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On Spectra and Almost Similarity of Operators in Hilbert Spaces

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Joseph Mutuku Matheka

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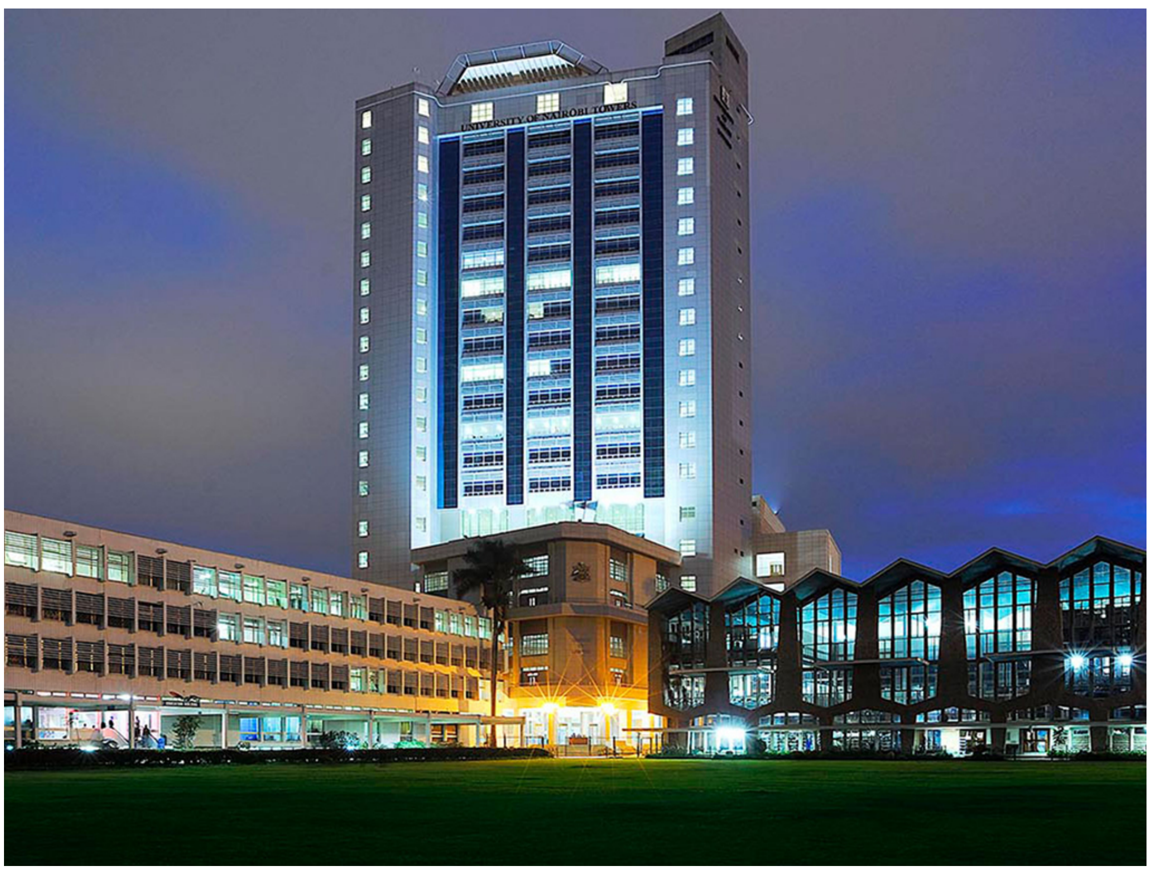

School of Mathematics

Submitted to the School of Mathematics in partial fulfilment for a degree in Master of Science in Pure Mathematics

Master Project in Pure Mathematics University of Nairobi
June 2019

On Spectra and Almost Similarity of Operators in Hilbert Spaces

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Master Thesis
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Abstract

This project is on spectra and almost similarity of operators in Hilbert spaces.
In chapter one we discuss the meaning and the structure of a Hilbert space. Here the linear structure, the norm, the inner product structure and convergence of sequences in a set of vectors are discussed to yield the meaning of a Hilbert space.

In chapter two, transformation of elements in a Hilbert space is discussed. The nature of transformations are also discussed in this chapter i.e. the preservation of linear structure, boundedness and the norm. The Banach algebra of bounded linear operators is also established. We use the linear operator to define invariant subspaces of a Hilbert space. We also define the spectra of operators on Hilbert spaces. The structure and the subsets of the spectrum are discussed in this chapter. We also discuss the spectrum of some classes of operators.

The third chapter is on similarity and quasi-similarity of operators. We show that unitary equivalence, similarity and quasi-similarity of operators are equivalence relations. Also unitary equivalence implies similarity and similarity implies quasi-similarity. Unitary equivalent and Similar operators have equal spectra in general. Quasi-similar operators on a finite dimensional Hilbert space have equal spectra but on infinite dimensional Hilbert spaces, quasi similar operators have equal spectra if the operators are hypo-normal.

The fourth chapter is on almost similarity of operators. We discuss the relationship of cartesian and polar decomposition of operators with almost similarity of operators. We show that almost similarity of operators is an equivalence relation. Almost similar operators which are Hermitian or projections have equal spectra.

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Declaration and Approval

I the undersigned declare that this dissertation is my original work and to the best of my knowledge, it has not been submitted in support of an award of a degree in any other university or institution of learning.

Signature Date

Joseph Mutuku Matheka
Reg No. I56/7735/2017

In my capacity as a supervisor of the candidate's dissertation, I certify that this dissertation has my approval for submission.

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Dedication

I would like to dedicate this project to my parents Mr. and Mrs. Matheka Kaleli

Contents

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I acknowledge the almighty God for giving me the vision and guidance to pursue it

....

Joseph Mutuku Matheka

Nairobi, 2017.

List of Notations

$=$	is equal to
$\neq$	is not equal to
$\equiv$	is equivalent to
$\in$	is an element in
$\exists$	there exists
$\nexists$	there does not exists
$\forall$	for all
$\subseteq$	is a Subset or equal to
$\subset$	Proper subset of
$\leq$	Less than or equal to
$\geq$	Greater than or equal to
$<$	Less than
$>$	Greater than
$\cup$	Union
$\cap$	Intersection
$\Rightarrow$	Implies
$\Leftrightarrow$	if an only if
s.t	such that
w.r.t	with respect to
iff	if and only if
$\rightarrow$	tends to
$\leftrightarrow$	Commutes with
$\perp$	is perpendicular to
$\cong$	is unitarily equivalent to
$\sim$	is similar to
$\approx$	is quasi-similar to
$\overset{a.s}{\sim}$	is almost similar to

1 PRELIMINARIES

1.1 Introduction

A set is a well defined collection of objects.
Sets are denoted by capital letters, e.g.set $\mathbb{X}$.
The objects in a set are called elements.
If ϱ is one of the objects in $\mathbb{X}$, then this is denoted $\varrho \in \mathbb{X}$ and read as ϱ belongs to $\mathbb{X}$.
If every element ϑ of $\mathbb{Y}$ is also a member in $\mathbb{X}$, then $\mathbb{Y} \subseteq \mathbb{X}$
e.g $\mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$
A set can be empty (i.e. having no elements), having a finite number of elements or have infinite number of elements.
Examples of infinite sets include: $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$

A function is a rule f that uniquely associates members of one set, say $\mathbb{X}$ with members of another set, say $\mathbb{Y}$ denoted $f : \mathbb{X} \to \mathbb{Y}$.
A domain D of f denoted $D(f)$ contains all the values at which a function f is defined.
The collection $\{f(\varrho) \in \mathbb{Y} : \varrho \in \mathbb{X}\}$ of values that function f can produce is called the range of f and denoted $Ran(f)$.
If an object $\varrho \in \mathbb{X}$ is mapped to an object $f(\varrho) \in \mathbb{Y}$ by a mapping f, this is denoted $f : \varrho \mapsto f(\varrho)$

The term vector was first used by 18^{th} century Astronomers investigating planetary revolution around the sun.
A Euclidean vector is a Geometric object that has magnitude and direction.
In pure mathematics, vectors are abstract entities which may or may not be characterised by a magnitude and direction.
Thus euclidean vectors are special kind of vectors which are elements of a special kind of a vector space called the euclidean space. The inner product associates each pair of elements in a vector space with a scalar.

A metric is a distance function on vectors while a norm gives the notion of a vector's length. A linear space $\mathbb{X}$ with norm function defined on $\mathbb{X}$ given $\mathbb{X}$ is complete is a Banach space.

The earliest Hilbert spaces were worked on in the early twentieth century by David Hilbert. A complete normed linear space $\mathbb{H}$ whose norm is induced by the inner product function is a Hilbert space.

A linear transformation on a linear space preserves the operation of addition and scalar multiplication of vectors. This transformation may be bounded or unbounded,invertible or not invertible, symmetric or not symmetric among other properties. Various classes of operators are studied which include normal, unitary, hypo-normal,quasi-normal among other classes.

In mathematics the spectrum of a linear transformation is a generalized collection of eigenvalues of a given matrix. specifically a $\lambda \in \mathbb{C}$ is an object contained in the spectrum of a bounded linear Transformation Λ when $\Lambda - \lambda I$ does not have an inverse where I is identity operator. If the linear space has dimension less than infinity then a transformation on this space has a spectrum which is just given by its eigenvalues. In the case where the dimension of the linear space is infinite we will have some other objects in the spectrum of a linear transformation of this space added to its eigen-values

Two Operators on a Hilbert Space such that the Operators are intertwined by another invertible operator are similar and are equivalent to each other in terms of eigenvalues, trace and spectrum among others.
Quasi Similarity was first studied by Foias and S. Nagy(4). Two Operators are Quasi Similar when each is a quasi-affine transform of each other. Quasi-similar operators on a finite dimensional Hilbert spaces have equal spectra but in case of infinite dimensional Hilbert spaces, Sz Nagy(4) has shown that the Operators may be quasi-similar but have spectra which are not equal. Clary(17) proved the condition under which two Quasi Similar transformaions will be having spectra being equal, i.e. if the operators are hyponormal.

Almost similarity was first introduced by A.A.S Jibril(21). He proved various results that relate almost similarity and other classes of operators. In 2008

Nzimbi et al(13) results are also handy in enriching almost similarity where he attempts to classify those operators where quasi-similarity implies almost similarity.
A bounded linear operator Λ can be expressed like a complex number $\xi = \varrho + i\vartheta$ into real and imaginary parts by decomposing it into two unique hermitian operators Υ and Ψ such that $\Lambda = \Upsilon + i\Psi$.
Λ can also be expressed like a complex number $\xi = re^{i\theta}$ into polar form as $\Lambda = UG$ given U is a partially isometric and G being non-negative self adjoint operator.
If $\Lambda = UG = \Upsilon + i\Psi$ then $G^2 = \Lambda^*\Lambda$ and $2\Upsilon = \Lambda^* + \Lambda$.
Definition of almost similarity makes use of Cartesian and polar decomposition of operators. Two operators Λ and Ξ are almost similar if $\Lambda^*\Lambda = \Upsilon^{-1}(\Xi^*\Xi)\Upsilon$ and $\Lambda^* + \Lambda = \Upsilon^{-1}(\Xi^* + \Xi)\Upsilon$ where Υ is invertible operator.
We investigate the conditions under which two almost similar operators have equal spectra, i.e. if the operators are Hermitian or projections.

1.2 Notation and terminology

In this project, Vector spaces and vector subspaces will be denoted by

$$\mathbb{X}, \mathbb{X}_1, \mathbb{X}_2, \mathbb{Y}, \mathbb{Y}_1, \mathbb{Y}_2$$

Hilbert spaces will be denoted by

$$\mathbb{H}, \mathbb{H}_1, \mathbb{H}_2, \mathbb{H}_3, \mathbb{H}_4$$

and closed subspaces by

$$\mathbb{M}, \mathbb{M}_1, \mathbb{M}_2.$$

We will denote linear operators by

$$\Lambda, \Xi, \Psi, \Upsilon, \Gamma, \Omega, G, I, L, P, Q, R, S, T, U$$

For Λ on $\mathbb{H}$ and $\varrho \in \mathbb{H}$, then Λ of ϱ will be denoted $\Lambda(\varrho)$ or $\Lambda\varrho$.
$\Lambda(\mathbb{H})$ is bounded if $\| \Lambda\varrho \| \leq \alpha \| \varrho \|, \forall \varrho \in \mathbb{H}$ and $\alpha > 0$
$\mathbb{B}(\mathbb{H})$ will be the class of linear transformations on $\mathbb{H}$, which are linear and bounded.
The domain, co-domain, range, image, co-image, kernel and co-kernel of Λ will be denoted by
$D(\Lambda), \mathrm{CoD}(\Lambda), \mathrm{Ran}(\Lambda), \mathrm{Im}(\Lambda), \mathrm{CoIm}(\Lambda), \mathrm{Ker}(\Lambda), \mathrm{Coker}(\Lambda)$ respectively.
The dense range of $\Lambda \in \mathbb{B}(\mathbb{H}_1, \mathbb{H}_2)$ will be denoted by $\overline{\mathrm{Ran}(\Lambda)} = \mathbb{H}_2$
The norm of $\Lambda \in \mathbb{B}(\mathbb{H})$ will be denoted by

$$\|\Lambda\| = inf\{\alpha : \| \Lambda\varrho \| \leq \alpha \| \varrho \|, \forall(\varrho \neq 0) \in \mathbb{H}, \alpha > 0\}$$

$\Lambda^* \in \mathbb{B}(\mathbb{H}_2, \mathbb{H}_1)$ will denote the Hilbert ad-joint operator of $\Lambda \in \mathbb{B}(\mathbb{H}_1, \mathbb{H}_2)$.
The space of bounded invertible operators from $\mathbb{H}_1$ into $\mathbb{H}_2$ will be denoted by $\mathbb{G}(\mathbb{H}_1, \mathbb{H}_2)$.
$\Lambda \in \mathbb{B}(\mathbb{H})$ is said to be:

- idempotent if $\Lambda^2 = \Lambda$
- an involution if $\Lambda^2 = I$
- self ad-joint or Hermitian if $\Lambda^* = \Lambda$

- a projection if $\Lambda^2 = \Lambda = \Lambda^*$
- normal if $\Lambda^*\Lambda = \Lambda\Lambda^*$
- hypo-normal if $\Lambda^*\Lambda \geq \Lambda\Lambda^*$
- co-hyponormal if Λ^* is hyponormal
- seminormal if either hypornormal or co-hyponormal
- quasi-hypo-normal if $\Lambda^*(\Lambda^*\Lambda - \Lambda\Lambda^*)\Lambda \geq 0$
- p-quasi-hyponormal if $\Lambda^*[(\Lambda^*\Lambda)^p - (\Lambda\Lambda^*)^p]\Lambda \geq 0$
- paranormal if $\| \Lambda\varrho \|^2 \leq \| \Lambda^2\varrho \| \| \varrho \| \; \forall \varrho \in \mathbb{H}$
- unitary if $\Lambda^*\Lambda = \Lambda\Lambda^* = 1$
- isometry if $\Lambda^*\Lambda = I$
- partial isometry if $\Lambda\Lambda^*\Lambda = \Lambda$ ie if $\Lambda^*\Lambda$ is a projection
- co-isometry if $\Lambda\Lambda^* = I$

If $\Lambda \in \mathbb{B}(\mathbb{H})$ is
$Unitary \Rightarrow Normal \Rightarrow Quasi - normal \Rightarrow hypo - normal$
$\Rightarrow Paranormal$
The trace of Λ will be denoted

$$tr(\Lambda) = \sum_{i=1}^{n} \varrho_{ii}$$

where $\{\varrho_{ii} : i = 1, 2, ..., n\}$ are elements in the main diagonal.
$N(\Lambda - \lambda I)$ will denote the eigen space.
$\rho(\Lambda), \sigma(\Lambda), \sigma_\Lambda(\Lambda), \sigma_C(\Lambda), \sigma_R(\Lambda), \pi_(\Lambda), \tau_(\Lambda)$ will denote resolvent set, spectrum, point spectrum, continuous spectrum, residual spectrum, approximate point spectrum and compression spectrum of an operator $\Lambda \in \mathbb{B}(\mathbb{H})$ respectively.
$R_\lambda(\Lambda) = (\Lambda - \lambda I)^{-1}$ will denote the resolvent of an operator $\Lambda \in \mathbb{B}(\mathbb{H})$ at λ
$W(\Lambda)$ will be the numerical range and $w(\Lambda)$ numerical radius of an operator Λ.

Two operators $\Lambda \in \mathbb{B}(\mathbb{H}_1)$ and $\Xi \in \mathbb{B}(\mathbb{H}_2)$ which are:
unitary equivalent will be denoted $\Lambda \cong \Xi$,
similar will be denoted $\Lambda \sim \Xi$,
quasi-similar will be denoted $\Lambda \approx \Xi$,
almost similar will be denoted $\Lambda \overset{a.s}{\sim} \Xi$
Note:

$$unitary - equivalence \Rightarrow similarity \Rightarrow quasi - similarity$$

and

$$unitary - equivalence \Rightarrow similarity \Rightarrow almost - similarity$$

.

1.3 Hilbert spaces

Definition 1.3.1. : *Linear space*
A linear space $(L.S)$ *over a scalar field* $\mathbb{K}$ *is a set* $\mathbb{X}$ *of vectors together with a rule* " + " *for adding any two elements* ϱ *and* ι *of* $\mathbb{X}$ *to form an element* $\varrho + \iota$ *of* $\mathbb{X}$ *(called vector addition) and another rule* "." *for multiplying any element* ϱ *of* $\mathbb{X}$ *by an element* κ *of* $\mathbb{K}$ *to form an element* $\kappa\varrho$ *of* $\mathbb{X}$ *(called scalar multiplication). Moreover the rules must satisfy the following familiar algebraic properties:*

1. *closure of vector addition.*
   $\varrho + \iota \in \mathbb{X}, \forall \varrho, \iota \in \mathbb{X}$
2. *Addition of vectors is commutative*
   $\varrho + \iota = \iota + \varrho, \forall \varrho, \iota \in \mathbb{X}$
3. *Addition of vectors is associative.*
   $\varrho + (\iota + \zeta) = (\varrho + \iota) + \zeta, \forall \varrho, \iota, \zeta \in \mathbb{X}$
4. *Existence of addition identity.*
   $\exists 0 \in \mathbb{X}$ *such that* $\varrho + 0 = \varrho, \forall \varrho \in \mathbb{X}$
5. *Existence of inverse elements of addition.*
   $\exists - \varrho \in \mathbb{X}$ *such that* $\varrho + (-\varrho) = (-\varrho) + \varrho = 0, \forall \varrho \in \mathbb{X}$
6. *Closure of scalar multiplication.*
   $\kappa\varrho \in \mathbb{X}, \forall \varrho \in \mathbb{X}, \forall \kappa \in \mathbb{K}$
7. *Distributivity of* "." *w.r.t scalar addition.*
   $\kappa(\varrho + \iota) = \kappa\varrho + \kappa\iota, \forall \varrho, \iota \in \mathbb{X}, \forall \kappa \in \mathbb{K}$
8. *Distributivity of* "." *w.r.t scalar addition.*
   $(\kappa_1 + \kappa_2)\varrho = \kappa_1\varrho + \kappa_2\varrho, \forall \varrho \in \mathbb{X}, \forall \kappa_1, \kappa_2 \in \mathbb{K}$
9. *Compatibility of* "." *with field multiplication.*
   $(\kappa_1\kappa_2)\varrho = \kappa_1(\kappa_2\varrho), \forall \varrho \in \mathbb{X}, \forall \kappa_1, \kappa_2 \in \mathbb{K}$
10. *Existence of identity element of scalar multiplication.*
    $\exists I \in \mathbb{X}$ *such that* $I\varrho = \varrho, \forall \varrho \in \mathbb{X}$

Definition 1.3.2.

For a subset $\mathbb{Y}$ of $\mathbb{X}$ to be a subspace the following are necessary and sufficient conditions.

$$\varrho + \iota \in \mathbb{Y}, \forall \varrho, \iota \in \mathbb{Y}$$

$$\kappa\varrho \in \mathbb{Y}, \forall \varrho \in \mathbb{Y}, \forall \kappa \in \mathbb{K}$$

Remark 1.3.3.
Note, $\{\bar{0}\}$ and $\mathbb{X}$ are trivial subspaces of $L.S$ $\mathbb{X}$.

Proposition 1.3.4.
let $\mathbb{X}$ be $L.S$ over a scalar field $\mathbb{K}$ and $\mathbb{Y}_1, \mathbb{Y}_2$ be subspaces. then $\mathbb{Y}_1 + \mathbb{Y}_2$ and $\mathbb{Y}_1 \cap \mathbb{Y}_2$ are linear subspace of $\mathbb{X}$

Proof.

1. Let $\varrho \in \mathbb{Y}_1$ and $\iota \in \mathbb{Y}_2$, then $(\varrho + \iota \in \mathbb{Y}_1 + \mathbb{Y}_2)$
   Let $\varrho_1, \varrho_2 \in \mathbb{Y}_1$ and $\iota_1, \iota_2 \in \mathbb{Y}_2$ then $(\varrho_1 + \iota_1), (\varrho_2 + \iota_2) \in \mathbb{Y}_1 + \mathbb{Y}_2$
   and
   $$(\varrho_1 + \iota_1) + (\varrho_2 + \iota_2) = (\varrho_1 + \varrho_2) + (\iota_1 + \iota_2) \in \mathbb{Y}_1 + \mathbb{Y}_2$$
   because $(\varrho_1 + \varrho_2) \in \mathbb{Y}_1$ and $(\iota_1 + \iota_2) \in \mathbb{Y}_2$ since $\mathbb{Y}_1$ and $\mathbb{Y}_2$ are linear spaces
   If $\kappa \in \mathbb{K}$ and $\varrho + \iota \in \mathbb{Y}_1 + \mathbb{Y}_2$ then $\kappa(\varrho + \iota) = \kappa\varrho + \kappa\iota$
   but $\kappa\varrho \in \mathbb{Y}_1$ and $\kappa\iota \in \mathbb{Y}_2 \Rightarrow \kappa(\varrho + \iota) \in \mathbb{Y}_1 + \mathbb{Y}_2$

2. Let $\varrho, \iota \in \mathbb{Y}_1 \cap \mathbb{Y}_2$ then $\varrho, \iota \in \mathbb{Y}_1$ and $\varrho, \iota \in \mathbb{Y}_2$
   $\Rightarrow \varrho + \iota \in \mathbb{Y}_1$ and $\varrho + \iota \in \mathbb{Y}_2 \Rightarrow \varrho + \iota \in \mathbb{Y}_1 \cap \mathbb{Y}_2$
   Let $\varrho \in \mathbb{Y}_1 \cap \mathbb{Y}_2$ and $\kappa \in \mathbb{K}$ then $\varrho \in \mathbb{Y}_1$ and $\varrho \in \mathbb{Y}_2$
   also $\kappa\varrho \in \mathbb{Y}_1$ and $\kappa\varrho \in \mathbb{Y}_2 \Rightarrow \kappa\varrho \in \mathbb{Y}_1 \cap \mathbb{Y}_2$

$\square$

Definition 1.3.5. : $Inner\ product\ spaces\ (I.P.S)$
A mapping $\langle , \rangle : \mathbb{X} \times \mathbb{X} \to \mathbb{K}$ is an inner product function on $\mathbb{X}$ if $\forall \varrho, \vartheta \in \mathbb{X}$, and $\kappa \in \mathbb{K}$, it satisfies the axioms below.

1. *Positivity axiom.*
   $\langle \varrho, \varrho \rangle \geq 0, \langle \varrho, \varrho \rangle = 0 \Leftrightarrow \varrho = 0, \varrho \in \mathbb{X}$

2. *Conjugate symmetry.*
   $\langle \varrho, \vartheta \rangle = \overline{\langle \vartheta, \varrho \rangle}$

3. *Homogeneity property.*
   $\langle \kappa\varrho, \vartheta \rangle = \kappa \langle \varrho, \vartheta \rangle$

4. *Distributive property.*
   $\langle \varrho + \vartheta, \zeta \rangle = \langle \varrho, \zeta \rangle + \langle \vartheta, \zeta \rangle$

Remark 1.3.6.
If $\mathbb{K} = \mathbb{R}$ *then the conjugate symmetry reduces to symmetry thus* $\langle \varrho, \vartheta \rangle = \langle \vartheta, \varrho \rangle$

Definition 1.3.7.
An $(I.P.S)$ *is a linear space endowed with* $I.P$ *structure.*

Examples 1.3.8.

1. *The euclidean vector space* $\mathbb{X} = \mathbb{R}^n$ *with the* $I.P$ *function*
   $\langle \varrho, \vartheta \rangle = \varrho.\vartheta = \varrho_1\vartheta_1 + \varrho_2\vartheta_2 + ... + \varrho_n\vartheta_n, \forall \varrho, \vartheta \in \mathbb{X}$
   such that $\varrho = (\varrho_1, \varrho_2, ..., \varrho_n)$ *and* $\vartheta = (\vartheta_1, \vartheta_2, ..., \vartheta_n)$
   is an IPS.

2. *The space* $\mathbb{X} = \mathbb{M}_{m,n}(\mathbb{R})$ *of* $m \times n$ *matrices with the inner product function*
   $\langle \Lambda, \Xi \rangle = \Sigma_{i=1}^{n} \Sigma_{j=1}^{m} \varrho_{ji}\vartheta_{ji}, \forall \Lambda, \Xi \in X$
   is an IPS.

3. *The space* $\mathbb{X} = C[p, q]$ *of continuous functions in the closed interval* $[p, q]$ *with the inner product function*
   $\langle f_1, f_2 \rangle = \int_p^q f_1(\varrho) f_2(\varrho) d\varrho, \forall f_1, f_2 \in \mathbb{X}$
   is an IPS.

Definition 1.3.9. : $Normed\ linear\ spaces$
A norm on $\mathbb{X}$ *is a rule* $\|, \|: \mathbb{X} \to \mathbb{R}^+$ *which meets the following requirements.*

1. *None negativity property*
   $\| \varrho \| \geq 0, \forall \varrho \in \mathbb{X}$

2. *Continuity property*
   $\| \varrho \| = 0 \Leftrightarrow \varrho = 0, \forall \varrho \in \mathbb{X}$

3. *Homogeneity property*
   $\| \kappa\varrho \| = | \kappa | \| \varrho \|, \forall \varrho \in \mathbb{X}, \forall \kappa \in \mathbb{K}$

4. *Triangular inequality*
   $\parallel \varrho + \vartheta \parallel \leq \parallel \varrho \parallel + \parallel \vartheta \parallel, \forall \varrho, \vartheta \in \mathbb{X}$

Remark 1.3.10.
If continuity property is relaxed i.e. $\parallel \varrho \parallel = 0$ *for some* $\varrho \neq 0$ *then* $\|, \|$ *is a semi-norm.*

Definition 1.3.11.
The pair $(\mathbb{X}, \|, \|)$ *where* $\|, \|$ *is a norm defined on* $\mathbb{X}$ *is a normed linear space* $(N.L.S)$.

Examples 1.3.12.

1. *The n-dimensional euclidean space with the euclidean norm of point* $\varrho = (\varrho_1, \varrho_2, ..., \varrho_n) \in \mathbb{R}^n$ *defined by* $\|\varrho\| = \sqrt{\varrho_1^2 + \varrho_2^2 + ... + \varrho_n^2}$.
2. *The linear space* $\mathbb{X} = \mathbb{R}^n$ *with the norm of point* $\varrho = (\varrho_1, \varrho_2, ..., \varrho_n) \in \mathbb{X}$ *defined by* $\parallel \varrho \parallel = {}^{sup}_{1 \leq i \leq n} \{| \varrho_i |\}$.
3. *The space* $\mathbb{X}$ *of integrable functions in the interval* $[p, q]$ *with* $\|f\| = (\int_p^q | f(t) |^p \, dt)^{\frac{1}{p}}$.

Theorem 1.3.13. *:The Cauchy Bunyakosky theorem (CBS)*
Let $\mathbb{X}$ *be an* $I.P.S$ *and* $\varrho, \vartheta \in \mathbb{X}$. *Then*

$$| \langle \varrho, \vartheta \rangle | \leq \sqrt{\langle \varrho, \varrho \rangle} \sqrt{\langle \vartheta, \vartheta \rangle}$$

Theorem 1.3.14.
Suppose $\langle \varrho, \vartheta \rangle$ *is an inner product on* $\mathbb{X}$, *then*

$\|\varrho\| = \sqrt{\langle \varrho, \varrho \rangle}$ *is a norm.*

`Proof.` $\langle \varrho, \varrho \rangle \geq 0 \Rightarrow \sqrt{\langle \varrho, \varrho \rangle} \geq 0 \Rightarrow \parallel \varrho \parallel \geq 0$,
hence the positivity property.
$\parallel \varrho \parallel = 0 \Leftrightarrow \sqrt{\langle \varrho, \varrho \rangle} = 0 \Leftrightarrow \langle \varrho, \varrho \rangle = 0 \Leftrightarrow \varrho = 0$,
the continuity property
$\parallel \varrho, \vartheta \parallel^2 = \langle \varrho + \vartheta, \varrho + \vartheta \rangle = \langle \varrho, \varrho + \vartheta \rangle + \langle \vartheta, \varrho + \vartheta \rangle$

$$= \langle \varrho, \varrho \rangle + \langle \varrho, \vartheta \rangle + \langle \vartheta, \varrho \rangle + \langle \vartheta, \vartheta \rangle$$

$$\leq \langle \varrho, \varrho \rangle + |\langle \varrho, \vartheta \rangle| + |\langle \vartheta, \varrho \rangle| + \langle \vartheta, \vartheta \rangle$$

$\leq \|\varrho\|^2 + 2\|\varrho\|\|\vartheta\| + \|\vartheta^2\|$ by CBS

$$(\|\varrho\| + \|\vartheta\|)^2$$

i.e. $\|\varrho + \vartheta\|^2 \leq (\|\varrho\| + \|\vartheta\|)^2 \Rightarrow \|\varrho + \vartheta\| \leq \|\varrho\| + \|\vartheta\|$
Therefore the sub additivity property is satisfied.
Consequently $\|\varrho\| = \sqrt{\langle \varrho, \varrho \rangle}$ is a norm. □

Remark 1.3.15.
The norm on ϱ defined by $\|\varrho\| = \sqrt{\langle \varrho, \varrho \rangle}$ is a norm.

Theorem 1.3.16.
Let $\mathbb{X}$ be an $I.P.S$ and $\varrho, \vartheta \in \mathbb{X}$. Then

$$\|\varrho + \vartheta\|^2 + \|\varrho - \vartheta\|^2 = 2(\|\varrho\|^2 + \|\vartheta\|^2)$$

Theorem 1.3.17.
Let $\mathbb{X}$ be an $I.P.S$ and $\varrho, \vartheta \in \mathbb{X}$. Then

$$\langle \upsilon, \vartheta \rangle = \frac{1}{4}(\|\varrho + \vartheta\|^2 - \|\varrho - \vartheta\|^2)$$

if $\mathbb{K} = \mathbb{R}$

$$\langle \varrho, \vartheta \rangle = \frac{1}{4}(\|\varrho + \vartheta\|^2 - \|\varrho - \vartheta\|^2 + i\|\varrho + i\vartheta\|^2 - i\|\varrho - i\vartheta\|^2)$$

if $\mathbb{K} = \mathbb{C}$

Definition 1.3.18.
A non empty subset $\mathbb{Y}$ of $I.P.S$ $\mathbb{X}$ is an orthogonal set if $\forall \varrho, \vartheta \in \mathbb{Y}$ and $\varrho \neq \vartheta$ then $\langle \varrho, \vartheta \rangle = 0$.
An orthogonal subset $\mathbb{Y}$ of $\mathbb{X}$ is called an orthonormal set if $\|\varrho\| = 1$ for any $\varrho \in \mathbb{Y}$

Theorem 1.3.19. *:Pythagorean law*
Let $\mathbb{X}$ be an $I.P.S$ and $\varrho, \vartheta \in \mathbb{X}$ where ϱ is orthogonal to ϑ denoted $\varrho \perp \vartheta$ i.e. $\langle \varrho, \vartheta \rangle = 0$. Then $\|\varrho + \vartheta\|^2 = \|\varrho\|^2 + \|\vartheta\|^2$

Proof.

$$\|\varrho+\vartheta\|^2 = \langle\varrho+\vartheta, \varrho+\vartheta\rangle = \langle\varrho, \varrho+\vartheta\rangle+\langle\vartheta, \varrho+\vartheta\rangle = \langle\varrho, \varrho\rangle+\langle\varrho, \vartheta\rangle+\langle\vartheta+\varrho\rangle+\langle\vartheta, \vartheta\rangle$$

$$\langle\varrho, \varrho\rangle + \langle\vartheta, \vartheta\rangle = \|\varrho\|^2 + \|\vartheta\|^2$$

□

Theorem 1.3.20.
A $N.L.S$ is a metric space with the metric

$$d(\varrho, \vartheta) = \|\varrho - \vartheta\|$$

Proof. $d(\varrho, \vartheta) = \|\varrho - \vartheta\| \geq 0$
$d(\varrho, \vartheta) = 0 \Leftrightarrow \| \varrho - \vartheta \| = 0 \Leftrightarrow \varrho - \vartheta = 0 \Leftrightarrow \varrho = \vartheta$
$d(\varrho, \vartheta) = \|\varrho, \vartheta\| = \|(-1)(\vartheta - \varrho)\| = |-1|\|\vartheta - \varrho\| = \|\vartheta - \varrho\| = d(\vartheta, \varrho)$
$d(\varrho, \iota) = \|\varrho - \iota\| = \|(\varrho - \vartheta) + (\vartheta - \iota)\| \leq \| \varrho - \vartheta \| + \| \vartheta - \iota \|$

$$= d(\varrho, \vartheta) + d(\vartheta, \iota)$$

□

Definition 1.3.21. *:Cauchy sequence A sequence $\{a_n\}$ in $\mathbb{X}$ is Cauchy if given a positive ε we can find $N \in J^+$ satisfying*

$$\|\varrho_m - \varrho_n\| < \varepsilon, \forall m, n > N$$

i.e $d(\varrho_m, \varrho_n) \to 0$ *as* $m, n \to \infty$

Definition 1.3.22.
A $N.L.S$ $\mathbb{X}$ is complete (a Banach space) if every Cauchy sequence in $\mathbb{X}$ also converges to a finite limit in $\mathbb{X}$.

Definition 1.3.23.
A Banach Space $\mathbb{H}$ is a Hilbert space if the norm on elements in $\mathbb{H}$ is induced by the $I.P$ function.

Examples 1.3.24.

1. *Euclidean space* $\mathbb{R}^3$ *equipped with inner product function defined by*

$$\langle \varrho, \vartheta \rangle = (\varrho_1, \varrho_2, \varrho_3).(\vartheta_1, \vartheta_2, \vartheta_3) = \vartheta_1\vartheta_1 + \varrho_2\vartheta_2 + \varrho_3\vartheta_3$$

   is a Hilbert space.

2. *The Lebesgue space* $(X, M.\mu)$ *of functions with inner product of functions* $f_1, f_2 \in (X, M, \mu)$ *defined as*

$$\langle f_1, f_2 \rangle = \int_{\times} f_1(s)\overline{f_2(s)}d\mu(s)$$

   is a Hilbert space.

3. *For* s *a non-negative integer and* $\Omega \subset \mathbb{R}^n$*, the sobolev space* $H^s(\Omega)$ *which contains* L^2 *functions whose weak derivatives of order upto* s *are also* L^2*, equiped with the inner product function defined by*

$$\langle f_1, f_2 \rangle = \int_\Omega f_1(\varrho)\overline{f_2(\varrho)}d\varrho + \int_\Omega Df_1(\varrho)D\overline{f_2(\varrho)}d\varrho + ... + \int_\Omega D^s f_1(\varrho)D^s\overline{f_2(\varrho)}d\varrho$$

   (with D *as a set which is open) is a Hilbert space.*

2 LINEAR OPERATORS AND THEIR SPECTRAL PROPERTIES

2.1 Some properties of linear operators

Definition 2.1.1.
A mapping $\Lambda : \mathbb{H}_1 \to \mathbb{H}_2$ *such that,*

$$\Lambda(\varrho + \vartheta) = \Lambda\varrho + \Lambda\vartheta, \forall \varrho, \vartheta \in \mathbb{H}_1$$

$$\Lambda(\kappa\varrho) = \kappa\Lambda\varrho, \forall \varrho \in \mathbb{H}_1, \forall \kappa \in \mathbb{K}$$

is called a $linear\ operator.$

Definition 2.1.2.
A mapping $F : \mathbb{H} \to \mathbb{K}$ *defined by* $F(\kappa_1\varrho + \kappa_2\vartheta) = \kappa_1 F\varrho + \kappa_2 F\vartheta$ $\forall \varrho, \vartheta \in \mathbb{H}$ *and* $\kappa_1, \vartheta, \kappa_1 F\varrho, \kappa_2 F\vartheta \in \mathbb{K}$ *is called a* $linear\ functional.$

Definition 2.1.3.
The kernel of an operator $\Lambda : \mathbb{H}_1 \to \mathbb{H}_2$ *is given by*

$$Ker(\Lambda) = \{\varrho \in \mathbb{H}_1 : \Lambda(\varrho) = 0\}$$

The image of Λ *is given by*

$$Im(\Lambda) = \{\Lambda\varrho \in \mathbb{H}_2 : \varrho \in \mathbb{H}_1\}$$

Theorem 2.1.4.
The kernel of $P : \mathbb{H}_1 \to \mathbb{H}_2$ *is a subspace of* $\mathbb{H}_1$

`Proof.` Let $\varrho, \vartheta \in Ker(\Lambda) \subseteq \mathbb{H}_1$ and $\kappa \in \mathbb{K}$ Then

$$\Lambda(\varrho + \vartheta) = \Lambda(\varrho) + \Lambda(\vartheta) = 0 + 0 = 0 \Rightarrow \varrho + \vartheta \in Ker(\Lambda)$$

$$\Lambda(\kappa\varrho) = \kappa\Lambda(\varrho) = \kappa.0 = 0 \Rightarrow \kappa\varrho \in Ker(\Lambda)$$

$$\Lambda(0) = 0 \Rightarrow 0 \in Ker(\Lambda)$$

Hence $Ker(\Lambda)$ is a subspace of $\mathbb{H}_1$ □

Theorem 2.1.5.
Let $P : \mathbb{H}_1 \to \mathbb{H}_2$ *then* $Im(P)$ *is a linear subspace of* $\mathbb{H}_2$*.*

`Proof.` Let $\vartheta_1, \vartheta_2 \in Im(\Lambda)$ where

$\vartheta_1 = \Lambda\varrho_1, \vartheta_2 = \Lambda\varrho_2$ for $\varrho_1, \varrho_2 \in \mathbb{H}_1$.
Then

$\varrho_1 + \varrho_2 \in H_1 \Rightarrow \Lambda(\varrho_1 + \varrho_2) = \Lambda\varrho_1 + \Lambda\varrho_2 = \vartheta_1 + \vartheta_2$
$\Rightarrow \vartheta_1 + \vartheta_2 \in Im(P)$

$\kappa\varrho \in \mathbb{H}_1 \Rightarrow \Lambda(\kappa\varrho) = \kappa\Lambda(\varrho) = \Lambda\vartheta \in Im(\Lambda)$

$$0 \in \mathbb{H}_1 \Rightarrow \Lambda(0) = 0 \Rightarrow 0 \in Im(\Lambda)$$

Hence $Im(\Lambda)$ is a subspace of $\mathbb{H}_2$. □

Definition 2.1.6.
The co-kernel of $\Lambda : \mathbb{H}_1 \to \mathbb{H}_2$ *is given by*

$$Coker(\Lambda) = \mathbb{H}_2 / Im(\Lambda)$$

The co-image of Λ *is given by*

$$Coim(\Lambda) = \mathbb{H}_1 / Ker(\Lambda)$$

Remark 2.1.7.
The following are the basic properties of mappings from $\mathbb{H}_1$ *into* $\mathbb{H}_2$

1. *The sum of* Λ *and* Ξ *is given by*

$$(\Lambda + \Xi)\varrho = \Lambda\varrho + \Xi\varrho, \forall \varrho \in \mathbb{H}_1$$

2. *The product of* Λ *and* Ξ *is given by*

$$(\Lambda\Xi)\varrho \equiv \Lambda(\Xi\varrho), \forall \varrho \in \mathbb{H}_1$$

3. Λ *and* Ξ *are equal if* $\Lambda\varrho = \Xi\varrho, \forall \varrho \in \mathbb{H}_1$

4. *The identity operator I is defined by $I\varrho = \varrho, \forall \varrho \in \mathbb{H}_1$*
5. *The associative law holds for operators Λ, Ξ and Υ i.e. $\Lambda(\Xi\Upsilon) = (\Lambda\Xi)\Upsilon$*
6. *The commutative law does not generally hold for operators Λ and Ξ i.e. $\Lambda\Xi \neq \Xi\Lambda$. If Λ and Ξ commute, then $\Lambda\Xi = \Xi\Lambda$ and $[\Lambda, \Xi] = 0$*
7. *The n^{th} power of an operator Λ denoted Λ^n is n successive applications of the operator. e.g. $\Lambda^2\varrho = \Lambda\Lambda\varrho$*
8. *The exponential of an operator Λ denoted e^Λ is defined via power series*

$$e^\Lambda = I + \Lambda + \frac{\Lambda^2}{2!} + \frac{\Lambda^3}{3!} + ...$$

Definition 2.1.8.
$\Lambda : \mathbb{H}_1 \to \mathbb{H}_2$ is a $bounded$ if we can have $\kappa > 0$ satisfying

$$\|\Lambda\varrho\|_{\mathbb{H}_2} \leq \kappa\|\varrho\|_{\mathbb{H}_1}, \forall \varrho \in \mathbb{H}_1$$

Definition 2.1.9.
$\Lambda : \mathbb{H}_1 \to \mathbb{H}_2$ is $continuous$ if for every $\varepsilon > 0, \exists\delta > 0$ where

$\|\varrho - \vartheta\| < \delta \Rightarrow \|\Lambda\varrho - \Lambda\vartheta\| < \varepsilon, \forall \varrho, \vartheta \in \mathbb{H}_1$

Theorem 2.1.10.
If $\Lambda : \mathbb{H}_1 \to \mathbb{H}_2$ is bounded ,then Λ is continuous.

`Proof.` Let Λ be bounded and $\varrho, \vartheta \in \mathbb{H}_1$ with $\varrho \neq \vartheta$.
Then $\|\Lambda\varrho - \Lambda\vartheta\| = \|\Lambda(\varrho - \vartheta)\| < \kappa\|(\varrho - \vartheta)\| < \varepsilon$
If $\kappa = 0$ the relation holds for all $\delta > 0$
If $\kappa > 0$, let $\|\varrho - \vartheta\| < \delta$ then $\delta = \frac{\varepsilon}{\kappa} > 0$
Then Λ is continuous. □

Remark 2.1.11.
if $\Lambda : \mathbb{H}_1 \to \mathbb{H}_2$ is continuous, then Λ is continuous at $0 \in \mathbb{H}_1$, or at some point $a_0 \in \mathbb{H}_1$ and therefore continuous everywhere in $\mathbb{H}_1$.

Theorem 2.1.12.
An operator $\Lambda : \mathbb{H}_1 \to \mathbb{H}_2$ is bounded iff Λ maps bounded sets into bounded sets.

`Proof.` Suppose Λ is a bounded. Then $\exists \alpha > 0$ satisfying

$$\|\Lambda\varrho\| \leq \alpha\|\varrho\|, \forall \varrho \in \mathbb{H}_1$$

. Let $\mathbb{M} \subset \mathbb{H}_1$ where $\mathbb{M}$ is a bounded set.
Then $\|\varrho\| \leq k, \forall \varrho \in \mathbb{M}$ and some constant k.
to have $\|\Lambda\varrho\| \leq \alpha\|\varrho\| \leq \alpha k, \forall \varrho \in \mathbb{M} \Rightarrow \Lambda(\mathbb{M})$ is a bounded set.
Conversely suppose Λ maps bounded sets onto bounded sets and consider $\varrho \neq 0$.
Then $\| \frac{\varrho}{\|\varrho\|} \|= 1$
Let $\vartheta = \frac{\varrho}{\|\varrho\|}$. Then all ϑ are bounded for all ϱ
so $\| \Lambda(\vartheta) \| \leq \alpha$ for some constant α i.e. $\| \Lambda(\frac{\varrho}{\|\varrho\|}) \| \leq \alpha$ i.e. $\| \Lambda\varrho \| \leq \alpha \| \varrho \|$
Thus Λ is a bounded. □

Definition 2.1.13.
We define the norm of $\Lambda : \mathbb{H}_1 \to \mathbb{H}_2$ *as*

$$\|\Lambda\| = inf\{\kappa : \|\Lambda\varrho\| \leq \kappa\|\varrho\|, \forall \varrho \in \mathbb{H}_1\}$$

Remark 2.1.14.
If $\|\Lambda\| \leq 1$ *then* Λ *is a contraction.*

Proposition 2.1.15.
If Λ *and* Ξ *are bounded linear operators from* $\mathbb{H}_1$ *into* $\mathbb{H}_2$ *over the same scalar field* $\mathbb{K}$ *then:*

1. $\Lambda + \Xi$ *is a bounded linear operator*
2. κ *is a bounded linear operator ,*$\forall \kappa \in \mathbb{K}$
3. $\Lambda\Xi$ *and* $\Xi\Lambda$ *are bounded linear operators.*

`Proof.` 1. Let $\varrho, \vartheta \in \mathbb{H}_1$, $\lambda, \mu \in \mathbb{K}$ and $\alpha, \beta > 0$.

$$(\Lambda + \Xi)(\lambda\varrho + \mu\vartheta) = \Lambda(\lambda\varrho + \mu\vartheta) + \Xi(\lambda\varrho + \mu\vartheta)$$

$$= \lambda\Lambda\varrho + \mu\Lambda\vartheta + \lambda\Xi\varrho + \mu\Xi\vartheta = \lambda(\Lambda + \Xi)\varrho + \mu(\Lambda + \Xi)\vartheta$$

$\Rightarrow \Lambda + \Xi$ is linear
Using triangular inequality $\|(\Lambda + \Xi)\varrho\| = \|\Lambda\varrho + \Xi\varrho\|$
$\leq \|\Lambda\varrho\| + \|\Xi\varrho\| \leq \alpha\|\varrho\| + \beta\|\varrho\| = (\alpha + \beta)\|\varrho\|$
but $(\alpha + \beta) > 0 \Rightarrow \Lambda + \Xi$ is linear and bounded.

2. Let $\varrho, \vartheta \in \mathbb{H}_1$ and $\lambda, \mu, \alpha, \kappa \in \mathbb{K}$

$$(\kappa\Lambda)(\lambda\varrho + \mu\vartheta) = \kappa(\Lambda\lambda\varrho + \Lambda\mu\vartheta) = \lambda\Lambda\varrho + \kappa\mu\Lambda\vartheta$$

$\Rightarrow \kappa\Lambda$ is linear.
Now $\|(\kappa\Lambda)\varrho\| = \|\kappa(\Lambda\varrho)\| = |\varrho|\|\Lambda\varrho\| \leq |\kappa|\alpha\|\varrho\|$
but $(|\kappa|\alpha) > 0 \in \mathbb{K} \Rightarrow \kappa\Lambda$ is linear and bounded.

3. $\Lambda\Xi(\lambda\varrho + \mu\vartheta) = \Lambda[\Xi(\lambda\varrho + \mu\vartheta)] = \Lambda[\lambda\Xi\varrho + \mu\Xi\vartheta] = \lambda(\Lambda\Xi)\varrho + \mu(\Lambda\Xi)\vartheta$
$\Rightarrow \Lambda\Xi$ is linear.
Now using continuity of Λ and Ξ we have

$$\|(\Lambda\Xi)\varrho\| = \|\Lambda(\Xi\varrho)\| \leq \alpha\|\Xi\varrho\| \leq \alpha\beta\|\varrho\|$$

but $\alpha\beta > 0 \in \mathbb{K} \Rightarrow \Lambda\Xi$ is a linear and bounded.

$\square$

Remark 2.1.16.
By the results of the proposition 2.1.15, we have shown that the set of bounded linear operators from $\mathbb{H}_1$ into $\mathbb{H}_2$ is a linear space.

Proposition 2.1.17.
The space $\mathbb{B}(\mathbb{H}_1, \mathbb{H}_2)$ of all bounded linear operators from $\mathbb{H}_1$ into $\mathbb{H}_2$ is a $N.L.S.$

`Proof.` Let $\Lambda, \Xi \in \mathbb{B}(\mathbb{H}_1, \mathbb{H}_2), \varrho \in \mathbb{H}_1, \vartheta \in \mathbb{H}_2, \kappa \in \mathbb{K}$ then the norm of Λ is defined by

$$\|\Lambda\| = inf\{\kappa : \|\Lambda\varrho\| \leq \kappa\|\varrho\|\} = sup\{\frac{\|\Lambda\varrho\|}{\|\varrho\|}\} = sup\{\|\Lambda\varrho\| : \|\varrho\| \leq 1\}$$

1. To prove the non-negativity axiom
$\|\Lambda\| \geq 0$ from the definition of the norm of Λ
Now let $\mathbb{D} = \{\varrho \in \mathbb{H}_1 : \|\varrho\| \leq 1\}$ and let $\vartheta = \frac{\varrho}{\|\varrho\|}$ then $\vartheta \in \mathbb{D}$
but
$\|\Lambda\| = 0 \Leftrightarrow \Lambda\varrho = 0 \Leftrightarrow \Lambda\vartheta = 0 \Leftrightarrow \Lambda(\frac{\varrho}{\|\varrho\|}) = 0 \Leftrightarrow \Lambda = 0$
i.e. $\|\Lambda\| = 0$ iff $\Lambda = 0$

2. To prove the homogeneity property

$$\|\kappa\Lambda\| = sup\{\|\kappa\Lambda\varrho\| : \|\varrho\| = 1\} = sup\{|\kappa|\|\Lambda\varrho\| : \|\varrho\| = 1\}$$

$$= |\kappa| sup\{\|\Lambda\varrho\| : \|\varrho\| = 1\} = |\kappa|\|\Lambda\|$$

3. To prove the triangular in equality

$$\|(\Lambda + \Xi)\varrho\| = \|\Lambda\varrho + \Xi\varrho\| \leq \|\Lambda\varrho\| + \|\Xi\varrho\|$$

taking supremum on right hand side for $\|\varrho\| = 1$ we have

$$\|(\Lambda + \Xi)\varrho\| \leq sup\|\Lambda\varrho\| + sup\|\Xi\varrho\| \leq \|\Lambda\| + \|\Xi\|$$

Taking supremum on left hand side for $\|\varrho\| = 1$ we have

$$\|\Lambda + \Xi\| \leq \|\Lambda\| + \|\Xi\|$$

□

Theorem 2.1.18.
The space $\mathbb{B}(\mathbb{H}_1, \mathbb{H}_2)$ is a banach space.

`Proof.` We have already shown in propositions 2.1.17 that the set $\mathbb{B}(\mathbb{H}_1, \mathbb{H}_2)$ is $N.L.S.$ It now remains to show that it‘s complete.
Let Λ_n be a Cauchy in $\mathbb{B}(\mathbb{H}_1, \mathbb{H}_2)$, then for $\varepsilon > 0,\ \exists N \in j^+$ satisfying

$$(\|\Lambda_n - \Lambda_m)\varrho\| \leq \varepsilon, \forall m, n > N$$

So $\|(\Lambda_n + \Lambda_m)\varrho\| \leq \|\Lambda_n + \Lambda_m\|\|\varrho\| \leq \varepsilon\|\varrho\|$
hence $\{\Lambda_n(\varrho)\}$ is a Cauchy sequence in $\mathbb{H}_2$.
but $\mathbb{H}_2$ is a banach space, so every sequence $\{\Lambda_n\}$ in $\mathbb{H}_2$ also converges in $\mathbb{H}_2$.
so from $\|\Lambda_n(\varrho) - \Lambda_m(\varrho)\| \leq \varepsilon\|\varrho\|$ we let $m \to \infty$ to obtain

$$\|\Lambda_n(\varrho) - \Lambda\varrho\| \leq \varepsilon\|\varrho\| \Rightarrow \|(\Lambda_n - \Lambda)\varrho\| \leq \varepsilon\|\varrho\| \Rightarrow (\Lambda_n - \Lambda) \in \mathbb{B}(\mathbb{H}_1, \mathbb{H}_2)$$

such that $\Lambda = \Lambda_n - (\Lambda_n - \Lambda) \in \mathbb{B}(\mathbb{H}_1, \mathbb{H}_2)$
Now the norm of $(\Lambda_n - \Lambda)$ is given by
$\|\Lambda_n - \Lambda\| = sup\{\|\Lambda_n(\varrho) - \Lambda(\varrho)\|\}$

$\leq sup\{\|\Lambda_n - \Lambda\|\|\varrho\| : \|\varrho\| = 1\} \leq \varepsilon$
i.e. $\|\Lambda_n - \Lambda\| \leq \varepsilon$ and therefore Λ_n converges uniformly to Λ
So $\mathbb{B}(\mathbb{H}_1, \mathbb{H}_2)$ is complete. □

Definition 2.1.19.
Let $\Lambda \in \mathbb{B}(\mathbb{H}_1, \mathbb{H}_2)$ then Λ is invertible if we have $\Lambda^{-1} : \mathbb{H}_2 \to \mathbb{H}_1$ to have $\Lambda^{-1}\Lambda(\varrho) = I\varrho = \varrho, \forall \Lambda(\varrho) \in \mathbb{H}_2$

Remark 2.1.20.
Note that Λ is invertible if it is injective i.e. (1-1) and surjective i.e. onto. If Λ is injective the $Ker(\Lambda) = 0$ and if Λ is surjective then $Im(\Lambda) = \mathbb{H}_2$

Theorem 2.1.21.
Let $\Lambda \in \mathbb{B}(\mathbb{H}_1, \mathbb{H}_2)$ then Λ^{-1} exists and $\Lambda^{-1} \in \mathbb{B}(\mathbb{H}_2, \mathbb{H}_1)$ iff we have $\alpha > 0$ and $\|\Lambda\varrho\| \geq \alpha\|\varrho\|$ i.e. Λ is bounded below.

`Proof.` suppose Λ is bounded below to have $\|\Lambda\varrho\| \geq \alpha\|\varrho\|$ and $\varrho_1, \varrho_2 \in Ker(\Lambda)$.
Then $\|\Lambda(\varrho_1 - \varrho_2)\| \geq \alpha\|\varrho_1 - \varrho_2\|$
but $\Lambda(\varrho_1 - \varrho_2) = \Lambda\varrho_1 + \Lambda\varrho_2 = 0 + 0 = 0$
i.e.
$\|0\| \geq \alpha\|\varrho_1 - \varrho_2\| \Rightarrow 0 \geq \alpha\|\varrho_1 - \varrho_2\| \Rightarrow \|\varrho_1 - \varrho_2\| = 0 \Rightarrow \varrho_1 - \varrho_2 = 0$
i.e. $\Lambda(0) = 0 \Rightarrow Ker(\Lambda) = 0 \Rightarrow \Lambda$ is injective.
Now Λ is injective and Λ^{-1} exist.
Let $\vartheta \in \Lambda(\mathbb{H}_1)$ so that $\vartheta = \Lambda(\varrho)$ for $\varrho \in \mathbb{H}_1$. then $\Lambda^{-1}(\vartheta) = \Lambda^{-1}\Lambda(\varrho) = \varrho$
So

$$\|\Lambda\varrho\| \geq \alpha\|\varrho\| \Rightarrow \|\Lambda(\Lambda^{-1}\vartheta)\| \geq \alpha\|\Lambda^{-1}\vartheta\|$$

i.e.

$$\|\vartheta\| \geq \alpha\|\Lambda^{-1}\vartheta\| \Rightarrow \|\Lambda^{-1}\vartheta\| \leq \frac{1}{\alpha}\|\vartheta\|.$$

Hence Λ^{-1} is a bounded.
Conversely, suppose P^{-1} exists and is bounded.
Then $\exists\ \alpha > 0$ satisfying

$$\|\Lambda^{-1}\vartheta\| \leq \alpha\|\vartheta\|$$

Let $\vartheta = \Lambda(\varrho)$ for some $\varrho \in \mathbb{H}_1$
Then

$\|\Lambda^{-1}(\Lambda\varrho)\| \leq \alpha\|\Lambda\varrho\| \Rightarrow \|\varrho\| \leq \alpha\|\Lambda\varrho\| \Rightarrow \|\Lambda\varrho\| \geq \frac{1}{\alpha}\|\varrho\|$
Hence Λ is bounded below. □

Remark 2.1.22.
In mathematics each bounded linear operator on a complex Hilbert space has a corresponding $ad-joint\ operator$ *also called Hermitian ad-joint named after Charles Hermite.*

Definition 2.1.23.
If to every operator $\Lambda \in \mathbb{B}(\mathbb{H}_1, \mathbb{H}_2)$ *we associate another unique operator* $\Lambda^* \in \mathbb{B}(\mathbb{H}_2, \mathbb{H}_1)$ *to satisfy*

$$\langle \Lambda\varrho, \vartheta \rangle = \langle \varrho, \Lambda^*\vartheta \rangle, \forall \varrho \in \mathbb{H}_1, \forall \vartheta \in \mathbb{H}_2$$

. Then Λ^* *is called an Hermitian ad-joint of* Λ

Remark 2.1.24.
Note that the ad-joint of Λ^* *denoted* $(\Lambda^*)^* = \Lambda$
i.e. for each $\Lambda^* : \mathbb{H}_2 \to \mathbb{H}_1$ *we associate by a unique operator*

$\Lambda^{**} : \mathbb{H}_1 \to \mathbb{H}_2$ *satisfying* $\langle \Lambda^*\vartheta, \varrho \rangle = \langle \vartheta, \Lambda^{**}\varrho \rangle = \langle \vartheta, \Lambda\varrho \rangle$

Remark 2.1.25.
If an operator Λ *is a square matrix* $[\lambda_{ij}]$*, then its ad-joint operator* Λ^**is the conjugate transpose of* $[\lambda_{ij}]$*.*

Proposition 2.1.26.
Let $\Lambda, \Xi \in \mathbb{B}(\mathbb{H}_1, \mathbb{H}_2)$*. Then*

1. $(\Lambda + \Xi)^* = \Lambda^* + \Xi^*$
2. $(\lambda\Lambda)^* = \bar{\lambda}\Lambda^*$
3. $(\Lambda\Xi)^* = \Xi^*\Lambda^*$
4. $(\Lambda^*)^{-1} = (\Lambda^{-1})^*$

`Proof.` 1. For all $\varrho, \vartheta \in \mathbb{H}_1$. Then

$$\langle (\Lambda + \Xi)\varrho, \vartheta \rangle = \langle \varrho, (\Lambda + \Xi)^*\vartheta \rangle \text{..........}(i)$$

but $\langle(\Lambda+\Xi)\varrho,\vartheta\rangle=\langle\Lambda\varrho+\Xi\varrho,\vartheta\rangle=\langle\Lambda\varrho,\vartheta\rangle+\langle\Xi\varrho,\vartheta\rangle$

$$=\langle\varrho,\Lambda^*\vartheta\rangle+\langle\varrho,\Xi^*\vartheta\rangle=\langle\varrho,\Lambda^*\vartheta+\Xi^*\vartheta\rangle=\langle\varrho,(\Lambda^*+\Xi^*)\vartheta\rangle$$

i.e. $\langle(\Lambda+\Xi)\varrho,\vartheta\rangle=\langle\varrho,(\Lambda^*+\Xi^*)\vartheta\rangle$..........$(ii)$
The results (i) and (ii) imply that
$\langle\varrho,(\Lambda+\Xi)^*\vartheta\rangle=\langle\varrho,(\Lambda^*+\Xi^*)\vartheta\rangle\Rightarrow(\Lambda+\Xi)^*=\Lambda^*+\Xi^*$

2. let $\varrho,\vartheta\in\mathbb{H}_1$ and $\lambda\in\mathbb{K}$. Then

$$\langle(\lambda\Lambda)\varrho,\vartheta\rangle=\langle\varrho,(\lambda\Lambda)^*\vartheta\rangle...........(iii)$$

but $\langle(\lambda\Lambda)\varrho,\vartheta\rangle=\lambda\langle\Lambda\varrho,\vartheta\rangle=\lambda\langle\varrho,\Lambda^*\vartheta\rangle=\langle\varrho,\bar{\lambda}\Lambda^*\vartheta\rangle$.....$(iv)$
From (iii) and (iv),

$$\langle\varrho,(\lambda\Lambda)^*\vartheta\rangle=\langle\varrho,\bar{\lambda}\Lambda^*\vartheta\rangle\Rightarrow(\lambda\Lambda)^*=\bar{\lambda}\Lambda^*$$

3. $\langle(\Lambda\Xi)\varrho,\vartheta\rangle=\langle\varrho,(\Lambda\Xi)^*\vartheta\rangle$................$(v)$
but $\langle\Lambda\Xi\varrho,\vartheta\rangle=\langle\Xi\varrho,\Lambda^*\vartheta\rangle=\langle\varrho,\Xi^*\Lambda^*\vartheta\rangle$..............$(vi)$
From (v) and (vi),

$$\langle\varrho,(\Lambda\Xi)^*\vartheta\rangle=\langle\varrho,\Xi^*\Lambda^*\vartheta\rangle\Rightarrow(\Lambda\Xi)^*=\Xi^*\Lambda^*$$

4. Note
$\langle\varrho,\vartheta\rangle=\langle I\varrho,\vartheta\rangle=\langle\varrho,I^*\vartheta\rangle\Rightarrow I\vartheta=\vartheta=I^*\vartheta\Rightarrow I=I^*$
Λ is invertible $\Rightarrow\Lambda^{-1}\in\mathbb{B}(\mathbb{H}_2,\mathbb{H}_1)$
and $\Lambda^{-1}\Lambda=I=I^*=\Lambda\Lambda^{-1}$

$$\Rightarrow(\Lambda^{-1}\Lambda)^*=I^*=I=(\Lambda\Lambda^{-1})^*\Rightarrow\Lambda^*(\Lambda^{-1})^*=I=(\Lambda^{-1})^*\Lambda^*$$

i.e. $(\Lambda^{-1})^*$ is the inverse of $\Lambda^*\Rightarrow(\Lambda^*)^{-1}=(\Lambda^{-1})^*$

$\square$

Definition 2.1.27.
$\Lambda\in\mathbb{B}(\mathbb{H})$ *is* $self\ ad-joint$ *(also hermitian) if* $\Lambda^*=\Lambda$ *i.e. if*

$$\langle\Lambda\varrho,\vartheta\rangle=\langle\varrho,\Lambda\vartheta\rangle,\forall\varrho,\vartheta\in\mathbb{H}$$

Theorem 2.1.28.
Let $\Lambda, \Xi \in \mathbb{B}(\mathbb{H})$ be self ad-joint. Then $\Lambda\Xi$ is self ad-joint iff $\Lambda \leftrightarrow \Xi$ i.e. if Λ commutes with Ξ

`Proof.` Let $\Lambda \leftrightarrow \Xi$, then $\Lambda\Xi = \Xi\Lambda$
thus $(\Lambda\Xi)^* = \Xi^*\Lambda^* = \Xi\Lambda = \Lambda\Xi \Rightarrow \Lambda\Xi$ is self ad-joint.
conversely let $\Lambda\Xi$ be self ad-joint, then $(\Lambda\Xi)^* = \Lambda\Xi$
thus $\Xi\Lambda = \Xi^*\Lambda^* = (\Lambda\Xi)^* = \Lambda\Xi \Rightarrow \Lambda \leftrightarrow \Xi$ □

Theorem 2.1.29.
Let $\Lambda \in \mathbb{B}(\mathbb{H})$ be self ad-joint. Then the real polynomial $\Phi(\Lambda) = \alpha_0 I + \alpha_1\Lambda + \alpha_2\Lambda^2 + ... + \alpha_n\Lambda^n$ is also self ad-joint.

`Proof.` Since Λ commutes with itself we have $\Lambda\Lambda = \Lambda^2$ is self ad-joint.
Also $\Lambda^2 \leftrightarrow \Lambda \Rightarrow \Lambda^3$ is self ad-joint thus inductively we have
$\Lambda^n = \Lambda^{n-1}\Lambda$ is self ad-joint for all $n \in J^+$
If $\alpha_i \in \mathbb{R}$ the for $\lambda = \alpha_i$ we have $(\lambda\Lambda)^* = \lambda\Lambda^* = \lambda\Lambda \Rightarrow \lambda\Lambda$ is self ad-joint
Therefore every term of the polynomial $\Phi(\Lambda)$ is self ad-joint and their finite sum is also self ad-joint. □

Definition 2.1.30.
$\Lambda \in \mathbb{B}(\mathbb{H})$ *is idempotent if*

$\Lambda^2(\varrho) = \Lambda(\varrho), \forall \varrho \in \mathbb{H}$

Definition 2.1.31.
$\Lambda : \mathbb{H} \to \mathbb{M}$ *is a projection if* Λ *is* $i.e\Lambda^2 = \Lambda = \Lambda^*$

Remark 2.1.32.
If $\Lambda \in \mathbb{B}(\mathbb{H})$ is a projection then $\|\Lambda\| \leq 1$ i.e. Λ is a contraction.

Definition 2.1.33.
Let $\Lambda \in \mathbb{B}(\mathbb{H}_1, \mathbb{H}_2)$ then Λ is an isometry if $\|\Lambda\varrho\| = \|\varrho\|, \forall \varrho \in \mathbb{H}_1$

Remark 2.1.34.
Clearly an isometry is injective.

Proposition 2.1.35.
If $\Lambda \in \mathbb{B}(\mathbb{H})$ is an isometry then $\Lambda^*\Lambda = I$
Note $\|\Lambda\varrho\| = \|\varrho\| \Rightarrow \|\Lambda\varrho\|^2 = \|\varrho\|^2 \Rightarrow \langle\Lambda\varrho, \Lambda\varrho\rangle = \langle\varrho, \varrho\rangle$

$\Rightarrow \langle\varrho, \Lambda^*\Lambda\varrho\rangle = \langle\varrho, I\varrho\rangle \Rightarrow \Lambda^*\Lambda = I$

Definition 2.1.36.
Let $\Lambda \in \mathbb{B}(\mathbb{H}_1, \mathbb{H}_2)$ *then* Λ *is an isomorphism if* Λ *is surjective and*

$$\langle \Lambda\varrho, \Lambda\vartheta \rangle = \langle \varrho, \vartheta \rangle, \forall \varrho, \vartheta \in \mathbb{H}_1$$

Remark 2.1.37.
Clearly an isometry is an isomorphism, since

$$\|\Lambda\varrho\|^2 = \langle \Lambda\varrho, \Lambda\varrho \rangle = \langle \varrho, \varrho \rangle = \|\varrho\|^2 \Rightarrow \|\Lambda\varrho\| = \|\varrho\|$$

2.2 Invariant subspaces

Definition 2.2.1.
A closed subspace $\mathbb{M}$ of $\mathbb{H}$ is invariant under transformation $\Upsilon \in \mathbb{B}(\mathbb{H})$ if

$$\Upsilon\varrho \in \mathbb{M}, \forall\varrho \in \mathbb{M}$$

i.e. $\Upsilon\mathbb{M} \subseteq \mathbb{M}$ where $\Upsilon\mathbb{M} = \{\Upsilon\varrho : \varrho \in \mathbb{M}\}$

Remark 2.2.2.
If $\mathbb{M} \subseteq \mathbb{H}$ is $\Upsilon - invariant$ we can restrict Υ to $\mathbb{M}$ to arrive at a new linear transformation

$$\Upsilon/\mathbb{M} : \mathbb{M} \to \mathbb{M}$$

Proposition 2.2.3.
The subspaces $\{0\}$ and $\mathbb{H}$ of $\mathbb{H}$ are invariant subspaces under any linear operator $\Upsilon : \mathbb{H} \to \mathbb{H}$

`Proof.` $\Upsilon(0) = 0 \Rightarrow \Upsilon\{0\} \subseteq \{0\}$ thus $\{0\}$ is $\Upsilon - invariant$
$\Upsilon(\varrho) \in \mathbb{H}, \forall\varrho \in \mathbb{H}$ and so $\Upsilon(\mathbb{H}) \subseteq \mathbb{H}$ thus $\mathbb{H}$ is $\Upsilon - invariant$ □

Remark 2.2.4.
The subspaces $\{0\}$ and $\mathbb{H}$ of $\mathbb{H}$ under linear operator Υ are called trivial $\Upsilon - invariant$ subspaces.
If $\mathbb{H}$ has no non-trivial $\Upsilon - invariant$ subspaces then Υ is simple.

Proposition 2.2.5.
Both kernel and the image of Υ on $\mathbb{H}$ are $\Upsilon - invariant$ subspaces.

`Proof.` If $\varrho \in Ker(\Upsilon)$ then $\Upsilon\varrho = 0$ thus $\Upsilon(\Upsilon\varrho) = \Upsilon(0) = 0$ and so $\Upsilon\varrho \in Ker(\Upsilon)$
i.e. $\Upsilon(Ker(\Upsilon)) \subseteq Ker(\Upsilon) \Rightarrow Ker(\Upsilon)$ is $\Upsilon - invariant$
If $\vartheta \in Im(\Upsilon)$ then $\vartheta \in \Upsilon(\varrho)$ for $\varrho \in \mathbb{H}$ and then $\Upsilon(\varrho) = \Upsilon(\Upsilon\varrho) \in Im(\Upsilon)$
i.e. $\Upsilon[Im(\Upsilon)] \subseteq Im(\Upsilon) \Rightarrow Im(\Upsilon)$ is $\Upsilon - invariant.$ □

Proposition 2.2.6.
Let $\mathbb{M}_1$ and $\mathbb{M}_2$ be $\Upsilon - invariant$ subspaces in Hilbert space $\mathbb{H}$.
Then $\mathbb{M}_1 + \mathbb{M}_2$ and $\mathbb{M}_1 \cap \mathbb{M}_2$ are $\Upsilon - invariant$ subspaces in $\mathbb{H}$

`Proof.` Clearly $\mathbb{M}_1+\mathbb{M}_2$ and $\mathbb{M}_1\cap\mathbb{M}_2$ are subspaces of $\mathbb{H}$. Now let $\varrho \in \mathbb{M}_1$ and $\vartheta \in \mathbb{M}_2$, then $\varrho+\vartheta \in \mathbb{M}_1+\mathbb{M}_2$ and $\Upsilon(\varrho+\vartheta) = \Upsilon\varrho + \Upsilon\vartheta \in \mathbb{M}_1+\mathbb{M}_2$ since $\Upsilon\varrho \in \mathbb{M}_1$ and $\Upsilon\vartheta \in \mathbb{M}_2$ thus implying $\Upsilon(\varrho+\vartheta) \subseteq \mathbb{M}_1+\mathbb{M}_2$
hence $\mathbb{M}_1+\mathbb{M}_2$ is $\Upsilon - invariant$
Now let $\varrho \in \mathbb{M}_1 \cap \mathbb{M}_2$, then $\varrho \in \mathbb{M}_2$ and $\varrho \in \mathbb{M}_2$
Also $\Upsilon\varrho \in \mathbb{M}_1$ and $\Upsilon\varrho \in \mathbb{M}_2 \Rightarrow \Upsilon\varrho \in \mathbb{M}_1\cap\mathbb{M}_2 \Rightarrow \Upsilon(\mathbb{M}_1\cap\mathbb{M}_2) \subseteq \mathbb{M}_1\cap\mathbb{M}_2$
and hence $\mathbb{M}_1 \cap \mathbb{M}_2$ is $\Upsilon - invariant$ □

Remark 2.2.7.
$Lat(\Upsilon)$ deenotes the lattice of all $\Upsilon - invariant$ subspaces of $\mathbb{H}$. Thus

$$Lat(\Upsilon) = \{\mathbb{M} \subseteq \mathbb{H} : \Upsilon\mathbb{M} \subseteq \mathbb{M}\}$$

Definition 2.2.8.
$\mathbb{M} \subseteq \mathbb{H}$ reduces $\Upsilon \in \mathbb{B}(\mathbb{H})$ if $\mathbb{M}$ is $\Upsilon - invariant$ and $\Upsilon^* - invariant$ i.e. if $\Upsilon(\mathbb{M}) \subseteq \mathbb{M}$ and $\Upsilon^*(\mathbb{M}) \subseteq \mathbb{M}$

Remark 2.2.9.
$Red(\Upsilon)$ denotes the lattice of all $\Upsilon - reducing$ subspaces of $\mathbb{H}$. Thus

$$Red(\Upsilon) = \{\mathbb{M} \subseteq \mathbb{H} : \Upsilon\mathbb{M} \subseteq \mathbb{M}, \Upsilon^*\mathbb{M} \subseteq \mathbb{M}\}$$

Remark 2.2.10.
Recall,two operators $\Upsilon, \Psi \in \mathbb{B}(\mathbb{H})$ are said to commute if
$\Upsilon\Psi = \Psi\Upsilon$ i.e. if the commutator of Υ and Ψ denoted $[\Upsilon, \Psi] = \Upsilon\Psi - \Psi\Upsilon = 0$
$\{R\}'$ denotes the set of $\Psi \in \mathbb{B}(\mathbb{H})$ such that $\Upsilon\Psi = \Psi\Upsilon$

Definition 2.2.11.
$\mathbb{M} \subseteq \mathbb{H}$ is $\Upsilon - hyperinvariant$ if $\mathbb{M}$ is $\Psi - invariant$ for all $\Psi \in \{\Upsilon\}'$ i.e. if $\Psi(\mathbb{M}) \subseteq \mathbb{M}$ for $\Psi\Upsilon = \Upsilon\Psi$

Remark 2.2.12.
$Hyperlat(\Upsilon)$ denotes $Hyper - invariant$ subspaces under transformation Υ

Remark 2.2.13.
Note that $Red(\Upsilon) \subseteq Lat(\Upsilon)$ since every $\Upsilon - reducing$ subspace is $\Upsilon - invariant$
Also $Hyperlat(\Upsilon) \subseteq Lat(\Upsilon)$ because $\Upsilon \in \{\Upsilon\}'$ since $[\Upsilon, \Upsilon] = \Upsilon\Upsilon - \Upsilon\Upsilon = 0$

2.3 Some classes of linear operators

Definition 2.3.1. *: $Positive\ operators$*
$\Lambda \in \mathbb{B}(\mathbb{H})$ *is* $positive$ *if* $\Lambda^* = \Lambda$ *and*
$\langle \Lambda\varrho, \varrho\rangle \geq 0, \forall \varrho \in \mathbb{H}$

Proposition 2.3.2.
If $\Lambda \in \mathbb{B}(\mathbb{H})$ *is self ad-joint, then* Λ^2 *is positive.*

`Proof.` $\langle \Lambda^2\varrho, \varrho\rangle = \langle \Lambda\varrho, \Lambda^*\varrho\rangle = \langle \Lambda\varrho, \Lambda\varrho\rangle = \|\Lambda\varrho\|^2 \geq 0, \forall \varrho \in \mathbb{H}$
hence Λ^2 is positive. □

Corollary 2.3.3.
Any $projection$ Λ *is a positive operator since* $\Lambda = \Lambda^*$ *and* $\Lambda^2 = \Lambda$
So

$$\langle \Lambda\varrho, \varrho\rangle = \langle \Lambda^2\varrho, \varrho\rangle = \langle \Lambda\varrho, \Lambda^*\varrho\rangle = \langle \Lambda\varrho, \Lambda\varrho\rangle = \|\Lambda\varrho\|^2 \geq 0, \forall \varrho \in \mathbb{H}$$

Theorem 2.3.4. *:Remainder theorem*
If $\Lambda \in \mathbb{B}(\mathbb{H}) = \Upsilon^*\Upsilon$ *(i.e. the composite of* $\Upsilon^*, \Upsilon \in \mathbb{B}(\mathbb{H})$*), then* Λ *is positive.*

`Proof.` $\Lambda^* = (\Upsilon^*\Upsilon)^* = \Upsilon^*\Upsilon^{**} = \Upsilon^*\Upsilon = \Lambda \Rightarrow \Lambda$ is self ad-joint.
Also
$\langle \Lambda\varrho, \varrho\rangle = \langle \Upsilon^*\Upsilon\varrho, \varrho\rangle = \langle \Upsilon\varrho, \Upsilon\varrho\rangle = \|\Upsilon\varrho\|^2 \geq 0, \forall \varrho \in \mathbb{H}$
Hence Λ is positive □

Definition 2.3.5.
$\Lambda \in \mathbb{B}(\mathbb{H})$ *is* $normal$ *if*
$\Lambda^*\Lambda = \Lambda\Lambda^*$ *i.e. if* $[\Lambda^*, \Lambda] = 0$

Theorem 2.3.6.
$\Lambda \in \mathbb{B}(\mathbb{H})$ *is* $normal$ *iff*

$$\|\Lambda\varrho\| = \|\Lambda^*\varrho\|, \forall \varrho \in \mathbb{H}$$

`Proof.` Let Λ be normal and $\varrho \in \mathbb{H}$, then

$$\|\Lambda\varrho\|^2 = \langle \Lambda\varrho, \Lambda\varrho\rangle = \langle \Lambda^*\Lambda\varrho, \varrho\rangle$$

$$\|\Lambda^*\varrho\|^2 = \langle \Lambda^*\varrho, \Lambda^*\varrho\rangle = \langle \Lambda\Lambda^*\varrho, \varrho\rangle$$

but Λ is normal $\Rightarrow \Lambda^*\Lambda = \Lambda\Lambda^* \Rightarrow \langle \Lambda^*\Lambda\varrho, \varrho\rangle = \langle \Lambda\Lambda^*\varrho, \varrho\rangle$

$$\Rightarrow \|\Lambda\varrho\|^2 = \|\Lambda^*\|^2 \Rightarrow \|\Lambda\varrho\| = \|\Lambda^*\varrho\|$$

Conversely let $\|\Lambda\varrho\| = \|\Lambda^*\varrho\|$, then
$\|\Lambda\varrho\|^2 = \|\Lambda^*\varrho\|^2 \Rightarrow \langle \Lambda\varrho, \Lambda\varrho\rangle = \langle \Lambda^*\varrho, \Lambda^*\varrho\rangle \Rightarrow \langle \Lambda^*\Lambda\varrho, \varrho\rangle = \langle \Lambda\Lambda^*\varrho, \varrho\rangle \Rightarrow$
$\Lambda^*\Lambda = \Lambda\Lambda^* \Rightarrow \Lambda$ is normal. □

Definition 2.3.7. *: $Quasi - normal\ operators$*
$\Lambda \in \mathbb{B}(\mathbb{H})$ *is $quasinormal$ if*

$$\Lambda(\Lambda^*\Lambda) = (\Lambda\Lambda^*)\Lambda$$

Remark 2.3.8.
Every normal operator is quasi-normal.
Note , if Λ is normal then $\Lambda^*\Lambda = \Lambda\Lambda^* \Rightarrow \Lambda(\Lambda^*\Lambda) = \Lambda(\Lambda\Lambda^*) \Rightarrow \Lambda(\Lambda^*\Lambda) = (\Lambda\Lambda^*)\Lambda \Rightarrow \Lambda$ is quasi normal

Remark 2.3.9.
Every isometry is quasi-normal but not every isometry is normal
Note, if Λ is an isometry then $\Lambda^*\Lambda = I \Rightarrow \Lambda(\Lambda^*\Lambda)(\Lambda^*\Lambda) = \Lambda I(\Lambda^*\Lambda) \Rightarrow \Lambda\Lambda^*\Lambda = \Lambda\Lambda^*\Lambda \Rightarrow \Lambda$ is quasi-normal
But Λ is an isometry implies that Λ is normal iff Λ commutes with Λ^* and so not true in general

Definition 2.3.10.
$\Lambda \in \mathbb{B}(\mathbb{H})$ *is $bi - normal$ if*
$(\Lambda^*\Lambda)\Lambda\Lambda^* = (\Lambda\Lambda^*)\Lambda^*\Lambda$ *i.e. if $\Lambda^*\Lambda$ commutes with $\Lambda\Lambda^*$*

Definition 2.3.11.
$\Lambda \in \mathbb{B}(\mathbb{H})$ *is hypo-normal if $\Lambda^*\Lambda \geq \Lambda\Lambda^*$*
Λ *is $p - hypornormal$ if $(\Lambda^*\Lambda)^p \geq (\Lambda\Lambda^*)^p$ thus Λ is hyponormal if $p = 1$*

Remark 2.3.12.
Clearly if Λ is hypo-normal operator then $\Lambda^*\Lambda - \Lambda\Lambda^* \geq 0 \Rightarrow \Lambda^*\Lambda - \Lambda\Lambda^*$ is a positive.

Definition 2.3.13.
$\Lambda \in \mathbb{B}(\mathbb{H})$ *is semi-normal if either Λ or Λ^* is hypo-normal*

Definition 2.3.14.
$U \in \mathbb{B}(\mathbb{H})$ *is a unitary if* $U^*U = UU^* = I$

Remark 2.3.15.
The weaker condition $U^*U = I$ *defines an isometry while the other condition* $UU^* = I$ *defines co-isometry.*
So a unitary operator is a bounded linear operator which is both an isometry and co-isometry.

Definition 2.3.16.
$\Lambda \in \mathbb{B}(\mathbb{H})$ *is* $paranormal$ *if* $\|\Lambda^2\| \geq \|\Lambda\|^2$

Remark 2.3.17.
Note, if Λ *is*
$unitary \Rightarrow normal \Rightarrow quasi{-}normal \Rightarrow subnormal \Rightarrow hyponormal \Rightarrow semi - normal \Rightarrow paranormal$

Definition 2.3.18.
$\Lambda \in \mathbb{B}(\mathbb{H})$ *is a normaloid if* $\|\Lambda\| = sup\{|\langle \Lambda\varrho, \varrho\rangle| : \|\varrho\| = 1\}$

2.4 Spectra of linear operators

Definition 2.4.1.
An $eigenvector$ is a non zero vector $\varrho \in \mathbb{H}$ that changes by only a scalar factor λ when a linear transformation $\Lambda \in \mathbb{B}(\mathbb{H})$ is applied to it, i.e.$\Lambda\varrho = \lambda\varrho$

Definition 2.4.2.
Let $\Lambda \in \mathbb{B}(\mathbb{H})$, a value $\lambda \in \mathbb{C}$ is an $eigen - value$ if $\exists$ a non-zero vector $\varrho \in \mathbb{H}$ satisfying $\Lambda\varrho = \lambda\varrho$.

Remark 2.4.3.
The pair ϱ, λ where ϱ is an eigenvector and λ is its associated eigenvalue is called an $eigen\ pair$.

Definition 2.4.4.
Let $\Lambda \in \mathbb{B}(\mathbb{H})$. The equation $\Lambda\varrho = \lambda\varrho$ where ϱ is an eigenvector and λ is its associated eigenvalue is called the $eigen - equation$.

.

Remark 2.4.5.
If $\mathbb{H}$ is a continuous function space, then Λ can be a differential operator like d/da while f is an eigen function which is scaled by d/da.
hence $(d/da)f = \lambda f$ is the eigen equation.

Proposition 2.4.6.
The eigen equation $\Lambda\varrho = \lambda\varrho$ where Λ is an $n \times n$ matrix and ϱ is an n-dimensional column vector can be re-written as

$$\Lambda\varrho = \lambda\varrho \Leftrightarrow \Lambda\varrho - \lambda\varrho = 0 \Leftrightarrow (\Lambda - \lambda)\varrho = 0$$

Let I be an $n \times n$ identity matrix, then

$$(\Lambda - \lambda I)\varrho = 0 \ldots\ldots\ldots\ldots\ldots\ldots.(i)$$

Equation (i) has non-zero solutions iff determinant of $(\Lambda - \lambda I)$ is equal to zero, i.e.

$$\mid \Lambda - \lambda I \mid = 0 \ldots\ldots\ldots\ldots\ldots\ldots(ii)$$

Equation (ii) is the $characteristic\ equation$ whose left side is the $characteristic\ polynomial$.

Remark 2.4.7.
The characteristic polynomial $\mid \Lambda - \lambda I \mid$ *is of degree* n *such that the equation* (ii) *has* n *roots,* $(i.e. \lambda_1, \lambda_2, \lambda_3, ..., \lambda_n)$ *which are the eigenvalues of the operator* Λ

Definition 2.4.8.
Let $\Lambda \in \mathbb{B}(\mathbb{H})$*, then the union of the zero vector and all the eigenvectors associated with an eigenvalue* λ *such that* $\Lambda\varrho = \lambda\varrho$ *is a linear subspace of* $\mathbb{H}$ *called the* $eigen - space$ *associated with* λ*.*

Remark 2.4.9.
We can find out all the vectors associated with an eigenvalue λ *by solving the equation*

$$(\Lambda - \lambda I)\varrho = 0$$

Lemma 2.4.10.
The eigen-values of a bounded self ad-joint operator are real and the eigen-vectors associated with different eigenvalues are orthogonal.

`Proof.` If $\Lambda : \mathbb{H} \to \mathbb{H}$ is self ad-joint and $\Lambda\varrho = \lambda\varrho$ with $\varrho \neq 0$
then $\lambda\langle\varrho, \varrho\rangle = \langle\lambda\varrho, \varrho\rangle = \langle\Lambda\varrho, \varrho\rangle = \langle\varrho, \Lambda\varrho\rangle = \langle\varrho, \lambda\varrho\rangle = \bar{\lambda}\langle\varrho, \varrho\rangle \Leftrightarrow \lambda = \bar{\lambda}$
$\Rightarrow \lambda$ is real.
Let λ and μ be real eigenvalues such that $\Lambda\varrho = \lambda\varrho$ and $\Lambda\vartheta = \mu\vartheta$
Then

$\lambda\langle\varrho, \vartheta\rangle = \langle\lambda\varrho, \vartheta\rangle = \langle\Lambda\varrho, \vartheta\rangle = \langle\varrho, \Lambda\vartheta\rangle = \langle\varrho, \mu\vartheta\rangle = \mu\langle\varrho, \vartheta\rangle$.
i.e. $\lambda\langle\varrho, \vartheta\rangle = \mu\langle\varrho, \vartheta\rangle \Rightarrow (\lambda - \mu)\langle\varrho, \vartheta\rangle = 0$
since $\lambda \neq \mu$ then we have $\langle\varrho, \vartheta\rangle = 0 \Rightarrow \varrho \perp \vartheta$ □

Definition 2.4.11.
Resolvent set, $\rho(\Lambda)$ *of* $\Lambda \in \mathbb{B}(\mathbb{H})$ *is the set of complex numbers* λ *satisfying:*
(i)$\Lambda - \lambda I$ *is injective thus* $(\Lambda - \lambda I)^{-1}$ *exists.*
(ii)Both $\Lambda - \lambda I$ *and* $(\Lambda - \lambda I)^{-1}$ *are in* $\mathbb{B}(\mathbb{H})$*.*
(iii)$\overline{Ran(\Lambda - \lambda I)} = \mathbb{H}$*.*

Remark 2.4.12.
Note that if $\lambda \in \rho(\Lambda)$ *then* λ *is a regular value of* Λ

i.e. $\rho(\Lambda) = \{\lambda \in \mathbb{C} : \lambda$ *is a regular value* $\}$

Definition 2.4.13.
The spectrum of $\Lambda \in \mathbb{B}(\mathbb{H})$ *is* $\sigma(\Lambda) = \{\lambda \in \mathbb{C} : \lambda \in [\rho(\Lambda)]^C\}$ *ie* $\sigma(\Lambda) = \mathbb{C}/\rho(\Lambda)$

Remark 2.4.14.
If λ *does not meet any of the three conditions of the resolvent set of an operator* Λ *then* $\lambda \in \sigma(\Lambda)$.
Thus depending on the condition which is not met we can decompose the spectrum of an operator Λ *in to:*

1. *The* $point\ spectrum$ *of* Λ

   $\sigma_P(\Lambda) = \{\lambda \in \mathbb{C} : (\Lambda - \lambda I)^{-1}$ *does not exist* $\}$.

2. $Continuous\ \ spectrum$ *of* Λ

   $\sigma_C(\Lambda) = \{\lambda \in \mathbb{C} : \overline{Ran(\Lambda - \lambda I)} = \mathbb{H}$ *and* $(\Lambda - \lambda I)^{-1}$ *exists but not bounded*$\}$

3. $Residual\ spectrum$ *of* Λ

   $\sigma_R(\Lambda) = \{\lambda \in \mathbb{C} : (\Lambda - \lambda I)^{-1}$ *exists but* $\overline{Ran(\Lambda - \lambda I)} \neq \mathbb{H}\}$

Theorem 2.4.15.
$\lambda \in \mathbb{C}$ *is an element of* $\sigma_P(\Lambda)$ *iff the eigen equation* $\Lambda\varrho = \lambda\varrho$ *has a non-zero solution of* $\varrho \in \mathbb{H}$.

Proof. Let $\lambda \in \sigma_P(\Lambda)$ then $(\Lambda - \lambda I)^{-1}$ does not exist.
i.e. $\Lambda - \lambda I$ is not $1-1$ so a singular matrix $\Rightarrow (\Lambda - \lambda I)\varrho = 0$ for some $\varrho \neq 0$
$\Rightarrow \Lambda\varrho - \lambda I\varrho = 0 \Rightarrow \Lambda\varrho - \lambda\varrho = 0 \Rightarrow \Lambda\varrho = \lambda\varrho$
Therefore $\Lambda\varrho = \lambda\varrho$ has a non-zero solution of ϱ.
Conversely let $\Lambda\varrho = \lambda\varrho$ for some $\varrho \neq 0$
Then $(\Lambda - \lambda)\varrho = 0 \Rightarrow (\Lambda - \lambda I)\varrho = 0$ for some $\varrho \neq 0 \Rightarrow \Lambda - \lambda I$ is not $1-1$
therefore $(\Lambda - \lambda I)^{-1}$ does not exist $i.e.\lambda \in \sigma_P(\Lambda)$ □

Remark 2.4.16.
In the above prove such a λ *is called an eigen value of* Λ *and a vector* $\varrho \neq 0$ *satisfying the eigen equation* $\Lambda\varrho = \lambda\varrho$ *is called the eigenvector of* Λ *corresponding to* λ.

The set consisting of all eigenvectors corresponding to λ is the eigen space and denoted $N(\Lambda - \lambda I)$.
The dimension of $N(\Lambda - \lambda I)$ is called the multiplicity of the eigenvalue λ.

Theorem 2.4.17.
If $\mathbb{H}$ has a finite dimension and $\Lambda \in \mathbb{B}(\mathbb{H})$, then $\sigma\Lambda = \sigma_P(\Lambda)$ i.e. $\sigma_C(\Lambda)$ and $\sigma_R(\Lambda)$ are empty sets.

`Proof.` An operator on a $\mathbb{H}$ is always bounded. So $\forall\lambda \in \mathbb{C}$, $(\Lambda - \lambda I)^{-1}$ is bounded if it exists and therefore $\sigma_C(\Lambda) = \phi$
If $\lambda \in \sigma_R(\Lambda)$ then $(\Lambda - \lambda I)^{-1}$ exists and is bounded since $\mathbb{H}$ has a finite dimension. Therefore $(\Lambda - \lambda I)^{-1}$ is injective on $\mathbb{H}$.
Now let $\{\varrho_1, \varrho_2, ..., \varrho_n\}$ be a basis for $\mathbb{H}$, then it follows $\{(\Lambda - \lambda I)\varrho_1, (\Lambda - \lambda I)\varrho_2, ..., (\Lambda - \lambda I)\varrho_n\}$ which spans the $Ran(\Lambda - \lambda I)$ is linearly independent and therefore $\overline{Ran(\Lambda - \lambda I)} = \mathbb{H}$.
This contradicts the definition of $\sigma_R(\Lambda)$ and therefore there is no such λ in $\sigma_R(\Lambda) \Rightarrow \sigma_R(\Lambda) = \phi$
Hence we have $\sigma(\Lambda) = \sigma_P(\Lambda) \cup \sigma_C(\Lambda) \cup \sigma_R(\Lambda) = \sigma_P(\Lambda) \cup \phi \cup \phi = \sigma_P(\Lambda)$ □

Definition 2.4.18.
A complex number $\lambda \in \pi(\Lambda)$ if for $\varepsilon > 0, \exists\varrho \in D(\Lambda)$ such that $\| \varrho \| = 1$ and $\| (\Lambda - \lambda I)\varrho \| < \varepsilon$

Remark 2.4.19.
Note that the points $\lambda \in [\sigma(\Lambda) - \pi(\Lambda)]$ form the $compresion\ spectrum$ of Λ denoted $\tau(\Lambda)$

Theorem 2.4.20.
A complex number $\lambda \in \pi(\Lambda)$ iff $(\Lambda - \lambda I)$ does not have a bounded inverse.

`Proof.` Suppose $\lambda \in \pi(\Lambda)$ then for each $n \in J^+, \exists a\varrho_n \in D(\Lambda)$ with $\|\varrho_n\| = 1$ satisfying

$$\| (\Lambda - \lambda I)\varrho_n \| \leq \frac{1}{n}$$

i.e. we can not find $\kappa > 0$ such that $\| (\Lambda - \lambda I)\varrho \| \geq \kappa \| \varrho \| \ \forall\varrho \in D(\Lambda)$
Suppose $\kappa > 0$ satisfying $\| (\Lambda - \lambda I)\varrho \| \geq \kappa \| \varrho \|$ does not exist.
This means for any $\varepsilon > 0$ and $\varrho \in D(\Lambda)$ with $\| \varrho \| = 1$ is such that $\| (\Lambda - \lambda I)\varrho \| < \varepsilon \Rightarrow \lambda \in \pi(\Lambda)$ □

Corollary 2.4.21.
Let $\Lambda \in \mathbb{B}(\mathbb{H})$ *then* $\pi(\Lambda) \subset \sigma(\Lambda)$

`Proof.` Let λ be not in $\sigma(\Lambda)$ then $\lambda \in \rho(\Lambda) \Rightarrow \Lambda - \lambda I$ has a bounded inverse which implies that λ is not an element in $\pi(\Lambda) \Rightarrow \pi(\Lambda) \subseteq \sigma(\Lambda)$ $\square$

Theorem 2.4.22.
For $\Lambda \in \mathbb{B}(\mathbb{H})$ *,* $\sigma_C(\Lambda) \cup \sigma_P(\Lambda) \subseteq \pi(\Lambda)$

`Proof.` Let $\lambda \in \sigma_P(\Lambda)$ then $(\Lambda - \lambda I)^{-1}$ does not exist and so $\Lambda - \lambda I$ is not 1-1
There is no $\kappa > 0$ such that $\| (\Lambda - \lambda I)\varrho \| \geq \kappa \| \varrho \|$
This implies for $\varepsilon > 0 \exists \varrho \in D(\Lambda)$ with $\| \varrho \| = 1$ such that
$\| (\Lambda - \lambda I)\varrho \| < \varepsilon \Rightarrow \lambda \in \pi(\Lambda) \Rightarrow \sigma_P(\Lambda) \subseteq \pi(\Lambda)$................$(i)$
Also let $\lambda \in \sigma_C(\Lambda)$ then $(\Lambda - \lambda I)^{-1}$ exists and $\overline{Ran(\Lambda - \lambda I)} = \mathbb{H}$ but $(\Lambda - \lambda I)^{-1}$ is not bounded.
Therefore for any $\kappa > o$ we can find $\vartheta \in Ran(\Lambda - \lambda I)$ such that
$\| (\Lambda - \lambda I)^{-1}\vartheta \| \geq \kappa \| \vartheta \|$
i.e. $\| \varrho \| \geq \kappa \| (\Lambda - \lambda I)\varrho \| \Rightarrow \| (\Lambda - \lambda I)\varrho \| \leq \frac{1}{\kappa} \| \varrho \|$
Thus for $\| \varrho \| = 1$ it is equivalent to say $\| (\Lambda - \lambda I)\varrho \| < \varepsilon$ i.e. $\lambda \in \pi(\Lambda)$ and so $\sigma_C(\Lambda) \subseteq \pi(\Lambda)$..............$(ii)$
Hence by (i) and (ii) we have $\sigma_P(\Lambda) \cup \sigma_C(\Lambda) \subseteq \pi(\Lambda)$ $\square$

Definition 2.4.23.
The $spectral\ radius$ *of* $\Lambda \in \mathbb{B}(\mathbb{H})$ *denoted* $\gamma(\Lambda)$ *is the radius of the smallest disc whose center is at zero that contains* $\sigma(\Lambda)$*.*

So

$$\gamma(\Lambda) = Sup\{|\lambda| : \lambda \in \sigma(\Lambda)\}$$

Theorem 2.4.24.
Let $\Lambda^* = \Lambda$ *and* $f : \sigma(\Lambda) \to \mathbb{C}$ *a continuous function then*

$$\sigma(f(\Lambda)) = f(\sigma(\Lambda))$$

$proof$
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Definition 2.4.25.
Numerical range of $\Lambda \in \mathbb{B}(\mathbb{H})$ *is*

$$W(\Lambda) = \{\langle \Lambda\varrho, \varrho\rangle : \varrho \in \mathbb{H}, \| \varrho \|= 1\}$$

Theorem 2.4.26.
If $\Lambda \in \mathbb{B}(\mathbb{H})$ *then* $\sigma_P(\Lambda) \subseteq W(\Lambda)$

Proof. Let $\lambda \in \sigma_P(\Lambda)$ then $\exists \varrho \in \mathbb{H}$ with $\| \varrho \|= 1$ such that $\Lambda\varrho = \lambda\varrho$
Now $\langle \varrho, \varrho\rangle = 1 \Rightarrow \lambda = \lambda\langle \varrho, \varrho\rangle = \langle \lambda\varrho, \varrho\rangle = \langle \Lambda\varrho, \varrho\rangle \in W(\Lambda)$
i.e. $\lambda \in \sigma_P(\Lambda) \Rightarrow \lambda \in W(\Lambda) \Rightarrow \sigma_P(\Lambda) \subseteq W(\Lambda)$ $\square$

2.5 Spectra of some classes of linear operators

Remark 2.5.1.
The following proposition gives some results on spectra of some classes of operators on $\mathbb{H}$ according to Paul Garrett (11)page 18.

Proposition 2.5.2.
Let $\Lambda \in \mathbb{B}(\mathbb{H})$.

Then:

1. *if Λ is self ad-joint then $\sigma(\Lambda) \subseteq \mathbb{R}$*
2. *if Λ is positive then $\sigma(\Lambda)$ is a non-negative real number.*
3. *if Λ is a projection the $\sigma(\Lambda) \subseteq \{0, 1\}$*
4. *if Λ is unitary then $\sigma(\Lambda) \subseteq z \in \mathbb{C} :| z |= 1$*

Theorem 2.5.3.
If $\Lambda \in \mathbb{B}(\mathbb{H})$ is a normal operator then

$$\sigma_R(\Lambda) = \phi$$

`Proof.` Let $\lambda \in \sigma_R(\Lambda)$ then

$$\overline{Ran(\Lambda - \lambda I)} \neq \mathbb{H} \Rightarrow \overline{Ran(\Lambda - \lambda I)}^{\perp} \neq \{0\}$$

$\Rightarrow$ the null set $N(\Lambda^* - \bar{\lambda} I) \neq \{0\} \Rightarrow (\Lambda^* - \bar{\lambda} I)\varrho = 0$ for some $\varrho \neq 0$
Let Λ be normal, then $\Lambda - \lambda I$ is normal and

$$\| (\Lambda - \lambda I)\varrho \| = \| (\Lambda^* - \bar{\lambda} I)\varrho \|, \forall \varrho \in \mathbb{H}$$

but $(\Lambda^* - \bar{\lambda} I)\varrho = 0 \Rightarrow \| (\Lambda^* - \bar{\lambda} I)\varrho \| = 0 \Rightarrow \| (\Lambda - \lambda I)\varrho \| = 0$
$\Rightarrow (\Lambda - \lambda I)\varrho = 0$ for some $\varrho \neq 0$
This is only possible if $| \ \Lambda - \lambda I \ | = 0$ i.e. if λ is an eigenvalue i.e. if $\lambda \in \sigma_P(\Lambda)$
This is a contradiction since $\sigma_P(\Lambda)$ and $\sigma_R(\Lambda)$ must be disjoint sets.
Therefore there is no such λ and hence $\sigma_R(\Lambda) = \phi$ $\square$

Corollary 2.5.4.
If Λ is normal,

$$\pi(\Lambda) = \sigma(\Lambda)$$

`Proof.` Recall, always $\pi(\Lambda) \subseteq \sigma(\Lambda)$.....................$(i)$
Also recall $\sigma_R(\Lambda) = \phi$ for normal Λ implying $\sigma(\Lambda) = \sigma_P(\Lambda) \cup \sigma_C(\Lambda)$
but $\sigma_P(\Lambda) \cup \sigma_C(\Lambda)$ is always a subset of $\pi(\Lambda)$
So $\sigma(\Lambda) = \sigma_P(\Lambda) \cup \sigma_C(\Lambda) \subseteq \pi(\Lambda)$
i.e. $\sigma(\Lambda) \subseteq \pi(\Lambda)$.................$(ii)$
(i) and $(ii) \Rightarrow \sigma(\Lambda) = \pi(\Lambda)$. □

3 SIMILARITY AND QUASI-SIMILARITY OF OPERATORS AND THEIR SPECTRAL PROPERTIES

3.1 Some results on similarity of operators

Definition 3.1.1.
$\Upsilon \in \mathbb{B}(\mathbb{H}_1, \mathbb{H}_2)$ has inverse if it is injective i.e. (1-1) and surjective i.e.(onto). Equivalently Υ is invertible if Υ has a trivial kernel in $\mathbb{H}_1$ and $\mathbb{H}_2$ is the range of Υ
i.e. $Ker(\Upsilon) = 0$ and $Ran(R) = \mathbb{H}_2$

Definition 3.1.2.
Let $\Lambda : D(\Lambda) \to \mathbb{H}_1, \Xi : D(\Xi) \to \mathbb{H}_2$ be two linear operators on dense subspaces $D(\Lambda)$ of $\mathbb{H}_1$ and $D(\Xi)$ of $\mathbb{H}_2$ respectively. Then $\Upsilon : \mathbb{H}_1 \to \mathbb{H}_2$ intertwines Λ and Ξ if

$$\Upsilon : D(\Lambda) \to D(\Xi)$$

$$\Upsilon\Lambda\varrho = \Xi\Upsilon\varrho, \forall\varrho \in D(\Lambda)$$

Remark 3.1.3.
The intertwining operator Υ for
$\Lambda \in \mathbb{B}(\mathbb{H}_1)$ and $\Xi \in \mathbb{B}(\mathbb{H}_2)$ is invertible and in $\mathbb{B}(\mathbb{H}_1, \mathbb{H}_2)$ and called an affinity of Λ and Ξ

Remark 3.1.4.
We denote by $\mathbb{G}(\mathbb{H}_1, \mathbb{H}_2)$ the class of all invertible operators from $\mathbb{H}_1$ into $\mathbb{H}_2$

Proposition 3.1.5.
If $\Upsilon \in \mathbb{G}(\mathbb{H}_1, \mathbb{H}_2)$ intertwines $\Lambda \in \mathbb{B}(\mathbb{H}_1)$ and $\Xi \in \mathbb{B}(\mathbb{H}_2)$, then $\Upsilon^* \in \mathbb{G}(\mathbb{H}_2, \mathbb{H}_1)$ intertwines $\Xi^* \in \mathbb{B}(\mathbb{H}_2)$ and $\Lambda^* \in \mathbb{B}(\mathbb{H}_1)$.

`Proof.` Let Υ intertwine Λ and Ξ, then $\Upsilon\Lambda = \Xi\Upsilon$
Let $\varrho \in \mathbb{H}_1$ and $\vartheta \in \mathbb{H}_2$
then $\langle \Upsilon\Lambda\varrho, \vartheta\rangle = \langle \Lambda\varrho, \Upsilon^*\vartheta\rangle = \langle \varrho, \Lambda^*\Upsilon^*\vartheta\rangle$................(i)
and $\langle \Xi\Upsilon\varrho, \vartheta\rangle = \langle \Upsilon\varrho, \Xi^*\vartheta\rangle = \langle \varrho, \Upsilon^*\Xi^*\vartheta\rangle$................(ii)
So from (i) and (ii) we have

$\langle \varrho, \Upsilon^*\Xi^*\vartheta \rangle = \langle \varrho, \Lambda^*\Upsilon^*\vartheta \rangle$ since $\langle \Xi\Upsilon\varrho, \vartheta \rangle = \langle \Upsilon\Lambda\varrho, \vartheta \rangle$
Thus $\Upsilon^*\Xi^* = \Lambda^*\Upsilon^*$ which implies that Υ^* intertwines Ξ^* and Λ^* $\square$

Definition 3.1.6.
Let $\Lambda \in \mathbb{B}(\mathbb{H}_1)$ *and* $\Xi \in \mathbb{B}(\mathbb{H}_2)$.
Then Λ *is similar to* Ξ *denoted* $\Lambda \sim \Xi$ *if* $\exists\ \Upsilon \in \mathbb{G}(\mathbb{H}_1, \mathbb{H}_2)$ *satisfying* $\Lambda = \Upsilon^{-1}\Xi\Upsilon$

Remark 3.1.7.
Note, if $\Lambda \sim \Xi$ *and* $\Xi \sim \Lambda$ *then* Λ *and* Ξ *are similar.*

Corollary 3.1.8.
Let $\Lambda \in \mathbb{B}(\mathbb{H}_1)$ *and* $\Xi \in \mathbb{B}(\mathbb{H}_2)$ *be similar, i.e.* $\Lambda \sim \Xi$ *and* $\Xi \sim \Lambda$ *then* $\exists$ *an intertwining operator* $\Upsilon \in \mathbb{G}(\mathbb{H}_1, \mathbb{H}_2)$ *for* Λ *and* Ξ *satisfying* $\Upsilon\Lambda = \Xi\Upsilon$.
Moreover $\Upsilon^{-1} \in \mathbb{G}(\mathbb{H}_2, \mathbb{H}_1)$ *is an intertwining operator for* Ξ *and* Λ *satisfying* $\Upsilon^{-1}\Xi = \Lambda\Upsilon^{-1}$

Proof. Since $\Lambda \sim \Xi$ then

$\Lambda = \Upsilon^{-1}\Xi\Upsilon \Leftrightarrow \Upsilon\Lambda = \Upsilon\Upsilon^{-1}\Xi\Upsilon \Leftrightarrow \Upsilon\Lambda = \Xi\Upsilon$
Hence Υ is an intertwining operator for Λ and Ξ
Now

$$\Upsilon\Lambda = \Xi\Upsilon \Leftrightarrow \Upsilon^{-1}\Upsilon\Lambda\Upsilon^{-1} = \Upsilon^{-1}\Xi\Upsilon\Upsilon^{-1} \Leftrightarrow \Lambda\Upsilon^{-1} = \Upsilon^{-1}\Xi$$

Hence Υ^{-1} is an intertwining operator for Ξ and Λ $\square$

Definition 3.1.9.
Let the relation $\Re$ *on objects* ϱ, ϑ *and* ι *be such that:*

1. $\varrho\Re\varrho$ *i.e.* $\Re$ *reflexive.*
2. *if* $\varrho\Re\vartheta$ *then* $\vartheta\Re\varrho$ *i.e.* $\Re$ *is symmetric*
3. *if* $\varrho\Re\vartheta$ *and* $\vartheta\Re\iota$ *then* $\varrho\Re\iota$ *i.e.* $\Re$ *is transitive*

Then the relation $\Re$ *satisfying i,ii and iii above is called an equivalence relation.*

Theorem 3.1.10.
Similarity of operators is an equivalence relation.

Proof. Let similarity be represented by $\sim$ and let $\Lambda \in \mathbb{B}(\mathbb{H}_1)$.
Consider the identity operator $I \in \mathbb{B}(\mathbb{H}_1)$
Note $I \in \mathbb{G}(\mathbb{H}_1)$and $I = I^{-1}$
Now $I^{-1}\Lambda I = I\Lambda I = \Lambda$ implying $\Lambda = I^{-1}\Lambda I \Leftrightarrow \Lambda \sim \Lambda$
So $\sim$ is reflexive.
Let $\Xi \in \mathbb{B}(\mathbb{H}_2)$ and $\Lambda \sim \Xi$
Then we can find $\Upsilon \in \mathbb{G}(\mathbb{H}_1, \mathbb{H}_2)$ satisfying $\Lambda = \Upsilon^{-1}\Xi\Upsilon$
$\Rightarrow \Upsilon\Lambda\Upsilon^{-1} = \Upsilon\Upsilon^{-1}\Xi\Upsilon\Upsilon^{-1} = \Xi$ i.e. $\Xi = \Upsilon\Lambda\Upsilon^{-1}$ i.e. $\Xi = (\Upsilon^{-1})^{-1}\Lambda\Upsilon^{-1}$
i.e. we can find $\Upsilon^{-1} \in \mathbb{G}(\mathbb{H}_2, \mathbb{H}_1)$ satisfying $\Xi = (\Upsilon^{-1})^{-1}\Lambda\Upsilon^{-1}$

$\Rightarrow \Xi \sim \Lambda$
So $\Lambda \sim \Xi \Rightarrow \Xi \sim \Lambda$ and therefore $\sim$ is symmetric.
Let $\Gamma \in \mathbb{B}(\mathbb{H}_3)$ with $\Lambda \sim \Xi$ and $\Xi \sim \Gamma$
Then can find $\Upsilon \in \mathbb{G}(\mathbb{H}_1, \mathbb{H}_2)$ and $\Psi \in \mathbb{G}(\mathbb{H}_2, \mathbb{H}_3)$ satisfying
$\Lambda = \Upsilon^{-1}\Xi\Upsilon$ and $\Xi = \Psi^{-1}\Gamma\Psi$

$\Rightarrow \Lambda = \Upsilon^{-1}(\Psi^{-1}\Gamma\Psi)\Upsilon = (\Psi\Upsilon)^{-1}\Gamma\Psi\Upsilon$
i.e. we can find $\Psi\Upsilon \in \mathbb{G}(\mathbb{H}_1, \mathbb{H}_3)$ satisfying

$$\Lambda = (\Psi\Upsilon)^{-1}\Gamma\Psi\Upsilon \Rightarrow \Lambda \sim \Gamma$$

So $\Lambda \sim \Xi, \Xi \sim \Gamma \Rightarrow \Lambda \sim \Gamma$ and therefore $\sim$ is transitive.
Hence $\sim$ is an equivalence relation. □

Definition 3.1.11.
$\Lambda \in \mathbb{B}(\mathbb{H}_1)$ *is unitarily equivalent to* $\Xi \in \mathbb{B}(\mathbb{H}_2)$ *denoted* $\Lambda \cong \Xi$ *if we can find a unitary operator* $U \in \mathbb{B}(\mathbb{H}_1, \mathbb{H}_2)$ *satisfying* $\Lambda = U^*\Xi U$

Theorem 3.1.12.
Unitary equivalence is an equivalence relation.

Proof. Let $\cong$ denote unitary equivalence and let $\Lambda \in \mathbb{B}(\mathbb{H}_1)$.
Consider the identity operator $I \in \mathbb{B}(\mathbb{H}_1)$
Note $I^* = I \Rightarrow I^*I = II = I = II^*$ i.e. I is unitary.
Now $I^*\Lambda I = I\Lambda I = \Lambda \Rightarrow \Lambda \cong \Lambda$ thus we have $\cong$ is reflexive.
Let $\Xi \in \mathbb{B}(\mathbb{H}_2)$ and $\Lambda \cong \Xi$
Then we can find $U_1 \in \mathbb{B}(\mathbb{H}_1, \mathbb{H}_2)$ satisfying

$$\Lambda = {U_1}^*\Xi U_1 \Rightarrow U_1\Lambda {U_1}^* = U_1{U_1}^*\Xi U_1{U_1}^* \Rightarrow \Xi = U_1\Lambda {U_1}^*$$

Let $U_2 = U_1{}^*$ then U_2 is unitary and $\Xi = U_2{}^*\Lambda U_2$
i.e. we can find $U_2 \in \mathbb{B}(\mathbb{H}_2, \mathbb{H}_1)$ satisfying $\Xi = U_2{}^*\Lambda U_2 \Rightarrow \Xi \cong \Lambda$
so $\Lambda \cong \Xi \Rightarrow \Xi \cong \Lambda$ and thus we have $\cong$ is symmetric.
Let $\Gamma \in \mathbb{B}(\mathbb{H}_3)$ with $\Lambda \cong \Xi$ and $\Xi \cong \Gamma$
then we can find $U_1 \in \mathbb{B}(\mathbb{H}_1, \mathbb{H}_2)$ and $U_2 \in \mathbb{B}(\mathbb{H}_2, \mathbb{H}_3)$ satisfying
$\Lambda = U_1{}^*\Xi U_1$ and $\Xi = U_2{}^*\Gamma U_2$
So $\Lambda = U_1{}^*U_2{}^*\Gamma U_2 U_1 = (U_2U_1)^*\Gamma(U_2U_1)$
Let $U_3 = U_2U_1$ then U_3 is unitary and $\Lambda = U_3{}^*\Gamma U_3 \Rightarrow \Lambda \cong \Gamma$
So $\Lambda \cong \Xi, \Xi \cong \Gamma \Rightarrow \Lambda \cong \Gamma$ thus we have $\cong$ is transitive.
Hence $\cong$ is an equivalence relation. □

Remark 3.1.13.
Recall $\Lambda \in \mathbb{B}(\mathbb{H})$ *is positive if* $\Lambda^* = \Lambda$ *and* $\langle \Lambda\varrho, \varrho\rangle \geq 0, \forall \varrho \in \mathbb{H}$
Note that Λ *is strictly positive if we also have* $\langle \Lambda\varrho, \varrho\rangle = 0 \Leftrightarrow \varrho = 0$

Definition 3.1.14.
$\Lambda \in \mathbb{B}(\mathbb{H}_1, \mathbb{H}_2)$ *which is self ad-joint and strictly positive is called a metric operator.*

Remark 3.1.15.
We now give the following results on metric operators according to Antoine and Trapani (14)

Proposition 3.1.16.

1. *A metric operator* $\Lambda : \mathbb{H}_1 \to \mathbb{H}_2$ *is invertible and densely defined but may be bounded or unbounded.*

2. *If* Λ *is metric then* Λ^α *is metric for all* $\alpha \in \mathbb{R}$

3. *if* Ξ *and* Ξ^* *are densely defined on* $\mathbb{H}_1$ *and* $\mathbb{H}_2$ *respectively. Then* Ξ *is similar to* Ξ^* *with intertwining metric* Λ *satisfying* $\Xi = \Lambda^{-1}\Xi^*\Lambda$

4. *If* $\Xi \in \mathbb{B}(\mathbb{H}_1)$ *and* $\Gamma \in \mathbb{B}(\mathbb{H}_2)$ *which are self adjoint, then* Ξ *is metrically similar to* Γ *satisfying* $\Xi = \Lambda^{-(\frac{1}{2})}\Gamma\Lambda^{(\frac{1}{2})}$

3.2 Spectral properties of similar operators

Remark 3.2.1.
Recall, Λ represented by a square $n \times n$ matrix $\Lambda = [\lambda_{ij}]$ is diagonal if $\lambda_{ij} = 0 \ \forall \ i \neq j$ and the elements $\lambda_{ii} = \{\lambda_1, \lambda_2, ..., \lambda_n\}$ in the main diagonal are all eigenvalues of Λ (multiplicity included).

Definition 3.2.2.
An $n \times n$ matrix $\Lambda = [\lambda_{ij}]$ is a triangular if all the elements above the main diagonal are all zeroes.

Remark 3.2.3.
Note that the elements $\lambda_{11}, \lambda_{22}, ..., \lambda_{nn}$ given by $\lambda_1, \lambda_2, ..., \lambda_n$ in the main diagonal of a triangular matrix Λ are the eigenvalues of Λ hence the points $\sigma_P(\Lambda)$.

Theorem 3.2.4.
Let $\mathbb{H} = \mathbb{K}^n$.
If a matrix operator Λ on $\mathbb{K}^n$ is such that all its eigenvalues are in $\mathbb{K}$ then we can find Υ on $\mathbb{K}^n$ to have $\Upsilon\Lambda\Upsilon^{-1} = \Xi$ as a triangular matrix
i.e. we can find Ξ, a triangular matrix satisfying $\Lambda \sim \Xi$.

Definition 3.2.5.
Let $\Lambda = [\lambda_{ij}]$ be an $n \times n$ square matrix and let $\{\lambda_{ii}\}$ be the elements in the main diagonal for $i = 1, 2, ...n$. Then the trace of Λ denoted

$$tr(\Lambda) = \Sigma_{i=1}^{n} \lambda_{ii}$$

Proposition 3.2.6.
The matrix operators $\Lambda, \Xi \in \mathbb{B}(\mathbb{H})$ which are are similar have the same characteristic polynomial. Moreover Λ and Ξ have the same trace.

`Proof.` Since Λ is similar to Ξ then we can have an invertible matrix Υ satisfying

$$\Lambda = \Upsilon^{-1}\Xi\Upsilon$$

The characteristic polynomial of Λ i.e. $\Phi_\Lambda\lambda = det(\Lambda - \lambda I)$
So we have $\Phi_\Lambda\lambda = det(\Lambda - \lambda I) = det[\Upsilon^{-1}(\Xi - \lambda I)\Upsilon]$

$$= det(\Upsilon^{-1})det(\Xi - \lambda I)det(\Upsilon)$$

$$= det(\Upsilon)^{-1}det(\Xi - \lambda I)det(\Upsilon) = det(\Xi - \lambda I) = \Phi_\Xi \lambda$$

Thus we have $\Phi_\Lambda \lambda = \Phi_\Xi \lambda$ i.e. Λ and Ξ have the same characteristic polynomial.
So Λ and Ξ have the same eigenvalues (multiplicity included).
But the trace of an $n \times n$ matrix is given by addition of its eigenvalues with multiplicities and therefore Λ and Ξ have the same trace. □

Corollary 3.2.7.
If $\Lambda, \Xi \in \mathbb{B}(\mathbb{H})$ *are similar then*

$$\sigma_P(\Lambda) = \sigma_P(\Xi)$$

Proof. We have already established that similar operators have the same characteristic polynomial and hence the same eigenvalues.
The collection of eigen-values is the σ_P and therefore
$\Lambda \sim \Xi \Rightarrow \sigma_P(\Lambda) = \sigma_P(\Xi)$ □

Theorem 3.2.8.
Let $\Lambda \in \mathbb{B}(\mathbb{H}_1)$ *and* $\Xi \in \mathbb{B}(\mathbb{H}_2)$ *where* $\mathbb{H}_1$ *and* $\mathbb{H}_2$ *are finite dimensional with* $\Lambda \sim \Xi$*. Then* Λ *and* Ξ *have the same spectrum i.e.* $\sigma(\Lambda) = \sigma(\Xi)$

Proof. Since $\mathbb{H}_1$ is finite dimensional we have $\sigma_C(\Lambda) = \phi$ and $\sigma_R(\Lambda) = \phi \Rightarrow \sigma(\Lambda) = \sigma_P(\Lambda)$.
Similarly $\sigma_C(\Xi) = \phi$ and $\sigma_R(\Xi) = \phi \Rightarrow \sigma(\Xi) = \sigma_P(\Xi)$
but Λ and Ξ are similar, so by corollary 3.2.7 ,

$$\sigma_P(\Lambda) = \sigma_P(\Xi) \Rightarrow \sigma(\Lambda) = \sigma(\Xi)$$

□

Remark 3.2.9.
According to Halmos(2) the equality of spectra of similar operators can be extended to infinite dimensional Hilbert spaces. So for $\Lambda, \Xi \in \mathbb{B}(\mathbb{H})$ *and* $\Lambda \sim \Xi$ *given* $\mathbb{H}$ *has infinite dimension , then:*

$$\sigma_P(\Lambda) = \sigma_P(\Xi)$$

$$\sigma_C(\Lambda) = \sigma_C(\Xi)$$

$$\sigma_R(\Lambda) = \sigma_R(\Xi)$$

and hence

$$\sigma(\Lambda) = \sigma(\Xi)$$

3.3 Some results on quasi-similarity of operators

Remark 3.3.1.
Recall if $\Lambda \in \mathbb{B}(\mathbb{H}_1)$ *and* $\Xi \in \mathbb{B}(\mathbb{H}_2)$ *are similar, then we can find an affinity* $\Upsilon \in \mathbb{B}(\mathbb{H}_1, \mathbb{H}_2)$ *satisfying* $\Upsilon\Lambda = \Xi\Upsilon$

Definition 3.3.2.
$\Upsilon \in \mathbb{B}(\mathbb{H}_1, \mathbb{H}_2)$ *is quasi invertible (also called a quasi-affinity) if* $1 - 1$ *and the range of* Υ *is dense in* $\mathbb{H}_2$

Definition 3.3.3.
Let $\Lambda \in \mathbb{B}(\mathbb{H}_1)$ *and* $\Xi \in \mathbb{B}(\mathbb{H}_2)$
Then Λ *is quasi-affine transform of* Ξ *denoted* $\Lambda \approx \Xi$ *if we can find a quasi-affinity* $\Upsilon \in \mathbb{B}(\mathbb{H}_1, \mathbb{H}_2)$ *satisfying* $\Upsilon\Lambda = \Xi\Upsilon$

Remark 3.3.4.
If $\Lambda \in \mathbb{B}(\mathbb{H}_1)$ *and* $\Xi \in \mathbb{B}(\mathbb{H}_2)$ *are quasi-affine transforms of each other i.e. we have quasi-affinities* $\Upsilon \in \mathbb{B}(\mathbb{H}_1, \mathbb{H}_2)$ *and* $\Psi \in \mathbb{B}(\mathbb{H}_2, \mathbb{H}_1)$ *satisfying*

$$\Upsilon\Lambda = \Xi\Upsilon$$

$$\Psi\Xi = \Lambda\Psi$$

then Λ *and* Ξ *are said to be quasi-similar.*

Theorem 3.3.5.
If Υ *is a quasi-affinity from* $\mathbb{H}_1$ *to* $\mathbb{H}_2$ *and* Ψ *is a quasi-affinity from* $\mathbb{H}_2$ *to* $\mathbb{H}_3$ *then* $\Psi\Upsilon$ *is a quasi-affinity from* $\mathbb{H}_1$ *to* $\mathbb{H}_3$ *and* $\Upsilon\Psi$ *is a quasi-affinity from* $\mathbb{H}_3$ *to* $\mathbb{H}_1$.

`Proof.` Since for quasi-similar operators $\Lambda \in \mathbb{B}(\mathbb{H}_1)$ and $\Xi \in \mathbb{B}(\mathbb{H}_2)$ we can find quasi-affinities $\Upsilon \in \mathbb{B}(\mathbb{H}_1, \mathbb{H}_2)$ and $\Psi \in \mathbb{B}(\mathbb{H}_2, \mathbb{H}_1)$ then we can have the following commutative diagram which illustrates this relationship.

$$\begin{tikzcd}
\mathbb{H}_1 \arrow[r, "R"] \arrow[d, "P"'] \arrow[dr, "R"] & \mathbb{H}_2 \arrow[r, "S"] \arrow[d, "Q"] \arrow[dr, "S"] & \mathbb{H}_1 \arrow[r, "R"] \arrow[d, "P"] \arrow[dr, "R"] & \mathbb{H}_2 \arrow[d, "Q"] \\
\mathbb{H}_1 \arrow[r, "R"'] & \mathbb{H}_2 \arrow[r, "S"'] & \mathbb{H}_1 \arrow[r, "R"'] & \mathbb{H}_2
\end{tikzcd}$$

Now Υ is a quasi-affinity from $\mathbb{H}_1$ to $\mathbb{H}_2$ means that Υ is $1-1$ and $\overline{Ran(\Upsilon)} = \mathbb{H}_2$.
Similarly Ψ is $1-1$ and $\overline{Ran(\Psi)} = \mathbb{H}_3$
Then $\Psi\Upsilon$ is a $1-1$ operator since is a composition of $1-1$ operators.
The range of $\Psi\Upsilon$ is subset of $\mathbb{H}_1$ i.e. $Ran(\Psi\Upsilon) \subseteq \mathbb{H}_1$
Then it follows $\overline{\Psi\Upsilon(\mathbb{H}_1)} = \overline{\Psi\overline{\Upsilon(\mathbb{H}_1)}}$.............$(i)$
but $\overline{Ran(\Upsilon)} = \mathbb{H}_2 \Rightarrow \overline{\Upsilon(\mathbb{H}_1)} = \mathbb{H}_2$ since Υ is a quasi-affinity
so (i) becomes $\overline{\Psi\overline{\Upsilon(\mathbb{H}_1)}} = \overline{\Psi(\mathbb{H}_2)}$
but $\overline{Ran(\Psi)} = \mathbb{H}_1 \Rightarrow \overline{\Psi(\mathbb{H}_2)} = \mathbb{H}_1$ since Ψ is a quasi-affinity
consequently $\overline{\Psi\Upsilon\Upsilon(\mathbb{H}_1)} = \mathbb{H}_1$ i.e. $\overline{Ran(\Psi\Upsilon)} = \mathbb{H}_1$
Now that we have shown that $\Psi\Upsilon$ is $1-1$ and has a dense range in $\mathbb{H}_1$, it implies that $\Psi\Upsilon$ is a quasi-affinity.
Similarly $\Upsilon\Psi$ is $1-1$ operator because is a composition of $1-1$ operators.
So $Ran(\Upsilon\Psi) \subseteq \mathbb{H}_2$
Then it follows $\overline{\Upsilon\Psi(\mathbb{H}_2)} = \overline{\Upsilon\overline{\Psi(\mathbb{H}_2)}}$...............(ii)
but $\overline{Ran(\Psi)} = \mathbb{H}_1 \Rightarrow \overline{\Psi(\mathbb{H}_2)} = \mathbb{H}_1$ since Ψ is a quasi-affinity.
so (ii) becomes $\overline{\Upsilon\overline{\Psi(\mathbb{H}_2)}} = \overline{\Upsilon(\mathbb{H}_1)}$
but $\overline{Ran(\Upsilon)} = \mathbb{H}_2 \Rightarrow \overline{\Upsilon(\mathbb{H}_1)} = \mathbb{H}_2$ since Υ is a quasi-affinity.
consequently $\overline{\Upsilon\Psi(\mathbb{H}_2)} = \mathbb{H}_2$ i.e. $\overline{Ran(\Upsilon\Psi)} = \mathbb{H}_2$
Now since we have shown that $\Upsilon\Psi$ is $1-1$ and $\overline{Ran(\Upsilon\Psi)} = \mathbb{H}2$ then it implies that $\Upsilon\Psi$ is a quasi-affinity. $\square$

Theorem 3.3.6.
Let $\Lambda \in \mathbb{B}(\mathbb{H})$ be a quasi-affinity, then Λ^* is a quasi-affinity.

`Proof.` Given $\Lambda \in \mathbb{B}(\mathbb{H})$ is a quasi-affinity, then Λ is $1-1$ and $\overline{Ran(\Lambda)} = \mathbb{H}$
Since Λ is $1-1$ i.e. $Ker(\Lambda) = \{\bar{0}\}$
but $Ker(\Lambda) = Ran(\Lambda^*) \Rightarrow Ran(\Lambda^*) = \{\bar{0}\} = \overline{Ran(\Lambda^*)}$
so Λ^* has a dense range in $\{\bar{0}\}$
clearly $Ker(\Lambda^*) = \{\bar{0}\}$ i.e. Λ^* is $1-1$
Now that Λ^* is $1-1$ and has dense range, it follows that Λ^* is a quasi-affinity. □

Theorem 3.3.7.
Quasi-similarity of operators is an equivalence relation.

`Proof.` Let $\Lambda \in \mathbb{B}(\mathbb{H}_1)$ and denote quasi-similarity by $\approx$.
Now let Υ and Ψ be quasi-affinities where $\Upsilon = \Psi = I$ without lose of generality.
then $\Upsilon\Lambda = I\Lambda = \Lambda I = \Lambda\Psi$ i.e.$\Psi\Upsilon\Lambda = \Lambda\Psi$
also $\Psi\Lambda = I\Lambda = \Lambda I = \Lambda\Upsilon$ i.e.$\Psi\Lambda = \Lambda\Upsilon$
hence we have $\Lambda \approx \Lambda$,thus $\approx$ is reflexive.
Let $\Xi \in \mathbb{B}(\mathbb{H}_2)$ and $\Lambda \approx \Xi$
then we can find quasi-affinities $\Upsilon \in \mathbb{B}(\mathbb{H}_1, \mathbb{H}_2)$ and $\Psi \in \mathbb{B}(\mathbb{H}_2, \mathbb{H}_1)$ satisfying
$\Upsilon\Lambda = \Xi\Upsilon$ and $\Psi\Xi = \Lambda\Psi$
i.e. $\Psi\Xi = \Lambda\Psi$ and $\Upsilon\Lambda = \Xi\Upsilon \Rightarrow \Xi \approx \Lambda$
i.e. $\Lambda \approx \Xi \Rightarrow \Xi \approx \Lambda$ thus $\approx$ is symmetric.
Let $\Gamma \in \mathbb{B}(\mathbb{H}_3)$ where $\Lambda \approx \Xi$ and $\Xi \approx \Gamma$
We can find quasi-affinities $\Upsilon \in \mathbb{B}(\mathbb{H}_1, \mathbb{H}_2)$ and $\Psi \in \mathbb{B}(\mathbb{H}_2, \mathbb{H}_1)$ satisfying

$$\Upsilon\Lambda = \Xi\Upsilon$$

$$\Psi\Xi = \Lambda\Psi$$

and quasi-affinities $\Theta \in \mathbb{B}(\mathbb{H}_2, \mathbb{H}_3)$ and $\Omega \in \mathbb{B}(\mathbb{H}_3, \mathbb{H}_2)$ such that

$$\Theta\Xi = \Gamma\Theta$$

$$\Omega\Gamma = \Xi\Omega$$

Now $\Omega\Theta\Psi\Upsilon$ is a composition of $1-1$ operators and hence a $1-1$ operator.

but
$\Omega\Theta\Psi\Upsilon\Lambda = \Omega\Theta\Lambda\Psi\Upsilon$ since $\Psi\Upsilon\Lambda = \Lambda\Psi\Upsilon$
$\Omega\Theta\Lambda\Psi\Upsilon = \Omega\Theta\Psi\Xi\Upsilon$ since $\Lambda\Lambda\Psi = \Psi\Xi$
$\Omega\Theta\Psi\Xi\Upsilon = \Omega\Xi\Theta\Psi\Upsilon$ since $\Theta\Psi\Xi = \Xi\Theta\Psi$
$\Omega\Xi\Theta\Psi\Upsilon = \Gamma\Omega\Theta\Psi\Upsilon$ since $\Omega\Xi = \Gamma\Omega$
Thus $\Omega\Theta\Psi\Upsilon\Lambda = \Gamma\Omega\Theta\Psi\Upsilon$
i.e. $\Omega\Theta\Psi\Upsilon$ is a quasi-affinity of Λ and Γ Also
$\Lambda\Psi\Upsilon\Theta\Omega = \Psi\Upsilon\Lambda\Theta\Omega$ since $\Lambda\Psi\Upsilon = \Psi\Upsilon\Lambda$
$\Psi\Upsilon\Lambda\Theta\Omega = \Psi\Xi\Upsilon\Theta\Omega$ since $\Upsilon\Lambda - \Xi\Upsilon$
$\Psi\Xi\Upsilon\Theta\Omega = \Psi\Upsilon\Theta\Xi\Omega$ since $\Xi\Upsilon\Theta = \Upsilon\Theta\Xi$
$\Psi\Upsilon\Theta\Xi\Omega = \Psi\Upsilon\Theta\Omega\Gamma$ since $\Xi\Omega = \Omega\Gamma$
Thus $\Psi\Upsilon\Theta\Omega\Gamma = \Lambda\Psi\Upsilon\Theta\Omega$
i.e. $\Psi\Upsilon\Theta\Omega$ is a quasi-affinity of Γ and Λ
Now that $\Omega\Theta\Psi\Upsilon \in \mathbb{B}(\mathbb{H}_1, \mathbb{H}_3)$ is a quasi-affinity of Λ and Γ
and $\Psi\Upsilon\Theta\Omega \in \mathbb{B}(\mathbb{H}_3, \mathbb{H}_1)$ is a quasi-affinities of Γ and Λ then it follows
$\Lambda \approx \Gamma$
i.e.$\Lambda \approx \Xi, \Xi \approx \Gamma \Rightarrow \Lambda \approx \Gamma$, thus $\approx$ is transitive.
Hence it follows that $\approx$ is an equivalence relation. $\square$

Theorem 3.3.8.
Let $\Lambda \in \mathbb{B}(\mathbb{H}_1)$,$\Xi \in \mathbb{B}(\mathbb{H}_2)$ be similar, then Λ and Ξ are quasi-similar.

Proof. If Λ and Ξ are similar, we can find $\Upsilon \in \mathbb{G}(\mathbb{H}_1, \mathbb{H}_2)$ satisfying

$$\Lambda = \Upsilon^{-1}\Xi\Upsilon \Rightarrow \Upsilon\Lambda = \Xi\Upsilon \Rightarrow \Lambda\Upsilon^{-1} = \Upsilon^{-1}\Xi$$

Let $\Psi = \Upsilon^{-1}$ then $\Lambda\Psi = \Psi\Xi$ i.e. there exists a quasi-invertible operator
$\Psi \in \mathbb{B}(\mathbb{H}_2, \mathbb{H}_1)$ satisfying $\Psi\Xi = \Lambda\Psi$
Now that there exists quasi-affinities Υ and Ψ satisfying

$$\Upsilon\Lambda = \Xi\Upsilon$$

$$\Psi\Xi = \Xi\Psi$$

it now follows that Λ and Ξ are quasi-similar $\square$

Proposition 3.3.9.
Let $\mathbb{H}_1, \mathbb{H}_2$ and $\mathbb{H}_3$ be finite dimensional and

$\Upsilon \in \mathbb{B}(\mathbb{H}_1, \mathbb{H}_2)$, $\Psi \in \mathbb{B}(\mathbb{H}_2, \mathbb{H}_3)$ be quasi-affinities.
Then the inverse $(\Psi\Upsilon)^{-1} \in \mathbb{B}(\mathbb{H}_3, \mathbb{H}_1)$ of composite operator

$\Psi\Upsilon \in \mathbb{B}(\mathbb{H}_1, \mathbb{H}_3)$ *exists and* $(\Psi\Upsilon)^{-1} = \Upsilon^{-1}\Psi^{-1}$

Proof. Recall the composite $\Psi\Upsilon \in \mathbb{B}(\mathbb{H}_1, \mathbb{H}_3)$ is a quasi-affinity and is a bijection and so $(\Psi\Upsilon)^{-1}$ exists.
Now
$(\Psi\Upsilon)(\Psi\Upsilon)^{-1} = I_{\mathbb{H}_3}$ is the identity operator in $\mathbb{H}_3$
then applying Ψ^{-1} we have

$$\Psi^{-1}S\Upsilon(\Psi\Upsilon)^{-1} = \Psi^{-1}I_{\mathbb{H}_3} \Rightarrow \Upsilon(\Psi\Upsilon)^{-1} = \Psi^{-1}$$

and applying Υ^{-1} we have

$$\Upsilon^{-1}\Upsilon(\Psi\Upsilon)^{-1} = \Upsilon^{-1}\Psi^{-1} \Rightarrow (\Psi\Upsilon)^{-1} = \Upsilon^{-1}\Psi^{-1}$$

$\square$

Lemma 3.3.10.
If $\Lambda \in \mathbb{B}(\mathbb{H}_1)$ *is quasi-affine transform of* $\Xi \in \mathbb{B}(\mathbb{H}_2)$ *then*

$\Xi^* \in \mathbb{B}(\mathbb{H}_2)$ *is a quasi-affine transform of* $\Lambda^* \in \mathbb{B}(\mathbb{H}_1)$

Lemma 3.3.11.
Let Υ *be a quasi-affinity from* $\mathbb{H}_1$ *to* $\mathbb{H}_2$
Then $\mid \Upsilon \mid = \sqrt{\Upsilon^*\Upsilon}$ *is a quasi-affinity from* $\mathbb{H}_2$ *to* $\mathbb{H}_1$ *and* $\Upsilon \mid \Upsilon \mid^{-1}$ *by continuity extends to a unitary transformation* $U : \mathbb{H}_1 \to \mathbb{H}_2$

Theorem 3.3.12.
Let $\Lambda \in \mathbb{B}(\mathbb{H}_1)$ *be a quasi-affine transform of* $\Xi \in \mathbb{B}(\mathbb{H}_2)$ *given* Λ *and* Ξ *are unitary. Then* Λ *and* Ξ *are unitarily equivalent.*

Proof. Recall if Λ and Ξ are unitary, then
$\Lambda^*\Lambda = \Lambda\Lambda^* = I \Rightarrow \Lambda^* = \Lambda^{-1}$

and $\Xi^*\Xi = \Xi\Xi^* = I \Rightarrow \Xi^* = \Xi^{-1}$
Let $\Upsilon \in \mathbb{B}(\mathbb{H}_1, \mathbb{H}_2$ be a quasi-affinity such that $\Upsilon\Lambda = \Xi\Upsilon$..................$(i)$
Then applying Λ^{-1} on (i),

$$\Upsilon\Lambda\Lambda^{-1} = \Xi\Upsilon\Lambda^{-1} \Rightarrow \Upsilon = \Xi\Upsilon\Lambda^{-1}$$

Applying Ξ^{-1} ,

$$\Xi^{-1}\Upsilon = \Xi^{-1}\Xi\Upsilon\Lambda^{-1} \Rightarrow \Xi^{-1}\Upsilon = \Upsilon\Lambda^{-1} \Rightarrow \Xi^{*}\Upsilon = \Upsilon\Lambda^{*}................(ii)$$

Note that Υ^{*} is also a quasi-affinity and by lemma 3.3.10 ,

$$\mid \Upsilon \mid = \sqrt{\Upsilon^{*}\Upsilon} \Rightarrow \mid \Upsilon \mid^{2} = \Upsilon^{*}\Upsilon$$

So by (i) above, $\mid \Upsilon \mid^{2} \Lambda = \Upsilon^{*}\Upsilon\Lambda = \Upsilon^{*}\Xi\Upsilon$
Then applying (ii) , $\Upsilon^{*}\Xi\Upsilon = \Lambda\Upsilon^{*}\Upsilon = \Lambda \mid \Upsilon \mid^{2}$
thus $\mid \Upsilon \mid^{2} \Lambda = \Lambda \mid \Upsilon \mid^{2}$
and by iteration $\mid \Upsilon \mid^{2n} \Lambda = \Lambda \mid \Upsilon \mid^{2n}$ where $n = 0, 1, 2, 3, ...$
so for every polynomial $\Phi(\Upsilon)$ we have $\Phi(\mid \Upsilon \mid^{2})\Lambda = \Lambda\Phi(\mid \Upsilon \mid^{2})$
Let $\{\Phi_n(\Upsilon)\}$ be a sequence of polynomials tending to $\mid \Upsilon \mid^{\frac{1}{2}}$ uniformly on the interval $0 < \Upsilon \leq \parallel \Upsilon \parallel^{\frac{1}{2}}$ then $\Phi_n(\mid \Upsilon \mid^{2})$ converges to $\mid \Upsilon \mid$ so that $\mid \Upsilon \mid \Lambda = \Lambda \mid \Upsilon \mid ....(iii)$
Note $\Upsilon \mid \Upsilon \mid^{-1} = U$ by lemma 3.3.11
thus it follows that $\Xi U \mid \Upsilon \mid = \Xi\Upsilon \mid \Upsilon \mid^{-1} \mid \Upsilon \mid = \Xi\Upsilon$ since $\mid \Upsilon \mid^{-1} \mid \Upsilon \mid = I$
but $\Xi\Upsilon = \Upsilon\Lambda$ by (i) above
so $\Xi U \mid \Upsilon \mid = \Upsilon\Lambda$ and applying $\mid \Upsilon \mid^{-1}$,

$$\Xi U \mid \Upsilon \mid \mid \Upsilon \mid^{-1} = \Upsilon\Lambda \mid \Upsilon \mid^{-1} = \Upsilon \mid \Upsilon \mid^{-1} \Lambda \Rightarrow \Xi U = U\Lambda$$

i.e. we can have a unitary operator $U \in \mathbb{B}(\mathbb{H}_1, \mathbb{H}_2)$ satisfying

$U\Lambda = \Xi U \Rightarrow \Lambda = U^{*}\Xi U$
hence Λ and Ξ are unitarily equivalent. $\square$

3.4 Spectral properties of quasi-similar operators

Remark 3.4.1.
On finite dimensional Hilbert space, quasi-similar operators are also similar. But in infinite dimensional spaces, quasi-similarity is a weak relationship, leading to the operators to have spectra which are not equal in some cases (Sz-Nagy(3)).
However there are some conditions under which two quasi-similar operators on infinite dimensional Hilbert spaces will have equal spectra.

Lemma 3.4.2.
Let $\Xi \in \mathbb{B}(\mathbb{H}_2)$ *be hypo-normal operator and*

$$\{\vartheta_n\}_{n=0}^{\infty}$$

a sequence in $\mathbb{H}_2$ *where* $\Xi\vartheta_{n+1} = \vartheta_n, \forall n \geq 0$
Then either $\| \vartheta_0 \| \geq \| \vartheta_1 \| \geq \| \vartheta_2 \| \geq, \ldots\ldots.$
or $\| \vartheta_n \| \to \infty$ *as* $n \to \infty$

`Proof.` Recall an operator Λ is hypo-normal if $\Lambda^*\Lambda \leq \Lambda\Lambda^*$
For any $\vartheta \in \mathbb{H}_2$ we have

$$\| \Xi\vartheta \| = \langle \Xi\vartheta, \Xi\vartheta \rangle^{\frac{1}{2}} \leq (\| \Xi^*\Xi\vartheta \| \| \vartheta \|)^{\frac{1}{2}}$$

$$\leq (\| \Xi^2\vartheta \| \| \vartheta \|)^{\frac{1}{2}} \leq \frac{1}{2}(\| \Xi^2\vartheta \| + \| \vartheta \|)$$

letting $\vartheta = \vartheta_{n+2}$ we have $\Xi\vartheta = \vartheta_{n+1}$ and $\Xi^2\vartheta = \vartheta_n$ satisfying

$$\| \vartheta_{n+1} \| \leq \frac{1}{2}(\| \vartheta_n \| + \| \vartheta_{n+2} \|)$$

so $\{\|\vartheta_n\|\}$ is convex, hence the result. □

Lemma 3.4.3.
Let $\Lambda \in \mathbb{G}(\mathbb{H}_1)$ *and* $\Xi \in \mathbb{B}(\mathbb{H}_2)$ *be hypo-normal.*
Let $\Upsilon \in \mathbb{B}(\mathbb{H}_1, \mathbb{H}_2)$ *be such that* $\Upsilon\Lambda = \Xi\Upsilon$
then $\| \Upsilon\Lambda^{-1}\varrho \| \leq \| \Lambda^{-1} \| \| \Upsilon\varrho \|$ *for all* $\varrho \in \mathbb{H}_1$

`Proof.` Assume $dim\mathbb{H}_1 \geq 1$ and let $\lambda = \| \Lambda^{-1} \| > 0$
Fix $\varrho \in \mathbb{H}_1$ and define $\vartheta_n = \lambda^{-n}\Upsilon\Lambda^{-n}\varrho$ for $n > 0$
then $\lambda\Xi\vartheta_{n+1} = \vartheta_n$ and $\| \vartheta_n \| \leq \| \Upsilon \| \| \varrho \|$ for all $n > 0$
since $\lambda\Xi$ is hypo-normal then
$\| \vartheta_0 \| \geq \| \vartheta_1 \| \geq \| \vartheta_2 \| \geq$ .... by lemma above.
The first inequality in this chain $\| \vartheta_1 \| \leq \| \vartheta_0 \|$ shows that

$$\| \Upsilon\Lambda^{-1}\varrho \| \leq \| \Lambda^{-1} \| \| \Upsilon\varrho \|$$

□

Theorem 3.4.4.
Let $\Lambda \in \mathbb{G}(\mathbb{H}_1)$ and $\Xi \in \mathbb{B}(\mathbb{H}_2)$ be hypo-normal. Let Λ and Ξ be quasi-similar operators i.e. we can find $\Upsilon \in \mathbb{B}(\mathbb{H}_1, \mathbb{H}_2)$ where $Ker(\Upsilon) = 0$ and $\overline{Ran(\Upsilon)} = \mathbb{H}_2$ and satisfies $\Upsilon\Lambda = \Xi\Upsilon$. Then $\Xi \in \mathbb{G}(\mathbb{H}_2)$.

`Proof.` Assume $dim\mathbb{H}_1 \geq 1$.
Clearly $Ran(\Xi) \supseteq Ran(\Upsilon)$ and so $\overline{Ran(\Xi)} = \mathbb{H}_2$
Now of we set $\varrho_1 = P^{-1}\varrho$ in lemma 3.4.3 we have the inequality

$$\| \Xi(\Upsilon\varrho_1) \| \geq \frac{\| \Upsilon\varrho_1 \|}{\| \Lambda^{-1} \|}$$

i.e.

$$\| \Xi(\Upsilon\varrho_1) \| \geq \frac{1}{\| \Lambda^{-1} \|} \| \Upsilon\varrho_1 \|$$

thus Ξ is bounded below in the range of Υ and hence $\Xi \in \mathbb{G}(\mathbb{H}_2)$ □

Theorem 3.4.5.
Let $\Lambda \in \mathbb{B}(\mathbb{H}_1)$ and $\Xi \in \mathbb{B}(\mathbb{H}_2)$ be quasi-similar hypo-normal operators. Then $\sigma(\Lambda) = \sigma(\Xi)$

`Proof.` If Λ and Ξ hypo-normal with $\Lambda \approx \Xi$ then for any $\lambda \in \mathbb{C}$ we have $\Lambda - \lambda I \approx \Xi - \lambda I$ are also hypo-normal operators.
So by theorem (3.4.4), $\Lambda \in \mathbb{G}(\mathbb{H}_1)$,$\Xi \in \mathbb{G}(\mathbb{H}_2)$ or both are non invertible and therefore the $\sigma(\Lambda) = \sigma(\Xi)$ □

Remark 3.4.6.
Note there is proper inclusion relation of operators i.e.

$$Normal \subset Hypo - normal \subset Paranormal$$

Then in view of theorem above, quasi-similar normal operators have equal spectra since every normal operator is hypo-normal.
Now every hypo-normal operator is paranormal.
So equality of quasi-similar paranormal operators is an area for further research.

4 ALMOST SIMILARITY OF OPERATORS AND THEIR SPECTRAL PROPERTIES

4.1 Cartesian and polar decomposition of operators

Remark 4.1.1.
Recall if $\Lambda, \Xi \in \mathbb{B}(\mathbb{H})$ are self ad-joint , then $\Lambda\Xi$ is self ad-joint iff Λ commutes with Ξ.
Also if $\Phi(\Lambda)$ is a polynomial of a self ad-joint operator Λ with real coefficients then $\Phi(\Lambda)$ is a self ad-joint operator.
These results show that bounded self ad-joint operator may be comparable to generalised real numbers.
Then it must be feasible to look upon any bounded linear operator as a generalised complex number.
An object $\xi \in \mathbb{C}$ has a unique representation $\xi = \varrho + i\vartheta$ with $\varrho, \vartheta \in Upsilon\Upsilon$.
It must then be feasible to express $\Lambda \in \mathbb{B}(\mathbb{H})$ as $\Lambda = \Upsilon + i\Psi$ with $\Upsilon, \Psi \in \mathbb{B}(\mathbb{H})$ being unique and self ad-joint.

Theorem 4.1.2.
For $\Lambda \in \mathbb{B}(\mathbb{H})$ we can find a unique self ad-joint operators $\Upsilon, \Psi \in \mathbb{B}(\mathbb{H})$ to have

$$\Lambda = \Upsilon + i\Psi$$

`Proof.` Since $\Lambda = \Upsilon + i\Psi$ it implies

$$\Lambda^* = (\Upsilon + i\Psi)^* = \Upsilon^* + (i\Psi)^* = \Upsilon^* + \Psi^* i^* = \Upsilon^* - i\Psi^* = \Upsilon - i\Psi$$

Then

$$\Lambda + \Lambda^* = \Upsilon + i\Psi + \Upsilon - i\Psi = 2\Upsilon \Rightarrow \Upsilon = \frac{\Lambda + \Lambda^*}{2}$$

and

$$\Lambda - \Lambda^* = \Upsilon + i\Psi - \Upsilon + i\Psi = 2i\Psi \Rightarrow \Psi = \frac{\Lambda - \Lambda^*}{2i}$$

so Υ and Ψ are uniquely determined.
Thus

$$\Lambda = \Upsilon + i\Psi = \frac{\Lambda + \Lambda^*}{2} + i\frac{\Lambda - \Lambda^*}{2i} = \frac{\Lambda + \Lambda^*}{2} + \frac{\Lambda - \Lambda^*}{2}$$

Now $\Upsilon^* = (\frac{\Lambda+\Lambda^*}{2})^* = \frac{1}{2}(\Lambda + \Lambda^*)^* = \frac{1}{2}(\Lambda^* + \Lambda^{**})$

$$= \frac{1}{2}(\Lambda + \Lambda^*) = \frac{\Lambda + \Lambda^*}{2} = \Upsilon$$

i.e.
$\Upsilon^* = \Upsilon \Rightarrow \Upsilon$ is self ad-joint.
Also $\Psi^* = (\frac{\Lambda-\Lambda^*}{2i})^* = -\frac{1}{2i}(\Lambda - \Lambda^*)^* = -\frac{1}{2i}(\Lambda^* - \Lambda^{**})$

$$= -\frac{1}{2i}(\Lambda^* - \Lambda) = \frac{\Lambda - \Lambda^*}{2i} = \Psi$$

i.e. $\Psi^* = \Psi \Rightarrow \Psi$ is self ad-joint.
therefore Υ and Ψ are unique self ad-joint operators □

Remark 4.1.3.
However if a complex $z = a + ib$ then a and b always commute.
But in the operator theory if $P \in \mathbb{B}(\mathbb{H})$ and $P = R + iS$ where $R, S \in \mathbb{B}(\mathbb{H})$ are self ad-joint operators, then we need not have R commute with S.
But in a special case we may have R commute with S as expressed below.

Theorem 4.1.4.
Let $\Lambda \in \mathbb{B}(\mathbb{H})$ and $\Lambda = \Upsilon + i\Psi$ with $\Upsilon, \Psi \in \mathbb{B}(\mathbb{H})$ as self ad-joint. Then Υ commutes with Ψ iff Λ is a normal.

$$ie\Upsilon \leftrightarrow S \Leftrightarrow \Lambda^*\Lambda = \Lambda\Lambda^*$$

`Proof.` Let Υ commute with Ψ then

$$\Upsilon\Psi = \Psi\Upsilon \Rightarrow (\frac{\Lambda + \Lambda^*}{2})(\frac{\Lambda - \Lambda^*}{2i}) = (\frac{\Lambda - \Lambda^*}{2i})(\frac{\Lambda + \Lambda^*}{2})$$

$$\Rightarrow (\Lambda + \Lambda^*)(\Lambda - \Lambda^*) = (\Lambda - \Lambda^*)(\Lambda + \Lambda^*)$$

$$\Rightarrow \Lambda\Lambda - \Lambda\Lambda^* + \Lambda^*\Lambda - \Lambda^*\Lambda^* = \Lambda\Lambda + \Lambda\Lambda^* - \Lambda^*\Lambda - \Lambda^*\Lambda^*$$

$$\Rightarrow \Lambda^*\Lambda + \Lambda^*\Lambda = \Lambda\Lambda^* + \Lambda\Lambda^*$$
$$\Rightarrow 2\Lambda^*\Lambda = 2\Lambda\Lambda^*$$
$$\Rightarrow \Lambda^*\Lambda = \Lambda\Lambda^*$$

i.e. Λ is a normal operator.
Conversely let Λ be a normal.
Then

$$\Lambda^*\Lambda = \Lambda\Lambda^* \Rightarrow (\Upsilon - i\Psi)(\Upsilon + i\Psi) = (\Upsilon + i\Psi)(\Upsilon - i\Psi)$$

$$\Rightarrow \Upsilon^2 + \Psi^2 + i\Upsilon\Psi - i\Psi\Upsilon = \Upsilon^2 + \Psi^2 + i\Psi\Upsilon - i\Upsilon\Psi$$
$$\Rightarrow i(\Upsilon\Psi - \Psi\Upsilon) = i(\Psi\Upsilon - \Upsilon\Psi)$$
$$\Rightarrow \Upsilon\Psi - \Psi\Upsilon = \Psi\Upsilon - \Upsilon\Psi$$
$$\Rightarrow 2\Upsilon\Psi = 2\Psi\Upsilon$$
$$\Rightarrow \Upsilon\Psi = \Psi\Upsilon$$

ie $\Upsilon\Psi$ commutes with $\Psi\Upsilon$. □

Remark 4.1.5.
$\xi \in \mathbb{C}$ can be decomposed into polar form as $\xi = re^{i\theta}$ where r is the absolute value of ξ and $e^{i\theta}$ is called the complex sign of ξ.
In mathematics the polar decomposition of a linear operator is a factorization analogous to $\xi \in \mathbb{C}$.

Definition 4.1.6.
The polar decomposition of $\Lambda \in \mathbb{B}(\mathbb{H})$ is a canonical factorization $\Lambda = UG$ where U is partialy isometric and G is a positive.

Remark 4.1.7.
The non-negative self ad-joint operator G is such that

$$G = (\Lambda^*\Lambda)^{\frac{1}{2}} \Rightarrow G^2 = \Lambda^*\Lambda$$

while U must be partialy isometric, since if U is unitary and Λ is a one sided shift operator on $\mathbb{L}^2(\mathbb{N})$ then
$(\Lambda^*\Lambda)^{\frac{1}{2}} = I \Rightarrow \Lambda = U(\Lambda^*\Lambda)\frac{1}{2} \Rightarrow U$ must be Λ but not unitary
hence a contradiction.

4.2 Some results on almost similarity of operators

Remark 4.2.1.
Almost similarity is a new class in operator theory and was first introduced by A.A. Jibril (21). He proved various results that relate almost similarity and other classes of operators.

Definition 4.2.2.
Two operators $\Lambda, \Xi \in \mathbb{B}(\mathbb{H})$ are almost similar denoted $\Lambda \overset{a.s}{\sim} \Xi$ if we can find an operator $\Upsilon \in \mathbb{G}(\mathbb{H})$ satisfying

$$\Lambda^*\Lambda = \Upsilon^{-1}(\Xi^*\Xi)\Upsilon$$

$$\Lambda^* + \Lambda = \Upsilon^{-1}(\Xi^* + \Xi)\Upsilon$$

Remark 4.2.3.
The definition of almost similarity above we have made use of both Cartesian and polar decomposition of Λ and Ξ.
Thus if $\Lambda = UG = \Upsilon + i\Psi$ then $G = \sqrt{\Lambda^*\Lambda} \Rightarrow \Lambda^*\Lambda = G^2$ and $\Lambda^* + \Lambda = 2\Upsilon$

Theorem 4.2.4.
Almost similarity of operators is an equivalence relation.

`Proof.` Let $\overset{a.s}{\sim}$ denote almost similarity and let $\Lambda \in \mathbb{B}(\mathbb{H})$.
Let Υ be an identity operator I on $\mathbb{H}$. Then

$$\Upsilon^{-1}(\Lambda^*\Lambda)\Upsilon = I(\Lambda^*\Lambda)I = \Lambda^*\Lambda$$

$$\Upsilon^{-1}(\Lambda^* + \Lambda)\Upsilon = I(\Lambda^* + \Lambda)I = \Lambda^* + \Lambda$$

Thus $\Lambda \overset{a.s}{\sim} \Lambda$ i.e. almost similarity of operators is reflexive.
Let $\Xi \in \mathbb{B}(\mathbb{H})$ and $\Lambda \overset{a.s}{\sim} \Xi$
Then we can find $\Upsilon \in \mathbb{G}(\mathbb{H})$ satisfying

$$\Lambda^*\Lambda = \Upsilon^{-1}(\Xi^*\Xi)\Upsilon$$

$$\Lambda^* + \Lambda = \Upsilon^{-1}(\Xi^* + \Xi)\Upsilon$$

applying Υ and Υ^{-1} we have

$$\Upsilon(\Lambda^*\Lambda)\Upsilon^{-1} = \Upsilon\Upsilon^{-1}(\Xi^*\Xi)\Upsilon\Upsilon^{-1} = \Xi^*\Xi$$

$$\Upsilon(\Lambda^* + \Lambda)\Upsilon^{-1} = \Upsilon\Upsilon^{-1}(\Xi^* + \Xi)\Upsilon\Upsilon^{-1} = \Xi^* + \Xi$$

Let $\Psi = \Upsilon^{-1}$ then $\Psi \in \mathbb{G}(\mathbb{H})$ and $\Psi^{-1} = \Upsilon$ and hence we have

$$\Xi^*\Xi = \Psi^{-1}(\Lambda^*\Lambda)\Psi$$

$$\Xi^* + \Xi = \Psi^{-1}(\Lambda^* + \Lambda)\Psi$$

i.e $\Xi \overset{a.s}{\sim} \Lambda$ and hence almost similarity of operators is symmetric.
Let $\Gamma \in \mathbb{B}(\mathbb{H})$ and let $\Lambda \overset{a.s}{\sim} \Xi$ and $\Xi \overset{a.s}{\sim} \Gamma$.
Then we can find $\Upsilon, \Psi \in \mathbb{G}(\mathbb{H})$ satisfying

$$\Lambda^*\Lambda = \Upsilon^{-1}(\Xi^*\Xi)\Upsilon$$

$$\Lambda^* + \Lambda = \Upsilon^{-1}(\Xi^* + \Xi)\Upsilon$$

and

$$\Xi^*\Xi = \Psi^{-1}(\Gamma^*\Gamma)\Psi$$

$$\Xi^* + \Xi = \Psi^{-1}(\Gamma^* + \Gamma)\Psi$$

This implies

$$\Lambda^*\Lambda = \Upsilon^{-1}\Psi^{-1}(\Gamma^*\Gamma)\Psi\Upsilon$$

$$\Lambda^* + \Lambda = \Upsilon^{-1}\Psi^{-1}(\Gamma^* + \Gamma)\Psi\Upsilon$$

which implies

$$\Lambda^*\Lambda = (\Psi\Upsilon)^{-1}(\Gamma^*\Gamma)\Psi\Upsilon$$

$$\Lambda^* + \Lambda = (\Psi\Upsilon)^{-1}(\Gamma^* + \Gamma)\Psi\Upsilon$$

but $\Psi\Upsilon \in \mathbb{G}(\mathbb{H})$ since it is a composition of operators in $\mathbb{G}(\mathbb{H})$.
thus $\Lambda \overset{a.s}{\sim} \Gamma$ and hence almost similarity of operators is transitive. $\square$

Proposition 4.2.5.
Let $\Lambda, \Xi \in \mathbb{B}(\mathbb{H})$.
Then:

1. *if* $\Lambda \overset{a.s}{\sim} 0$ *then* $\Lambda = 0$

2. *if* $\Lambda \overset{a.s}{\sim} I$ *then* $\Lambda = I$

3. *if* $\Lambda \overset{a.s}{\sim} \Xi$ *and* Ξ *is isometric then* Λ *is also isometric.*

`Proof.` 1. Let $\Lambda \overset{a.s}{\sim} 0$
Then we can find $\Upsilon \in \mathbb{G}(\mathbb{H})$ satisfying

$$\Lambda^*\Lambda = \Upsilon^{-1}0\Upsilon = 0$$

$$\Lambda^* + \Lambda = \Psi^{-1}0\Psi = 0 \Rightarrow \Lambda^* = -\Lambda$$
$$\Rightarrow \Lambda^*\Lambda = -\Lambda^2 = 0 \Rightarrow \Lambda^2 = 0 \Rightarrow \Lambda = 0$$

2. Let $\Lambda \overset{a.s}{\sim} I$
Then we can have $\Upsilon \in \mathbb{G}(\mathbb{H})$ satisfying

$$\Lambda^*\Lambda = \Psi^{-1}I^*I\Psi = \Psi^{-1}I\Psi = I$$

$$\Lambda^* + \Lambda = \Psi^{-1}(I^* + I)\Psi = \Psi^{-1}(I + I)\Psi = 2I$$

Now $\Lambda^* + \Lambda = 2I \Rightarrow \Lambda^*\Lambda + \Lambda\Lambda = 2\Lambda$ i.e. by applying Λ
but $\Lambda^*\Lambda = I \Rightarrow I + \Lambda^2 = 2\Lambda \Rightarrow \Lambda^2 - 2\Lambda + I = 0$
$\Rightarrow (\Lambda - I)(\Lambda - I) = 0$
Let $\varrho \in \mathbb{H}$ then $(\Lambda - I)(\Lambda - I)\varrho = 0$
Let $(\Lambda - I)\varrho = \vartheta$ then $(\Lambda - I)\vartheta = 0 \Rightarrow \Lambda\vartheta = \vartheta$
Consequently $\Lambda\varrho = \varrho \Rightarrow \Lambda = I\varrho$ and hence $\Lambda = I$

3. Let $\Lambda \overset{a.s}{\sim} \Xi$ then we can find $\Upsilon \in \mathbb{G}(\mathbb{H})$ satisfying

$$\Lambda^*\Lambda = \Upsilon^{-1}(\Xi^*\Xi)\Upsilon$$

$$\Lambda^* + \Lambda = \Upsilon^{-1}(\Xi^* + \Xi)\Upsilon$$

Let Ξ be an isometry then $\Xi^*\Xi = I \Rightarrow \Lambda^*\Lambda = \Upsilon^{-1}I\Upsilon = I$
i.e. $\Lambda^*\Lambda = I \Rightarrow \Lambda$ is an isometry.

□

Definition 4.2.6.
$\Lambda \in \mathbb{B}(\mathbb{H})$ *is called a* $\theta - operator$ *if* $\Lambda^* + \Lambda$ *commutes with* $\Lambda^*\Lambda$

Remark 4.2.7.
Note, the set of $\theta - operators$ *in* $\mathbb{B}(\mathbb{H})$ *is denoted* $\theta(\mathbb{H})$
so

$$\theta(\mathbb{H}) = \{\Lambda \in \mathbb{B}(\mathbb{H}) : [\Lambda^*\Lambda, \Lambda^* + \Lambda] = 0\}$$

Proposition 4.2.8.
If $\Lambda, \Xi \in \mathbb{B}(\mathbb{H})$ *with* $\Lambda \overset{a.s}{\sim} \Xi$ *and* $\Xi \in \theta(\mathbb{H})$ *then* $\Lambda \in \theta(\mathbb{H})$

Proof. Since $\Lambda \overset{a.s}{\sim} \Xi$ we can find $\Upsilon \in \mathbb{G}(\mathbb{H})$ satisfying

$$\Lambda^*\Lambda = \Upsilon^{-1}(\Xi^*\Xi)\Upsilon$$

$$\Lambda^* + \Lambda = \Upsilon^{-1}(\Xi^* + \Xi)\Upsilon$$

so we have
$(\Lambda^*\Lambda)(\Lambda^* + \Lambda) = [\Upsilon^{-1}(\Xi^*\Xi)\Upsilon][\Upsilon^{-1}(\Xi^* + \Xi)\Upsilon]$
$= \Upsilon^{-1}[(\Xi^*\Xi)(\Xi^* + \Xi)]\Upsilon$............$(i)$
we also have
$(\Lambda^* + \Lambda)(\Lambda^*\Lambda) = [\Upsilon^{-1}(\Xi^* + \Xi)\Upsilon][\Upsilon^{-1}(\Xi^*\Xi)\Upsilon]$
$= \Upsilon^{-1}[(\Xi^* + \Xi)(\Xi^*\Xi)]\Upsilon$..........$(ii)$
but $\Xi \in \theta \Rightarrow (\Xi^*\Xi)(\Xi^* + \Xi) = (\Xi^* + \Xi)(\Xi^*\Xi)$
so from (i) and (ii)

$$(\Lambda^*\Lambda)(\Lambda^* + \Lambda) = (\Lambda^* + \Lambda)(\Lambda^*\Lambda)$$

i.e. $\Lambda^*\Lambda$ commutes with $\Lambda^* + \Lambda$ and hence $\Lambda \in \theta(\mathbb{H})$ $\square$

Theorem 4.2.9.
Let $\Lambda \in \mathbb{B}(\mathbb{H})$, *then* Λ *is Hermitian iff* $(\Lambda^* + \Lambda)^2 \geq 4\Lambda^*\Lambda$

Proof. we proof for the case where equality sign holds i.e Λ is Hermitian iff $(\Lambda + \Lambda^*)^2 = 4\Lambda^*\Lambda$
Suppose Λ is hermitian
Then

$$(\Lambda + \Lambda^*)^2 = (\Lambda + \Lambda)^2 = (2\Lambda)^2 = 4\Lambda^2$$
$$4\Lambda^*\Lambda = 4\Lambda\Lambda = 4\Lambda^2$$

So $(\Lambda + \Lambda^*)^2 = 4\Lambda^*\Lambda$

Conversely suppose $(\Lambda+\Lambda^*)^2 = 4\Lambda^*\Lambda$
and let $\Lambda = \Upsilon + i\Psi$ be the Cartesian decomposition of Λ
Then

$$(\Lambda+\Lambda^*)^2 = (\Upsilon + i\Psi + \Upsilon - i\Psi)^2 = (2\Upsilon)^2 = 4\Upsilon^2$$

$$4\Lambda^*\Lambda = 4[(\Upsilon - i\Psi)(\Upsilon + i\Psi)] = 4[\Upsilon^2 + \Psi^2 + i(\Upsilon\Psi - \Psi\Upsilon)] = 4\Upsilon^2 + 4\Psi^2$$

So $4\Upsilon^2 = 4\Upsilon^2 + 4\Psi^2 \Rightarrow 4\Psi^2 = 0 \Rightarrow \Psi^2 = 0 \Rightarrow \Psi = 0$
so that $\Upsilon = \Upsilon + i\Psi \Rightarrow \Lambda = \Upsilon + 0 \Rightarrow \Lambda = \Upsilon$
Now since Υ is Hermitian it follows Λ is Hermitian. □

Proposition 4.2.10.
If $\Lambda, \Xi \in \mathbb{B}(\mathbb{H})$ such that $\Lambda \overset{a.s}{\sim} \Xi$ and Ξ is Hermitian, then Λ is Hermitian.

`Proof.` Since $\Lambda \overset{a.s}{\sim} \Xi$, we can have $\Upsilon \in \mathbb{G}(\mathbb{H})$ so that

$$\Lambda^*\Lambda = \Upsilon^{-1}(\Xi^*\Xi)\Upsilon \Rightarrow 4\Lambda^*\Lambda = \Upsilon^{-1}(4\Xi^*\Xi)R............(i)$$

$$\Lambda^* + \Lambda = \Upsilon^{-1}(\Xi^* + \Xi)\Upsilon \Rightarrow (\Lambda+\Lambda^*)^2 = [\Upsilon^{-1}(\Xi+\Xi^*)\Upsilon][\Upsilon^{-1}(\Xi+\Xi^*)\Upsilon]$$
$$\Rightarrow (\Lambda+\Lambda^*)^2 = \Upsilon^{-1}(\Xi+\Xi^*)^2\Upsilon..............(ii)$$

Since Ξ is Hermitian then $(\Xi+\Xi^*)^2 = 4\Xi^*\Xi$ and then (ii) becomes

$$(\Lambda+\Lambda^*)^2 = \Upsilon^{-1}4\Xi^*\Xi\Upsilon..................(iii)$$

but from (i) we have $\Upsilon^{-1}(4\Xi^*\Xi)\Upsilon = 4\Lambda^*\Lambda$ then (iii) becomes
$(\Lambda+\Lambda^*)^2 = 4\Lambda\Lambda^*$ i.e. Λ is Hermitian. □

Remark 4.2.11.
Recall if $\Lambda \in \mathbb{B}(\mathbb{H})$ and Λ is an isometry then $\Lambda^*\Lambda = I$

Definition 4.2.12.
$\Lambda \in \mathbb{B}(\mathbb{H})$ *is a partial isometry if* $\Lambda\Lambda^*\Lambda = \Lambda$
i.e. if $\Lambda^*\Lambda\Lambda^*\Lambda = \Lambda^*\Lambda \Rightarrow (\Lambda^*\Lambda)^2 = \Lambda^*\Lambda$
and $(\Lambda^*\Lambda)^* = \Lambda^*\Lambda$ *implying* $\Lambda^*\Lambda$ *is a projection.*

Proposition 4.2.13.
Let $\Lambda, \Xi \in \mathbb{B}(\mathbb{H})$ and $\Lambda \overset{a.s}{\sim} \Xi$ with Λ partially isometric. Then Ξ is also a partial isometry.

`Proof.` If $\Lambda \overset{a.s}{\sim} \Xi$ then we can have $\Upsilon \in \mathbb{G}(\mathbb{H})$ so that

$$\Lambda^*\Lambda = \Upsilon^{-1}\Xi^*\Xi\Upsilon.................(i)$$

So

$$(\Lambda^*\Lambda)^2 = [\Upsilon^{-1}(\Xi^*\Xi)\Upsilon][\Upsilon^{-1}(\Xi^*\Xi)\Upsilon] = \Upsilon^{-1}(\Xi^*\Xi)^2\Upsilon.......(ii)$$

Now $(\Lambda^*\Lambda)^2 = \Lambda^*\Lambda$ and from (i) and (ii) we have

$$\Upsilon^{-1}\Xi^*\Xi\Upsilon = \Upsilon^{-1}(\Xi^*\Xi)^2\Upsilon \Rightarrow (\Xi^*\Xi)^2 = \Xi^*\Xi...........(iii)$$

Also $(\Xi^*\Xi)^* = \Xi^*\Xi^{**} = \Xi^*\Xi...........(iv)$ i.e.
(iii) and (iv) imply that $\Xi^*\Xi$ is a projection and hence Ξ is partially isometric. □

Proposition 4.2.14.
Let $\Lambda \in \mathbb{B}(\mathbb{H})$ and $\Lambda \overset{a.s}{\sim} \Xi$. If Λ is a projection, then Ξ also.

`Proof.` If $\Lambda \overset{a.s}{\sim} \Xi$ we can find $\Upsilon \in \mathbb{G}(\mathbb{H})$ so that

$$\Lambda^*\Lambda = \Upsilon^{-1}\Xi^*\Xi\Upsilon.........(i)$$

$$\Lambda^* + \Lambda = \Upsilon^{-1}(\Xi^* + \Xi)\Upsilon..........(ii)$$

Since Λ is a projection then Λ is Hermitian thus $\Lambda^* = \Lambda$ and by proposition 4.2.10, Ξ is also Hermitian thus $\Xi^* = \Xi$
Now from i

$$\Lambda^*\Lambda = \Lambda\Lambda = \Lambda^2 = \Upsilon^{-1}\Xi^*\Xi\Upsilon = \Upsilon^{-1}\Xi\Xi\Upsilon = \Upsilon^{-1}\Xi^2\Upsilon$$

i.e. $\Lambda^2 = \Upsilon^{-1}\Xi^2\Upsilon$
but Λ is a projection implying that $\Lambda^2 = \Lambda$ so that

$$\Lambda = \Upsilon^{-1}\Xi^2\Upsilon................(iii)$$

Also from ii

$$\Lambda^* + \Lambda = \Lambda + \Lambda = 2\Lambda = \Upsilon^{-1}(\Xi^* + \Xi)\Upsilon = \Upsilon^{-1}(\Xi + \Xi)\Upsilon = \Upsilon^{-1}2\Xi\Upsilon$$

i.e. $2\Lambda = \Upsilon^{-1}2\Xi\Upsilon \Rightarrow \Lambda = \Upsilon^{-1}\Xi\Upsilon..........(iv)$
From iii and iv, $\Upsilon^{-1}\Xi\Upsilon = \Upsilon^{-1}\Xi^2\Upsilon \Rightarrow \Xi^2 = \Xi$ and since Ξ is Hermitian,then Ξ is a projection. $\square$

Proposition 4.2.15.
Let $\Lambda, \Xi \in \mathbb{B}(\mathbb{H})$. If Λ and Ξ are unitarily equivalent, then $\Lambda \overset{a.s}{\sim} \Xi$

`Proof.` Let Λ and Ξ be unitarily equivalent, then

$$\Lambda = U^*\Xi U \Rightarrow \Lambda^* = (U^*\Xi U)^* = U^*\Xi^* U$$

So

$$\Lambda^*\Lambda = (U^*\Xi^* U)(U^*\Xi U) = U^*\Xi^*\Xi U = U^{-1}\Xi^*\Xi U$$

i.e. we can find $U \in \mathbb{G}(\mathbb{H})$ satisfying

$$\Lambda^*\Lambda = U^{-1}\Xi^*\Xi U..............(i)$$

Also

$$\Lambda^* + \Lambda = U^*\Xi^* U + U^*\Xi U = U^*(\Xi^* + \Xi)U = U^{-1}(\Xi^* + \Xi)U$$

ie there is $U \in \mathbb{G}(\mathbb{H})$ which satisfies

$$\Lambda^* + \Lambda = U^{-1}(\Xi^* + \Xi)U..............(ii)$$

The results (i) and (ii) imply that $\Lambda \overset{a.s}{\sim} \Xi$ $\square$

Remark 4.2.16.
The following proposition gives the condition under which quasi-similarity implies almost similarity.

Proposition 4.2.17.
Let $\Lambda, \Xi \in \mathbb{B}(\mathbb{H})$ and $\Lambda \approx \Xi$ with equal unitary quasi-affinities and that $\mathbb{H}$ has finite dimension. Then $\Lambda \overset{a.s}{\sim} \Xi$

`Proof.` Let $\Lambda \approx \Xi$, then we can have two quasi-affinities $\Upsilon, \Psi \in \mathbb{B}(\mathbb{H})$ which satisfy

$$\Upsilon\Lambda = \Xi\Upsilon$$

$$\Psi\Xi = \Lambda\Psi$$

Assume Υ and Ψ are unitary and $\Upsilon = \Psi$
then $\Upsilon^*\Upsilon = \Upsilon\Upsilon^* = I \Rightarrow \Upsilon^* = \Upsilon^{-1}$
but $\Upsilon\Lambda = \Xi\Upsilon \Rightarrow \Lambda = \Upsilon^{-1}\Xi\Upsilon = \Upsilon^*\Xi\Upsilon \Rightarrow \Lambda^* = (\Upsilon^*\Xi\Upsilon)^* = \Upsilon^*\Xi^*\Upsilon$
So $\Lambda^*\Lambda = (\Upsilon^*\Xi^*\Upsilon)\Upsilon^*\Xi\Upsilon = \Upsilon^*\Xi^*\Xi\Upsilon = \Upsilon^{-1}\Xi^*\Xi\Upsilon ....(i)$
and $\Lambda^* + \Lambda = \Upsilon^*\Xi^*\Upsilon + \Upsilon^*\Xi\Upsilon = \Upsilon^*(\Xi^* + \Xi)\Upsilon$
$= \Upsilon^{-1}(\Xi^* + \Xi)\Upsilon ......(ii)$
The results (i) and (ii) imply that $\Lambda \overset{a.s}{\sim} \Xi$ □

Proposition 4.2.18.
Let $\Lambda, \Xi \in \mathbb{B}(\mathbb{H})$ and $\Lambda \overset{a.s}{\sim} \Xi$. If Λ is Hermitian, then $\Lambda \sim \Xi$.

`Proof.` If $\Lambda \overset{a.s}{\sim} \Xi$, we can find $\Upsilon \in \mathbb{G}(\mathbb{H})$ satisfying

$$\Lambda^* + \Lambda = \Upsilon^{-1}(\Xi^* + \Xi)\Upsilon$$

Since Λ is Hermitian then Ξ is Hermitian by proposition $4.2.10$
Thus

$$\Lambda + \Lambda = \Upsilon^{-1}(\Xi + \Xi)\Upsilon \Rightarrow 2\Lambda = \Upsilon^{-1}2\Xi\Upsilon \Rightarrow \Lambda = \Upsilon^{-1}\Xi\Upsilon$$

Hence we have $\Lambda \sim \Xi$ □

Remark 4.2.19.
Note that in the above proposition, since Λ and Ξ are Hermitian then both Λ and Ξ are normal and so $\Lambda \cong \Xi$.

Proposition 4.2.20.
Let $\Lambda \in \mathbb{B}(\mathbb{H})$, if Λ is normal then $\Lambda \overset{a.s}{\sim} \Lambda^*$

`Proof.` Since Λ is normal then $\Lambda^*\Lambda = \Lambda\Lambda^*$
So $\Lambda^*\Lambda = I^{-1}\Lambda\Lambda^* I$

but $\Lambda\Lambda^* = (\Lambda^*)^*\Lambda^*$ thus $\Lambda^*\Lambda = I^{-1}(\Lambda^*)^*\Lambda^* I$.............$(i)$
Now $\Lambda^* + \Lambda = \Lambda + \Lambda^* = (\Lambda^*)^* + \Lambda^*$

$$\Rightarrow \Lambda^* + \Lambda = I^{-1}(\Lambda^*)^* + \Lambda^* I\text{............}(ii)$$

The results (i) and (ii) imply that $\Lambda \overset{a.s}{\sim} \Lambda^*$ $\square$

Proposition 4.2.21.
Let $U \in \mathbb{B}(\mathbb{H})$, if U is a unitary, then $\Lambda \in \mathbb{B}(\mathbb{H})$ is isometric iff $\Lambda \overset{a.s}{\sim} U$

`Proof.` Suppose $\Lambda \overset{a.s}{\sim} U$, then there is $\Upsilon \in \mathbb{G}(\mathbb{H})$ which satisfy

$$\Lambda^*\Lambda = \Upsilon^{-1}U^*U\Upsilon = \Upsilon^{-1}I\Upsilon = I$$

i.e. $\Lambda^*\Lambda = I$ thus Λ is isometric.
Now let Λ be isometric then $\Lambda \in \theta$
thus we can find $\Upsilon \in \mathbb{G}(\mathbb{H})$ with $\Lambda \overset{a.s}{\sim} \Upsilon$ implying Υ is isometric by proposition 4.2.5, thus Υ is unitary and hence $\Lambda \overset{a.s}{\sim} U$ $\square$

4.3 Spectral properties of almost similar operators

Proposition 4.3.1.
Let $\Lambda, \Xi \in \mathbb{B}(\mathbb{H})$ and $\Lambda \overset{a.s}{\sim} \Xi$. Then $(\Lambda + \lambda I) \overset{a.s}{\sim} (\Xi + \lambda I)$ for all real λ

Proof. If $\Lambda \overset{a.s}{\sim} \Xi$, we can find $\Upsilon \in \mathbb{G}(\mathbb{H})$ satisfying

$$\Lambda^*\Lambda = \Upsilon^{-1}\Xi^*\Xi\Upsilon..................(i)$$

$$\Lambda^* + \Lambda = \Upsilon^{-1}(\Xi^* + \Xi)\Upsilon.................(ii)$$

Now from (ii) we have

$$\Lambda^* + \Lambda = \Upsilon^{-1}\Xi^*\Upsilon + \Upsilon^{-1}\Xi\Upsilon$$

So that

$$\Lambda^* + \Lambda + 2\lambda = \Upsilon^{-1}\Xi^*\Upsilon + \Upsilon^{-1}\Xi\Upsilon + \Upsilon^{-1}2\lambda\Upsilon = \Upsilon^{-1}(\Xi^* + \Xi + 2\lambda)\Upsilon$$

$$\Rightarrow (\Lambda^* + \lambda I) + (\Lambda + \lambda I) = \Upsilon^{-1}[(\Xi^* + \lambda I) + (\Xi + \lambda I)]\Upsilon$$
$$\Rightarrow (\Lambda + \lambda I)^* + (\Lambda + \lambda I) = \Upsilon^{-1}[(\Xi + \lambda I)^* + (\Xi + \lambda I)]\Upsilon..........(iii)$$

So that

$$\lambda\Lambda^* + \lambda\Lambda + \lambda^2 = \Upsilon^{-1}\lambda\Xi^*\Upsilon + \Upsilon^{-1}\lambda\Xi\Upsilon + \Upsilon^{-1}\lambda^2\Upsilon............(iv)$$

Thus adding (i) and (iv) we have

$$\Lambda^*\Lambda + \lambda\Lambda^* + \lambda\Lambda + \lambda^2 = \Upsilon^{-1}\Xi^*\Xi\Upsilon + \Upsilon^{-1}\lambda\Xi^*\Upsilon + \Upsilon^{-1}\lambda\Xi\Upsilon + \Upsilon^{-1}\lambda^2\Upsilon$$

$$\Rightarrow \Lambda^*\Lambda + \lambda\Lambda^* + \lambda\Lambda + \lambda^2 = \Upsilon^{-1}(\Xi^*\Xi + \lambda\Xi^* + \lambda\Xi + \lambda^2)\Upsilon$$
$$\Rightarrow (\Lambda^* + \lambda I)(\Lambda + \lambda I) = \Upsilon^{-1}(\Xi^* + \lambda I)(\Xi + \lambda I)\Upsilon.............(v)$$

but $(\Lambda + \lambda I)^* = \Lambda^* + \lambda I$ and $(\Xi + \lambda I)^* = \Xi^* + \lambda I$ for $\lambda \in \mathbb{R}$
Thus (v) becomes

$$(\Lambda + \lambda I)^*(\Lambda + \lambda I) = \Upsilon^{-1}(\Xi + \lambda I)^*(\Xi + \lambda I)\Upsilon...........(vi)$$

The results (iii) and (vi) imply that $(\Lambda + \lambda I) \overset{a.s}{\sim} (\Xi + \lambda I)$ □

Remark 4.3.2.
The following corollary gives the conditions under which the spectra of almost similar operators are equal.

Corollary 4.3.3.
Let $\Lambda, \Xi \in \mathbb{B}(\mathbb{H})$ and $\Lambda \overset{a.s}{\sim} \Xi$ such that $(\Lambda + \lambda I) \overset{a.s}{\sim} (\Xi + \lambda I)$ for all real λ. If Λ and Ξ are Hermitian or projections then

$$\sigma(\Lambda) = \sigma(\Xi)$$

Proof. Since $\Lambda \overset{a.s}{\sim} \Xi$ then we can find $\Upsilon \in \mathbb{G}(\mathbb{H})$ which satisfy

$$\Lambda^*\Lambda = \Upsilon^{-1}\Xi^*\Xi\Upsilon................(i)$$

$$\Lambda^* + \Lambda = \Upsilon^{-1}(\Xi^* + \Xi)\Upsilon.................(ii)$$

If Λ and Ξ are Hermitian, i.e. $\Lambda^* = \Lambda$ and $\Xi^* = \Xi$ then ii becomes

$$\Lambda + \Lambda = \Upsilon^{-1}(\Xi + \Xi)R \Rightarrow 2\Lambda = \Upsilon^{-1}2\Xi\Upsilon$$

$$\Rightarrow \Lambda = \Upsilon^{-1}\Xi\Upsilon$$

i.e. $\Lambda \sim \Xi$ and therefore

$$\sigma(\Lambda) = \sigma(\Xi)$$

Similarly if Λ and Ξ are projections then $\Lambda^2 = \Lambda = \Lambda^*$ and $\Xi^2 = \Xi = \Xi^*$ so (i) becomes

$$\Lambda\Lambda = \Upsilon^{-1}\Xi\Xi\Upsilon \Rightarrow \Lambda^2 = \Upsilon^{-1}\Xi^2\Upsilon \Rightarrow \Lambda = \Upsilon^{-1}\Xi\Upsilon$$

i.e. $\Lambda \sim \Xi$ and hence

$$\sigma(\Lambda) = \sigma(\Xi)$$

□

5 CONCLUSION AND RECOMMENDATION

5.1 Conclusion

In this project we have seen that a Hilbert space is a Banach space $\mathbb{H}$ where the norm on objects in $\mathbb{H}$ is induced by the $I.P$ function.
The set of all operators in $\mathbb{B}(\mathbb{H})$ is a Banach space.

$\sigma(\Lambda)$ is the $compliment$ of the $\rho(\Lambda)$ and subset of $\mathbb{C}$ i.e $\sigma(\Lambda) = \{\rho(\Lambda)\}^C \subseteq \mathbb{C}$.
If $\Lambda \in \mathbb{B}(\mathbb{H})$ the $\sigma(\Lambda) = \sigma_P(\Lambda) \cup \sigma_C(\Lambda) \cup \sigma_R(\Lambda)$ but if $\mathbb{H}$ is finite dimensional we have $\sigma(\Lambda) = \sigma_P(\Lambda)$ since $\sigma_C(\Lambda)$ and $\sigma_R(\Lambda)$ are empty sets.
To $\Lambda \in \mathbb{B}(\mathbb{H})$, the $\pi(\Lambda) \subseteq \sigma(\Lambda)$ but if Λ is normal, $\pi(\Lambda) = \sigma(\Lambda)$.
In general $\sigma_P(\Lambda) \cup \sigma_C(\Lambda) \subseteq \pi(\Lambda)$.

$Similarity$ of operators is an $equivalence\ relation$ and similar operators have equal spectra.
$\Upsilon \in \mathbb{B}(\mathbb{H})$ is invertible if it is left or right invertible i.e. if $\Upsilon\Psi = \Psi\Upsilon = I, \forall \Psi \in \mathbb{B}(\mathbb{H})$. Therefore a unitary operator U is invertible since $U^*U = UU^* = I$.
Unitary equivalent operators are also similar operators, i.e. Unitary equivalence $\Rightarrow$ similarity.

If Υ and Ψ are quasi-affinities, then there composites, inverses and there ad-joints are also quasi-affinities.
Quasi-similarity is an equivalence relation and the spectra of quasi-similar hypo-normal operators are equal since quasi-similar hypo-normal operators are similar.
$Normal$ operators are also $hypo-normal$ and therefore the spectra of quasi-similar normal operators are also equal.
Two operators which are similar are also quasi-similar, so that we have a chain of implication: $unitary\ equivalence\ \Rightarrow\ similarity \Rightarrow\ quasi-similarity$.

Almost similarity of operators is an equivalence relation and $Hermitian$ almost similar operators have equal spectra since they are similar.
Also two projections which are almost similar have $equal\ spectra$.
Two $quasi - similar$ operators $\Lambda, \Xi \in \mathbb{B}(\mathbb{H})$ are also almost similar if they have equal unitary quasi-affinities and $\mathbb{H}$ has finite dimension.
A normal operator is almost similar to its ad-joint operator.

5.2 Recommendations for further research

1. We have seen that quasi-similar operators $\Lambda, \Xi \in \mathbb{B}(\mathbb{H})$ are almost similar if they have equal unitary quasi-affinities and $\mathbb{H}$ is finite dimensional.

   - We need to investigate whether there are other conditions under which quasi-similar operators can be almost similar.
   - We also need to find out the conditions under which we can have the converse.

2. If an operator Λ is normal then we have the implication: $\Lambda \, is \, normal \Rightarrow \Lambda \, is \, quasi - normal \Rightarrow \Lambda \, is \, hypo - normal \Rightarrow \Lambda \, is \, paranormal$
   We have seen that quasi-similar hypo-normal operators have equal spectra.
   We have similar results for quasi-similar normal operators.

   - We need to investigate whether there are some conditions under which paranormal operators can have equal spectra.

3. Almost similar operators have equal spectra if they are Hermitian or projections.

   - We need to research the behaviour of subsets of their spectra.

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