

- study the existence of Ricci solitons on Para- Kenmotsu manifolds satisfying some curvature tensor conditions,
- study the existence of Ricci solitons on Lorentzian Para- Sasakian manifolds satisfying some curvature tensor conditions,
- Compare the results obtained in two different cases.

1.13.4 Significance of study

This study has been important endeavor in promoting research in differential geometry and study of Ricci flows stability. This study has also been beneficial to the students and researchers in the domain of differential geometry. It has provided important tools in understanding the structures of Ricci solitons in contact geometry, especially their symmetries. Moreover, this study has been helpful to the Geometers who are interested in manifolds with special metrics especially Einstein and quasi- Einstein metrics on odd dimensional manifolds. It hs also served as a future reference for researchers in the domain of Riemannian Geometry especially those, involved in the study of Ricci flows stability and their symmetries.

1.14 Summary

η- Ricci solitons are used to investigate the conditions of Ricci solitons existence on the manifolds under study as follows.

1.14.1 on Para- Kenmotsu Manifolds defined with W_8- Curvature tensor

In chapter three,computations of Ricci tensor, Riemannian and W_8 curvature tensor allow us to classify antisymmetric and semisymmetric Sasakian para- Kenmotsu manifolds with respect to W_8 by analyzing the Ricci solitons existence conditions on these spaces. Also the same computations give us the possibility of classifying para- Kenmotsu manifolds with Ricci cyclic tensor.

1.14.2 on Lorentzian Para- Sasakian manifolds defined with W_8- Curvature tensor

The results of chapter four have been obtained by computing Ricci tensor, Riemannian and W_8 curvature tensor on semi symmetric Lorentzian Para- Sasakian manifolds with respect to W_8 and studying Ricci solitons existence conditions on these spaces.

1.14.3 on Lorentzian Para- Sasakian manifolds defined with W_2- Curvature tensor

After computations of Ricci tensor, Riemannian and W_2 curvature tensor, on antisymmetric Lorentzian para- Sasakian manifolds with respect to W_2, semi Symmetric Lorentzian para- Sasakian manifolds with respect to W_2, semisymmetric para- Sasakian manifolds with respect to W_2 and studying the conditions of Ricci solitons existence on them, we get the results of chapter five.

Chapter 2

Literature Review

2.1 Generalities

The Poincaré Conjecture has been one of the entente century problem which have taken long time before being proved by Perelman. The solution of this problem has been possible due to the method introduced by Hamilton [1982] . From this time to now different authors have studied Ricci solitons on various manifolds. Among them Chow [1991] , who considered Ricci flow on the 2-sphere and showed that the Gaussian curvature of any metric on S^2 becomes positive in finite time. Chow et al. [2007] used Maximum principle to control various geometric quantities associated to the metric under Ricci flow and Chandra et al. [2015] used Second order parallel tensors to find the conditions of Ricci solitons on Lorentzian concircular structure n-manifolds to be shrinking , steady and expending. It is noted that Ricci soliton are the solutions of Ricci flows, which move only by one parameter group of diffeomorphism and scaling, that is, a Ricci soliton (g, v, λ) on Riemannian manifold (M, g) is generalization of Einstein metric such that

$$L_v g + 2S + 2\lambda g = 0. \tag{2.1.1}$$

Where L_V is the Lie derivative along the vector V on M, S is Ricci tensor, λ is a scalar and g is Riemannian metric on M.
After Perelman [2003] , used Ricci flow and its surgery to prove the Poincaré Conjecture, most mathematicians have been interested in the study of Ricci solitons. Huisken [1985] , was the first to study the Ricci flows on a manifold of dimensions greater than four basing his analysis on the decomposition of the Riemann curvature tensor as follows.

$$R_{ijkl} = U_{ijkl} + V_{ijkl} + W_{ijkl},$$

where U_{ijkl} is curvature tensor associated with the scalar curvature, V_{ijkl} is curvature tensor associated with trace free curvature and W_{ijkl} is Weyl tensor. From this time more mathematicians have investigated the properties of Ricci flows solutions especially the existence of Ricci solitons in some particular directions under certain conditions. Due to the results obtained by Huisken and noting that Pokhariyal [1982a] has defined W_8 curvature tensor with help of Weyl's projective tensor, we have classified Ricci solitons on para-Kenmotsu manifolds satisfying some conditions with respect to W_8 in direction of characteristic vector. Blaga [2015], have studied $\eta-$Ricci solitons on para -Kenmotsu geometry manifolds satisfying $(\xi, \cdot)_R \cdot S = 0$,$(\xi, \cdot)_S \cdot R = 0$,$(\xi, \cdot)_{W_2} \cdot S = 0$ and $(\xi, \cdot)_S \cdot W_2 = 0$. Also Nagaraja and Venu [2016], obtained some results on Rcci solitons satisfying $(\xi, \cdot)_H \cdot S = 0$,$(\xi, \cdot)_{\tilde{C}} \cdot S = 0$,$(\xi, \cdot)_R \cdot \tilde{C} = 0$, $(\xi, \cdot)_P \cdot \tilde{C} = 0$ and Bagewadi et al. [2013] have considered the cases of $(\xi, \cdot)_R \cdot B = 0$, $(\xi, \cdot)_B \cdot S = 0$, $(\xi, \cdot)_S \cdot R = 0$, $(\xi, \cdot)_R \cdot \bar{P} = 0$ and $(\xi, \cdot)_{\bar{P}} \cdot S = 0$.

2.2 On Para- Kenmotsu manifolds

In the present research Almost para contact manifolds are considered and precisely η- Ricci soliton are studied on para- Kenmotsu manifold satisfying $(\xi,\cdot)_{W_8}\cdot S=0$, $(\xi,\cdot)_S\cdot W_8=0$ and those with cyclic Ricci tensor. That why we have to cite some of documents these inspire us in this research. Para-Kenmotsu structure has been introduced Wełyczko [2009] for 3- dimensional normal almost paracontact metric structure. In the following we give the fundamental properties of this structure , as they have been given by Blaga [2015].

Proposition 2.2.1. *On Para-Kenmotsu manifolds the followings hold:*

$$\nabla\xi X = X - \eta\otimes\xi(x), \tag{2.2.1}$$

$$\eta(\nabla_X\xi) = 0, \tag{2.2.2}$$

$$\nabla_\xi\xi = 0, \tag{2.2.3}$$

$$R(X,Y)\xi = -\eta(X)Y - \eta(y)X, \tag{2.2.4}$$

$$R(X,Y)Z = -Xg(Y,Z) + Yg(X,Z), \tag{2.2.5}$$

$$\nabla\eta = g - \eta\otimes\eta, \tag{2.2.6}$$

$$\nabla_\xi\eta = 0, \tag{2.2.7}$$

$$L_\xi\varphi = 0, \tag{2.2.8}$$

$$L_\xi\eta = 0, \tag{2.2.9}$$

$$L_\xi(\eta\otimes\eta) = 0, \tag{2.2.10}$$

$$L_\xi g = 2(g - \eta \otimes \eta) \tag{2.2.11}$$

$$\nabla_X \xi = X - \eta(X)\xi, \tag{2.2.12}$$

where R is Riemann curvature tensor field and ∇ is Levi- Civita connection of g. Example of para- Kenmotsu structure can be found Blaga [2015]. If $(M, \varphi, \xi, \eta, g)$ is an almost paracontact metric manifold, (g, ξ, λ, μ) satisfying

$$L_\xi g + 2S + 3\lambda g + 2\mu\eta \otimes \eta = 0, \tag{2.2.13}$$

where L_ξ is the lie derivative along the characteristic vector field , S is the Ricci curvature tensor of g, and λ and μ are constant, then (g, ξ, λ, μ) is called $\eta-$ Ricci soliton structure on M. Note that (2.2.13) become equation of Ricci soliton for $\mu = 0$ and it is called shrinking , steady or expending according to λ is negative, zero or positive , respectively. In terms of Levi Civita connection (2.2.13) becomes:

$$2S(X, Y) = -g(\nabla_X \xi, Y) - g(X, \nabla_Y \xi) - 2\lambda g(X, Y) - 2\mu\eta(X)\eta(Y). \tag{2.2.14}$$

It has been showed that one of important geometrical object to the study of Ricci Solitons is a symmetric tensor $(0, 2)$, which is parallel with respect to the Levi- Civita connection Blaga [2015] and some of geometric properties of such tensor field are defined by Bejan and Crasmareanu [2010]. Considering such symmetric $(0, 2)$ tensor field α and from the Ricci identity

$$\nabla^2\alpha(X, Y; Z, V) - \nabla^2\alpha(X, Y; V, Z) = 0, \tag{2.2.15}$$

we get

$$\alpha(R(X,Y)Z,V)+\alpha(Z,R(X,Y)V)=0,\forall X,Y,Z,V\in\mathfrak{X}(M). \tag{2.2.16}$$

Taking $Z=V=\xi$ and using (2.2.4), we get

$$\alpha(R(X,y)\xi,\xi)=0,\forall X,Y\in\mathfrak{X}(M).$$

Using (2.2.12) , (2.2.14) becomes

$$S(X,Y)=-(\lambda+1)g(X,Y)-(\mu-1)\eta(X)\eta(Y). \tag{2.2.17}$$

The following theorem has been proved by Blaga [2015] .

Theorem 2.2.2. *Let (M,φ,ξ,η,g) be a para-Kenmotsu manifold. assume that the symmetric $(0,2)$- tensor field $\beta=L_\xi g+2s+2\mu\otimes\eta$ is parallel with respect to the Levi- Civita connection associated to g. Then (g,ξ,μ) yields an $\eta-$ Ricci soliton.*

From this theorem , for $\mu=0$, the following corollaries have been deduced.

Corollary 2.2.3. *On para-Kenmotsu manifold (M,φ,ξ,η,g) with the property that the symmetric $(0,2)$- tensor field $\alpha=L_\xi g+2s$ is parallel with respect to Levi-Civita connection associated to g the relation*(2.2.13) *, defines a Ricci soliton on M. for $\lambda=2n$ and $\mu=0$*

Corollary 2.2.4. *For $\mu=1$ and $\lambda=2n-1$, (M,g) is quasi- Einstein.*

In the following we shall study η-Ricci solitons whose curvature satisfies $(\xi,.)_S.W_8=0$ and $(\xi,.)_{W_8}.S=0$ respectively, where the W_8-curvature tensor has been introduced by Pokhariyal [1982a], and it is given by :

$$W_8(X,Y)Z = R(X,Y)Z + \frac{1}{2n}(S(X,Y)Z - S(Y,Z)X), \tag{2.2.18}$$

for a $(2n+1)-$ dimensional Para-Kenmotsu manifold. Using (1.12.26) and (1.12.24), we get

$$g(\xi,\xi) = -1, \tag{2.2.19}$$

$$\eta(X) = g(X,\xi) \tag{2.2.20}$$

and

$$g(\varphi X, Y) = g(X, \varphi Y) \quad X, Y \in \mathfrak{X}(M). \tag{2.2.21}$$

2.3 On Lorentzian Para- Sasakian manifolds

The Lorentzian Para- Sasakian manifolds have been one of our interested field of our study and that is why their important properties, as they have been proved by Blaga [2016], are here presented.

Proposition 2.3.1. *On Lorentzian Para- Sasakian manifolds, the followings hold:*

$$\nabla_X \xi = \varphi X, \tag{2.3.1}$$

$$\eta(\nabla_X \xi) = 0, \quad \nabla_\xi \xi = 0, \tag{2.3.2}$$

$$R(X,Y)\xi = -\eta(X)Y + \eta(Y)X \tag{2.3.3}$$

$$\eta(R(X,Y)Z) = \eta(X)g(Y,Z) - \eta(y)g(X,Z), \quad \eta(R(X,Y)\xi) = 0 \tag{2.3.4}$$

$$(\nabla_X \eta)Y = (\nabla_Y \eta)X = g(\varphi X, Y), \quad \nabla_\xi \eta = 0, \tag{2.3.5}$$

$$L_\xi\varphi = 0, \quad L_\xi\eta = 0, L_\xi g = 2g(\varphi.,.) \tag{2.3.6}$$

where R is Riemannian curvature tensor field and ∇ is Levi- Civita connection associated to g.

From this proposition, we get that $g(X, \varphi Y)$ is symmetric,

$$(\nabla g(X,\varphi))(Y,Z) = \eta(Y)(X,Z) + \eta(Z)g(X,Y) + 2\eta(X)\eta(Y)\eta(Z), \tag{2.3.7}$$

and

$$g(\varphi X, \varphi^2 Y) = g(X, \varphi Y). \tag{2.3.8}$$

Let $(M, \varphi, \xi, \eta, g)$ be a Lorentzian manifold. The data satisfying the equation

$$L_\xi g + 2s + 2\lambda g + 2\mu\eta \otimes \eta \tag{2.3.9}$$

where L_ξ is the Lie derivative operator along the vector field ξ, S is the Ricci tensor field of the metric g, and λ and μ are scalars is said to be $\eta-$Ricci soliton on M. Writing L_ξ in terms of the Levi- Civita connection (2.3.9) becomes:

$$2S(X,Y) = -g(\nabla_X\xi, Y) - g(X, \nabla_Y\xi) - 2\lambda g(X,Y) - \mu\eta(X)\eta(Y) \quad \forall X, Y \in \mathfrak{X}(M). \tag{2.3.10}$$

Using (2.3.1) in (2.3.9), we get

$$S(X,Y) = -g(\varphi X, Y) - \lambda g(X,Y) - \mu\eta(X)\eta(Y). \tag{2.3.11}$$

Matsumoto and Mihai [1988] proved that on Lorentzian para- Sasakian manifold $(M, \varphi, \xi, \eta, g)$ the Ricci tensor satisfies

$$S(X, \xi) = (dim(M) - 1))\eta(X) \tag{2.3.12}$$

and

$$S(\varphi X, \varphi Y) = S(X, Y) + (dim(M) - 1)\eta(X)\eta(Y). \tag{2.3.13}$$

Putting $Y = \xi$ in (2.3.11), we get

$$\mu - \lambda = 2n, \tag{2.3.14}$$

for a $(2n + 1)-$dimensional Lorentzian Para- Sasakian manifold M. Blaga [2016] studied the Lorentzian Para - Sasakian Manifolds having cyclic Ricci tensor and those with cyclic $\eta-$ recurrent Ricci tensor and proved that there is no Ricci soliton with potential vector field ξ. In the same paper the author discussed the Lorentzian Para- Sasakian manifolds satisfying $(\xi, .)_R.S = 0$,and $(\xi, .)_S.R = 0$ and proved that in such cases M is Einstein.

Chapter 3

Ricci Solitons on Para- Kenmotsu Manifolds defined with W_8- Curvature tensor

This chapter contains the results obtained by analyzing antisymmetric and semisymmetric Sasakian para-Kenmotsu manifolds with respect to W_8 and these with Ricci cyclic tensor.

3.1 Ricci solitons on para- Kenmotsu manifolds satisfying $(\xi, .)W_8.S = 0$.

The condition to be satisfied by S is:

$$S(W_8(\xi, X)Y, Z) + S(Y, W_8(\xi, X)Z) = 0, \forall X, Y, Z \in \mathfrak{X}(M). \tag{3.1.1}$$

Using (2.2.17) and (2.2.18) in (3.1.1), we get :

$$(\mu-\lambda-2)\{(g(x,y)\eta(z)+g(X,Z)\eta(Y))-2g(Y,Z)\eta(x)-2(\mu-1)\eta(x)\eta(y)\eta(z)\} = 0, \forall X, Y, Z \tag{3.1.2}$$

Making $Z = \xi$ we get

$$(\mu - \lambda - 2)(g(X,Y) - \eta(X)\eta(Y)) = 0 \tag{3.1.3}$$

or

$$(\mu - \lambda - 2)g(\varphi X, \varphi Y) = 0, \forall X, Y \in \mathfrak{X}(M). \tag{3.1.4}$$

But $\lambda + \mu = 2n$, so $\mu - (2n - \mu) - 2 = 0$, or $2\mu = 2n + 2$. Hence, we have the following theorem.

Theorem 3.1.1. *If (φ, ξ, η, g) is a para- Kenmotsu structure on the $(2n+1)$-dimensional manifold M, $(g, \xi, \eta, \lambda, \mu)$ is an η-Ricci soliton on M satisfying $(\xi, .)_{W_8}.S = 0$, then $\mu = n + 1$ and $\lambda = n - 1$.*

Proof. Using equations (2.2.1) to (2.2.11) and the fact that $\mu + \lambda = 2n$, the results in theorem are obtained by solving (3.1.4). □

Corollary 3.1.2. *" If (φ, ξ, η, g) is a para- Kenmotsu structure on the $(2n+1)$- dimensional manifold M, $(g, \xi, \eta, \lambda, \mu)$ is an η-Ricci soliton on M satisfying $(\xi, .)_{W_8}.S = 0$, then M is quasi- Einstein.*

Proof. As $\mu \neq 0$ and by using definition (1.12.12), the Corollary is deduced from the above theorem by using the expression 2.2.13. □

Corollary 3.1.3. *On a para-Kenmotsu manifold $(M, \varphi, \xi, \eta, g)$satisfying $(\xi, .)_{W_8}.S = 0$, there is no Ricci solitons with potential vector field ξ.*

Proof. The relation (2.2.13) is Ricci soliton if $\mu = 0$ and $\lambda \neq 0$ but $\mu = 0$ implies $n = -1$,that is, $\lambda = -2$ which is a contradiction as dimension of manifold is always positive and $\mu + \lambda = 2n$. □

3.2 Ricci solitons on para-Kenmotsu manifolds satisfying $(\xi,.)_S.W_8=0$.

The condition to be satisfied by S is :

$$\begin{aligned}
S(X,W_8(Y,Z)V)\xi-S(\xi,W_8(Y,Z)V)X+S(X,Y)W_8(\xi,Z)V\\
-S(\xi,Y)W_8(X,Z)V+S(X,Z)W_8(Y,\xi)V-S(\xi,Z)W_8(Y,X)V\\
+S(X,V)W_8(Y,Z)\xi-S(\xi,V)W_8(Y,Z)X=0\quad\forall X,Y,Z,V\in\mathfrak{X}(M).
\end{aligned}\tag{3.2.1}$$

Making inner product with ξ,the relation (3.2.1) becomes:

$$\begin{aligned}
S(X,W_8(Y,Z)V)-S(\xi,W_8(Y,Z)V)\eta(X)+S(X,Y)\eta(W_8(\xi,Z)V)\\
-S(\xi,Y)\eta(W_8(X,Z)V)+S(X,Z)\eta(W_8(Y,\xi)V)-S(\xi,Z)\eta(W_8(Y,X)V)\\
+s(X,V)\eta(W_8(Y,Z)\xi)-S(\xi,v)\eta(W_8(Y,Z)X)\quad\forall\quad X,Y,Z,V\in\mathfrak{X}(M).
\end{aligned}\tag{3.2.2}$$

Expanding the terms of (3.2.2) and using(2.2.17) in(2.2.18) , we get :

$$(\mu+\lambda)\{\frac{1}{2n}S(X,Y)+\eta(X)\eta(Y)+g(X,Y)-\eta(X)\eta(Y)\}=0.\tag{3.2.3}$$

Hence,

$$\frac{1}{2n}S(X,Y)=-g(X,Y)\tag{3.2.4}$$

$$S(X,Y)=-2ng(X,Y).\tag{3.2.5}$$

Thus , we have the following theorem

Theorem 3.2.1. *If (φ,ξ,η,g) is a para- Kenmotsu structure on the* $(2n+1)$-

dimensional manifold M, (g,ξ,η,λ,μ) is an η-Ricci soliton on M satisfying $(\xi,.)_S.W_8 = 0$, then $\mu = 0$ and $\lambda = 2n$ or $\mu = 2n$ and $\lambda = 2n$.

Proof. Using equations(2.2.1) to (2.2.11) and from the fact that, $\lambda + \mu = 2n$, the results in above theorem are obtained from (3.2.3). □

Corollary 3.2.2. *If (φ,ξ,η,g) is a para- Kenmotsu structure on the $(2n+1)$-dimensional manifold M, (g,ξ,η,λ,μ) is an η-Ricci soliton on M satisfying $(\xi,.)_S.W_8 = 0$, then M is Einstein manifold.*

Proof. From the above theorem, the Ricci tensor is collinear with metric. Hence by definition (1.12.12) we deduce the corollary. □

Corollary 3.2.3. *On a para-Kenmotsu manifold (M,φ,ξ,η,g)satisfying $(\xi,.)_S.W_8 = 0$, there is no Ricci solitons with potential vector field ξ.*

Proof. The relation (3.2.4) implies that $L_\xi g = 0$ and (2.1.1) becomes:

$$2S(X,Y) + 2\lambda g(X,Y) = 0 \quad \forall X,Y \in \mathfrak{X}(M) \tag{3.2.6}$$

That is, $L_\xi g = 0$ and hence, g is invariant in direction ξ. □

3.3 Ricci solitons on para- Kenmotsu manifolds having cyclic Ricci tensor.

A Riemannian manifold (M, g) is said to have a cyclic Ricci tensor if:

$$(\nabla_X S)(Y,Z) + (\nabla_Y S)(Z,X) + (\nabla_Z S)(X,Y) = 0, \tag{3.3.1}$$

where:

$$(\nabla_X S)(Y,Z) = X(S(Y,Z)) - S(\nabla_X Y, Z) - S(Y, \nabla_X Z). \tag{3.3.2}$$

Using (2.2.17), we get:

$$\begin{aligned}(\nabla_X S)(Y,Z) = -(\lambda+1)[Xg(Y,Z) - g(\nabla_x Y, Z)\\ -g(Y, \nabla_x Z)] - (\mu-1)[X\eta(Y)\eta(Z)\\ -\eta(\nabla_X Y)\eta(Z) - \eta(Y)\eta(\nabla_X Z)].\end{aligned} \tag{3.3.3}$$

As $\nabla g = 0$, (3.3.3) becomes:

$$\begin{aligned}(\nabla_X S)(Y,Z) = -(\mu-1)[X\eta(Y)\eta(Z) - \eta(\nabla_X Y)\eta(Z)\\ -\eta(Y)\eta(\nabla_X Z)].\end{aligned} \tag{3.3.4}$$

Hence,

$$
\begin{aligned}
(\nabla_X S)(Y,Z)+(\nabla_Y S)(Z,X)+(\nabla_Z S)(X,Y) = -(\mu-1)[X\eta(Y)\eta(Z) \\
-\eta(\nabla_X Y)\eta(Z)-\eta(Y)\eta(\nabla_X Z) \\
+Y\eta(X)\eta(Z)-\eta(\nabla_Y Z)\eta(X)-\eta(Z)\eta(\nabla_Y X) \\
+Z\eta(X)\eta(Y)-\eta(\nabla_Z X)\eta(Y)-\eta(X)\eta(\nabla_Z Y)].
\end{aligned}
\tag{3.3.5}
$$

After computations, the condition (3.3.1) becomes:

$$
\begin{aligned}
-2(\mu-1)[((g(X,Y)-\eta(X)\eta(Y))\eta(Z)+(g(X,Z) \\
-\eta(X)\eta(Z))\eta(Y)+(g(Y,Z)-\eta(Y)\eta(Z))\eta(X)]=0.
\end{aligned}
\tag{3.3.6}
$$

Putting $Z=\xi$, we get:

$$
-2(\mu-1)[g(X,Y)-\eta(X)\eta(Y)]=0. \tag{3.3.7}
$$

Hence, $\mu=1$ and we state the following theorem.

Theorem 3.3.1. *If (φ,ξ,η,g) is a para- Kenmotsu structure on $(2n+1)-$ dimensional manifold M, and manifold (M,g) has Cyclic Ricci tensor, then $\mu=1$ and $\lambda=2n-1$.*

Proof. From equations(2.2.1) to (2.2.11) and knowing that from (3.3.7) we have $\mu-1=0$,the results in the theorem are obtained from the fact that $\mu+\lambda=2n$. □

Corollary 3.3.2. *If (φ,ξ,η,g) is a para- Kenmotsu structure on $(2n+1)-$ dimensional manifold M, and manifold (M,g) has Cyclic Ricci tensor, then*

M is quasi- Einstein.

Proof. Using the same argument as in corollary (3.1.2), the result follows from the above theorem. □

Corollary 3.3.3. *On a para-Kenmotsu manifold $(M, \varphi, \xi, \eta, g)$ having Ricci cyclic tensor , there is no Ricci solitons with potential vector field ξ.*

Proof. The relation (2.2.13) is Ricci soliton if $\mu = 0$ and $\lambda \neq 0$ but $\mu = 0$ implies $n = -1$,that is, $\lambda = -2$ and this is a contradiction as dimension of manifold is always positive and $\mu + \lambda = 2n$. □

Chapter 4

Ricci solitons on Lorentzian Para- Sasakian manifolds defined with W_8- Curvature tensor

In this chapter, antisymmetric and semisymmetric Lorentzian Para- Sasakian manifolds with respect to W_8 are discussed.

4.1 Ricci Solitons on Lorentzian Para- Sasakian Manifolds satisfying $(\xi,.)_{W_8}.S=0$

The condition to be satisfied by S is

$$S(W_8(\xi,X)Y,Z)+S(Y,W_8(\xi,X)Z)=0 \tag{4.1.1}$$

Using the above expression in (2.3.11), we get

$$\begin{aligned} S(W_8(\xi,X)Y,Z) &= -[g(\varphi Y,Z)\eta(X)-g(\varphi X,Z)\eta(Y)] \\ &-\lambda[\frac{\mu}{2n}g(X,Y)\eta(Z)-g(X,Z)\eta(Y)+g(Y,Z)\eta(X) \\ &+\frac{1}{2n}g(\varphi X,Y)\eta(Z)+\frac{\mu}{2n}\eta(x)\eta(Y)\eta(Z)] \\ -\mu[\frac{-\mu}{2n}g(X,Y)\eta(Z) &-\frac{1}{2n}g(\varphi X,Y)\eta(z)-\frac{\mu}{2n}\eta(X)\eta(Y)\eta(Z)]. \end{aligned} \tag{4.1.2}$$

and

$$\begin{aligned}S(Y,W_8(\xi,X)Z) &= -[g(\varphi Z,Y)\eta(X)-g(\varphi X,Y)\eta(Z)]\\ &-\lambda[\frac{\mu}{2n}g(X,Z)\eta(Y)-g(X,Y)\eta(Z)+g(Y,Z)\eta(X)\\ &+\frac{1}{2n}g(\varphi X,Z)\eta(Y)+\frac{\mu}{2n}\eta(X)\eta(Y)\eta(Z)]\\ -\mu[\frac{-\mu}{2n}g(X,Z)\eta(Y)&-\frac{1}{2n}g(\varphi X,Z)\eta(Y)-\frac{\mu}{2n}\eta(X)\eta(Y)\eta(Z)].\end{aligned} \tag{4.1.3}$$

Putting $Z=\xi$, the condition (4.1.1) becomes :

$$g(\varphi X,Y)[\frac{\lambda}{2n}-\frac{\mu}{2n}-1]+[g(X,Y)+\eta(X)\eta(Y)][\frac{\lambda\mu}{2n}-\frac{\mu^2}{2n}-\lambda]=0. \tag{4.1.4}$$

Introducing (2.3.14) in (4.1.4), we get:

$$2g(\varphi X,Y)+(\lambda+\mu)g(\varphi X,\varphi Y)=0. \tag{4.1.5}$$

Putting $Y=\varphi Y$ in (4.1.5), we get :

$$2g(\varphi X,\varphi Y)+(\lambda+\mu)g(\varphi X,Y)=0. \tag{4.1.6}$$

Adding (4.1.5) and (4.1.6), we get :

$$(2+\lambda+\mu)(g(\varphi X,Y)+g(\varphi X,\varphi Y))=0. \tag{4.1.7}$$

That is,

$$2+\lambda+\mu=0 \tag{4.1.8}$$

Thus, by (2.3.14), the following theorem is stated.

Theorem 4.1.1. *If (φ, ξ, η, g) is a Lorentzian para- Sasakian structure on the $(2n+1)-$dimensional manifold M satisfying $(\xi, .)_{W_8}.S = 0$, (g, ξ, λ, μ) is $\eta-$Ricci soliton on M, then $\mu = n - 1$ and $\lambda = -n - 1$.*

Proof. Using properties defined by equations (2.3.1) to (2.3.14),the above theorem is obtained from definition (5.1.8). □

From (4.1.5) and (2.3.11), we get

$$S(X, Y) = ng(X, Y) - n\eta(X)\eta(Y) \tag{4.1.9}$$

Thus, we have the following theorem.

Theorem 4.1.2. *If (φ, ξ, η, g) is a Lorentzian para- Sasakian structure on the $(2n+1)-$dimensional manifold M satisfying $(\xi, .)_{W_8}.S = 0$, (g, ξ, λ, μ) is $\eta-$Ricci soliton on M, then M is quasi-Einstein*

Proof. Using properties defined by equations (2.3.6) to (2.3.14) and from definition (1.12.12),the above theorem is obtained. □

Corollary 4.1.3. *On Lorentzian Para- Sasakian manifold $(M, \varphi, \xi, \eta, g)$ satisfying $(\xi, .)_{W_8}.S = 0$, there is no Ricci soliton with potential vector ξ.*

Proof. Since M is Quasi-Einstein , g is invariant in direction ξ. □

4.2 Ricci Solitons on Lorentzian Para-Sasakian Manifolds satisfying $(\xi,.)_S.W_8 = 0$

The condition to be satisfied by S is:

$$\begin{aligned} S(X,W_8(Y,Z)V)\xi - S(\xi,W_8(Y,Z)V)X + S(X,Y)W_8(\xi,Z)V \\ -S(\xi,Y)W_8(X,Z)V + S(X,Z)W_8(Y,Z)V - S(\xi,Z)W_8(Y,X)V \\ +S(X,V)W_8(Y,Z)\xi - S(\xi,V)W_8(Y,Z)X = 0. \end{aligned} \tag{4.2.1}$$

Taking inner product with ξ, the relation (4.2.1) becomes:

$$\begin{aligned} -S(X,W_8(Y,Z)V) - S(\xi,W_8(Y,Z)V)\eta(X)+ \\ S(X,Y)\eta(W_8(\xi,Z)V) - S(\xi,Y)\eta(W_8(X,Z)V) \\ +S(X,Z)\eta(W_8(Y,\xi)V) - S(\xi,Z)\eta(W_8(Y,X)V) \\ +S(X,V)\eta(W_8(Y,Z)V) - S(\xi,V)\eta(W_8(Y,Z)X) = 0. \end{aligned} \tag{4.2.2}$$

After using (2.3.11) in expanding the terms in (4.2.2) and putting $V = Z = \xi$ the condition to be satisfied by S becomes:

$$\frac{1}{2n}g(\varphi X,Y) + (\frac{\lambda}{2n}+1)[g(X,Y)+\eta(X)\eta(Y)] = 0, \tag{4.2.3}$$

using (2.3.14)

$$g(\varphi X,Y) + \mu g(\varphi X,\varphi Y) = 0. \tag{4.2.4}$$

Putting $Y = \varphi Y$, we get

$$g(\varphi X,\varphi Y) + \mu g(\varphi X,Y) = 0, \tag{4.2.5}$$

and adding (4.2.4) and (4.2.5), we get

$$(1+\mu)(g(\varphi X, Y) + g(\varphi X, \varphi Y)) = 0. \tag{4.2.6}$$

Thus, we get the following theorem.

Theorem 4.2.1. *If (φ, ξ, η, g) is a Lorentzian para- Sasakian structure on the $(2n+1)-$dimensional manifold M satisfying $(\xi, .)_S.W_8 = 0$, (g, ξ, λ, μ) is $\eta-$Ricci soliton on M, then $\mu = -1$ and $\lambda = -(2n+1)$.*

Proof. We have used properties defined by equations (2.3.1) to (2.3.14) to obtain the results in above theorem through (4.2.6). □

Using (4.2.4), we get:

$$g(\varphi X, y) = -\mu(g(X, Y) - \eta(X)\eta(Y)). \tag{4.2.7}$$

Introducing (4.2.7) in (2.3.11), we get:

$$S(X, Y) = -2ng(X, Y). \tag{4.2.8}$$

Hence,we state the following theorem.

Theorem 4.2.2. *If (φ, ξ, η, g) is a Lorentzian para- Sasakian structure on the $(2n+1)-$dimensional manifold M satisfying $(\xi, .)_S.W_8 = 0$, (g, ξ, λ, μ) is $\eta-$Ricci soliton on M, then M is Einstein.*

Proof. the results in above theorem are due to definition (1.12.12). □

From this theorem we can deduce :

Corollary 4.2.3. *On Lorentzian Para- Sasakian manifold* $(M, \varphi, \xi, \eta, g)$ *satisfying* $(\xi, .)_S.W_8 = 0$*, there is no Ricci soliton with potential vector* ξ*.*

Proof. According to the above theorem g is invariant in direction ξ, as M is Einstein. □

Chapter 5

Ricci solitons on Lorentzian Para- Sasakian manifolds defined with W_2- Curvature tensor

This chapter discuss antisymmetric and semisymmetric Lorentzian Para-Sasakian manifolds with respect to W_2.

5.1 $\eta-$Ricci Solitons on Lorentzian Para- Sasakian Manifolds satisfying $(\xi, .)_{W_2}.S = 0$.

The condition to be satisfied by S is

$$S(W_2(\xi, X)Y, Z) + S(Y, W_2(\xi, X)Z) = 0. \tag{5.1.1}$$

Using the above expression in (2.3.11), we get

$$S(W_2(\xi, X)Y, Z) = \eta(Y)\{g(\varphi X, Z) + \lambda g(X, Z) - \mu\eta(X)\eta(Z)$$
$$+\frac{1}{2n}[g(X, Z) + \eta(X)\eta(Z) + 2\lambda g(\varphi X, Z) + \lambda^2 g(X, Z) + 2\lambda\mu\eta(X)\eta(Z) - \mu^2\eta(X)\eta(Z)]\}. \tag{5.1.2}$$

That is,

$$S(W_2(\xi, X)Y, Z) = \frac{1}{2n}[(\lambda+\mu)g(\varphi X, Z)\eta(Y)+(\lambda\mu+1)(g(X, Z)\eta(Y)+\eta(x)\eta(Y)\eta(Z))]. \tag{5.1.3}$$

and

$$S(Y, W_2(\xi, X)Z) = \frac{1}{2n}[(\lambda+\mu)g(\varphi X, Y)\eta(Z) \\ +(\lambda\mu+1)(g(X,Y)\eta(Z)+\eta(X)\eta(Y)\eta(Z))]. \tag{5.1.4}$$

Putting $Z = \xi$, the condition (5.1.1) becomes :

$$(\lambda+\mu)g(\varphi X, Y) + (\lambda\mu+1)g(\varphi X, \varphi Y) = 0. \tag{5.1.5}$$

Putting $Y = \varphi Y$ in (5.1.5), we get :

$$(\lambda+\mu)g(\varphi X, \varphi Y) + (\lambda\mu+1)g(\varphi X, Y) = 0. \tag{5.1.6}$$

Adding (5.1.5) and (5.1.6), we get :

$$(\mu+\lambda+\lambda\mu+1)(g(\varphi X, \varphi Y) + g(\varphi X, Y) = 0. \tag{5.1.7}$$

That is,

$$(\lambda+1)(\mu+1) = 0. \tag{5.1.8}$$

Thus , the following theorem is stated.

Theorem 5.1.1. *If (φ, ξ, η, g) is a Lorentzian para- Sasakian structure on the $(2n+1)-$dimensional manifold M satisfying $(\xi, .)_{W_2}.S = 0$, (g, ξ, λ, μ) is $\eta-$Ricci soliton on M, then $\mu = -1$ and $\lambda = -2n-1$ or $\lambda = -1$ and $\mu = 2n-1$.*

Proof. The results in the theorem are obtained by solving (5.1.8) and using properties defined by equations (2.3.1) to (2.3.14). □

For $\mu = -1$ we get from 5.1.5 that

$$(\lambda\mu + 1)g(\varphi X, Y) = g(\varphi X, \varphi Y), \tag{5.1.9}$$

and using 5.1.5 in 2.3.11, we get

$$S(X, Y) = -2ng(X, X). \tag{5.1.10}$$

Thus, we have the following theorem.

Theorem 5.1.2. *If (φ, ξ, η, g) is a Lorentzian para- Sasakian structure on the $(2n+1)-$dimensional manifold M satisfying $(\xi, .)_{W_2}.S = 0$, (g, ξ, λ, μ) is $\eta-$Ricci soliton on M, then for $\mu = -1$, M is Einstein.*

Proof. Using definition (1.12.12) of Einstein and quasi- Einstein and the above results (5.1.10) we prove the theorem. □

For $\lambda = -1$ we get from 5.1.5 that

$$g(\varphi X, Y) = g(\varphi X, \varphi Y) \tag{5.1.11}$$

and from 5.1.5 and 2.3.11, we get

$$S(X, Y) = -2n\eta(X)\eta(Y) \tag{5.1.12}$$

Thus, we have the following theorem.

Theorem 5.1.3. *If (φ, ξ, η, g) is a Lorentzian para- Sasakian structure on the $(2n+1)-$dimensional manifold M satisfying $(\xi, .)_{W_2}.S = 0$, (g, ξ, λ, μ) is $\eta-$Ricci soliton on M, then for $\lambda = -1$ M is Einstein.*

Proof. Using the definition(1.12.12) of Einstein and quasi- Einstein and the above results (5.2.9) we prove the theorem. $\square$

The following corollary is deduced from the above two theorems.

Corollary 5.1.4. *On Lorentzian Para- Sasakian manifold $(M, \varphi, \xi, \eta, g)$ satisfying $(\xi, .)_{W_2}.S = 0$, there is no Ricci soliton with potential vector ξ.*

5.2 $\eta-$Ricci Solitons on Lorentzian Para-Sasakian Manifolds satisfying $(\xi, .)_S.W_2 = 0$.

The condition to be satisfied by S is:

$$\begin{aligned} S(X, W_2(Y, Z)V)\xi - S(\xi, W_2(Y, Z)V)X \\ +S(X, Y)W_2(\xi, Z)V - S(\xi, Y)W_2(X, Z)V \\ +S(X, Z)W_2(Y, Z)V - S(\xi, Z)W_2(Y, X)V \\ +S(X, V)W_2(Y, Z)\xi - S(\xi, V)W_2(Y, Z)X = 0. \end{aligned} \tag{5.2.1}$$

Taking inner product with ξ the relation (5.2.1) becomes:

$$\begin{aligned} -S(X, W_2(Y, Z)V) - S(\xi, W_2(Y, Z)V)\eta(X)+ \\ S(X, Y)\eta(W_2(\xi, Z)V) - S(\xi, Y)\eta(W_2(X, Z)V) \\ +S(X, Z)\eta(W_2(Y, \xi)V) - S(\xi, Z)\eta(W_2(Y, X)V) \\ +S(X, V)\eta(W_2(Y, Z)V) - S(\xi, V)\eta(W_2(Y, Z)X) = 0. \end{aligned} \tag{5.2.2}$$

After using (2.3.11) and expanding the terms in (5.2.2) and putting $V = Z = \xi$ the condition to be satisfied by S becomes:

$$(\lambda + \mu)g(\varphi X, Y) + (\lambda\mu + 1) = 0. \tag{5.2.3}$$

Putting $Y = \varphi Y$, we get

$$(\lambda + \mu)g(\varphi X, \varphi Y) + (\lambda\mu + 1)g(\varphi X, Y) = 0. \tag{5.2.4}$$

After adding (5.2.3) and (5.2.4), we get

$$(\lambda + \lambda\mu + \mu + 1)[g(\varphi X, \varphi Y) + g(\varphi X, Y)] = 0. \tag{5.2.5}$$

Thus, the following theorem is stated.

Theorem 5.2.1. *If (φ, ξ, η, g) is a Lorentzian para- Sasakian structure on the $(2n+1)-$dimensional manifold M satisfying $(\xi, .)_S.W_2 = 0$, (g, ξ, λ, μ) is $\eta-$Ricci soliton on M, then $\mu = -1$ and $\lambda = -2n-1$ or $\lambda = -1$ and $\mu = 2n-1$.*

Proof. The results in above theorem are due to properties given by equations (2.3.1) to (2.3.14) by solving (5.2.5). □

For $\mu = -1$ we get from 5.2.3 that

$$(\lambda\mu + 1)g(\varphi X, Y) = g(\varphi X, \varphi Y), \tag{5.2.6}$$

and using 5.2.3 in 2.3.11, we get

$$S(X, Y) = -2ng(X, X). \tag{5.2.7}$$

Thus, we have the following theorem.

Theorem 5.2.2. *If (φ, ξ, η, g) is a Lorentzian para- Sasakian structure on the $(2n+1)-$dimensional manifold M satisfying $(\xi, .)_S.W_2 = 0$, (g, ξ, λ, μ) is $\eta-$Ricci soliton on M, then for $\mu = -1$ M is Einstein.*

Proof. The theorem is obtained by using properties in equations (2.3.1) to (2.3.14) and solving (5.2.6). □

For $\lambda = -1$ we get from 5.2.3 that

$$g(\varphi X, Y) = g(\varphi X, \varphi Y), \tag{5.2.8}$$

and from 5.2.3 and 2.3.11, we get

$$S(X, Y) = -2n\eta(X)\eta(Y). \tag{5.2.9}$$

Thus, we have the following theorem.

Theorem 5.2.3. *If (φ, ξ, η, g) is a Lorentzian para- Sasakian structure on the $(2n+1)-$dimensional manifold M satisfying $(\xi, .)_S.W_2 = 0$, (g, ξ, λ, μ) is $\eta-$Ricci soliton on M, then for $\lambda = -1$ M is Quasi- Einstein.*

Proof. the results in theorem are obtained using properties given by equations (2.3.1) to (2.3.14) and solving (5.2.9). □

From the above two theorems the following corollary is deduced.

Corollary 5.2.4. *On Lorentzian Para- Sasakian manifold $(M, \varphi, \xi, \eta, g)$ satisfying $(\xi, .)_S.W_8 = 0$, there is no Ricci soliton with potential vector ξ.*

Chapter 6

Results discussion and recommendations

6.1 Results discussion

Several mathematicians have investigated the properties of Ricci flows solutions especially the existence of Ricci solitons in some particular directions under certain conditions. Thus, for example, Blaga [2015], have studied $\eta-$Ricci solitons on para -Kenmotsu geometry manifolds satisfying $(\xi,\cdot)_R{\cdot}S=0$,$(\xi,\cdot)_S\cdot R=0$,$(\xi,\cdot)_{W_2}\cdot S=0$ and $(\xi,\cdot)_S\cdot W_2=0$. Also Nagaraja and Venu [2016], obtained some results on Rcci solitons satisfying $(\xi,\cdot)_H\cdot S=0$,$(\xi,\cdot)_{\tilde{C}}\cdot S=0$,$(\xi,\cdot)_R\cdot\tilde{C}=0$, $(\xi,\cdot)_P\cdot\tilde{C}=0$ and Bagewadi et al. [2013]have considered the cases of $(\xi,\cdot)_R{\cdot}B=0$, $(\xi,\cdot)_B{\cdot}S=0$, $(\xi,\cdot)_S{\cdot}R=0$, $(\xi,\cdot)_R{\cdot}\bar{P}=0$ and $(\xi,\cdot)_{\bar{P}}\cdot S=0$. Comparing our results to the previous results in the same field, we can conclude that our objectives has been reached as in chapter three we proved that the Para- Kenmotsu manifolds satisfying $(\xi,.)_{W_8}.S=0$. are quasi- Einstein Manifolds and those satisfying $(\xi,.)_S.W_8=0$, are Einstein Manifolds.

At the end of this chapter,we proved that the para- Kenmotsu manifolds with cyclic Ricci tensor and $\eta-$ Ricci soliton structure are quasi-Einstein manifolds .

Also in chapter four it has been proved that Lorentzian Para- Sasakian mani-

folds satisfying $(\xi, .)_s.W_8 = 0$ and having $\eta-$ Ricci soliton structure$L_\xi g + 2s + 2\lambda g + 2\mu\eta \otimes \eta$ are quasi-Einstein manifolds and those satisfying $(\xi, .)_{W_8}.S = 0$ are Einstein manifolds while the results in chapter five show that Lorentzian Para- Sasakian manifolds satisfying $(\xi, .)_s.W_2 = 0$ and having $\eta-$ Ricci soliton structure are Einstein or quasi-Einstein manifolds according to the value μ and λ.
The same results have been established on Manifolds satisfying $(\xi, .)_{W_2}.S = 0$.
Briefly this research has established the existence of strong relation between Ricci solitons and quasi-Einstein metrics on Lorentzian para- Sasakian and para- Kenmotsu manifolds and classification of these manifolds with the properties under our study, has been established.

6.2 Recommendations

The field of Ricci solitons and Ricci flows is among richest field of research interest. It is still open domain of research for multiple discipline, including Geometry, Calculus, algebra among others.
We recommend the consideration of the connection between Ricci solitons and Einstein metrics on Spaces of any dimensions and any structure, for future coming research.

The main problem when those Ricci solitons are taken as solutions of Ricci flows is the study of their stability.
That is why it is recommended for future research, to investigate different cases by starting on some of them with high degree of symmetry, in future researches.

The properties of complex Riemannian foliations on complex Kähler Ricci solitons of some special type of symmetries is up to now one of open problem in Applied mathematics field.

That is why, future research can be conducted in the investigation of Ricci flows stability. Also deep geometric analysis are particular techniques to study flows such as Ricci flow especially the dynamical stability of Kahller Ricci flow at its critical point. Locally Riemannian symmetric spaces are our hope to develop an explicit theory for foliations and submanifold giving a generalization a classical study of surface in $\Re^3$ to a large class of fundamental manifolds by using the strong control over the Riemannian curvature tensor. Another problem here to study is the case of Riemannian foliations of symmetric spaces and spaces with positive sectional curvature.

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