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Effects of Mortality and Longevity Risk in Risk Management in Life Insurance Companies

Research Report in Mathematics, Number 46, 2019

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November 2019

Submitted to the School of Mathematics in partial fulfillment for a degree in Master of Science in Actuarial Science

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Mugi Irene Wanjiku

School of Mathematics
College of Biological and Physical sciences
Chiromo, off Riverside Drive
30197-00100 Nairobi, Kenya

Master of Science Project

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Prepared for The Director
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Abstract

Mortality risk posses a huge risk in insurance companies.It is therefore crucial to continuously carry out research relating to this risk. In this paper mortality risk components systematic, unsystematic,adverse selection and basis risk which results from hedging longevity risk are studied broadly. The effects of this risk's components in risk managements in life insurance company is demonstrated.

An insurance company with a portfolio of annuity and term life assurance is used in the study. U.S.A data has been used to carry out various analysis, it is obtained from Human Mortality Database(HMD) and projected through demographic package available in R programming language.The projections and Actuarial Present Values (APV) are computed using the extension of the Lee-Carter (1992) model proposed by Brouhns, Denuit, and Vermunt (2002a).Lee-Carter (1992) propose to fit an appropriate Auto Regressive Integrated Moving Average(ARIMA) process on the estimated time series of the time index.The projections are done using ARIMA (0, 1,0) model(random walk with a drift). Various risk management tools are widely discussed. This includes the transfer of mortality risk to capital market through Mortality Contingent Bond(MCB) and natural hedging.

The default risk measurement tools which comprise of Probability of default which measures the frequency of default and the Expected Mean loss which reflects the amount by which the assets are not sufficient to cover liabilities(average amount of money necessary for funding a case of default during the contract term)are discussed. Probability of default and the MCB are used to display the effects of mortality risk in risk management. From this study, it is evident that the various mortality risk components have an effect in pricing and valuation if they are misestimated.

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Declaration and Approval

I the undersigned declare that this project report is my original work and to the best of my knowledge, it has not been submitted in support of an award of a degree in any other university or institution of learning.

Signature   Date

IRENE WANJIKU MUGI
Reg No. I56/89026/2016

In my capacity as a supervisor of the candidate, I certify that this report has my approval for submission.

Signature   Date

Prof. Patrick Guge O. Weke
School of Mathematics,
University of Nairobi,
Box 30197, 00100 Nairobi, Kenya.
E-mail: pweke@uonbi.ac.ke

Dedication

I would like to thank God for this far.

This project is dedicated to my family especially my parents Joe & Joyce Kamau and Victor Mosoti who have been very supportive and stood by me.
I would also wish to dedicate this work to my siblings Kamau Mugi,Grace Mugi,Njau Mugi,my nephew Delvin Mugi and friends.

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First, I thank God for giving me knowledge and patience to carry out this research. ....

My family,lecturers, friends and relatives, your support is highly appreciated, i will forever be indented.

A special thank you to my lecturer and supervisor Pro. Patrick Weke for guiding me through the entire project.Your assistance is highly appreciated.

IRENE WANJIKU MUGI

Nairobi, 2019.

Contents

1 INTRODUCTION

1.1 Background to the study

Mortality risk is a key risk factor for life insurance companies and can have a crucial impact on its risk situation. It can be divided into different subcategories, among them unsystematic risk, adverse selection, systematic risk, anti-selection risk, basis risk may arise in case of hedging for example longevity risk
Mortality risk refers to a risk that an insurance company can suffer financially because too many of their life insurance policyholders die before their expected lifespans.
A mortality risk may arise from selective withdrawals. The policy holders highly likely to withdraw from the contract are those in good health,leaving the insurance company with a sub-standard group of lives. The company can try to allow for this in the premiums it charges and in the terms it offers to withdrawing policyholders, but it cannot completely eliminate the risk. Therefore the need to continuously study ways of coping with this risk

Where a life insurance company has mortality risk under a contract, it will obtain evidence about the health of the applicant so as to assess whether he or she attains the company's required standard of health and, if not, what their state of health is relative to that standard.
Anti-selection risk is associated with Mortality risk. This is the risk that those who take out the policy are the ones who expect to have heavier than average mortality. Without underwriting, the insurance company would suffer earlier claims, on average, resulting in lower profits.
The extent of this risk will depend on the extent of the actual, or perceived, choice the policyholder had in effecting the contract. Anti-selection can involve, for example, an applicant for insurance using knowledge he or she has about their own state of health to gain favorable terms from an insurer.
Longevity risk is a major risk associated with annuities, mostly with regard to understating the rate of improvement of life expectancy.The insurance company will make a loss if annuitants live longer on average than the life insurance company has allowed for in pricing.
Mortality risk is therefore high for term assurances and longevity risk for annuities.
A longevity risk is any potential risk attached to the increasing life expectancy of pensioners and policy holders, which can eventually result in higher pay-out ratios than expected for many pension funds and insurance companies
Anti-selection risk is associated with longevity risk just like in Mortality risk, the extent of which will depend on the extent of free choice available to the policyholder regarding the purchase of the contract.
For contracts where there is only a longevity risk, evidence of health could also be obtained,

if the life insurance company intends to offer different terms according to the health of the applicant.
If a significant benefit is payable on death then the risk is of higher death rates than expected thus mortality. If no benefit is given on death then the risk is of longevity.
There are different ways of mitigating mortality risk which include;
Capital market solution which involves transferring the risk to the capital market through mortality securitization survivor bonds and swaps.
Natural hedging through product mix.
Mortality projection aimed at providing accurate estimate of mortality processes
Longevity and mortality risk can be hedged from the liability or asset side, the latter may be more flexible and cost-effective.

The company will be required to study the experience of its related contracts in order to minimize the risk of modelling or parameterizing mortality risks incorrectly. Term insurance and whole life insurance should be a good start point, and endowment assurance should be closer than it would be with a standard term insurance contract because of the expected Socio-economic profile of the new policyholders.

Reinsurers' and industry statistics will also be a useful point of reference. Pricing risk on the wrong mortality basis could be reduced by offering the contract as unit-linked with a variable mortality charge, rather than conventional with a guaranteed mortality charge.
The company should ensure that atleast all of the rating factors used by competition such as age,sex,smoker/non-smoker,medical etc are taken into consideration.
It should also ensure that the policyholders undergo underwriting at least as strict as that enjoyed by the policyholders underlying the mortality investigations above. Alternatively the mortality advantage of belonging to the socio-economic group might be deemed sufficient to compensate for a weakening of the underwriting procedures,for instance;a raising of the sums assured at which detailed medical evidence is required.

The company could use individual surplus reinsurance in order to deal with risk of random fluctuations. The extent of reinsurance required will depend on the company's total portfolio of similar policies,the extent to which smooth financial results are desirable which is more important for a proprietary than for a mutual company and its free reserves. An alternative to this would be to set up a mortality fluctuations reserve, which would require a certain amount of capital.
There different ways of mitigating longevity risk which include; Longevity Risk Transfer For many institutions, the need for relief from liabilities exposed to longevity risk has created an emerging market with innovative market-based risk transfer solutions. Longevity risk transfer mechanisms are divided into a longevity swap, a buy-out and a buy-in. Se-

curities such as longevity bonds and indexes Insurers provide a wide range of products designed to help them manage the risk that they will outlive their assets. Those without defined benefit plans can ensure lifetime income by purchasing annuities within their defined contribution plans and personal retirement accounts.A single premium immediate annuity can also be purchased by taking a full or partial distribution defined contribution plan upon retirement or through other lump sum savings.
Insurers also provide products to accommodate the growing demand for lifetime income.Most of this innovation came from adding variable annuity living benefit riders, such as guaranteed minimum income benefits and guaranteed lifetime withdrawal benefits. These products have the advantage of providing income protection and investment flexibility. As a way to isolate the longevity risk protection,contingent deferred annuities (CDAs) were introduced to the market in 2008. This product benefits are similar to variable annuities with guaranteed lifetime withdrawal benefits as they provide protection against outliving ones assets.

1.2 Statement of the problem.

Mortality risk is a key risk factor for life insurance companies and can have a crucial impact on its risk situation. It can be divided into different subcategories, among them unsystematic risk, adverse selection, systematic risk, anti-selection risk, basis risk may arise in case of hedging, e.g.longevity risk
Mortality risk can result from pricing due to misleading information obtained from the client It may also arise from selective withdrawals in that the policyholders most likely to withdraw from the contract are those in good health, leaving sub-standard group of lives in the insurance company. The company can try to allow for this in the premiums charged and terms it offered to withdrawing policyholders, but it cannot completely eliminate the anti-selection risk associated with mortality risk.
Longevity risk which is basis risk that may arise from hedging mortality risk. This is any potential risk attached to the increasing life expectancy of pensioners and policy holders, which can eventually result in higher pay-out ratios than expected for pension funds and insurance companies
Annuities are associated with longevity risk, particularly with regard to understating the rate of improvement of life expectancy. If annuitants live longer on average than the life insurance company has allowed for in its pricing, then the insurance company will make a loss. Mortality risk is therefore high for term assurances and longevity risk for annuities. According to Association of Kenya insurers (AKI) 2015 annual report 28.36 percent of the total incurred claims accounted for ordinary life claims which are majorly as a result of mortality based risks. 40.34 percent accounts for pension claims as a result of longevity risks. It is therefore prudent to study Mortality risk and Longevity risk together as they are both a major risk factor in insurance companies and adversely affect the cash flows.

1.3 Objective

1.3.1 General Objective

The overall objective of the study is to forecast mortality risk, explore and analyze the impact of different types of mortality risk in risk management in life insurance companies in Kenya. Our focus will be on insurer holding a portfolio of annuities and term life insurance contracts.

1.3.2 Specific Objective

The specific objectives of the study include :-

i. Study the different types of mortality risks.

ii. Model and forecast mortality risk

iii. Study ways of mitigating mortality risk alongside longevity risk that results from mortality risk hedging.

iv. Explore and analyze the impact of different types of mortality risks on risk management process in life insurance companies.
Main focus will be on insurer holding a portfolio of annuities and term life insurance contracts.

v. Analyze mortality risk hedging strategies.

1.4 Justification of the study

Impressive and comprehensive research study has been done in this field but there is need to continuously study the effects of mortality risk and longevity risk due to continuously improving and unstable demographic risk as a result of improved social-economic factors e.g. access to information, improved living standards etc.
Mortality and longevity risk have a humongous effect on the risk management of any insurance company thus the need to study new and cost effective ways of dealing with it.This risk can result to negative cash flows.
According to Association of Kenya insurers (AKI) 2015 report over 65 percent of the claims incurred in Kenya insurance companies since 2011 were as a result of ordinary life and pensions therefore the need to study mortality and longevity risk concurrently.

The research will benefit the insurance companies maintaining a portfolio of whole/term life and annuities.

2 LITERATURE REVIEW

2.1 Introduction

This chapter reviews various types of mortality risk and risk management tools in relation to life insurance companies. Cox and Lin (2007) indicate that mortality risk may not be hedgeable in financial markets but it may be reduced or otherwise eliminated by insurers by specific means such as natural hedging,reinsurance,asset-liability management,or mortality swaps.

2.1.1 Theoretical review

Mortality risk possess a major risk in insurance companies and associated with mortality risk is longevity risk that affect insurance companies holding a portfolio of annuities. It is majorly divided into different subcategories: unsystematic risk, systematic risk, adverse selection and basis risk.

Mortality risk is modeled comprehensively to gain deeper insight into the interaction among the different types of risk, incorporating the different types of mortality risk with respect to the risk management instruments. Population mortality is forecasted using the extension of the Lee-Carter (1992) model proposed by Brouhns, Denuit, and Vermunt (2002a).Adverse selection is modeled based on an extension of the Brass-type relational model by Brouhns, Denuit, and Vermunt (2002a) and estimated based on data from the Continuous Mortality Investigation (CMI).

The Brass relational model (Brass, 1971, 1974) was developed for estimation purposes in data-poor contexts, it can however be used more widely. The two-parameter relational model expresses the logit transformation of the observed survival function as a linear function of the logit of an empirical standard schedule. It is thus a special case of graduation by reference to a standard table (Benjamin Pollard, 1980).

Fitting to annual data yields time series of the two parameters may be forecast using time series methods. Keyfitz (1991) applied the method to Canadian data with limited success (see also Pollard,1987).

Other extensions of the model include Zaba (1979) and Ewbank et al. (1983) who developed separate four-parameter models to increase flexibility at the youngest and oldest ages as depicted in Zaba Paes (1995). Congdon (1993) fitted the Zaba model to mortality for Greater London in 1971 to 1990, obtaining relatively clear trends for three of the four parameters, and forecast all parameters using univariate ARIMA models. Brouhns, Denuit, and Vermunt

(2002a) devised Brass-type relational used by Gatzert and Wesker (2011), which allows a difference in the level and trend of annuitant in comparison to mortality of the population.

2.2 Mortality Risk categories

mortality risk can be divided into different subcategories, among them and the most common ones are unsystematic risk, systematic risk, adverse selection and basis risk which may arise in case of hedging, e.g., longevity risk.

2.2.1 Unsystematic mortality risk

Biffis, Denuit, and Devolder (2009)defined unsystematic risk the individual time of death is a random variable with a certain probability distribution.Even if the true mortality rate is known, the number of deaths will be random. The larger the population,the smaller the unsystematic mortality risk.
This risk can be diversified through natural hedging or transfer to capital market through Mortality Contingent Bonds (MCB),survivor swaps, q-Forwards, Longevity bonds among others.

2.2.2 Systematic mortality risk

This is the risk of unexpected changes in the underlying population mortality, for instance as a result of common factors impacting the mortality of the population as a whole which causes dependencies between lives. This risk cannot be diversified through enlarging the insurance portfolio. It is the risk of unexpected deviations from the expected mortality rates applying to all individuals which can result from a common factor unexpectedly impacting mortality at all ages (see, e.g., Wills and Sherris (2010)). This can be attributed to unexpected environmental or social influences impacting mortality positively or negatively. It can also be as a result of wrong expectations about future mortality due to estimation errors. Unexpected common factors that influence lives in a similar way induce dependencies and thus destroy diversification benefits of large pool sizes.

Wang et al.,(2010) describe systematic risk as a constant shock to the force of mortality, thus accounting for unexpected changes in mortality rates similarly to Milevsky and Promislow (2003) and Gründl,Post,and Schulze (2006).

Whereas mortality risk may not be hedge-able in financial markets,it may be eliminated or reduced by natural hedging,asset-liability management,reinsurance, mortality linked swaps(see,Cox and Lin (2007).

2.2.3 Adverse selection risk

Irregularity in details maintained by the insurance company about the different individuals' mortality experience and that of the life assured as well as mortality heterogeneity results to adverse selection.
Mortality rates are not equivalent for persons baring the same age x, they differ depending on behavior,genetic susceptibility among other factors this is referred to as Mortality heterogeneity.
This information is generally not accessible to insurance companies making it impossible to directly differentiate between individuals baring below or above average health. Such situations give rise to information asymmetries which leads to adverse selection since the level of mortality rates and growth over time is different between population in general and that of annuitants.

Adverse selection indicates basis risk while hedging against the risk of longevity as a result of difference between the mortality rates of the referenced population used as basis for hedging and the mortality rates for annuity holders.
Adverse selection may lead to misestimation of annuitants mortality resulting to a difference between actual and expected mortality used in reserving, pricing and other fields if the data on average annuitant mortality is insufficient and the insurance company is unable to observe the insured's individual mortality.(see,Gatzert and Wesker (2011))
Some information about individuals mortality can be obtained through general health situation or family history which may have an influence on insurance decisions made by them (see Finkelstein and Poterba (2002)).

2.2.4 Basis risk

This risk results from difference in population mortality underlying the hedge and the hedged portfolio mortality Gatzert and Wesker (2011).
A system for evaluating basis risk in hedging longevity used broadly was introduced by Coughlan et al. (2010)and made a conclusion that this risk can be significantly lowered by use of their framework for purposes of hedge calibration

2.3 Risk Management

Insurance companies can improve their risk management by implementing framework based approach and governance structure in the company so that all risks are assessed, understood and controlled- see Shriram Gokte.
These companies use different risk management approaches including:
• Retain mortality risk as a legitimate business risk. This assumes that the company is able to achieve an adequate expected rate of return relative to the level of risk being carried.
• Diversify their mortality risk across product ranges, regions and socio-economic groups. For example natural hedging (see,Cox and Lin, 2004) where gains on the life book will balance losses on the annuity book.
• Enter in to variety of forms of full or partial reinsurance, in order to hedge downside mortality risk.
• Pension plans can arrange a full or partial buyout of their liabilities by a specialist insurer an example is Paternoster in the UK. Small pension plans in the UK are exposed to considerable non-systematic mortality risk and thus purchase annuities from a life office for employees at the time of their retirement, thereby removing the tail mortality risk.
• For future annuity and pension provision, non-profit contracts could be replaced by participating annuities. Such annuities might share mortality profits or losses by adjusting the amounts of pensions in payment or by linking the date of retirement to current life expectancy.
• Assurers can securitize a line of business an example is Cowley and Cummins, 2005).
• Mortality risk can be managed through the use of mortality-linked securities and derivatives.
The risk management tools include natural hedging & mortality-linked securities and Derivatives that are used to transfer mortality risk in capital market.

2.3.1 Natural hedging

Natural hedging refers to prevaricating systematic mortality risk through portfolio composition.This hedging strategy uses the opposed reaction towards changes in mortality rates of term life insurances and annuities to immunize a life insurer against systematic mortality risk.
To stabilize aggregate liability cash flows the values of life insurance and annuity liabilities moving in opposite directions is utilized in response to a change in the underlying mortality.
If future mortality improves relative to current expectations, life insurer liabilities decrease because death benefit payments will be lesser than expected. However, annuity writers have a loss relative to current expectations because they have to pay annuity benefits longer than expected. If the mortality deteriorates,the situation is reversed i.e life insurers have losses and annuity writers have gains.

Mortality variance due to mortality fluctuations can significantly be reduced by 99% through portfolio composition(see Wetzel and Zwiesler(2008))

Lin and Cox (2005)) find empirical evidence implying that annuity writing insurers using natural hedging also charge lower premiums than other similar insurers who do not which may increase their market share. This indicates that insurers able to adopt natural hedging have a much higher competitive edge over those unable to.
Natural hedging may be difficult to implement due to below reasons.

(i)Life insurance and annuity products duration cannot be easily equated in order for suitable hedges to be realized. This is because annuities are mostly bought by clients of older age unlike life insurance policies generally bought by young clients.
(ii)Insurance companies may be forced to make alterations in their portfolio composition in order for effective hedging. This could increase operational cost or result to the insurer incurring additional expenses.Insurers may be required to reduce or increase annuity or life insurance prices in order to achieve an optimal hedging strategy by making the products more or less attractive. This pricing adjustment could reduce natural hedging effect.
(iii)Some insurers have a specialized production line and may be required to alter their internal business portfolio composition providing annuities solutions and life insurance products.
Wang et al.(2010) propose an immunization model which incorporates a stochastic mortality dynamic in order to evaluate the optimal level of a product mix which includes both annuities and life insurance.This assists life insurers attain an improved natural hedging effect.

2.3.2 Mortality-linked securities and derivatives

Mortality catastrophe/contingent bonds(MCB)

MCB provides insurers with the possibilities to assignment of mortality risk to the capital market. There has been successful issues of short-dated MCBs.The bond's payments are directly linked to a mortality index.
Swiss-Re-bond also known as Vita(1) was the first bond of this type to be issued in December of 2003. It had a term of three years maturing on 1st January 2017 with a principal at risk of 400M dollars designed for secularization of Swiss -Re's exposure to mortality risk.(see, Cairns, Blake and Dowd (2007, 2008))

Mortality or survivor swaps.

A mortality swap refers to an agreement to exchange cash flows in the future based on the outcome of at least one random mortality or survivor index. In return to continuous number of payments directly connected with survivors in a particular cohort,the counter parties swap a fixed series of payments.
They are key derivative of interest.

Its fist of the kind was publicly announced mortality in April 2007 between SwissRe and a United Kingdom life assurer Friends' Provident.The swap was meant for purely longevity risk transfers. **Advantages**
There are particular advantages relating to Mortality swaps over longevity bonds which include.

• Lower transactions cost reinsurance treaties,bonds and can easily be canceled.
• More flexible than reinsurance treaties,bonds and other traditional arrangements and can therefore be customized to address diverse circumstances.
• Readiness of different counter parties to experience their trade view of comparative advantages on the mortality development over time suffices.They therefore do not necessitate a liquid market.

Longevity or survivor bonds

Longevity bond coupon payments are based on rate of survivor-ship of the referenced birth cohort and terminate after the death of the last survivor. They are generally classified under two categories.
(a)Coupon-based longevity bonds: Coupon payments are dependent on mortality e.g(EIB)/BNP.
(b)Principal at risk longevity bonds: The principal is dependent on mortality event. All or part of the principal is lost if the specified event occurs.(e.g Swiss-Re)

The first longevity bond was issued in the United Kingdom in November 2004.

Mortality forwards or q-forwards

This is an instrument that exchanges a fixed rate of mortality agreed upon at contract inception for a future period's realized mortality rate.
The launch of the first mortality forward was announced by JPMorgan in July 2007.

3 METHODOLOGY

3.1 Basic Mortality Functions

3.1.1 Initial Rate Of Mortality

q-type mortality rate at age x denoted by q_x**(0)** measures the probability that a life currently aged x dies over the subsequent year.

$$q_x = \frac{d_x}{\ell_x} \tag{1}$$

where:- $\ell_x$is the number number of survivors at age x
d_x the number of deaths within the next one year $(x+1)$.

3.1.2 Probability of survival

p_x measures the probability that a life currently aged x survives in the subsequent one year

$$P_x = \frac{l_{x+1}}{l_x} \tag{2}$$

3.1.3 Central Rate Of Mortality

Central rate of mortality denoted by m_x is the ratio of deaths recorded between ages x and $x+1$ to the mean population alive at that age.

$$m_x = \frac{d_x}{L_x} \tag{3}$$

L_x indicates the persons years lived between ages x and $x+1$

$L_x = \int_0^1 l_{x+t}\, dt$

$=\int_x^{x+1} l_y\, dy$

$=\int_x^{\infty} l_y\, dy - \int_{x+1}^{\infty} l_x\, dy$

$=T_x - T_{x+1}$

$m_x = \frac{d_x}{L_x}$

$m_x = \frac{\int_0^1 l_{x+t}\mu_{x+t}\, dt}{\int_0^1 l_{x+t}\, dt}$

$=\mu_{x+\frac{1}{2}}$

where T_x and T_{x+1} are future lifetimes of an individual aged x and $x+1$ respectively.

3.1.4 Force of Mortality

μ_x also known as instantaneous rate force of mortality. It is the instantaneous death rate at exact time t for an persons aged (x+t) at time t

$\mu_x(t) = lim_{\Delta x \to 0} \frac{[x < T_0,(t-x) \leq x + \Delta x / T_0,(t-x) > x]}{\Delta x}$

where $T_{0,(t-x)}$ is a persons' born at time t_x outstanding lifetime.

The most famous law of mortality is that of Gompertz (1825), who postulated that μ_x satisfies the following simple differential equation:

$$\frac{d\mu_x}{dx} = k\mu_x \quad x \geq \alpha$$

$$\mu_x = BC^x \quad x \geq \alpha \tag{4}$$

In 1860 Makeham suggested the addition of a constant term to Gompertz' formula for μ_x, giving Makeham's law:

$$\mu_x = A + BC^x \tag{5}$$

3.2 Mortality Risk Models

Mortality risk projection is done through different models. In this subsection criteria for assessing a suitable mortality model as discussed by (Cairns, Blake and Dowd (2006a) and Cairns et al.(2007, 2008) & the different types of models used in systematic risk,unsystematic risk and adverse selection risk will be reviewed.
Various mortality models are derived and discussed in details.

3.2.1 Mortality model selection criteria

To determine that the mortality model used is fit, the criteria against which a model can be assessed, along the lines proposed by Cairns, Blake and Dowd (2006a) and Cairns et al. (2007, 2008) include:-

i. Mortality rates should be positive.
ii. Consistent with historical data.
A good model should be consistent with the historic patterns of mortality. Much greater doubt must be placed on the validity of any forecasts produced by the model.
Cairns et al. (2007) carried out a detailed comparison test based on maximum likelihoods using criteria that penalize over-parameterized models and established that a good model should be consistent with the historical data. Historical analysis has been performed by Dowd et al. (2008a,b) who use a variety of back testing procedures to evaluate out-of-sample performance of a range of models.

iii. Biologically reasonable.

Long-term dynamics under the model used should be biologically reasonable. A biologically reasonable model in general should be:-
• A forecasting model that produces period mortality tables with results that align with the past mortality tables displaying increase in mortality with age at higher ages. • short-term

mean reversion can be places in this categorized due to ecological changes.

iv. Robustness.

A model is categorized as robust if it exhibits revised parameters and forecasts with changes in the age and years considered.It is therefore crucial to stipulate the past years used and the age range.

v. Plausibility
Central trends and predictions uncertainty levels should be reasonable and aligned with changes in past mortality data and trajectories.

vi. Easy to Implement
It should be easy to execute various analysis (e.g analytic methods and numerical algorithms) using the model.

vii. Parsimony

Excessively parameterized models should be avoided. This can be be achieved through the use of Bayes Information Criterion(BIC) method that ensures that additional parameters are incorporated if there is notable increase in the model fit.//

viii. Sample paths generation and prediction intervals calculation
According to Cairns et al.,(2008),Most models with an exception of P-spines generate a sample path and therefore accommodate evaluation of future mortality-linked uncertainty and cash flows pricing.

ix. Incorporate Parameter uncertainty
According parameter uncertainty inclusion has a notable effect on forecast levels of uncertainty in mortality rates and future expected lifetimes particularly at longer times horizons.

x. Cohort effect
It should include a stochastic cohort effect for some countries. According to Cairns et al. (2007) addition of a cohort effect gives a significant better fit.Thus it is expected that such

impacts will continue into the future and inclusion of a cohort effect will lead to forecasts improvement.

xi. Correlation term structure

The model should not have an identical correlated structure. Rates of development at different ages have not been the same from one year to the next and also over a long period of time.

3.2.2 Lee-Carter (1992) model(LC)

Lee-Carter (1992) model uses stochastiv process to model future uncertainty.It is the earliest,most popular and frequently used mortality mode. The model comprise of a time series and demographic part.The secular deviations in mortality is described as a function of a single time index.

The logarithm of the observed mortality rate for age x and year τ ,$m_{x,\tau}$ are described as the sum of an age specific component ,a_x which is independent of time,k_τ and another component reflecting the general level of mortality which is the product of a time-varying parameter and b_x an age-specific component representing how mortality at each age varies with changes in the general level of mortality.

The force of mortality or central death rate $m_{x,\tau}/\mu_x(\tau)$ is modeled as:-

$$ln[\mu_x(\tau)] = a_x + b_x.k_\tau + \varepsilon_{x,\tau} \Leftrightarrow \mu_x(\tau) = e^{a_x+b_x.k_\tau+\varepsilon_{x,\tau}} \quad (6)$$

$\varepsilon_{x,}\tau \sim N(0,\sigma^2)$
where
a_x-time constant parameter reflecting the general shape of mortality over age
b_x-time constant parameter indicating the rate of mortality sensitivity at age x to change in k_τ
k_τ- time varying index which reflects mortality development over time.
$\varepsilon_{x,}\tau-$ The error term.

Since the parameters in the model are not fully identified,below constraints were enforced.

$$\sum_{x=1}^{\omega} b_x = 1 \ \sum_{\tau=1}^{n} k_\tau = 0 \quad (7)$$

The constraint $\sum_{\tau=1}^{n} k_t = 0$ implies that by summing over the years t $\hat{a}_x$ and parameters estimates are given by the force of mortality averages over the time period.i.e

$$\hat{a}_x = \frac{1}{n} \sum_{\tau=t_1}^{t_n} ln(\mu_x(\tau)) \tag{8}$$

$\hat{b}$ is obtained by differentiating both sides of the equation(Lee-Carter (1992) model)

$$\hat{b}_x = \frac{\frac{\partial ln \mu_x(\tau)}{\partial t}}{\frac{\partial \hat{k}}{\partial t}} \tag{9}$$

K_τ is generally modeled as a random walk(an example of Lee-Carter (1992)) or as an ARIMA process(see CMI,(2007))
Lee and Carter (1992) propose to fit an appropriate Auto Regressive Integrated Moving Average(ARIMA) process on kτestimated time series.

$$k_\tau = \phi + \alpha_1.k_{\tau-1} + \alpha_2.k_{\tau-2} + ... + \alpha_p.k_{\tau-p} + \sigma_1.\varepsilon_{\tau-1} + \sigma_2.\varepsilon_{\tau-2} + \sigma_q.\varepsilon_{\tau-q} + \varepsilon_\tau \tag{10}$$

$$= \hat{k}_\tau + \varepsilon_\tau$$

The various approaches used in parameter estimation have been discussing in a subsequent section of this chapter.

Advantages of the Lee and Carter (1992) model include:

i. Gives a good fit to historical data,the age function in the model allows it to be used across all ages.Additionally term captures the dominant trend in the evolution of mortality
ii. Simplicity in fitting and projecting. It provides an easy way of fitting and projecting since the parameters in the model are relatively few compared to the other models. The singular value decomposition and Poisson likelihood methods are likewise simple to put into practice.
iii.It is easy to project since the linear trend in the parameters is common in most of the data used. The random walk with drift time series time structure is widely used to give estimates of future central mortality rates.

The model has several drawbacks which include:-

i. Being a one factor model,mortality improvement is perfectly correlated at all ages.

ii. The b$_x$ is primarily calculated as the average development rate at age$x$, it also sets uncertainity levels in futu
$x : Var[\ln \mu_x(\tau)/\mu] = b_x^2[k_\tau/k_t]$ which implies that improvement rates cannot be used to decouple uncertainty in future mortality rates. Past research findings results illustrate that improved rates have been lower at high ages which indicates that forecasted death rates uncertainity will be considerably smaller at older ages.
iii. Lee and Miller (2001) identifies that the model is biased in forecasts.
iv.It can result in lack of smoothness in estimation b$_x$.

3.2.3 BDV (2002a) model/ Poisson log-bilinear model

This is an improvement of the lee-Carter(1992) model by Brouhns, Denuit, and Vermunt (BDV) (2002a) with modifications that results in slightly better theoretical properties. Number of deaths Poisson random variation are substituted for an extra error term on mortality rates logarithm.(McDonald (1996a,b,c)) indicates Poisson distribution to be quite suited to mortality analyses.
The observed number of deaths for age x and time τ, x, D$_\tau$ are modeled as follows:-

$$D_{x,\tau} \sim Poisson(E_{x\tau}.\mu_{x,\tau}) \text{ with } \mu_x(\tau) = e^{a_x+b_x.\hat{k}_\tau+\varepsilon_{x,}\tau} \quad \textbf{(11)}$$

E$_{x\tau} = \frac{n_{x-1}(\tau-1)+n_x(\tau)}{2}$

where

$\widehat{k}_\tau$: The forecasted time index
E$_{x\tau}$: Exposure to risk at age x and time τ
n$_x(\tau)$: Population size aged x years old alive end of year τ.

Advantages
i.Homoscedastic errors which is a restrictive assumption is dropped.
ii. Poison distribution is an effective measure of counting variables such as the number of deaths.

3.2.4 Other models

3.2.5 Lee-Miller (2001) model

This is another extension of the Lee-Carter (1992) model.
Lee and Miller (2001) took a different approach in parameter estimation and proposed that the main focus should be on the goodness of fit in the final years since the model is purposed for mortality rates projection.

Lee and Miller (2001) proposed the following modifications to Lee-Carter (1992) model

i. Fitted period limited to 1950 in order to decrease structural shift.
ii. Projecting forward rather than fitted rates in order to get rid of the jump-off error.
iii. kt adjustment done by matching life expectancy;

This model reduced bias in forecast through these modifications.

Renshaw and Haberman (2003) model

Renshaw and Haberman(2003) proposed a muti factor age-period model expressed as:

$$\log M_{(\tau,x)} = b_x^{(2)} + b_x^{(2)} k_\tau^{(2)} + b_x^{(3)} k_{\tau^{(3)}} \quad \textbf{(12)}$$

k$_\tau^{(2)}$ and $k_\tau^{(3)}$ are dependent period effects .

Being multi-factor it provides notable qualitative advantages over the LC model but still does not address the cohort effect.

The Renshaw-Haberman cohort (2006) model

This model is an improvement of the lee-carter (1992) model with an extra parameter that incorporates the cohort effect. Renshaw and Haberman (2006) proposed one of the initial population mortality stochastic models with a cohort effect. It is expressed as

$$logM_{(\tau,x)} = b_x^{(2)} + b_x^{(2)} k_\tau^{(2)} + b_x^{(3)} \upsilon^{(3)}{}_{\tau-x} \quad (13)$$

$k_\tau^{(2)}$- indicates a random effect
$\upsilon^{(3)}{}_{\tau-x}$ appears to have a deterministic linear or possibly quadratic trend in the year of birth(Cairns et al.(2008).

Cairns, Blake and Dowd (2006b) model.

This model concentrates on older ages (60 to 89). It was less parametirized based on the logistic transform of the mortality rate as opposed to LC model that is based on the log of the death rate.
Being multi-factor it provides notable qualitative advantages over the LC model but still does not address the cohort effect:-

$$logitq(\tau,x) = \frac{\log(q(\tau,x))}{1 - q(\tau,x)}$$

$$= k_\tau^{(1)} + k_\tau^{(2)}(x - \bar{x})$$

$k_\tau^{(1)}$ and $k_\tau^{(2)}$ are assumed to be a bivariate random walk with drift

The Cairns-Blake-Dowd model with cohort effect.

The model has been proven to generate outcome that combine multi-factor age-period structure with a cohort effect and are less parameterized (see Cairns et al.,2007). It is expressed by

$$q(\tau,x) = k_\tau^{(1)} + k_\tau^{(2)}(x - \bar{x}) + k_\tau^{(3)}((x - \bar{x})^2 - \sigma_x^2) + \upsilon_{\tau-x}^{(4)} \quad (15)$$

$$\bar{x} = (x_\mu - x_\iota + 1)^{-1} \sum_{x=x_1}^{x_\mu} x \text{ is the mean in the range of ages } x_1 \text{ to } x_\mu$$

to be fitted and

$\sigma_x^2 = (x_\mu - x_\iota + 1)^{-1} \sum_{x=x_1}^{x_\mu} (x - \bar{x})^2$ the variance
This model is an extension of et al.(2006b) with two additional components i.e
(a) age period effect quadratic in age $k_\tau^{(3}) ((x - \bar{x})^2 - \sigma_x^2))$ and
(b) cohort effect $(\upsilon_{\tau-x}^{(4)})$: a function birth year estimate $\tau - x$
This additional age effect term was proven to deliver statistically important enhancements for England& Wales and US males it is however termed insignificant as compared to the initial two-age-period effects.

Penalised spines (P-splines) model

The use of this model is common in the United Kingdom. It is expressed as

$$\log M_{(\tau,x)} = \sum_{f,g} \theta_{f,g} \beta_{f,g} (\tau, x) \tag{16}$$

where $\beta_{f,g} (\tau, x)$- specified basis functions with regularly-spaced knots and

$\theta_{f,g}$ -parameters to be estimated.

The use of the model can lead to over-fitted functions resulting in fitted mortality surfaces that are unreasonably lumpy. P-splines however avoid this problem by penalising roughness in the $\theta_{f,g}$
The approach has proven to be very effective at producing globally a good fit (CMI,2006).

P-Spine model drawbacks include:-

i. Excessive smoothing in the period dimension.(Cairnset al.,2007).
ii. The model fits a deterministic surface to the data and extends this into the future rather than allowing future rates to be generated by a stochastic process.
iii.The model does not allow for cohort effects. It can however be reformulated from an

age/period to an age/cohort model if desired. This however removes the period effects which are usually felt to be dominant giving rise to problems due to limited observation of cohorts.

3.3 Estimation of parameters

In order to use Lee-carter (1992) model for purposes of forecasting,it should be fitted first i.e. in order for its parameters estimation a$_x : x = 1,2,...,N, b_x : 1,2,...,N$ and $k_t : 1,2,...,T$.

3.3.1 Singular Value Decomposition (SVD) Estimation Method

Lee and Carter applied SVD of the matrix -
G$_{x,t} = [ln(\mu_{x,t}) - \hat{a}_x]$.
The vector was used for a$_x$ and k$_t$ estimates(Lawson,Hanson, 1974) of the equation

$$ln(\mu_{x,t}) = a_x + b_x k_t + \varepsilon_{x,t}$$

.
The parameter vector â$_x$is calculated as the average overtime of the central death logarithm.

$$\hat{a}_x = \frac{1}{n} \sum_{\tau=t_1}^{n} ln(\mu_{(x,t)}$$

Applying the SVD on matrix G;-

$$SDV(G) = MRN'$$

where

$$MRN'$$

denotes the age component matrix,Singular values arranged in descending order and time component matrix respectively.

$$MRN' = \begin{bmatrix} M_{1,1}...M_{1,x} \\ . \\ . \\ . \\ M_{x,1}...M_{x,x} \end{bmatrix} \begin{bmatrix} R_1...0 \\ . \\ . \\ . \\ 0...0 \end{bmatrix} \begin{bmatrix} N_{1,1}...N_{1,t} \\ . \\ . \\ . \\ N_{t,1}...N_{t,t} \end{bmatrix} \quad (17)$$

Below steps illustrate how the parameters are derived from the the matrix.
$\hat{b}_k$ is obtained from the age component matrix first vector:-

$$\hat{b}_k = (M_{1,1}; M_{21}; ...M_{x,1} = M_{x,1})$$

$\hat{k}_t$ is obtained from the first singular value multiplied by the the first vector of the time component.i.e -

$$\hat{k}_t = (R_1) * (N_{1,1}; N_{2,1}...N_{t,1})$$

To fulfill the Lee and Carter (1992) enforced constraints as per equation (7) the quotient transformation is used. parameter $\hat{b}$ and $\hat{k}_t$ therefore become: –

$$\hat{b}_x = \frac{1}{\sum_x} M_{x,1\,,1} * M_{x,1}$$

$$\hat{k}_t = \sum_x M_{x,1\,,1} * (R_1) * N_{t,1}$$

3.3.2 Weighted least squares

(Wilmoth,1993) proposed fitting Lee-Carter model (1992) parameters using the Weighted Least Squares (WLS) : Using the least squares method,the sum of the least square errors $\varepsilon_{x,t}$is minimized.

$$\sum_{x=1}^{N}\sum_{t=1}^{T}((ln(\mu_{x,t}) - a_x - b_x.k_t)^2 \tag{18}$$

This generates a series of system equations

$\text{T}a_x + b_x \sum_{t=1}^{T} k_t - \sum_{t=1}^{T} ln(\mu_{x,t}) = 0 \;\; x = 1,2,...,N'$

$\text{a}_x \sum_{t=0}^{T} k_t + b_x \sum_{t=1}^{T} k_t^2 - \sum_{t=1}^{T} k_t.ln(\mu_{x,t}) = 0 \;\; x = 1,2,...,N'$

$\sum_{x=1}^{N} a_x.b_x + k_t \sum_{x=1}^{N} b_x^2 - \sum_{x=1}^{N} b_x.ln(\mu_{x,t}) = 0 \; t = 1,2,...,T$

The estimate of a_x which minimizes the sum of the least of errors given the Lee and Carter conditions $\sum_1^{N'} b_x = 1$ and $\sum_{t=1}^{T} k_t = 0$ is the average of $\mathfrak{m}_{(x,t)}$
i.e

$$\hat{a}_x = \frac{1}{N'} \sum_t \mu_{x,t} \; x = 1, 2, ..., N'$$

N' indicates the total no. of calendar years.The difference in the matrix $D_{x,t}$ is formed as $D_{x,t} = \mu_{x,t} - a_x$and it satisfies $\sum_t k_t = 0$and $\sum_x (b_x)^2 = 1$

$$Z = \sum_{x,t} (k_t b_x - D_{x,t})^2$$

To find the value that minimizes Z we introduce the langrangers c and d that minimizes

$$R = Z - c \sum_t k - d \sum_x b_x^2$$

$$\frac{dR}{dk_t} = 2 \sum_x b_x (b_x k_t - D_{x,t}) - c$$

$$\frac{dR}{b_x} = 2 \sum_t k_t (b_x k_t - D_{x,t}) - 2d$$

$$\frac{c}{2} = k_t \sum_x b_x^2 - \sum_x b_x D_{x,t}$$

Adding the sum with respect to a=0, we then solve for $k_t$and b a system of equations to get

$$k_t = \sum_x b_x D_{x,t}$$

$$b_x = \frac{\sum_t k_t D_{x,t}}{\sqrt{\sum_x (\sum_t k_t D_{x,t})^2}}$$

3.3.3 Maximum likelihood estimate (MLE)

(Wilmoth,1993) and (Alho,2000) proposed using MLE to find the parameters in the LC model. This approach is based on $D_{(x,t)}$(number deaths),poisson approximation presented by (Brillinger,1986):

$$D_{(x,t)} \sim Poison(m_{(x,t)}E_{(x,t)}$$

where:-

$$\mu_{(x,t)} = exp(a_x + b_x k_t)$$

The parameters $a_x, b_x and k_t$ are estimated through the full log likelihood maximization(Wilmoth, 1993) $L(a,b,k,D) = log\prod_{(x,t)} f(D_{(x,t)};a,b,k)$

$$= log\prod_{(x,t)} exp - E_{x,t}\mu_{(x,t)}\frac{(E_{(x,t)}\mu_{(x,t)})D_{(x,t)}}{D_{(x,t)}}$$

=$\sum_{x=x_1}^{xA}\sum_{t=t_1}^{t_1+T-1}[D_{(x,t)}ln(E_{(x,t)}\mu_{(x,t)}) - E_{(x,t)}exp(a_x + b_x k_t) - ln(D_{(x,t)}!)]$

l(θ)which is the likelihood function is defined as l(θ) $= ln(L(a,b,c,d)$,equals

$$l(\theta) = \sum_{65}^{100} d_x lnq_x(\theta) + (l_x - d_x)lnp_x(\theta))$$

We can find the MLE (θ)numerically, either by maximizing directly the log-likelihood function l(θ)or by solving the system of equations.

$$\frac{\delta l(\theta)}{\delta\theta_j} = 0, j = 1, ..., p,$$

where $\theta_j$is the $j_t$component of θ

3.4 Forecasting Mortality index kt

Mortality forecasting is the process of projecting the future in reference to the historic and present data. k_t (the time factor) is basically viewed as a stochastic process. The Box-Jenkins techniques are used in this parameter's estimation and forecast k_t within an ARIMA (p, d, q) times series model. Lee–Carter (1992) utilized an ARIMA (0, 1,0) model (random walk with a drift) to define, i.e.,

$$\hat{k}_t = \hat{k}_{(t-1)} + \vartheta + \varepsilon_t$$

where:
$\varepsilon \sim N(, \sigma^2_{r\,w})$is the error term assumed to be independent with the same variance.
$\theta$is the drift parameter. It's MLE is expressed as

$$\hat{\vartheta} = \frac{(\hat{k}_T - \hat{k}_1)}{T-1}$$

it's reliant on the first and last of the k estimates only.
$\hat{k}_t$represents the estimated mortality index at time t.

The drift parameter estimate $\hat{\vartheta}$ is plugged in and the definition of $\hat{k}_{(t-1)}$ is substituted and shifted back in time one period in order to forecast two periods ahead:

$$\hat{k}_t = \hat{k}_{t-1} + \hat{\vartheta} + \varepsilon_t$$

$$= (\hat{k}_{t-2} + \hat{\vartheta} + \varepsilon_{t-1}) + \hat{\vartheta} + \varepsilon_t$$

$$= \hat{k}_{t-2} + 2\hat{\vartheta} + (\varepsilon_{t-1} + \varepsilon_t)$$

With data availability up to period T same procedure is followed iteratively (Δt)times to forecast $\hat{k}_t$at time T + (Δt)

$$\hat{k}_{T+(\Delta t)} = \hat{k} + (\Delta t)\hat{\vartheta} + \sum_{t=1}^{(\Delta t)} \varepsilon_{T+l-1}$$

$$= \hat{k}_t + (\Delta(t)\hat{\vartheta} + \sqrt{(\Delta t)}\varepsilon_t$$

This indicates a proportional increase in forecast conditional standard errors with increase in square root of the distance to (Δt)(forecast horizon). The conditional standard errors would therefore be larger if estimation uncertainty is included. Forecast point estimates following a straight line as a function of $(\Delta t)$with slope $\hat{\vartheta}$can be arrived as as follows:

$$E[\hat{k}_{T+(\Delta t)}|\hat{k},....,\hat{k}_T] \equiv \mu_{T+(\Delta t)} = \hat{k}_T + (\Delta t)\hat{\vartheta}$$

LC model extrapolates from a straight through the first $\hat{k}_1$and the last $\hat{k}_T$ the rest of the $\hat{k}'s$ are ignored.The force of mortality is expressed as:

$$\mu_{T+(\Delta t)} = \hat{a} + \hat{b}\hat{k}_{T+(\Delta t)}$$

$$= \hat{a} + \hat{b}[\hat{k}_T + (\Delta t)\hat{\vartheta}]$$

3.5 life insurance liabilities modeling

3.5.1 Life Assurance company modeling

Balance sheet for Life assurance company selling immediate annuity and term life assurance company at time t=0 can be demonstrated as:-

Asset	Liabilities
$S_{Lo}(t_0)$ – Low risk assets Market value.	$M_A(t_0)$ – Annuity value.
$S_H(t_0)$ – High risk assets Market Value.	$M_L(t_0)$ – Term life assurance liability value.
$M_{bond}(t_0)$ – Mortality contingent bond value.	$E(t_0)$ – Initial Equity contributed by share holders.

A constant fraction $f_e$from the positive earnings is received by the shareholders each year as a dividend as return for their investments. This is given by:

$$divid(t) = f_e.Max(E(t) - E(t-1);0)$$

where: E(t) indicates difference between assets and liabilities.

Assuming the insurance company sells term insurance policy that pays a constant benefit on death denoted by D_b and immediate annuities paying annual annuity in arrears denoted by a annually given that the life insured is still alive and in turn purchases a mortality contingent bond,a premium, $\pi_{x,T}$ must remitted at the beginning of the contract in t = 0

. The annuity is financed through a single premium and the term life assurance contract is assumed to be financed through constant annual premiums. The first capital can be expressed as:

$$S(t_0) = E(t_0) + n_A(t_0).P_A + n_L(t_0).P_L - \pi_{x,T}$$

where below symbols represent:
$n_A(t_0)$ Number of sold annuities at time t=0.
P_A Single premium used for funding the annuity.
$n_L(t_0)$ Number of life assurance policies sold at time t=0.
P_L Annual premium for funding the term life assurance policies.

The assets total value A(t)at time t captured in the balance sheet thus has a proportional rise with the the Mortality Contingent Bond market value as illustrated below.

$$A(t_0) = S(t_0) + M_{bond}(t_0)$$

According to Gatzert and Wesker (2011)the assets market value $S_j(t)$ is assumed to follow a geometric Brownian motion with drift μ_j and volatility σ_j.If we denote $B^{plow}$and B^{phigh} denote two Brownian motions with correlation ρ under the real-world measure P on the probability space $(\Omega, \mathcal{F}, P)$, where $\mathcal{F}$ denotes the filtration generated by the Brownian motion as defined by Gatzert and Wesker (2011). Hence, $S_j(t)$ can be therefore be expressed as follows (see Björk (2004),Gatzert and Wesker (2011)

$$S_j(t) = S_j(s).exp([\mu_j - \frac{1}{2}\sigma_j^2].(t-s).B^p_{j,t} - B^p_{j,s}))j = high, low$$

The Capital investment value S(t) can be calculated as:

$$S^i(t) = S^i_H(t) + S^i_{Lo}(t) + n_L(t).P_L - n_A(t).a - d_L(t).D_b + X(t) - divid(t)\ i = systematic, unsystematic\ risk$$

where the below denotes
X(t) Mortality Bond coupon Payments in year t.
$n_L(t)$ Term life assurance policyholders alive at the end of year t.
$d_L(t)$ Number of deaths recorded relating to term life assurance policy holders in year t.
$n_A(t)$ Annuity policy holders alive at the end of year t.
D_b Benefit paid on death for the term life assurance policies.
$S^i_{Lo}(t-s)=\alpha.S^i(t-s)$ and $S_H(t-s)=(1-\alpha).S^i(t-s)$(see,Gatzert and Wesker (2011))
The total value of the assets at time t can be expressed as:

$$A^i(t)=S^i(t)+M^i_{bond}(t) \quad (19)$$

3.5.2 Life Insurance liabilities valuation.

The annuity and term life market values are given as follows if the contracts are evaluated through risk neutral valuation and assuming market independence and mortality risk.The assumption is that the annuity is paid in arrears at end of each year.

$$M^i_A(t)=n^i_A(t).\sum_{s=1}^{T_A-t}.a._s p^A_{x+t}.(1+r)^{-s}$$

$$M^i_L(t)=n^i_L(t).[\sum_{s=1}^{T_L-t-1}.D_b.{}_s p^L_{x+t}.q^L_{x+t+s}.(1+r)^{-(s+1)}-P^i_L.{}_s p^L_{x+t}.(1+r)^{-s}]\ i=unsystematic,systematic$$

where:
$T_L$and $T_A$denote the maximum period for the life insurance and annuity contracts respectively. (A denotes Annuity, L denotes term Life)
D_b, nd P^i_L are evaluated by actuarial equivalent principle.
$P^f_A$and P^f_L.denotes the premiums
The parameters are calculated by:

$$V=\sum_{t=0}^{T_L-1} D_b.{}_s p^L_x.q^L_{x+t}.(1+r)^{-(t+1)}=\sum_{t=0}^{T_L-1} P^f_L.{}_t p^L_x.(1+r)^{-t} \quad (20)$$

$$V=\sum_{t=0}^{T_A-1} a._{t+1}p^A_x.(1+r)^{-(t+1)}=P^f_A \quad (21)$$

A risk premium (1+δ)is demanded.In the presence of systematic risk we shall denote these premiums by $P^{systematic}_L$ *and* $P^{systematic}_A$
Thus without systematic risk $P^{unsystematic}_L=P^f_L$and $P^{unsystematic}_A=P^f_A$

In the presence of systematic risk:

$P_L^{systematic} = P_L^f(1+\delta)$ and $P_A^{systematic} = P_A^f(1+\delta)$ where superscripts "systematic" and "unsystematic" denote systematic and unsystematic risks respectively.
The above results are achieved by assuming that:
(a)The risk premium is equal for both life insurance product and annuities.
(b) The expected benefit payouts present value is V for both policies.
Overall the liabilities value at time t:

$$L^i(t) = M_L^i(t) + M_A^i(t) \quad i = unsystematic,\ systematic\ risk$$

3.6 Risk management and risk measurements models

3.6.1 Simple mortality contingent bond modeling and valuation

Blake and Burrows (2001) proposed the use of a survivor bond which is a simple coupon-based mortality contingent bond (MCB)for risk management,it provides an efficient hedge against longevity risk.
We shall therefore consider this risk management tool. According to Gatzert and Wesker (2011) Mortality contigent bond can be used to effectively hedge mortlaity risk.
Lin and Cox (2005) and Dowd et al. (2006) applied the Wang transform to price mortality risk bonds; Wang (2000, 2002) introduced a class of distortion operator. This method will be used to price the survivor bond. we consider a time horizon [0, T] and an insurer's liability X.
Let $\phi(x)$ be the standard normal cumulative distribution function with a probability density function:
$\phi(x) = \frac{1}{\sqrt{2\pi}} e^{x^2/2}$ for all x. Wang (2000, 2002) defines the distortion operator as

$g_\theta(\mu) = \phi[\phi^{-1}(\mu) - \theta]$ for $0 < \mu < 1$ and a parameter θ market price of risk,it depicts the extent of diversifiable mortality risk proportional to portfolio expansion i.e systematic risk.

A "distorted" distribution $H^*(t)$ is determined by θ as illustrated below given H(t)("cumulative density function")

$$H^*(t) = g_\theta(H)(t) = g_\theta(H(t))$$

The value or fair price of the liability is the expected value under the distribution obtained from the distortion operator discounted at time 0. The price of the MCB omitting the discount can be given as follows:

$$G(X,\theta) = E^*(X) = \int xdH^*(x)$$

where

$$H^*(x) = g_\theta(H)(x) = \phi[\phi^{-1}(H(x)) - \theta]$$

Using this approach and assuming that $X_i(t)$("A variable coupon payable at the end of every year"), for i = unsystematic,systematic, at the end of each year t = 0, ..., (T-1) is equivalent to the proportion of the referenced cohort still alive at time t.
According to Gatzert and Wesker (2011) in their analysis on MCBs in risk management,The premium required to be paid by the life assurance company at the beginning of the contract ,$W^i_{x,T}$for i =unsystematic, systematic risk, x denotes the age of the population in reference relating to the MCB and t is the term of the bond can be evaluated by:

$$W^i_{x,T} = \sum_{t=0}^{T-1} E(X^{unsystematic}(t).(1+r-\theta)^{-(t+1)}$$

r is the risk-free interest rate of return, $\theta$is the risk premium for systematic mortality risk, if i = systematic and $\theta = 0$ if i = unsystematic.
The cash flow at time t defined as $X_i(t)$ is dependent on referenced cohort mortality and if systematic risk is considered or not.(i = unsystematic, systematic).

The number of individuals in the study cohort alive by end year t,$n^i_{reference}$can be iteratively evaluated as:

$$n^i_{reference} = (n^i_{reference}(t-1) - d^i_{reference})$$

where $d^i_{reference}$ is the number deaths in year t which can be measured as

$$d^i_{reference} \sim Poisson(E^{reference,i}_{x,t}.\mu^{population,i}_x(t))$$

and

$$\mu^{population,i}_x(t) = e^{a_x+b_x.k^i_t} \quad i = unsystematic, systematic\ risk$$

The risk exposure of the reference population ,$E_{x,t}^{reference,i}$ is expresses as ;

$$E_{x,t}^{reference,i} = \frac{-(n_{reference}^{i}(t-1).q_x^{popupation,i})}{ln\, p_x^{population,i}} \; (seeBDV(2002b)).$$

The $X_i(t)$, annual pay off is therefore equals to:

$$X_i(t) = \frac{n_{reference}^{i}(t)}{n_{reference}(0)}.C$$

where C is the initial coupon and $n_{reference}(0)$which is equal to an arbitrary number. $X^{systematic}(t)$ is considered in risk measurement by means of actual cash flows. $X^{unsystematic}(t)$ for valuation. Mortality Contingent Bond worth at time t is calculated as the number of Mortality contingent Bonds purchased at time 0,multiplied by the cash-flows expected value discounted to time t and information available at time t as illustrated below.

$$M_{bond}^{i}(t) = n_B.\sum_{t=0}^{T-1} E(X^{unsystematic}(j)).(1+r-\theta)^{-(j-t+1)}\, t = 0,1,...,T-1\,, \theta = 0 \text{ if } i = unsystematic\ risk$$

3.6.2 Natural hedging

According to Gatzert and Wesker (2011, this risk management tool uses the contradicting reaction towards deviations in mortality rates of annuities and term life insurance contracts to hedge systematic mortality risk in life assurance companies.
Since the effects of changes in mortality on the liability of life insurers is similar to that of an interest rate change,Wang et al.(2010) used the extended the immunization theory proposed by Redington(1952) to deal with longevity risk.
They expressed the variations of total liability ΔS by considering both effective mortality convexity and effective mortality duration:
The total liability of an insurance company S obtained by assuming an insurer sells two types of contract Term Life assurance and Annuity can be expressed as:

$$S = S^L + S^A$$

where S^L is the expected liability of the life insurance and S^A is the expected liability of the Annuity.
By speculation and assuming a constant force of mortality (μ) the impact of mortality rate on S can be calculated as.

$$D_{\mu}^{S} = \frac{ds}{d\mu}.\frac{1}{s}$$

The impact of mortality variations ΔS on S can be expressed through Taylor expansion demonstrated below. This has been achieved by extending the immunization theory according to Redington (1952).

$$\Delta S = (\frac{dS^L}{d\mu}.\frac{dS^A}{d\mu})(\Delta\mu) + \frac{1}{2}(\frac{d^2S^L}{d\mu}.\frac{d^2S^A}{d\mu})(\Delta\mu)^2 + ..$$

Optimum contract mix is obtained by setting this equation to zero in order to ensure that the immunization strategy for the mortality variations is realized.This can be achieved as follows,first order approximation is considered in this case.

$$(\frac{dS^L}{d\mu}.\frac{dS^A}{d\mu})(\Delta\mu) = 0$$

It can also be written as:

$D^L_\mu.\omega_L - D^A_\mu.\omega_A = 0$

where $D^L_\mu = \frac{dS^L}{d\mu}.\frac{1}{S^L}$is the mortality term of S^L*and*
$D^A_\mu = -\frac{dS^A}{d\mu}.\frac{1}{S^A}$ is the mortality term for S^A;
$\omega_L = \frac{dS^L}{S}$; $\omega_A = \frac{dS^A}{S}$and $\omega_L + \omega_A = 1$

The above equation can be expressed as as follows by considering effective mortality duration which captures the mortality dynamic more precisely(future mortality changes or improvements)

$D^L_{e\mu}.\omega_L - D^A_{e\mu}.\omega_A = 0$ where:
$D^L_{e\mu} = \frac{S^{L+} - S^{L-}}{2*S^L*\Delta\mu'}$ and $D^A_{e\mu} = -\frac{S^{A+} - S^{A-}}{2*S^A*\Delta\mu'}$:

$S^{L+}$and S^{A+} represent the liability value at high mortality $(\mu + \Delta\mu)$;
S^{L-} and S^{A-} represent the liability value at high mortality $(\mu - \Delta\mu)$

The impact of mortality variations on S can be calculated by use of mortality convexity:

$$C^S_\mu = \frac{d^2S}{d\mu^2}.\frac{1}{S}$$

Thus $C^L_{e\mu}$and $C^L_{e\mu}$) ("the effective mortality convexity for life insurance and annuity")

$$C_{e\mu}^{L} = \frac{S^{L-} - S^{L+} - 2S^{L}}{S^{L} * (\Delta\mu)^2}$$

and

$$C_{e\mu}^{A} = \frac{S^{A-} - S^{A+} - 2S^{A}}{S^{A} * (\Delta\mu)^2}$$

Considering $D_{e\mu}^{L}L, D_{e\mu}^{A}, C_{e\mu}^{L}$ and $C_{e\mu}^{A}$ as expressed above; $(\Delta S)$is as follows:

$$\Delta S = (D_{e\mu}^{L}.\omega_L - D_{e\mu}^{A}.\omega_A)(\Delta\mu) + \frac{1}{2}(C_{e\mu}^{L}.\omega_L + C_{e\mu}^{A}.\omega_A)(\Delta\mu)^2$$

Proportions optimum product mix for a life insurance liability can be obtained as follows.

$$\omega_L^{*} = \frac{D_{e\mu}^{A} + \frac{\Delta\mu}{2}C_{e\mu}^{A}}{D_{e\mu}^{A} + D_{e\mu}^{L} + \frac{\Delta\mu}{2}(C_{e\mu}^{A} - C_{e\mu}^{L})}$$

The impact of natural hedging can therefore be examine as illustrated above. Gatzert and Wesker (2011) approach considered insurance company as a whole.).
Using the probability of default (risk measurement tool) to be defined in the subsequent subsection as an optimal portfolio composition and not considering adverse selection risk e.g f_L^{*} is defined as:

$$g(f_L) = \Delta Probabilityof Default(f_L; \mu_x^{population}; \mu_x^{population,systematic})$$
$$= Probabilityof Default(f_L; \mu_x^{population}) - Probabilityof Default(f_L; \mu_x^{population,systematic})$$
$$= Probablityof Default^{unsystematic}(f_L) - Probabilityof Default^{systematic}(f_L) = 0$$

This result therefore implies that the probability of default does not change if systematic risk is considered or in its absence.

3.6.3 Risk Measurement Models

The effects of risk of mortality on the insurer's risk is analyzed by considering two downside default risk measures.Gatzert and Wesker (2011) examines the probability of default as risk measures insurer's risk situation These measures reflect probability of default (PD*) and the mean loss (ML*)which basically correspond to the Lower Partial Moments (LPM) of order zero and explain the long tern of the policies.

Probability Of Default (PD^{*i})

The probability of default (ruin probability) is given as follows under the real-world measure P, :

$$PD^{*}i = P(t^{i}_{default} \leq t), \quad i = unsystematic, systematic\ risk. \tag{22}$$

where:
$t^{i}_{default} = inf\left\{t_1 : A^{i}(t_1) < L^{i}(t_1)\right\}$ $t_1 = 1, ..., t$ is the is the time of default.This implies that this risk measure only accounts for the default frequency.

The Mean Loss(ML$^{*}i$)

This is defined as Lower Partial Moments (LPM)of order one at default time,discounted to time 0. It is calculated as the discounted expected loss in case of default. Therefore in contrast to PD that only measures the frequency of default this risk measure accounts for the extent of the default.
In other words it accounts for the how much assets would not suffice to cater for liabilities. This can also be explained as the average cash of cash requires in case of a default before end of the term. see,Gatzert and Wesker (2011) It is expressed as:

$$ML^{i} = E[(L^{i}(t_{default}) - A^{i}(t_{default})).(1+r)^{-t^{i}_{default}}.1\left\{t^{i}_{default} \leq t\right\}] \ i = unsystematic, systematic\ risk \tag{23}$$

where: $1\left\{t^{i}_{default} \leq t\right\}$reflects the indicator function equals to one if the condition displayed in the brackets is satisfied.

4 DATA ANALYSIS

This section illustrates results of the numeric analysis as follows:

i. U.S.A general past and forecast mortality is analyzed and different demographic parameters modeled through Lee-Carter (1992) model.
ii. Different input parameters, assumptions and resulting premium and annuity values are calculated using Actuarial Projected Values(APV.
iii. The effects of mortality risk is illustrated with the main focus on adverse selection and probability of default as the measure of risk used.

4.1 Source of data and overview

4.1.1 Source of data

United States Of America(USA)has been selected for the study due to the availability of mortality rates deduced from actuarial mortality tables originating from actual insurance companies.
The data used for estimation of mortality for both groups of insured (annuitant data and population data used for term life policy holders) is the number of deaths and exposure to risk for U.S.A from 1933 to 2017 available from Human Mortality database(H.M.D) and U.S.A annuitant mortality from Society of Actuaries (SOA)1996 Individual Annuity Mortality (IAM) Basic Table – Male. Minimum Age: 5 Maximum Age: 115.

4.1.2 U.S.A Mortality overview analysis

U.S.A data is downloaded through demographic package from HMD. The figures below depict the logarithm of death rates according to age and time for male, female and entire population.

From these figures we can confirm that mortality has been on a decrease with a different behaviors observed according to different ages.Mortality rate in 2017 is lower compared to 1933. we can also observe that mortality decrease over age but there is a notable change in mortality trend between the age of 20 to 45 years this could be attributed to drug use,cancer, heart disease,suicide,homicide,accidents among other factors.

Figure 1. Death rates according to age

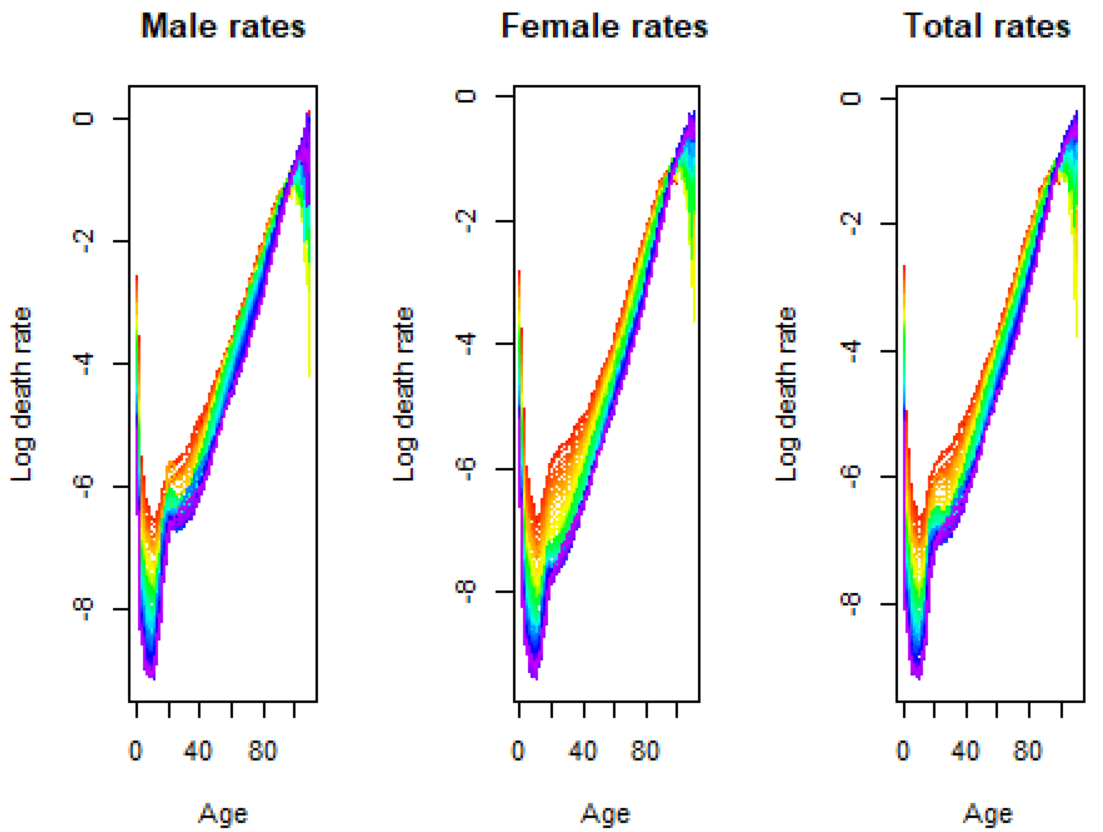
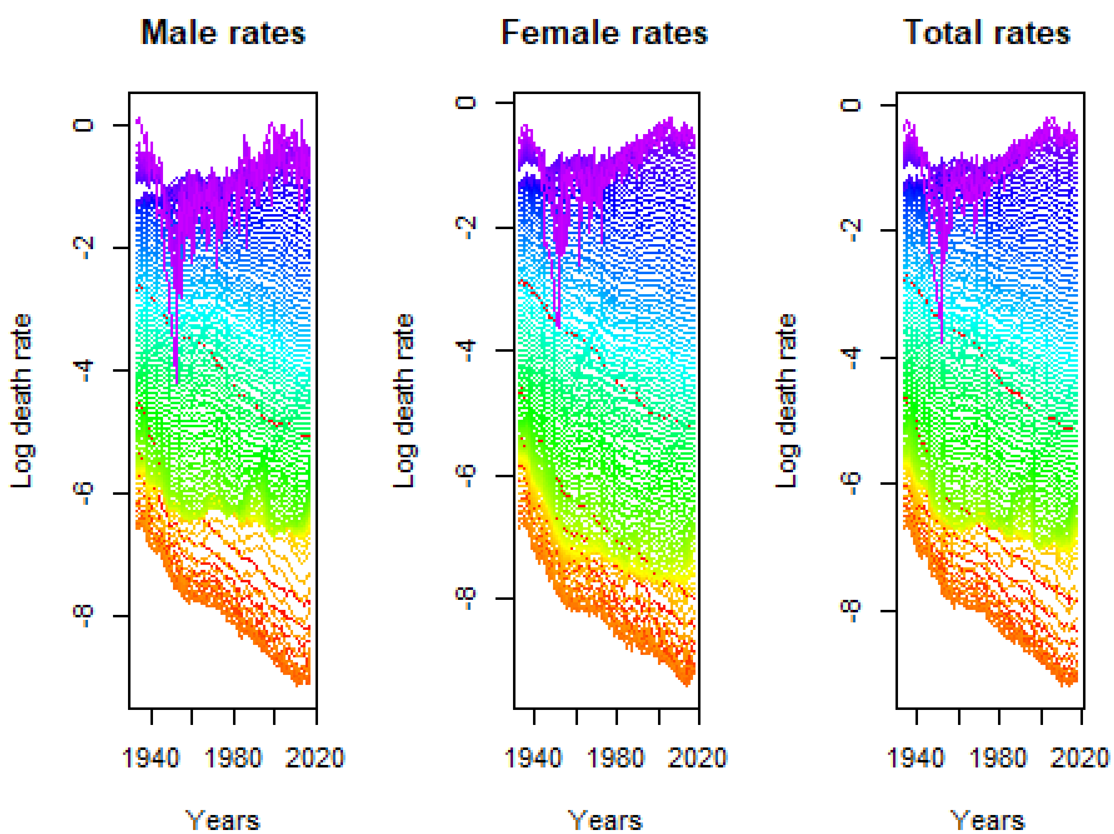


Figure 2. Deathrates according to time



4.2 Fitting Lee-Carter model and Estimating demographic Parameters

4.2.1 Fitting Lee-Carter Model

Lee-Carter model has been fitted by considering a maximum age of 100 years. Below figure represents the values of fitted demographic parameters of $a_x, b_x and k_t$.

Figure 3. Estimated parameters ax,bx and kt

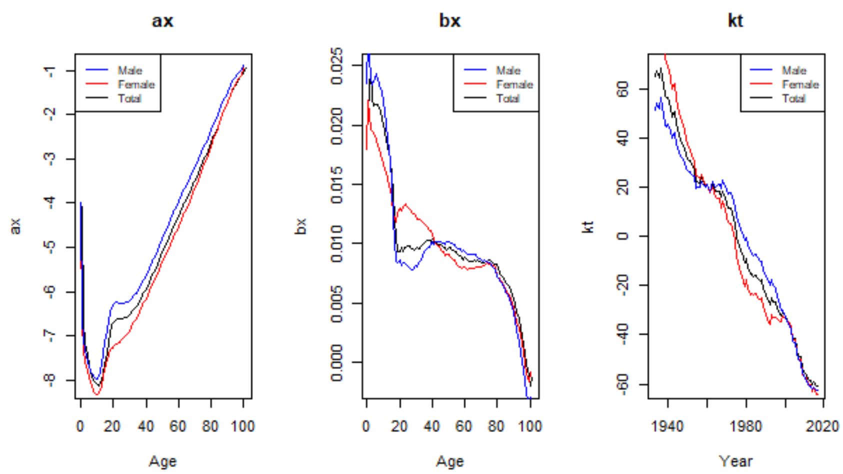
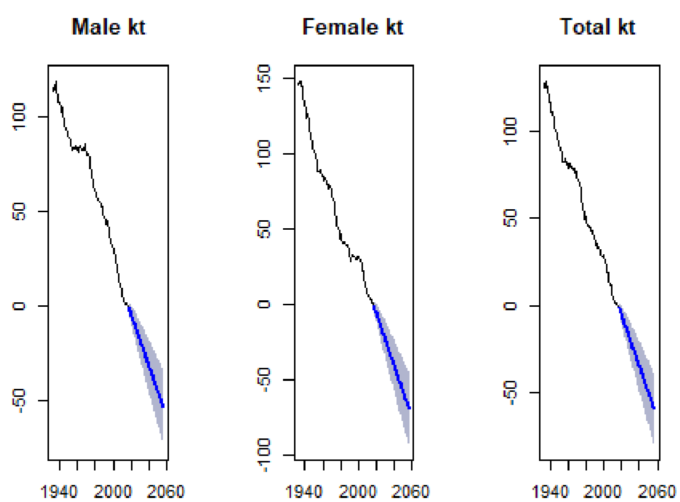


Figure 4. Projected values of kt for 40years



The above figure depicts the projected values of $k_t s$ for 40 years. k_t According to Gatzert and Wesker (2011) values are obtained by applying Box-Jenkins time series analysis techniques, which indicated an ARIMA (0,1,0) with drift $\phi = -1.3508$(standard error 0.2172) ; the standard error of ε_τ is estimated as 1.32295 .

4.2.2 Input parameters definition, assumptions and estimations

Term life insurance and Annuity Contract

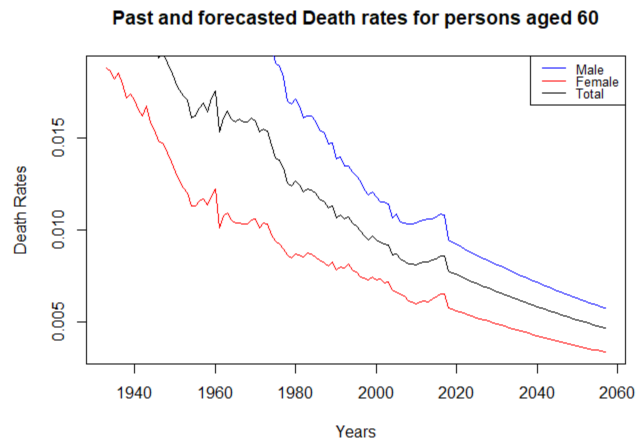
Calendar year is set as 2017. The assumption done is that term life insurance policies are consumed by male policy holders aged 30 years (x=30) for a term of T=40 years.Age of the annuitant is assumed to be x=60 years and the maximum age attained is assumed to be 100 years this corresponds to an annuity term of T=40 years. V is assumed to be $10,500 for each of the contract types. "We follow Gründl, Post, and Schulze (2006) by assuming a loading of $\delta = 1\%$"in calculating premiums for systematic risk due to unavailability of data concerning the size of the loading for purely systematic risk.
using Equation (20) the assumption in the life contract results to a fair Death Benefit=$210,000. The term life contract annual premiums used under systematic risk for the term life policy is $P_L^{systematic} = 452$ and $P_A^{unsystematic} = 457$using the premium loading 1%.These figures are calculated through Actuarial Present Values (APV) from the males population mortality and projected values for the policy duration T=40 years. The APVs are provided in the appendices.
The single premium under systematic and unsystematic risk for the annuity is set at $P_A^{unsystematic} = \$10,500$ since the value of v is assumed as $\$10,500$ for both contracts ,$P_A^{systematic} = \$11,550$ respectively.
Equation (21) will be used to calculate the fair annuity, this amount is dependent on whether adverse selection risk is considered in pricing and reserving or not.The figures will be illustrated in the subsequent section.

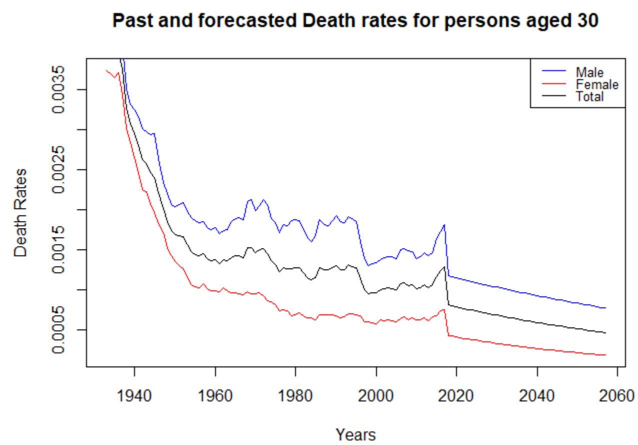
Investments

Initial equity E(0) contributed by the share holders is set to $50 Million dollars and dividend rate of payment $r_e = 20\%$. A constant risk free rate 3% is assumed.

High and low risk asset drift and volatility are assumed to be 10%(5%) and 20%(8%), a correlation of 0.1 and a fraction $\alpha = 80\%$.Sensitivity analysis is conducted on these figures for robustness purposes. The total number of contracts sold at time 0(n(0)) are also fixed and equal to 10,500 to ensure comparability in portfolios.

Below figures reflect the past and projected death rates for term life and annuitants based on the above assumptions.





From the above figures we can deduce that death rates have been on decrease but rose in the mid 80s through to mid 90s for persons aged 30 years.This could have been attributed to HIV-AIDs pandemic and drug abuse. There is also notable rise between 2010 and 2017 for both age groups, this could be attributed to high rise in deaths resulting from cancer, accidents, homicides, suicide among others.

4.3 Effects of Mortality risk on life assurance company risk state

In this section we demonstrate the different cases with respect to mortality risk for pricing and risk measurement. The first case will be purely in consideration to systematic and unsystematic risks. The second scenario will be considered in the presence and absence of adverse selection consideration under unsystematic risk.
The effects of adverse selection will be modeled through an extension of the brass-type relational model used by Brouhns,Denuit,and Vermunt(2002a),Gatzert and Wesker (2011), among others.

$$ln[\mu_x(\tau)^{\text{annuitant mortality}}] = \alpha + \beta . ln[\mu_x(\tau)^{\text{Population Mortality}}] \qquad \textbf{(24)}$$

β is the annuity policy holders mortality development as compared to general population. If $\beta > 1 \sim$ greater development.
weshallassumethattheinsurancecompanyisunabletoaccuratelyquantifyforadverseselectionbytakingthe
0 (negative figure) and $\beta = 1$ and calculate the fair annuity required to be paid.The annuity is assumed to be immediate payable in advance. We shall also do a further analysis to demonstrate the fair annuity if the insurance company is able to perfectly account for adverse section by taking values of $\alpha = 0$ and the value of $\beta = 0.81\%$ obtained from SOA Mortality improvement scale MP-2019 through an annual update to the $RPEC_2014$ model and corresponding mortality improvement scales reflects historical U.S. population mortality experience through 2017.This report is based on Social Security Administration (SSA) 2019 trustee's report intermediate projection for Males for year range 2055-2064.

Figure 5. Effects of adverse selection on annuity payable

Scenario	Notation	Pricing and reserving Notations	Fair Annuity	Risk Measurement
Without Adverse selection	"unsystematic risk"	$q_x^A = p_x^{pop}$ $q_x^L = p_x^{pop}$ $P_L^{unsystematic} = P_L^{fair}$; $P_A^{unsyatematic} = P_A^{fair}$	580	$q_x^A = p_x^{pop}$ $q_x^L = p_x^{pop}$
	"unsystematic risk + systematic risk"	$q_x^A = p_x^{pop}$ $q_x^L = p_x^{pop}$ $P_L^{Systematic} = (1+\delta)P_L^{fair}$; $P_A^{systematic} = (1+\delta)P_A^{fair}$	580	$q_x^A = p_x^{pop,systematic}$ $q_x^L = p_x^{pop,systematic}$
With Adverse selection	"unsystematic risk +Adverse selection"	$q_x^A = \{ {q_x^{ann*\alpha\neq0,\beta_1=1,\beta_2=0}}, misestimated \atop {q_x^{ann}} \text{ perfectly estimated}$ $q_x^L = p_x^{pop}$ $P_L^{unsystematic} = P_L^{fair}$; $P_A^{unsystematic} = P_A^{fair}$	573 449	$q_x^A = p_x^{ann}$ $q_x^L = p_x^{pop}$
	"unsystematic risk + systematic risk+adverse selection"	$q_x^A = \{ {q_x^{ann*\alpha\neq0,\beta_1=1,\beta_2=0}}, misestimated \atop {q_x^{ann}} \text{ perfectly estimated}$ $q_x^L = p_x^{pop}$ $P_L^{systematic} = (1+\delta)P_L^{fair}$; $P_A^{systematic} = (1+\delta)P_A^{fair}$	573 449	$q_x^A = p_x^{ann,Systematic}$ $q_x^L = p_x^{pop,systematic}$

From the numerical analysis we can deduce that if the insurer is not capable of perfectly measuring adverse selection then mortality rates improvement for both type of policies cannot be completely considered in premiums and benefits calculation. All the calculations of the fair annuity are done through actuarial tables created through the historic data and various APV for annuitants aged x=60 years.

4.3.1 Effect of Mortality risk in risk management in life insurance companies

The below table illustrates the effectiveness of MCB in risk management.

Figure 6. Effects of adverse selection on annuity payable

	PROBABILITY OF DEFAULT		
Portfolio with annuities only	Adverse selection not considered	Adverse selection Misestimated	Adverse selection Perfectly estimated
Without MCB	0.29%	0.37%	0.30%
With MCB	0.24%	0.32%	0.25%
Relative reduction by MCB	**21%**	**16%**	**20%**
Relative ={PD(without MCB)-PD(with MCB)}/PD(with MCB)			

The results demonstrate that MCB can reduce the probability of default by 20% if perfectly estimated under a portfolio comprising of annuity products only.

5 CONCLUSION AND RECOMMENDATIONS

5.1 Summary

The effects of the mortality components are discussed broadly i.e unsystematic, systematic , adverse selection and basis risk that results from longevity risk while hedging mortality risk. Different risk management tools have also been discussed at length in two categories natural hedging and Mortality Contingent bond/other derivatives used to transfer mortality risk to capital market. Two default risk measures Expected mean loss and probability of default have been explained. However,in this paper the probability of default has been used as the risk measure. In chapter three different mortality projection models have been discussed with the main focus on Lee-Carter (1992) model and the extension of the model by Brouhns, Denuit, and Vermunt. Life insurance company has been modeled and valuation of insurance liabilities broadly described. Finally U.S.A past mortality is modeled and different analysis done in relation to past and projected data. The prices of fair annuity have been derived from the Actuarial Projected Values(APV), figures in the appendices.
We can conclude that mortality risk is a major risk component for life insurance companies and can be minimized through MCB. From the analysis we can deduce that the probability of default has greatly reduced if adverse selection is perfectly estimated and in the presence of a MCB.The annuity amount payable to an individual when adverse selection is perfectly estimated is lower as compared to that payable if the risk is misestimated.

5.2 Recommendations

In this project only one mortality model has been explored and the APVs are based on the extension of the Lee-Carter (1992) model. Other mortality models can be explored and results compared. Only one risk measure Probability of default which measures the frequency of default has been used in numerical analysis. Future analysis can be done using the expected mean loss to establish the the average amount of cash necessary for funding a case of default during the contract term. My initial aim was to evaluate the effects of mortality risk in risk management in Kenyan life insurance companies, this was however not possible due to inaccessible of mortality data for annuitants.Future research can be done based on Kenyan Mortality.

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7 APPENDICES

7.1 Appendix 1: Lee-Carter Analysis

Years in fit: 1933 - 2017
Ages in fit: 0 - 100
Male
Percentage variation explained: 94.6%
ERROR MEASURES BASED ON MORTALITY RATES
Averages across ages:
ME MSE MPE MAPE
0.00001 0.00007 0.01136 0.08414
Averages across years:
IE ISE IPE IAPE
0.00258 0.00549 1.13330 8.34414
ERROR MEASURES BASED ON LOG MORTALITY RATES
Averages across ages:
ME MSE MPE MAPE
0.00471 0.01341 -0.00035 0.02049
Averages across years:
IE ISE IPE IAPE
0.47050 1.32295 -0.04385 1.97864
Female
Percentage variation explained: 96.6%
ERROR MEASURES BASED ON MORTALITY RATES
Averages across ages:
ME MSE MPE MAPE
-0.00005 0.00005 0.00579 0.07348
Averages across years:
IE ISE IPE IAPE
-0.00417 0.00391 0.58059 7.29335
ERROR MEASURES BASED ON LOG MORTALITY RATES
Averages across ages:
ME MSE MPE MAPE
0.00008 0.01152 0.00039 0.01654

Averages across years:

IE ISE IPE IAPE

0.00806 1.14298 0.03385 1.60220

Total Population

Percentage variation explained: 96.1%

ERROR MEASURES BASED ON MORTALITY RATES

Averages across ages:

ME MSE MPE MAPE

-0.00001 0.00005 0.00763 0.07028

Averages across years:

IE ISE IPE IAPE

0.00013 0.00384 0.76442 6.98597

ERROR MEASURES BASED ON LOG MORTALITY RATES

Averages across ages:

ME MSE MPE MAPE

0.00278 0.00973 -0.00012 0.01668

Averages across years:

IE ISE IPE IAPE

0.27775 0.96634 -0.01713 1.61878

7.2 Appendix 11 : Lee-Carter summary for Forecast

Forecasted k_t for 40 years.
Years: 2018 - 2057 Ages: 0 - 100

Figure 7

MALE	FEMALE	TOTAL POPULATION
ERROR MEASURES BASED ON MORTALITY RATES	ERROR MEASURES BASED ON MORTALITY RATES	ERROR MEASURES BASED ON MORTALITY RATES
Averages across ages:	Averages across ages:	Averages across ages:
ME MSE MPE MAPE	ME MSE MPE MAPE	ME MSE MPE MAPE
0.00001 0.00007 0.01136 0.08414	-0.00005 0.00005 0.00579 0.07348	-0.00001 0.00005 0.00763 0.07028
Averages across years:	Averages across years:	Averages across years:
IE ISE IPE IAPE	IE ISE IPE IAPE	IE ISE IPE IAPE
0.00258 0.00549 1.13330 8.34414	-0.00417 0.00391 0.58059 7.29335	0.00013 0.00384 0.76442 6.98597
ERROR MEASURES BASED ON LOG MORTALITY RATES	ERROR MEASURES BASED ON LOG MORTALITY RATES	ERROR MEASURES BASED ON LOG MORTALITY RATES
Averages across ages:	Averages across ages:	Averages across ages:
ME MSE MPE MAPE	ME MSE MPE MAPE	ME MSE MPE MAPE
0.00471 0.01341 -0.00035 0.02049	0.00008 0.01152 0.00039 0.01654	0.00278 0.00973 -0.00012 0.01668
Averages across years:	Averages across years:	Averages across years:
IE ISE IPE IAPE	IE ISE IPE IAPE	IE ISE IPE IAPE
0.47050 1.32295 -0.04385 1.97864	0.00806 1.14298 0.03385 1.60220	0.27775 0.96634 -0.01713 1.61878

APVs used in calculations

Figure 8

APV for Life Assurance premium projected for 40 years for a life aged 30years

```
For cohort  1987  the expected lifetime at birth is 79.25  and the APV is : 22.95
For cohort  1992  the expected lifetime at birth is 80.09  and the APV is : 23
For cohort  1997  the expected lifetime at birth is 80.83  and the APV is : 23.05
For cohort  2002  the expected lifetime at birth is 81.56  and the APV is : 23.09
For cohort  2007  the expected lifetime at birth is 82.27  and the APV is : 23.14
For cohort  2012  the expected lifetime at birth is 82.9  and the APV is : 23.18
For cohort  2017  the expected lifetime at birth is 83.42  and the APV is : 23.22
```

APV for annuity in advance for a life aged 30 years projected for 40 years used in Life premiums calculations

```
For cohort  1987  the expected lifetime at birth is 79.25  and the APV is : 0.08
For cohort  1992  the expected lifetime at birth is 80.09  and the APV is : 0.07
For cohort  1997  the expected lifetime at birth is 80.83  and the APV is : 0.07
For cohort  2002  the expected lifetime at birth is 81.56  and the APV is : 0.06
For cohort  2007  the expected lifetime at birth is 82.27  and the APV is : 0.06
For cohort  2012  the expected lifetime at birth is 82.9  and the APV is : 0.06
For cohort  2017  the expected lifetime at birth is 83.42  and the APV is : 0.05
```

APV for annuity in advance for a life aged 60 years projected for 40 years used Annuity payment calculations.

```
For cohort  1987  the expected lifetime at birth is 79.25  and the APV is : 17.05
For cohort  1992  the expected lifetime at birth is 80.09  and the APV is : 17.23
For cohort  1997  the expected lifetime at birth is 80.83  and the APV is : 17.42
For cohort  2002  the expected lifetime at birth is 81.56  and the APV is : 17.59
For cohort  2007  the expected lifetime at birth is 82.27  and the APV is : 17.76
For cohort  2012  the expected lifetime at birth is 82.9  and the APV is : 17.93
For cohort  2017  the expected lifetime at birth is 83.42  and the APV is : 18.09
```