COMMUTANTS AND SPECTRAL PROPERTIES OF OPERATORS IN HILBERT SPACES

BY

MUTIE KAVILA

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DECLARATION

This thesis is my original work and has not been presented for a degree in any other University.

Signature: ………………………….. **Date**:…………………………..

Mutie Kavila I80/7503/2005

This thesis has been submitted for examination with our approval as University supervisors.

Signature: ………………………….. **Date**:…………………………..

Prof. J.M. Khalagai

Signature: ………………………….. **Date**:…………………………..

Prof. G.P. Pokhariyal

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ABSTRACT

This thesis is devoted to the study of commutants and spectral properties of operators in Hilbert spaces. This is done via the following operator equations: $AB = \lambda BA$, where $\lambda \in \mathbb{C}$, $AX = XB$ and $AXB = X$. In the operator equation $AB = \lambda BA$, conditions on A and B under which $\lambda = 1$ are investigated. This indeed is a sufficient condition for the operators A and B to belong to the commutant of each other. In the operator equations $AX = XB$ and $AXB = X$, conditions that ensure the existence of the operator equations $C(A,B)X = 0$ and $R(A,B)X = 0$ are given. Finally, in the operator equation $AB = \lambda BA$, the equality of the general spectra and other subsets of the spectra, namely essential and approximate point spectra of AB and BA or B and λB, are established. This final bit justifies the spectral properties part of our thesis.

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This thesis is dedicated to the four pillars of my life. My parents, the late Mr. Geoffrey Katiku Kavila and the late Mrs. Eva Mukite Tumwa, my loving wife, Eva Kiragu and my lovely daughter Makenna Mukite Mutie.

CHAPTER ONE

PRELIMINARIES

1.0 Introduction

The mathematical concept of a Hilbert space, named after David Hilbert who formulated Hilbert spaces, generalizes the notion of a Euclidean space. It extends the methods of vector algebra and calculus from the two dimensional Euclidean plane and three dimensional space to spaces with any finite or infinite number of dimensions.

Before the development of Hilbert spaces, other generalizations of Euclidean spaces were known to mathematicians and physicists. One of them was realized towards the end of the 19th century. The idea of a space whose elements can be added together and multiplied by scalars, known as an abstract linear space, gained some traction here. At the turn of the 20th century functions were added together or multiplied by constant scalars. In the first decade of the 20th century, parallel developments led to the introduction of Hilbert spaces.

The first development arose during David Hilbert and Erhard Schmidt's study of integral equations. Here, they showed that two square integrable real-valued functions f and g on an interval $[a,b]$ have an inner product given by:

$$\langle f,g\rangle = \int_a^b f(x)\overline{g(x)}dx.$$

The second development was the Lebesgue integral, an extension to the Riemann integral introduced by Heine Lebesgue in 1904. The Lebesgue integral made it possible to integrate a much broader class of functions. In 1907, Frigyes Riesz and Ernst Sigismund Fischer independently proved that the space L^2 of square integrable functions is a complete metric space. Thereafter, the 19th century results of Joseph Fourier, Friedrich Bessel and Marc-Antoine Parseval on trigonometric series easily carried over to more general spaces.

Further basic results were proved in the early 20th century by other scholars and in particular, John von Neumann coined the term abstract Hilbert space in his work on unbounded hermittian operators. He gave the first complete and axiomatic treatment of these Hilbert spaces, using them in his work on the foundations of quantum mechanics. The name Hilbert space was soon adopted by others, like Weyl in 1931. The significance of the concept of a Hilbert space was underlined with the realization that it offers one of the best mathematical formulations of quantum mechanics. The states of a quantum mechanical system are vectors in a certain Hilbert space, the observables are self adjoint operators on that space, the symmetries of the system are unitary operators and measurement are orthogonal projections. The relation between quantum mechanical symmetries and unitary operators provided an impetus for the development of the unitary representation theory of groups initiated by Weyl in 1931. See [44].

With this historical background in mind, a connection between the importance of this thesis to areas like quantum mechanics and physical chemistry was made. As a result it has been a vibrant area for recent study.

Operators are commonly used to perform a specific mathematical operation on another function. The operation can be to take the derivative or integrate with respect to a particular term, or to multiply, divide, add or subtract a number or term with regards to the initial function. Operators are commonly used in physics, mathematics and chemistry, often to simplify complicated equations such as the Hamiltonian operator used to solve the Schrodinger equation in order to figure out the energy of a wave function. See [40]

This thesis is devoted to the study of commutants and spectral properties of some operators in the algebra of all bounded linear operators on an infinite dimensional complex Hilbert space H, denoted by $B(H)$. The operator equations $AB = \lambda BA$, $AX = XB$ and $AXB = X$ are used. To begin with, the importance of this research to quantum mechanics and physical chemistry is given briefly, followed by an account of the work done by several authors as far as these operator equations are concerned and the above subject matter.

Certain problems in quantum theory have motivated research in pure mathematics and more so in matrix and operator theory. One of these mathematical problems is the commutator. The study of $\lambda -$ commuting operators is important for the interpretation of quantum mechanical observables and the analysis of their spectra.

Operators in these application areas are generally characterized by a hat. Thus, they generally appear like the following equation with $\hat{E}$ being the operator operating on $f(x)$.

$\hat{E}f(x) = g(x)$...(i).

For instance if $\hat{E} = \frac{d}{dx}$, then (i) above becomes:

$g(x) = f'(x)$.

One property of operators is that the order of operation matters. $\hat{A}\hat{E}f(x) \neq \hat{E}\hat{A}f(x)$ unless the two operators $\hat{A}$ and $\hat{E}$ commute.

For two physical quantities to be simultaneously observable in quantum mechanics, their operator representations must commute. Notable pairs for observation are position and momentum, and energy and time. If the operators do not commute, they cannot be measured simultaneously to precision. See [40].

In physical chemistry, if two operators commute then both quantities can be measured at the same time. Otherwise, there will be a tradeoff in the accuracy in the measurement for one quantity versus the other. See [31].

1.1 Literature review

Wintner (1947), made the first important contribution to the study of commutators when he proved that the identity operator is not a commutator. This result has its roots in physics where the quantum-mechanical momentum and position satisfy the commutation relation $PQ-QP=\left(\frac{-ih}{2\pi}\right)I$, where h is the Plank's constant, I is the identity operator, P the quantum-mechanical momentum and Q the position. The problem here was to investigate the structure of a commutator and a non-commutator. This notion makes the study of the operator equations $AX-XB=C$, $AX-XA=C$ and others an important study to physicists.

Embry (1970) proved that if A, J and $K\in B(H)$ such that $AJ=KA$, with J and K commuting normal operators and $0\notin W(A)$, then $J=K$. This result acted like a lemma to the key result, namely Theorem 2.5 in chapter two, where an improvement of Brooke et al.'s (2002) result was made. It should be noted here that Embry's (1970) equation is similar to the main operator equation $AB=\lambda BA$. Allowing $J=B$ and $K=\lambda B$ in $AJ=KA$, we end up with the operator equation $AB=\lambda BA, AB\neq 0$, where $\lambda\in\mathbb{C}$, which has its applicability in interpretation of quantum mechanical observables and analysis of their spectra.

Moajil (1976) proved that if N is a normal operator such that $N^2X=XN^2$ and $N^3X=XN^3$, for some $X\in B(H)$, then $NX=XN$. Kittaneh (1986) generalized this result to cover subnormal operators by taking A and B^* to be subnormal operators. In other words, if $A^2X=XB^2$ and $A^3X=XB^3$, then $AX=XB$. Bachir

(2004) generalized the above two results further to cover the classes of dominant and p-hyponormal operators and proved that, if A is a dominant operator and B^* is a p-hyponormal or log-hyponormal operator, then $A^2X = XB^2$ and $A^3X = XB^3$ implies $AX = XB$. Thus, the same results were achieved despite going on to bigger classes of operators. Similar results have been obtained in the third chapter of this thesis, with no class specification being made to the operators A and B. Instead the property of A and B being quasiaffinities was used to achieve the same result.

Hlandik and Omladic (1988) had picked two operators A and B on a Hilbert space H and showed that $\sigma(AB) = \sigma(BA) = \sigma(PAP)$, if the operator B is positive and P is the positive square root of the operator B. We show similar results, but working with two $\lambda-$commuting operators A and B. Furthermore, we look at subsets of the spectra. Diverting to this situation, Williams (1981), proved that if T is a pure and dominant operator, K compact and having dense range with $KT = TK$, then the essential spectrum of T and the spectrum of T are equal. It is noted that the spectrum of an operator, say T, is simply its eigen- values in finite dimensional spaces and that the essential spectrum is a subset of the spectrum. With A and B, $\lambda-$commuting non-trivially, conditions under which AB and BA or B and λB have same spectrum or same essential spectrum, were reported in chapter four.

Brooke et al. (2002) showed that, if $A, B \in B(H)$ such that $AB \neq 0$ and $AB = \lambda BA$ for $\lambda \in \mathbb{C}$, with A and B self-adjoint and one of them positive, then $\lambda = 1$. Looking at bigger classes of operators in this thesis like the normal operators, this result becomes a mere corollary thus, making an improvement on this theorem.

Khalagai and Nyamai (2006) studied the operator equations $AXB = X$ and $A^*XB^* = X$. Their study, motivated parallel results to be obtained in chapter three such as $A^3XB^3 = X$ and $A^2XB^2 = X$, to imply $AXB = X$, without necessarily specifying the classes in which A and B belong to. These properties are one of the operators being injective and the other having dense range.

1.2 Definitions and Notations

- Inner product space: Let X be a vector space over the complex scalars $\mathbb{C}$. If there exists a complex number $\langle x,y\rangle$, for each pair of vectors $x, y \in X$ satisfying the following:
  i. $\langle x,x\rangle \geq 0$ for all x in X and $\langle x,x\rangle = 0$ if and only if $x = 0$.
  ii. $\langle y,x\rangle = \overline{\langle x,y\rangle}$ for all x and y in X.
  iii. $\langle x+y,z\rangle = \langle x,z\rangle + \langle y,z\rangle$ for all x, y and z in X.
  iv. $\langle \lambda x,y\rangle = \lambda\langle x,y\rangle$ for all x and y in X and all $\lambda \in \mathbb{C}$.

  Then $\langle x,y\rangle$ is an inner product space.
- A complex vector space X having the inner product is called an inner product space or a **pre-Hilbert space**.
- Complete space: a space M is said to be complete, if every Cauchy sequence of points in M has a sequence that converges in M.
- Hilbert space: A complete inner product space is said to be a **Hilbert space**.
- A linear operator T on a Hilbert space is said to be bounded, if there exists $c > 0$ such that $\|Tx\| \leq c\|x\|$, for all $x \in H$ where;

  $\|T\| = \inf\{c > 0: \|Tx\| \leq c\|x\| \text{ for all } x \in H \}$.

- If H and K are (complex) Hilbert spaces, the (bounded linear) operator $X : H \to K$ is said to be quasi-invertible, if and only if it is injective and has dense range.
- Two operators A and B are quasi-similar provided there exists quasi-invertible operators $X : H \to K$ and $Y : K \to H$ such that $XA = BX$ and $YB = AY$.
- Let $B(H)$ denote the algebra of bounded linear operators on an infinite dimensional complex Hilbert Space H to itself.
- $C(A,B)X$ and $R(A,B)X$ are defined by the following relations: $C(A,B)X = AX - XB$ and $R(A,B) = AXB - X$.
- The range of an operator A is denoted by $R(A)$.
- $R(A) = \{Ax : x \in H\}$.
- Spectrum of an operator $A \in B(H)$ denoted by $\sigma(A)$ is the set $\sigma(A) = \{\lambda \in \mathbb{C}: A - \lambda I$ is not invertible}.
- $\lambda \in \sigma_p(A)$, where $\sigma_p(A)$ denotes the point spectrum of A, if λ is such that $(A - \lambda I)^{-1}$ does not exist. It is the $\sigma(A)$ in a finite dimensional space, otherwise it is a decomposition of the $\sigma(A)$.
- The essential and approximate point spectrum of an operator A, is also a decomposition of $\sigma(A)$, denoted by $\sigma_e(A)$ and $\sigma_\pi(A)$ respectively.
- $\ker A$ = kernel of A $= \{v \in V : A(v) = 0\}$, where $A : V \to W$ is any linear transformation.
- Cokernel of A is given by: $co\ker A = \ker A^* = \{v \in V : A^*(v) = 0\}$.
- Let $AA^*B = B$. Then B has dense range if $AA^* = I$.
- An operator A is said to be Fredholm, if its range is closed and both $\ker A$ and $\ker A^*$ are finite dimensional.

- $\lambda \in \sigma_e(A)$, where $\sigma_e(A)$ denotes the essential spectrum of A, if λ is such that $A-\lambda I$ is not Fredholm.
- $\lambda \in \sigma_\pi(A)$, where $\sigma_\pi(A)$denotes the approximate point spectrum of A, if λ is such that $A-\lambda I$ is not bounded below.
- Numerical range of A, denoted by $W(A)$, is the set $\{\langle Ax,x\rangle : \|x\|=1\}$ for $x \in H$.
- The adjoint of an operator A will be denoted by A^*.
- An operator $C \in B(H)$ is a commutator, if there exists $A,B \in B(H)$ such that $C = AB - BA.$.
- The commutator of A and B will be denoted by $[A,B]$ where;

  $[A,B] = AB - BA.$
- If the commutator $C = [A,B] = 0,$ then A is said to be in the commutant of $B,$ while B is said to be in the commutant of A. This is denoted by $A \in \{B\}'$ and $B \in \{A\}'$ respectively. Thus, the commutant of A will be denoted $\{A\}'$ and is given by the set $\{A\}' = \{X \in B(H) : [A,X] = 0\}.$
- The class $\mathcal{H} \cup (p)$ denotes the class of p-hyponormal operators A with the polar decomposition $A = U|A|,$ where $|A| = (A^*A)^{\frac{1}{2}}$ and U is unitary.

The operator A will be said to be:

- Self-adjoint if $A = A^*.$
- A projection if $A^2 = A$ and $A^* = A.$
- An involution if $A^2 = I$. In other words a projectivity of period 2.
- Normal if $A^*A = AA^* \ or \ \|Ax\| = \|A^*x\| \forall x \in H.$

- Dominant if to each $\lambda \in \mathbb{C}$ there corresponds a number $M_\lambda \geq 1$ such that for all $x \in H$, $\left\|(A-\lambda I)^* x\right\| \leq M_\lambda \left\|(A-\lambda I)x\right\|$.
- $M-$hyponormal if there is a constant M such that $M_\lambda \leq M$ for all $\lambda \in \mathbb{C}$ such that $\left\|(A-\lambda I)^* x\right\| \leq M \left\|(A-\lambda I)x\right\|$.
- Alternatively, A is $M-$hyponormal if there exists a positive number M such that: $M^2(A-\lambda)^*(A-\lambda) \geq (A-\lambda)(A-\lambda)^* \, \forall \lambda \in \mathbb{C}$.
- Hyponormal if from above $M=1$ or $A^*A \geq AA^*$ or $\|Ax\| \geq \|A^*x\| \forall x \in H$.
- P-hyponormal if $\left(A^*A\right)^p \geq \left(AA^*\right)^p$ for $0<p<1$.
- Log-hyponormal if A is an invertible operator such that $\log\left(A^*A\right) \geq \log\left(AA^*\right)$.
- Paranormal if $\|Ax\|^2 \leq \left\|A^2x\right\|$ for any unit vector $x \in H$ or $\|Ax\|^2 \leq \left\|A^2x\right\|\|x\|$.
- Subnormal if A has a normal extension. More precisely, an operator A on a Hilbert space H is subnormal if there exists a normal operator B on a Hilbert space K such that H is a subspace of K.
- Quasinormal if $\left[A^*A, A\right]=0$.
- 2 – normal if $A^*A^2 = A^2A^*$ and binormal if $\left[A^*A, AA^*\right]=0$.
- Posinormal if there exists a positive operator $P \in B(H)$ called the interrupter such that $AA^* = A^*PA$.
- Totally posinormal if the translates $A-\lambda$ are posinormal $\forall \lambda \in \mathbb{C}$.
- Partial isometry if $A = AA^*A$.
- Isometry if $A^*A = I$ or $\|Ax\| = \|x\|$.
- Co-Isometry if $AA^* = I$.
- Unitary if $A^*A = AA^* = I$.

- Compact if for each bounded sequence $\{x_n\}$ in the domain H, the sequence $\{Ax_n\}$ contains a subsequence converging to some limit in the range.
- Contraction if $\|A\| \le 1$.
- Positive if A is self adjoint and $\langle Ax, x \rangle \geq 0$ for $0 \neq x \in H$. Positive definite (strictly positive) if in addition $\langle Ax, x \rangle$ is real.
- Quasinilpotent if $\sigma(A) = \{0\}$.
- Nilpotent if $A^n = 0$ for some positive integer n.
- a class $\mathcal{Y}_\alpha$ operator for $\alpha \ge 1$ if there exists a positive number k_α such that $|AA^* - A^*A|^\alpha \le k_\alpha (A-\lambda)^* (A-\lambda)$ for all $\lambda \in \mathbb{C}$. Note that $\mathcal{Y}_\alpha \subset \mathcal{Y}_\beta$ if $1 \le \alpha \le \beta$. Also note that $\mathcal{Y} = \bigcup_{\alpha \ge 1} \mathcal{Y}_\alpha$ so as to have a class $\mathcal{Y}$ operator.

A Hilbert space H is said to be infinite dimensional, if the space is too big to be spanned by any finite set of vectors.

Other terminologies are given below.

- The direct sum, denoted by $\oplus$, of any pair of matrices A of size $m \times n$ and B of size $p \times q$, is a matrix of size $(m+p) \times (n+q)$ defined as $A \oplus B = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$.
- A symmetric matrix is a square matrix that is equal to its transpose.
- The adjoint of a square matrix is its conjugate transpose.

1.3 Inclusions of Classes of Operators in Hilbert Spaces

We have the following inclusions of classes of operators:

- Projection$\subseteq$Self-adjoint $\subseteq$ Normal $\subseteq$ Hyponormal $\subseteq$ Dominant.
- Self-adjoint$\subseteq$ Unitary$\subseteq$ Normal$\subseteq$ Hyponormal$\subseteq$ Paranormal.
- Normal$\subseteq$Quasinormal$\subseteq$ Subnormal$\subseteq$ Hyponormal$\subseteq$ M-hyponormal$\subseteq$ Dominant.
- Self-adjoint$\subseteq$ Unitary$\subseteq$ Normal$\subseteq$ Quasinormal$\subseteq$ Binormal.
- Self-adjoint $\subseteq$ Normal $\subseteq$Hyponormal $\subseteq$M-hyponormal $\subseteq$Dominant
- Unitary$\subseteq$ Isometry$\subseteq$ Partial Isometry$\subseteq$ Contraction.
- Unitary$\subseteq$ Isometry$\subseteq$2-Normal$\subseteq$ Binormal.
- Normal$\subseteq$ Quasinormal$\subseteq$Subnormal$\subseteq$ Hyponormal$\subseteq$Paranormal.
- Normal$\subseteq$Quasinormal$\subseteq$Subnormal$\subseteq$Hyponormal$\subseteq$P-hyponormal$\subseteq$log-hyponormal.
- Normal $\subseteq$ Hyponormal $\subseteq$ P-hyponormal $\subseteq\omega$-Hyponormal.
- Hyponormal$\subseteq$M-hyponormal$\subseteq$Quasi M-hyponormal.
- Normal $\subseteq$Quasinormal $\subseteq$Subnormal $\subseteq$ Hyponormal $\subseteq$ M-hyponormal.
- Hyponormal $\subseteq$ M-hyponormal $\subseteq$ Dominant $\subseteq$ Posinormal.

It is also important to note that among some classes of operators, it is impossible to relate them in terms of subsets of each other. For instance, unitary$\subseteq$ normal and unitary$\subseteq$ isometry, but for normal and isometric operators we cannot put them in terms of subsets. Other pairs of operators that behave like this are: partial isometric with 2-normal, quasinormal with partial isometric and quasinormal with 2-normal.

1.4 Berberian's technique

Let N and M be normal operators. If $L = \begin{pmatrix} N & 0 \\ 0 & M \end{pmatrix}$, $Y = \begin{pmatrix} 0 & 0 \\ X & 0 \end{pmatrix}$, then L is normal on $H \oplus H$ and $LY = YL$. Hence, using Fuglede's theorem $L^*Y = YL^*$. Berberian (1959), used the entries of $L^*Y = YL^*$, commonly known as "Berberian's trick", to show that the Putnam-Fuglede theorem follows from the Fuglede theorem. These two theorems are given below:

Theorem 1.4.1- Fuglede [Fuglede (1950)]
Let T and N be in $B(H)$ such that $TN = NT$ where N is normal then $TN^* = N^*T$.

Theorem 1.4.2- Putnam-Fuglede [Putnam (1951)]
Let A and B be normal operators and X be an operator such that $XA = BX$. Then $XA^* = B^*X$.

Remark 1.4.3
For the entire thesis, unless stated otherwise, all the operators dealt with belong to $B(H)$ the set of all bounded linear operators on a complex Hilbert space H. The fundamental results on linear operators in this chapter are mostly based on matrix theory, since $B(H)$, the set of all bounded linear operators on a complex Hilbert space H, is regarded as an extension of the set of all 2×2 matrices. See [17] and [21].

CHAPTER TWO

ON λ-COMMUTING OPERATORS IN HILBERT SPACES

2.0 Introduction

Two operators A and B in $B(H)$ are said to $\lambda-$commute for a scalar $\lambda \in \mathbb{C}$, provided $AB = \lambda BA \neq 0$. In this chapter, some properties satisfied by the operators A and B, so that $\lambda = 1$ are studied. When $\lambda = 1$, then A and B are said to commute. It is shown among other results that; if one of the operators raised to some power is normal and 0 does not belong to the numerical range of the other operator, then $\lambda = 1$.

To begin with, some results by other authors that will be useful in this chapter are given.

Theorem 2.1 [Brooke et al. (2002), Theorem 1.1, p. 110]

Let $AB = \lambda BA \neq 0, \lambda \in \mathbb{C}$. Then;

(i) if A or B is self-adjoint then $\lambda \in \mathbb{R}$.

(ii) if both A and B are self-adjoint then $\lambda \in \{-1,1\}$.

(iii) if A and B are self-adjoint and one of them is positive then $\lambda = 1$.

Theorem 2.2 [Brooke et al. (2002), Theorem 1.2, p. 110]
Let A and B be self-adjoint operators. Then the following statements are equivalent;

(i) $AB = UBA$ for some unitary U.

(ii) $AB^2 = B^2A$ and $BA^2 = A^2B$.

If A or B is positive then (i) is equivalent to $AB = BA$.

Theorem 2.3 [Embry (1970), Theorem 1, p. 331]

Let A, J and K be operators such that $AJ = KA.$ If J and K are commuting normal operators and $0 \notin W(A),$ then $J = K.$

Theorem 2.4 [Sheth and Khalagai (1987), Theorem 3, p. 34]

Let A and B be operators such that $[B, A^2] = 0.$ Then $[B, A] = 0,$ under any one of the following conditions;

(i) $\sigma(A) \cap \sigma(-A) = \varnothing.$

(ii) A is normal and $0 \notin W(A).$

(iii) $\{A\}' = \{A^{2m}\}'$ for a positive integer $m.$

(iv) A is normal and $0 \notin W(\operatorname{Re} A).$

(v) A is normal and $\sigma(\operatorname{Re} A) \cap \sigma(-\operatorname{Re} A) = \varnothing.$

The following theorem is an improvement of Theorem 2.1 and Theorem 2.2 above.

Theorem 2.5 [Khalagai and Kavila (2012a), Theorem 1, p.28]

Let $AB = \lambda BA \neq 0, \lambda \in \mathbb{C}$. Then we have:

(i) If $B^n \neq 0$ is normal for some positive integer n and $0 \notin W(A),$ then $[B^n, A] = 0.$

(ii) If $A^n \neq 0$ is normal for some positive integer n and $0 \notin W(B),$ then $[A^n, B] = 0.$

Proof

(i) Given $AB = \lambda BA,$ we have, $AB^2 = \lambda BAB = \lambda B \lambda BA = \lambda^2 B^2 A.$ That is, $AB^2 = \lambda^2 B^2 A.$

Post multiplying by B again yields; $AB^3 = \lambda^2 B^2 AB = \lambda^2 B^2 \lambda BA = \lambda^3 B^3 A.$

Thus, by induction, $AB^n = \lambda^n B^n A$ is true for all $n \in J^+$. Now B^n and $\lambda^n B^n$ are commuting normal operators. Hence, by Theorem 2.3, $B^n = \lambda^n B^n$ since $0 \notin W(A)$.

This implies:

$B^n - \lambda^n B^n = 0.$ Factoring out B^n gives, $\left(1-\lambda^n\right)B^n = 0.$

But $B^n \neq 0$ meaning $1-\lambda^n = 0$ which implies $\lambda^n = 1$. Thus, $AB^n = B^n A$ meaning $\left[B^n, A\right] = 0.$

(ii) Given $AB = \lambda BA,$ we have $A^2B = A\lambda BA = \lambda ABA = \lambda\lambda BAA = \lambda^2 BA^2$. That is $A^2B = \lambda^2 BA^2$. Pre-multiplying by A yields;

$A^3B = A\lambda^2 BA^2 = A\lambda\lambda BAA = \lambda^2 ABAA = \lambda^2\lambda BAAA = \lambda^3 BA^3.$

Thus, by induction, $A^nB = \lambda^n BA^n$ is true for all $n \in J^+$ which means $B\left(\lambda^n A^n\right) = A^n B.$ But $\lambda^n A^n$ and A^n are commuting normal operators. Since $0 \notin W(B),$ by Theorem 2.3, $A^n = \lambda^n A^n$. Thus, $A^n - \lambda^n A^n = 0.$ Factoring out A^n gives $\left(1-\lambda^n\right)A^n = 0.$ But $A^n \neq 0$ meaning $1-\lambda^n = 0$ which implies $\lambda^n = 1.$ Thus, $A^nB = BA^n$ meaning $\left[A^n, B\right] = 0.$

Remark 2.6
Theorem 2.5 gives the equivalent result that if one of the operators raised to some power is normal and 0 does not belong to the numerical range of the other operator, then $\lambda^n = 1$ in $AB = \lambda BA \neq 0$. The following illustration, using matrices, shows that indeed $AB = \lambda BA$ implies $AB^n = \lambda^n B^n A$ for $n \in J^+$.

Example 2.7
Let $A = \begin{pmatrix} 1 & i \\ -i & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & -i \\ i & 1 \end{pmatrix}$. Then $AB = BA$. Thus $\lambda = 1$.

Let $n = 3$. Then $AB^n = \lambda^n B^n A$. That is, $AB^3 = \lambda^3 B^3 A$.

$AB^3 = \begin{pmatrix} 5 & -3i \\ 3i & 2 \end{pmatrix}$ and $\lambda^3 B^3 A = B^3 A = \begin{pmatrix} 5 & -3i \\ 3i & 2 \end{pmatrix}$. This works for all values of $\lambda \neq 1$.

For let $A = \begin{pmatrix} 2 & 0 & 0 \\ 3 & 6 & 0 \\ 4 & 9 & 18 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$. Then $AB = \begin{pmatrix} 0 & 0 & 0 \\ 6 & 0 & 0 \\ 9 & 18 & 0 \end{pmatrix}$ and $BA = \begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 3 & 6 & 0 \end{pmatrix}$.

Therefore, $AB = 3BA$.

Let $n = 2$. Then, $AB^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 18 & 0 & 0 \end{pmatrix}$ and $\lambda^2 B^2 A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 18 & 0 & 0 \end{pmatrix}$.

In Theorem 2.5 above, if we put $n = 1$ the following corollary follows.

Corollary 2.8 [Khalagai and Kavila (2012a), Corollary 1, p.29]

Given $AB = \lambda BA \neq 0, \lambda \in \mathbb{C}$, then $[A,B] = 0$ under any one of the following conditions:

(i) A is normal and $0 \notin W(B)$.

(ii) B is normal and $0 \notin W(A)$.

Proof

By putting $n = 1$ in Theorem 2.5 it follows;

(i) If $B \neq 0$ is normal and $0 \notin W(A)$, then $[B,A] = 0$.

(ii) If $A \neq 0$ is normal and $0 \notin W(B)$, then $[A,B] = 0$.

Since $[A,B]=[B,A]$, then $[A,B]=[B,A]=0$. Thus, $[A,B]=0$ takes care of (i) and (ii).

Remark 2.9
The above corollary gives the equivalent result that, if one of the operators is normal and 0 does not belong to the interior of the numerical range of the other operator, then $\lambda=1$. This is since the condition $[A,B]=0$ implies $AB=BA$. The corollary has summarily shown that what has been achieved using self adjoint operators in Theorem 2.1, has been achieved using a bigger class of operators specifically the normal operator under further hypothesis, namely $0\notin W(A)$ or $0\notin W(B)$. Note also, Theorem 2.1's condition that A or B be positive is more stringent than a mere requirement that 0 does not belong to the numerical range of an operator. More precisely, the following corollary is an improvement of Theorem 2.1.

Corollary 2.10 [Khalagai and Kavila (2012a), Corollary 2, p.29]
Let A and B be self-adjoint operators such that $AB=\lambda BA\neq 0, \lambda\in\mathbb{C}$. Then $\lambda=1$ under any one of the following conditions:

(i) $\sigma(A)\cap\sigma(-A)=\varnothing$.

(ii) $0\notin W(A)$.

(iii) $\sigma(\operatorname{Re}A)\cap\sigma(-\operatorname{Im}A)=\varnothing$.

(iv) $\sigma(\operatorname{Re}A)\cap\sigma(-\operatorname{Re}A)=\varnothing$.

(v) $\{A\}'=\{A^{2m}\}'$ for a positive integer m.

(vi) $\sigma(B)\cap\sigma(-B)=\varnothing$.

(vii) $0\notin W(B)$.

(viii) $\sigma(\operatorname{Re}B)\cap\sigma(-\operatorname{Im}B)=\varnothing$.

(ix) $\sigma(\operatorname{Re}B)\cap\sigma(-\operatorname{Re}B)=\varnothing$.

(x) $\{B\}' = \{B^{2m}\}'$ for a positive integer $m.$

Proof

Given $AB = \lambda BA,$ we have: $A^2B = A\lambda BA = \lambda ABA = \lambda.\lambda BAA = \lambda^2 BA^2.$ By Theorem 2.1, $\lambda^2 = 1$. Thus, $A^2B = BA^2$. Which means $\left[B, A^2\right] = 0.$ In view of Theorem 2.4, each of the conditions (i) to (v) in this corollary, imply $\left[A, B\right] = 0$ and consequently $\lambda = 1$. Also $AB^2 = \lambda BAB = \lambda B\lambda BA = \lambda^2 B^2 A.$ By Theorem 2.1 again**,** $\lambda^2 = 1$. Thus, $AB^2 = B^2 A.$ This means $\left[A, B^2\right] = 0.$ Similarly in view of Theorem 2.4 above, each of the conditions (vi) to (x) in this corollary implies $\left[A, B\right] = 0$ and consequently $\lambda = 1$.

Remark 2.11

Commutators give rise to commutants, (see Sec. 1.2). With this in mind, we have the following theorem that is crucial in justifying the title of the thesis. This theorem establishes Corollaries 2.13 and 2.14 that make further improvements to Theorem 2.1.

Theorem 2.12 [Khalagai and Kavila (2012a), Theorem 2, p.29]

Let $AB = \lambda BA \neq 0, \lambda \in \mathbb{C}$. Then we have:

(i) A is self-adjoint implies $B^*B \in \{A\}'$ and $BB^* \in \{A\}'$.

(ii) B is self-adjoint implies $A^*A \in \{B\}'$ and $AA^* \in \{B\}'$.

Proof

(i) Let $AB = \lambda BA.$ Taking adjoints on both sides of the operator equation yields; $B^*A^* = \overline{\lambda} A^*B^*$. Since A is self adjoint, we get; $B^*A = \overline{\lambda} AB^*$. From Theorem 2.1, since A is self-adjoint, then $\lambda \in \mathbb{R}$ meaning $\lambda = \overline{\lambda}$. Thus; $B^*A = \lambda AB^*$. Post multiplying by B gives; $B^*AB = \lambda AB^*B.$ But $AB = \lambda BA,$ which means

$B^*\lambda BA = \lambda AB^*B$. Since λ is a scalar, $\lambda B^*BA = \lambda AB^*B$. Factoring out the scalar yields; $\lambda\left(B^*BA\right) = \lambda\left(AB^*B\right)$.

Thus, $\lambda\left(B^*BA - AB^*B\right) = 0$.

Since $\lambda \neq 0$, $B^*BA - AB^*B = 0$. Hence, $\left[A, B^*B\right] = 0$. Thus, $B^*B \in \{A\}'$.

Similarly, starting by pre-multiplying $B^*A = \lambda AB^*$ by B gives;

$BB^*A = B\lambda AB^* = \lambda BAB^* = ABB^*$. That is, $BB^*A = ABB^*$ which implies that $\left[BB^*, A\right] = 0$. Hence $BB^* \in \{A\}'$.

(ii) Since B is self-adjoint, from $B^*A^* = \overline{\lambda}A^*B^*$ we have; $BA^* = \overline{\lambda}A^*B$. From Theorem 2.1, since B is self-adjoint, then $\lambda \in \mathbb{R}$ meaning $\lambda = \overline{\lambda}$. Thus, $BA^* = \lambda A^*B$. Post multiplying by A yields; $BA^*A = \lambda A^*BA$. Since λ is a scalar, $BA^*A = A^*\lambda BA$. But $AB = \lambda BA$. Thus, $BA^*A = A^*AB$. Hence, $\left[A^*A, B\right] = 0$. Thus, $A^*A \in \{B\}'$.

Also pre-multiplying $BA^* = \lambda A^*B$ by A gives; $ABA^* = \lambda AA^*B$. But $AB = \lambda BA$.

Hence, $\lambda BAA^* = \lambda AA^*B$. Thus, $\lambda\left(BAA^* - AA^*B\right) = 0$. But $\lambda \neq 0$. Thus, $BAA^* - AA^*B = 0$.

Hence, $\left[AA^*, B\right] = 0$. Thus, $AA^* \in \{B\}'$.

Corollary 2.13 [Khalagai and Kavila (2012a), Corollary 3, p.30]

Let $AB = \lambda BA \neq 0, \lambda \in \mathbb{C}$ with B having the polar decomposition $B = UP$, (Polar decomposition with U unitary). If A is self adjoint and $[A, U] = 0$, then $\lambda = 1$.

Proof

Since A is self adjoint, by Theorem 2.12, $B^*B \in \{A\}'$. This implies $P \in \{A\}'$. That is $AP = PA$. By hypothesis, $[A, U] = 0$ which means $AU = UA$. Post multiplying by P gives: $AUP = UAP$. But $AP = PA$ hence, $AUP = UPA$.

Thus, $\left[A,UP\right]=0.$ Since $B=UP,$ $\left[A,B\right]=0$ and the result $\lambda=1$ is attained.

Corollary 2.14 [Khalagai and Kavila (2012a), Corollary 4, p.30]
Let $AB=\lambda BA\neq 0,\lambda\in\mathbb{C}$ with A having the polar decomposition $A=UP,$ (Polar decomposition with U unitary). If B is self adjoint and $\left[B,U\right]=0,$ then $\lambda=1.$

Proof
Since B is self adjoint, by Theorem 2.12, $A^{*}A\in\{B\}'$. This implies $P\in\{B\}'$. That is, $BP=PB.$ By hypothesis, $\left[B,U\right]=0$ which means $BU=UB.$ Post multiplying by P gives: $BUP=UBP.$ But, $BP=PB$ hence, $BUP=UPB.$
Thus, $\left[B,UP\right]=0.$ Since $A=UP,$ $\left[B,A\right]=0$ and the result $\lambda=1$ is attained.

The following lemma assists in providing an alternative proof to Corollaries 2.13 and 2.14 above.

Lemma 2.15
For any operator A with polar decomposition $A=UP,$ where U is unitary and P is positive, $AA^{*}=UP^{2}U^{*}.$ Similarly, if $B=UP$ then $BB^{*}=UP^{2}U^{*}.$

Proof
Note, $AA^{*}=UP(UP)^{*}=UPP^{*}U^{*}=UPPU^{*}=UP^{2}U^{*}.$

Similarly, $BB^{*}=UP(UP)^{*}=UPP^{*}U^{*}=UPPU^{*}=UP^{2}U^{*}.$

Alternative proof to Corollary 2.13

Since A is self adjoint, we have by Theorem 2.12 $BB^* \in \{A\}'$. That is, $ABB^* = BB^*A.$ Since $AB = \lambda BA,$ we have by post multiplying by B^*, $ABB^* = \lambda BAB^*.$

This means $BB^*A = \lambda BAB^*$. But $BB^* = UP^2U^*$ from Lemma 2.15 and $B = UP.$

This means $UP^2U^*A = \lambda(UP)A(UP)^*$. This implies $UP^2U^*A = \lambda UPAPU^*$. But $AP = PA.$

$UP^2U^*A = \lambda UPPAU^*$. By hypothesis $AU = UA$ therefore, $UP^2AU^* = \lambda UP^2AU^*$.

This implies $\lambda U = U$. Thus, $\lambda U - U = 0.$ Factoring out U gives $(\lambda - 1)U = 0$

But $U \neq 0,$ hence $\lambda - 1 = 0$. Thus, $\lambda = 1$.

Alternative proof to Corollary 2.14

Since B is self-adjoint we have by Theorem 2.12 $AA^* \in \{B\}'$.

That is, $BAA^* = AA^*B.$ Since $AB = \lambda BA,$ post multiplying by A^* yields:

$ABA^* = \lambda BAA^*$. But $AA^* = UP^2U^*$ from Lemma 2.15 and $A = UP.$

This means $(UP)B(UP)^* = \lambda BUP^2U^*$. Thus, $UPBPU^* = \lambda BUP^2U^*$.

By hypothesis $BU = UB$. Thus $UPBPU^* = \lambda UBP^2U^*$. But $BP = PB.$

Thus, $UPBPU^* = \lambda UPBPU^*$. This implies $U = \lambda U$. Thus, $U - \lambda U = 0.$ Factoring out U gives $(1-\lambda)U = 0.$

But $U \neq 0,$ hence $1 - \lambda = 0$. Thus, $\lambda = 1$.

Similar results yielding $\lambda = 1$ in $AB = \lambda BA,$ are given in the theorem below. This is done by letting the operators A and B satisfy another operator equation.

Theorem 2.16

Let A satisfying $AB = \lambda BA \neq 0,$ where $\lambda \in \mathbb{C},$ be isometric such that $A^*B = BA^*.$ Let $B \neq 0$ be any operator, then $\lambda = 1$.

Proof

Let $AB = \lambda BA$. Pre-multiplying both sides by A^* yields $A^*AB = \lambda A^*BA$.

By hypothesis $A^*B = BA^*$. Therefore, $A^*AB = \lambda BA^*A$. But $A^*A = I$, since A is an isometry. Thus, $B = \lambda B$. This implies $B - \lambda B = 0$. Thus, $(1-\lambda)B = 0$.

But $B \neq 0$. Hence, $1 - \lambda = 0$. This means $\lambda = 1$.

Remark 2.17

Since Unitary$\subseteq$ Isometry, Theorem 2.16 will still hold with A being unitary. Remodelling the operator equation to $AB = BA^*$, the following theorem proved by Moore et al. (1981) was useful in giving the proof of Corollary 2.19 below. This time round, another value of λ is attained.

Theorem 2.18 [Moore et al. (1981), Theorem 1, p. 514]

Let A be $M-$hypornormal and suppose that $AB = BA^*$, then $A^*B = BA$.

Corollary 2.19

Let $AB = \lambda BA \neq 0,$ with A being $M-$hypornormal such that $AB = BA^*$, then, λ∈{1, -1}.

Proof

Let $AB = \lambda BA$..(i).

Since $AB = BA^*$, (i) becomes $BA^* = \lambda BA.$ Post multiplying by B gives $BA^*B = \lambda BAB$..(ii).

By Theorem 2.18**,** since A is $M-$hypornormal, $AB = BA^*$ implies $A^*B = BA.$ Thus (ii) becomes:

$BBA = \lambda BAB$...(iii).

But $AB = \lambda BA$. Hence, (iii) becomes $BBA = \lambda B\lambda BA = \lambda\lambda BBA$. This implies that $\lambda\lambda = 1$. Meaning that either $\lambda = 1$ or $\lambda = -1$.

The following theorem was required in order to prove Corollary 2.21.

Theorem 2.20 [Cho et al. (2010), p. 2630 – p. 2631]

A necessary and sufficient condition for normal operators $A = A_1 + iA_2$ and $B = B_1 + iB_2$ to $\lambda -$commute is that:

i. $AB = \lambda BA.$
ii. $A^* B = \overline{\lambda} BA^*.$
iii. $AB^* = \overline{\lambda} B^* A.$
iv. $A^* B^* = \lambda B^* A^*.$

Corollary 2.21
Let A and B be normal such that $AB = \lambda BA \neq 0, \lambda \in \mathbb{C}.$ Then $\lambda\overline{\lambda} = 1.$

Proof
Let $AB = \lambda BA \neq 0.$ Taking adjoints both sides we get:

$B^* A^* = \overline{\lambda} A^* B^*.$

Multiplying by λ both sides yields:

$\lambda B^* A^* = \lambda\overline{\lambda} A^* B^*.$

But by Theorem 2.20 part (iv), $A^* B^* = \lambda B^* A^*.$ Thus:

$\lambda B^* A^* = \lambda\overline{\lambda} A^* B^* = A^* B^*.$

Hence $\lambda\overline{\lambda} = 1.$

Remark 2.22
Recently, Zhang et al. (2011) had the following results on the operator equation $AB = \lambda BA.$ These results are stated to conclude this chapter and show that the equation continues to form a buzz in research. Starting from $\lambda = 1,$ they studied the properties of the product AB given certain conditions of the operators A and $B.$ On the other hand, given certain spectral properties of the product $AB,$ the

values attained by λ were also investigated. Thus, the theorems stated below will aid in suggesting areas for future research in chapter five.

Theorem 2.23 [Zhang et al. (2011), Theorem 5, p. 1689]
Let A be a paranormal operator and B be an isometry such that $AB = \lambda BA \neq 0, \lambda \in \mathbb{C}$. Then

i) $\lambda = 1 \Rightarrow AB$ is paranormal.

ii) $\sigma(AB) \neq \{0\} \Rightarrow |\lambda| = 1$.

Theorem 2.24 [Zhang et al. (2011), Theorem 4, p. 1688]
Let A be a hyponormal operator and B be a normal operator such that $AB = \lambda BA \neq 0, \lambda \in \mathbb{C}$. Then:

i) $\lambda = 1 \Rightarrow AB$ is hyponormal.

ii) $\sigma(AB) \neq \{0\} \Rightarrow |\lambda| = 1$.

Remark 2.25
Note that the above theorems by Zhang et al., remind us of the well-known result that states that the product of two normal operators that commute is again normal. This result is non-trivial and it follows from Putnam-Fuglede's theorem stated in Theorem 1.4.2. In general, if two normal operators A and B do not $\lambda-$commute, then the product AB fails to be normal. Specifically, Cho et al. (2011), Theorem 4, p.74, proved that for product invariance in normal operators to hold, we should have $|\lambda| = 1$ where $\lambda \neq 1$.

CHAPTER THREE

ON COMMUTANTS AND OPERATOR EQUATIONS

3.0 Introduction

This chapter is based on the operator equations $AX = XB$ and $AXB = X$. Some properties satisfied by the operators A and B, so that $C(A^2, B^2)X = 0$ and $C(A^3, B^3)X = 0$ imply $C(A, B)X = 0$, without necessarily specifying the classes of operators A and B belong to, are given. It is shown among other results that if A is one-one and B has dense range, then $C(A^2, B^2)X = 0$ and $C(A^3, B^3)X = 0$ imply $C(A, B)X = 0$. Similarly, if $R(A^2, B^2)X = 0$ and $R(A^3, B^3)X = 0$, then $R(A, B)X = 0$.

To start with, we look at some authors who have proved the above results and thereafter prove parallel results obtained in this subject matter.

Theorem 3.1 [Moajil (1976), Proposition 3.6, p.248]
If N is a normal operator, such that $N^2X = XN^2$ and $N^3X = XN^3$ for some operator X, then $NX = XN$.

The above Theorem 3.1, was generalized by the following Corollary to Theorem 2, p.48 of Kittaneh (1986) by looking at a larger class of operators.

Corollary 3.2 [Kittaneh (1986), Corollary 1, p.48]

If A and B^* are subnormal operators, such that $A^2X = XB^2$ and $A^3X = XB^3$ for some X, then $AX = XB$. Thus, if $C(A^2, B^2)X = 0$ and $C(A^3, B^3)X = 0$, then $C(A, B)X = 0$ for some X.

Remark 3.3

Since dominant and p-hyponormal operators are bigger classes of operators than subnormal operators, (see Sec. 1.3), Bachir (2004) further generalized Corollary 3.2 to cover the classes of dominant and p-hyponormal operators as follows:

Theorem 3.4 [Bachir (2004), Theorem 3.5, p. 115]

Let A be a dominant operator and B^* be a p-hyponormal operator or log-hyponormal such that $C(A^2, B^2)X = 0$ and $C(A^3, B^3)X = 0$, then $C(A, B)X = 0$, for some operator X.

Remark 3.5

The following results on the operator equation $R(A, B)X = 0$, which will be used to prove similar results to those of $C(A, B)X = 0$, were proved by the following authors:

Theorem 3.6 [Khalagai and Nyamai (2006), Theorem 1, p.15]

Let A, B and X be such that $R(A, B)X = 0$. Then B is one to one, whenever X is one to one.

Corollary 3.7 [Khalagai and Nyamai (2006), Corollary 1, p.15]

Let A, B and X be such that $R(A, B)X = 0$, where X is a quasiaffinity. Then both B and A^* are one to one.

Corollary 3.8 [Khalagai and Nyamai (2006), Corollary 2, p.15]

Let A, B and X be such that $R(A,B)X = 0$ implies $R(A^*, B^*)X = 0$, where X is a quasiaffinity. Then both A and B are also quasiaffinities.

Theorem 3.9 [Goya and Saito (1981), Theorem 2, p.128]

Let A be a paranormal contraction, B a co-isometry and X have dense range. Assume $C(A,B\ )X = 0$. Then A is a unitary operator. In particular, if X is injective and has a dense range, then B is also a unitary operator.

Remark 3.10

The following results show properties satisfied by the operators A and B so that $C(A^2, B^2)X = 0$ and $C(A^3, B^3)X = 0$ imply $C(A,B)X = 0$, without necessarily letting A and B belong to a particular class of operators.

Theorem 3.11 [Khalagai and Kavila (2012b), Theorem 1, p.101]

Let A and B be any pair of operators such that A is one-one and B has dense range. Then we have that $C(A^2, B^2)X = 0$ and $C(A^3, B^3)X = 0$ imply $C(A,B)X = 0$ for some operator X.

Proof

Let $T = AX$ and $S = XB$. Then from $A^2X = XB^2$ and $A^3X = XB^3$, we have the following two relations;

$AT = SB$ ....(i).

$A^2T = SB^2$ ....(ii).

Pre-multiplying (i) by A yields: $A(AT) = ASB$ ....(iii).

From (ii) and (iii) we get; $ASB = SB^2$. Thus; $ASB - SB^2 = 0$. Factoring out B gives; $(AS - SB)B = 0$. Since B has dense range, we have that $B \neq 0$ and hence $AS - SB = 0$. That is $AS = SB$. Using (i) gives; $AT = SB = AS$. This means $AT - AS = 0$. Thus, $A(T - S) = 0$. Since A is one to one, $A \neq 0$. Hence $T - S = 0$. This means $T = S$. Thus, $AX = XB$. This means $AX - XB = 0$. Hence, $C(A, B)X = 0$.

Corollary 3.12 [Khalagai and Kavila (2012b), Corollary 1, p.102]

If A and B are quasi-affinities such that $C(A^2, B^2)X = 0$ and $C(A^3, B^3)X = 0$, then $C(A, B)X = 0$, for some operator X.

Proof

If A and B are quasi-affinities then each one of them is both one-one and has dense range. Hence, by Theorem 3.11, the required result is achieved since in particular A can be one-one and B have dense range.

Corollary 3.13 [Khalagai and Kavila (2012b), Corollary 2, p.102]

If A is a quasi-affinity such that $C(A^2, A^{*2})X = 0$ and $C(A^3, A^{*3})X = 0$, then, $C(A, A^*)X = 0$, for some $X \in B(H)$.

Proof

Here it should be noted that if A is a quasi-affinity then A^* is also a quasi-affinity. Hence, by Corollary 3.12 the result follows by simply replacing A^* with B.

Corollary 3.14 [Khalagai and Kavila (2012b), Corollary 3, p.102]

Let $\Im$ be the class of operators defined as follows:

$\Im = \{A \in B(H) : 0 \notin W(A)\}$.

If $A, B \in \Im$ such that $C(A^2, B^2)X = 0$ and $C(A^3, B^3)X = 0,$ then $C(A, B)X = 0,$ for some operator X.

Proof

We only have to note that for any operator A and B with $0 \notin W(A)$ and $0 \notin W(B)$ respectively, then both A and B are one to one and have dense range. Once again by Theorem 3.11, $C(A^2, B^2)X = 0$ and $C(A^3, B^3)X = 0$ imply $C(A, B)X = 0.$

Corollary 3.15 [Khalagai and Kavila (2012b), Corollary 4, p.102]

If A is a quasi-affinity such that $A^2 \in \{X\}'$ and $A^3 \in \{X\}'$, then $A \in \{X\}'$ for some $X \in B(H)$. This is equivalent to putting it as, If A is a quasi-affinity, such that $C\left(A^2, A^2\right)X = 0$ and $C\left(A^3, A^3\right)X = 0$ then $C\left(A, A\right)X = 0.$

Proof

Put $A = B$ in Corollary 3.12.

Theorem 3.16 [Khalagai and Kavila (2012b), Theorem 2, p.102]

Let $A, B \in B(H)$ be a pair of operators, such that A is one-one and B has dense range. Then $R(A^2, B^2)X = 0$ and $R(A^3, B^3)X = 0$ imply $R(A, B)X = 0,$ for some operator X.

Proof

Given $A^2XB^2 = X$ and $A^3XB^3 = X$ we have $A^2XB^2 = A^3XB^3$.

This means that $A^3XB^3 - A^2XB^2 = 0.$

This imply $A(A^2XB^2 - AXB)B = 0$ ....$(*)$.

Since A is one-one and B has dense range, $(*)$ means that; $A^2XB^2 - AXB = 0.$ This implies $A(AXB - X)B = 0.$ Again, since *A* is one-one and B has dense range, $AXB - X = 0$ .Hence, $R(A, B)X = 0$.

Corollary 3.17 [Khalagai and Kavila (2012b), Corollary 5, p.103]

If $A, B \in B(H)$ are quasi-affinities, such that $R\left(A^2, B^2\right)X = 0$ and $R\left(A^3, B^3\right)X = 0$, then $R\left(A, B\right)X = 0.$

Proof

By Corollary 3.12, a quasi-affinity is both one to one and has dense range. Hence, result is immediate by Theorem 3.16 above.

Corollary 3.18

If $A, B \in \Im$ such that $R(A^2, B^2)X = 0$ and $R(A^3, B^3)X = 0,$ then $R(A, B)X = 0,$ for some operator X.

Proof

We only have to note that for any operator A and B with $0 \notin W(A)$, and $0 \notin W(B)$ respectively, then both A and B are one to one and have dense range. By Theorem 3.16, $R(A^2, B^2)X = 0$ and $R(A^3, B^3)X = 0$ imply $R(A, B)X = 0.$

Corollary 3.19 [Khalagai and Kavila (2012b), Corollary 6, p.103]

If A is a quasi-affinity such that: $R(A^2, A^{*2})X = 0$ and $R(A^3, A^{*3})X = 0,$ then $R(A, A^*)X = 0$ for some X.

Proof

It is immediate from Corollary 3.17 by having $B = A^*$ and using the fact that if A is a quasi-affinity, then A^* is also a quasi-affinity.

The following illustration will be useful in Corollary 3.21.

Illustration 3.20

If $A \in \Im$ then $A^* \in \Im$. This is equivalent to having that if $0 \notin W(A)$ then $0 \notin W(A^*)$. In Furuta (2001) pp. 98, we have that if $\lambda \notin \overline{W(A)}$ then $\lambda \notin \sigma(A)$. This implies $\sigma(A) \subset \overline{W(A)}$. Furthermore, Shapiro (2003), p. 12, in his proof to show that $\sigma(A) \subset \overline{W(A)}$, stated within the proof that if $0 \in W(A^*)$, then $0 \in W(A)$. We also know that every positive definite operator say A, has $0 \notin W(A)$. If A is positive definite then A^* is also positive definite. For let $A = \begin{bmatrix} 4 & -2 \\ -2 & 3 \end{bmatrix}$. Then this matrix is positive definite since it is symmetric and has positive eigenvalues namely 1.44 and 5.56, correct to two decimal places. The adjoint of A is $A^* = \begin{bmatrix} 4 & -2 \\ -2 & 3 \end{bmatrix}$. Thus, if A is positive definite, then A^* is also positive definite. Which cements the fact that if $0 \notin W(A)$, then $0 \notin W(A^*)$. With this in mind, the following corollary is established.

Corollary 3.21

If $A \in \Im$ such that $R(A^2, A^{*2})X = 0$ and $R(A^3, A^{*3})X = 0$, then $R(A, A^*)X = 0$, for some operator X.

Proof

Since $A \in \Im,$ we then have $A^* \in \Im.$ Hence A and A^* are one to one and have dense range. By having $B = A^*$ in Theorem 3.16, $R(A^2, A^{*2})X = 0$ and $R(A^3, A^{*3})X = 0$ imply $R\left(A, A^*\right)X = 0.$

Corollary 3.22 [Khalagai and Kavila (2012b), Corollary 7, p.103]

If $R(A\ ,B)X = 0$ implies $R(A^*, B^*)X = 0$ for some $X,$ which is a quasi-affinity, then $C(A^2, B^2)X = 0$ and $C(A^3, B^3)X = 0$ imply $C(A\ ,B\ )X = 0.$

Proof

$R(A\ ,B)X = 0$ Implying $R(A^*, B^*)X = 0,$ where X is a quasiaffinity implies A and B are quasiaffinities from Corollary 3.8. From Theorem 3.11, the result follows since quasiaffinities are both one to one and have dense range.

Corollary 3.23 [Khalagai and Kavila (2012b), Corollary 8, p.103]

Let A, B and X be operators with A a paranormal contraction, B a co-isometry and X a quasi-affinity. If $C(A, B^*)X = 0,$ then $R(A, B)X = 0 = R\left(A^*, B^*\right)X.$

Proof

First note that B is unitary from Theorem 3.9. By hypothesis $C(A, B^*)X = 0.$ This is equivalent to having $AX = XB^*.$ Post-multiplying by B yields; $AXB = XB^*B.$ Since B is unitary, $AXB = X.$ This is equivalent to having $AXB - X = 0.$ Thus, $R\left(A, B\right)X = 0.$

On the other hand, starting again from $AX = XB^*$ and pre-multiplying by A^* gives; $A^*AX = A^*XB^*.$ By Theorem 3.9, A is also unitary. Thus, $X = A^*XB^*.$ This is equivalent to having $A^*XB^* - X = 0.$ Thus, $R\left(A^*, B^*\right)X = 0.$

On the operator equation $C(A,B)X=0,$ the following theorem will be used to prove Corollary 3.25.

Theorem 3.24 [Moore et al. (1981), Theorem 2, p. 515]

Let A and B be $M-$hyponormal operators such that $C(A,B^*)X=0,$ then $C(A^*,B)X=0$ for some operator X.

Corollary 3.25

Let A and B be $M-$hyponormal operators with A positive, such that $C(A,B^*)X=0,$ then $C(A^n,B^n)X=0$ for some operator X.

Proof

By Theorem 3.24, $C(A^*,B)X=0.$ This means $A^*X=XB.$ Pre- multiplying by A yields; $AA^*X=AXB.$ Since A is positive, it is self adjoint. Thus; $A^2X=AXB.$ But $AX=XB$ therefore; $A^2X=XB^2.$ Thus, $C(A^2,B^2)X=0.$ Again pre-multiplying by A yields; $AA^2X=AXB^2.$ But $AX=XB.$ Thus, $A^3X=XB^3.$ Hence by induction $C(A^n,B^n)X=0.$

We end this section with a result, which is put as a proposition, to connect these two operator equations $C\left(A,B\right)X=0$ and $R\left(A,B\right)X=0.$

Proposition 3.26

Let A and B be operators with B being an involution, then $C\left(A,B\right)X=0$ if and only if $R\left(A,B\right)X=0.$

Proof

Since $C(A,B)X = 0,$ then $AX = XB.$ Post multiply by B yields:

$AXB = XB^2.$ Since B is an involution, $B^2 = I.$ Thus, $AXB = X.$ Hence, $R(A,B)X = 0.$

Conversely, let $R(A,B)X = 0.$ Then $AXB = X.$ Post multiplying by B gives; $AXB^2 = XB.$

But B is an involution, thus $B^2 = I$ which means $AX = XB.$ Hence, $C(A,B)X = 0.$

Remark 3.27

From the above Proposition 3.26, the following corollary follows.

Corollary 3.28

Let A and A^* be operators with A^* being an involution, then $C(A,A^*)X = 0$ if and only if $R(A,A^*)X = 0.$

Proof

Replace B with A^* in Proposition 3.26

CHAPTER FOUR

ON SPECTRAL PROPERTIES OF λ-COMMUTING OPERATORS

4.0 Introduction

In this final chapter the main operator equation $AB = \lambda BA,$ where $\lambda \in \mathbb{C},$ is looked at again and we investigate the conditions under which AB and BA or B and λB have same spectrum or same essential spectrum.

Remark 4.1

The following results by Williams (1981) will be required to establish some of our results in this section.

Theorem 4.2 [Williams (1981), Theorem 2.1, p.130]

Suppose that T is a pure and dominant operator and K a compact operator with dense range such that $KT = TK,$ then essential spectrum of T is equal to spectrum of $T.$ That is $\sigma_e(T) = \sigma(T).$

Theorem 4.3 [Williams (1981)), Theorem 2.2, p.130]

Suppose that T is a pure dominant operator and K a compact operator with $KT = TK,$ then K is quasinilpotent.

Remark 4.4

The following lemma and proposition by Brooke et al. (2002) will also be useful in proving some results in this chapter.

Lemma 4.5 [Brooke et al. (2002), Lemma 2.1, p.111]

Let $AB = \lambda BA,\ \lambda \in \mathbb{C},\ AB \neq 0.$ Then 0 is in either both or neither of $\sigma(AB)$ and $\sigma(BA)$. Hence $\sigma(AB) = \sigma(BA) = \lambda\sigma(AB)$.

Proposition 4.6 [Brooke et al. (2002), Proposition 2.2. p. 112]

Let $AB = \lambda BA,\ \lambda \in \mathbb{C},\ AB \neq 0$ and assume that A has a bounded inverse. Then; $\sigma(B) = \lambda\sigma(B)$.

The following theorem provides an **alternative prove** to part (ii) of Theorem 2.1 and aids in proving the corollary that follows.

Theorem 4.7 [24]

Let $AB = \lambda BA,\ \lambda \in \mathbb{C},\ AB \neq 0$ with A and B self-adjoint. Then:

(i) AB and BA are normal commuting operators.

(ii) $\lambda = \pm 1.$

Proof

Let $T = AB.$ Then $T^* = (AB)^* = B^*A^* = BA$ for self-adjoint A and $B.$

(i) Since by hypothesis $AB = \lambda BA,$ we have $T = \lambda T^*.$

Pre multiplying by T^* gives $T^*T = \lambda T^{*2}.$ On the other hand, post multiplying $T = \lambda T^*$ by T^* also gives $TT^* = \lambda T^{*2}.$ Thus, $T^*T = TT^*.$ Therefore, T is normal and $\left[T, T^*\right] = 0.$ Hence, AB and BA are normal commuting operators.

(ii) Since $T = \lambda T^*$ and λ is real, taking adjoints of each side gives: $T^* = \lambda T.$ Thus:

$$T = \lambda T^* \text{..} (1).$$

$$T^* = \lambda T \text{...} (2).$$

Adding (1) and (2) gives:

$$T + T^* = \lambda\left(T + T^*\right).$$

$$\mathrm{Re}T = \lambda\left(\mathrm{Re}T\right).$$

$$\left(1-\lambda\right)\mathrm{Re}T = 0 \text{..} (3).$$

But subtracting (2) from (1) gives:

$$T - T^* = \lambda\left(T^* - T\right).$$

$$= -\lambda\left(T - T^*\right).$$

That is, $\mathrm{Im}T = -\lambda\,\mathrm{Im}T.$

or

$$\left(1+\lambda\right)\mathrm{Im}T = 0 \text{...} (4).$$

But, since $T \neq 0,$ we have that either $\left(\mathrm{Re}T\right) \neq 0$ or $\left(\mathrm{Im}T\right) \neq 0$ or both $\left(\mathrm{Re}T\right) \neq 0$ and $\left(\mathrm{Im}T\right) \neq 0.$

Hence, from (3) and (4), $\lambda = \pm 1.$

Corollary 4.8 [24]

Let A and B be self-adjoint operators which $\lambda-$commute non-trivially. Then we have:

$\sigma_\pi\left(AB\right) = \sigma_\pi\left(BA\right) = \lambda\sigma_\pi\left(AB\right).$

Proof

Since A and B are self-adjoint operators which $\lambda-$commute non-trivially, AB and BA are normal commuting operators from Theorem 4.7. Hence, $T = AB = BA.$

Now for any normal operator T, $\sigma(T)=\sigma_\pi(T)$. (See Mattila (1978) Theorem 4.7). Therefore, $\sigma_\pi(AB)=\sigma_\pi(BA)=\lambda\sigma_\pi(AB)$ from Lemma 4.5.

The following theorem will be used to prove the corollary that follows.

Theorem 4.9 [24]

Let $A,B\in B(H)$ be such that $AB=\lambda BA,\ \lambda\in\mathbb{C},\ AB\neq 0$. We have:

(i) If A is normal and B is self-adjoint then $\lambda\in\{1,-1\}$.

(ii) If A is normal with $\operatorname{Re}A$ positive and B is self-adjoint then $\lambda=1$.

Proof

(i) Given A is normal, B is self-adjoint and $AB=\lambda BA$, we have by part (i) of Theorem 2.1 that λ is real. Thus, by Theorem 1.4.1 we have: $A^*B=\lambda BA^*$. Now from $AB=\lambda BA$ and $A^*B=\lambda BA^*$ we have:
$(A+A^*)B=\lambda B(A+A^*)$.
i.e. $\operatorname{Re}A.\ B=\lambda B.\ \operatorname{Re}A.$
Since both $\operatorname{Re}A$ and B are self-adjoint, we have by part (ii) of Theorem 2.1 that $\lambda\in\{1,-1\}$.

(ii) From proof of part (i) above we have: $\operatorname{Re}A.\ B=\lambda B.\ \operatorname{Re}A$, where both $\operatorname{Re}A$ and B are self-adjoint. Since $\operatorname{Re}A$ is positive, it follows from part (iii) of Theorem 2.1 above that $\lambda=1$.

Corollary 4.10 [24]

Let A be normal with $\operatorname{Re}A$ positive and B be self-adjoint such that $AB=\lambda BA$, $\lambda\in\mathbb{C},\ AB\neq 0$. Then, $AB=BA$ is normal. Thus:

$\sigma(AB)=\sigma(BA)=\sigma_\pi(AB)=\lambda\sigma_\pi(AB).$

Proof

We note that under the given hypothesis $\lambda=1$ by part (ii) of Theorem 4.9. Thus, $[A,B]=0.$ Letting $T=AB,\ T^*=BA^*.$ Therefore:

$T^*T=BA^*AB=BAA^*B=ABA^*B$..(1)

Applying Theorem 1.4.1, $A^*B=BA^*.$ Hence (1) becomes:

$T^*T=ABBA^*$..(2).

But $ABBA^*=TT^*.$ Therefore from (2) $T^*T=TT^*.$

Hence $T=AB$ is normal. Consequently applying Lemma 4.5 we have:

$\sigma(AB)=\sigma(BA)=\sigma_\pi(AB)=\lambda\sigma_\pi(AB).$

Let $\wp$ denote the class of all operators which satisfy Putnam-Fuglede property. Then we have the following result:

Theorem 4.11 [24]

Let $AB=\lambda BA,\ \lambda\in\mathbb{C},\ AB\neq 0,$ we have:

(i) if $B,\lambda B\in\wp$ with A a quasiaffinity, then λB and B are quasi-similar.

(ii) if $A,\lambda A\in\wp$ with B a quasiaffinity, then λA and A are quasi-similar.

Proof

(i) Given $AB=\lambda BA$...(1),

we have since $B,\lambda B\in\wp,\ AB^*=\overline{\lambda}B^*A.$ Taking adjoints on each side gives:

$BA^*=A^*\lambda B$..(2).

Since A is quasi-affinity, it follows that A^* is also a quasi-affinity. Now from (1) and (2) we have that the operators λB and B are quasi similar.

(ii) Note by hypothesis $AB = B\lambda A$..(1).

Similarly, since $A, \lambda A \in \wp$, $A^*B = B\overline{\lambda}A^*$. Taking adjoints of each side gives:

$$B^*A = \lambda AB^* \text{...} (2).$$

Since B is quasi-affinity, it follows that B^* is also a quasi-affinity. Now from (1) and (2) we have that the operators λA and A are quasi similar.

The following theorem by Duggal (1996) will assist in establishing the corollary that follows.

Theorem 4.12 [Duggal (1996), Theorem 7, p. 344]

If M^* and N are p-hyponormal such that $NX = XM$, then $N^*X = XM^*$ for some $X \in B(H)$.

Corollary 4.13 [24]

Let $AB = \lambda BA$, $\lambda \in \mathbb{C}$, $AB \neq 0$ with B and B^* p-hyponormal with A a quasiaffinity. Then we have λB and B are quasi similar.

Proof

Given $AB = \lambda BA$...(i),

we first note that λB and B^* are p-hyponormal and by Theorem 4.12, $AB^* = \overline{\lambda}B^*A$.

Taking adjoints of each side gives:

$$BA^* = A^*\lambda B \text{..} (ii)$$

Since A is quasi-affinity, it follows that A^* is also a quasi-affinity. Thus, from (i) and (ii) we have that the operators λB and B are quasi similar.

Corollary 4.14 [24]

Let $AB=\lambda BA,\ \lambda\in\mathbb{C},\ AB\neq 0$ with A and A^* p-hyponormal with B a quasiaffinity. Then we have λA and A are quasi similar.

Proof

Interchange A with B in Corollary 4.13.

Remark 4.15

We now note that equality of spectra or essential spectra for some classes of operators has been proved by a number of authors see [10], [13], [29], [45], [46], [49] and [50]. In particular, the following authors proved the following results that will assist in proving the corollaries that follow.

Theorem 4.16 [Williams (1980a), Theorem 3, p.65]

Suppose that T_1 and T_2 are quasi-similar quasi-normal operators, then:

$\sigma_e(T_1)=\sigma_e(T_2).$

Theorem 4.17 [Williams (1980b), Theorem 2, p.205]

Suppose that A and B are hyponormal operators and there exists quasi-affinities X and Y such that $XA=BX$ and $AY=YB$. If either X or Y is compact, then $\sigma_e(A)=\sigma_e(B)$.

Theorem 4.18 [Yang (1990)]

Quasi-similar subnormal operators have equal essential spectra.

Theorem 4.19 [Duggal (1996), Theorem 6, p. 343]

If A_1 and A_2 are quasi-similar and belong to $\mathcal{H} \cup (p)$, the class of p-hyponormal operators such that U_1 and U_2 are unitary in the polar decomposition $A_1 = U_1|A_1|$ and $A_2 = U_2|A_2|$, then:

$\sigma(A_1) = \sigma(A_2)$ and $\sigma_e(A_1) = \sigma_e(A_2)$.

Also Putnam-Fuglede commutativity theorem holds for the operators A_1 and A_2.

Theorem 4.20 [Duggal (1986), Theorem 1, p.354]

If M^* is m- hyponormal and N dominant such that $NX = XM$, then $N^*X = XM^*$ for some operator X.

Theorem 4.21 [Yang (1993), Theorem 2.10, p. 210]

Let M and N be quasisimilar m-hyponormal. Then M and N have equal essential spectra.

Theorem 4.22 [Gudder and Nagy (2001), Theorem 2, p. 1127]

Let A and B be self adjoint. The following statements are equivalent;

(i) $AB^2A = BA^2B.$

(ii) $AB^2 = B^2A$ and $BA^2 = A^2B.$

We now give the following corollaries involving essential spectra that will make use of some of the results stated above.

Remark 4.23

Note that in Proposition 4.6 above if we require that one of the operators say B in the equation $AB = \lambda BA,\ \lambda \in \mathbb{C},\ AB \neq 0$ belongs to some appropriate class of operators, then we can relax the condition on the other operator, say A, having a bounded inverse as is seen in Corollaries 4.24 and 4.25.

Corollary 4.24 [24]

Let $AB = \lambda BA$, $\lambda \in \mathbb{C}$, $AB \neq 0$ with A a quasi-affinity. Then we have:

(i) if B is quasi normal then $\sigma_e(B) = \sigma_e(\lambda B) = \lambda\sigma_e(B)$.

(ii) if B^* and B are hyponormal and A compact, then $\sigma_e(B) = \lambda\sigma_e(B)$.

(iii) If λB and $B \in \mathcal{H} \cup (p)$, then $\sigma_e(B) = \lambda\sigma_e(B)$ and $\sigma(B) = \lambda\sigma(B)$.

(iv) if B^* and λB are m-hyponormal, then $\sigma_e(B) = \sigma_e(\lambda B) = \lambda\sigma_e(B)$.

Proof

(i) By the inclusion {quasinormal} $\subset$ {p-hyponormal}, see Burnap et al. (2005), p. 382, and the fact that B^* is quasi-normal if B is quasi-normal, we have that B and B^* are indeed also p-hyponormal. Thus, applying Corollary 4.13, λB and B are quasi-similar. By Theorem 4.16 we have:

$\sigma_e(B) = \sigma_e(\lambda B) = \lambda\sigma_e(B)$. Also to authenticate our result further, by Theorem 4.18, quasisimilar subnormal operators have equal essential spectra and the inclusion {quasinormal} $\subset$ {subnormal} ensures that the result stands for quasinormal operators.

(ii) By the inclusion {hyponormal} $\subset$ {p-hyponormal} and Corollary 4.13, λB and B are quasi-similar. Hence, by Theorem 4.17 $\sigma_e(B) = \lambda\sigma_e(B)$

(iii) We only have to note that it was proved in Theorem 4.19 that for the operators in $\mathcal{H} \cup (p)$, they belong to $\wp$ and they have same spectrum and same essential spectrum.

(iv) By Theorem 4.20, λB and $B \in \wp$ due to the inclusion {m-hyponormal} $\subset$ {dominant}. Applying Theorem 4.11 we have that the operators λB

and B are quasi similar. By Theorem 4.21 we have $\sigma_e(B) = \sigma_e(\lambda B) = \lambda\sigma_e(B)$.

Corollary 4.25 [24]

Let $AB = \lambda BA,\ \lambda \in \mathbb{C},\ AB \neq 0$ with B a quasi-affinity. Then we have:

(i) if A is quasi normal then $\sigma_e(A) = \sigma_e(\lambda A) = \lambda\sigma_e(A)$.

(ii) if A^* and A are hyponormal and B compact, then $\sigma_e(A) = \lambda\sigma_e(A)$.

(iii) If λA and $A \in \mathcal{H} \cup (p)$, then $\sigma_e(A) = \lambda\sigma_e(A)$ and $\sigma(A) = \lambda\sigma(A)$.

(iv) if A^* and λA are m-hyponormal, then $\sigma_e(A) = \sigma_e(\lambda A) = \lambda\sigma_e(A)$.

Proof

We only have to interchange the conditions on the operators A and B in the Corollary 4.24 above.

In view of Theorem 4.2 we have the following theorem about essential spectrum of AB and spectrum of BA.

Theorem 4.26 [24]

Let $AB = \lambda BA,\ \lambda \in \mathbb{C},\ AB \neq 0$ with $\lambda \in \{1, -1\}$. If AB is pure dominant and BA is compact with dense range then: $\sigma_e(AB) = \sigma(AB) = \sigma(BA) = \lambda\sigma(AB)$.

Proof

Let $T = AB$ and $K = BA$. Then under the conditions that $\lambda \in \{1,-1\}$, we have: $[T,K] = 0$. For if $\lambda = -1$, then $AB = -BA$. Thus:

$ABBA = -BABA$ and $BAAB = -BABA$. This means AB and BA commute. Now by Theorem 4.2 above:

$\sigma_e(T) = \sigma(T)$.

That is $\sigma_e(AB) = \sigma(AB) = \sigma(BA) = \lambda\sigma(AB)$.

Theorem 4.27 [24]

Let $AB = \lambda BA$, $\lambda \in \mathbb{C}$, $AB \neq 0$ with $\lambda \in \{1,-1\}$. If BA is pure dominant and AB is compact with dense range, then: $\sigma_e(BA) = \sigma(BA) = \sigma(AB) = \lambda\sigma(AB)$.

Proof

Let $T = BA$ and $K = AB$. By Theorem 4.2 $\sigma_e(BA) = \sigma(BA) = \sigma(AB) = \lambda\sigma(AB)$.

Remark 4.28

Let us for convenience sake say that $T \in B(H)$ belongs to a class $\mathcal{M}$ of operators if:

(i) T is pure dominant.

(ii) T is compact.

(iii) T has dense range.

Thus, we have the following result.

Corollary 4.29 [24]

Let $AB = \lambda BA,\ \lambda \in \mathbb{C},\ AB \neq 0$ with $\lambda \in \{1,-1\}$. Also let $T = AB$ and $K = BA.$ If $T, K \in \mathcal{M},$ then $\sigma_e(T) = \sigma(T) = \sigma(K) = \sigma_e(K).$

Proof

In particular, T is pure dominant and K is compact with dense range. By Theorem 4.26 we have; $\sigma_e(T) = \sigma(T) = \sigma(K) = \lambda\sigma(T).$ Thus;

$\sigma_e(AB) = \sigma(AB) = \sigma(BA) = \lambda\sigma(AB) ...(i).$

Also in particular, K is pure dominant and T is compact with dense range.

By Theorem 4.27 we have; $\sigma_e(K) = \sigma(K) = \sigma(T) = \lambda\sigma(T).$

Thus;

$\sigma_e(BA) = \sigma(BA) = \sigma(AB) = \lambda\sigma(AB) ...(ii).$

From (i) and (ii) $\sigma_e(AB) = \sigma_e(BA).$

Remark 4.30

Note that it is a well-known fact that dominant operators that are compact are quasi-nilpotent hence, have zero as its essential spectra.

The next corollary is elicited by Theorem 4.2 above and in turn produces some corollaries thereafter.

Corollary 4.31

Let $AB = \lambda BA,\ \lambda \in \mathbb{C},\ AB \neq 0$ be such that $[B, A^2] = 0.$ Then if A is pure dominant and B is compact with dense range, satisfying the following conditions, then $\sigma_e(A) = \sigma(A).$

(i) $\sigma(A) \cap \sigma(-A) = \varnothing.$

(ii) $\{A\}' = \{A^{2m}\}'$ for a positive integer m.

(iii) A is normal and $0 \notin W(A)$.

(iv) A is normal and $0 \notin W(\operatorname{Re} A)$.

(v) A is normal and $\sigma(\operatorname{Re} A) \cap \sigma(-\operatorname{Re} A) = \varnothing$.

Proof

Note that from Theorem 2.4, in the operator equation $AB = \lambda BA$, $\lambda = 1$ under each one of the conditions here. From Theorem 4.2, $\sigma_e(A) = \sigma(A)$.

Corollary 4.32

Let $AB = \lambda BA$, $\lambda \in \mathbb{C}$, $AB \neq 0$ be such that $[A, B^2] = 0$. Then if B is pure dominant and A is compact with dense range, satisfying the following conditions, then $\sigma_e(B) = \sigma(B)$.

(i) $\sigma(B) \cap \sigma(-B) = \varnothing$.

(ii) $\{B\}' = \{B^{2m}\}'$ for a positive integer m.

(iii) B is normal and $0 \notin W(B)$.

(iv) B is normal and $0 \notin W(\operatorname{Re} B)$.

(v) B is normal and $\sigma(\operatorname{Re} B) \cap \sigma(-\operatorname{Re} B) = \varnothing$.

Proof

Interchange A with B in Corollary 4.31.

Remark 4.33

Note that if in Corollaries 4.31 and 4.32 above, we drop the condition that B and A has dense range respectively, then in view of Theorem 4.3, we have that A and B are quasinilpotent.

The next result is on the spectral properties of the product AB of operators.

Corollary 4.34

In the operator equation $AB = \lambda BA \neq 0,\ \lambda \in \mathbb{C}$, let A and B be self adjoint, such that $AB^2 = B^2A$ and $A^2B = BA^2$. Then $\sigma_e(AB) = \sigma(AB)$, provided AB is pure dominant and BA is compact with dense range.

Proof

Let $T = AB$ and $K = BA$. Then if $AB^2 = B^2A$ and $A^2B = BA^2$, we have $KT = TK$, from Theorem 4.22. Thus, applying now Theorem 4.2, we have that $\sigma_e(AB) = \sigma(AB)$.

Corollary 4.35

In the operator equation $AB = \lambda BA \neq 0,\ \lambda \in \mathbb{C}$, let A and B be self adjoint such that $AB^2 = B^2A$ and $A^2B = BA^2$. Then $\sigma_e(BA) = \sigma(BA)$ provided BA is pure dominant and AB is compact with dense range.

Proof

Let $T = BA$ and $K = AB$. Then proof similar to Corollary 4.34.

Remark 4.36

The following lemma aids in proving Corollary 4.38 and which in turn is used to prove Corollary 4.39. Corollary 4.38 also offers an **alternative proof** to Theorem 4.7 (i).

Lemma 4.37

Let $A, B \in B(H)$ be self adjoint such that $AB = \lambda BA \neq 0,\ \lambda \in \mathbb{C}$. Then, $BA^2 = A^2B$ and $AB^2 = B^2A.$

Proof

Since A and B are self adjoint, $\lambda \in \mathbb{R}$ by Theorem 2.1 (i). Thus, if $AB = \lambda BA,$ we have on taking adjoints that:

$BA = \overline{\lambda}AB = \lambda AB.$

Pre-multiplying by B yields: $B^2A = \lambda BAB = ABB = AB^2.$

Also from $BA = \lambda AB$ we have:

$BA^2 = \lambda ABA = AAB = A^2B.$

Corollary 4.38

If A, B are self adjoint with $AB = \lambda BA \neq 0,\ \lambda \in \mathbb{C}$, then $[AB, BA] = 0.$

Proof

Since by Lemma 4.37, $B^2A = AB^2$ and $A^2B = BA^2,$ we have by Theorem 4.22 that $[AB, BA] = 0.$

Thus, we have the following results.

Corollary 4.39

If A,B are self adjoint with $AB = \lambda BA \neq 0,\ \lambda \in \mathbb{C},$ then $\sigma_e(AB) = \sigma(AB)$ provided AB is pure dominant and BA is compact with dense range.

Proof

Let $T = AB$ and $K = BA.$ Then by Corollary 4.38, we have $KT = TK.$ Thus, applying now Theorem 4.2, we have that $\sigma_e(AB) = \sigma(AB).$

Corollary 4.40

If A,B are self adjoint with $AB = \lambda BA,$ then $\sigma_e(BA) = \sigma(BA)$ provided BA is pure dominant and AB is compact with dense range.

Proof

Let $T = BA$ and $K = AB.$ Then by Corollary 4.38, we have $KT = TK.$ Thus, applying now Theorem 4.2, we have that $\sigma_e(BA) = \sigma(BA).$

Remark 4.41

Note that by Theorem 2.2 (ii), in Corollary 4.39, $\lambda \in \{1,-1\}.$ Thus, in view of Theorem 4.26 and 4.9 (i), we have the following corollaries in particular.

Corollary 4.42

If A and B are self adjoint operators with $AB = \lambda BA,$

then $\sigma_e(AB) = \sigma(AB) = \sigma(BA) = \lambda\sigma(AB)$ provided AB is pure dominant and BA is compact with dense range and $\sigma_e(BA) = \sigma(BA) = \sigma(AB) = \lambda\sigma(AB),$ provided BA is pure dominant and AB is compact with dense range.

Corollary 4.43

If A is normal and B is self adjoint with $AB = \lambda BA$,

then $\sigma_e(AB) = \sigma(AB) = \sigma(BA) = \lambda\sigma(AB)$, provided AB is pure dominant and BA is compact with dense range and $\sigma_e(BA) = \sigma(BA) = \sigma(AB) = \lambda\sigma(AB)$, provided BA is pure dominant and AB is compact with dense range.

CHAPTER FIVE

SUMMARY

5.0 Conclusion

In this thesis, we have made several key contributions to the study of commutants and spectral properties for some classes of operators in Hilbert spaces. We have extended some results by looking at higher classes of operators, provided parallel results and established new results. We have also established certain parallel results without necessarily specifying the classes of operators the operators involved belong to.

The results in this thesis have shown conditions under which two $\lambda-$commuting operators A and B happen to be commuting. We also looked at the existence of the operator equations $C(A,B)X=0$ and $R(A,B)X=0$ under some given conditions. Finally, the spectral aspect in this thesis involved the general spectrum, approximate spectrum and essential spectrum that touched on the operator equation $AB=\lambda BA.$

5.1 Chapter wise summary

Chapter one was an introduction. Here a brief history of the concept of Hilbert spaces was given. Thereafter, tracing from the first contributor of commutators, we chronologically outlined the literature review on the subject matter mentioning the contribution that has been made in this thesis. Notations and terminologies that were used in the entire thesis were also defined in this chapter. The very important aspect of inclusions of classes of operators in Hilbert spaces, which were used in generalizing results or establishing parallel results from other scholars, was given towards the end of the chapter. Finally, we stated the

Fuglede and Putnam- Fuglede theorems that were applied in establishing some results in the thesis.

Chapter two was on $\lambda-$commutativity. Two operators A and B were said to $\lambda-$commute, for $\lambda\in\mathbb{C}$, provided that $AB=\lambda BA$. Some properties satisfied by the operators A and B in this equation so that $\lambda=1$ were investigated. Among other results, we showed in Theorem 2.5 that if one of the operators raised to some power is normal and 0 does not belong to the numerical range of the other operator, then $\lambda^{n}=1$ meaning that that the operators A^{n} and B for instance, belong to the commutant of each other. This theorem resulted in a corollary that stated summarily that if one of the operators is normal and 0 does not belong to the numerical range of the other operator, then $\lambda=1$ meaning this time that that the operators A and B, belong to the commutant of each other. Theorem 2.5 as a consequence of this was an improvement of Theorem 2.1 and Theorem 2.2 since the condition that the operators A or B be positive so that the two operators A and B belong to the commutant of each other was more stringent than a mere requirement that 0 does not belong to the numerical range of one operator with the other operator raised to some power being normal. Furthermore, a higher class of operator than self adjoint was used, namely the normal operator, thus extending the results in the above mentioned theorems.

In this same chapter, another key result of commutants was established. By letting A and B to $\lambda-$commute, we showed that If A is self adjoint, then $B^{*}B\in\{A\}'$ and $BB^{*}\in\{A\}'$. On the other hand, if B is self-adjoint then $A^{*}A\in\{B\}'$ and $AA^{*}\in\{B\}'$. This result generated two corollaries that were proved. One of the corollaries stated that if A and B do $\lambda-$commute, with B having the polar decomposition $B=UP$, where U is unitary, such that $A\in\{U\}'$ and $U\in\{A\}'$ with

A self adjoint, then $A \in \{B\}'$ and $B \in \{A\}'$. The other corollary just interchanged the two operators A and B. Last but not least, it was shown that if A and B are both normal then $\lambda\bar{\lambda} = 1$ provided A and B do $\lambda-$commute. All in all, in chapter two we have extended the study of $\lambda-$commuting operators to other classes of operators instead of limiting ourselves to self adjoint operators.

Chapter three was on commutants and operator equations in Hilbert spaces. The equations $AX = XB$ and $AXB = X$ were used. We investigated the conditions under which these two operator equations exist. It was shown among other results that if A is one-one and B has dense range then $C(A^2, B^2)X = 0$ and $C(A^3, B^3)X = 0$ imply $C(A, B)X = 0$. Similarly, if $R(A^2, B^2)X = 0$ and $R(A^3, B^3)X = 0$ then $R(A, B)X = 0$ for some operator X. Here, $C(A, B)X = AX - XB$ and $R(A, B)X = AXB - X$. These results are parallel results to Theorem 3.1, Corollary 3.2 and Theorem 3.4. Our contribution here is in the fact that we achieved the same results as those of Theorem 3.1, Corollary 3.2 and Theorem 3.4 without requiring any of the operators belong to a specific class of operators. Some properties that ensured both operators are one-one and have dense range were also given in this chapter and consequently the results given as corollaries. For instance the same was achieved with both A and B being quasiaffinities. To justify once again the title of our thesis, it was then established as a result of these parallel results that if A is a quasi-affinity such that $A^2 \in \{X\}'$ and $A^3 \in \{X\}'$, then $A \in \{X\}'$.

Chapter four was on spectral properties of $\lambda-$commuting operators. In the operator equation $AB = \lambda BA$ for $\lambda \in \mathbb{C}$, some conditions under which AB and BA or B and λB have same spectrum or same essential spectrum, were investigated. . On approximate spectrum it was shown among other results that if A and B are self adjoint operators that $\lambda-$commute non-trivially, then $\sigma_\pi(AB)=\sigma_\pi(BA)=\lambda\sigma_\pi(AB)$. Also, if A is normal with $\operatorname{Re}A$ positive and B self-adjoint then $\sigma_\pi(AB)=\lambda\sigma_\pi(AB)$. On essential spectrum it was shown that if A is a quasi-affinity, then we have:

(i) if B is quasi normal then $\sigma_e(B)=\sigma_e(\lambda B)=\lambda\sigma_e(B)$.

(ii) if B^* and B are hyponormal and A compact, then $\sigma_e(B)=\lambda\sigma_e(B)$.

(iii) If λB and $B\in\mathcal{H}\cup(p)$, then $\sigma_e(B)=\lambda\sigma_e(B)$ and $\sigma(B)=\lambda\sigma(B)$.

(iv) if B^* and λB are m-hyponormal, then $\sigma_e(B)=\sigma_e(\lambda B)=\lambda\sigma_e(B)$.

Still on essential spectrum, it was eventually established that $\sigma_e(AB)=\sigma_e(BA)$, where $\sigma_e(AB)$ denotes the essential spectrum of AB. This occurred by letting T and K belong to the class of operators that are pure dominant, compact and have dense range where $T=AB$ and $K=BA$.
It was also shown that if A and B be such that $\left[B,A^2\right]=0$, with A pure dominant and B compact with dense range, satisfying the conditions;

(i) $\sigma(A)\cap\sigma(-A)=\varnothing$.

(ii) $\{A\}'=\{A^{2m}\}'$ for a positive integer m.

(iii) A is normal and $0\notin W(A)$.

(iv) A is normal and $0\notin W(\operatorname{Re}A)$.

(v) A is normal and $\sigma(\operatorname{Re}A)\cap\sigma(-\operatorname{Re}A)=\varnothing$.

Then, $\sigma_e(A)=\sigma(A)$.

5.2 Future Research

In this last section, we shall briefly present some open problems which are of interest for possible future work.

- On commutants and higher classes of operators, by looking at higher classes of operators than normal operators for instance quasinormal or hyponormal, conditions that ensure A and B belong to the commutant of each other can be investigated. By doing this, improvements will be made for example on Theorem 2.5. For more on commutants see [11]. Also, [5] offers a connection between commutants and compact spaces that has not been covered in this thesis.

- Classification of product of operators that commute is also an area that needs further research and has not being covered in this thesis. For instance, Zhang et al. (2011) showed that if A is paranormal and B an isometry, then AB is paranormal. See Theorems 2.23 and 2.24. In both cases, they showed this with commuting operators. Of interest here will be to study other higher combinations of classes of operators for these two commuting operators in a bid to classify the class of operators the product operator AB falls in. For instance, with the combinations of A being paranormal and B binormal.

- Many problems in science and engineering have their mathematical formulation as an operator equation. For instance, we mentioned earlier that the operator equation $AB = \lambda BA$, is important in the interpretation of quantum mechanical observables. Some of the results established in this thesis, may assist in improving the formulation of quantum mechanics observables and solve open problems in science and engineering since it is known that commutators offer one of the best formulations in these application areas. For application of our work in other areas, one can refer to [3], [18], [37], [40] and [53].

- On operator equations, other properties that do not specify the class of operators other than injectivity and dense range that establish the existence of the operator equations $C(A,B)X=0$ and $R(A,B)X=0$, may be looked at. For instance it is known that if A and B are strictly positive, then $0 \notin W(A)$. As seen in chapter three, operators that have $0 \notin W(A)$ are one to one and have dense range which ensures the existence of the above equations.

- Higher classes of operators like posinormal and totally posinormal, offer a rich supply for new results that can be obtained on the subject matter of commutants and spectral properties of operators in Hilbert spaces. Posinormal operators were first introduced by Rhaly in 1994. See [38]. It is a larger class of operators than the dominant and many others as shown in [23] and [38]. For instance, by investigating whether these higher operators satisfy the PF property given certain conditions, one can establish quasisimilarity of these operators thereby enhancing the chances of equality of essential spectra being realized. This can make improvements to Corollaries 4.24 and 4.25. There is no known text touching on the PF property of these higher classes of operators and hence there is a wide area of research yet to be explored. On parallel results, Mecheri et al. (2006) showed if A is p-hyponormal and B a class $\mathcal{Y}$ operator, then A and B satisfy the PF property. Of interest here is to note that this class $\mathcal{Y}$ of operators may offer parallel results in our subject area. See [43].

- In this thesis, we have at most showed that $\sigma_e(A)=\sigma_e(\lambda A)=\lambda\sigma_e(A)$ and $\sigma_e(B)=\sigma_e(\lambda B)=\lambda\sigma_e(B)$ where A and B are $\lambda-$commuting operators. Other than that, there is no result here to show the equality of essential spectra of operators A and B. For instance, if one is able to build up a scenario where two commuting operators are quasisimilar $\omega-$hyponormal, then $\sigma_e(A)=\sigma_e(B)$. See [29]. Thus, we have the following open question. Under what conditions do

$\lambda-$commuting operators A and B have $\sigma_e(A)=\sigma_e(B)$? On the other hand if one of the operators is of index zero, we can realize $\sigma_e(AB)=\sigma_e(BA)$. What other conditions will ensure that $\sigma_e(AB)=\sigma_e(BA)$ other than those mentioned in this thesis? For further research on essential spectrum and general spectrum see [6],[13],[15],[45] and [46].

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On λ-Commuting Operators

J.M. Khalagai and M. Kavila
School of Mathematics, University of Nairobi, P.O. Box 30197 - 00100, Nairobi
Email: khalagai@uonbi.ac.keormutiekavila@gmail.com

Abstract: *Two bounded linear operators A and B on a complex Hilbert space are said to λ-commute for $\lambda \in \mathbf{C}$ provided that: $AB = \lambda BA$. In this paper we look for some properties satisfied by the operators A and B so that $\lambda = 1$. It is shown among other results that if one of the operators raised to some power is normal and 0 does not belong to the interior of the numerical range of the other operator then: $\lambda = 1$*

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1. Notation and Terminology

Given an operator A we shall denote the spectrum, the approximate point spectrum, the point spectrum and the closure of the numerical range by: $\sigma(A)$, $\sigma_{ap}(A)$, $\sigma_p(A)$ and $\overline{W(A)}$ respectively. For $A, B \in B(H)$ the commutator of A and B will be denoted by $[A, B]$. Thus $[A, B] = AB - BA$. The commutant of A will be denoted by $\{A\}'$.
Thus $\{A\}' = \{X \in B(H): [A, X] = 0\}$. Re$A$ and ImA will denote real and imaginary parts of A. The operator X is said to intertwine operators A and B if $AX = XB$.
The operator A will be said to be:

Dominant if to each $\lambda \in \mathbf{C}$ there corresponds a number $M_\lambda \geq 1$ such that $\|(A-\lambda)^* x\| \leq M_\lambda \|(A-\lambda)x\|$ for all $x \in H$.

Hyponormal if $A^*A \geq AA^*$.

Normal if $A^*A = AA^*$.

Self-adjoint if $A = A^*$.

We have the following inclusions of classes of operators:

$$\{\text{Self-adjoint}\} \subseteq \{\text{Normal}\} \subseteq \{\text{Hyponormal}\} \subseteq \{\text{Dominant}\}$$

2. Introduction

Let $B(H)$ denote the Banach algebra of bounded linear operators on a complex Hilbert space H. For any $T \in B(H)$ the numerical range of T denoted by $W(T)$ is the image of the unit sphere of H under the quadratic form $x \to \langle Tx, x\rangle$ associated with the operator.
More precisely, $W(T) = \{\langle Tx, x\rangle : \|x\| = 1\}$. Thus the numerical range of an operator, like the spectrum, is a subset of the complex plane whose topological properties carry some information about the operator. We now note that λ-commuting operators have been considered by a number of authors. Among them are Brook *et al*, (2002) who proved the following results:

Theorem A: Let $A, B \in B(H)$ such that $AB \neq 0$ and $AB = \lambda BA$ for $\lambda \in \mathbf{C}$. Then:

(i) If A or B is self-adjoint then $\lambda \in \mathbf{R}$..
(ii) If both A and B are self-adjoint then $\lambda \in \{1, -1\}$.
(iii) If A and B are self-adjoint and one of them is positive then $\lambda = 1$.

Theorem B: Let $A, B \in B(H)$ be self-adjoint. Then the following statements are equivalent:
(i) $AB = UBA$ for some unitary U.
(ii) $AB^2 = B^2A$ and $BA^2 = A^2B$.

If A or B is positive then (i) is equivalent to $AB = BA$.

Brown (1979) also proved the following result:

Theorem C: If T λ-commutes with a compact operator then T has a non trivial hyper invariant subspace.

In this paper our main task is to investigate the conditions under which λ-commuting operators turn out to be commuting operators. Thus in the equation $AB = \lambda BA$ we look for conditions under which $\lambda = 1$. We will make an improvement on each of the **theorems A** and **B** above in which the following results will be used.

Embry (1970) proved the following result:

Theorem D: Let A, J and $K \in B(H)$ such that $AJ = KA$. If J and K are commuting normal operators and $0 \notin W(A)$ then $J = K$.

Sheth and Khalagai (1987) proved the following result:
Theorem E: Let $A, B \in B(H)$ be such that $[B, A^2] = 0$. Then $[B, A] = 0$ under any one of the following conditions:
(i) $\sigma(A) \cap \sigma(-A) = \varnothing$
(ii) A is normal and $0 \notin W(A)$
(iii) $\{A\}'' = \{A^{2m}\}''$ for a positive integer m
(iv) A is normal and $0 \notin W(\mathrm{Re}A)$
(v) A is normal and $\sigma(\mathrm{Re}A) \cap \sigma(-\mathrm{Re}A) = \varnothing$

3. Main Results

Theorem 1: Let $A, B \in B(H)$ and $\lambda \in \mathbf{C}$ be such that: $AB = \lambda BA$. Then we have:
(i) If $B^n \neq 0$ is normal for some positive integer n and $0 \notin W(A)$ then $[B^n, A] = 0$
(ii) If $A^n \neq 0$ is normal for some positive integer n and $0 \notin W(B)$ then $[A^n, B] = 0$

Proof. (i) Given $AB = \lambda BA$ it immediately follows:

$$AB^n = \lambda^n B^n A \text{ for } n \in J^+$$

Now B^n and $\lambda^n B^n$ are commuting normal operators. Hence by **theorem D** above we have that:

$$B^n = \lambda^n B^n$$

Since $B^n \neq 0$ then $\lambda^n = 1$

Hence $\left[B^n, A\right] = 0$

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(ii) Similarly given $AB = \lambda BA$ we have:

$A^n B = \lambda^n BA^n$ for $n \in J^+$

i.e., $B\left(\lambda^n A^n\right) = A^n B$

But $\lambda^n A^n$ and A^n are commuting normal operators. Since $0 \notin W(B)$, we have by theorem D again that:

$A^n = \lambda^n A^n$

Since $A^n \neq 0$ then $\lambda^n = 1$

i.e., $\left[A^n, B\right] = 0$

Corollary 1: Given $AB = \lambda BA$ we have that $[A, B] = 0$ under any one of the following conditions:

(i) A is normal and $0 \notin W(B)$

(ii) B is normal and $0 \notin W(A)$

Proof: We put $n = 1$ in the proof of the theorem above.

Remarks: (i) We note that for the operator equation $AB = \lambda BA$ the condition $[A, B] = 0$ trivially implies that $\lambda = 1$.

(ii) We also note that the condition that A or B is positive is more stringent than a mere requirement that $0 \notin W(A)$ or $0 \notin W(B)$. More precisely the following corollary is an improvement of theorem A above.

Corollary 2: Let A and B be self-adjoint operators such that $AB = \lambda BA$. Then $[A, B] = 0$ under any one of the following conditions:

(i) $\sigma(A) \cap \sigma(-A) = \varnothing$

(ii) $0 \notin W(A)$

(iii) $\sigma(\mathrm{Re}A) \cap \sigma(-\mathrm{Im}A) = \varnothing$

(iv) $\sigma(B) \cap \sigma(-B) = \varnothing$

(v) $0 \notin W(B)$

(vi) $\sigma(\mathrm{Re}B) \cap \sigma(-\mathrm{Im}B) = \varnothing$

Proof: Given $AB = \lambda BA$ we have:

$$\begin{aligned} A^2B &= A\lambda BA \\ &= \lambda ABA \\ &= \lambda.\lambda BAA \\ &= \lambda^2 BA^2 \end{aligned}$$

Now by **theorem A** above we have that $\lambda^2 = 1$. Thus $A^2B = BA^2$ or $[B, A^2] = 0$. We also have:

$$\begin{aligned} AB^2 &= \lambda BAB \\ &= \lambda B\lambda BA \\ &= \lambda^2 B^2 A \end{aligned}$$

By **theorem A** again $\lambda^2 = 1$. Thus $AB^2 = B^2A$ or $\left[A, B^2\right] = 0$.

Now in view of **theorem E** above each of the conditions (i) to (vi) implies $[A, B] = 0$ and consequently $\lambda = 1$.

Theorem 2: Let $A, B \in B(H)$ be such that $AB = \lambda BA$. Then we have:

(i) A is self-adjoint implies $B^*B \in \{A\}'$ and $BB^* \in \{A\}'$

(ii) B is self-adjoint implies $A^*A \in \{B\}'$ and $AA^* \in \{B\}'$

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Proof: (i) Since $A=A^*$ it follows $\lambda \in \mathbf{R}$ hence:

$$AB = \lambda BA \Rightarrow B^*A = \lambda AB^*$$

i.e., $B^*AB = \lambda AB^*B$

i.e., $\lambda B^*BA = \lambda AB^*B$

i.e., $\lambda(B^*BA - AB^*B) = 0$

Since $\lambda \neq 0$ we have that:

$$B^*BA - AB^*B = 0$$

i.e., $[A, B^*B] = 0$. Hence $B^*B \in \{A\}'$

Similarly from the relation;

$$B^*A = \lambda AB^*$$

$$BB^*A = \lambda BAB^*$$

$$= ABB^*$$

Hence $[A, BB^*] = 0$. Thus $BB^* \in \{A\}'$

(ii) Since $B=B^*$ it follows $\lambda \in \mathbf{R}$ hence:

$$AB = \lambda BA \Rightarrow BA^* = \lambda A^*B$$

Thus: $BA^*A = \lambda A^*BA$

$$= A^*AB$$

i.e. $[B, A^*A] = 0$

Hence $A^*A \in \{B\}'$.

Also from the relation $BA^* = \lambda A^*B$ we have:

$$ABA^* = \lambda AA^*B$$

$\Rightarrow \lambda BAA^* = \lambda AA^*B$

i.e., $\lambda(BAA^* - AA^*B) = 0$

Since $\lambda \neq 0$, $BAA^* - AA^*B = 0$

i.e., $BAA^* = AA^*B$

i.e., $[B, AA^*] = 0$

Hence $AA^* \in \{B\}'$

Corollary 3: Let $A, B \in B(H)$ be such that $AB = \lambda BA$, $\lambda \in \mathbf{C}$ with $B = UP$, (Polar decomposition with U unitary). If A is self-adjoint and $[A, U] = 0$ then $[A, B] = 0$ and hence $\lambda = 1$.

Proof: By the theorem above, $B^*B \in \{A\}'$ implies $P \in \{A\}'$. Thus $[A, U] = 0 \Rightarrow [A, UP] = 0$, i.e., $[A, B] = 0$

Corollary 4: Let $A, B \in B(H)$ be such that $AB = \lambda BA$, $\lambda \in \mathbf{C}$ with $A = UP$, (Polar decomposition with U unitary). If B is self adjoint and $[B,U]=0$ then $[A,B]=0$

Proof: By the theorem above, $A^*A \in \{B\}'$ implies $P \in \{B\}'$. Thus $[B, U] = 0 \Rightarrow [B, UP] = 0$, i.e., $[A, B] = 0$

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ON COMMUTANTS AND OPERATOR EQUATIONS

J.M. Khalagai[1], M. Kavila[2 §]

[1,2]School of Mathematics
University of Nairobi
P.O. Box 30197, 00100, Nairobi, KENYA

Abstract: Let B(H) denote the algebra of bounded linear operators on a Hilbert Space H into itself. Given $A, B \in B(H)$ define $C(A,B)$ and $R(A,B) : B(H) \longrightarrow B(H)$ by $C(A,B)X = AX - XB$ and $R(A,B)X = AXB - X$. Our task in this note is to show that if A is one-one and B has dense range then $C(A^2, B^2)X = 0$ and $C(A^3, B^3)X = 0$ imply $C(A,B)X = 0$ for some $X \in B(H)$. Similarly, if $R(A^2, B^2)X = 0$ and $R(A^3, B^3)X = 0$ then $R(A,B)X = 0$ for some $X \in B(H)$.

AMS Subject Classification: 47B47, 47A30, 47B20
Key Words: commutant, quasiaffinity and normal operator

1. Introduction

Let $B(H)$ denote the algebra of operators, i.e. bounded linear transformations on the complex Hilbert space H into itself.

Given $A, B \in B(H)$, let $C(A,B) : B(H) \longrightarrow B(H)$ be defined by $C(A,B)X = AX - XB$ and $R(A,B)X = AXB - X$. Moajil [5] proved that if N is a normal operator such that $N^2X = XN^2$ and $N^3X = XN^3$ for some $X \in B(H)$, then $NX = XN$. Thus for a normal operator N, if $N^2 \in \{X\}'$ and $N^3 \in \{X\}'$, then $N \in \{X\}'$ for some $X \in B(H)$.

Kittaneh [4] generalized this result to cover subnormal operators by taking A and B^* to be subnormal operators, i.e. if $A^2X = XB^2$ and $A^3X = XB^3$ for some $X \in B(H)$, then $AX = XB$. Thus if $C(A^2, B^2)X = 0$ and $C(A^3, B^3)X = 0$ then $C(A,B) = 0$ for some $X \in B(H)$.

Bachir [1] generalized these results to cover the classes of dominant and p-hyponormal operators as follows:

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§Correspondence author

Theorem A. *Let A be a dominant operator and B^* be a p-hyponormal operator or log-hyponormal. If $A^2X = XB^2$ and $A^3X = XB^3$ then $AX = XB$, for some $X \in B(H)$. Thus we have that if A is dominant and B^* is either p-hyponormal or log-hyponormal then $C(A^2, B^2)X = 0$ and $C(A^3, B^3)X = 0$ imply $C(A, B)X = 0$*

In this note we consider any operator $A, B \in B(H)$ without necessarily specifying the classes in which they belong and look for other conditions under which we can get similar results on the operator equation $C(A, B)X = 0$. We will also investigate similar results on the operator equation $R(A, B)X = 0$. Khalagai & Nyamai, [3] also had the following theorem and corollaries on the operator equation $R(A, B)X = 0$.

Theorem B. *Let A, B and $X \in B(H)$ be such that $R(A, B)X = 0$. Then B is one to one whenever X is one to one.*

Corollary A. *Let A, B and $X \in B(H)$ be such that $R(A, B)X = 0$ where X is quasiaffinity. Then both B and A^* are one to one.*

Corollary B. *Let A, B and $X \in B(H)$ be such that $R(A, B)X = 0$ implies $R(A^*, B^*)X = 0$ where X is a quasiaffinity. Then both A and B are also quasiaffinities.*

Goya & Saito [2] had the following result:

Theorem C. *Let $A, B, X \in B(H)$ where A is a paranormal contraction, B a coisometry and X has a dense range. Assume $C(A, B)X = 0$. Then A is a unitary operator. In particular, if X is injective and has a dense range, then B is also a unitary operator.*

2. Notation and Terminology

Given an operator $A \in B(H)$ we shall denote the spectrum of A by $\sigma(A)$. Thus $\sigma(A) = \{\lambda \in \mathbb{C} : A - \lambda I \, is\, not\, invertible\}$. The numerical range of A is denoted by $W(A) = \{\langle Ax, x\rangle : \|x\| = 1\}$. The commutator of any two operators A and B is defined by $[A, B] = AB - BA$. The commutant of A is given by $\{A\}' = \{X \in B(H) : [A, X] = 0\}$. An operator A is said to be:

- Dominant if to each $\lambda \in \mathbb{C}$ there corresponds a number $M_\lambda \geq 1$ such that for all $x \in H, \|(A - \lambda I)^*x\| \leq M_\lambda \|(A - \lambda I)x\|$.
- M-hyponormal if there is a constant M such that $M_\lambda \leq M$ for all $\lambda \in \mathbb{C}$ such that $\|(A - \lambda I)^*x\| \leq M\|(A - \lambda I)x\|$
- Hyponormal if from above $M = 1$
- P-hyponormal if $(A^*A)^p \geq (AA^*)^p$ for $0 < p < 1$

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- Log-hyponormal if A is an invertible operator such that $log(A^*A) \geq log(AA^*)$
- Paranormal if $\|A^2x\| \leq \|Ax\|^2$ for any unit vector $x \in H$
- Normal if $A^*A = AA^*$
- Subnormal if A has a normal extension
- Partial isometry if $A = AA^*A$
- Isometry if $A^*A = I$
- Co-isometry if $AA^* = I$
- Unitary if $A^*A = AA^* = I$
- Compact if for each bounded sequence $\{x_n\}$ in the domain H, the sequence $\{Ax_n\}$ contains a sub sequence converging to some limit in the range.
- Contraction if $\|A\| \leq 1$.

3. Results

Theorem 1. *Let $A, B \in B(H)$ be any pair of operators such that A is one-one and B has a dense range. Then we have that $C(A^2, B^2)X = 0$ and $C(A^3, B^3)X = 0$ imply $C(A, B)X = 0$ for some $X \in B(H)$.*

Proof. Let $T = AX$ and $S = XB$. Then from $A^2X = XB^2$ and $A^3X = XB^3$, we have $AT = SB$ and $A^2T = SB^2$ and moreover:

$$A(AT) = ASB = (SB)B,$$

$$ASB - (SB)B = 0,$$

$$(AS - SB)B = 0.$$

Since B has dense range we have that $B \neq 0$ and hence $AS - SB = 0$. Therefore

$$AS = SB,$$

$$AT = SB = AS,$$

$$AT - AS = 0,$$

i.e. $T - S = 0$ since A is one-one, $T = S$. Thus $AX = XB$.

Hence $C(A, B)X = 0$. □

Corollary 1. *If A and B are quasi-affinities such that $C(A^2, B^2)X = 0$ and $C(A^3, B^3)X = 0$ then $C(A, B)X = 0$ for some $X \in B(H)$.*

Proof. If A and B are quasi-affinities then each one of them is both one-one and has dense range. Hence the proof of Theorem 1 can easily be traced to give the required result. □

Corollary 2. *If A is a quasi-affinity such that $C(A^2, A^{*2})X = 0$ and $C(A^3, A^{*3})X = 0$ then $C(A, A^*)X = 0$ for some $X \in B(H)$.*

Proof. If A is quasi-affinity then A^* is also quasi-affinity. Hence by Corollary 1 the result follows. □

Corollary 3. *Let $\wp$ be the class of operators defined as follows:*

$$\wp = \{A \in B(H) : 0 \notin W(A)\}.$$

If $A, B \in \wp$ such that $C(A^2, B^2)X = 0$ and $C(A^3, B^3)X = 0$ then $C(A, B)X = 0$ for some $X \in B(H)$.

Proof. We only have to note that for any operator A with $0 \notin W(A)$, A is both one-one and has a dense range. □

Corollary 4. *If A is a quasi-affinity such that $A^2 \in \{X\}'$ and $A^3 \in \{X\}'$ then $A \in \{X\}'$ for some $X \in B(H)$.*

Proof. We only have to note that in Theorem 1 we let $A = B$. □

Theorem 2. *Let $A, B \in B(H)$ be a pair of operators such that A is one-one and B has dense range. Then $R(A^2, B^2)X = 0$ and $R(A^3, B^3)X = 0$ imply $R(A, B)X = 0$ for some $X \in B(H)$.*

Proof. Given $A^2XB^2 = X$ and $A^3XB^3 = X$ we have $A^2XB^2 = A^3XB^3$,

$$A^3XB^3 - A^2XB^2 = 0,$$

$$A(A^2XB^2 - AXB)B = 0.$$

Since A is one-one and B has dense range we have:

$$A^2XB^2 - AXB = 0$$

i.e. $A(AXB - X)B = 0$. Since A is one-one and B has dense range we have that $AXB - X = 0$.

Hence $R(A, B)X = 0$. □

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Corollary 5. *If $A, B \in B(H)$ are quasi-affinity such that $R(A^2, B^2)X = 0$ and $R(A^3, B^3)X = 0$, then $R(A, B)X = 0$.*

Proof. We note that the quasi-affinity is both one to one and has dense range. Hence the result is immediate by Theorem 2 above. □

Corollary 6. *A is quasi-affinity such that:*

$R(A^2, A^{*2})X = 0$ and $R(A^3, A^{*3})X = 0$ then $R(A, A^*)X = 0$ for some $X \in B(H)$.

Proof. It is immediate from Theorem 2 above and the fact that if A is a quasi-affinity then A^* is also quasi-affinity. □

Corollary 7. *If $R(A, B)X = 0$ implies $R(A^*, B^*)X = 0$ for some X which is quasi-affinity then $C(A^2, B^2)X = 0$ and $C(A^3, B^3)X = 0$ imply $C(A, B)X = 0$.*

Proof. $R(A, B)X = 0$ implying $R(A^*, B^*)X = 0$ where X is a quasi-affinity implies A and B are quasi-affinities from Corollary B. From Theorem 1, the result follows since quasi-affinities are both one to one and have a dense range. □

Corollary 8. *Let $A, B, X \in B(H)$ where A is a paranormal contraction, B a coisometry and X is a quasi-affinity. If $C(A, B^*)X = 0$, then $R(A, B)X = 0 = R(A^*, B^*)X$.*

Proof. First note that B is unitary from Theorem C. Therefore

$$C(A, B^*)X = 0 \Rightarrow AX = XB^*,$$

$$\Rightarrow AXB = XB^*B, \Rightarrow AXB = X,$$

$$\Rightarrow AXB - X = 0,$$

$$R(A, B)X = 0.$$

We also have that A is unitary by theorem C. Thus:

$$C(A, B^*)X = 0 \Rightarrow AX = XB^*,$$

$$\Rightarrow A^*AX = A^*XB^*,$$

$$\Rightarrow X - A^*XB^* = 0,$$

$$\Rightarrow A^*XB^* - X = 0,$$

$$R(A^*, B^*)X = 0.$$

□

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Professor M. KAVILA
Kenyatta University
Department of Mathematics
P.O. Box 43844-00100, Nairobi, Kenya

Svetoslav Nenov
Prof. PhD, Managing Editor
Academic Publications, Ltd.
e-mail: eb@ijpam.eu
url: http://www.acadpubl.eu

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Dear Professor M. KAVILA,

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Sincerely Yours,
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On spectral properties of λ-commuting operators in Hilbert spaces

J. M. KHALAGAI *   M. KAVILA†

March 18, 2013

Abstract

Let $B(H)$ denote the Banach algebra of bounded linear operators on a complex Hilbert space H and let $A, B \in B(H)$ satisfying the equation; $AB = \lambda BA, \lambda \in \mathbb{C}, AB \neq 0$, where $\mathbb{C}$ denotes the complex number field. In this case A and B are said to be λ-commuting operators. In this paper we investigate the conditions under which either AB and BA or B and λB have same spectrum or same essential spectrum.

Keywords and Phrases: λ-*commuting operators and essential spectrum.*

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UNIVERSITY OF NAIROBI
SCHOOL OF MATHEMATICS
P.O. BOX 30197
NAIROBI.

Email: maths@uonbi.ac.ke or khalagai@uonbi.ac.ke or mutiekavila@gmail.com

*University of Nairobi, School of Mathematics, P.O. Box 30197, 00100 Nairobi. Email: maths@uonbi.ac.ke
†University of Nairobi, School of Mathematics, P.O. Box 30197, 00100 Nairobi. Email: mutiekavila@gmail.com

1 Introduction

Let $B(H)$ denote the Banach algebra of bounded linear operators on a complex Hilbert space H and $A, B \in B(H)$. Then A and B are said to λ-commute non-trivially if $AB = \lambda BA$, $\lambda \in \mathbb{C}$,$AB \neq 0$. These types of operators have been studied by a number of authors see [3], [5], [9]. It is of interest to determine for various classes of operators A and B what restriction this places on $\lambda \in \mathbb{C}$. Also note that this property of λ-commuting operators is important for the interpretation of quantum mechanical observables and the analysis of their spectra. See [13]. On the other hand, results on essential spectra aid in coming up with generalizations of Weyl's theorem. See [1], [2] and [18]. Brooke, A.J, Busch, P and Pearson, D.B. [3], proved the following results:

Theorem A *Let $A, B \in B(H)$ such that $AB \neq 0$ and $AB = \lambda BA$ for $\lambda \in \mathbb{C}$. Then:*

i. if A or B is self-adjoint then λ is real.

ii. if both A and B are self adjoint then $\lambda \in \{-1, 1\}$.

iii. if A and B are self adjoint and one of them is positive, then $\lambda = 1$.

Lemma B *Let $AB = \lambda BA$, $\lambda \in \mathbb{C}$, $AB \neq 0$. Then 0 is in either both or neither of $\sigma(AB)$ and $\sigma(BA)$. Hence: $\sigma(AB) = \sigma(BA) = \lambda\sigma(AB)$.*

Proposition C *Let $AB = \lambda BA$, $\lambda \in \mathbb{C}$, $AB \neq 0$ and assume that A has a bounded inverse. Then we have: $\sigma(B) = \lambda\sigma(B)$.*

In [5] the operator equation $AB = \lambda BA$, $\lambda \in \mathbb{C}$ was studied for normal operators A and B on a Banach space. In this paper, we will first make an improvement on Theorem A. above before we look at spectral properties including the essential spectrum of the operators AB and BA. We will make use of Putnam-Fuglede property, see [11] and [12], and the following result proved by [16].

Theorem D *Suppose T is a pure dominant operator, K is a compact operator with dense range such that $KT = TK$. Then essential spectrum of T is equal to spectrum of T, i.e. $\sigma_e(T) = \sigma(T)$.*

2 Notation and Terminology

- Given an operator A we shall denote the spectrum, the approximate point spectrum and essential spectrum of A by $\sigma(A)$, $\sigma_\pi(A)$ and $\sigma_e(A)$ respectively. Thus we have:

$$\sigma(A) = \{\lambda \in \mathbb{C} : A - \lambda I \; is\; not\; invertible\}$$

$$\sigma_\pi(A) = \{\lambda \in \mathbb{C} : A - \lambda I \; is\; not\; bounded\; below\}$$

$$\sigma_e(A) = \{\lambda \in \mathbb{C} : A - \lambda I \; is\; not\; Fredholm\}$$

- Two operators $A, B \in B(H)$ are said to be commuting operators if:

$$[A, B] = AB - BA = 0.$$

- The range of A and Kernel of A are denoted by $ranA$ and $kerA$ respectively.
- An operator A is said to be:
  - Fredholm if its range $ranA$ is closed and both $kerA$ and $kerA^*$ are finite dimensional.
  - Dominant if to each $\lambda \in \mathbb{C}$ there corresponds a number $M_\lambda \geq 1$ such that:

$$||(A - \lambda)^* x|| \leq M_\lambda \, ||(A - \lambda)x|| \, \forall x \in H.$$

  - M-hyponormal if $\exists$ a constant M with $M_\lambda \leq M$ for all $\lambda \in \mathbb{C}$ such that:

$$||(A - \lambda)^* x|| \leq M \, ||(A - \lambda)x|| \, \forall x \in H.$$

  - Hyponormal if $A^*A \geq AA^*$.
  - Normal if $A^*A = AA^*$.
  - Self-adjoint if $A = A^*$.
  - Pure dominant if it has no invariant subspace say N on which A/N is normal.
  - Compact if it maps a unit ball of H into a relatively compact set.
  - p-hypornormal, $0 < p \leq 1$ if $(A^*A)^p \geq (AA^*)^p$.
  - Quasi-affinity if it is both one-one and has a dense range.
  - a partial isometry if $A = AA^*A$.
- The class $\mathcal{H} \cup (p)$ denotes the class of p-hyponormal operators A for which the polar decomposition $A = U\,|A|$ is unitary where $|A| = (A^*A)^{1/2}$ and U is a partial isometry.
- Note that we have the following inclusions of classes of operators:
  - $\{self - adjoint\} \subset \{normal\} \subset \{hyponormal\} \subset \{m - hypornormal\} \subset \{dominant\}$
  - $\{normal\} \subset \{hyponormal\} \subset \{p - hyponormal\}$
  - $\{normal\} \subset \{quasinormal\} \subset \{p - hyponormal\}$
  - $\{normal\} \subset \{quasinormal\} \subset \{subnormal\} \subset \{hyponormal\} \subset \{m - hypornormal\}$

3 Results

We first show that the following result provides an alternative prove to part (ii) of Theorem A. above.

Lemma 1 *Let $AB = \lambda BA$, $\lambda \in \mathbb{C}$, $AB \neq 0$ with A and B self-adjoint. Then:*

(i) AB and BA are normal commuting operators.

(ii) $\lambda = \pm 1$

Proof

Let $T = AB$. Then $T^* = BA$ for self-adjoint A and B.

(i) Thus we have $T = \lambda T^*$, i.e. $T^*T = \lambda T^{*2}$ and $TT^* = \lambda T^{*2}$, i.e. $T^*T = TT^*$. Thus T is normal and $[T, T^*] = 0$. Hence AB and BA are normal commuting operators.

(ii) Since $T = \lambda T^*$ and λ is real, taking adjoint of each side gives: $T^* = \lambda T$. Thus:

(1) $$T = \lambda T^*$$

(2) $$T^* = \lambda T$$

Adding (1) and (2) gives:

$$T + T^* = \lambda(T + T^*)$$
$$ReT = \lambda(ReT)$$

(3) $$(1 - \lambda)ReT = 0$$

But subtracting (2) from (1) gives:

$$T - T^* = \lambda(T^* - T)$$
$$= -\lambda(T - T^*)$$
$$i.e. \quad ImT = -\lambda ImT.$$

(4) $$or \quad (1 + \lambda)ImT = 0$$

But, since $T \neq 0$, we have that either $(ReT) \neq 0$ or $(ImT) \neq 0$ or both $(ReT) \neq 0$ and $(ImT) \neq 0$.
Hence from (3) and (4), $\lambda = \pm 1$.

Corollary 1 *Let A and B be self-adjoint operators which λ-commute non-trivially. Then we have:*

$$\sigma_\pi(AB) = \sigma_\pi(BA) = \lambda \sigma_\pi(AB)$$

.

Proof

Since A and B are self adjoint operators which λ-commute non-trivially, AB and BA are normal commuting operators from Lemma 1. Hence $T = AB = BA$
Now for any normal operator T, $\sigma(T) = \sigma_\pi(T)$.

$$\sigma_\pi(BA) = \lambda \sigma_\pi(BA) \quad (from\ Lemma\ B).$$

Hence we have the end of the proof.

Theorem 1 *Let $A, B \in B(H)$ be such that $AB = \lambda BA$, $\lambda \in \mathbb{C}$, $AB \neq 0$. We have*

(i) If A is normal and B is self-adjoint then $\lambda \in \{1, -1\}$.

(ii) If A is normal with ReA positive and B is self-adjoint then $\lambda = 1$.

Proof

(i) Given A is normal, B is self-adjoint and $AB = \lambda BA$, we have by part (i) of Theorem A above, that λ is real. Thus by Putnam-Fuglede's theorem we have:

$$A^*B = \lambda BA^* \tag{5}$$

Now by adding the operator equation $AB = \lambda BA$ and (5) we have:

$$(A + A^*)B = \lambda B(A + A^*)$$

i.e. $\quad ReA.B = \lambda B.\, ReA$
Since both ReA and B are self-adjoint we have by part (ii) of Theorem A above that $\lambda \in \{1, -1\}$.

(ii) From proof of part (i) above we have: $ReA.B = \lambda B.ReA$, where both ReA and B are self-adjoint. Since ReA is positive it follows from part (iii) of Theorem A. above that $\lambda = 1$.

Corollary 2 *Let A be normal with ReA positive and B be self-adjoint such that $AB = \lambda BA$, $\lambda \in \mathbb{C}$, $AB \neq 0$. Then $AB = BA$ is normal. Thus:*

$$\sigma(AB) = \sigma(BA) = \sigma_\pi(AB) = \lambda\sigma_\pi(AB).$$

Proof

We note that under the given hypothesis $\lambda = 1$ by part (ii) of Theorem 1. above. Thus $[A, B] = 0$. Letting $T = AB$ and $T^* = BA^*$, we have:

$$T^*T = BA^*AB = BAA^*B = ABA^*B = ABBA^* = TT^*.$$

Hence $T = AB$ is normal. Consequently:

$$\sigma(AB) = \sigma(BA) = \sigma_\pi(AB) = \lambda\sigma_\pi(AB)$$

Remark 1 *We now note that in Proposition C. above, if we require that one of the operators say B in the equation $AB = \lambda BA$, $\lambda \in \mathbb{C}$, $AB \neq 0$ belongs to some appropriate class of operators then we can relax the condition on the operator A. But first we have the following result:*

Let $\wp$ denote the class of all operators which satisfy the Putnam-Fuglede property. Then we have the following results:

Theorem 2 *Let $AB = \lambda BA$, $\lambda \in \mathbb{C}$, $AB \neq 0$. Then we have:*

(i) If B and $\lambda B \in \wp$ with A a quasiaffinity, then λB and B are quasisimilar.

(ii) If A and $\lambda A \in \wp$ with B a quasiaffinity, then λA and A are quasisimilar.

Proof

(i) Given,

$$AB = \lambda BA \tag{6}$$

we have since B and λB are in $\wp$ then $AB^* = \bar{\lambda}B^*A$. Taking adjoints both sides we get:

$$BA^* = A^*\lambda B \tag{7}$$

Since A is quasi-affinity it follows that A^* is also a quasi-affinity. Now from (6) and (7) we have that the operators λB and B are quasi similar.

(ii) Similarly since A and λA are in $\wp$ then $A^*B = B\bar{\lambda}A^*$. Taking adjoints both sides we get:

$$B^*A = \lambda AB^* \tag{8}$$

Since B is quasi-affinity it follows that B^* is also a quasi-affinity. Now from (6) and (8) we have that the operators λA and A are quasi similar.

Corollary 3 *Let $AB = \lambda BA$, $\lambda \in \mathbb{C}$, $AB \neq 0$ with B and B^* p- hyponormal with A a quasiaffinity. Then we have λB and B are quasisimilar.*

Proof

Given $AB = \lambda BA$...(i), we first note that λB and B^* are p-hyponormal and by [8], $AB^* = \bar{\lambda}B^*A$ Taking adjoints on each side gives: $BA^* = A^*\lambda B$...(ii). Since A is a quasiaffintiy it follows that A^* is also a quasiaffinity. Thus form (i) and (ii) we have that λB and B are quasisimilar.

Remark 2 *We now note that equality of spectra or essential spectra for some classes of operators has been proved by a number of authors see [6],[8],[10],[14], and [15]. In our case we have the following corollary.*

Corollary 4 *Let $AB = \lambda BA$, $\lambda \in \mathbb{C}$, $AB \neq 0$ with A a quasi-affinity. Then we have:*

(i) if B is quasi normal then $\sigma_e(B) = \sigma_e(\lambda B) = \lambda\sigma_e(B)$.

(ii) if B and B^* are hyponormal and A compact, then $\sigma(B) = \lambda\sigma(B)$ and $\sigma_e(B) = \lambda\sigma_e(B)$.

(iii) if λB and $B \in \mathcal{H} \cup (p)$, then $\sigma_e(B) = \lambda\sigma_e(B)$ and $\sigma(B) = \lambda\sigma(B)$.

(iv) if B^* and λB are m-hyponormal, then $\sigma_e(B) = \sigma_e(\lambda B) = \lambda\sigma_e(B)$

Proof

(i) By the inclusion $\{quasinormal\} \subset \{p - hyponormal\}$, see [4], and the fact that B^* is quasinormal if B is quasinormal, we have that B and B^* are indeed also p-hypornormal. Thus applying Corollary 3, λB and B are quasisimilar. By [14] we have $\sigma_e(B) = \sigma_e(\lambda\, B) = \lambda\sigma_e(B)$. Also to authenticate our result further, [6] proved that quasisimilar subnormal operators have equal essential spectra and the inclusion $\{quasinormal\} \subset \{subnormal\}$ ensures that the result stands for quasinormal operators.

(ii) By the inclusion $\{hyponormal\} \subset \{p - hyponormal\}$ and Corollary 3, λB and B are quasisimilar. Hence by [6] and [15]: $\sigma(B) = \lambda\sigma(B)$ and $\sigma_e(B) = \lambda\sigma_e(B)$.

(iii) We only have to note that it was proved in [8] that for the operators in $\mathcal{H} \cup (p)$, they belong to $\wp$ and they have same spectrum and same essential spectrum.

(iv) From [7], λB and B belong to $\wp$ due to the inclusion $\{m-hyponormal\} \subset \{dominant\}$. Applying Theorem 2 we have that the operators λB and B are quasisimilar. By [17],we have $\sigma_e(B) = \sigma_e(\lambda B) = \lambda\sigma_e(B)$.

Remark 3 *We note that in Corollary 4 above the conditions on the operators A and B can be interchanged so that similar results can be obtained on spectrum and essential spectrum of A. We also note that in view of Theorem D. above we have the following result about essential spectra of the operators AB and BA.*

Theorem 3 *Let $AB = \lambda BA$, $\lambda \in \mathbb{C}$, $AB \neq 0$ with $\lambda \in \{1, -1\}$. If AB is pure dominant and BA is compact with dense range then: $\sigma_e(AB) = \sigma(AB) = \sigma(BA) = \lambda\sigma(AB)$.*

Proof

Let $T = AB$ and $K = BA$. Then under the conditions that $\lambda \in \{1, -1\}$, we have: $[T, K] = 0$ For if $\lambda = -1$, then $AB = -BA$. Thus: $ABBA = -BABA$ and $BAAB = -BABA$. This means AB and BA commute. Now by theorem D above: $\sigma_e(T) = \sigma(T)$. i.e $\sigma_e(AB) = \sigma(AB) = \sigma(BA) = \lambda\sigma(AB)$.

Remark 4 *Let us for convenience sake say that $T \in B(H)$ belong to a class $\mathcal{M}$ of operators if:*

(i) T is pure dominant

(ii) T is compact

(iii) T has a dense range

Then the following corollary to the theorem above is immediate.

Corollary 5 *Let $AB = \lambda BA$, $\lambda \in \mathbb{C}$, $AB \neq 0$ with $\lambda \in \{1, -1\}$. Also let $T = AB$ and $K = BA$. If $T, K \in \mathcal{M}$, then: $\sigma_e(T) = \sigma(T) = \sigma(K) = \sigma_e(K)$.*

Proof

Since the operators $T, K \in \mathcal{M}$,then from Theorem 3 above we get $\sigma_e(K) = \sigma(K) = \sigma(T) = \sigma_e(T)$. Hence $\sigma_e(AB) = \sigma_e(BA)$

Remark 5 *Note that it is a well known fact that dominant operators that are compact are quasinilpotent hence have zero as its essential spectra.*

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