FERTILITY MODELS BASED ON MIXTURES AND COMPOUND DISTRIBUTIONS

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Declaration

I the undersigned declare that this project is my original work and to the best of my knowledge, no portion of this work has been submitted in support of an application for another degree or qualification of this or any other university or any other institution of learning.

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Dedication

To My wife Carol, My Daughter Tabitha and My son Alvin

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Abstract

A considerable amount of work has been done regarding the fertility study in mathematical demography. Most researchers used Beta distribution since it a classic example of a distributions in the [0,1] domain.This project is prepared with the intention of reviewing some probability models generated through mixed and Compound distributions by different researchers, considering other distributions in the [0,1] as mixing distribution,as well as modifying some the distributions for applications to specific populations.

Contents

Chapter 1

GENERAL INTRODUCTION

1.1 Introduction

In the recent past, several probability models have been constructed to study the distribution between marriage and first conception, and between two parities of any order. In this project a review of these distributions will be discussed in details and further modifications to the distributions discussed. The following techniques will be used to generate the distributions

1.1.1 Distribution Mixtures techniques

In probability and statistics, a mixture distribution is the probability distribution of a random variable whose values can be interpreted as being derived in a simple way from an underlying set of other random variables. Let $f(t, \theta)$ be a Probability Density Function or Probability Mass Function of a random variable t then if θ is also a random variable then the Probability Density Function or Probability Mass Function of t becomes

$$f(t) = \int_{\theta} f(t \mid \theta)\, g(\theta)\, d\theta \tag{1.1}$$

if θ is continuous, where $g(\theta)$ the p.d.f. of random variable θ and ;

$$f(t) = \sum_{\theta} f(t \mid \theta)\, g(\theta) \tag{1.2}$$

if θ is discrete, where $g(\theta)$ is the p.m.f of random variable θ

If the p.d.f. or the p.m.f of a random variable can be expressed as a weighted sum, with non-negative weights that sum to 1 of other p.d.f.'s or p.m.f's, then the formed p.d.f. or p.m.f are also called mixture distribution . The individual distributions that are combined to form the mixture distribution are called the mixture components, and the probabilities (or weights) associated with each component are called the mixture weights. Let $f(t)$ be a p.d.f. or p.m.f of the random variable t, if;

$$f(t) = \sum_i p_i f(t_i) \text{ for } \sum_i p_i = 1 \text{ and } f(t_i) = \text{individual distributions} \tag{1.3}$$

then $f(t)$ is referred to as mixture distribution.

1.1.2 Convolution and Compound Distributions

Convolution

Discrete Consider three sequences of real numbers: $\{a_k\}$, $\{b_k\}$ and $\{c_k\}$ we say that $\{c_k\}$ is a convolution of $\{a_k\}$ and $\{b_k\}$ denoted by $\{c_k\} = \{a_k\} * \{b_k\}$ if

$$c_k = \sum_{r=0}^{k} a_r b_{k-r}$$

and

$$C(s) = A(s) * B(s)$$

Where :$A(s)$, $B(s)$ and $C(s)$ are generating functions of $\{a_k\}$, $\{b_k\}$ and $\{c_k\}$ respectively. If $S_N = X_1 + X_2 + ... + X_N$ and $X_i's$ are independent and identically distributed random variables, then;

$$\{prob(S_N = i)\} = \{P(X_1)\} * \{P(X_2)\} * ...... * \{P(X_N)\}$$

and since $X_i's$ are identical;

$$\{prob(S_N = i)\} = \{P(X_i)\} * \{P(X_i)\} * ...... * \{P(X_i)\} = \{P(X_i)\}^{N*} \tag{1.4}$$

This is referred to as an N^{th}-fold convolution of $\{P(X_i)\}$

Continuous Let X and Y be two identically and independently distributed continuous random variables with density functions $f(x)$ and $g(y)$ respectively, and $Z = X + Y$,then $f(z)$ is a convolution of $f(x)$ and $g(y)$ denoted by $f(z) = (f * g)(Z) = \int f(z-x) f(x)\,dx$ .More generally if $S_N = X_1 + \{X_2 + X_3 ..... + X_N\}$, the distribution of S_N is the sum of $(n-1)$ independent and identically distributed random variables with common parameter λ_2 convolved with that of an exponential random variable X_1 with parameter λ_1 i.e.

$$f^n(S_N) = \int_0^t f_1(t-x) f_{n-1}(\lambda_2 \ , t)\,dx \tag{1.5}$$

Compound Distribution

For $S_N = X_1 + X_2 + \cdots .... + X_N$ where N is a random variable independent of X_i's and with probability density function;

$$p(N = n) = g_n$$

And p.g.f,

$$g(s) = \sum_n g_n s^n$$

Then the probability distribution of S_N is given by;

$$h_i = prob(S_N = i) = \sum_{n=0}^{\infty} \{prob\,(N = n)\}\, probX_1 + X_2 + \cdots .... + X_N = j\}$$

Is called the Compound distribution of S_N. Using generating functions;

$$h(s) = \sum_{n=0}^{\infty} g_n [f(s)]^n \tag{1.6}$$

1.2 Problem Statement

The analysis of intervals between marriage and first birth and subsequent births is a another approach to the study of human fertility. The fact that birth intervals are obviously related to intervals between birth immediately suggests a reason for the potential interest of this analysis. There was limitations in this field since only Classic Beta distribution was considered to lie in the $[0, 1]$ domain. We seek to identify more distributions in this $[0, 1]$ domain and use them as mixing distributions to generate more alternative distribution in the study of human fertility.

1.3 Objectives

The main objective of this work is to derive several fertility models based on the following techniques

i) Distribution Mixtures

ii) Convolution

iii) Compound distribution

With modifications of others with the aim of constructing the best distribution to describe the waiting time intervals between marriage and first birth and consecutive births.

1.4 Significance

The study of birth intervals is increasingly attracting the attention of researchers on account of their possible use as sensitive indices to detect changes in the level of fertility which may be due to natural or unnatural causes,Mixtures and Compound models have become popular because , among other reasons they;

a) Provide a simple mechanism to incorporate extra variation and correlation in the model

b) Add model flexibility

c) Are a natural approach for modeling data that arise in multiple stages

In this research models of the distribution of births in human population will be discussed, these models are designed to describe , broadly, the main features common to a wide variety of communities. Their use for closer study of individual populations would require modifications accordingly.

1.5 Literature Review

Attempts to analyze first birth interval for the estimation of natural conception rate(fecundability) dates back to Gini(1924). Since then several attempts have been made to investigate this interval as well as interval for subsequent birth intervals and a number of demographers have formulated several stochastic models to describe this intervals under various sets of assumptions for related situations. In recent years due to a variety of reasons demographers have shown keen interest in the study of probability models relating to fertility in general and to birth interval in particular. While some have been attracted to it for finding practical solutions to problems connected with decision-making and evaluation of family planning action programmes, some are attracted to it due to the promise the field holds for analytic model building involving theories of stochastic processes. These demographers have used different approach in their research and the breakdown is as follows;

1.5.1 Initial study

Gini(1924) considered birth intervals as waiting time problem dependent on constant fecundability,In his approach, he considered that the probability that the conception does not occur in the first month but in the t^{th} month is;

$$h(t) = \theta(1-\theta)^{t-1}$$

giving a Geometric distribution with probability of conception given by θ.

1.5.2 Distribution Mixtures techniques

The assumption of constant fecundability by Gini(1924) was limiting since in reality this probability varies from woman to woman, The fecundability parameter is assumed to be a random variable that follows a certain distribution. The following demographers have used this technique to formulate different stochastic models.

i Henry(1958) originally proposed beta distribution for the probability of conception(fecundability) i.e. $f(\theta) = \frac{1}{B(\alpha,\beta)}\theta^{\alpha-1}(1-\theta)^{\beta-1}$, with the probability that the conception does not occur in the first month but in the t^{th} month given by;

$$h(t) = \theta(1-\theta)^{t-1}$$

resulting to a Beta-Geometric distribution.

ii Singh(1964) developed a probability model of waiting time to first conception with fecundability parameter following Gamma distribution i.e. $f(\theta) = \frac{\beta^\alpha e^{-\beta\theta}\theta^{\alpha-1}}{\Gamma(\alpha)}$ using Exponential distribution as the conditional distribution

$$f(t \mid \theta) = \theta e^{-\theta t}$$

,resulting to a Pareto distribution

iii Brass (1958) initially suggested the fecundability parameters to follow the same distribution(Gamma distribution) with truncated exponential .and Biswas and Shrestha(1986) extended this model by truncating the Exponential distribution at a fixed point ω i.e.

$$f(t \mid \theta) = \frac{\theta e^{-\theta t}}{1 - e^{-\theta\omega}}$$

with fecundability parameter following Gamma distribution i.e.,

$$f(\theta) = \frac{\beta^\alpha e^{-\beta\theta}\theta^{\alpha-1}}{\Gamma(\alpha)}$$

1.5.3 Compound Distributions Technique

In an effort to include more variables in the development of waiting time models, Compound distributions technique $S_N = X_1 + X_2 + \cdots\ldots + X_N$ was used. With this techniques the waiting time from marriage till first birth, denoted by T_i was expressed as a component of several variables such as non-susceptibility period,susceptibility period, gestation period among others which were all considered to be random variables. That is;

$$T_i = m + X_1 + X_2 + \ldots\ldots + X_{n+1} + L_1 + L_2 + \ldots. + L_n + 9$$

Where

- m postpartum amenorrhoea following the i^{th} birth
- $X_i, i = 1, 2, \ldots n + 1$denotes the number of months the mother goes on without conception in the susceptible state. $X_i's$ are independent and identically distributed random variables.

- $L_i, i = 1, 2, ..., n$ denotes the period of nonsusceptibility associated with a defective termination (including period of pregnancy and period of amenorrhoea following termination) for any women in the interval T_i . $L_i's$ are all independently and identically distributed random variables.

- '9' denotes the period of pregnancy leading to a live birth

with Probability Generating Function (p.g.f) of T_i given by

$$H(s) = E[s^m]\{E[s^{X_i}]\}^{n+1}\{E[s^{L_i}]\}^n s^9$$

several demographers have used this technique to generate statistical models which include;

i) Srinivasan (1966) developed a simple probability model describing the distribution of birth intervals between successive parities, In his research the distribution of n assumed a geometric distribution

$$prob(N = n) = g(n) = \theta(1-\theta)^n$$

with distribution of X_i given by $prob(X_i = x) = f(x) = pq^r$ and m assuming a triangular distribution. He applied the model generated to data on birth intervals of Indian women.

ii) Aleyamma George(1975) used Srinivasan approach but considered the case where the various distributions were both discrete and continuous. He considered a case where ,For a single woman of constant probability of conception and the distribution of n was geometric, given by $prob(N = n) = g(n) = \theta(1-\theta)^n$, and the distribution of X_i was Exponential given by $prob(X_i = x) = f(x) = \lambda e^{-\lambda x}$ and the distribution of L_i given by $prob(L_i = l) = f(x) = \lambda_1 e^{-\lambda_1 x}$,

iii) John Bongraats (1975) considered the waiting time from marriage to first live birth to be a random variable T with all variables in other two cases except the variable m (postpartum amenorrhoea following the i^{th} birth).With random variables $X_i's$, $L_i's$ and n assuming geometric distributions with different parameters. That is; $prob(N = n) = g(n) = \theta(1-\theta)^n$, $prob(X_i = x) = f(x) = f(1-f)^{x-1}$, $x = 1, 2, 3, ...$and the $prob(L_i = l) = g(l) = g(1-g)^{l-1}$ 1, 2, ...

1.5.4 Modifications to models

Pathak (1999) provided an analytical review of several distributions developed by earlier demographers, Pathak(2006) developed a model specific to rural parts of India factoring in their cultural practices.

This research project aims to review these distributions and extend the work to more distribution construction and specific modifications to some of the reviewed distributions.

Chapter 2

FERTILITY MODELS BASED ON MIXED DISTRIBUTION

2.1 Introduction

In this chapter we will review several distributions used to study the waiting time from marriage till conception. Gini(1924) first considered birth intervals as waiting time problem dependent on fecundability. His work deals with pregnancies and birth of first order under constant fecundability. Geometric distribution was the only distribution considered under the assumption of constant fecundability. However further studies have shown and proved that fecundability, the probability of conception varies from woman to woman, and that the fecundability parameter is assumed to follow a certain distribution. Under this study the Mixed distribution techniques given by,

$$f(t) = \int_{\theta} f(t \mid \theta)\, g(\theta)\, d\theta$$

where;

a θ denotes the probability of conception

b t is the waiting time from marriage till first conception

has been used. Under mixing distributions the following mixtures have been discussed;

1. $f(t \mid \theta)$ being geometric and $g(\theta)$ being Beta distribution
2. $f(t \mid \theta)$ being exponential and $g(\theta)$ being Gamma distribution

3. $f(t \mid \theta)$ being Truncated exponential and $g(\theta)$ being gamma distribution

4. $f(t \mid \theta)$ being Poisson and $g(\theta)$ being Beta distribution

In this chapter these distributions (considering both homogeneous and heterogeneous fecundability) and their applications on Srinivasan's data will be discussed in details.

2.2 Geometric Mixtures

These are distributions of the form,

$$f(x) = \int \theta (1-\theta)^{x-1} g(\theta) \, d\theta$$

when the mixing distribution $g(\theta)$ is continuos and

$$f(x) = \sum \theta (1-\theta)^{x-1} g(\theta)$$

when the mixing distribution $g(\theta)$ is discrete, the mixing p.d.f. or m.g.f. is $g(\theta)$ for any random variable between $[0, 1]$

2.2.1 Beta mixture of Geometric Distribution

The Distribution

In this case both θ (fecundability) and T (waiting time from marriage till conception) are considered to be random variables with the following distributions

$$f(t \mid \theta) = \theta (1-\theta)^{t-1} \text{ (Geometric distribution) and} \tag{2.1}$$

$$f(\theta) = \frac{1}{B(\alpha, \beta)} \theta^{\alpha-1} (1-\theta)^{\beta-1} \text{ (Beta distribution)} \tag{2.2}$$

The unconditional distribution of T is given by;

$$f(t) = \int_0^1 \left[\theta (1-\theta)^{t-1}\right] \left[\frac{1}{B(\alpha, \beta)} \theta^{\alpha-1} (1-\theta)^{\beta-1}\right] d\theta$$

$$f(t) = \frac{\alpha \Gamma(\beta + t - 1) \Gamma(\alpha + \beta)}{\Gamma(\alpha + \beta + t) \Gamma(\beta)} \text{ with} \tag{2.3}$$

$$\text{Mean} = E(T) = \frac{\alpha+\beta-1}{\alpha-1} \text{ and}$$
$$\text{Variance} = var(T) = \frac{\alpha\beta(\alpha+\beta-1)}{(\alpha-1)^2(\alpha-2)}$$

Proof.

$$f(t) = prob(T=t) = \int_\theta^1 f(t \mid \theta)\, g(\theta)\, d\theta$$
$$\Longrightarrow f(t) = \int_0^1 \left[\theta(1-\theta)^{t-1}\right]\left[\frac{1}{B(\alpha,\beta)}\theta^{\alpha-1}(1-\theta)^{\beta-1}\right] d\theta$$
$$\Longrightarrow f(t) = \frac{1}{B(\alpha,\beta)}\int_0^1 \theta^{\alpha}(1-\theta)^{t+\beta-2}\, d\theta$$
$$\Longrightarrow f(t) = \left\{\frac{1}{B(\alpha,\beta)}\right\}\{B(\alpha+1, t+\beta-1)\}$$
$$\Longrightarrow f(t) = \left[\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\right] * \frac{\Gamma(\alpha+1)\Gamma(t+\beta-2)}{\Gamma(\alpha+\beta+t-1)}$$
$$\Longrightarrow f(t) = \frac{\alpha\Gamma(t+\beta-2)\Gamma(\alpha+\beta)}{\Gamma(\alpha+\beta+t-1)\Gamma(\beta)}, t = 1,2,3,....$$

■

Parameter Estimations

Letting $E(T) = \mu$ and $var(T) = \delta^2$, and using the method of moments to estimate the parameters we have;

$$E(T) = \mu = \frac{\alpha+\beta-1}{\alpha-1} \text{ and } var(T) = \delta^2 = \frac{\alpha\beta(\alpha+\beta-1)}{(\alpha-1)^2(\alpha-2)} \text{ then}$$
$$\Longrightarrow \mu(\alpha-1) = (\alpha-1)+\beta = \beta = (\mu-1)(\alpha-1)$$
$$\Longrightarrow \delta^2 = \frac{\alpha[(\mu-1)(\alpha-1)][(\alpha-1)+(\mu-1)(\alpha-1)]}{(\alpha-1)^2(\alpha-2)}$$
$$\Longrightarrow \delta^2 = \frac{\alpha(\alpha-1)^2(\mu-1)[1+\mu-1]}{(\alpha-1)^2(\alpha-2)} = \frac{\alpha\mu(\mu-1)}{(\alpha-2)} = \delta^2$$
$$\Longrightarrow \alpha = \frac{2\delta^2}{\delta^2-\mu^2+\mu} \text{ and } \beta = (\mu-1)(\alpha-1) \qquad (2.4)$$

2.3 Exponential Mixtures

These are distributions of the form,

$$f(x) = \int \theta e^{-\theta x} g(\theta)\, d\theta$$

when the mixing distribution $g(\theta)$ is continuos and

$$f(x) = \sum \theta e^{-\theta x} g(\theta)$$

when the mixing distribution $g(\theta)$ is discrete, the mixing p.d.f. or m.g.f. is $g(\theta)$ for any random variable between $[0, \infty)$

2.3.1 Gamma mixture of Exponential Distribution(Pareto)

The Distribution

In this case both θ (fecundability) and T (waiting time from marriage till conception) are considered to be random variables with the following distributions;

$$f(t \mid \theta) = \theta e^{-\theta t} \text{ (Exponential) and} \tag{2.5}$$

$$f(\theta) = \frac{\beta^{\alpha} e^{-\beta\theta} \theta^{\alpha-1}}{\Gamma(\alpha)} \text{ (Gamma-with two parameters)} \tag{2.6}$$

Thus the unconditional density function of the waiting time T. is given by;

$$f(t) = \int_0^{\infty} \theta e^{-\theta t} \frac{\beta^{\alpha} e^{-\beta\theta} \theta^{\alpha-1}}{\Gamma(\alpha)} d\theta$$

$$f(t) = \frac{\alpha\beta^{\alpha}}{(\beta+t)^{\alpha+1}}, t \succ 0, \alpha, \beta \succ 0 \tag{2.7}$$

Proof.

$$\text{for } f(t) = \int_0^{\infty} \theta e^{-\theta t} \frac{\beta^{\alpha} e^{-\beta\theta} \theta^{\alpha-1}}{\Gamma(\alpha)} d\theta = \frac{\beta^{\alpha}}{\Gamma(\alpha)} \int_0^{\infty} \theta^{\alpha} e^{-\theta(\beta+t)} d\theta$$

letting $u = \theta(\beta+t) \Leftrightarrow \theta = \dfrac{u}{(\beta+t)}$ and $d\theta = \dfrac{du}{(\beta+t)}$ thus

$$\Longrightarrow f(t)=\frac{\beta^{\alpha}}{\Gamma(\alpha)}\int_{0}^{\infty}\left[\frac{u}{(\beta+t)}\right]^{\alpha}e^{-u}\frac{du}{(\beta+t)}$$

$$\Longrightarrow f(t)=\frac{\beta^{\alpha}}{\Gamma(\alpha)(\beta+t)^{\alpha+1}}\int_{0}^{\infty}u^{\alpha}e^{-u}du$$

$$\Longrightarrow f(t)=\frac{\beta^{\alpha}}{\Gamma(\alpha)(\beta+t)^{\alpha+1}}\Gamma(\alpha+1)=\frac{\alpha\beta^{\alpha}}{(\beta+t)^{\alpha+1}}$$

$$f(t)=\frac{\alpha\beta^{\alpha}}{(\beta+t)^{\alpha+1}}\ ,t\succ 0,\alpha,\beta\succ 0.$$

■

This is referred to as Pareto distribution with

$$\text{Mean}=E(T)=\frac{\beta}{\alpha-1}\text{ and variance}=var(T)=\frac{\alpha\beta^{2}}{(\alpha-1)^{2}(\alpha-2)}$$

Parameter Estimations

By method of moments let $E(T)=\mu$ and $var(T)=\delta^2$ then we have

$$E(T)=\mu=\frac{\alpha\beta}{\alpha-1}\text{ and }var(T)=\delta^{2}=\frac{\alpha\beta^{2}}{(\alpha-1)^{2}(\alpha-2)}$$

$$\Longrightarrow\beta=\frac{\mu(\alpha-1)}{\alpha}\text{ and }\delta^{2}=\frac{\alpha\left[\frac{\mu(\alpha-1)}{\alpha}\right]^{2}}{(\alpha-1)^{2}(\alpha-2)}$$

$$\delta^{2}=\frac{\alpha\mu^{2}(\alpha-1)^{2}}{\alpha^{2}(\alpha-1)^{2}(\alpha-2)}=\frac{\mu^{2}}{\alpha(\alpha-2)}=\frac{\mu^{2}}{\alpha^{2}-2\alpha}$$

$$\Longrightarrow\alpha(\alpha-2)=\frac{\mu^{2}}{\delta^{2}}\Leftrightarrow\alpha^{2}-2\alpha-\frac{\mu^{2}}{\delta^{2}}=0$$

$$\alpha=\frac{2\pm\sqrt[2]{2^{2}-4\left(-\frac{\mu^{2}}{\delta^{2}}\right)}}{2}=\frac{2\pm\sqrt{2^{2}+4\left(\frac{\mu^{2}}{\delta^{2}}\right)}}{2}$$

$$\alpha=1\pm\sqrt{1+\frac{\mu^{2}}{\delta^{2}}}\text{ and }\beta=\frac{\mu(\alpha-1)}{\alpha}\qquad(2.8)$$

2.4 Truncated Exponential Mixtures

These are of the form.distributions

$$f(x) = \int \frac{\theta e^{-\theta t}}{1 - e^{-\theta\omega}} g(\theta)\, d\theta \tag{2.9}$$

when the mixing distribution $g(\theta)$ is continuos and

$$f(x) = \sum \frac{\theta e^{-\theta t}}{1 - e^{-\theta\omega}} g(\theta) \tag{2.10}$$

when the mixing distribution $g(\theta)$ is discrete, the mixing p.d.f. or m.g.f. is $g(\theta)$ for any random variable between $[0, \infty)$

2.4.1 Beta mixture of Truncated Exponential

The distribution

In 1986 Biswas and Shrestha modified model in 2.4 above by considering the conditional distribution to be a exponential distribution truncated at ω, thus,

$$f(t \mid \theta) = \frac{\theta e^{-\theta t}}{1 - e^{-\theta\omega}}; t = 1, 2, 3....\omega, \alpha \succ 0, \beta \succ 0 \text{ (Truncated Exponential) and}$$

$$f(\theta) = \frac{\beta^{\alpha} e^{-\beta\theta} \theta^{\alpha-1}}{\Gamma(\alpha)} \text{ (Gamma-with two parameters)}$$

The unconditional density function of waiting time till first conception is given by;

$$f(t) = \int_0^{\infty} \left(\frac{\theta e^{-\theta t}}{1 - e^{-\theta\omega}} \right) \left(\frac{\beta^{\alpha} e^{-\beta\theta} \theta^{\alpha-1}}{\Gamma(\alpha)} \right) d\theta$$

$$f(t) = \alpha\beta^{\alpha} \sum_{c=0}^{\infty} \frac{1}{(\beta + t + c\omega)^{\alpha+1}} , t = 1, 2, 3, ...\omega, \alpha \succ 0, \beta \succ 0 \text{ with} \tag{2.11}$$

$$\text{Mean} = E(T) = \frac{\beta}{(\alpha - 1)} \left[1 - \left(\frac{\beta}{\beta + \omega} \right)^{\alpha-1} \right] \text{ and}$$

$$\text{variance} = var(T) = \frac{\beta}{(\alpha-1)^2}\left[\frac{\alpha\beta}{\alpha-2} - \left(\frac{\beta}{\beta+\omega}\right)^{\alpha-1}\left\{\frac{\alpha\omega(\alpha-1)+2\beta}{\alpha-2} + \beta\left(\frac{\beta}{\beta+\omega}\right)^{\alpha-1}\right\}\right]$$

Proof.

$$f(t) = \int_0^\infty \left(\frac{\theta e^{-\theta t}}{1-e^{-\theta\omega}}\right)\left(\frac{\beta^\alpha e^{-\beta\theta}\theta^{\alpha-1}}{\Gamma(\alpha)}\right) d\theta$$

$$\Longrightarrow f(t) = \frac{\beta^\alpha}{\Gamma(\alpha)}\int_0^\infty \left(\frac{\theta e^{-\theta t}}{1-e^{-\theta\omega}}\right)\left(e^{-\beta\theta}\theta^{\alpha-1}\right) d\theta$$

$$\Longrightarrow f(t) = \frac{\beta^\alpha}{\Gamma(\alpha)}\int_0^\infty \left(\frac{1}{1-e^{-\theta\omega}}\right)\left(e^{-\theta(t+\beta)}\theta^{\alpha}\right) d\theta$$

$$f(t) = \frac{\beta^\alpha}{\Gamma(\alpha)}\int_0^\infty \left(e^{-\theta(t+\beta)}\theta^{\alpha}\right)\left\{1 + e^{-\theta\omega} + \left(e^{-\theta\omega}\right)^2 + \left(e^{-\theta\omega}\right)^3 + ...\right\} d\theta;$$

$$\text{from } \frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + ...$$

$$f(t) = \frac{\beta^\alpha}{\Gamma(\alpha)}\left\{\int_0^\infty \left(e^{-\theta(t+\beta)}\theta^{\alpha}\right) d\theta + \int_0^\infty \left(e^{-\theta(t+\beta)}\theta^{\alpha}\right) e^{-\theta\omega} d\theta + \int_0^\infty \left(e^{-\theta(t+\beta)}\theta^{\alpha}\right) e^{-2\theta\omega} d\theta + ...\right\}$$

$$f(t) = \frac{\beta^\alpha}{\Gamma(\alpha)}\left\{\frac{1}{(t+\beta)^{\alpha+1}}\Gamma(\alpha+1) + \frac{1}{(t+\beta+\omega)^{\alpha+1}}\Gamma(\alpha+1) + \frac{1}{(t+\beta+2\omega)^{\alpha+1}}\Gamma(\alpha+1) + ...\right\}$$

$$f(t) = \frac{\beta^\alpha}{\Gamma(\alpha)}\Gamma(\alpha+1)\left\{\frac{1}{(t+\beta)^{\alpha+1}} + \frac{1}{(t+\beta+\omega)^{\alpha+1}} + \frac{1}{(t+\beta+2\omega)^{\alpha+1}} + ...\right\}$$

$$\Longrightarrow f(t) = \alpha\beta^\alpha \sum_{c=0}^{\infty} \frac{1}{(\beta+t+c\omega)^{\alpha+1}},, t \succ 0, \alpha, \beta \succ 0$$

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2.5 Application to Srinivasan's data

In this section a comparison is made by applying Geometric distribution and the distributions discussed above to Srinivasan's data. Under assumption of constant fecundability Geometric distribution was the only distribution considered. However further studies have shown and proved that fecundability, the probability of conception varies from woman to woman, and that the fecundability parameter is assumed to follow a certain distribution. In this section we discuss the application of Geometric distribution and the above mixed distributions to data from Srinivasan(1966)

Time interval in Months	Observed frequency	Mid-point x	fx	fx^2
1-13	121	7	847	5929
14-25	90	19.5	1755	34222.5
26-37	43	31.5	1354.5	42666.75
38-49	13	43.5	565.5	24599
50-61	18	55.5	999	55444.5
62-73	11	67.5	742.5	50118.75
74-156	19	115	2185	251275
$\sum_x$	315	339.5	8448.5	464255.8

From K. Srinivasan (1966) TABLE 1;

- sample mean $= \bar{X} = \frac{\sum fx}{\sum f} = \frac{8448.5}{315} = 26.82$ and
- sample variance $= \hat{S}^2 = \left(\frac{\sum fx^2}{\sum f}\right) - \left(\bar{X}\right)^2 = \frac{464255.8}{315} - 26.82^2 = 754.52$

2.5.1 Geometric Distribution

The Distribution

The probability model of waiting time from marriage till conception for women with constant fecundability is given by;

$$h(t) = prob(T = t)$$

The probability of conceiving in the first month is thus given by,

$$h(1) = prob(T = 1)$$

letting $h(1) = \theta$ denote the probability of conception for a woman in susceptible period. The probability that the conception does not occur in the first month but in the t^{th} month is;

$$h(t) = \theta(1-\theta)^{t-1}$$

this is the probability density function of a geometric distribution with;

$$\text{Mean} = E(T) = \frac{1}{\theta} \text{ and variance} = var(T) = \frac{1-\theta}{\theta^2}$$

Parameter Estimation

Maximum Likelihood Estimator for Grouped data

Consider the frequency function $f\left(x \mid \theta\right)$ depending on a single parameter θ. Let the real line, R, be partitioned into intervals of equal width h and centres x_i . A random sample of size N is now drawn from a population with frequency function f and the numbers falling in each interval counted. Let N_j be the number of observations falling in interval $[x_j - \frac{h}{2}, x_j + \frac{h}{2}]$, then the maximum likelihood estimate of θ from the grouped sample.

If we let

$$p_j\left(\theta\right) = prob(x_j - \frac{h}{2} \leq x \leq x_j + \frac{h}{2})$$

then,

$$p_j\left(\theta\right) = \int_{x_j-\frac{h}{2}}^{x_j+\frac{h}{2}} f\left(x \mid \theta\right) dx = F\left(x_j + \frac{h}{2} \mid \theta\right) - F\left(x_j - \frac{h}{2} \mid \theta\right) \qquad (2.12)$$

for continuos case and

$$p_j\left(\theta\right) = \sum_{x_j-\frac{h}{2}}^{x_j+\frac{h}{2}} f\left(x \mid \theta\right)$$

for discrete cases,then an application of standard techniques would lead us to maximize

$$L\left(\theta\right) = \prod_{j=1}^{k} \left[p_j\left(\theta\right)\right]^{N_i} = \prod_{j=1}^{k} F\left(x_j + \frac{h}{2} \mid \theta\right) - F\left(x_j - \frac{h}{2} \mid \theta\right) \qquad (2.13)$$

With respect to θ. These were the results obtained by G. M. Tallis(1967).

Similarly from the book Loss Models by Stuart A. Klugman, Harry H. Panjer, Gordon E. Willmot, page 341, 3rd Edition (December 9, 2008), from the sample , let n_j be the number of observations in the interval $(c_{j-1}, c_j]$ for such data, the Maximum likelihood function is given by;

$$L\left(\theta\right) = \prod_{j=1}^{k} \left[F(c_j \mid \theta) - F(c_{j-1} \mid \theta)\right]^{n_j} \qquad (2.14)$$

which is similar to the previous result;

$$L(\theta) = \prod_{i=1}^{k} F\left(x_j + \frac{h}{2} \mid \theta\right) - F\left(x_j - \frac{h}{2} \mid \theta\right)$$

with $x_j + \frac{h}{2} = c_j$ =upper limit and $x_j - \frac{h}{2} = c_{j-1}$ = lower limit of j^{th} class. With

$$\log L(\theta) = \sum_{j=1}^{k} n_j \log\left[F(c_j \mid \theta) - F(c_{j-1} \mid \theta)\right]$$

$$\frac{d}{d\theta}(\log L(\theta)) = \frac{\sum_{j=1}^{k} n_j \left[f(c_j \mid \theta) - f(c_{j-1} \mid \theta)\right]}{\sum_{j=1}^{k} n_j \left[F(c_j \mid \theta) - F(c_{j-1} \mid \theta)\right]} \tag{2.15}$$

To estimate θ from the data using the method of maximum likelihood based of grouped geometric data, the likelihood function is given by;

$$L = \prod_{j=1}^{k} \left[\sum_{c_{j-1}}^{c_j} \theta (1-\theta)^{t-1}\right]^{\eta_j} \tag{2.16}$$

where

k =number of classes,

η_j =frequency of the j^{th} class
c_{j-1} =lower bound of the j^{th} class
c_j =upper bound of the j^{th} class
Based on Srinivasan data above, then

$$\begin{aligned} L = & \left\{\sum_{t=1}^{13} \theta (1-\theta)^{t-1}\right\}^{121} \left\{\sum_{t=14}^{25} \theta (1-\theta)^{t-1}\right\}^{90} \left\{\sum_{t=26}^{37} \theta (1-\theta)^{t-1}\right\}^{43} \\ & \left\{\sum_{t=38}^{49} \theta (1-\theta)^{t-1}\right\}^{13} \ldots\ldots \left\{\sum_{t=74}^{85} \theta (1-\theta)^{t-1}\right\}^{19} \end{aligned}$$

$$L = \left[\sum_{t=1}^{13} \theta (1-\theta)^{t-1}\right]^{121} \prod_{r=1}^{6} \left[\sum_{2+12r}^{13+12r} \theta (1-\theta)^{t-1}\right]^{\eta_r}$$

where η_r is the frequency of conceptions in different age groups starting from the second class. Getting the Log of L and solving for θ we have

$$\log L = 121 \log\left\{\theta \sum_{t=1}^{13} (1-\theta)^{t-1}\right\} + \sum_{r=1}^{6} \eta_r \log\left\{\sum_{t=2+12r}^{13+12r} \theta (1-\theta)^{t-1}\right\}$$

$$\frac{d \log L}{d\theta} = 121\left\{\frac{\sum_{t=2}^{13} (1-\theta)^{t-2}(1-\theta t)}{\sum_{t=2}^{13} \theta (1-\theta)^{t-1}}\right\} + \sum_{r=1}^{6} \eta_r \left\{\frac{\sum_{t=2+12r}^{13+12r} (1-\theta)^{t-2}(1-\theta t)}{\sum_{t=2+12r}^{13+12r} \theta (1-\theta)^{t-1}}\right\}$$

Application to Srinivasan's data

From Srinivasan's data, the sample mean waiting time from marriage till conception is given by;

$$\bar{X} = 35.82$$

therefore the mean waiting time to first conception is $35.82 - 9 = 26.86$ months. Thus

$$\frac{1}{\theta} = \bar{X} \Longrightarrow \hat{\theta} = \frac{1}{26.82} \Longrightarrow \hat{\theta} = 0.03728$$

Using the method of maximum likelihood based on grouped Srinivasavan's data we have;

$$121\left\{\frac{\sum_{t=2}^{13}(1-\theta)^{t-2}(1-\theta t)}{\sum_{t=2}^{13}\theta(1-\theta)^{t-1}}\right\}+90\left\{\frac{\sum_{14}^{25}(1-\theta)^{t-2}(1-\theta t)}{\sum_{14}^{25}\theta(1-\theta)^{t-1}}\right\}+ \quad (2.17)$$

$$+43\left\{\frac{\sum_{26}^{37}(1-\theta)^{t-2}(1-\theta t)}{\sum_{26}^{37}\theta(1-\theta)^{t-1}}\right\}+13\left\{\frac{\sum_{38}^{49}(1-\theta)^{t-2}(1-\theta t)}{\sum_{38}^{49}\theta(1-\theta)^{t-1}}\right\}+$$

$$18\left\{\frac{\sum_{50}^{61}(1-\theta)^{t-2}(1-\theta t)}{\sum_{50}^{61}\theta(1-\theta)^{t-1}}\right\}+11\left\{\frac{\sum_{62}^{73}(1-\theta)^{t-2}(1-\theta t)}{\sum_{62}^{73}\theta(1-\theta)^{t-1}}\right\}+ \quad (2.18)$$

$$+19\left\{\frac{\sum_{74}^{156}(1-\theta)^{t-2}(1-\theta t)}{\sum_{74}^{156}\theta(1-\theta)^{t-1}}\right\}=0$$

and by using r program , we get;

$$\bar{\theta}=0.0414$$

therefore $\hat{\theta}$ and $\bar{\theta}$ are approximately equal.

2.5.2 Beta-Geometric

Given that $\hat{T}$ and $\hat{S}^2$ as the observed mean and variance from Srinivasan's data, the moment estimates of α and β are obtained by putting $\hat{T}=\mu$ and $\hat{S}^2=\delta^2$, resulting to

$$\hat{T}=26.82=\mu \text{ and } \hat{S}^2=754.52=\delta^2 \text{ by equation 2.4 ,then}$$

$$\alpha = \frac{2\,(754.\,52)}{754.\,52 - 26.82^2 + 26.82} = 24.33 \text{ and}$$
$$\beta = (24.33 - 1)\,(26.82 - 1) = 602.38$$

$$f\,(t) = \frac{\alpha\Gamma\,(t+\beta-1)\,\Gamma\,(\alpha+\beta)}{\Gamma\,(\alpha+\beta+t)\,\Gamma\,(\beta)} \simeq \frac{\alpha\,(\alpha+\beta)!}{(\beta-1)!}\left\{\frac{(t+\beta-2)!}{(\alpha+\beta+t-1)!}\right\}$$
$$f\,(t) = \frac{13.86\,(13.86+331.52)!}{(331.52-1)!}\left\{\frac{(t+\beta-2)!}{(\alpha+\beta+t-1)!}\right\}$$

2.5.3 Pareto

Given that $\bar{T}$ and $\bar{S}^2$ as the observed mean and variance from Srinivasan's data, the moment estimates of α and β are obtained by putting $\bar{T} = \mu$ and $\bar{S}^2 = \delta^2$ so that we have

$$\bar{T} = 26.82 \text{ and } \bar{S}^2 = 754.\,52$$

from equation 2.4.2.a we have

$$\alpha = 1 \pm \sqrt{1 + \frac{26.82^2}{812}} = 2.\,3733$$
$$\beta = \frac{26.86\,(2.\,3733 - 1)}{2.\,3733} = 15.\,561$$
$$f\,(t) = \frac{\alpha\beta^\alpha}{(\beta+t)^{\alpha+1}}, t \succ 0, \alpha, \beta \succ 0$$
$$f\,(t) = \frac{2.\,3733\,(15.\,561)^{2.\,374\,2}}{(\beta+t)^{\alpha+1}} = 1605.0\left(\frac{1}{(\beta+t)^{\alpha+1}}\right)$$

2.6 Summary

The table below observed and expected frequencies of waiting time distribution of conception of first order for the various distributions discussed above applicable to data from Srinivasan's 1966

1. **Geometric distribution;,**

$\theta = 0.0414$

$$p(t) = \theta(1-\theta)^{t-1}$$

$$prob(1 \leq t \leq 13) = 0.0414(0.9586)^0 + 0.0414(0.9586)^1 + ... + 0.0414(0.9586)^{12}$$
$$0.0414\left(1 + (0.9586)^1 + (0.9586)^2 + ... + (0.9586)^{12}\right)$$

$$\begin{aligned} prob(1 &\leq t \leq 13) = 0.0414\left\{\frac{1-0.9586^{13}}{0.0414}\right\} \\ &= 1 - 0.9586^{13} = 0.4228 \\ \text{frequency} &= 315\,(0.4228) = 133.20 \end{aligned}$$

$$prob(14 \leq t \leq 25) = 0.0414(0.9586)^{13}+0.0414(0.9586)^{14}+.......+0.0414(0.9586)^{24}$$

$$prob(14 \leq t \leq 25) = 0.0414(0.9586)^{13}\left\{1 + (0.9586)^1 + (0.9586)^2 + ... + (0.9586)^{11}\right\}$$
$$(0.9586)^{13}\left\{1 - (0.9586)^{11}\right\} = 0.214, \text{frequency}=0.214 * 315 = 67.62$$

....

$$\begin{aligned} prob(74 &\leq t \leq 156) = (0.9586)^{73}\left\{1 - 0.9586^{82}\right\} = 0.0442 \\ \text{frequency} &= 0.0423 * 315 = 13.93 \end{aligned}$$

2 **Beta Geometric,**

With p.m.f given by

$$p(t) = \frac{\alpha\Gamma(t+\beta-2)\,\Gamma(\alpha+\beta)}{\Gamma(\alpha+\beta+t-1)\,\Gamma(\beta)}$$

with $\alpha = 24.33$ and $\beta = 602.38$,

$$p(t) = \frac{24.33\Gamma(t+600.38)\Gamma(626.71)}{\Gamma(625.71+t)\Gamma(602.38)} = \frac{25.33!}{(t+25.44)!}$$

$$prob(1 \leq t \leq 13) = \frac{25.33!}{26.44!} + \frac{25.33!}{27.44!} + ... + \frac{25.33!}{38.44!} = 0.4228,$$

frequency= (0.4228) 315 = 133.2

$$prob(14 \leq t \leq 25) = \frac{25.33!}{(39.44)!} + \frac{25.33!}{(40.44)!} + ... + \frac{25.33!}{(50.44)!} = 0.231,$$

frequency= (0.231) 315 = 72.88

and so on

3 **Pareto;**

The p.d.f is given by;

$$f(t) = \frac{\alpha\beta^{\alpha}}{(\beta+t)^{\alpha+1}}$$

with $\alpha = 2.3976$ and $\beta = 15.657$, Therefore;

$$f(t) = \frac{2.3733\,(15.561)^{2.3742}}{(15.657+t)^{2.3976}} = \frac{1605.0}{(t+15.657)^{2.3976}}$$

$$prob(1 \leq t \leq 13) = \int_{1}^{13} \frac{1605.0}{(t+15.657)^{3.3733}} =$$

$$1605.0\left\{\left[-\frac{1}{2.3976(t+15.657)^{1.3976}}\right]_{1}^{13}\right\}$$

$$\begin{aligned} prob(1 \leq t \leq 13) &= 1605.0\left\{\frac{1}{2.3733\,(15.657)^{2.3733}} - \frac{1}{2.3733\,(28.657)^{2.3733}}\right\} \\ &= 0.507,\ frequency = 0.5066 * 315 = 159.59 \end{aligned}$$

$$prob\,(1 \leq t \leq 13) = \int_{14}^{25} \frac{1605.0}{(t + 15.657)^{3.3733}}$$

$$= 1605.0 \left\{ \frac{1}{2.3733\,(29.657)^{2.3733}} - \frac{1}{2.3733\,(40.657)^{2.3733}} \right\} = 0.21433$$

$$\text{frequency} = 0.21433 * 315 = 67.51$$

...and so on
Giving the table as below;

Time interval in Months	Observed frequency	Expected Frequency		
		Geometric $h(t)=\theta(1-\theta)^{t-1}$	Beta-Geometric $f(t)=\frac{\alpha\Gamma(t+\beta-1)\Gamma(\alpha+\beta)}{\Gamma(\alpha+\beta+t)\Gamma(\beta)}$	Pareto $f(t)=\frac{\alpha\beta^{\alpha}}{(\beta+t)^{\alpha+1}}$
1-13	121	133.29	133.20	159.59
14-25	90	67.62	72.88	67.51
26-37	43	40.71	44.59	37.96
38-49	13	24.51	27.74	23.57
50-61	18	14.75	17.59	15.63
62-73	11	8.88	11.17	10.90
74-156	19	13.33	6.59	7.89

Chapter 3

Geometric Mixtures with $[0, 1]$ Domain Mixing

In chapter two only classic Beta Distribution was considered as the mixing distribution. Special cases of Beta distributions and other distributions in the $[0, 1]$ domain have not been discussed. In this chapter our objective is to consider these distributions and construct the mixed distributions based on these distributions.

3.1 Special Cases of Beta Distribution

3.1.1 Case1

$\alpha > 1, \beta = 1$

Distribution

$$f(\theta) = \frac{1}{B(\alpha, \beta)} \theta^{\alpha-1} (1-\theta)^{\beta-1}$$

Becomes;

$$f(\theta) = \frac{1}{B(\alpha, 1)} \theta^{\alpha-1}, \alpha > 1, \beta = 1 \tag{3.1}$$

$$P_t = \int_0^1 \theta (1-\theta)^{t-1} \frac{1}{B(\alpha, 1)} \theta^{\alpha-1} d\theta$$

$$P_t = \frac{1}{B(\alpha,1)}\int_0^1 \theta^{\alpha}(1-\theta)^{t-1}\,d\theta = \frac{1}{B(\alpha,1)}\int_0^1 \theta^{(\alpha+1)-1}(1-\theta)^{t-1}\,d\theta$$

$$P_t = \frac{1}{B(\alpha,1)}.B(\alpha+1,t) = \frac{\Gamma(\alpha+1)}{\Gamma(1)\Gamma(\alpha)} * \frac{\Gamma(\alpha+1)\Gamma(t)}{\Gamma(t+\alpha+1)} = \frac{\alpha\Gamma(\alpha+1)\Gamma(t)}{\Gamma(t+\alpha+1)}$$

Therefore the distribution of T becomes;

$$P_t = \frac{\alpha\Gamma(\alpha+1)\Gamma(t)}{\Gamma(t+\alpha+1)}, t = 1,2,3,.... \text{ and } \alpha > 0 \tag{3.2}$$

Moments

First moment is given by;

$$M_1 = \sum_{t=1}^{\infty} tP_t$$

Using recussive relation technique,

$$P_t = \frac{\alpha\Gamma(\alpha+1)\Gamma(t)}{\Gamma(t+\alpha+1)}, P_{t-1} = \frac{\alpha\Gamma(\alpha+1)\Gamma(t-1)}{\Gamma(t+\alpha)}, P_1 = \frac{\alpha}{(\alpha+1)}$$

$$\frac{P_t}{P_{t-1}} = \frac{\alpha\Gamma(\alpha+1)\Gamma(t)\Gamma(t+\alpha)}{\Gamma(t+\alpha+1)\alpha\Gamma(\alpha+1)\Gamma(t-1)} = \frac{\Gamma(t)\Gamma(t+\alpha)}{\Gamma(t+\alpha+1)\Gamma(t-1)} = \frac{t-1}{t+\alpha}$$

$$\frac{P_t}{P_{t-1}} = \frac{t-1}{t+\alpha}, t = 2,3,4,...$$

$$(t+\alpha)P_t = (t-1)P_{t-1}$$

$$\sum_{t=2}^{\infty} t(t+\alpha)P_t = \sum_{t=2}^{\infty} t(t-1)P_{t-1}$$

$$\alpha\sum_{t=2}^{\infty} tP_t + \sum_{t=2}^{\infty} t^2P_t = \sum_{t=2}^{\infty}[(t-1)+1](t-1)P_{t-1}$$

$$\alpha(M_1 - P_1) + M_2 - P_1 = \sum_{t=2}^{\infty}\left\{(t-1)^2 + (t-1)\right\}P_{t-1}$$

$$\alpha M_1 - \alpha P_1 + M_2 - P_1 = M_2 + M_1$$

$$M_1(\alpha-1) = \alpha P_1 + P_1 = P_1(\alpha+1)$$

$$M_1(\alpha-1) = \frac{\alpha}{(\alpha+1)}(\alpha+1) = \alpha$$

$$M_1 = \frac{\alpha}{(\alpha - 1)} \tag{3.3}$$

Second moment is given by;

$$M_2 = \sum_{t=1}^{\infty} t^2 P_t$$

Using the recussive relation $(t + \alpha) P_t = (t - 1) P_{t-1}$ multiplying by t^2 and summing over t we have;

$$\sum_{t=2}^{\infty} t^2 (t + \alpha) P_t = \sum_{t=2}^{\infty} t^2 (t - 1) P_{t-1}$$

$$\sum_{t=2}^{\infty} t^3 P_t + \alpha \sum_{t=2}^{\infty} t^2 P_t = \sum_{t=2}^{\infty} \left[(t-1)^2 - 2t + 1\right] (t - 1) P_{t-1}$$

$$M_3 - P_1 + \alpha (M_2 - P_1) = \sum_{t=2}^{\infty} \left[(t-1)^2 - 2\left[(t-1) + 1\right] + 1\right] (t - 1) P_{t-1}$$

$$M_3 - P_1 + \alpha (M_2 - P_1) = \sum_{t=2}^{\infty} \left[(t-1)^2 - 2(t-1) - 2 + 1\right] (t - 1) P_{t-1}$$

$$M_3 - P_1 + \alpha (M_2 - P_1) = \sum_{t=2}^{\infty} (t-1)^3 P_{t-1} - 2 \sum_{t=2}^{\infty} (t-1)^2 P_{t-1} - \sum_{t=2}^{\infty} (t-1) P_{t-1}$$

$$M_3 - P_1 + \alpha (M_2 - P_1) = M_3 - 2M_2 - M_1$$

$$M_2 (\alpha - 2) = P_1 (\alpha - 1) - M_1 = \frac{\alpha (\alpha - 1)}{(\alpha + 1)} - \frac{\alpha}{(\alpha - 1)}$$

$$M_2 (\alpha - 2) = \frac{\alpha (\alpha - 1)(\alpha - 1) - \alpha (\alpha + 1)}{(\alpha + 1)(\alpha - 1)} = \frac{\alpha^2 (\alpha + 1)}{(\alpha - 1)(\alpha + 1)}$$

$$M_2 (\alpha - 2) = \frac{\alpha^2}{(\alpha - 1)}$$

$$M_2 = \frac{\alpha^2}{(\alpha - 1)(\alpha - 2)} \tag{3.4}$$

Therefore the distribution $P_t = \frac{\alpha\Gamma(\alpha+1)\Gamma(t)}{\Gamma(t+\alpha+1)}, t = 1, 2, 3, ...$ and $\alpha > 0$ has mean given by;

$$E(T) = M_1 = \frac{\alpha}{(\alpha - 1)}$$

and variance:

$$var(T) = M_2 - (M_1)^2 = \frac{\alpha^2}{(\alpha-1)^2(\alpha-2)}$$

3.1.2 Case2

$\alpha = \beta = \frac{1}{2}$

Distribution

$$f(\theta) = \frac{1}{B(\alpha,\beta)}\theta^{\alpha-1}(1-\theta)^{\beta-1}$$

Becomes;

$$f(\theta) = \frac{1}{B\left(\frac{1}{2},\frac{1}{2}\right)}\theta^{\frac{1}{2}-1}(1-\theta)^{\frac{1}{2}-1} \tag{3.5}$$

and

$$P_t = \int_0^1 \theta(1-\theta)^{t-1}\frac{1}{B\left(\frac{1}{2},\frac{1}{2}\right)}\theta^{\frac{1}{2}-1}(1-\theta)^{\frac{1}{2}-1}\,d\theta$$

$$P_t = \frac{1}{B\left(\frac{1}{2},\frac{1}{2}\right)}\int_0^1 \theta^{\frac{1}{2}-1+1}(1-\theta)^{\frac{1}{2}-1+t-1}\,d\theta = \frac{1}{B\left(\frac{1}{2},\frac{1}{2}\right)}\int_0^1 \theta^{\frac{1}{2}}(1-\theta)^{t-\frac{3}{2}}\,d\theta$$

$$P_t = \frac{1}{B\left(\frac{1}{2},\frac{1}{2}\right)}\int_0^1 \theta^{\frac{3}{2}-1}(1-\theta)^{\left(t-\frac{1}{2}\right)-1}\,d\theta = \frac{B\left(\frac{3}{2},t-\frac{1}{2}\right)}{B\left(\frac{1}{2},\frac{1}{2}\right)}$$

$$P_t = \frac{\Gamma(1)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)} * \frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(t-\frac{1}{2}\right)}{\Gamma\left(\frac{t-1}{2}+\frac{3}{2}\right)} = \left(\frac{\Gamma(1)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}\right) * \left(\frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(t-\frac{1}{2}\right)}{\Gamma\left(t-\frac{1}{2}+\frac{3}{2}\right)}\right)$$

Therefore the distribution of T becomes;

$$P_t = \left(\frac{1}{2\sqrt{\pi}}\right)\cdot\left\{\frac{\Gamma\left(t-\frac{1}{2}\right)}{\Gamma(t+1)}\right\} \tag{3.6}$$

Moments

First moment is given by;

$$M_1 = \sum_{t=1}^{\infty} tP_t$$

Using recussive relation technique,

$$\frac{P_t}{P_{t-1}} = \frac{\left(\frac{1}{2\sqrt{\pi}}\right)\cdot\left\{\frac{\Gamma(t-\frac{1}{2})}{\Gamma(t+1)}\right\}}{\left(\frac{1}{2\sqrt{\pi}}\right)\cdot\left\{\frac{\Gamma(t-\frac{3}{2})}{\Gamma(t)}\right\}} = \frac{\Gamma(t)}{\Gamma(t+1)} * \frac{\Gamma\left(t-\frac{1}{2}\right)}{\Gamma\left(t-\frac{3}{2}\right)}$$

$$P_1 = \left(\frac{1}{2\sqrt{\pi}}\right)\cdot\left\{\frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma(2)}\right\} = \frac{\sqrt{\pi}}{2\sqrt{\pi}} = \frac{1}{2}$$

$$\frac{P_t}{P_{t-1}} = \frac{\left(t-\frac{3}{2}\right)}{t}$$

$$tP_t = \left(t-\frac{3}{2}\right)P_{t-1}$$

Using the recussive relation $tP_t = \left(t-\frac{3}{2}\right)P_{t-1}$ multiplying by t and summing over t we have;

$$\sum_{t=2}^{\infty} t^2P_t = \sum_{t=2}^{\infty} t\left(t-\frac{3}{2}\right)P_{t-1}$$

$$M_2 - P_1 = \sum_{t=2}^{\infty}[(t-1)+1]\left[(t-1)-\frac{1}{2}\right]P_{t-1}$$

$$M_2 - P_1 = \sum_{t=2}^{\infty}(t-1)^2P_{t-1} - \frac{1}{2}\sum_{t=2}^{\infty}(t-1)P_{t-1} + \sum_{t=2}^{\infty}(t-1)P_{t-1} - \frac{1}{2}\sum_{t=2}^{\infty}P_{t-1}$$

$$M_2 - P_1 = M_2 - \frac{1}{2}M_1 + M_1 - \frac{1}{2}$$

$$-P_1 = \frac{1}{2}M_1 - \frac{1}{2}$$

$$M_1 = 0 \tag{3.7}$$

Second moment is given by;

$$M_2 = \sum_{t=1}^{\infty} t^2 P_t$$

Using the recussive relation $tP_t = \left(t - \frac{3}{2}\right) P_{t-1}$ multiplying by t^2 and summing over t we have;

$$\sum_{t=2}^{\infty} t^3 P_t = \sum_{t=2}^{\infty} t^2 \left(t - \frac{3}{2}\right) P_{t-1}$$

$$M_3 - P_1 = \sum_{t=2}^{\infty} \left[(t-1)^2 + 2t - 1\right] \left(t - \frac{3}{2}\right) P_{t-1}$$

$$M_3 - P_1 = \sum_{t=2}^{\infty} \left[(t-1)^2 + 2(t-1+1) - 1\right] \left[(t-1) - \frac{1}{2}\right] P_{t-1}$$

$$M_3 - P_1 = \sum_{t=2}^{\infty} \left[(t-1)^2 + 2(t-1) + 1\right] \left[(t-1) - \frac{1}{2}\right] P_{t-1}$$

$$\begin{aligned} M_3 - P_1 &= \sum_{t=2}^{\infty} (t-1)^3 P_{t-1} + 2\sum_{t=2}^{\infty} (t-1)^2 P_{t-1} + \sum_{t=2}^{\infty} (t-1) P_{t-1} \\ &\quad - \frac{1}{2}\sum_{t=2}^{\infty} (t-1)^2 P_{t-1} - \sum_{t=2}^{\infty} (t-1) P_{t-1} - \frac{1}{2}\sum_{t=2}^{\infty} P_{t-1} \end{aligned}$$

$$M_3 - P_1 = M_3 + 2M_2 + M_1 - \frac{1}{2}M_2 - M_1 - \frac{1}{2}$$

$$-P_1 = \frac{3}{2}M_2 - \frac{1}{2}$$

$$M_2 = 0 \tag{3.8}$$

3.1.3 Case 3

$\alpha = \frac{\mu}{\lambda}, \beta = \frac{1-\mu}{\lambda}$

Weinberg and Gladen modified this model by considering a beta distribution with parameters $\alpha = \frac{\mu}{\lambda}$ and $\beta = \frac{1-\mu}{\lambda}$ where μ is the mean parameter and λ "shape" parameter. Therefore the distributions becomes;

$$f(t \mid \theta) = \theta(1-\theta)^{t-1} \text{ (Geometric distribution) and}$$

$$f(p) = \frac{1}{B\left(\frac{\mu}{\lambda}, \frac{1-\mu}{\lambda}\right)} \theta^{\left(\frac{\mu}{\lambda}\right)-1} (1-\theta)^{\left(\frac{1-\mu}{\lambda}\right)-1} \text{ (Gamma distribution)} \tag{3.9}$$

And the unconditional distribution of T is given by;

$$f(t) = \int_0^1 \left[\theta(1-\theta)^{t-1}\right] \frac{1}{B\left(\frac{\mu}{\theta}, \frac{1-\mu}{\theta}\right)} \theta^{\left(\frac{\mu}{\theta}\right)-1} (1-\theta)^{\left(\frac{1-\mu}{\theta}\right)-1} d\theta$$

$$f(t) = \int_0^1 \left[\theta(1-\theta)^{t-1}\right] \frac{1}{B\left(\frac{\mu}{\lambda}, \frac{1-\mu}{\lambda}\right)} \theta^{\frac{\mu-\lambda}{\lambda}} (1-\theta)^{\frac{1-\mu-\lambda}{\lambda}} d\theta$$

$$f(t) = \frac{\mu\left(\frac{1-\mu}{\lambda} + t - 2\right)! \left(\frac{1}{\lambda}\right)!}{\left(\frac{1-\mu}{\lambda} - 1\right)! \left(\frac{1}{\lambda} + t - 1\right)!} \tag{3.10}$$

Proof.

$$f(t) = \int_0^1 \left[\theta(1-\theta)^{t-1}\right] \frac{1}{B\left(\frac{\mu}{\lambda}, \frac{1-\mu}{\lambda}\right)} \theta^{\frac{\mu-\lambda}{\lambda}} (1-\theta)^{\frac{1-\mu-\lambda}{\lambda}} d\theta$$

$$f(t) = \frac{1}{B\left(\frac{\mu}{\lambda}, \frac{1-\mu}{\lambda}\right)} \int_0^1 \theta^{\frac{\mu-\lambda}{\lambda}+1} (1-\theta)^{\frac{1-\mu-\lambda}{\lambda}+t-1} d\theta$$

$$f(t) = \frac{1}{B\left(\frac{\mu}{\lambda}, \frac{1-\mu}{\lambda}\right)} \int_0^1 \theta^{\frac{\mu-\lambda+\lambda}{\lambda}} (1-\theta)^{\frac{1-\mu-+t\lambda-\lambda}{\lambda}} d\theta$$

$$f(t) = = \frac{1}{B\left(\frac{\mu}{\lambda}, \frac{1-\mu}{\lambda}\right)} \int_0^1 \theta^{\frac{1}{\lambda}\mu} (1-\theta)^{\frac{1}{\lambda}(t\lambda-\mu-2\lambda+1)} d\theta$$

$$f(t) = \frac{1}{B\left(\frac{\mu}{\lambda}, \frac{1-\mu}{\lambda}\right)} B\left[\left(\frac{1}{\lambda}\mu+1\right), \left(\frac{1}{\lambda}(t\lambda-\mu-2\lambda+1)+1\right)\right]$$

$$f(t) = = \frac{1}{B\left(\frac{\mu}{\lambda}, \frac{1-\mu}{\lambda}\right)} B\left[\frac{1}{\lambda}(\mu+\lambda), \left(\frac{1}{\lambda}(t\lambda-\mu-\lambda+1)\right)\right]$$

$$f(t) = \frac{\Gamma\left(\frac{\mu}{\lambda}+\frac{1-\mu}{\lambda}\right)}{\Gamma\left(\frac{\mu}{\lambda}\right)\Gamma\left(\frac{1-\mu}{\lambda}\right)} * \frac{\Gamma\left(\frac{1}{\lambda}(\mu+\lambda)\right)\Gamma\left(\frac{1}{\lambda}(t\lambda-\mu-\lambda+1)\right)}{\Gamma\left(\frac{1}{\lambda}(\mu+\lambda)+\frac{1}{\lambda}(t\lambda-\mu-\lambda+1)\right)}$$

$$f(t) = \frac{\Gamma\left(\frac{1}{\lambda}\right)}{\Gamma\left(\frac{\mu}{\lambda}\right)\Gamma\left(\frac{1-\mu}{\lambda}\right)} * \frac{\Gamma\left(\frac{\mu}{\lambda}+1\right)\Gamma\left(\frac{1-\mu}{\lambda}+t-1\right)}{\Gamma\left(\frac{1}{\lambda}+t\right)}$$

$$f(t) = \frac{\left(\frac{1}{\lambda}-1\right)!\left(\frac{\mu}{\lambda}\right)!\left(\frac{1-\mu}{\lambda}+t-2\right)!}{\left(\frac{\mu}{\lambda}-1\right)!\left(\frac{1-\mu}{\lambda}-1\right)!\Gamma\left(\frac{1}{\lambda}+t-1\right)!}$$

$$f(t) = = \frac{\mu\left(\frac{1}{\lambda}\right)\left(\frac{1}{\lambda}-1\right)!\left(\frac{1-\mu}{\lambda}+t-2\right)!}{\left(\frac{1-\mu}{\lambda}-1\right)!\Gamma\left(\frac{1}{\lambda}+t-1\right)!} = \frac{\mu\left(\frac{1}{\lambda}\right)!\left(\frac{1-\mu}{\lambda}+t-2\right)!}{\left(\frac{1-\mu}{\lambda}-1\right)!\Gamma\left(\frac{1}{\lambda}+t-1\right)!}$$

Therefore

$$f(t) = \frac{\mu\left(\frac{1}{\lambda}\right)!\left(\frac{1-\mu}{\lambda}+t-2\right)!}{\left(\frac{1-\mu}{\lambda}-1\right)!\Gamma\left(\frac{1}{\lambda}+t-1\right)!}, t = 1, 2, 3, ...$$

■

3.1.4 Generalized Beta

Distribution

$$f(p) = \frac{1}{B(\alpha,\beta)} p^{\alpha-1}(1-p)^{\beta-1}$$

For $p = \theta^c$ then the distribution of θ is given by;

$$f(\theta) = f(p)\left|\frac{dp}{d\theta}\right| = \frac{1}{B(\alpha,\beta)}\theta^{c(\alpha-1)}(1-\theta^c)^{\beta-1}\cdot\left|c\theta^{c-1}\right|$$

$$f(\theta) = \frac{1}{B(\alpha,\beta)} c\theta^{c\alpha-c+c-1}(1-\theta^c)^{\beta-1}$$

$$f(\theta) = \frac{1}{B(\alpha,\beta)} c\theta^{c\alpha-1}(1-\theta^c)^{\beta-1} \tag{3.11}$$

Which is the generalized beta distribution. To get the distribution of T given by;

$$f(t) = \int_0^1 f(t|\theta) f(\theta)\, d\theta$$

$$f(t) = \int_0^1 f(t|\theta) f(\theta)\, d\theta = \int_0^1 \theta(1-\theta)^{t-1}\frac{1}{B(\alpha,\beta)}c\theta^{c\alpha-1}(1-\theta^c)^{\beta-1}\, d\theta$$

$$f(t) = \frac{1}{B(\alpha,\beta)}c\int_0^1 \theta^{c\alpha}(1-\theta)^{t-1}(1-\theta^c)^{\beta-1}\, d\theta$$

$$\text{Rewriting} = (1-\theta)^{t-1} = \sum_{x=0}^{t-1}\binom{t-1}{x}(-\theta)^x = \sum_{x=0}^{t-1}\binom{t-1}{x}(-1)^x\theta^x$$

$$(1-\theta)^{t-1} = \sum_{x=0}^{t-1}\binom{t-1}{x}(-1)^x\theta^x$$

$$f(t) = \frac{c}{B(\alpha,\beta)}\int_0^1 \theta^{c\alpha}(1-\theta^c)^{\beta-1}\sum_{x=0}^{t-1}\binom{t-1}{x}(-1)^x\theta^x d\theta$$

$$f(t) = \frac{c}{B(\alpha,\beta)}\sum_{x=0}^{t-1}\binom{t-1}{x}(-1)^x\int_0^1 \theta^x\theta^{c\alpha}(1-\theta^c)^{\beta-1}\, d\theta$$

$$f(t) = \frac{c}{B(\alpha,\beta)}\sum_{x=0}^{t-1}\binom{t-1}{x}(-1)^x\int_0^1 \theta^{c\alpha+x}(1-\theta^c)^{\beta-1}\, d\theta$$

$$f(t)=\frac{c}{B(\alpha,\beta)}\sum_{x=0}^{t-1}\binom{t-1}{x}(-1)^x\int_0^1\theta^{c\left(\alpha+\frac{x}{c}\right)}(1-\theta^c)^{\beta-1}d\theta$$

$$f(t)=\frac{c}{B(\alpha,\beta)}\sum_{x=0}^{t-1}\binom{t-1}{x}(-1)^x B\left(\alpha+\frac{x}{c}+1,\beta\right)$$

$$f(t)=\sum_{x=0}^{t-1}\binom{t-1}{x}(-1)^x\frac{cB\left(\alpha+\frac{x}{c}+1,\beta\right)}{B(\alpha,\beta)}$$

$$f(t)=c\sum_{x=0}^{t-1}\binom{t-1}{x}(-1)^x\frac{B\left(\alpha+\frac{x}{c}+1,\beta\right)}{B(\alpha,\beta)},t=1,2,3,\ldots.\alpha,\beta\succ 0 \quad (3.12)$$

Mean

To get the mean of the distribution of t from the relationship;

$$E(T)=E\{E(T|\theta)\}$$

$$E(T|\theta)=\sum_{t=1}^{\infty}t\theta(1-\theta)^{t-1}=\frac{1}{\theta}$$

$$E(T)=E\left(\frac{1}{\theta}\right)=\int_0^1\frac{1}{\theta}\frac{1}{B(\alpha,\beta)}c\theta^{c\alpha-1}(1-\theta^c)^{\beta-1}d\theta$$

$$E(T)=\frac{c}{B(\alpha,\beta)}\int_0^1\theta^{c\alpha-2}(1-\theta^c)^{\beta-1}d\theta$$

$$let\ u=\theta^c \iff du=c\theta^{c-1}d\theta,\theta=u^{\frac{1}{c}}$$

$$E(T)=\frac{c}{B(\alpha,\beta)}\int_0^1\theta^{c\alpha-2}(1-u)^{\beta-1}\frac{du}{c\theta^{c-1}}$$

$$E(T)=\frac{1}{B(\alpha,\beta)}\int_0^1\theta^{c\alpha-2}(1-u)^{\beta-1}\frac{du}{\theta^{c-1}}=\frac{1}{B(\alpha,\beta)}\int_0^1\theta^{c\alpha-1-c}(1-u)^{\beta-1}du$$

$$E(T)=\frac{1}{B(\alpha,\beta)}\int_0^1u^{\frac{1}{c}(c\alpha-1-c)}(1-u)^{\beta-1}du$$

$$E(T)=\frac{1}{B(\alpha,\beta)}\int_0^1u^{\alpha-\frac{1}{c}-1}(1-u)^{\beta-1}du=\frac{1}{B(\alpha,\beta)}B\left(\alpha-\frac{1}{c},\beta\right)$$

$$E(T) = \frac{1}{B(\alpha,\beta)} B\left(\alpha - \frac{1}{c}, \beta\right)$$

$$E(T) = \frac{B\left(\alpha - \frac{1}{c}, \beta\right)}{B(\alpha,\beta)} \tag{3.13}$$

Variance

To get the variance of the distribution of t from the relationship;

$$var(T) = var\{E(T|\theta)\} + E\{var(T|\theta)\}$$

$$var(T) = var\left\{\frac{1}{\theta}\right\} + E\left\{\frac{1-\theta}{\theta^2}\right\}$$

$$var(T) = \left\{E\left(\frac{1}{\theta^2}\right) - \left[E\left(\frac{1}{\theta}\right)\right]^2\right\} + E\left\{\frac{1-\theta}{\theta^2}\right\}$$

$$\begin{aligned} var(T) &= \left\{\int_0^1 \frac{1}{\theta^2}\frac{1}{B(\alpha,\beta)} c\theta^{c\alpha-1}(1-\theta^c)^{\beta-1} d\theta - \left[\int_0^1 \frac{1}{\theta}\frac{1}{B(\alpha,\beta)} c\theta^{c\alpha-1}(1-\theta^c)^{\beta-1} d\theta\right]^2\right\} \\ &+ \int_0^1 \frac{1-\theta}{\theta^2}\frac{1}{B(\alpha,\beta)} c\theta^{c\alpha-1}(1-\theta^c)^{\beta-1} d\theta \end{aligned}$$

$$\begin{aligned} var(T) &= \left\{\int_0^1 \frac{1}{B(\alpha,\beta)} c\theta^{c\alpha-3}(1-\theta^c)^{\beta-1} d\theta - \left[\frac{B\left(\frac{c\alpha-1}{c},\beta\right)}{B(\alpha,\beta)}\right]^2\right\} \\ &+ \int_0^1 \frac{1}{B(\alpha,\beta)} c\theta^{c\alpha-3}(1-\theta^c)^{\beta-1}(1-\theta) d\theta \end{aligned}$$

$$let\ u = \theta^c \iff du = c\theta^{c-1} d\theta, \theta = u^{\frac{1}{c}}$$

$$\begin{aligned} \int_0^1 \frac{1}{B(\alpha,\beta)} c\theta^{c\alpha-3}(1-\theta^c)^{\beta-1} d\theta &= \frac{c}{B(\alpha,\beta)}\int_0^1 \theta^{c\alpha-3}(1-u)^{\beta-1}\frac{du}{c\theta^{c-1}} \\ &= \frac{1}{B(\alpha,\beta)}\int_0^1 u^{\frac{1}{c}(c\alpha-2-c)}(1-u)^{\beta-1} du \\ &= \frac{1}{B(\alpha,\beta)}\int_0^1 u^{\left(\alpha-\frac{2}{c}\right)-1}(1-u)^{\beta-1} du = \frac{B\left(\alpha-\frac{2}{c},\beta\right)}{B(\alpha,\beta)} \end{aligned}$$

$$var(T) = \frac{B\left(\alpha - \frac{2}{c}, \beta\right)}{B(\alpha, \beta)} - \left[\frac{B\left(\alpha - \frac{1}{c}, \beta\right)}{B(\alpha, \beta)}\right]^2 + \frac{B\left(\alpha - \frac{2}{c}, \beta\right)}{B(\alpha, \beta)} - \frac{B\left(\alpha - \frac{1}{c}, \beta\right)}{B(\alpha, \beta)} \tag{3.14}$$

3.2 Triangular Distribution

3.2.1 Distribution

For Triangular distribution with lower limit $a = 0$, upper limit $b = 1$, and the mode $c = \frac{1}{2}$ then;

$$f(\theta) = \begin{cases} 4\theta, & 0 < \theta < \frac{1}{2} \\ 4(1-\theta), & \frac{1}{2} < \theta < 1 \end{cases} \tag{3.15}$$

and the unconditional distribution of T is given by;

$$f(t) = \int_0^{\frac{1}{2}} f(t|\theta) f(\theta)\, d\theta + \int_{\frac{1}{2}}^1 f(t|\theta) f(\theta)\, d\theta$$

This becomes;

$$f(t) = \int_0^{\frac{1}{2}} \theta (1-\theta)^{t-1} 4\theta d\theta + \int_{\frac{1}{2}}^1 \theta (1-\theta)^{t-1} 4(1-\theta)\, d\theta$$

$$f(t) = 4\int_0^{\frac{1}{2}} \theta^2 (1-\theta)^{t-1} d\theta + 4\int_{\frac{1}{2}}^1 \theta (1-\theta)^{t-1} (1-\theta)\, d\theta$$

$$f(t) = 4\left\{\int_0^{\frac{1}{2}} \theta^2 (1-\theta)^{t-1} d\theta + \int_{\frac{1}{2}}^1 \theta (1-\theta)^{t} d\theta\right\}$$

$$f(t) = 4\left\{\int_{\frac{1}{2}}^1 \theta (1-\theta)^{t} d\theta + \int_0^{\frac{1}{2}} \theta^2 (1-\theta)^{t-1} d\theta\right\}$$

$$\textit{letting } u = (1-\theta) \Leftrightarrow \theta = 1-u,\ d\theta = -du$$

$$f(t) = 4\left\{\int_{\frac{1}{2}}^1 (1-u)\, u^t (-du) + \int_0^{\frac{1}{2}} (1-u)^2 u^{t-1} (-du)\right\}$$

$$f(t)=4\left\{\int_{\frac{1}{2}}^{1} u^{t}-u^{t+1}(-du)+\int_{0}^{\frac{1}{2}}\left(1-2u+u^{2}\right)u^{t-1}(-du)\right\}$$

$$f(t)=4\left\{\left[\int_{c}^{1} u^{t+1}-u^{t}du\right]+\left[\int_{0}^{\frac{1}{2}} 2u^{t}-u^{t-1}-u^{t+1}du\right]\right\}$$

$$4\left\{\left[\frac{u^{t+2}}{t+2}-\frac{u^{t+1}}{t+1}\right]_{\frac{1}{2}}^{1}+\left[\frac{2u^{t+1}}{t+1}-\frac{u^{t}}{t}-\frac{u^{t+2}}{t+2}\right]_{0}^{\frac{1}{2}}\right\}$$

$$f(t)=4\left\{\left[\frac{(1-\theta)^{t+2}}{t+2}-\frac{(1-\theta)^{t+1}}{t+1}\right]_{\frac{1}{2}}^{1}+\left[\frac{2(1-\theta)^{t+1}}{t+1}-\frac{(1-\theta)^{t}}{t}-\frac{(1-\theta)^{t+2}}{t+2}\right]_{0}^{\frac{1}{2}}\right\}$$

$$f(t)=4\left\{\begin{array}{c}\left[\frac{(t+1)(1-\theta)^{t+2}-(t+2)(1-\theta)^{t+1}}{(t+2)(t+1)}\right]_{\frac{1}{2}}^{1}+\\ \left[\frac{t(t+2)2(1-\theta)^{t+1}-(t+1)(t+2)(1-\theta)^{t}-t(t+1)(1-\theta)^{t+2}}{t(t+1)(t+2)}\right]_{0}^{\frac{1}{2}}\end{array}\right\}$$

$$f(t)=4\left\{\begin{array}{c}\left[\frac{(1-\theta)^{t+1}}{(t+2)(t+1)}\left((t+1)(1-\theta)-(t+2)\right)\right]_{\frac{1}{2}}^{1}\\ +\left[\frac{t(t+2)2(1-\theta)^{t+1}-(t+1)(t+2)(1-\theta)^{t}-t(t+1)(1-\theta)^{t+2}}{t(t+1)(t+2)}\right]_{0}^{\frac{1}{2}}\end{array}\right\}$$

$$f(t)=4\left\{\begin{array}{c}\left[\frac{(1-\theta)^{t+1}}{(t+2)(t+1)}(-t\theta-\theta-1)\right]_{\frac{1}{2}}^{1}\\ +\left[\frac{(1-\theta)^{t}}{t(t+1)(t+2)}\left(2t(t+2)(1-\theta)-(t+1)(t+2)-t(t+1)(1-\theta)^{2}\right)\right]_{0}^{\frac{1}{2}}\end{array}\right\}$$

$$f(t)=4\left\{\left[\frac{(1-\theta)^{t+1}}{(t+2)(t+1)}(-t\theta-\theta-1)\right]_{\frac{1}{2}}^{1}+\left[\frac{(1-\theta)^{t}}{t(t+1)(t+2)}\left(-2t\theta-2-t^{2}\theta^{2}-t\theta^{2}\right)\right]_{0}^{\frac{1}{2}}\right\}$$

$$f(t)=4\left\{0-\frac{\left(\frac{1}{2}\right)^{t+1}}{(t+2)(t+1)}\left(-\frac{t}{2}-\frac{1}{2}-1\right)+\left[\frac{(1-\theta)^{t}}{t(t+1)(t+2)}\left(-2t\theta-2-t^{2}\theta^{2}-t\theta^{2}\right)\right]_{0}^{\frac{1}{2}}\right\}$$

$$f(t)=4\left\{0-\frac{\left(\frac{1}{2}\right)^{t+1}}{(t+2)(t+1)}\left(-\frac{t}{2}-\frac{1}{2}-1\right)+\frac{\left(\frac{1}{2}\right)^{t}}{t(t+1)(t+2)}\left(-2t\frac{1}{2}-2-t^{2}\left(\frac{1}{2}\right)^{2}-t\left(\frac{1}{2}\right)^{2}\right)\right.$$

$$f(t)=4\left\{\frac{2}{t(t+1)(t+2)}-\frac{\left(\frac{1}{2}\right)^{t+1}}{(t+2)(t+1)}\left(-\frac{t}{2}-\frac{1}{2}-1\right)-\frac{\left(\frac{1}{2}\right)^{t}}{t(t+1)(t+2)}\left(-2t\frac{1}{2}-2-t^{2}\left(\frac{1}{2}\right)\right.\right.$$

$$f(t)=\frac{4}{(t+1)(t+2)}\left\{\frac{2}{t}+\left(\frac{1}{2}\right)^{t+1}\left(\frac{t}{2}+\frac{3}{2}\right)-\left(\frac{1}{2}\right)^{t}\left(1+\frac{2}{t}+\frac{t}{4}+\frac{1}{4}\right)\right\}$$

$$f(t)=\frac{4}{(t+1)(t+2)}\left\{\frac{2}{t}-\left(\frac{1}{2}\right)^{t}\right\} \tag{3.16}$$

Mean

To get the mean of the distribution of t from the relationship;

$$E(T)=E\{E(T|\theta)\}$$

$$E(T|\theta)=\sum_{t=1}^{\infty}t\theta(1-\theta)^{t-1}=\frac{1}{\theta}$$

$$E(T)=4\int_{0}^{c}1d\theta+4\int_{c}^{1}\frac{1}{\theta}(1-\theta)\,d\theta$$

$$E(T)=4\left\{[\theta]_{0}^{c}+[\ln\theta-\theta]_{c}^{1}\right\}$$

$$E(T)=4\{c+[\ln 1-1-\ln c+c]\}$$

$$E(T)=4\{2c-\ln c-1\}$$

Thus the mean waiting time is given by,

$$E(T)=\frac{2}{c}\{2c-\ln c-1\} \tag{3.17}$$

Variance

To get the variance of the distribution of t from the relationship;

$$var(T)=var\{E(T|\theta)\}+E\{var(T|\theta)\}$$

$$var(T)=var\left\{\frac{1}{\theta}\right\}+E\left\{\frac{1-\theta}{\theta^{2}}\right\}$$

$$var(T)=\left\{E\left(\frac{1}{\theta^{2}}\right)-\left[E\left(\frac{1}{\theta}\right)\right]^{2}\right\}+E\left\{\frac{1-\theta}{\theta^{2}}\right\}$$

$$\begin{aligned} E\left(\frac{1}{\theta^2}\right) &= \int_0^c \frac{1}{\theta^2}\left(\frac{2\theta}{c}\right) d\theta + \int_c^1 \frac{1}{\theta^2}\left(\frac{2-2\theta}{c}\right) d\theta = \frac{2}{c}\left\{\int_0^c \frac{1}{\theta} d\theta + \int_c^1 \left(\frac{1}{\theta^2} - \frac{1}{\theta}\right) d\theta\right\} \quad (3.18) \\ &= \frac{2}{c}\left\{[\ln\theta]_0^c + \left[-\frac{1}{\theta}\right]_c^1 - [\ln\theta]_c^1\right\} = \frac{2}{c}\left\{\ln c - \ln 0 - 1 + \frac{1}{c} - 1 + \ln c\right\} \\ &= \frac{2}{c}\left\{2\ln c - \ln + \frac{1}{c} - 2\right\} \end{aligned}$$

3.3 Kumaraswamy Distribution

The distribution of θ is as follows;

$$f(\theta) = \alpha\beta\theta^{\alpha-1}(1-\theta^{\alpha})^{\beta-1} \qquad (3.19)$$

and the conditional distribution of T given θ is given by;

$$f(t|\theta) = \theta(1-\theta)^{t-1}$$

Therefore the unconditional distribution of T is given by;

$$f(t) = \int_0^1 f(t|\theta) f(\theta)\, d\theta = f(t) = \int_0^1 \theta(1-\theta)^{t-1} \alpha\beta\theta^{\alpha-1}(1-\theta^{\alpha})^{\beta-1} d\theta$$

$$f(t) = \alpha\beta \int_0^1 \theta^{\alpha}(1-\theta)^{t-1}(1-\theta^{\alpha})^{\beta-1} d\theta$$

Rewriting $= (1-\theta)^{t-1} = \sum_{x=0}^{t-1}\binom{t-1}{x}(-\theta)^x = \sum_{x=0}^{t-1}\binom{t-1}{x}(-1)^x \theta^x$

$$f(t) = \alpha\beta \int_0^1 \theta^{\alpha}(1-\theta^{\alpha})^{\beta-1} \sum_{x=0}^{t-1}\binom{t-1}{x}(-1)^x \theta^x d\theta$$

$$f(t) = \alpha\beta \sum_{x=0}^{t-1}\binom{t-1}{x}(-1)^x \int_0^1 \theta^{\alpha}(1-\theta^{\alpha})^{\beta-1} \theta^x d\theta$$

$$f(t) = \alpha\beta \sum_{x=0}^{t-1}\binom{t-1}{x}(-1)^x \int_0^1 \theta^{\alpha+x}(1-\theta^{\alpha})^{\beta-1} d\theta$$

$$f(t) = \alpha\beta \sum_{x=0}^{t-1} \binom{t-1}{x} (-1)^x \int_0^1 \theta^{\alpha\left(1+\frac{x}{\alpha}\right)} (1-\theta^\alpha)^{\beta-1} d\theta$$

$$f(t) = \alpha\beta \sum_{x=0}^{t-1} \binom{t-1}{x} (-1)^x B\left(\frac{x}{\alpha}+2, t\right) \tag{3.20}$$

For $\alpha, \beta \succ 0$ and $t = 1, 2, 3, \ldots..$

Mean

To get the mean of the distribution of t from the relationship;

$$E(T) = E\{E(T|\theta)\}$$

$$E(T|\theta) = \sum_{t=1}^{\infty} t\theta(1-\theta)^{t-1} = \frac{1}{\theta}$$

$$E(T) = E\left(\frac{1}{\theta}\right) = \int_0^1 \frac{1}{\theta}\alpha\beta\theta^{\alpha-1}(1-\theta^\alpha)^{\beta-1} d\theta$$

$$E(T) = \int_0^1 \alpha\beta\theta^{\alpha-2}(1-\theta^\alpha)^{\beta-1} d\theta$$

$$E(T) = \alpha\beta \int_0^1 \theta^{\alpha-2}(1-\theta^\alpha)^{\beta-1} d\theta$$

$$let\ u = \theta^\alpha \Leftrightarrow \frac{du}{d\theta} = \alpha\theta^{\alpha-1},\ \theta = u^{\frac{1}{\alpha}}$$

$$E(T) = \alpha\beta \int_0^1 \theta^{\alpha-2}(1-u)^{\beta-1} \frac{du}{\alpha\theta^{\alpha-1}}$$

$$E(T) = \beta \int_0^1 \frac{1}{\theta}(1-u)^{\beta-1} du = \beta \int_0^1 u^{-\frac{1}{\alpha}}(1-u)^{\beta-1} du$$

$$E(T) = \beta \int_0^1 u^{\left(1-\frac{1}{\alpha}\right)-1}(1-u)^{\beta-1} du$$

$$E(T) = \beta\left\{B\left(1-\frac{1}{\alpha}, \beta\right)\right\} = \beta\left\{B\left(\frac{\alpha-1}{\alpha}, \beta\right)\right\}$$

Therefore the mean waiting time to conception is given by;

$$E(T) = \beta\left\{B\left(1-\frac{1}{\alpha}, \beta\right)\right\} \tag{3.21}$$

Variance

To get the variance of the distribution of t from the relationship;

$$var(T) = var\{E(T|\theta)\} + E\{var(T|\theta)\}$$

$$var(T) = var\left\{\frac{1}{\theta}\right\} + E\left\{\frac{1-\theta}{\theta^2}\right\}$$

$$var(T) = \left\{E\left(\frac{1}{\theta^2}\right) - \left[E\left(\frac{1}{\theta}\right)\right]^2\right\} + E\left\{\frac{1-\theta}{\theta^2}\right\}$$

$$\begin{aligned} var(T) &= \int_0^1 \frac{1}{\theta^2}\alpha\beta\theta^{\alpha-1}(1-\theta^\alpha)^{\beta-1}d\theta - \left[\beta\left\{B\left(1-\frac{1}{\alpha},\beta\right)\right\}\right]^2 \\ &+ \int_0^1 \frac{1-\theta}{\theta^2}\alpha\beta\theta^{\alpha-1}(1-\theta^\alpha)^{\beta-1}d\theta \end{aligned}$$

$$\begin{aligned} var(T) &= \int_0^1 \alpha\beta\theta^{\alpha-3}(1-\theta^\alpha)^{\beta-1}d\theta - \left[\beta\left\{B\left(1-\frac{1}{\alpha},\beta\right)\right\}\right]^2 \\ &+ \int_0^1 \alpha\beta\theta^{\alpha-3}(1-\theta^\alpha)^{\beta-1}(1-\theta)\,d\theta \end{aligned}$$

$$letting\ u = \theta^\alpha \Leftrightarrow du = \alpha\theta^{\alpha-1}d\theta \Leftrightarrow d\theta = \frac{du}{\alpha\theta^{\alpha-1}}$$

$$\begin{aligned} \alpha\beta\int_0^1 \theta^{\alpha-3}(1-u)^{\beta-1}\frac{du}{\alpha\theta^{\alpha-1}} &= \beta\int_0^1 \theta^{\alpha-3-\alpha+1}(1-u)^{\beta-1}du \\ &= \beta\int_0^1 u^{-\frac{2}{\alpha}}(1-u)^{\beta-1}du \\ &= \beta\left\{B\left(1-\frac{2}{\alpha},\beta\right)\right\} \end{aligned}$$

$$\int_0^1 \alpha\beta\theta^{\alpha-3}(1-\theta^\alpha)^{\beta-1}(1-\theta)\,d\theta = \int_0^1 \alpha\beta\theta^{\alpha-3}(1-\theta^\alpha)^{\beta-1}d\theta - \int_0^1 \alpha\beta\theta^{\alpha-2}(1-\theta^\alpha)^{\beta-1}d\theta$$

$$\beta\left\{B\left(1-\frac{2}{\alpha},\beta\right)\right\} - \beta\left\{B\left(1-\frac{1}{\alpha},\beta\right)\right\}$$

$$var(T)=\beta\left\{B\left(1-\frac{2}{\alpha},\beta\right)\right\}-\left[\beta\left\{B\left(1-\frac{1}{\alpha},\beta\right)\right\}\right]^{2}+\beta\left\{B\left(1-\frac{2}{\alpha},\beta\right)\right\}-\beta\left\{B\left(1-\frac{1}{\alpha},\beta\right)\right\}$$

$$var(T)=2\beta\left\{B\left(1-\frac{2}{\alpha},\beta\right)\right\}-\left[\beta\left\{B\left(1-\frac{1}{\alpha},\beta\right)\right\}\right]^{2}-\beta\left\{B\left(1-\frac{1}{\alpha},\beta\right)\right\} \tag{3.22}$$

3.4 Van Dorp and Kotz's(2004b)

The distribution of θ is given by

$$\begin{aligned} f(\theta) &= \alpha-2(\alpha-1)\theta, \\ 0 &\leq \alpha\leq 2,\ 0\prec\theta\prec 1 \end{aligned} \tag{3.23}$$

and

$$f(t|\theta)=\theta(1-\theta)^{t-1}$$

The unconditional distribution of T is given by;

$$f(t)=\int_0^1 f(t|\theta)\,f(\theta)\,d\theta$$

$$f(t)=\int_0^1 \theta(1-\theta)^{t-1}\{\alpha-2(\alpha-1)\theta\}\,d\theta$$

$$f(t)=\alpha\int_0^1 \theta(1-\theta)^{t-1}-2(\alpha-1)\int_0^1 \theta^2(1-\theta)^{t-1}\,d\theta$$

$$f(t)=\alpha B(2,t)-2(\alpha-1)B(3,t)$$

$$f(t)=\alpha\frac{\Gamma(2)\Gamma(t)}{\Gamma(t+2)}-2(\alpha-1)\frac{\Gamma(3)\Gamma(t)}{\Gamma(t+3)}$$

$$f(t)=\frac{\alpha(t-1)!}{(t+1)!}-\frac{2(\alpha-1)2(t-1)!}{(t+2)!}$$

$$f(t)=\frac{\alpha}{t(t+1)}-\frac{4(\alpha-1)}{t(t+1)(t+2)}=\frac{\alpha(t+2)-4(\alpha-1)}{t(t+1)(t+2)}$$

$$\begin{aligned} f(t) &= \frac{\alpha(t-2)+4}{t(t+1)(t+2)} \\ 0 &\leq \alpha\leq 2, t=1,2,3,... \end{aligned} \tag{3.24}$$

Mean
To get the mean of the distribution of t from the relationship;

$$E(T) = E\{E(T|\theta)\}$$

$$E(T|\theta) = \sum_{t=1}^{\infty} t\theta(1-\theta)^{t-1} = \frac{1}{\theta}$$

$$E(T) = E\left(\frac{1}{\theta}\right) = \int_0^1 \frac{1}{\theta}\{\alpha - 2(\alpha-1)\theta\}\,d\theta$$

$$\int_0^1 \left\{\frac{\alpha}{\theta} - 2(\alpha-1)\right\} d\theta$$

$$E(T) = \int_0^1 \left\{\frac{\alpha}{\theta} - 2(\alpha-1)\right\} d\theta$$

$$E(T) = [\alpha \ln\theta - 2(\alpha-1)\theta]_0^1$$

3.5 Truncated Exponential

Let the distribution of y be given by;

$$f(y) = \int_0^k \alpha e^{-\alpha y} dy = \alpha \int_0^k e^{-\alpha y} dy$$

$$\alpha\left[\frac{-e^{-\alpha y}}{\alpha}\right]_0^k = \left[-e^{-\alpha y}\right]_0^k$$

$$f(y) = \frac{\alpha e^{-\alpha y}}{1-e^{-\alpha k}},\ 0 < y < k$$

letting $y = k\theta$, then;

$$f(\theta) = f(y)\left|\frac{dy}{dp}\right| = \frac{\alpha e^{-\alpha k\theta}}{1-e^{-\alpha k}}(k)$$

$$f(\theta) = \frac{k\alpha e^{-\alpha k\theta}}{1-e^{-\alpha k}},\ 0 < \theta < 1 \tag{3.25}$$

Now using this distribution and mixing it with geometric we have;

$$f(t) = \int_0^1 \theta(1-\theta)^{i-1}\frac{k\alpha e^{-\alpha k\theta}}{1-e^{-\alpha k}}d\theta$$

$$f(t) = k\alpha \int_0^1 \theta (1-\theta)^{t-1} \frac{e^{-\alpha k\theta}}{1-e^{-\alpha k}} d\theta$$

$$f(t) = \frac{k\alpha}{1-e^{-\alpha k}} \int_0^1 \theta (1-\theta)^{t-1} e^{-\alpha k\theta} d\theta, \text{ but } e^{-\alpha k\theta} = \sum_{z=0}^{\infty} (-1)^z \frac{(\alpha k\theta)^z}{z!}$$

$$f(t) = \frac{k\alpha}{1-e^{-\alpha k}} \int_0^1 \theta (1-\theta)^{t-1} \sum_{z=0}^{\infty} (-1)^z \frac{(\alpha k\theta)^z}{z!} d\theta$$

$$f(t) = \frac{k\alpha}{1-e^{-\alpha k}} \sum_{z=0}^{\infty} (-1)^z \frac{(\alpha k)^z}{z!} \int_0^1 \theta^{z+1} (1-\theta)^{t-1} d\theta$$

$$f(t) = \frac{k\alpha}{1-e^{-\alpha k}} \sum_{z=0}^{\infty} (-1)^z \frac{(\alpha k)^z}{z!} B(z+2,t)$$

$$f(t) = \sum_{z=0}^{\infty} (-1)^z \frac{(\alpha k)^{z+1}}{(1-e^{-\alpha k})\, z!} B(z+2,t)$$

$$f(t) = \sum_{z=0}^{\infty} (-1)^z \frac{(\alpha k)^{z+1}}{z!\,(1-e^{-\alpha k})} B(z+2,t) \qquad (3.26)$$

Mean
To get the mean of the distribution of t from the relationship;

$$E(T) = E\{E(T|\theta)\}$$

$$E(T|\theta) = \sum_{t=1}^{\infty} t\theta (1-\theta)^{t-1} = \frac{1}{\theta}$$

$$E(T) = E\left(\frac{1}{\theta}\right) = \int_0^1 \frac{1}{\theta} \frac{k\alpha e^{-\alpha k\theta}}{1-e^{-\alpha k}} d\theta$$

$$E(T) = \frac{k\alpha}{1-e^{-\alpha k}} \int_0^1 \frac{1}{\theta} e^{-\alpha k\theta} d\theta$$

$$E(T) = \frac{k\alpha}{1-e^{-\alpha k}} \int_0^1 \frac{1}{\theta} \sum_{z=1}^{\infty} (-1)^{z-1} \frac{(\alpha k\theta)^{z-1}}{(z-1)!} d\theta$$

$$E(T) = \frac{k\alpha}{1-e^{-\alpha k}} \sum_{z=1}^{\infty} \frac{(-1)^{z-1}}{(z-1)!} (\alpha k)^{z-1} \int_0^1 \theta^{z-2} d\theta$$

$$E(T) = \frac{k\alpha}{1-e^{-\alpha k}} \sum_{z=1}^{\infty} \frac{(-1)^{z-1}}{(z-1)!} (\alpha k)^{z} \left[\frac{\theta^{z}}{z}\right]_{0}^{1}$$

$$E(T) = \frac{k\alpha}{1-e^{-\alpha k}} \sum_{z=0}^{\infty} \frac{(-1)^{z}}{z!z} (\alpha k)^{z} \tag{3.27}$$

Chapter 4

FERTILITY MODELS BASED ON COMPOUND DISTRIBUTIONS

4.1 Introduction: Sums of Independent Random Variables

Let $S_N = X_1 + X_2 + ... + X_N$ where $X_i's$ for $i = 1, 2, ...N$ are independent and identically distributed random variables.

a When N is fixed then the distribution of S_N is called a Convolution of $X_i's$

b When N is also a random variable independent of $X_i's$, then the distribution is called a Compound distribution of N.

4.1.1 Convolution

Consider three sequences of real numbers: $\{a_k\}$, $\{b_k\}$ and $\{c_k\}$ we say that $\{c_k\}$ is a convolution of $\{a_k\}$ and$\{b_k\}$ denoted by $\{c_k\} = \{a_k\} * \{b_k\}$ if

$$c_k = \sum_{r=0}^{k} a_r b_{k-r}$$

and

$$C(s) = A(s) * B(s)$$

Where :$A(s), B(s)$ and $C(s)$ are generating functions of $\{a_k\}$, $\{b_k\}$ and $\{c_k\}$ respectively. More generally; If $S_N = X_1 + X_2 + ... + X_N$ and $X_i's$ are independent and identically distributed random variables, then;

$$\{prob(S_N = i)\} = \{P(X_1)\} * \{P(X_2)\} * ...... * \{P(X_N)\}$$

and since $X_i's$ are identical;

$$\{prob(S_N = i)\} = \{P(X_i)\} * \{P(X_i)\} * \ldots\ldots * \{P(X_i)\} = \{P(X_i)\}^{N*}$$

This is referred to as an N^{th}-fold convolution of $\{P(X_i)\}$.Using the generating functions

$$H(s) = G_{(X_1)}(s)G_{(X_2)}(s)\ldots\ldots..G_{(X_1)}(s) = [G_i(s)]^N \qquad (4.1)$$

Where $G_i(s)$ is the probability generating functions of each of $X_i's$. For $S_N = X_1 + \{X_2 + X_3 \ldots.. + X_N\}$, the distribution of S_N is the sum of $(n-1)$ independent and identically distributed exponential random variables with common parameter λ_2 convolved with that of an exponential random variable X_1 with parameter λ_1 i.e.

$$f^n(S_N) = \int_0^t f_1(t-x)\, f_{n-1}(\lambda_2\ ,t)$$

where;

$$f_1(x) = \lambda_1 e^{-\lambda_1 x} \text{ and } F_1(t) = \int_0^t \lambda_1 e^{-\lambda_1 x} = 1 - e^{-\lambda_1 t}$$

and

$$F_1(t-x) = 1 - e^{-\lambda_1(t-x)}$$

For $N \succeq 2$, the distribution of the sum $X_2 + X_3 \ldots.. + X_N$ of Independent and identically distributed exponential random variables is a gamma function given by;

$$f_{n-1}(\lambda_2\ ,t) = \frac{\lambda_2^{n-1}\ x^{n-2}e^{-\lambda_2 x}}{\Gamma(n-1)}, \text{ and } F_{n-1}(\lambda_2\ ,t) = \int_0^t \frac{\lambda_2^{n-1}\ x^{n-2}e^{-\lambda_2 x}}{\Gamma(n-1)}$$

$$F_{n-1}(\lambda_2\ ,t) = \frac{\lambda_2^{n-1}}{\Gamma(n-1)} \int_0^t x^{n-2}e^{-\lambda_2 x}$$

Therefore the convolution of the two distributions becomes;

$$F^n(S_N) = \int_0^t F_1(t-x)\, F_{n-1}(\lambda_2\ ,t)$$

$$F^n(S_N) = \int_0^t \left(1 - e^{-\lambda_1(t-x)}\right) \frac{\lambda_2^{n-1}}{\Gamma(n-1)} x^{n-2}e^{-\lambda_2 x}dx$$

$$F^n(S_N) = \frac{\lambda_2^{n-1}}{\Gamma(n-1)} \int_0^t \left(1 - e^{-\lambda_1(t-x)}\right) x^{n-2} e^{-\lambda_2 x} dx$$

$$F^n(S_N) = \frac{\lambda_2^{n-1}}{\Gamma(n-1)} \int_0^t x^{n-2} e^{-\lambda_2 x} dx - \frac{\lambda_2^{n-1}}{\Gamma(n-1)} \int_0^t e^{-\lambda_1(t-x)} x^{n-2} e^{-\lambda_2 x} dx$$

$$F^n(S_N) = \Gamma_{n-1}(\lambda_2, t) - \frac{\lambda_2^{n-1}}{\Gamma(n-1)} e^{-\lambda_1 t} \int_0^t x^{n-2} e^{-x(\lambda_2 - \lambda_1)} dx$$

Letting $s = x(\lambda_2 - \lambda_1) \Leftrightarrow x = \frac{s}{(\lambda_2 - \lambda_1)}$

$$dx = \frac{ds}{(\lambda_2 - \lambda_1)}$$

$$F^n(S_N) = \Gamma_{n-1}(\lambda_2, t) - \frac{\lambda_2^{n-1}}{\Gamma(n-1)} e^{-\lambda_1 t} \int_0^t \left[\frac{s}{(\lambda_2 - \lambda_1)}\right]^{n-2} e^{-s} \frac{ds}{(\lambda_2 - \lambda_1)}$$

$$F^n(S_N) = \Gamma_{n-1}(\lambda_2, t) - \frac{\lambda_2^{n-1}}{\Gamma(n-1)(\lambda_2 - \lambda_1)^{n-1}} e^{-\lambda_1 t} \int_0^t s^{n-2} ds$$

$$F^n(S_N) = \Gamma_{n-1}(\lambda_2, t) - e^{-\lambda_1 t} \left(\frac{\lambda_2}{\lambda_2 - \lambda_1}\right)^{n-1} \Gamma_{n-1}(s, t)$$

$$F^n(S_N) = \Gamma_{n-1}(\lambda_2, t) - e^{-\lambda_1 t} \left(\frac{\lambda_2}{\lambda_2 - \lambda_1}\right)^{n-1} \Gamma_{n-1}(s, t)$$

4.1.2 Compound distribution

General introduction

Let $\{X_i\}$ be a sequence of mutually independent random variables with identical distributions(i.i.d). Let $p(X_i = j) = f_i$ be the probability density function of X_i with probability generating function,

$$f(s) = \sum_i f_i s^i$$

For $S_N = X_1 + X_2 + \cdots\ldots + X_N$ where N is a random variable independent of X_i's and with probability density function;

$$p(N = n) = g_n$$

And p.g.f,

$$g(s) = \sum_n g_n s^n$$

Then the probability distribution of S_N is given by;

$$h_i = prob(S_N = i) = \sum_{n=0}^{\infty} \{prob(N = n)\} probX_1 + X_2 + \cdots\ldots + X_N = j\}$$

using generating functions;

$$h(s) = \sum_{n=0}^{\infty} g_n [f(s)]^n$$

this is referred to as Compound distribution of S_N.

Compound Geometric Distribution

Theorem 4.1.1 *Let $N, X_1, X_2, \ldots\ldots$ be independent discrete random variables. If the $\{X_i\}$ are identically distributed each with Probability Generating Function $G_X(s)$, then*

$$S_N = X_1 + X_2 + \cdots\ldots + X_N$$

has Probability Generating Function given by

$$G_{s_N}(s) = G_N(G_X(s))$$

If X_i is geometrically distributed, i.e.

$$g_n = prob(N = n) = pq^n, n = 0, 1, 2, \ldots.$$

with probability generating function,

$$g(s) = \sum_{n=0}^{\infty} pq^n s^n = p \sum_{n=0}^{\infty} q^n s^n = p\left\{(qs)^0 + (qs)^1 + (qs)^2 + (qs)^3 + \ldots\ldots\right\}$$

$$g(s) = \frac{p}{1 - qs}$$

The probability generating function of $S_N = X_1 + X_2 + \cdots\ldots + X_N$ is given by,

$$H(s) = g\{f(s)\} = \frac{p}{1 - q\{f(s)\}}$$

considering the various cases of $X_i's$ we have the following;

$X_i's$ **Being Geometric** For

$$p(X_i = x) = g_n = pq^{n-1}, n = 1, 2, 3, ...$$

Then the p.g.f of X_i is given by;

$$f(s) = \sum_{n=1}^{\infty} pq^{n-1}s^n = ps\sum_{n=1}^{\infty}(qs)^{n-1} = ps\left\{(qs)^0 + (qs)^1 + (qs)^2 + ...\right\}$$

$$f(s) = \frac{ps}{1-qs}$$

and

$$H(s) = \frac{p}{1-q\left\{\frac{ps}{1-qs}\right\}} = \frac{p}{\frac{1-qs-pqs}{1-qs}}$$

$$H(s) = \frac{p(1-qs)}{1-qs-pqs}$$

Using the method of moments to determine the mean and variance of the distribution of S_N we have;

$$\frac{d}{ds}H(s) = \frac{d}{ds}\left\{\frac{p(1-qs)}{1-qs-pqs}\right\} = \frac{(1-qs-pqs)(-pq)-(p-pqs)\{-(q+pq)\}}{(1-qs-pqs)^2}$$

$$H'(s) = \frac{-pq+pq^2+p^2q^2s+pq+p^2q-pq^2s-p^2q^2s}{(1-qs-pqs)^2}$$

$$H'(s) = \frac{p^2q}{(1-qs-pqs)^2}$$

$$H'(1) = \frac{p^2q}{(1-q-pq)^2}$$

$$H'(s) = \frac{p^2q}{(p-pq)^2} = \frac{q}{p^2}$$

Getting the second derivative of $H(s)$ we have $\frac{d^2}{ds^2}H(s) = \frac{d^2}{ds^2}\left\{\frac{p(1-qs)}{1-qs-pqs}\right\}$

$$H''(s) = \frac{d}{ds}\left\{\frac{p^2q}{(1-qs-pqs)^2}\right\}$$

$$H''(s) = \frac{d}{ds}\left\{p^2q\left(1 - qs - pqs\right)^{-2}\right\}$$

$$H''(s) = p^2q(-2)\left(1 - qs - pqs\right)^{-3}(-)(q + pq)$$

$$H''(s) = \frac{2p^2q(q + pq)}{(1 - qs - pqs)^3}$$

$$H''(1) = \frac{2p^2q(q + pq)}{(1 - q - pq)^3} = \frac{2p^2q(q + pq)}{p^3(1 - q)^3} = \frac{2p^2q^2(1 + p)^2}{p^6} = \frac{2q^2(1 + p)}{p^4}$$

$$var(S_N) = H''(1) - \{H'(s)\}^2 + H'(s)$$

$$var(S_N) = \frac{2q^2(1 + p)}{p^4} - \left\{\frac{q}{p^2}\right\}^2 + \frac{q}{p^2} = \frac{q^2 + pq(2q + p)}{p^4}$$

Thus S_N has the mean given $\frac{q}{p^2}$ and variance given by $\frac{q^2 + pq(2q + p)}{p^4}$

$X_i's$ **Being Bernoulli** For

$$p(X_i = x) = g_n = p^n q^{1-n}, n = 0, 1$$

Then the p.g.f of X_i is given by;

$$f(s) = \sum_{n=0}^{1} p^n q^{1-n} s^n = q + ps$$

and

$$H(s) = \frac{p}{1 - q(q + ps)} = \frac{p}{1 - q^2 - pqs}$$

Using the method of moments to determine the mean and variance of the distribution of S_N we have;

$$\frac{d}{ds}H(s) = \frac{d}{ds}\left\{\frac{p}{1 - q^2 - pqs}\right\}$$

$$H'(s) = p\left(1 - q^2 - pqs\right)^{-1} = p(-1)(-pq)\left(1 - q^2 - pqs\right)^{-2} = \frac{p^2q}{(1 - q^2 - pqs)^2}$$

$$H'(1) = \frac{p^2q}{(1-q^2-pq)^2} = \frac{p^2q}{\{1-q(q+p)\}^2} = \frac{p^2q}{(1-q)^2} = \frac{p^2q}{p^2} = q \quad (3.2.7)$$

Getting the second derivative of $H(s)$ we have $\frac{d^2}{ds^2}H(s) = \frac{d^2}{ds^2}\left\{\frac{p}{1-q^2-pqs}\right\}$

$$H''(s) = \frac{d}{ds}\left\{\frac{p^2q}{(1-q^2-pqs)^2}\right\} = \frac{d}{ds}\left\{p^2q\left(1-q^2-pqs\right)^{-2}\right\} = p^2q(-2)\left(1-q^2-pqs\right)^{-3}(-pq)$$

$$H''(s) = \frac{2p^3q^2}{(1-q^2-pqs)^3}$$

$$H''(1) = \frac{2p^3q^2}{\{1-q(q+p)\}^3} = \frac{2p^3q^2}{(1-q)^3} = \frac{2p^3q^2}{p^3} = q^2$$

$$var(S_N) = H''(s) - \{H'(1)\}^2 - H'(1)$$

$$q^2 - \{q\}^2 + q = q$$

$$var(S_N) = q$$

$X_i's$ **Being Binomial** For

$$p(X_i = x) = g_n = \binom{n}{k}p^kq^{n-k}, k = 0, 1, \ldots..n$$

with p.g.f of X_i is given by;

$$f(s) = \sum_{k=0}^{n}\binom{n}{k}p^kq^{n-k}s^k = (q+ps)^n$$

and therefore,

$$H(s) = \frac{p}{1-q\{f(s)\}} = \frac{p}{1-q(q+ps)^n}$$

Using the method of moments to determine the mean and variance of the distribution of S_N we have;

$$\frac{d}{ds}H(s) = \frac{d}{ds}\left\{\frac{p}{1-q(q+ps)^n}\right\} = \frac{0-p\left[0-nq(q+ps)^{n-1}p\right]}{[1-q(q+ps)^n]^2} = \frac{np^2q(q+ps)^{n-1}}{[1-q(q+ps)^n]^2}$$

$$H'(1) = \frac{np^2q\,(q+p)^{n-1}}{\left[1-q\,(q+p)^n\right]^2} = \frac{np^2q}{(1-q)^2} = nq$$

Getting the second derivative of $H(s)$ we have $\frac{d^2}{ds^2}H(s) = \frac{d^2}{ds^2}\left\{\frac{p}{1-q\,(q+ps)^n}\right\}$

$$H''(s) = \frac{d}{ds}\left\{\frac{np^2q\,(q+ps)^{n-1}}{\left[1-q\,(q+ps)^n\right]^2}\right\}$$

letting $u = np^2q\,(q+ps)^{n-1}$ and $v = \left[1-q\,(q+ps)^n\right]^2$

$$H''(s) = \frac{n(n-1)\,p^3q\,(q+ps)\left[1-q\,(q+ps)^n\right]^2 + 2npq\left[1-q\,(q+ps)^n\right](q+ps)^{n-1}\left[np^2q\,(q+ps)^{n-1}\right]}{\left[1-q\,(q+ps)^n\right]^4}$$

$$\begin{aligned} H''(1) &= \frac{n(n-1)\,p^3q\,(q+p)\left[1-q\,(q+p)^n\right]^2 + 2npq\left[1-q\,(q+p)^n\right](q+p)^{n-1}\left[np^2q\,(q+p)^{n-1}\right]}{\left[1-q\,(q+p)^n\right]^4} \\ \text{but } q+p &= 1 \end{aligned}$$

$$H''(1) = \frac{n(n-1)\,p^3q\,(1-q)^2 + 2n^2p^3q^2\,(1-q)}{(1-q)^4}$$

$$H''(1) = \frac{p^4\left\{n(n-1)\,q + 2n^2q^2\right\}}{p^4} = n^2q - nq + 2n^2q^2$$

$$var\,(S_N) = H''(s) - \left\{H'(1)\right\}^2 - H'(1)$$

$$var\,(S_N) = n^2q - nq + 2n^2q^2 - n^2q^2 + nq = n^2q + n^2q^2$$

$$var\,(S_N) = n^2q\,(1+q)$$

Table of Summary

Distribution of N	Distribution of X_i	$H(s)$	Mean	Variance
Geometric	Geometric($g_n = pq^{n-1}, n = 1, 2, $)	$\frac{p(1-qs)}{1-qs-pqs}$	$\frac{q}{p^2}$	$\frac{q^2 + pq\,(2q + p}{p^4}$
Geometric	Bernoulli($g_n = p^n q^{1-n}, n = 0, 1$)	$\frac{p}{1-q^2-pqs}$	q	q
Geometric	Binomial($g_n = \binom{n}{k} p^k q^{n-k}, k = 0, 1, ..n$)	$\frac{p}{1-q(q+ps)^n}$	nq	$n^2q\,(1+q)$

4.2 Waiting Time Distribution for the nth Live Birth

4.2.1 Description of Waiting time process

Consider a time interval T_i between the $i^{th}$and $(i+1)^{th}$ live births for any married woman for $i = 1, 2, 3, ...$

Therefore

$$T_i = m + X_1 + X_2 + ... + X_{n+1} + L_1 + L_2 + ... + L_n + 9 \qquad (4.2)$$

Where

- m denotes the period of non-susceptibility (postpartum amenorrhoea) following the i^{th} birth
- $X_i, i = 1, 2, ...n+1$denotes the number of months the mother goes on without conception in the susceptible state. $X_i's$ are independent and identically distributed random variables.
- $L_i, i = 1, 2, ..., n$ denotes the period of non-susceptibility associated with a defective termination (including period of pregnancy and period of amenorrhoea following termination) for any women in the interval T_i . $L_i's$ are all independently and identically distributed random variables.
- $'9'$ denotes the period of pregnancy leading to a live birth

4.2.2 The Probability Generating Function (p.g.f) of T_i

Theorem 4.2.1 *Let X and Y be independent discrete random variables with p.g.f $G_X(s)$ and $G_Y(s)$ respectively, and let $Z = X + Y$;then*

$$G_Z(s) = G_{X+Y}(s) = G_X(s)\,G_Y(s)$$

More generally if

$$U = W_{(j)} + X_{(m)} + Y_{(n)} + Z_{(l)} + k \tag{4.3}$$

Where;

- $W_{(j)} = w_1 + w_2 + ... + w_j$ with p.g.f $G_W(s)$
- $X_{(m)} = x_1 + x_2 + ... + x_m$ with p.g.f $G_X(s)$
- $Y_{(n)} = y_1 + y_2 + ... + y_n$ with p.g.f $G_Y(s)$
- $Z_{(l)} = z_1 + z_2 + ... + z_n$ with p.g.f $G_Z(s)$

and k is a constant, then the p.g.f of U denoted by $G_U(s)$ is given by

$$G_U(s) = S^k [G_W(s)]^w [G_X(s)]^m [G_Y(s)]^n [G_Z(s)]^l \tag{4.4}$$

let $H(s)$ denote the p.g.f of T_i , then

$$H(s) = E\left[s^{T_i}\right]$$

$$H(s) = E\left[s^{m+X_1+X_2+......+X_{n+1}+L_1+L_2+....+L_n+9}\right]$$

Since all the random variables are independent of each other;

$$H(s) = E[s^m]\,E\left[s^{X_1+X_2+......+X_{n+1}}\right]E\left[s^{L_1+L_2+....+L_n+}\right]E\left[s^9\right]$$

and since $X_i's$ are identically distributed and $L_i's$ are also identically distributed, then

$$H(s) = E[s^m]\left\{E\left[s^{X_i}\right]\right\}^{n+1}\left\{E\left[s^{L_i}\right]\right\}^n s^9 \tag{4.5}$$

Letting

i) $E[s^m]$ the probability generating function of m be denoted by $\Phi_m(s)$

ii) $E\left[s^{X_i}\right]$ the probability generating function of X_i be denoted by $\Phi_x(s)$

iii) $E\left[s^{L_i}\right]$ the probability generating function of L_i be denoted by $\Phi_l(s)$

then

$$H(s) = \Phi_m(s)\,[\Phi_x(s)]^{n+1}\,[\Phi_l(s)]^n s^9 \tag{4.6}$$

This is the Probability density function of T_i for a fixed n.

Hence knowing the distribution functions of m, $X_i's$, $L_i's$ and N we will be able to obtain the distribution function of T_i

4.2.3 Special Cases

Case 1

For a single woman of constant probability of conception p and the distribution of N given by

$$prob\,(N = n) = g\,(n) = \theta\,(1-\theta)^n \tag{4.7}$$

and distribution of X_i given by

$$prob\,(X_i = x) = f\,(x) = pq^x \tag{4.8}$$

with probability generating function

$$\Phi_x(s) = \frac{p}{1-qs} \tag{4.9}$$

$$H\,(s) = \sum_n \theta\,(1-\theta)^n\,\Phi_m(s)\left[\frac{p}{1-qs}\right]^{n+1}[\Phi_l(s)]^n\,s^9$$

$$H\,(s) = \frac{p\theta\Phi_m(s)s^9}{1-qs-p\,(1-\theta)\,\Phi_l(s)} \tag{4.10}$$

Proof.

$$H\,(s) = \sum_n \theta\,(1-\theta)^n\,\Phi_m(s)\left[\frac{p}{1-qs}\right]^{n+1}[\Phi_l(s)]^n\,s^9$$

$$H\,(s) = \frac{p\theta}{(1-qs)}\Phi_m(s)s^9\sum_n(1-\theta)^n\left[\frac{p}{1-qs}\right]^n[\Phi_l(s)]^n$$

$$H\,(s) = \frac{p\theta}{(1-qs)}\Phi_m(s)s^9\sum_n^n\left[\frac{p\,(1-\theta)\,\Phi_l(s)}{1-qs}\right]^n$$

$$H\,(s) = \frac{p\theta}{(1-qs)}\Phi_m(s)s^9\left\{1+\left(\frac{p\,(1-\theta)\,\Phi_l(s)}{1-qs}\right)+\left(\frac{p\,(1-\theta)\,\Phi_l(s)}{1-qs}\right)^2+\left(\frac{p\,(1-\theta)\,\Phi_l(s)}{1-qs}\right)^3+\ldots\right.$$

$$H\,(s) = \frac{p\theta}{(1-qs)}\Phi_m(s)s^9\left\{\frac{1}{1-\frac{p(1-\theta)\Phi_l(s)}{1-qs}}\right\}$$

$$H\,(s) = \frac{p\theta}{(1-qs)}\Phi_m(s)s^9\left\{\frac{1}{\frac{1-qs-p(1-\theta)\Phi_l(s)}{1-qs}}\right\} = \frac{p\theta}{(1-qs)}\Phi_m(s)s^9\left\{\frac{1-qs}{1-qs-p\,(1-\theta)\,\Phi_l(s)}\right\}$$

$$H(s) = \frac{p\theta\Phi_m(s)s^9}{1 - qs - p(1-\theta)\Phi_l(s)}$$

■

Equation 3.4.2.d is the p.g.f for any single woman. If we consider data from a group of women, then p (fecundability) is also found to be a random variable. K. Srinivasan (1966) considered a case where p assumed a bivariate Beta distribution with parameters α and β. For this case the probability generating functions becomes;

$$H(s) = \int_0^1 \frac{p^\alpha q^\beta \theta\Phi_m(s)s^9}{B(\alpha,\beta)\left[1 - qs - p(1-\theta)\Phi_l(s)\right]} dp \tag{4.11}$$

$$H(s) = \frac{\theta\Phi_m(s)s^9}{B(\alpha,\beta)\Phi_l(s)} \int_0^1 \frac{p^\alpha q^\beta}{\left[1 - qs - p(1-\theta)\right]} dp \tag{4.12}$$

$$H(s) = \left\{\frac{\theta\Phi_m(s)s^9}{\Phi_l(s)}\right\}\left\{\int_0^1 \frac{p^\alpha q^\beta}{B(\alpha,\beta)\left[1 - qs - p(1-\theta)\right]}\right\} \tag{4.13}$$

It can be seen that $H(s)$ is of the form $L(s).M(s)$, where $L(1) = 1$ and $M(1) = 1$.Hence $L(s)$ and $M(s)$ represents the p.g.f's of two independent random variables Y and Z, such that $T_i = Y + Z$. where Y stands for the sum total of all nosusceptible periods in the interval T_i and Z stands for the susceptible periods(ovulation cycle) spent without conception in T_i. Knowing the p.g.f we can use the same for derivation of the moments of any order for the random variable T_i.These theoretical expressions for the moments which will be functions of the parameters can be equated to the observed moments of the distribution of birth intervals and consistent estimates of the parameters arrived at.

The expression for the probability generating function for the interval from marriage (date of consummation) to the first live birth differ from other birth intervals only in the component m(the postpartum amenorrhoea) associated with a live birth. So that

$$T_0 = X_1 + X_2 + \dots\dots + X_{n+1} + L_1 + L_2 + \dots. + L_n + 9$$

Therefore;

$$G_0(x+9) = \sum_n \theta(1-\theta)^n \left[F(x)\right]^{(n+1)*} \left[Q(l)\right]^{n*}$$

and therefore,

$$H(s) = \left\{ \frac{\theta s^9}{\Phi_l(s)} \right\} \left\{ \int_0^1 \frac{p^\alpha q^\beta}{B(\alpha,\beta)\left[1 - qs - p(1-\theta)\right]} \right\}$$

Case 2

Aleyamma George (1973) considered a case where .For a single woman of constant probability of conception and the distribution of N given by

$$prob(N = n) = g(n) = \theta(1-\theta)^n \tag{4.14}$$

,distribution of X_i given by;

$$prob(X_i = x) = f(x) = \lambda e^{-\lambda x} \tag{4.15}$$

with characteristic function;

$$\Phi_x(s) = \frac{\lambda}{(\lambda - it)} \tag{4.16}$$

and the distribution of L_i given by

$$prob(L_i = l) = f(x) = \lambda_1 e^{-\lambda_1 x} \tag{4.17}$$

with characteristic function given by;

$$\Phi_l(s) = \frac{\lambda_1}{(\lambda_1 - it)} \tag{4.18}$$

then the Characteristic function of T_i ;

$$H(s) = \sum_n g(n)\,\Phi_m(s)\left[\Phi_x(s)\right]^{n+1}\left[\Phi_l(s)\right]^n s^9$$

becomes;

$$H(s) = \frac{\Phi_m(s) s^9 \lambda\lambda_1\theta\left(1 - \frac{it}{\lambda_1}\right)}{it(it - \lambda - \lambda_1) + \lambda\lambda_1\theta}$$

$$H(s)=\sum_n \theta(1-\theta)^n \Phi_m(s)\left[\frac{\lambda}{(\lambda-it)}\right]^{n+1}\left[\frac{\lambda_1}{(\lambda_1-it)}\right]^n s^9$$

$$H(s)=\left[\frac{\lambda}{(\lambda-it)}\right]\Phi_m(s)s^9\sum_n(1-\theta)^n\left[\frac{\lambda}{(\lambda-it)}\right]^n\left[\frac{\lambda_1}{(\lambda_1-it)}\right]^n$$

$$H(s)=\frac{\lambda\theta}{(\lambda-it)}\Phi_m(s)s^9\sum_n^n\left[\frac{\lambda(1-\theta)\lambda_1}{(\lambda-it)(\lambda_1-it)}\right]^n$$

$$H(s)=\frac{\lambda\theta}{(\lambda-it)}\Phi_m(s)s^9\left\{1+\left(\frac{\lambda(1-\theta)\lambda_1}{(\lambda-it)(\lambda_1-it)}\right)+\left(\frac{\lambda(1-\theta)\lambda_1}{(\lambda-it)(\lambda_1-it)}\right)^2+........\right\}$$

$$H(s)=\frac{\lambda\theta}{(\lambda-it)}\Phi_m(s)s^9\left\{\frac{1}{1-\frac{\lambda(1-\theta)\lambda_1}{(\lambda-it)(\lambda_1-it)}}\right\}$$

$$H(s)=\frac{\lambda\theta}{(\lambda-it)}\Phi_m(s)s^9\left\{\frac{1}{\frac{(\lambda-it)(\lambda_1-it)-\lambda\lambda_1(1-\theta)}{(\lambda-it)(\lambda_1-it)}}\right\}$$

$$H(s)=\frac{\lambda\theta}{(\lambda-it)}\Phi_m(s)s^9\left\{\frac{(\lambda-it)(\lambda_1-it)}{(\lambda-it)(\lambda_1-it)-\lambda\lambda_1(1-\theta)}\right\}$$

$$H(s)=\frac{\lambda\theta\Phi_m(s)s^9(\lambda_1-it)}{it(it-\lambda-\lambda_1)+\lambda\lambda_1\theta}=\frac{\Phi_m(s)s^9\lambda\lambda_1\theta\left(1-\frac{it}{\lambda_1}\right)}{it(it-\lambda-\lambda_1)+\lambda\lambda_1\theta} \quad (4.19)$$

This is the PGF for any single woman. If we consider data from a group of women, λ is considered to be a random variable. A. George considered a case where λ assumed a Gamma distribution with parameters α and β i.e.

$$g(\lambda)=\frac{\beta^\alpha}{\Gamma(\alpha)}\lambda^{\alpha-1}e^{-\beta\lambda}$$

For this case the probability generating functions becomes;

$$H(s)=\int_0^\infty\left(\frac{\beta^\alpha}{\Gamma(\alpha)}\lambda^{\alpha-1}e^{-\beta\lambda}\right)\left(\frac{\Phi_m(s)s^9\lambda\lambda_1\theta\left(1-\frac{it}{\lambda_1}\right)}{it(it-\lambda-\lambda_1)+\lambda\lambda_1\theta}\right)d\lambda$$

$$H(s) = \frac{\beta^\alpha \theta \Phi_m(s) s^9}{\Gamma(\alpha)} \int_0^\infty \lambda^{\alpha-1} e^{-\beta\lambda} \left(\frac{\lambda\lambda_1\theta\left(1 - \frac{it}{\lambda_1}\right)}{it\,(it - \lambda - \lambda_1) + \lambda\lambda_1\theta} \right) d\lambda$$

$$H(s) = \frac{\beta^\alpha \theta \Phi_m(s) s^9 \lambda_1 \left(1 - \frac{it}{\lambda_1}\right)}{\Gamma(\alpha)} \int_0^\infty \left(\frac{\lambda^\alpha e^{-\beta\lambda}}{it\,(it - \lambda - \lambda_1) + \lambda\lambda_1\theta} \right) d\lambda \quad (4.20)$$

Case 3

John Bongraats (1975) considered the waiting time from marriage to first live birth to be a random variable T with all variables in other two cases except the variable m (postpartum amenorrhoea following the i^{th} birth). I.e.

$$\begin{aligned} T_0 &= X_1 + X_2 + \ldots\ldots + X_{n+1} + L_1 + L_2 + \ldots. + L_n + 9 \text{ or} \\ T_0 - 9 &= X_1 + X_2 + \ldots\ldots + X_{n+1} + L_1 + L_2 + \ldots. + L_n \end{aligned}$$

letting $T_0 - 9 = Z$ then,

$$Z = X_1 + X_2 + \ldots\ldots + X_{n+1} + L_1 + L_2 + \ldots. + L_n \quad (4.21)$$

He considered a case where the random variables $X_i's$, $L_i's$ and n assumed geometric distributions with different parameters. That is;

$$prob\,(N = n) = g\,(n) = \theta\,(1 - \theta)^n \quad (4.22)$$

,distribution of X_i given by;

$$prob\,(X_i = x) = f\,(x) = f\,(1 - f)^{x-1}, x = 1, 2, 3, \ldots. \quad (4.23)$$

with characteristic function;

$$\Phi_x(s) = \frac{fs}{1 - (1 - f)\,s} \quad (4.24)$$

and the distribution of L_i given by

$$prob\,(L_i = l) = g\,(l) = g\,(1 - g)^{l-1}, l = 1, 2, \ldots. \quad (4.25)$$

with characteristic function given by;

$$\Phi_l(s) = \frac{ls}{1 - (1 - l)\,s} \quad (4.26)$$

hen the Characteristic function of Z ;

$$H_z(s) = \sum_n g(n) [\Phi_x(s)]^{n+1} [\Phi_l(s)]^n$$

becomes;

$$H_z(s) = \frac{\theta f s [1-(1-l)s]}{1-(2-f-l)s+[1-f-l+fl\theta]s^2} \tag{4.27}$$

Proof.

$$H(s) = \sum_{n=0}^{\infty} g(n) [\Phi_x(s)]^{n+1} [\Phi_l(s)]^n = \sum_{n=0}^{\infty} \theta (1-\theta)^n [\Phi_x(s)]^{n+1} [\Phi_l(s)]^n$$

$$H(s) = \theta \Phi_x(s) \sum_{n=0}^{\infty} \{(1-\theta)\Phi_x(s)\Phi_l(s)\}^n$$

$$H(s) = \frac{\theta\Phi_x(s}{1-(1-\theta)\Phi_x(s)\Phi_l(s)} = \frac{\theta\left(\frac{fs}{1-(1-f)s}\right)}{1-(1-\theta)\left(\frac{fs}{1-(1-f)s}\right)\left(\frac{ls}{1-(1-l)s}\right)}$$

$$H(s) = \frac{\frac{\theta f s}{1-(1-f)s}}{\frac{(1-(1-f)s)(1-(1-l)s)-(1-\theta)fls^2}{(1-(1-f)s)(1-(1-l)s)}}$$

$$H(s) = \frac{\theta f s [1-(1-f)s][1-(1-l)s]}{[1-(1-f)s][(1-(1-f)s)(1-(1-l)s)-(1-\theta)fls^2]}$$

$$H(s) = \frac{\theta f s [1-(1-l)s]}{[(1-(1-f)s)(1-(1-l)s)-(1-\theta)fls^2]}$$

$$H(s) = \frac{\theta f s [1-(1-l)s]}{1-(2-f-l)s+[1-f-l+fl\theta]s^2}$$

■

Chapter 5

FURTHER MODIFICATIONS TO PARTICULAR DISTRIBUTIONS

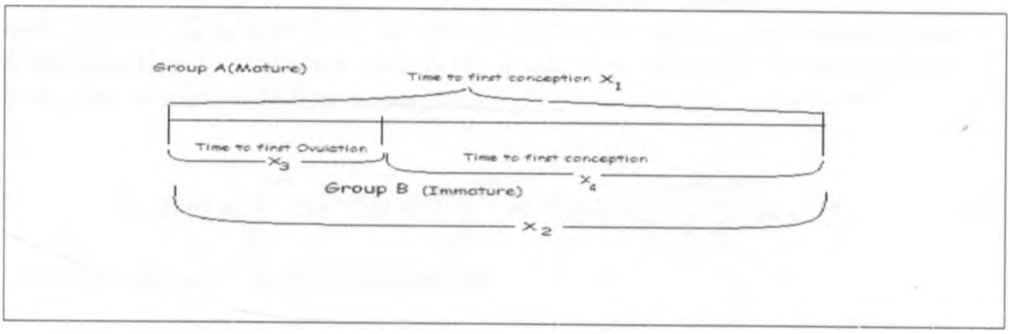
Another way of mixing two of more distribution if by use of the formulae

$$f(t) = \alpha f_1(t) + (1 - \alpha) f_2(t) \tag{5.1}$$

or more generally

$$f(t) = \sum_i p_i f_i \text{ iff } \sum_i p_i = 1 \tag{5.2}$$

Consider the illustration below



With the probability of falling in Group A given by α and that of falling in Group B given by $1 - \alpha$,then;

$$f(x) = \alpha f(x_1) + (1 - \alpha) f(x_2) \tag{5.3}$$

In this chapter we will be Constructing distributions based on this distribution mixing technique.

5.1 Exponential Mixture

5.1.1 Distribution

A model to estimate adolescent sterility among married women is presented using the principle of convex combination of two or more probability density functions, For a group of women married when they under 20 years, there are two groups;

a) Group A-those who are biologically mature at the time of marriage, with proportion α and p.d.f $f_1(t)$

b) Group B- those who are biologically immature at the time of marriage, with proportion $(1-\alpha)$ and p.d.f $f_2(t)$

If $f(t)$ is the p.d.f of the interval between marriage and first conception, among a given group of women not stratified into groups A and B, then

$$f(t) = \alpha f_1(t) + (1-\alpha) f_2(t) \tag{5.4}$$

$f_1(t)$ was considered to be a negative binomial with parameter λ i.e. $\lambda e^{-\lambda t}$ and $f_2(t) = \int_0^t f_4(t/x) f_3(x)\, dx$ where $f_4(t/x)$ is the p.d.f of waiting time from marriage to ovulation and $f_3(t) = \mu e^{-\mu t}$ is the p.d.f of waiting time from first ovulation to first conception

$$f_4(t/x) = \int_x^{\infty} \lambda e^{-\lambda t} dt = \lambda \int_x^{\infty} e^{-\lambda t} dt = \lambda \left[\frac{e^{-\lambda t}}{\lambda}\right]_x^{\infty} = e^{-\lambda x}$$

therefore the p.d.f. $f_4(t/x)$ is given by

$$f_4(t/x) = \frac{\lambda e^{-\lambda t}}{e^{-\lambda x}} = \lambda e^{-\lambda(t-x)} \tag{5.5}$$

and

$$f_2(t) = \int_0^t \mu e^{-\mu x} \lambda e^{-\lambda(t-x)} dx$$

$$f_2(t) = \lambda\mu \int_0^t e^{-x(\mu-\lambda)} e^{-\lambda t} dx = \lambda\mu e^{-\lambda t} \int_0^t e^{-x(\mu-\lambda)} dx$$

$$\frac{\lambda\mu e^{-\lambda t}}{\mu-\lambda}\left[-e^{-x(\mu-\lambda)}\right]_0^t=\frac{\lambda\mu e^{-\lambda t}}{\mu-\lambda}\left[1-e^{-t(\mu-\lambda)}\right]$$

$$f_2(t)=\frac{\lambda\mu}{\mu-\lambda}\left[e^{-\lambda t}-e^{-\mu t}\right] \tag{5.6}$$

and therefore;

$$f(t)=\alpha\ \lambda e^{-\lambda t}+(1-\alpha)\frac{\lambda\mu}{\mu-\lambda}\left[e^{-\lambda t}-e^{-\mu t}\right],\ \text{if } \lambda\neq\mu \tag{5.7}$$

and if $\lambda=\mu$

$$f_2(t)=\int_0^t \mu e^{-\mu x}\lambda e^{-\lambda(t-x)}dx$$

$$f_2(t)=\int_0^t e^{-\lambda x}\lambda^2 e^{-\lambda(t-x)}dx=\lambda^2\int_0^t e^{-\lambda t}dx=\lambda^2 e^{-\lambda t}\int_0^t dx=\lambda^2 e^{-\lambda t}\left[x\right]_0^t$$

$$f_2(t)=\lambda^2 te^{-\lambda t}$$

$$f(t)=\alpha\ \lambda e^{-\lambda t}+(1-\alpha)\lambda^2 te^{-\lambda t},\ \text{if } \lambda=\mu \tag{5.8}$$

5.1.2 Parameter Estimation

To estimate α, λ, μ using the method of moments for $\lambda\neq\mu$ we get the first three moments as follows;

First moment

$$M_1=\int tf(t)\,dt=\int_0^\infty t\left\{\alpha\ \lambda e^{-\lambda t}+(1-\alpha)\frac{\lambda\mu}{\mu-\lambda}\left[e^{-\lambda t}-e^{-\mu t}\right]\right\}dt \tag{5.9}$$

$$M_1=\alpha\ \lambda\int_0^\infty te^{-\lambda t}+\frac{(1-\alpha)\lambda\mu}{(\mu-\lambda)}\int_0^\infty t\left[e^{-\lambda t}-e^{-\mu t}\right]dt$$

Using integration by parts

$$M_1=\alpha\ \lambda\left[\frac{1}{\lambda^2}\right]+\frac{(1-\alpha)\lambda\mu}{(\mu-\lambda)}\left[\frac{1}{\lambda^2}-\frac{1}{\mu^2}\right]=\frac{\alpha}{\lambda}+\frac{(1-\alpha)}{(\mu-\lambda)}\left[\frac{\mu}{\lambda}-\frac{\lambda}{\mu}\right]$$

$$M_1=\frac{\alpha}{\lambda}+\frac{(1-\alpha)}{(\mu-\lambda)}\frac{(\mu-\lambda)(\mu+\lambda)}{\lambda\mu}=\frac{\alpha}{\lambda}+\frac{(1-\alpha)(\mu+\lambda)}{\lambda\mu}$$

$$M_1=\frac{1}{\lambda\mu}\left[\alpha\mu+\mu-\alpha\mu+\lambda-\alpha\lambda\right]$$

$$M_1 = \frac{1}{\lambda\mu}\left[\mu + \lambda - \alpha\lambda\right] \tag{5.10}$$

Second moment

$$M_2 = \int t^2 f(t)\,dt = \int_0^\infty t^2 \left\{\alpha\ \lambda e^{-\lambda t} + (1-\alpha)\frac{\lambda\mu}{\mu-\lambda}\left[e^{-\lambda t} - e^{-\mu t}\right]\right\} dt \tag{5.11}$$

$$M_2 = \int t^2 f(t)\,dt = \int_0^\infty t^2\alpha\ \lambda e^{-\lambda t} + \int_0^\infty t^2(1-\alpha)\frac{\lambda\mu}{\mu-\lambda}\left[e^{-\lambda t} - e^{-\mu t}\right] dt$$

using integration by part

$$M_2 = \alpha\ \lambda\left(\frac{2}{\lambda^3}\right) + \frac{(1-\alpha)\lambda\mu}{(\mu-\lambda)}\left(\frac{2}{\lambda^3} - \frac{2}{\mu^3}\right)$$

$$M_2 = \frac{2\alpha}{\lambda^2} + \frac{2(1-\alpha)}{\lambda^2\mu^2}\left(\lambda^2 + \mu^2 + \lambda\mu\right) = \frac{2}{\lambda^2\mu^2}\left[\alpha\mu^2 + (1-\alpha)\left(\lambda^2 + \mu^2 + \lambda\mu\right)\right]$$

$$M_2 = \frac{2}{\lambda^2\mu^2}\left(\lambda\mu - \alpha\lambda\mu + \lambda^2 + \mu^2 - \alpha\lambda^2\right) = \frac{2}{\lambda^2\mu^2}\left[(\lambda+\mu)^2 - \lambda\mu - \alpha\lambda(\lambda+\mu)\right]$$

$$M_2 = \frac{2}{\lambda^2\mu^2}\left[(\lambda+\mu)^2 - \lambda\mu - \alpha\lambda(\lambda+\mu)\right] \tag{5.12}$$

and ,Third moment

$$M_3 = \int t^3 f(t)\,dt = \int_0^\infty t^3 \left\{\alpha\ \lambda e^{-\lambda t} + (1-\alpha)\frac{\lambda\mu}{\mu-\lambda}\left[e^{-\lambda t} - e^{-\mu t}\right]\right\} dt \tag{5.13}$$

$$M_3 = \int_0^\infty t^3\alpha\ \lambda e^{-\lambda t} + \int_0^\infty t^3(1-\alpha)\frac{\lambda\mu}{\mu-\lambda}\left[e^{-\lambda t} - e^{-\mu t}\right] dt$$

Using integration by parts

$$M_3 = \alpha\ \lambda\left(\frac{6}{\lambda^4}\right) + \frac{(1-\alpha)\lambda\mu}{(\mu-\lambda)}\left(\frac{6}{\lambda^4} - \frac{6}{\mu^4}\right)$$

$$M_3 = \frac{6\alpha}{\lambda^3\mu^3}\left(\alpha\mu^3 + \lambda^3 + \mu^3 - \alpha\lambda - \alpha\mu^3 + \lambda\mu^2 + \lambda^2\mu - \alpha\lambda\mu^2 - \alpha\lambda^2\mu\right)$$

$$M_3 = \frac{6\alpha}{\lambda^3\mu^3}\left(\lambda^3 + \mu^3 + \lambda\mu(\mu+\lambda) - \alpha\lambda\left(1 + \mu^2 + \lambda\mu\right)\right) \tag{5.14}$$

5.2 Geometric Mixture

In a population like India where pre-puberty marriages have been practiced, certain proportion of women remain adolescent infertile in the beginning but become fecund in subsequent months, there are two groups in the married women category;

a) Group A-those who are biologically mature at the time of marriage, with proportion α and p.d.f. $f_1(t)$

b) Group B- those who are biologically immature at the time of marriage, with proportion $(1-\alpha)$ and p.d.f. $f_2(t)$

If $f(t)$ is the p.d.f. of the interval between marriage and first conception, among a given group of women not stratified into groups A and B, then

$$f(t) = \alpha f_1(t) + (1-\alpha) f_2(t)$$

Let $f_1(t)$ be a geometric distribution with parameter p i.e.

$$f_1(t) = pq^{t-1}, t = 1, 2, 3... \tag{5.15}$$

and

$$f_2(t) = \sum_{x=1}^{t} f_4(t/x) f_3(x)$$

where $f_4(t/x)$ is the p.m.f of waiting time from marriage to ovulation and $f_3(t) = \Phi(1-\Phi)^{x-1}$ is the p.d.f. of waiting time from first ovulation to first conception

$$f_4(t/x) = pq^{t-x} \tag{5.16}$$

and therefore,

$$f_2(t) = \sum_{x=1}^{t} pq^{t-x}\Phi(1-\Phi)^{x-1} = \frac{q^t p\Phi}{(1-\Phi)} \sum_{x=1}^{t} q^{-x}(1-\Phi)^x$$

$$f_2(t) = \frac{q^t p\Phi}{(1-\Phi)} \sum_{x=1}^{t} \left(\frac{1-\Phi}{q}\right)^x$$

$$f_2(t) = \frac{q^t p\Phi}{(1-\Phi)} \left\{\frac{1-\Phi}{q} + \left(\frac{1-\Phi}{q}\right)^2 + \left(\frac{1-\Phi}{q}\right)^3 + \ldots\ldots + \left(\frac{1-\Phi}{q}\right)^t\right\}$$

$$f_2(t) = \frac{q^t p\Phi}{(1-\Phi)}\left(\frac{1-\Phi}{q}\right)\left\{1+\frac{1-\Phi}{q}+\left(\frac{1-\Phi}{q}\right)^2+\left(\frac{1-\Phi}{q}\right)^3+......+\left(\frac{1-\Phi}{q}\right)^{t-1}\right\}$$

$$f_2(t) = \frac{q^t p\Phi}{(1-\Phi)}\left(\frac{1-\Phi}{q}\right)\left\{\frac{1-\left(\frac{1-\Phi}{q}\right)^t}{1-\frac{1-\Phi}{q}}\right\}$$

$$f_2(t) = \frac{q^t p\Phi}{(1-\Phi)}\left(\frac{1-\Phi}{q}\right)\left\{\frac{\frac{q^t-(1-\Phi)^t}{q^t}}{\frac{q-(1-\Phi)}{q}}\right\}$$

$$f_2(t) = \frac{q^t p\Phi\left[q^t-(1-\Phi)^t\right]}{q^t\left(q-(1-\Phi)\right)}$$

$$f_2(t) = \frac{p\Phi\left[q^t-(1-\Phi)^t\right]}{[q-(1-\Phi)]}$$

$$f_2(t) = \left(\frac{p\Phi}{[q-(1-\Phi)]}\right)\left[q^t-(1-\Phi)^t\right],\ p \neq \Phi \tag{5.17}$$

$$f(t) = \alpha f_1(t) + (1-\alpha) f_2(t)$$

$$f(t) = \alpha p q^{t-1} + (1-\alpha)\left(\frac{p\Phi}{[q-(1-\Phi)]}\right)\left[q^t-(1-\Phi)^t\right] \tag{5.18}$$

If $p = \Phi$ then

$$f_2(t) = \sum_{x=1}^{t} p q^{t-x}\Phi(1-\Phi)^{x-1}$$

$$f_2(t) = \sum_{x=1}^{t} p q^{t-x}\left(p q^{x-1}\right) = p^2 \sum_{x=1}^{t} q^{t-x+x-1}$$

$$f_2(t) = p^2 t q^{t-1}$$

$$f(t) = \alpha f_1(t) + (1-\alpha) f_2(t)$$

$$f(t) = \alpha p q^{t-1} + (1-\alpha) p^2 t q^{t-1}$$

$$f(t) = \begin{cases} \alpha p q^{t-1} + (1-\alpha)\left(\frac{p\Phi}{[q-(1-\Phi)]}\right)\left[q^t-(1-\Phi)^t\right],\ p \neq \Phi \\ \alpha p q^{t-1} + (1-\alpha) p^2 t q^{t-1},\ p = \Phi \end{cases} \tag{5.19}$$

5.2.1 Parameter Estimation

To estimate α, p and Φ using the method of moments for $\lambda \neq \mu$ we get the first three moments as follows;

$$f(t) = \alpha p q^{t-1} + (1-\alpha)\left(\frac{p\Phi}{[q-(1-\Phi)]}\right)\left[q^t - (1-\Phi)^t\right]$$

$$\text{First moment} = M_1 = \sum_{t=1}^{\infty} t\left\{\alpha p q^{t-1} + (1-\alpha)\left(\frac{p\Phi}{[q-(1-\Phi)]}\right)\left[q^t - (1-\Phi)^t\right]\right\} \tag{5.20}$$

$$\text{let } (1-\Phi) = \Psi$$

$$M_1 = \sum_{t=1}^{\infty} t\alpha p q^{t-1} + \sum_{t=1}^{\infty} t(1-\alpha)\left(\frac{p\Phi}{[q-(1-\Phi)]}\right)\left[q^t - \Psi^t\right]$$

$$M_1 = \alpha p \sum_{t=1}^{\infty} t q^{t-1} + (1-\alpha)\left(\frac{p\Phi}{[q-(1-\Phi)]}\right)\sum_{t=1}^{\infty} t\left[q^t - \Psi^t\right]$$

$$\begin{aligned} M_1 &= \alpha p\left\{1 + 2q + 3q^2 + 4q^3 + ...\right\} + \\ &\quad (1-\alpha)\left(\frac{p\Phi}{[q-(1-\Phi)]}\right)\left\{\left[q + 2q^2 + 3q^3 + ...\right] - \left[\Psi + 2\Psi^2 + 3\Psi^3 + ...\right]\right\} \end{aligned}$$

$$1 + 2q + 3q^2 + 4q^3 + ... = \frac{d}{dq}\left(q + q^2 + q^3 + ..\right) = \frac{1}{(1-q)^2}$$

and

$$q + 2q^2 + 3q^3 + ... = q\left(1 + 2q + 3q^2 + 4q^3\right) = \frac{q}{(1-q)^2}$$

and

$$\Psi + 2\Psi^2 + 3\Psi^3 + ... = \frac{\Psi}{(1-\Psi)^2}$$

therefore

$$M_1 = \alpha p\left\{\frac{1}{(1-q)^2}\right\} + (1-\alpha)\left(\frac{p\Phi}{[q-\Psi]}\right)\left\{\frac{q}{(1-q)^2} - \frac{\Psi}{(1-\Psi)^2}\right\}$$

$$M_1 = \frac{\alpha}{(1-q)} + \frac{(1-\alpha)}{q-(1-\Phi)}\left\{\frac{q\Phi}{p} - \frac{p(1-\Phi)}{\Phi}\right\} \tag{5.21}$$

Second moment $= M_2 = \sum_{t=1}^{\infty} t^2 \left\{ \alpha p q^{t-1} + (1-\alpha)\left(\frac{p\Phi}{[q-(1-\Phi)]}\right)\left[q^t - (1-\Phi)^t\right] \right\}$ (5.22)

$$\text{let } (1-\Phi) = \Psi$$

$$M_2 = \alpha p \sum_{t=1}^{\infty} t^2 q^{t-1} + (1-\alpha)\left(\frac{p\Phi}{[q-(1-\Phi)]}\right)\sum_{t=1}^{\infty} t^2\left[q^t - \Psi^t\right]$$

$$\begin{aligned} M_2 &= \alpha p\left\{1 + 4q + 9q^2 + 16q^3 + \ldots.\right\} + \\ &\quad (1-\alpha)\left(\frac{p\Phi}{[q-(1-\Phi)]}\right)\left\{\left[q + 4q^2 + 9q^3 + \ldots.\right] - \left[\Psi + 4\Psi^2 + 9\Psi^3 + \ldots.\right]\right\} \end{aligned}$$

but

$$\begin{aligned} 1 + 4q + 9q^2 + 16q^3 + \ldots. &= \frac{d}{dq}\left(q + 2q^2 + 3q^3 + 4q^4\right) \\ &= \frac{d}{dq}\left[\frac{q}{(1-q)^2}\right] = \frac{1+q}{(1-q)^3} \end{aligned}$$

and

$$q + 4q^2 + 9q^3 + \ldots. = q\left(1 + 4q + 9q^2 + 16q^3\right) = \frac{q(1+q)}{(1-q)^3}$$

Therefore

$$M_2 = \alpha p\left\{\frac{1+q}{(1-q)^3}\right\} + (1-\alpha)\left(\frac{p\Phi}{[q-(1-\Phi)]}\right)\left\{\frac{q(1+q)}{(1-q)^3} - \frac{\Psi(1+\Psi)}{(1-\Psi)^3}\right\}$$

$$M_2 = \frac{\alpha(1+q)}{(1-q)^2} + \frac{(1-\alpha)}{q-(1-\Phi)}\left\{\frac{q\Phi(1+q)}{(1-q)^2} - \frac{p\Psi(1+\Psi)}{(1-\Psi)^2}\right\}$$

$$M_2 = \frac{\alpha(1+q)}{(1-q)^2} + \frac{(1-\alpha)}{q-(1-\Phi)}\left\{\frac{q\Phi(2-p)}{p^2} - \frac{p(1-\Phi)(2-\Phi)}{\Phi^2}\right\} \quad (5.23)$$

and the Third moment

$$M_3 = \sum_{t=1}^{\infty} t^3 \left\{ \alpha p q^{t-1} + (1-\alpha)\left(\frac{p\Phi}{[q-(1-\Phi)]}\right)\left[q^t - (1-\Phi)^t\right] \right\} \quad (5.24)$$

$$M_3 = \alpha p \sum_{t=1}^{\infty} t^3 q^{t-1} + (1-\alpha)\left(\frac{p\Phi}{[q-(1-\Phi)]}\right)\sum_{t=1}^{\infty} t^3\left[q^t - \Psi^t\right]$$

$$M_3 = \alpha p\left\{1+8q+27q^2+....\right\}+(1-\alpha)\left(\frac{p\Phi}{[q-(1-\Phi)]}\right)$$
$$\left\{\left[q+8q^2+27q^2+...\right]-\left[\Psi+8\Psi^2+27\Psi^2+..\right]\right\}$$

But

$$1+8q+27q^2+.... = \frac{d}{dq}\left\{q+4q^2+9q^3+\right\}$$
$$= \frac{d}{dq}\left\{\frac{q(1+q)}{(1-q)^3}\right\} = \frac{1+4q+q^2}{(1-q)^4}$$

and

$$q+8q^2+27q^2+... = q\left(1+8q+27q^2+....\right) = \frac{q\left(1+4q+q^2\right)}{(1-q)^4}$$

Therefore

$$M_3 = \alpha p\left\{\frac{1+4q+q^2}{(1-q)^4}\right\}+(1-\alpha)\left(\frac{p\Phi}{[q-(1-\Phi)]}\right)\left\{\frac{q\left(1+4q+q^2\right)}{(1-q)^4}-\frac{\Psi\left(1+4\Psi+\Psi^2\right)}{(1-\Psi)^4}\right\}$$

$$M_3 = \alpha\left\{\frac{1+4q+q^2}{(1-q)^3}\right\}+\frac{(1-\alpha)}{[q-(1-\Phi)]}\left\{\frac{q\Phi\left(1+4q+q^2\right)}{p^3}-\frac{p\Psi\left(1+4\Psi+\Psi^2\right)}{\Phi^3}\right\}$$

The first three moments are give by

$$M_1 = \frac{\alpha}{(1-q)}+\frac{(1-\alpha)}{q-(1-\Phi)}\left\{\frac{q\Phi}{p}-\frac{p(1-\Phi)}{\Phi}\right\},$$

$$M_2 = \frac{\alpha(1+q)}{(1-q)^2}+\frac{(1-\alpha)}{q-(1-\Phi)}\left\{\frac{q\Phi(2-p)}{p^2}-\frac{p(1-\Phi)(2-\Phi)}{\Phi^2}\right\},\text{ and}$$

$$M_3 = \alpha\left\{\frac{1+4q+q^2}{(1-q)^3}\right\}+\frac{(1-\alpha)}{[q-(1-\Phi)]}\left\{\frac{q\Phi\left(1+4q+q^2\right)}{p^3}-\frac{p\Psi\left(1+4\Psi+\Psi^2\right)}{\Phi^3}\right\} \tag{5.25}$$

5.3 Modified Beta Geometric

5.3.1 The Distribution

This is a extension of the previous distribution that is taking care of the fact that some women conceive prior to the marriage and report to have conceived in the first month of marriage. Letting $(1-\beta)$ be the probability that the woman is pregnant before marriage, therefore

$$f(t \mid \theta) = \theta(1-\theta)^{t-1}$$

becomes;

$$f(t=1 \mid \theta) = (1-\beta) + \beta\left[\theta(1-\theta)^{1-1}\right] = (1-\beta) + \beta\theta \qquad (5.26)$$

and

$$f(t) = \int_0^1 f(t \mid \theta)\, g\theta d\theta$$

becomes;

$$f(t) = (1-\beta) + \beta\frac{B(\alpha+1,\beta)}{B(\alpha,\beta)} \text{ for } T=1 \qquad (5.27)$$

Proof.

$$f(t) = \int_0^1 [(1-\beta)+\beta\theta]\left[\frac{1}{B(\alpha,\beta)}\theta^{\alpha-1}(1-\theta)^{\beta-1}\right]d\theta$$

$$f(t) = (1-\beta)\int_0^1 \frac{1}{B(\alpha,\beta)}\theta^{\alpha-1}(1-\theta)^{\beta-1}d\theta + \beta\int_0^1 \theta\left[\frac{1}{B(\alpha,\beta)}\theta^{\alpha-1}(1-\theta)^{\beta-1}\right]d\theta$$

$$f(t) = (1-\beta)\,1 + \frac{\beta}{B(\alpha,\beta)}\int_0^1 \theta^{\alpha}(1-\theta)^{\beta-1}d\theta$$

$$f(t) = (1-\beta) + \frac{\beta}{B(\alpha,\beta)}B(\alpha+1,\beta)$$

■

If the pregnancy was not achieved in the first month, then;

$$f(t \mid \theta) = \theta(1-\theta)^{t-1}$$

becomes;

$$f(t \mid \theta) = \beta\theta(1-\theta)^{t-1}$$

and the unconditional distribution of T becomes

$$f(t) = \int_0^1 \beta\theta(1-\theta)^{t-1}\frac{1}{B(\alpha,\beta)}\theta^{\alpha-1}(1-\theta)^{\beta-1}d\theta$$

$$f(t) = \frac{\beta}{B(\alpha,\beta)} \int_0^1 \theta^{\alpha} (1-\theta)^{\beta+t-1} \, d\theta$$

$$f(t) = \beta \frac{B(\alpha+1, \beta+t-1)}{B(\alpha,\beta)} \tag{5.28}$$

Therefore

$$f(t) = \begin{Bmatrix} (1-\beta) + \frac{\beta}{B(\alpha,\beta)} B(\alpha+1,\beta) \ , t=1 \\ \beta \frac{B(\alpha+1,\beta+t-1)}{B(\alpha,\beta)} \ , t = 2,3,4, \ldots\ldots\ldots \end{Bmatrix} \tag{5.29}$$

End

Chapter 6

Conclusion

The application of stochastic models so far have been confined towards their validation to describe the data from different populations and demonstrate various estimation procedures to estimate different parameters and thereafter fitting them to data. Earlier works have been confined to mixture of geometric distribution with beta distribution since these were the only distributions considered to be in the [0,1] domain, but further research have shown more distribution within this [0,1] domain and distributions from mixtures of these distributions with geometric distributions has been constructed. However further work is required on estimating the parameters of these distribution and application to data of birth intervals to test their usefulness to this study. Conducting of National Family Health surveys in India and Demographic Health Surveys in other countries have certainly opened up new areas for exploring the uses of these models. There is need to validate many of these models in their original or modified form for different populations and prepare comparison of estimates of different parameters of reproduction. These models have also been applied to find out the extent of infertility among married women after each birth(Postpartum Amenorrhea)

Most of these models are applicable for analyzing reproduction process of a cohort of married women but none of them is suitable for studying period effects n the biosocial parameters affecting conception and birth intervals.

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