BINOMIAL MIXTURES WITH CONTINUOUS MIXING PRIORS

BY

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DECLARATION

I declare that this is my original work and has not been presented for an award of a degree in any other University.

Signature.. **Date** 30/07/2012..........

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This thesis is submitted for examination with my approval as the University supervisor:

Signature.. **Date** 30/07/2012..........

Prof. J.A.M Ottieno

DEDICATION

This work is dedicated to my father,

Mr. Naftali Ogal ,

for always believing in me.

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ABSTRACT

In this work, we consider a class of mixture distributions generated by randomizing the success parameter p of a Binomial distribution. We derive the density functions of the mixture distributions, and in some cases, give their simple properties, such as the mean and variance. We also derive the density functions of mixing distributions that may be new to the reader. This is important because these densities are then used in the mixing procedure. The reader can then follow through with the process and calculations therein.

We find that the Beta distribution is a most tractable mixer for the Binomial, and therefore dedicate a whole Chapter to various Beta generalizations in the unit interval, and their derived Binomial mixtures. Chapter three then looks at viable alternatives to the Beta in this regard.

The remaining Chapters are dedicated to various transformations that enable us to move beyond the unit interval in which the parameter p is restricted. A summary of our findings and further areas of research is given in Chapter eight.

CHAPTER 1

INTRODUCTION

1.1 Probability Distributions

One major area of statistics is Probability Distributions.

Let $f(x)$be a function of a random variable X . If

$$f(x) \geq 0 \quad and \quad \int_{-\infty}^{\infty} f(x)dx = 1 \tag{1.1}$$

then $f(x)$is called a probability density function of a continuous random variable X . If

$$0 \leq f(x) \leq 1 \quad and \quad \sum_{-\infty}^{\infty} f(x) = 1 \tag{1.2}$$

then $f(x)$is called a probability mass function of a discrete random variable X .

Various methods have been developed for constructing $f(x)$, such as:

- Power series based distributions
- Transformations based distributions
- Distributions based on mixtures
- Distributions based on recursive relations in probabilities
- Lagrangian distributions
- Distributions based on hazard functions of survival analysis
- Distributions emerging from stochastic processes
- Sum of independent random variables

1.2 Constructing Binomial distribution

1.2.1 Power series based distribution

Consider the binomial expansion of

$$(a+b)^n = \sum_{k=0}^{\infty} \binom{n}{k} a^k b^{n-k} \tag{1.3}$$

i. When $b = p$ and $a = q$ such that $p + q = 1$, then (1.3) becomes

$$1 = \sum_{k=0}^{n} \binom{n}{k} p^k q^{n-k} \tag{1.4}$$

for $k = 0,1, \dots \dots, n; p + q = 1$
This is a binomial distribution with parameters n and p.

ii. Put $a = 1$ and $b = \theta$
Then

$$(1+\theta)^n = \sum_{k=0}^{\infty} \binom{n}{k} \theta^k$$

Therefore

$$1 = \sum_{k=0}^{\infty} \binom{n}{k} \frac{\theta^k}{(1+\theta)^n}$$

And,

$$\begin{aligned} p_k &= \binom{n}{k} \frac{\theta^k}{(1+\theta)^n} \\ &= \binom{n}{k} \frac{\theta^k}{(1+\theta)^k (1+\theta)^{n-k}} \end{aligned}$$

$$p_k = \binom{n}{k} \frac{\theta^k}{(1+\theta)^k (1+\theta)^{n-k}} \quad , \quad k = 0,1, \dots, n \tag{1.5}$$

Which is a Binomial distribution with parameters n and $\frac{\theta}{1+\theta}$.

1.2.2 Binomial distributions based on mixtures

In (1.4) we have a Binomial distribution with parameters n and p, where $n = 1,2,3, \dots$, that is, n is a positive integer, and $0 < p < 1$.

Suppose n is varying, and p is fixed, then we have

$$p_{k|n} = Prob\,(X = k|\, N = n) = \binom{n}{k} p^k q^{n-k}$$

for $k = 1,2,\ldots,n$

Thus

$$p_k = \sum_{n=1}^{\infty} Prob\,(X = k|\, N = n) Prob(N = n)$$

Thus

$$p_k = \sum_{n=1}^{\infty} \binom{n}{k} p^k q^{n-k} p_n \tag{1.6}$$

1.2.2.1 Binomial-Binomial distribution

If

$$p_n = \binom{M}{n} \theta^n (1-\theta)^{M-n} \quad ; \quad n = 0,1,2,\ldots,M$$

Then

$$p_k = \sum_{n=0}^{M} \binom{n}{k} p^k q^{n-k} \binom{M}{n} \theta^n (1-\theta)^{M-n}$$

$$= \sum_{n=0}^{M} \binom{n}{k} \binom{M}{n} p^k q^{n-k} \theta^n (1-\theta)^{M-n}$$

$$= \sum_{n=0}^{M} \binom{n}{k} \binom{M}{n} (\theta p)^k (\theta q)^{n-k} (1-\theta)^{M-n}$$

$$= (\theta p)^k \sum_{n=0}^{M} \binom{n}{k} \binom{M}{n} (\theta)^{n-k} (1-p)^{n-k} (1-\theta)^{M-n}$$

$$= (\theta p)^k \sum_{n=0}^{M} \binom{n}{k} \binom{M}{n} \left(\theta(1-p)\right)^{n-k} (1-\theta)^{M-n}$$

$$= (\theta p)^k \sum_{n=0}^{M} \frac{n!}{k!\,(n-k)!} \frac{M!}{n!\,(M-n)!} \left(\theta(1-p)\right)^{n-k} (1-\theta)^{M-n}$$

$$= (\theta p)^k \sum_{n=0}^{M} \frac{1}{k!\,(n-k)!} \frac{M!}{(M-n)!} \big(\theta(1-p)\big)^{n-k} (1-\theta)^{M-n}$$

$$= (\theta p)^k \sum_{n=0}^{M} \frac{(M-k)!}{k!\,(M-k)!\,(n-k)!} \frac{M!}{(M-n)!} \big(\theta(1-p)\big)^{n-k} (1-\theta)^{M-n}$$

$$= (\theta p)^k \frac{M!}{k!\,(M-k)!} \sum_{n=0}^{M} \frac{(M-k)!}{(n-k)!\,(M-n)!} \big(\theta(1-p)\big)^{n-k} (1-\theta)^{M-n}$$

$$= (\theta p)^k \binom{M}{k} \sum_{n=0}^{M} \binom{M-k}{n-k} \big(\theta(1-p)\big)^{n-k} (1-\theta)^{M-n}$$

$$= (\theta p)^k \binom{M}{k} (\theta - \theta p + 1 - \theta)^{M-k}$$

$$= (\theta p)^k \binom{M}{k} (1 - \theta p)^{M-k}$$

Thus,

$$p_k = (\theta p)^k \binom{M}{k} (1-\theta p)^{M-k} \quad , \quad k = 0,1,\dots,M \tag{1.7}$$

Using the pgf technique, we have

$$G(s) = \sum_{k=0}^{M} p_k s^k$$

$$= \sum_{k=0}^{M} \sum_{n=0}^{M} \binom{n}{k} p^k (1-p)^{n-k} \binom{M}{n} \theta^n (1-\theta)^{M-n} s^k$$

$$= \sum_{k=0}^{M} (ps)^k \sum_{n=0}^{M} \binom{n}{k} (1-p)^{n-k} \binom{M}{n} \theta^{n-k} \theta^k (1-\theta)^{M-n}$$

$$= \sum_{k=0}^{M} (\theta ps)^k \sum_{n=0}^{M} \binom{n}{k} \big(\theta(1-p)\big)^{n-k} \binom{M}{n} (1-\theta)^{M-n}$$

$$= \sum_{k=0}^{M} (\theta ps)^k \sum_{n=0}^{M} \frac{n!}{k!\,(n-k)!} \big(\theta(1-p)\big)^{n-k} \frac{M!}{n!\,(M-n)!} (1-\theta)^{M-n}$$

$$= \sum_{k=0}^{M} (\theta ps)^k \sum_{n=0}^{M} \frac{M!\,(M-k)!}{k!\,(n-k)!\,(M-k)!\,(M-n)!} \big(\theta(1-p)\big)^{n-k} (1-\theta)^{M-n}$$

$$= \sum_{k=0}^{M} (\theta ps)^k \frac{M!}{(M-k)!\,k!} \sum_{n=0}^{M} \frac{(M-k)!}{(n-k)!\,(M-n)!} \big(\theta(1-p)\big)^{n-k} (1-\theta)^{M-n}$$

$$= \sum_{k=0}^{M} (\theta ps)^k \binom{M}{k} \sum_{n=0}^{M} \binom{M-k}{n-k} \big(\theta(1-p)\big)^{n-k} (1-\theta)^{M-n}$$

$$= \sum_{k=0}^{M} (\theta ps)^k \binom{M}{k} [\theta - \theta p + 1 - \theta]^{M-k}$$

$$= \sum_{k=0}^{M} (\theta ps)^k \binom{M}{k} [1 - \theta p]^{M-k}$$

$$G(s) = \binom{M}{k} [\theta ps + 1 - \theta p]^M$$

Hence,

$$G(s) = \binom{M}{k} [\theta ps + 1 - \theta p]^M \qquad (1.8)$$

Which is the pgf of a Binomial distribution with parameters θp and M . Therefore, the Binomial-Binomial distribution is a Binomial distribution.

1.2.2.2 Hypergeometric-Binomial distribution

Let

$$Prob(X = k|n, M, N) = \frac{\binom{M}{k}\binom{N-M}{n-k}}{\binom{N}{n}} \quad , \ k = 0,1,\dots n$$

And M be a random variable with probability function

$$Prob(M) = \binom{N}{M} p^M q^{N-M} \ , \ M = 0,1,\dots,N$$

Then

$$p_k = \sum_{M=0}^{N} \frac{\binom{M}{k}\binom{N-M}{n-k}}{\binom{N}{n}} \binom{N}{M} p^M q^{N-M}$$

$$= \sum_{M=0}^{N} \frac{M!\,(N-M)!\,n!\,(N-n)!\,N!\;p^M q^{N-M}}{k!\,(n-k)!\,(M-k)!\,N!\,(N-M-n+k)!\,(N-M)!\,M!}$$

$$= \sum_{M=0}^{N} \frac{n!\,(N-n)!\;p^M q^{N-M}}{k!\,(n-k)!\,(M-k)!\,(N-M-n+k)!}$$

$$= \frac{n!\,(N-n)!}{k!\,(n-k)!} \sum_{M=0}^{N} \frac{p^M q^{N-M}}{(M-k)!\,(N-M-n+k)!}$$

$$= \binom{n}{k}(N-n)! \sum_{M=0}^{N} \frac{p^M q^{N-M}}{(M-k)!\,(N-M-n+k)!}$$

Therefore

$$p_k = \binom{n}{k} \sum_{M=0}^{N} \binom{N-n}{M-k} p^M q^{N-M} \;, \qquad k = 0,1,\dots,n$$

$$= \binom{n}{k} \sum_{M=0}^{N} \binom{N-n}{M-k} p^{M-k} p^k q^{N-M}$$

$$= \binom{n}{k} p^k \sum_{M=0}^{N} \binom{N-n}{M-k} p^{M-k} q^{N-M}$$

$$= \binom{n}{k} p^k \sum_{M=0}^{N} \binom{N-n}{M-k} p^{M-k} q^{(N-n)-(M-k)}\, q^{n-k}$$

$$= \binom{n}{k} p^k q^{n-k} \sum_{M=0}^{N} \binom{N-n}{M-k} p^{M-k} q^{(N-n)-(M-k)}$$

Put $j = M - k \Rightarrow M = j + k$

Then

$$p_k = \binom{n}{k} p^k q^{n-k} \sum_{j+k=0}^{j+k=N} \binom{N-n}{j} p^j q^{(N-n)-j}$$

$$= \binom{n}{k} p^k q^{n-k} \sum_{j=-k}^{j=N-k} \binom{N-n}{j} p^j q^{(N-n)-j}$$

$$= \binom{n}{k} p^k q^{n-k} \sum_{j=0}^{j=N-k} \binom{N-n}{j} p^j q^{(N-n)-j}$$

For $\binom{N-n}{j}$ to hold, $N-k \leq N-n$

Thus $n \leq k$

and $k \geq n$.

Alternatively, $j \leq N-n = N-k$,thus, $k = n$

Therefore

$$p_k = \binom{n}{k} p^k q^{n-k}.1$$

and

$$p_k = \binom{n}{k} p^k q^{n-k} \quad , \quad k = 0,1,\dots,n \tag{1.9}$$

This is a Binomial distribution with parameters n and k .

1.2.3 Sums of Independent Random Variables

Let $S_N = X_1 + X_2 + \cdots + X_N$, where the $X_i's$ are iid random variables.

Let

$G(s) = E(S^x)$, the pgf of X

$F(s) = E(S^N)$, the pgf of N

$H(s) = E(S^{S_N})$, the pgf of S_N

Case 1: When N is fixed

Here,

$$H(s) = E(S^{S_N}) = (ES^{X_i})^N = \left(G(s)\right)^N$$

If X_i is $Bin(1,p)$ then $G(s) = q + ps$

Therefore,

$$H(s) = (q+ps)^N$$

S_N is $Bin(N,p)$, that is

$$h_j = Prob(S_N = j) = \binom{N}{j} p^j q^{N-j} \quad , \ j = 0,1,\dots,n$$

If $X_i \sim Bin(n,p)$, then

$$G(s) = (q+ps)^n$$

$$H(s) = (q+ps)^{nN}$$

And

$$S_N \sim Bin(nN, p)$$

That is

$$h_j = \binom{Nn}{j} p^j q^{Nn-j} \quad , \ j = 0,1,\dots,nN \tag{1.10}$$

Case 2: N is also a random variable independent of $X_i's$

$$H(s) = F_N\left(G_X(s)\right)$$

If $F_N(s) = (q+ps)^n$

Then

$$H(s) = \left(q + pG(s)\right)^n$$

If $X_i \sim Bin(1,\theta)$, then

$$G(s) = [(1-\theta)+\theta s]$$

Therefore

$$H(s) = (q + p[(1-\theta)+\theta s])^n$$
$$= (q + p - p\theta + p\theta s)^n$$
$$= (1 - p + p - p\theta + p\theta s)^n$$
$$= (1 - p\theta + p\theta s)^n$$

Thus,

$$H(s) = (1 - p\theta + p\theta s)^n \qquad (1.11)$$

This is the pgf of the Binomial, with parameters n and $p\theta$.

1.2.4 Conditional probability distribution given sum of two Poisson random variables

Let X and Y be two independent random variables from a Poisson distribution, i.e.

$X \sim Poi(\lambda)$ and $Y \sim Poi(\mu)$

Then

$$Prob(X = x \mid X + Y = x + y)$$
$$= \frac{Prob(X = x \mid X + Y = x + y)}{Prob(X + Y = x + y)}$$
$$= \frac{\left(\frac{\lambda^x e^{-\lambda}}{x!}\right)\left(\frac{\mu^y e^{-\mu}}{y!}\right)}{\frac{(\lambda+\mu)^{x+y} e^{-(\lambda+\mu)}}{(x+y)!}}$$
$$= \frac{\lambda^x \mu^y (x+y)!\ e^{-(\lambda+\mu)}}{x!\,y!\,(\lambda+\mu)^{x+y} e^{-(\lambda+\mu)}}$$
$$= \frac{(x+y)!\ \lambda^x \mu^y}{x!\,y!\,(\lambda+\mu)^{x+y}}$$
$$\binom{x+y}{x}\left(\frac{\lambda}{\lambda+\mu}\right)^x\left(\frac{\mu}{\lambda+\mu}\right)^y$$

Let $x + y = n$

Therefore

$$Prob(X = x \mid X + Y = n) = \binom{n}{x}\left(\frac{\lambda}{\lambda+\mu}\right)^{x}\left(\frac{\mu}{\lambda+\mu}\right)^{n-x}$$

for $x = 0,1,\ldots,n$

which is $B\left(n, \frac{\lambda}{\lambda+\mu}\right)$.

1.3 Mixtures

From Feller (1957) we can develop a class of probability distributions in the following manner.

Let F_X be a distribution function depending on the parameter θ, and let F be another distribution function. Then

$$F_Y(y) = \int_{-\infty}^{\infty} F_X(y|\theta)dF(\theta)$$

is also a distribution function.

Feller calls distributions generated in this manner, mixtures.

Mixtures can thus be generated by randomizing a parameter(s) in a parent distribution.

1.3.1 Binomial Mixtures

The Binomial distribution has two parameters n and p, either, or both of which may be randomized, to give a binomial mixture.

This project discusses cases in which the parameter p has a continuous mixing distribution with probability density $g(p)$ so that

$$f(x) = \int \binom{n}{x} p^{x}(1-p)^{n-x} g(p)dp$$

where $f(x)$ is a Binomial mixture distribution.

1.4 Literature Review

We now look at the various works that has been done in the literature on Binomial mixtures.

Skellam (1948) came up with an expression for the standard Beta-Binomial distribution. He studied various properties of this distribution, and described an iterative procedure for obtaining Maximum likelihood estimates of the parameters of the Beta mixing distribution using the digamma function.

Ishii and Hayakawa (1960) also derived the Beta-Binomial distribution, and further examined its various properties and extensions.

Bhattacharya (1968) used truncations of the Beta and Gamma distributions to derive mixing densities for the Binomial in the interval$[0, 1]$. He went on to give expressions for the mean and higher moments, including its characteristic function, and Bayes estimates of the mixing densities.

Grassia(1977) derived new distributions from the two parameter Gamma distribution by using log-inverse transformations of the type $t = \ln\left(\frac{1}{v}\right)$or $t = \ln\left(\frac{1}{1-u}\right)$. He noted that these distributions presented many of the characteristics of the Beta, and could therefore be used as mixers with the Binomial. He obtained the Binomial mixture densities, studied their shape properties and simple moments. He also considered their applicability in problems in which the inoculation approach was used to estimate bacteria or virus density in dilution assays with host variability to infection.

Bowman et al (1992) also derived a large number of new Binomial mixture distributions by assuming that the probability parameter p varied according to some laws, mostly derived from frullani integrals. They used the transformation $p = e^{-t}$and considered various densities for the transformed variables. They also gave graphical representations for some of the more significant distributions.

Alanko and Duffy(1996) developed a class of Binomial mixtures arising from transformations of the Binomial parameter p as $1 - e^{-\lambda}$where λ was treated as a random variable. They showed that this formulation provided closed forms for the marginal probabilities in the compound distribution if the Laplace transform of the mixing distribution could be written in a closed form. They gave examples of the derived compound Binomial distributions; simple properties, and parameter estimates from Moments and Maximum likelihood estimation. They further illustrate the use of these models by examples from consumption processes.

Karlis and Xekalaki (2006) studied the triangular distribution as a tractable alternative to the Beta. They noted its limited applicability, attributing this to its inflexibility in acquiring many shapes. They went ahead to examine its use as a prior to the Binomial, deriving the Binomial-Triangular distribution.

Gerstenkorn (2004) studied a mixture of the Binomial-as a special case of the Negative Binomial-with a four parameter generalized Beta distribution using a transformation for the parameter p . He obtained expressions for the factorial and crude moments.

Li Xiaohu et al (2011) studied the mixed Binomial model with probability of success having the Kumaraswamy distribution. They considered two models for this distribution, derived their density functions and other simple properties. They also discussed their stochastic orders and dependence properties. They employed the Kumaraswamy-Binomial models to real data sets on incidents of international terrorism and made comparisons with the Beta-Binomial model.

1.5 Statement of the problem and objectives of the study

From the literature, we find that the Binomial distribution can be expressed in two forms, namely:

i. By the direct use of p

$$\binom{n}{x} p^x (1-p)^{n-x} \;;\quad x = 0,1,\dots,n$$

ii. By the substitution $p = e^{-t}$we have

$$\binom{n}{x} e^{-tx} (1-e^{-t})^{n-x} \;;\quad x = 0,1,\dots,n$$

iii. By the substitution $p = 1 - e^{-t}$we have

$$\binom{n}{x} e^{-t(n-x)} (1-e^{-t})^{x} \;;\quad x = 0,1,\dots,n$$

The corresponding mixed Binomial distributions are:

i. $f(x) = \int_0^1 \binom{n}{x} p^x (1-p)^{n-x} g(p)dp$

ii. $f(x) = \int_0^\infty \binom{n}{x} e^{-tx} (1-e^{-t})^{n-x} g(p)dp$

iii. $f(x) = \int_0^\infty \binom{n}{x} e^{-t(n-x)} (1-e^{-t})^{x} g(p)dp$

- Since p is restricted in the domain$[0,1\,]$the choice of mixing distributions seems limited. The commonly used one being the classical beta.
- Most of the integrands between [0, 1] and$[0,\infty]$cannot be evaluated in closed forms.

Research Questions:

1. How can we generate tractable mixers for the Binomial, in the [0, 1] and $[0, \infty]$ domain?
2. What are other forms of expressing the Binomial mixtures?

Objectives

Main Objectives

1. To generate more mixers
2. To examine alternative forms of the Binomial mixed distributions.

Specific Objectives

1. To use the classical beta distribution and its generalizations in the interval [0, 1] as prior distributions
2. To use distributions beyond the classical beta, lying in the interval [0, 1], as mixers.
3. To construct new distributions in the [0, 1] domain, based on the log-inverse transformations $y = -\ln v$ or $y = -\ln(1-u)$ for $y > 0$.
4. To apply distributions in the $[0, \infty]$ domain as mixers for the binomial, when the success parameter is transformed, such that $p = e^{-t}$ or $p = 1 - e^{-t}$.

1.6 Applications

1.6.1 Academic pass rate

Consider X the number of subjects passed by every student at a university during an academic year. X depends on n , the number of subjects for which the student was registered (which is known a priori). For this type of data, the binomial distribution provides a poor fit since it would be unreasonable to consider the probability of success p to be constant for all students. Considering the students as having different probability of success according to their ability is more natural. Thus, binomial mixtures can be used to analyze the overdispersion of X and its relationship with n (see Karlis & Xekalaki, (2006))

1.6.2 Daily alcohol consumption

Binomial mixtures can be used to analyze sociological experiments about the number of days per week X , in which alcohol is consumed (see Alanko & Duffy, (1996)).

CHAPTER 2

BINOMIAL MIXTURES BASED ON CLASSICAL AND [0, 1] DOMAIN GENERALIZED BETA DISTRIBUTIONS

2.1 Introduction

The classical Beta pdf is given by

$$f(x) \quad = \quad \frac{x^{\alpha-1}(1-x)^{\beta-1}}{B(\alpha,\beta)}, \qquad 0 < x < 1; \alpha, \beta > 0$$

Special cases of this pdf are:

- Uniform distribution
- Power function distribution
- Arc sine distribution

The classical Beta density is a two parameter distribution. Libby and Novick () [] extended it to 3 parameters, and McDonald suggested a 2 and 3 parameter generalization for it.

The Beta distribution is a premier prior to the Binomial density. It allows many differently shaped mixing density functions, hence is a very flexible tool in modeling a variety of continuous distributions for the success probability p.

In deriving the Binomial mixtures, we use the following methods:

- Moments method

  This Method follows from the definition of a binomial mixture $f(x)$, in the domain [0, 1], so that

$$f(x) = \binom{n}{x} \int_0^1 p^x (1-p)^{n-x} g(p)dp$$

  Further evaluation gives us

$$f(x) = \binom{n}{x} \sum_{k=0}^{n-x} \binom{n-x}{k} (-1)^k \int_0^1 p^{x+k} g(p)dp$$

$$= \binom{n}{x} \sum_{j=x}^{n-x} \binom{n-x}{j-x} (-1)^{j-x} \int_0^1 p^j g(p)dp$$

$$f(x) = \sum_{j=x}^{n} \frac{n!}{x!\,(n-j)!\,(j-x)!} E(P^j) \quad (2.1)$$

for $j \geq x$ and 0 if $j < x$

where$E(P^j)$ is the jth moment of the mixing distribution (see Sivaganesan & Berger, (1993))

- Direct substitution and integration.
- Recursive relations.

2.2 Classical Beta-Binomial distribution

2.2.1 Classical Beta distribution

Construction

Let X_1 and X_2 be 2 stochastically independent random variables that have Gamma distributions and joint pdf

$$f(x_1, x_2) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} {x_1}^{\alpha-1} {x_2}^{\beta-1} e^{-x_1} e^{-x_2}$$

With$(0 < x_1 < \infty)$, $(0 < x_2 < \infty)$, $\alpha, \beta > 0$, and zero elsewhere.

Let $Y = X_1 + X_2$ and $P = \frac{X_1}{X_1+X_2}$

Then

$$g_1(p, y) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} (py)^{\alpha-1} [y(1-p)]^{\beta-1} e^{-py} e^{-y+py} |J|$$

Where $|J| = \begin{vmatrix} \frac{dx_1}{dy} & \frac{dx_1}{dp} \\ \frac{dx_2}{dy} & \frac{dx_2}{dp} \end{vmatrix} = \begin{vmatrix} p & y \\ 1-p & -y \end{vmatrix} = |-y| = y$

$$g_1(p, y) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} p^{\alpha-1} y^{\alpha+\beta-1} (1-p)^{\beta-1} e^{-y}$$

with $0 < y < \infty;\quad 0 < p < 1$

The marginal pdf of p is

$$g_2(p) = p^{\alpha-1}(1-p)^{\beta-1}\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\int_0^{\infty}\frac{y^{\alpha+\beta-1}e^{-y}}{\Gamma(\alpha+\beta)}dy$$

Thus

$$g_2(p) = \frac{p^{\alpha-1}(1-p)^{\beta-1}}{B(\alpha,\beta)}, \qquad 0 < p < 1; \alpha, \beta > 0 \tag{2.2}$$

$g_2(p)$Is called a Classical Beta distribution with parameters $\alpha$and β .

Properties

The jth moment of this distribution,$E(P^j)$ is

$$E(P^j) = \int_0^1 \frac{p^{j+\alpha-1}(1-p)^{\beta-1}}{B(\alpha,\beta)}dp$$

$$= \frac{B(j+\alpha,\beta)}{B(\alpha,\beta)}$$

Therefore

$$E(P^j) = \frac{(j+\alpha-1)!\ (\alpha+\beta-1)!}{(j+\alpha+\beta-1)!\ \ (\alpha-1)!} \tag{2.3}$$

Thus the mean is

$$E(P) = \frac{\alpha}{\alpha+\beta}$$

and the variance is

$$Var(P) = \left\{E(P^2) - \left(E(P)\right)^2\right\} = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$$

2.2.2 Classical Beta mixing distribution

The Binomial probability function being

$$B(x) = \binom{n}{x} p^x (1-p)^{n-x}, \qquad x = 0,1,\dots,n; 0 < p < 1$$

Then the Classical Beta-Binomial probability function becomes

$$f(x) = \int_0^1 \binom{n}{x} p^x (1-p)^{n-x} g(p) dp$$

where $g(p)$ is the pdf of the Classical Beta distribution.

Thus,

$$f(x) = \int_0^1 \binom{n}{x} p^x (1-p)^{n-x} \frac{p^{\alpha-1}(1-p)^{\beta-1}}{B(\alpha,\beta)} dp \qquad (2.4)$$

$$= \binom{n}{x} \frac{B(x+\alpha, n-x+\beta)}{B(\alpha,\beta)} \int_0^1 \frac{p^{x+\alpha-1}(1-p)^{n-x+\beta-1}}{B(x+\alpha, n-x+\beta)} dp$$

$$f(x) = \begin{cases} \binom{n}{x} \frac{B(x+\alpha, n-x+\beta)}{B(\alpha,\beta)}, & x = 0,1,\dots,n;\ \alpha,\beta > 0 \\ 0, & elsewhere \end{cases} \qquad (2.5)$$

From the Moments method, this distribution can also be written as

$$f(x) = \sum_{j=x}^{n} \frac{n!}{x!\,(n-j)!\,(j-x)!} E(P^j)$$

for $j \geq x$ and 0 if $j < x$

The jth moment of the Classical Beta distribution, from (2.3) is

$$E(P^j) = \frac{B(j+\alpha,\beta)}{B(\alpha,\beta)}$$

Thus

$$f(x) = \begin{cases} \sum_{j=x}^{n} \frac{n!\ B(j+\alpha,\beta)}{x!\ (n-j)!\ (j-x)!\, B(\alpha,\beta)} & , x = 0,1,\ldots,n\,;\alpha,\beta > 0 \\ 0\ , & elsewhere \end{cases} \qquad (2.6)$$

From(2.4), a recursive expression for $f(x)$can be determined as follows:

$$f(x) = \int_0^1 \binom{n}{x} p^x (1-p)^{n-x} \frac{p^{\alpha-1}(1-p)^{\beta-1}}{B(\alpha,\beta)} dp$$

Then

$$f(x) = \binom{n}{x} \frac{1}{B(\alpha,\beta)} \int_0^1 p^{x+\alpha-1}(1-p)^{n-x+\beta-1} dp$$

$$\frac{f(x)B(\alpha,\beta)}{\binom{n}{x}} = \int_0^1 p^{x+\alpha-1}(1-p)^{n-x+\beta-1} dp$$

Let

$$I_x = \frac{f(x)B(\alpha,\beta)}{\binom{n}{x}} = \int_0^1 p^{x+\alpha-1}(1-p)^{n-x+\beta-1} dp$$

Using integration by parts to evaluate the integral on the RHS, we put

$$u = (1-p)^{n-x+\beta-1} \Rightarrow du = -1(n-x+\beta-1)(1-p)^{n-x+\beta-2} dp$$

And

$$dv = p^{x+\alpha-1} \Rightarrow v = \int p^{x+\alpha-1} dp = \frac{p^{x+\alpha}}{x+\alpha}$$

Therefore

$$I_x = \left|(1-p)^{n-x+\beta-1}\frac{p^{x+\alpha}}{x+\alpha}\right|_0^1 + \frac{n-x+\beta-1}{x+\alpha}\int_0^1 p^{x+\alpha}(1-p)^{n-x+\beta-2}\,dp$$

$$I_x = \frac{n-x+\beta-1}{x+\alpha} I_{x+1}$$

$$\frac{f(x)B(\alpha,\beta)}{\binom{n}{x}} = \left(\frac{n-x+\beta-1}{x+\alpha}\right)\frac{f(x+1)B(\alpha,\beta)}{\binom{n}{x+1}}$$

$$f(x) = \left(\frac{n-x+\beta-1}{x+\alpha}\right)\frac{n!\,(n-x-1)!\,(x+1)!}{x!\,n!\,(n-x)!} f(x+1)$$

$$f(x+1) = \left(\frac{x+\alpha}{n-x+\beta-1}\right)\left(\frac{n-x}{x+1}\right) f(x) \qquad (2.7)$$

Properties of the Classical Beta-Binomial distribution

The mean is

$$E(X) = nE(P) = n\left(\frac{\alpha}{\alpha+\beta}\right)$$

and the variance, is

$$Var(X) = nE(P) - nE(P^2) + n^2Var(P)$$

$$= n\left(\frac{\alpha}{\alpha+\beta}\right) - n\frac{\alpha(\alpha+1)}{(\alpha+\beta)(\alpha+\beta+1)} + \frac{n^2\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$$

$$= \frac{n\alpha\beta(\alpha+\beta+n)}{(\alpha+\beta+1)(\alpha+\beta)^2}$$

2.3 McDonald's Generalized Beta-Binomial distribution

2.3.1 McDonald's Generalized Beta distribution

Construction

Given a Classical Beta distribution with pdf

$$g(x) = \frac{x^{a-1}(1-x)^{b-1}}{B(a,b)}$$

for $0 < x < 1; a, b > 0$

Let $X = Y^P$

Then

$$g(y) = \frac{(y^p)^{a-1}(1-y^p)^{b-1}}{B(a,b)}|J|$$

where $|J| = \left|\frac{dx}{dy}\right| = py^{p-1}$

Therefore

$$g(y) = \frac{p\,y^{ap-1}(1-y^p)^{b-1}}{B(a,b)}, \quad 0 < y < 1; a, b, p > 0 \tag{2.8}$$

This is the McDonald's Generalized Beta pdf, with parameters a, b and p

(See Nadarajah & Kotz(2007))

The jth moment of McDonald's Generalized Beta distribution is

$$E(Y^j) = \int_0^1 \frac{y^{j+ap-1}p(1-y^p)^{b-1}\,dy}{B(a,b)}$$

Let $z = y^p$

Then

$$E(Y^j) = \int_0^1 \frac{z^{\frac{1}{p}(j+ap-1)}p(1-z)^{b-1}z^{\frac{1}{p}-1}\,dz}{p\,B(a,b)}$$

$$= \int_0^1 \frac{z^{\frac{1}{p}(j+ap-p)}(1-z)^{b-1}}{B(a,b)}dz$$

$$E(Y^j) = \frac{B\left(\frac{j+ap}{p}, b\right)}{B(a,b)} \tag{2.9}$$

2.3.2 McDonald's Generalized Beta mixing distribution

The Binomial-McDonald's Generalized Beta distribution is given by

$$f(x) = \int_0^1 \binom{n}{x} y^x (1-y)^{n-x} g(y)dy$$

where$g(p)$ is the pdf of the McDonald's Generalized Beta distribution.

Thus,

$$f(x) = \frac{p}{B(a,b)} \int_0^1 \binom{n}{x} y^x (1-y)^{n-x} y^{ap-1} (1-y^p)^{b-1} dy$$

$$= \frac{p}{B(a,b)} \int_0^1 \binom{n}{x} y^{x+ap-1} (1-y)^{n-x} (1-y^p)^{b-1} dy$$

$$= \binom{n}{x} \frac{p}{B(a,b)} \sum_{k=0}^{b-1} \binom{b-1}{k} (-1)^k \int_0^1 y^{x+ap+pk-1} (1-y)^{n-x} dy$$

$$f(x) = \binom{n}{x} \frac{p}{B(a,b)} \sum_{k=0}^{b-1} \binom{b-1}{k} (-1)^k B(x+ap+pk, n-x+1) \tag{2.10}$$

for $x = 0,1, \dots\dots, n\,; a, b, p > 0$

From the Moments method, this distribution can also be written as

$$f(x) = \sum_{j=x}^{n} \frac{n!}{x!\,(n-j)!\,(j-x)!} E(P^j)$$

for $j \geq x$ and 0 if $j < x$

The jthmoment of McDonald'sGB distribution from (2.9) is

$$E(P^j) = \frac{B\left(\frac{j+ap}{p}, b\right)}{B(a,b)}$$

Thus

$$f(x) = \frac{1}{B(a,b)} \sum_{j=x}^{n} \frac{n!\; B\left(\frac{j+ap}{p}, b\right)}{x!\; (n-j)!\; (j-x)!} \tag{2.11}$$

for $x = 0,1, \dots, n\,; a, b, p > 0$

2.4 Libby and Novick's Generalized Beta-Binomial distribution

2.4.1 Libby and Novick's Generalized Beta distribution

This pdf due to Libby & Novick is given by

$$g(p) = \frac{c^a p^{a-1}(1-p)^{b-1}}{B(a,b)\{1-(1-c)p\}^{a+b}} \tag{2.12}$$

for $0 < p < 1\,; a, b, c > 0$

(seeLibby & Novick, (1982)).

The jth moment about the origin of this distribution is

$$E(P^j) = \frac{c^a}{B(a,b)} \int_0^1 \frac{p^{j+a-1}(1-p)^{b-1}}{\{1-(1-c)p\}^{a+b}}\, dp$$

$$= \frac{c^a B(j+a,b)}{B(a,b)} \int_0^1 \frac{p^{j+a-1}(1-p)^{b-1}}{B(j+a,b)\{1-(1-c)p\}^{a+b}}\, dp$$

But

$$2F1(a,\gamma;a+b;z) = \int_0^1 \frac{t^{a-1}(1-t)^{b-1}}{B(a,b)\{1-zt\}^{\gamma}}dt$$

Thus

$$E(P^j) = \frac{c^a B(j+a,b)}{B(a,b)} 2F1(j+a,a+b;b+j+a;1-c) \tag{2.13}$$

2.4.2 Libby and Novick's Generalized Beta mixing distribution

The Binomial-Libby and Novick's Generalized Beta distribution is given by

$$f(x) = \int_0^1 \binom{n}{x} p^x(1-p)^{n-x} g(p)dp$$

where$g(p)$ is the pdf of the Libby and Novick GB distribution.

Thus

$$f(x) = \frac{c^a}{B(a,b)}\binom{n}{x}\int_0^1 p^x(1-p)^{n-x}\frac{p^{a-1}(1-p)^{b-1}}{\{1-(1-c)p\}^{a+b}}dp$$

$$= \frac{c^a}{B(a,b)}\binom{n}{x}\int_0^1 \frac{p^{x+a-1}(1-p)^{n-x+b-1}}{\{1-(1-c)p\}^{a+b}}dp$$

$$= \frac{c^a B(x+a,n-x+b)}{B(a,b)}\binom{n}{x}\int_0^1 \frac{p^{x+a-1}(1-p)^{n-x+b-1}}{B(x+a,n-x+b)\{1-(1-c)p\}^{a+b}}dp$$

But

$$2F1(a,\gamma;a+b;z) = \int_0^1 \frac{t^{a-1}(1-t)^{b-1}}{B(a,b)\{1-zt\}^{\gamma}}dt$$

Thus

$$f(x) = \binom{n}{x}\frac{c^a B(x+a, n-x+b)}{B(a,b)} 2F1(x+a, a+b; n+a+b; 1-c) \tag{2.14}$$

for $x = 0, 1, \dots, n\,; a, b, c > 0$

From the Moments method, this distribution can also be written as

$$f(x) = \sum_{j=x}^{n} \frac{n!}{x!\,(n-j)!\,(j-x)!} E(P^j)$$

for $j \geq x$ and 0 if $j < x$

The jthmoment of Libby and Novick's GB distribution from (2.13)is

$$E(P^j) = \frac{c^a B(j+a, b)}{B(a,b)} 2F1(j+a, a+b; b+j+a; 1-c)$$

Thus

$$f(x) = \frac{c^a}{B(a,b)} \sum_{j=x}^{n} \frac{n!\,B(j+a,b)}{x!\,(n-j)!\,(j-x)!} 2F1(j+a, a+b; b+j+a; 1-c) \tag{2.15}$$

for $x = 0, 1, \dots, n\,; a, b, c > 0$

2.5 Gauss Hypergeometric-Binomial distribution

2.5.1 Gauss Hypergeometric distribution

Construction

Given the Gauss Hypergeometric function

$$2F1(a, \gamma; a+b; -z) = \int_0^1 \frac{t^{a-1}(1-t)^{b-1}}{B(a,b)\{1+zt\}^{\gamma}} dt$$

for $0 < t < 1\,; a, b > 0\ ; -\infty < \gamma < \infty$

Dividing both sides by $2F1(a,\gamma;a+b;-z)$we have

$$\int_0^1 \frac{t^{a-1}(1-t)^{b-1}}{B(a,b)\{1+zt\}^{\gamma}2F1(a,\gamma;a+b;-z)}dt = 1$$

Thus, the pdf of the Gauss Hypergeometric (GH) distribution is

$$g(t) = \frac{t^{a-1}(1-t)^{b-1}}{B(a,b)\{1+zt\}^{\gamma}2F1(a,\gamma;a+b;-z)} \tag{2.16}$$

for $0 < t < 1\,; a, b > 0\ ;\ -\infty < \gamma < \infty$

(seeArmero & Bayarri, (1994)).

The jth moment of the GH distribution is

$$E(T^j) = \frac{1}{B(a,b)2F1(a,\gamma;a+b;-z)}\int_0^1 \frac{t^{j+a-1}(1-t)^{b-1}\,dt}{\{1+zt\}^{\gamma}}$$

$$= \frac{B(j+a,b\,)}{B(a,b)2F1(a,\gamma;a+b;-z)}\int_0^1 \frac{t^{j+a-1}(1-t)^{b-1}\,dt}{B(j+a,b\,)\{1+zt\}^{\gamma}}$$

Thus

$$E(T^j) = \frac{B(j+a,b\,)}{B(a,b)2F1(a,\gamma;a+b;-z)}2F1(j+a,\gamma;b+j+a;\ -z) \tag{2.17}$$

2.5.2 Gauss Hypergeometric (GH) mixing distribution

The Binomial-GH distribution is given by

$$f(x) = \int_0^1 \binom{n}{x} p^x(1-p)^{n-x}g(p)dp$$

where$g(p)$ is the pdf of the GH distribution.

Thus,

$$f(x) = \frac{1}{B(a,b)2F1(a,\gamma;a+b;-z)}\binom{n}{x}\int_0^1 p^x(1-p)^{n-x}\frac{p^{a-1}(1-p)^{b-1}}{\{1+zp\}^\gamma}dp$$

$$= \frac{B(x+a,n-x+b)}{B(a,b)2F1(a,\gamma;a+b;-z)}\binom{n}{x}\int_0^1 \frac{p^{x+a-1}(1-p)^{n-x+b-1}}{B(x+a,n-x+b)\{1+zp\}^\gamma}dp$$

$$f(x) = \binom{n}{x}\frac{B(x+a,n-x+b)}{B(a,b)2F1(a,\gamma;a+b;-z)}2F1(x+a,\gamma;n+b+a;\ -z) \quad (2.18)$$

for $x = 0, 1, \dots, n\,; a, b > 0\ ;\ -\infty < \gamma < \infty$

From the **Moments** method, this distribution can also be written as

$$f(x) = \sum_{j=x}^{n}\frac{n!}{x!\ (n-j)!\ (j-x)!}E(P^j)$$

for $j \geq x$ and 0 if $j < x$

The jthmoment of the GH distribution from (2.17) is

$$E(P^j) = \frac{B(j+a,b)}{B(a,b)2F1(a,\gamma;a+b;-z)}2F1(j+a,\gamma;b+j+a;\ -z)$$

Thus

$$f(x) = \frac{1}{B(a,b)2F1(a,\gamma;a+b;-z)}\sum_{j=x}^{n}\frac{n!\ B(j+a,b)}{x!\ (n-j)!\ (j-x)!}2F1(j+a,\gamma;b+j+a;\ -z) \quad (2.19)$$

for $x = 0, 1, \dots, n\,; a, b > 0\ ;\ -\infty < \gamma < \infty$.

2.6 Confluent Hypergeometric (CH)-Binomial distribution

2.6.1 Confluent Hypergeometric (CH) distribution

Given the Confluent Hypergeometric function

$$1F1(a, a+b; -x) = \int_0^1 \frac{t^{a-1}(1-t)^{b-1}e^{-tx}\, dt}{B(a,b)}$$

Dividing both sides by $1F1(a, a+b; -x)$we have

$$\int_0^1 \frac{t^{a-1}(1-t)^{b-1}e^{-tx}\, dt}{B(a,b)1F1(a, a+b; -x)} = 1$$

Thus the pdf of the Confluent Hypergeometric distribution is

$$g(t) = \frac{t^{a-1}(1-t)^{b-1}e^{-tx}}{B(a,b)1F1(a, a+b; -x)} \tag{2.20}$$

for $0 < t < 1\,; a, b > 0\ ;\ -\infty < x < \infty$.

(seeNadarajah & Kotz, (2007)).

The jth moment of the CH distribution is given by

$$E(T^j) = \frac{B(a+j,b)}{B(a,b)1F1(a, a+b; -x)} \int_0^1 \frac{t^{a+j-1}(1-t)^{b-1}e^{-tx}}{B(a+j,b)} dt$$

But

$$1F1(a, a+b; -x) = \int_0^1 \frac{y^{a-1}(1-y)^{b-1}e^{-yx}\, dy}{B(a,b)}$$

Thus

$$E(T^j) = \frac{B(a+j,b)}{B(a,b)1F1(a, a+b; -x)} 1F1(a+j, b+a+j; -x\,) \tag{2.21}$$

2.6.2 Confluent Hypergeometric (CH) mixing distribution

The Binomial-CH distribution is given by

$$f(x) = \int_0^1 \binom{n}{x} p^x (1-p)^{n-x} g(p) dp$$

where$g(p)$ is the pdf of the CH distribution.

Thus,

$$f(x) = \binom{n}{x} \frac{B(a+x, n-x+b)}{B(a,b)1F1(a, a+b; -x)} \int_0^1 \frac{p^{a+x-1}(1-p)^{n-x+b-1}e^{-px}}{B(a+x, n-x+b)} dp$$

$$f(x) = \binom{n}{x} \frac{B(a+x, n-x+b)}{B(a,b)1F1(a, a+b; -x)} 1F1(a+x, n+b+a; -x) \tag{2.22}$$

for $x = 0, 1, \dots, n;\ a, b > 0\ ;\ -\infty < x < \infty$.

From the Moments method, this distribution can also be written as

$$f(x) = \sum_{j=x}^{n} \frac{n!}{x!\ (n-j)!\ (j-x)!} E(P^j)$$

for $j \geq x$ and 0 if $j < x$

The jthmoment of the CH distribution from (2.21)is

$$E(P^j) = \frac{B(a+j, b)}{B(a,b)1F1(a, a+b; -x)} 1F1(a+j, b+a+j; -x)$$

Thus

$$f(x) = \frac{1}{B(a,b)1F1(a, a+b; -x)} \sum_{j=x}^{n} \frac{n!\ B(a+j, b)1F1(a+j, b+a+j; -x)}{x!\ (n-j)!\ (j-x)!} \tag{2.23}$$

for $x = 0, 1, \dots, n;\ a, b > 0\ ;\ -\infty < x < \infty$.

2.7 Binomial-Uniform distribution

2.7.1 Uniform distribution

Construction

Given a random variable P on $[0,1]$ that is Beta (α, β) distributed with pdf given in (2.2.1) as

$$g(p) = \frac{p^{\alpha-1}(1-p)^{\beta-1}}{B(\alpha,\beta)}, \qquad 0 < p < 1; \alpha, \beta > 0$$

Let $\alpha = \beta = 1$

Then we have the Uniform density $[0,1]$ given by,

$$g(p) = \begin{cases} 1, & 0 < p < 1 \\ 0, & elsewhere \end{cases}$$

Properties

The moment of order j about the origin of $g(p)$ is

$$E(P^j) = \int_0^1 p^j \, dp$$

$$= \left[\frac{p^{j+1}}{j+1}\right]_0^1$$

$$E(P^j) = \frac{1}{j+1} \tag{2.24}$$

Thus the mean is

$$E(P) = \frac{1}{2}$$

and the variance is

$$Var(P) = \{E(P^2) - (E(P))^2\} = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$$

2.7.2 Uniform mixing distribution

The Binomial-Uniform distribution is given by

$$f(x) = \int_0^1 \binom{n}{x} p^x (1-p)^{n-x} g(p) dp$$

where$g(p)$ is the pdf of the Uniform distribution.

Thus,

$$f(x) = \int_0^1 \binom{n}{x} p^x (1-p)^{n-x} dp \qquad (2.25)$$

$$= \binom{n}{x} B(x+1, n-x+1) \int_0^1 \frac{p^x (1-p)^{n-x}}{B(x+1, n-x+1)} dp$$

$$= \frac{n! \;\; x! \;\; (n-x)!}{(n-x)!\, x! \;\; (n+1)!}$$

$$= \frac{1}{(n+1)}$$

$$f(x) = \begin{cases} \dfrac{1}{(n+1)}, & x = 0,1,\dots,n \\ 0\,, & elsewhere \end{cases} \qquad (2.26)$$

$f(x)$is a discrete type of uniform distribution, also called the discrete rectangular distribution (See Johnson, Kotz and Kemp (1992) pp. 272-274.).

From the Moments method, this distribution can also be written as

$$f(x) = \sum_{j=x}^{n} \frac{n!}{x! \; (n-j)! \; (j-x)!} E(P^j)$$

for $j \geq x$ and 0 if $j < x$

The jthmoment of the Uniform distribution from (2.24) is

$$E(P^j) = \frac{1}{j+1}$$

Thus

$$f(x) = \sum_{j=x}^{n} \frac{n!}{x!\,(n-j)!\,(j-x)!\,(j+1)} \qquad (2.27)$$

with $x = 0,1,\dots,n$; and zero elsewhere.

From (2.25) a recursive expression for $f(x)$ can be determined as follows:

$$f(x) = \int_0^1 \binom{n}{x} p^x (1-p)^{n-x} dp$$

Then

$$f(x) = \binom{n}{x} \int_0^1 p^x (1-p)^{n-x} dp$$

$$\frac{f(x)}{\binom{n}{x}} = \int_0^1 p^x (1-p)^{n-x} dp$$

Let

$$I_x = \frac{f(x)}{\binom{n}{x}} = \int_0^1 p^x (1-p)^{n-x} dp$$

Using integration by parts to evaluate the integral on the RHS, we put

$$u = (1-p)^{n-x} \Rightarrow du = -1(n-x)(1-p)^{n-x-1} dp$$

And

$$dv = p^x \Rightarrow v = \int p^x dp = \frac{p^{x+1}}{x+1}$$

Therefore

$$I_x = \left|(1-p)^{n-x}\frac{p^{x+1}}{x+1}\right|_0^1 + \frac{n-x}{x+1}\int_0^1 p^{x+1}(1-p)^{n-x-1}\,dp$$

$$I_x = \frac{n-x}{x+1}I_{x+1}$$

$$\frac{f(x)}{\binom{n}{x}} = \left(\frac{n-x}{x+1}\right)\frac{f(x+1)}{\binom{n}{x+1}}$$

$$f(x) = \left(\frac{n-x}{x+1}\right)\frac{n!\,(n-x-1)!\,(x+1)!}{x!\,n!\,(n-x)!}f(x+1)$$

$$f(x) = \left(\frac{n-x}{x+1}\right)\frac{(x+1)}{(n-x)}f(x+1)$$

$$f(x+1) = f(x) \tag{2.28}$$

Properties of the Binomial-Uniform distribution

The mean is

$$E(X) = nE(p) = \frac{n}{2}$$

and the variance, is

$$Var(X) = nE(P) - nE(P^2) + n^2Var(P)$$

$$= \frac{n}{2} - \frac{n}{3} + \frac{n^2}{12}$$

$$= \frac{n^2+2n}{12}$$

2.8 Binomial-Power function distribution

2.8.1 Power function distribution

Construction

Consider the Beta distribution

$$g(p) = \frac{p^{\alpha-1}(1-p)^{\beta-1}}{B(\alpha,\beta)}\,, \qquad 0<p<1;\ \alpha,\beta>0$$

and zero elsewhere.

Letting $\beta = 1$, we have

$$g(p) = \frac{p^{\alpha-1}(\alpha)!}{(\alpha-1)!}$$

$$g(p) = \begin{cases} \alpha p^{\alpha-1}, & 0 < p < 1;\ \alpha > 0 \\ 0\ , & elsewhere \end{cases} \qquad (2.29)$$

This pdf is that of the power function distribution, with parameter α.

Properties

The moment of order j about the origin of the Power function distribution is given by

$$E(P^j) = \int_0^1 p^j \alpha p^{\alpha-1}\, dp$$

$$= \alpha \int_0^1 p^{j+\alpha-1}\, dp$$

$$= \alpha \left[\frac{p^{j+\alpha}}{j+\alpha}\right]_0^1$$

$$E(P^j) = \frac{\alpha}{\alpha + j} \qquad (2.30)$$

The mean is

$$E(P) = \frac{\alpha}{\alpha+1}$$

and the variance is

$$Var(P) = \{E(P^2) - (E(P))^2\} = \frac{\alpha}{\alpha+2} - \frac{\alpha^2}{(\alpha+1)^2} = \frac{\alpha}{(\alpha+2)(\alpha+1)^2}$$

2.8.2 Power function mixing density

The Binomial-Power function probability function is given by

$$f(x) = \int_0^1 \binom{n}{x} p^x (1-p)^{n-x} g(p)dp$$

Where $g(p)$ is the pdf of the Power function distribution.

Therefore,

$$f(x) = \alpha \int_0^1 \binom{n}{x} p^x (1-p)^{n-x} p^{\alpha-1} dp \qquad (2.31)$$

$$= \alpha \int_0^1 \binom{n}{x} p^{x+\alpha-1} (1-p)^{n-x} dp$$

$$f(x) = \alpha \binom{n}{x} B(x+\alpha, n-x+1) \qquad (2.32)$$

with$x = 0,1, \ldots, n$; $\alpha > 0$ and zero elsewhere.

This is the density function of the binomial-power function distribution.

From the Moments method, the Binomial-Power function distribution can also be written as,

$$f(x) = \sum_{j=x}^{n} \frac{n!\,(-1)^{j-x}}{x!\,(j-x)!\,(n-j)!} E(P^j) \ , \qquad x = 0,1, \ldots, n$$

for $j \geq x$ and 0 if $j < x$

The Jth moment,$E(P^j)$ of the Power function distribution is$\frac{\alpha}{\alpha+j}$ from (2.30).

Thus,

$$f(x) = \sum_{j=x}^{n} \frac{n!\ (-1)^{j-x}}{x!\ (j-x)!\,(n-j)!} \frac{\alpha}{\alpha + j} \tag{2.33}$$

With $x = 0, 1, \dots, n\ ;\ \alpha > 0$ and $0,\ elsewhere$

From (2.31) a recursive expression for $f(x)$is

$$f(x) = \alpha \int_0^1 \binom{n}{x} p^x (1-p)^{n-x} p^{\alpha-1} dp$$

Then

$$f(x) = \binom{n}{x} \alpha \int_0^1 p^{x+\alpha-1} (1-p)^{n-x} dp$$

$$\frac{f(x)}{\binom{n}{x}\alpha} = \int_0^1 p^{x+\alpha-1} (1-p)^{n-x} dp$$

Let

$$I_x = \frac{f(x)}{\binom{n}{x}\alpha} = \int_0^1 p^{x+\alpha-1} (1-p)^{n-x} dp$$

Using integration by parts to evaluate the integral on the RHS, we put

$$u = (1-p)^{n-x} \Rightarrow du = -1(n-x)(1-p)^{n-x-1} dp$$

And

$$dv = p^{x+\alpha-1} \Rightarrow v = \int p^{x+\alpha-1} dp = \frac{p^{x+\alpha}}{x+\alpha}$$

Therefore

$$I_x = \left[(1-p)^{n-x} \frac{p^{x+\alpha}}{x+\alpha} \right]_0^1 + \frac{n-x}{x+\alpha} \int_0^1 p^{x+\alpha} (1-p)^{n-x-1} dp$$

$$I_x = \frac{n-x}{x+\alpha} I_{x+1}$$

$$\frac{f(x)}{\binom{n}{x}\alpha} = \left(\frac{n-x}{x+\alpha}\right)\frac{f(x+1)}{\binom{n}{x+1}\alpha}$$

$$f(x) = \left(\frac{n-x}{x+\alpha}\right)\frac{n!\,(n-x-1)!\,(x+1)!\,\alpha}{x!\,n!\,(n-x)!\,\alpha}f(x+1)$$

$$f(x+1) = \frac{(x+\alpha)}{(x+1)}f(x) \tag{2.34}$$

Properties of the Binomial-Power function distribution

The mean is

$$E(X) = nE(P) = n\left(\frac{\alpha}{\alpha+1}\right)$$

and the variance, is

$$Var(X) = nE(P) - nE(P^2) + n^2Var(P)$$

$$n\left(\frac{\alpha}{\alpha+1}\right) - n\left(\frac{\alpha}{\alpha+2}\right) + n^2\left(\frac{\alpha}{(\alpha+2)(\alpha+1)^2}\right)$$

$$= \frac{n\alpha}{(\alpha+1)(\alpha+2)} + \frac{n^2\alpha}{(\alpha+2)(\alpha+1)^2}$$

$$= \frac{n\alpha(\alpha+n+1)}{(\alpha+2)(\alpha+1)^2}$$

2.9 Binomial-Truncated Beta distribution

2.9.1 Truncated Beta distribution

Construction

Consider a two-sided truncated Beta function given by

$$\int_{\alpha}^{\beta} p^{a-1}(1-p)^{b-1}dp$$

Where $0 < \alpha < p < \beta < 1; a, b > 0$

This function can be expressed in terms of incomplete Beta functions as

$$\int_{\alpha}^{\beta} p^{a-1}(1-p)^{b-1}dp = \int_{0}^{\beta} p^{a-1}(1-p)^{b-1}dp - \int_{0}^{\alpha} p^{a-1}(1-p)^{b-1}dp$$

$$= B_{\beta}(a,b) - B_{\alpha}(a,b)$$

Thus,

$$\int_{\alpha}^{\beta} \frac{p^{a-1}(1-p)^{b-1}}{B_{\beta}(a,b) - B_{\alpha}(a,b)}dp = 1$$

This gives us the Truncated-Beta distribution, $g(p)$with parameters α, β, a, and b.

$$g(p) = \begin{cases} \dfrac{p^{a-1}(1-p)^{b-1}}{B_{\beta}(a,b) - B_{\alpha}(a,b)}, & 0 < \alpha < p < \beta < 1\,;\ a,b > 0 \\ 0, & elsewhere \end{cases} \tag{2.35}$$

Properties

The moment of order j about the origin of the Truncated-Beta distribution is given by

$$E(P^j) = \int_{\alpha}^{\beta} \frac{p^j p^{a-1}(1-p)^{b-1}}{B_{\beta}(a,b) - B_{\alpha}(a,b)}dp$$

$$= \frac{B_{\beta}(j+a,b) - B_{\alpha}(j+a,b)}{B_{\beta}(a,b) - B_{\alpha}(a,b)} \tag{2.36}$$

2.9.2 Truncated-Beta mixing distribution

The Binomial-Truncated Beta distribution is given by

$$f(x) = \binom{n}{x} \int_{\alpha}^{\beta} p^{x}(1-p)^{n-x} g(p)dp$$

Where $g(p)$ is the Truncated-Beta distribution.

Thus,

$$f(x) = \binom{n}{x} \int_{\alpha}^{\beta} \frac{p^{x+a-1}(1-p)^{n+b-1-x}}{B_\beta(a,b) - B_\alpha(a,b)} \, dp$$

$$= \binom{n}{x} \frac{B_\beta(x+a, n-x+b) - B_\alpha(x+a, n-x+b)}{B_\beta(a,b) - B_\alpha(a,b)}$$

$$f(x) = \begin{cases} \binom{n}{x} \dfrac{B_\beta(x+a, n-x+b) - B_\alpha(x+a, n-x+b)}{B_\beta(a,b) - B_\alpha(a,b)}, & x = 0,1,\dots,n \,;\, a,b > 0 \,;\, 0 < \alpha < \beta < 1 \\ 0\,, & elsewhere \end{cases} \quad (2.37)$$

asobtained by Bhattacharya, (1968).

From the Moments method, this probability distribution can also be written as,

$$f(x) = \sum_{j=x}^{n} \frac{n!\ (-1)^{j-x}}{x!\ (j-x)!\,(n-j)!} E(P^j) \ , \quad x = 0,1,\dots,n$$

for $j \geq x$ and 0 if $j < x$

The Jth moment, $E(P^j)$ of the Truncated-Beta distribution is

$\frac{B_\beta(j+a,b) - B_\alpha(j+a,b)}{B_\beta(a,b) - B_\alpha(a,b)}$from.(2.36).

Thus,

$$f(x) = \sum_{j=x}^{n} \frac{n!\ (-1)^{j-x}}{x!\ (j-x)!\,(n-j)!} \frac{B_\beta(j+a,b) - B_\alpha(j+a,b)}{B_\beta(a,b) - B_\alpha(a,b)} \quad (2.38)$$

with $x = 0,1,\dots,n \,;\, a,b > 0 \,;\, 0 < \alpha < \beta < 1$ and $a,b > 0$.

2.10 Binomial-Arcsine distribution

2.10.1 Arc-sine distribution

Consider a Beta distribution with pdf

$$g(p) = \frac{p^{a-1}(1-p)^{b-1}}{B(a,b)}$$

where $0 < p < 1 ; a, b > 0$.

Putting $a = b = \frac{1}{2}$we have

$$g_1(p) = \frac{p^{-\frac{1}{2}}(1-p)^{-\frac{1}{2}}}{B\left(\frac{1}{2},\frac{1}{2}\right)}$$

$$B\left(\frac{1}{2},\frac{1}{2}\right) = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(1)} = \pi$$

Thus

$$g_1(p) = \frac{1}{\pi\sqrt{p(1-p)}} \tag{2.39}$$

for $0 < p < 1 ; a, b > 0$.

This is the pdf of an arc-sine distribution.

2.10.2 Arc-sine mixing distribution

The Binomial-arc sine distribution is given by

$$f(x) = \binom{n}{x}\int_0^1 p^x(1-p)^{n-x}g(p)dp$$

Where $g(p)$ is the arc-sine density function.

Thus,

$$f(x) = \binom{n}{x}\frac{1}{\pi}\int_0^1 p^x(1-p)^{n-x}p^{-\frac{1}{2}}(1-p)^{-\frac{1}{2}}dp \tag{2.40}$$

$$= \binom{n}{x}\frac{1}{\pi}\int_0^1 p^{x-\frac{1}{2}}(1-p)^{n-x-\frac{1}{2}}dp$$

$$f(x) = \binom{n}{x}\frac{1}{\pi} B\left(x+\frac{1}{2}, n-x+\frac{1}{2}\right) \qquad (2.41)$$

From (2.40) a recursive expression for $f(x)$can be determined as follows:

$$f(x) = \binom{n}{x}\frac{1}{\pi}\int_0^1 p^x(1-p)^{n-x}p^{-\frac{1}{2}}(1-p)^{-\frac{1}{2}}\,dp$$

Then

$$\frac{f(x)\,\pi}{\binom{n}{x}} = \int_0^1 p^{x-\frac{1}{2}}(1-p)^{n-x-\frac{1}{2}}\,dp$$

Let

$$I_x = \frac{f(x)\,\pi}{\binom{n}{x}} = \int_0^1 p^{x-\frac{1}{2}}(1-p)^{n-x-\frac{1}{2}}\,dp$$

Using integration by parts to evaluate the integral on the RHS, we put

$$u = (1-p)^{n-x-\frac{1}{2}} \Rightarrow du = -1(n-x-\frac{1}{2})(1-p)^{n-x-\frac{3}{2}}\,dp$$

And

$$dv = p^{x-\frac{1}{2}} \Rightarrow v = \int p^{x-\frac{1}{2}}\,dp = \frac{p^{x+\frac{1}{2}}}{x+\frac{1}{2}}$$

Therefore

$$I_x = \left[(1-p)^{n-x-\frac{1}{2}}\frac{p^{x+\frac{1}{2}}}{x+\frac{1}{2}}\right]_0^1 + \frac{n-x-\frac{1}{2}}{x+\frac{1}{2}}\int_0^1 (1-p)^{n-x-\frac{3}{2}}p^{x+\frac{1}{2}}\,dp$$

$$I_x = \left(\frac{n-x-\frac{1}{2}}{x+\frac{1}{2}}\right)I_{x+1}$$

$$\frac{f(x)\,\pi}{\binom{n}{x}} = \left(\frac{n - x - \frac{1}{2}}{x + \frac{1}{2}}\right)\frac{f(x+1)\pi}{\binom{n}{x+1}}$$

$$f(x+1) = \left(\frac{x + \frac{1}{2}}{n - x - \frac{1}{2}}\right)\left(\frac{n-x}{x+1}\right) f(x) \tag{2.42}$$

CHAPTER 3

BINOMIAL MIXTURES BASED ON DISTRIBUTIONS BEYOND BETA

3.1 Introduction

In this chapter, we considermixing distributions that are not based on Beta distributions, but whose random variables are defined between 0 and 1.

They include the following distributions:

- Triangular distribution
- Kumaraswamy (I) and (II) distribution
- Truncated Exponential
- Truncated Gamma
- Minus log distribution.

The mixing is done using the following methods;

- Moments method
- Direct integration and substitution.

3.2 Binomial-Triangular Distribution

3.2.1 Triangular distribution

Construction

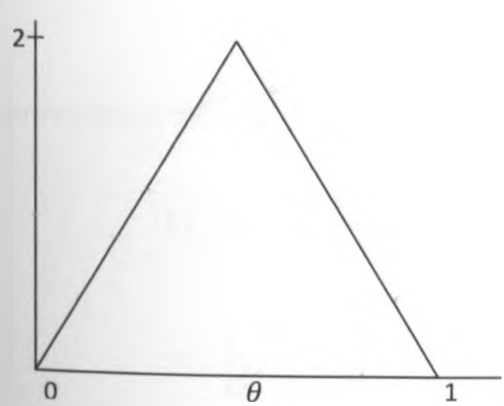


Figure 1:

The Triangular distribution $T(0,1,\theta)$,arises from the conjunction of two lines which share the same vertex (see figure1).

The points$(0,0)$and$(1,0)$determine the initial points of two hypotenuses that intersect at$(\theta,2)$forming a triangle with vertices at$(0,0)$, $(1,0)$and$(\theta,2)$.

The density function of the Triangular $T(0,1,\theta)$distribution is defined as

$$g(p) = \begin{cases} g_1(p), & if\ 0<p<\theta \\ g_2(p), & if\ \theta<p<1 \\ 0\ , & elsewhere \end{cases}$$

Where,

$g_1(p)$is the equation of the line$(0,0)$, $(\theta,2)$computed as

$\frac{g_1(p)}{p} = \frac{2}{\theta}$, giving $g_1(p) = \frac{2p}{\theta}$

$g_2(p)$is the equation of the line$(1,0)$, $(\theta,2)$computed as

$\frac{g_2(p)}{p-1} = \frac{-2}{1-\theta}(p-1)$, giving $g_2(p) = \frac{2(1-p)}{1-\theta}$.

Thus the density function of the Triangular $T(0,1,\theta)$is

$$g(p,\theta) = \begin{cases} \frac{2p}{\theta}, & 0<p<1 \\ \frac{2(1-p)}{1-\theta}, & \theta<p<1 \\ 0\ , & elsewhere \end{cases} \tag{3.1}$$

Properties

The moment of order$j$ about the origin of the triangular distribution is given by

$$E(P^j) = \int_0^\theta \frac{2p}{\theta} p^j dp + \int_\theta^1 \frac{2(1-p)}{(1-\theta)} p^j dp$$

$$= \left[\frac{2p^{j+2}}{\theta(j+2)}\right]_0^\theta + \frac{2}{1-\theta}\left[\frac{p^{j+1}}{j+1} - \frac{p^{j+2}}{j+2}\right]_\theta^1$$

$$= \frac{2\theta^{j+2}}{\theta(j+2)} + \frac{2}{1-\theta}\left[\frac{1}{(j+1)(j+2)} - \frac{\theta^{j+1}}{j+1} + \frac{\theta^{j+2}}{j+2}\right]$$

$$= \frac{2\theta^{j+2}(1-\theta) + 2\theta^{j+3}}{(1-\theta)\theta(j+2)} + \frac{2}{1-\theta}\left[\frac{1}{(j+1)(j+2)} - \frac{\theta^{j+1}}{j+1}\right]$$

$$= \frac{2\,\theta^{j+2}}{(1-\theta)\theta(j+2)} + \frac{2}{1-\theta}\left[\frac{1}{(j+1)(j+2)} - \frac{\theta^{j+1}}{j+1}\right]$$

$$= \frac{2}{1-\theta}\left[\frac{1-\theta^{j+1}(j+2)+\theta^{j+1}(j+1)}{(j+1)(j+2)}\right]$$

$$E(P^j) = \frac{2}{1-\theta}\left[\frac{1-\theta^{j+1}}{(j+1)(j+2)}\right] \quad (3.2)$$

Thus the mean is

$$E(P) = \frac{1+\theta}{3}$$

and the variance is

$$Var(P) = \{E(P^2) - (E(P))^2\} = \frac{\theta^2 - \theta + 1}{18}$$

3.2.2 Triangular mixing distribution

The Binomial-Triangular probability function is given by

$$f(x) = \int_0^1 \binom{n}{x} p^x (1-p)^{n-x} g(p)dp$$

Where $g(p)$ is the pdfof the Triangular distribution.

Therefore,

$$f(x) = \int_0^\theta p^x(1-p)^{n-x}\frac{2p}{\theta}dp + \int_\theta^1 \binom{n}{x} p^x(1-p)^{n-x}\frac{2(1-p)}{1-\theta}dp$$

$$= 2\binom{n}{x}\left[\frac{1}{\theta}\int_0^{\theta} p^{x+1}(1-p)^{n-x}dp + \frac{1}{1-\theta}\int_{\theta}^{1} p^{x}(1-p)^{n-x+1}dp\right]$$

$$= 2\binom{n}{x}\left[\frac{1}{\theta}B_{\theta}(x+2, n-x+1) + \frac{1}{1-\theta}\{B(x+1, n-x+2) - B_{\theta}(x+1, n-x+2)\}\right]$$

Where

$$B_{\theta}(\alpha, \beta) = \int_0^{\theta} t^{\alpha-1}(1-t)^{\beta-1}dt$$

is an incomplete Beta function.

Thus,

$$f(x) = 2\binom{n}{x}\left[\frac{1}{\theta}B_{\theta}(x+2, n-x+1) + \frac{1}{1-\theta}\{B(x+1, n-x+2) - B_{\theta}(x+1, n-x+2)\}\right] \qquad (3.3)$$

For $x = 0,1, \ldots, n\,; 0 < \theta < 1$

asobtained by Karlis & Xekalaki, (2006).

From the Moments method, this probability distribution can also be written as,

$$f(x) = \sum_{j=x}^{n} \frac{n!\,(-1)^{j-x}}{x!\ (j-x)!\,(n-j)!} E(P^{j}) \quad , \qquad x = 0,1, \ldots, n$$

for $j \geq x$ and 0 if $j < x$

The Jth moment, $E(P^{j})$ of the Triangular distribution is

$$\frac{2}{1-\theta}\left[\frac{1-\theta^{j+1}}{(j+1)(j+2)}\right]$$

from (3.2).

Thus,

$$f(x;n,\theta) = \sum_{j=x}^{n} \frac{n!\;(-1)^{j-x}}{x!\;(j-x)!\,(n-j)!} \frac{2}{1-\theta} \left[\frac{1-\theta^{j+1}}{(j+1)(j+2)}\right]$$

$$= \frac{2}{1-\theta} \sum_{j=x}^{n} \frac{n!\;(-1)^{j-x}}{x!\;(j-x)!\,(n-j)!} \left[\frac{1-\theta^{j+1}}{(j+1)(j+2)}\right] \qquad (3.4)$$

With $x = 0, 1, \dots, n\,; 0 < \theta < 1$

Properties of the Binomial-Triangular distribution

The mean is

$$E(X) = nE(P) = n\left(\frac{1+\theta}{3}\right)$$

and the variance, is

$$Var(X) = nE(P) - nE(P^2) + n^2 Var(P)$$

$$= n\left(\frac{1+\theta}{3}\right) - n\left(\frac{1-\theta^3}{6(1-\theta)}\right) + n^2\left(\frac{\theta^2-\theta+1}{18}\right)$$

$$= \frac{n(n+3)}{18} - \frac{n(n-3)\theta(1-\theta)}{18}$$

3.3 Binomial-Kumaraswamy (II) distribution

3.3.1 Kumaraswamy (II) distribution

Construction

One major disadvantage of the Beta distribution is that it involves a special function, and its cdf is an incomplete beta ratio, which can not be expressed in a closed form.

To circumvent this special function, Kumaraswamy (1980) modified the Beta function to

$$x^{a-1}(1-x^a)^{b-1}\,dx \quad , \qquad 0 < x < 1$$

Whose integral is

$$I = \int_0^1 x^{a-1}(1-x^a)^{b-1}\;dx$$

Putting $t = 1 - x^a$

We have

$$I = -\int_1^0 t^{b-1}(1-t)^{\frac{1}{a}(a-1)}\frac{1}{a}(1-t)^{\frac{1}{a}-1}\,dt$$

$$= \int_0^1 \frac{t^{b-1}}{a}\,dt = \frac{1}{ab}$$

Therefore

$$\int_0^1 x^{a-1}(1-x^a)^{b-1}\,dx = \frac{1}{ab}$$

and $\int_0^1 abx^{a-1}(1-x^a)^{b-1}\,dx = 1$

Thus

$$g(x) = abx^{a-1}(1-x^a)^{b-1} \quad , \quad 0 < x < 1 \qquad (3.5)$$

this is called a Kumaraswamy (Kw) distribution, with the parameters a and b . It is also called the Minimax distribution (see Jones (2007)).

This p.d.f. can also be constructed as the minimum of a random sample from the power function distribution, as is shown below:

Let $P_1 < P_2 < \cdots < P_\beta$ denote the order statistics of a random sample of size β from the power function distribution, having pdf

$$g(p) = \begin{cases} \alpha p^{\alpha-1} \,, & 0 < p < 1;\ \alpha > 0 \\ 0 \,, & elsewhere \end{cases}$$

Then the pdf of P_1 is

$$g_1(p_1) = \begin{cases} \beta[1-F(p_1)]^{\beta-1}f(p_1) \,, & 0 < p_1 < 1 \\ 0 \,, & elsewhere \end{cases}$$

where $F(p_1) = \int_0^{p_1} \alpha t^{\alpha-1}\,dt$

$$= p_1{}^{\alpha}$$

Thus,

$$g_1(p_1;\, \alpha, \beta) \;=\; \alpha\beta[1 - p_1{}^{\alpha}]^{\beta-1} p_1{}^{\alpha-1} \tag{3.6}$$

with $0 < p_1 < 1\,;\ \alpha, \beta > 0$ and $0\,, elsewhere.$

We denote this distribution as the Kw (II) distribution.

(seeJones (2009), (2007)).

Properties

The moment of order$j$about the origin of this distribution is given by

$$E(P^j) = \alpha\beta \int_0^1 p^{j+\alpha-1}(1 - p^{\alpha})^{\beta-1} dp$$

Letting$p^{\alpha} = t$ for $0 < t < 1\,;\ \alpha > 0$

We have

$$E(P^j) = \alpha\beta \int_0^1 t^{\frac{j}{\alpha}+1-\frac{1}{\alpha}}(1 - t)^{\beta-1} \frac{1}{\alpha} t^{\frac{1}{\alpha}-1} dt$$

$$= \beta \int_0^1 t^{\frac{j}{\alpha}}(1 - t)^{\beta-1} dt$$

$$E(p^j) = \beta B\left(\frac{j}{\alpha} + 1, \beta\right) \tag{3.7}$$

The mean is

$$E(P) = \frac{\beta!\ \left(\frac{1}{\alpha}\right)!}{\left(\frac{1}{\alpha} + \beta\right)!}$$

and the variance is

$$Var(P) = \{E(P^2) - (E(P))^2\} = \beta B\left(\frac{2}{\alpha} + 1, \beta\right) - \beta^2 B^2\left(\frac{1}{\alpha} + 1, \beta\right)$$

3.3.2 Kumaraswamy (II) mixing distribution

The Binomial-Kw (II) probability function is given by

$$f(x) = \int_0^1 \binom{n}{x} p^x (1-p)^{n-x} g(p)dp$$

Where $g(p)$ is the pdf of the Kw (II) distribution.

Therefore,

$$f(x) = \int_0^1 \binom{n}{x} p^x (1-p)^{n-x} \beta\alpha p^{\alpha-1} (1-p^\alpha)^{\beta-1}\, dp$$

$$= \alpha\beta \int_0^1 \binom{n}{x} p^{x+\alpha-1} (1-p)^{n-x} (1-p^\alpha)^{\beta-1}\, dp$$

$$f(x) = \alpha\beta \binom{n}{x} \sum_{k=0}^{\infty} \binom{\beta-1}{k} (-1)^k B(x+\alpha+\alpha k, n-x+1) \qquad (3.8)$$

with$x = 0,1,\dots,n;\ \alpha,\beta > 0$ and $0, elsewhere$

asobtained by Li Xiaohu et al (2011)

From the Moments method, this probability distribution can also be written as,

$$f(x) = \sum_{j=x}^{n} \frac{n!\ (-1)^{j-x}}{x!\ (j-x)!\,(n-j)!} E(P^j)\ , \quad x = 0,1,\dots,n$$

for $j \geq x$ and 0 if $j < x$

The Jth moment,$E(P^J)$ of the Kw (II) distribution from (3.7) is

$$\beta B\left(\frac{j}{\alpha}+1, \beta\right)$$

Thus,

$$f(x) = \sum_{j=x}^{n} \frac{n!\,(-1)^{j-x}}{x!\,\,(j-x)!\,(n-j)!} \beta\, B\left(\frac{j}{\alpha}+1,\beta\right) \tag{3.9}$$

with$x = 0, 1, \ldots, n\,;\ \alpha, \beta > 0$ and $0,\ elsewhere$.

Properties of the Binomial-Kumaraswamy (II) distribution

The mean is

$$E(X) = nE(P) = \frac{n\,\beta!\,\left(\frac{1}{\alpha}\right)!}{\left(\frac{1}{\alpha}+\beta\right)!}$$

and the variance, is

$$Var(X) = \ nE(P) - \ nE(P^2) + \ n^2 Var(P)$$

$$= \frac{n\,\beta!\,\left(\frac{1}{\alpha}\right)!}{\left(\frac{1}{\alpha}+\beta\right)!} - \frac{n\,\beta!\,\left(\frac{2}{\alpha}\right)!}{\left(\frac{2}{\alpha}+\beta\right)!} + n^2\left[\beta B\left(\frac{2}{\alpha}+1,\beta\right) - \beta^2 B^2\left(\frac{1}{\alpha}+1,\beta\right)\right]$$

3.4 Binomial-Kumaraswamy (I) distribution

3.4.1 Kumaraswamy (I) distribution

Construction

Given the Kw (II) distribution with pdf

$$g(p) \ = \ \alpha\beta[1-p^{\alpha}]^{\beta-1}p^{\alpha-1}\ ,\ \ 0 < p < 1\,;\ \alpha, \beta > 0$$

Let$P \ = \ U^{\frac{1}{\alpha}},\quad 0 < u < 1;\ \alpha > 0$

Then$g(u) \ = \ \alpha\beta(1-u)^{\beta-1}\ u^{1-\frac{1}{\alpha}}|J|$

where$|J| \ = \ \left|\frac{dp}{du}\right| \ = \ \frac{1}{\alpha}u^{\frac{1}{\alpha}-1}$

Thus

$$g(u) = \beta(1-u)^{\beta-1}\ , \quad 0 < u < 1;\ \beta > 0 \tag{3.10}$$

and zero elsewhere.

This pdf is that of the Kumaraswamy (I) distributionwith parameterβ.

We denote it as the Kw (I) distribution

Properties

The moment of order$j$ about the origin of this distribution is given by

$$E(U^j) = \beta\int_0^1 u^j(1-u)^{\beta-1}dp$$

$$= \beta B(j+1,\beta) \tag{3.11}$$

The mean$E(U)$is

$$\frac{1}{\beta+1}$$

and the variance$Var(U)$ is

$$\{E(U^2)-(E(U))^2\} = \frac{2}{(\beta+2)(\beta+1)} - \frac{1}{(\beta+1)^2} = \frac{\beta}{(\beta+2)(\beta+1)^2}$$

3.4.2 Kumaraswamy (I) mixing distribution

The Binomial-Kw (I) density function is given by

$$f(x) = \int_0^1 \binom{n}{x} p^x(1-p)^{n-x} g(p)dp$$

Where $g(p)$ is the pdf of the Kw (I) distribution.

Therefore,

$$f(x) = \int_0^1 \binom{n}{x} p^x (1-p)^{n-x} g(p) dp$$

$$= \binom{n}{x} \beta \int_0^1 p^x (1-p)^{n-x+\beta-1}\, dp$$

$$f(x) = \binom{n}{x} \beta B(x+1, n-x+\beta) \ , \qquad x = 0,1, \dots, n \ ; \ \beta > 0 \tag{3.12}$$

and zero elsewhere.

asobtained by Li Xiaohu et al (2011).

From the Moments method, the Binomial-Kw (I) distribution can also be written as,

$$f(x) = \sum_{j=x}^{n} \frac{n! \ (-1)^{j-x}}{x! \ (j-x)! \, (n-j)!} E(P^j) \quad , \quad x = 0,1, \dots, n$$

for $j \geq x$ and 0 if $j < x$

The Jth moment, $E(P^j)$ of the Kw (I) distribution from (3.11) is

$$\beta B(j+1, \beta)$$

Thus,

$$f(x) = \beta \sum_{j=x}^{n} \frac{n! \ (-1)^{j-x}}{x! \ (j-x)! \, (n-j)!} B(j+1, \beta) \tag{3.13}$$

With $x = 0, 1, \dots, n \ ; \ \beta > 0$ and $0, \ elsewhere$

Properties of the Binomial-Kumaraswamy (I) distribution

The mean is

$$E(X) = nE(P) = \frac{n}{\beta + 1}$$

and the variance, is

$$Var(X) = nE(P) - nE(P^2) + n^2 Var(P)$$

$$= \frac{n}{\beta + 1} - \frac{2n}{(\beta + 2)(\beta + 1)} + \frac{n^2\beta}{(\beta + 2)(\beta + 1)^2}$$

$$= \frac{n\beta(\beta + n + 1)}{(\beta + 2)(\beta + 1)^2}$$

3.5 Binomial-Truncated Exponential distribution

3.5.1 Truncated Exponential distribution

Construction

Let Y be a one sided truncated exponential, $TEX(\lambda, b)$random variable, then, the pdf of Y can be evaluated as,

$$\int_0^b \frac{e^{-\frac{y}{\lambda}}}{\lambda} dy = \left[1 - e^{-\frac{b}{\lambda}}\right]$$

Dividing throughout by $\left[1 - e^{-\frac{b}{\lambda}}\right]$ we have

$$1 = \int_0^b \frac{\frac{1}{\lambda} e^{-\frac{y}{\lambda}}}{\left[1 - e^{-\frac{b}{\lambda}}\right]} d\lambda$$

Letting $y = pb$ we have,

$$1 = \int_0^1 \frac{\frac{b}{\lambda} e^{-\frac{pb}{\lambda}}}{1 - e^{-\frac{b}{\lambda}}} dp$$

Thus,

$$g(p) = \begin{cases} \dfrac{\frac{b}{\lambda} e^{-\frac{pb}{\lambda}}}{1 - e^{-\frac{b}{\lambda}}}, & 0 < p < 1; b > 0, \lambda > 0 \\ 0, & elsewhere \end{cases} \qquad (3.14)$$

3.5.2 Truncated Exponential $TEX(\lambda, b)$ mixing distribution

The Binomial-TruncatedExponential distribution is given by

$$f(x) = \binom{n}{x} \int_0^1 p^x (1-p)^{n-x} g(p) dp$$

Where $g(p)$ is the pdf of the TruncatedExponential distribution

Thus

$$f(x) = \binom{n}{x} \int_0^1 p^x (1-p)^{n-x} \frac{\frac{b}{\lambda} e^{-\frac{pb}{\lambda}}}{1-e^{-\frac{b}{\lambda}}} dp$$

$$= \binom{n}{x} \frac{b}{\lambda\left(1-e^{-\frac{b}{\lambda}}\right)} \int_0^1 p^x (1-p)^{n-x} e^{-\frac{pb}{\lambda}} dp$$

But,

$$\int_0^1 e^{xt} t^{a-1} (1-t)^{c-a-1} dt = 1F1(a, c; x) \frac{\Gamma(a)\Gamma(c-a)}{\Gamma(c)}$$

where $1F1(a, c; x)$ is a confluent hypergeometric function.

Thus,

$$f(x) = \binom{n}{x} \frac{b\, \Gamma(x+2)\Gamma(n-x+1)}{\lambda\left(1-e^{-\frac{b}{\lambda}}\right) \Gamma(n+2)} F_1\left(x+1, n+2; \frac{-b}{\lambda}\right)$$

Therefore the Binomial-TEX(λ, b)distribution has the pdf

$$f(x) = \begin{cases} \binom{n}{x} b \dfrac{\Gamma(x+2)\Gamma(n-x+1)}{\lambda\left(1-e^{-\frac{b}{\lambda}}\right) \Gamma(n+2)} F_1\left(x+1, n+2; \dfrac{-b}{\lambda}\right), & x = 0,1, \dots, n\,;\ b, \lambda > 0 \\ 0\ , & elsewhere \end{cases} \quad (3.15)$$

3.6 Binomial-Truncated Gamma distribution

3.6.1 Truncated Gamma distribution

Construction

Consider an incomplete gamma function given by

$$\gamma(\beta, a) = \int_0^a t^{\beta-1} e^{-t}\, dt$$

So that

$$1 = \int_0^a \frac{t^{\beta-1} e^{-t}}{\gamma(\beta, a)}\, dt$$

Putting $t = ap$, we have,

$$1 = \int_0^1 \frac{a^{\beta} p^{\beta-1} e^{-ap}}{\gamma(\beta, a)}\, dp$$

Therefore

$$g(p) = \frac{a^{\beta} p^{\beta-1} e^{-ap}}{\gamma(\beta, a)} \qquad (3.16)$$

for $0 < p < 1; a, \beta > 0$

This is the pdfof a truncated-gamma distribution, with parameters a , β

3.6.2 Truncated Gamma mixing distribution

This distribution is given by

$$f(x) = \binom{n}{x} \int_0^1 p^x (1-p)^{n-x} g(p) dp$$

Where $g(p)$ is the distribution of the Truncated Gamma

Thus

$$f(x) = \binom{n}{x}\int_0^1 p^x(1-p)^{n-x}\frac{a^\beta p^{\beta-1}e^{-ap}}{\gamma(\beta,a)}\,dp$$

$$= \binom{n}{x}\frac{a^\beta}{\gamma(\beta,a)}\int_0^1 p^{x+\beta-1}(1-p)^{n-x}e^{-ap}\,dp$$

But $\int_0^1 e^{-ap}p^{x+\beta-1}(1-p)^{n-x}\,dp = \frac{\Gamma(x+\beta)\,\Gamma(n-x+1)}{\Gamma(n+\beta+1)}\,1F1\big((x+\beta),(n+\beta+1);\,-a\big)$

Thus

$$f(x) = \binom{n}{x}\frac{a^\beta}{\gamma(\beta,a)}\frac{\Gamma(x+\beta)\,\Gamma(n-x+1)}{\Gamma(n+\beta+1)}\,1F1\big((x+\beta),(n+\beta+1);\,-a\big) \qquad (3.17)$$

with $x = 0,1,\dots,n\,;\, a,\beta > 0$

asobtained by Bhattacharya, (1968).

3.7 Binomial-Minus Log distribution

3.7.1 Minus Log distribution

Construction

Let the random variable X have the uniform pdf $U\,[0,1]$and letX_1, X_2denote a random sample from this distribution. The joint pdf of$X_1$and$X_2$is then

$$\varphi(x_1x_2) = \begin{cases} f(x_1)\,f(x_2), & 0 < x_1 < 1, \quad 0 < x_2 < 1 \\ 0, & elsewhere \end{cases}$$

Consider the two random variables $P = X_1X_2$and $Y = X_2$.

The joint pdf of P and Y is

$$g(p,y) = 1\,|J|$$

where$|J| = \begin{vmatrix} \frac{dx_1}{dp} & \frac{dx_1}{dy} \\ \frac{dx_2}{dp} & \frac{dx_2}{dy} \end{vmatrix} = \begin{vmatrix} \frac{1}{y} & \frac{-p}{y^2} \\ 0 & 1 \end{vmatrix} = \frac{1}{y}$

Thus,

$$g(p,y) = \begin{cases} \dfrac{1}{y} \, , & 0 < p < y < 1 \\ 0 \, , & elsewhere \end{cases}$$

The marginal pdf of P is

$$g(p) = \int_p^1 \frac{1}{y}\, dy$$

$$= |\log y|_p^1$$

$$= 0 - \log p$$

Thus

$$g(p) = \begin{cases} -\log p \, , & 0 < p < 1 \\ 0 \, , & elsewhere \end{cases} \qquad (3.18)$$

We refer to this distribution as the Minus-Log distribution.

Properties

The moment of order j about the origin of the Minus-log distribution is given by

$$E(P^j) = \int_0^1 p^j(-\log p)\, dp$$

Letting $a = -\log p$

We have,

$$E(P^j) = -\int_\infty^0 e^{-a(j+1)} a\, da$$

Using the negative outside the integral to swap the limits, we have

$$E(P^j) = \int_0^\infty e^{-a(j+1)} a\, da$$

$$= \frac{1}{(j+1)^2} \tag{3.19}$$

Thus the mean is

$$E(P) = \frac{1}{4}$$

and the variance is

$$Var(P) = \{E(P^2) - (E(P))^2\} = \frac{1}{9} - \frac{1}{16} = \frac{7}{144}$$

3.7.2 Minus-Log mixing distribution

The Binomial-Minus Log distribution is given by

$$f(x) = \binom{n}{x} \int_0^1 p^x (1-p)^{n-x} g(p) dp$$

Where $g(p)$ is the pdf of the Minus Log distribution

Thus

$$f(x) = \binom{n}{x} \int_0^1 p^x (1-p)^{n-x} (-\log p)\, dp$$

Letting $a = -\log p$

We have,

$$f(x) = \binom{n}{x} \sum_{k=0}^{n-x} \binom{n-x}{k} (-1)^k \int_0^\infty e^{-a(x+k+1)} a\, da$$

Thus,

$$f(x) = \binom{n}{x} \sum_{k=0}^{n-x} \binom{n-x}{k} (-1)^k \frac{1}{(x+k+1)^2}$$

and the Binomial-Minus Log density function is

$$f(x) = \begin{cases} \binom{n}{x}\sum_{k=0}^{n-x}\binom{n-x}{k}(-1)^k \frac{1}{(x+k+1)^2}, & x = 0,1,\dots,n \\ 0 \quad , & elsewhere \end{cases} \quad (3.20)$$

From the **Moments** method, this probability distribution can also be written as,

$$f(x) = \sum_{j=x}^{n} \frac{n!\ (-1)^{j-x}}{x!\ (j-x)!\,(n-j)!} E(P^j) \quad , \qquad x = 0,1,\dots,n$$

for $j \geq x$ and 0 if $j < x$

The Jth moment, $E(P^j)$ of the Minus Log distribution from(3.19) is

$$\frac{1}{(j+1)^2}$$

Thus,

$$f(x) = \begin{cases} \sum_{j=x}^{n} \frac{n!\ (-1)^{j-x}}{x!\ (j-x)!\,(n-j)!\,(j+1)^2}, & x = 0,1,\dots,n;\ \beta > 0 \\ 0 \quad , & elsewhere \end{cases} \quad (3.21)$$

Properties of the Binomial-Minus log distribution

The mean is

$$E(X) = nE(P) = \frac{n}{4}$$

and the variance, is

$$Var(X) = nE(P) - nE(P^2) + n^2Var(P)$$

$$= \frac{n}{4} - \frac{n}{9} + \frac{7n^2}{144}$$

$$= \frac{20n + 7n^2}{144}$$

3.8 Binomial-Standard Two sided power distribution

3.8.1 Two sided power distribution

Construction

The Standard Two sided power distribution can be viewed as a particular case of the general two sided continuous family with support [0, 1] given by the density

$$g\{p|\theta, z(.|\psi)\} = \begin{cases} z\left(\frac{p}{\theta}|\psi\right), & 0 < p < \theta \\ z\left(\frac{1-p}{1-\theta}|\psi\right), & \theta < p < 1 \end{cases} \tag{3.22}$$

where $z(.|\psi)$is an appropriately selected continuous pdf on [0, 1] with parameter(s) ψ.

The density $z(.|\psi)$is called a generating density, so that, when

$$z(y) = \; ky^{k-1} \;, \qquad 0 < y < 1\,; k > 0$$

(a power function distribution)

Then

$$g(p) = \begin{cases} k\left(\frac{p}{\theta}\right)^{k-1}, & 0 < p < \theta; k > 0 \\ k\left(\frac{1-p}{1-\theta}\right)^{k-1}, & \theta < p < 1\,; k > 0 \end{cases} \tag{3.23}$$

This is the pdf of the Standard Two sided power distribution, with parameters k and θ

(seeDorp & Kotz, (2003)).

3.8.2 Standard Two sided (STSP) powermixing distribution

The Binomial-Standard two sided power distribution is given by

$$f(x) = \binom{n}{x} \int_0^1 p^x (1-p)^{n-x} g(p) dp$$

Where $g(p)$ is the pdf of the STSP distribution

Thus

$$f(x) = \binom{n}{x} k \int_0^{\theta} p^x (1-p)^{n-x} \left(\frac{p}{\theta}\right)^{k-1} dp + \binom{n}{x} k \int_{\theta}^{1} p^x (1-p)^{n-x} \left(\frac{1-p}{1-\theta}\right)^{k-1} dp$$

$$= \binom{n}{x} k \left\{ \frac{1}{\theta^{k-1}} \int_0^{\theta} p^{x+k-1} (1-p)^{n-x}\, dp + \frac{1}{(1-\theta)^{k-1}} \int_{\theta}^{1} p^x (1-p)^{n-x+k-1} dp \right\}$$

$$f(x) = \binom{n}{x} k \left\{ \frac{B_{\theta}(x+k, n-x+1)}{\theta^{k-1}} + \frac{1}{(1-\theta)^{k-1}} \left(B(x+1, n-x+k) - B_{\theta}(x+1, n-x+k) \right) \right\} \quad (3.24)$$

3.9 Binomial-Ogivedistribution

3.9.1 The Ogive mixing distribution

The general form of the ogive distribution is given by the pdf

$$g(p) = \frac{2m(m+1)}{3m+1} p^{\frac{m-1}{2}} + \frac{1-m^2}{3m+1} p^m, \qquad 0 < p < 1; \; m > 0$$

(seeDorp & Kotz, (2003)).

The Binomial-Ogivedistribution is given by

$$f(x) = \binom{n}{x} \int_0^1 p^x (1-p)^{n-x} g(p) dp$$

Where $g(p)$ is the pdf of the Ogive distribution

Thus

$$f(x) = \binom{n}{x} \int_0^1 p^x (1-p)^{n-x} \frac{2m(m+1)}{3m+1} p^{\frac{m-1}{2}} dp + \binom{n}{x} \int_0^1 p^x (1-p)^{n-x} \frac{1-m^2}{3m+1} p^m dp$$

$$= \binom{n}{x}\frac{2m(m+1)}{3m+1}\int_0^1 p^{x+\frac{m-1}{2}}(1-p)^{n-x}dp + \binom{n}{x}\frac{1-m^2}{3m+1}\int_0^1 p^{x+m}(1-p)^{n-x}dp$$

$$= \binom{n}{x}\frac{2m(m+1)}{3m+1}B\left(\frac{2x+m+1}{2}, n-x+1\right) + \binom{n}{x}\frac{1-m^2}{3m+1}B(x+m+1, n-x+1)$$

$$f(x) = \binom{n}{x}\frac{2m(m+1)}{3m+1}B\left(\frac{2x+m+1}{2}, n-x+1\right) + \binom{n}{x}\frac{1-m^2}{3m+1}B(x+m+1, n-x+1) \quad (3.25)$$

for $x = 0,1,\ldots,n\,; m > 0$

3.10 Binomial-Two sided ogivedistribution

3.10.1 The Two sided ogivedistribution

Construction

The Two sided ogive distribution can be viewed as a particular case of the general two sided continuous family with support [0, 1] given by the density (3.22) as

$$g\{p|\theta, z(.|\psi)\} = \begin{cases} z\left(\frac{p}{\theta}|\psi\right), & 0 < p < \theta \\ z\left(\frac{1-p}{1-\theta}|\psi\right), & \theta < p < 1 \end{cases}$$

where $z(.|\psi)$is an appropriately selected continuous pdf on [0, 1] with parameter(s) ψ.

The density $z(.|\psi)$is called a generating density, so that, when

$$z(y) = \frac{2m(m+1)}{3m+1}y^{\frac{m-1}{2}} + \frac{1-m^2}{3m+1}y^m, \qquad 0 < y < 1\,; m > 0$$

(anogive distribution).

Then

$$
g(p) = \begin{cases} \dfrac{2m(m+1)}{3m+1}\left(\dfrac{p}{\theta}\right)^{\frac{m-1}{2}} + \dfrac{1-m^2}{3m+1}\left(\dfrac{p}{\theta}\right)^{m}, & 0 < p < \theta; m > 0 \\ \dfrac{2m(m+1)}{3m+1}\left(\dfrac{1-p}{1-\theta}\right)^{\frac{m-1}{2}} + \dfrac{1-m^2}{3m+1}\left(\dfrac{1-p}{1-\theta}\right)^{m}, & \theta < p < 1; m > 0 \end{cases} \tag{3.26}
$$

This is the pdf of the Two sided ogive distribution, with parameters m and θ

The Two sided ogive distribution is smooth at the reflection point$(p = \theta)$. This is in contrast to the Two sided power distribution (see (3.8.1)).

3.10.2 The Two sided ogive mixing distribution

The Binomial-Two sided ogivedistribution is given by

$$
f(x) = \binom{n}{x} \int_0^1 p^x (1-p)^{n-x} g(p) dp
$$

Where $g(p)$ is the pdf of the Two sided ogive distribution

Thus

$$
\begin{aligned}
f(x) = {} & \binom{n}{x} \int_0^\theta p^x (1-p)^{n-x} \left(\frac{2m(m+1)}{3m+1}\left(\frac{p}{\theta}\right)^{\frac{m-1}{2}} + \frac{1-m^2}{3m+1}\left(\frac{p}{\theta}\right)^{m} \right) dp \\
& + \binom{n}{x} \int_\theta^1 p^x (1-p)^{n-x} \left(\frac{2m(m+1)}{3m+1}\left(\frac{1-p}{1-\theta}\right)^{\frac{m-1}{2}} \right. \\
& \left. + \frac{1-m^2}{3m+1}\left(\frac{1-p}{1-\theta}\right)^{m} \right) dp
\end{aligned}
$$

$$f(x) = \binom{n}{x}\frac{2m(m+1)}{3m+1}\frac{1}{\theta^{\frac{m-1}{2}}}\int_0^\theta p^{x+\frac{m-1}{2}}(1-p)^{n-x}\,dp$$
$$+\frac{1-m^2}{3m+1}\binom{n}{x}\frac{1}{\theta^m}\int_0^\theta p^{x+m}(1-p)^{n-x}\,dp$$
$$+\binom{n}{x}\frac{2m(m+1)}{3m+1}\frac{1}{(1-\theta)^{\frac{m-1}{2}}}\int_\theta^1 p^x(1-p)^{n-x+\frac{m-1}{2}}\,dp$$
$$+\frac{1-m^2}{3m+1}\binom{n}{x}\frac{1}{(1-\theta)^m}\int_\theta^1 p^x(1-p)^{n-x+m}\,dp$$

$$\begin{aligned}
f(x) \\
= \binom{n}{x}\Bigg[&\frac{2m(m+1)}{3m+1}\frac{1}{\theta^{\frac{m-1}{2}}}B_\theta\left(x+\frac{m+1}{2}, n-x+1\right) \\
&+\left(\frac{1-m^2}{3m+1}\right)\frac{1}{\theta^m}B_\theta(x+m+1, n-x+1) \\
&+\left(\frac{2m(m+1)}{3m+1}\right)\frac{B\left(x+1, n-x+\frac{m+1}{2}\right)-B_\theta\left(x+1, n-x+\frac{m+1}{2}\right)}{(1-\theta)^{\frac{m-1}{2}}} \\
&+\left(\frac{1-m^2}{3m+1}\right)\frac{B(x+1, n-x+m+1)-B_\theta(x+1, n-x+m+1)}{(1-\theta)^m}\Bigg]
\end{aligned} \tag{3.27}$$

for $x = 0,1,\ldots,n;\ m > 0,\ 0 < \theta < 1$

CHAPTER 4

BINOMIAL MIXTURES BASED ON TRANSFORMATION OF THE PARAMETERp

4.1 Introduction

In this chapter, we investigate Binomial mixtures obtained by transforming the parameter$p$to include mixing distributions in the interval$[0, \infty]$.

The following transformations are used:

- $p = e^{-t}$or $p = 1 - e^{-t}$for $t > 0$.
- $p = cy$, $0 < cy < 1$, where Y is a random variable.

When $p = e^{-t}$, the Binomial mixture is

$$f(x) = \binom{n}{x} \int_0^\infty e^{-tx} (1 - e^{-t})^{n-x} g(t) dt$$

Further evaluation gives

$$f(x) = \binom{n}{x} \sum_{k=0}^{n-x} \binom{n-x}{k} (-1)^k \int_0^\infty e^{-t(x+k)} g(t)\, dt$$

where$\int_0^\infty e^{-t(x+k)} g(t)\, dt$ is the Laplace transform $E\left(e^{-(x+k)t}\right)$of $g(t)$

denoted by$L_t(x+k)$.

Thus

$$f(x) = \binom{n}{x} \sum_{k=0}^{n-x} \binom{n-x}{k} (-1)^k L_t(x+k) \qquad (4.1)$$

With $x = 0, 1, \ldots, n$

(See Bowman et al (1992))

The mean of $f(x)$is

$$E(X^j) = E_t\{E(X^j | T)\}$$

where E_t denotes expectation with respect to the distribution of T.

Hence

$$E(X) = E[E(X|P)] = nE(P)$$

$$= n\,E(e^{-t}) = nL_t(1) \quad (4.2)$$

And the Variance $Var(X)$ is

$$nE(P) - nE(P^2) + n^2Var(P)$$
$$= nE(e^{-t}) - nE(e^{-2t}) + n^2Var(e^{-t})$$
$$= nL_t(1) - nL_t(2) + n^2[E(e^{-2t}) - [E(e^{-t})]^2]$$

$$= nL_t(1) - nL_t(2) + n^2[L_t(2) - [L_t(1)]^2] \quad (4.3)$$

When $p = 1 - e^{-t}$

$$f(x) = \binom{n}{x}\int_0^\infty e^{-t(n-x)}(1-e^{-t})^x g(t)dt$$

$$= \binom{n}{x}\sum_{k=0}^{x}\binom{x}{k}(-1)^k\int_0^\infty e^{-t(n-x+k)}\,g(t)\,dt$$

$$= \binom{n}{x}\sum_{k=0}^{x}\binom{x}{k}(-1)^k L_t(n-x+k) \quad (4.4)$$

The mean is

$$E(X) = E[E(X|P)] = nE(P)$$

$$= n\,E(1-e^{-t})$$

$$= n\big(1 - L_t(1)\big) \quad (4.5)$$

And the Variance is

$$Var(P) = nE(P) - nE(P^2) + n^2Var(P)$$

$$= nE(1-e^{-t}) - nE((1-e^{-t})^2) + n^2Var(1-e^{-t})$$

$$= n[1 - L_t(1)] - n[1 - 2L_t(1) + L_t(2)] + n^2\left[1 - 2L_t(1) + L_t(2) - \left(1 - L_t(1)\right)^2\right]$$

$$= nL_t(1) - nL_t(2) + n^2\left[L_t(2) - \left(L_t(1)\right)^2\right] \qquad (4.6)$$

asobtained byAlanko and Duffy (1996).

4.2 Using$p = \exp(-t)$

4.2.1 Binomial- Exponential distribution

4.2.1.1 Exponential distribution

The p.d.f. of the Exponential distribution is given by

$$g(t) = \beta e^{-t\beta}\ , \qquad t > 0;\ \beta > 0$$

Its Laplace transform is

$$L_t(s) = E(e^{-ts}) = \int_0^\infty \beta\, e^{-t(s+\beta)}\, dt$$

$$= \frac{\beta}{(s+\beta)} \int_0^\infty e^{-t(s+\beta)}(s+\beta)dt$$

$$= \frac{\beta}{(s+\beta)} \qquad (4.7)$$

4.2.1.2 Exponential mixing distribution

The Binomial-Exponential distribution is given by

$$f(x) = \binom{n}{x}\sum_{k=0}^{n-x}\binom{n-x}{k}(-1)^k L_t(x+k)$$

$L_t(x+k) = \frac{\beta}{(x+k+\beta)}$from(4.7).

Thus

$$f(x) = \beta \binom{n}{x} \sum_{k=0}^{n-x} \binom{n-x}{k} \frac{(-1)^k}{(x+k+\beta)} \qquad (4.8)$$

with $x = 0,1,\ldots,n\,;\ \beta > 0$

Properties of the Binomial-Exponential distribution

The mean is

$$E(X) = nL_t(1) = \frac{n\beta}{\beta+1}$$

And the Variance is

$$\begin{aligned}
Var(X) &= nL_t(1) - nL_t(2) + n^2[L_t(2) - [L_t(1)]^2] \\
&= \frac{n\beta}{\beta+1} - \frac{n\beta}{\beta+2} + n^2\left[\frac{\beta}{\beta+2} - \left(\frac{\beta}{\beta+1}\right)^2\right] \\
&\quad \frac{n\beta}{(\beta+1)(\beta+2)} + \frac{n^2\beta}{(\beta+2)(\beta+1)^2} \\
&= \frac{n\beta(\beta+1) + n^2\beta}{(\beta+2)(\beta+1)^2}
\end{aligned}$$

4.2.2 Binomial-Gamma with 1 parameter distribution

4.2.2.1 Gamma with 1 parameter distribution

The general form of the Gamma with 1 parameter (Gamma (I)) distribution is

$$g(t) = \begin{cases} \dfrac{e^{-t}t^{\alpha-1}}{\Gamma(\alpha)}, & t > 0; \alpha > 0 \\ 0\ , & elsewhere \end{cases}$$

Its Laplace transform $L_t(s)$ is

$$E(e^{-ts}) = \int_0^\infty \frac{t^{\alpha-1}}{\Gamma(\alpha)} e^{-t(s+1)}\, dt$$

$$= \frac{1}{(s+1)^{\alpha}} \int_{0}^{\infty} \frac{t^{\alpha-1}}{\Gamma(\alpha)} e^{-t(s+1)} (s+1)^{\alpha} dt$$

$$= \frac{1}{(s+1)^{\alpha}} \tag{4.9}$$

4.2.2.2 Gamma with 1 parameter mixing distribution

The Binomial- Gamma (I) distribution is given by

$$f(x) = \binom{n}{x} \sum_{k=0}^{n-x} \binom{n-x}{k} (-1)^k L_t(x+k)$$

$L_t(x+k) = \frac{1}{(x+k+1)^{\alpha}}$ from(4.9).

Thus

$$f(x) = \binom{n}{x} \sum_{k=0}^{n-x} \binom{n-x}{k} (-1)^k \frac{1}{(x+k+1)^{\alpha}} \tag{4.10}$$

with $x = 0,1, \dots, n$; $\alpha > 0$

Properties of the Binomial-Gamma (I) distribution

The mean is

$$E(X) = nL_t(1) = \frac{n}{2^{\alpha}}$$

And the Variance is

$$Var(X) = nL_t(1) - nL_t(2) + n^2[L_t(2) - [L_t(1)]^2]$$

$$= \frac{n}{2^{\alpha}} - \frac{n}{3^{\alpha}} + n^2 \left| \frac{1}{3^{\alpha}} - \frac{1}{2^{2\alpha}} \right|$$

4.2.3 Binomial- Gamma with 2 parameters distribution

4.2.3.1 Gamma with 2 parameters distribution

The general form of the Gamma with 1 parameter distribution is

$$g(y) = \begin{cases} \dfrac{e^{-y}y^{\alpha-1}}{\Gamma(\alpha)}, & y > 0; \alpha > 0 \\ 0, & elsewhere \end{cases}$$

Putting $y = t\beta$ we have

$$g(t) = \frac{(t\beta)^{\alpha-1}}{\Gamma(\alpha)} e^{-t\beta}|J|$$

with $t > 0;\ \alpha, \beta > 0$

where$|J| = \left|\frac{dy}{dt}\right| = \beta$

Thus

$$g(t) = \begin{cases} \dfrac{e^{-t\beta}t^{\alpha-1}\beta^{\alpha}}{\Gamma(\alpha)}, & t > 0; \alpha, \beta > 0 \\ 0, & elsewhere \end{cases} \tag{4.11}$$

This is the p.d.f. of the Gamma distribution, with 2 parameters α and β . We denote it as the Gamma (II) distribution.

Its Laplace transform$L_t(s)$is

$$E(e^{-ts}) = \int_0^{\infty} \frac{t^{\alpha-1}}{\Gamma(\alpha)} e^{-t(s+\beta)}\beta^{\alpha}\, dt$$

$$= \frac{\beta^{\alpha}}{(s+\beta)^{\alpha}} \int_0^{\infty} \frac{t^{\alpha-1}}{\Gamma(\alpha)} e^{-t(s+\beta)}(s+\beta)^{\alpha} dt$$

$$L_t(s) = \left(\frac{\beta}{s+\beta}\right)^{\alpha} \tag{4.12}$$

4.2.3.2 Gamma with 2 parameters mixing distribution

The Binomial- Gamma (II) distribution is given by

$$f(x) = \binom{n}{x} \sum_{k=0}^{n-x} \binom{n-x}{k} (-1)^k L_t(x+k)$$

$L_t(x+k) = \left(\frac{\beta}{x+k+\beta}\right)^{\alpha}$ from(4.12).

Thus

$$f(x) = \begin{cases} \binom{n}{x}\sum_{k=0}^{n-x}\binom{n-x}{k}(-1)^k\left(\frac{\beta}{x+k+\beta}\right)^{\alpha}, & x = 0,1,\dots,n;\ \alpha,\beta > 0 \\ 0\ , & elsewhere \end{cases} \qquad (4.13)$$

Properties of the Binomial-Gamma (II) distribution

The mean is

$$E(X) = nL_t(1) = n\left(\frac{\beta}{\beta+1}\right)^{\alpha}$$

And the Variance is

$$Var(X) = nL_t(1) - nL_t(2) + n^2[L_t(2) - [L_t(1)]^2]$$

$$= n\left(\frac{\beta}{\beta+1}\right)^{\alpha} - n\left(\frac{\beta}{\beta+2}\right)^{\alpha} + n^2\left[\left(\frac{\beta}{\beta+2}\right)^{\alpha} - \left(\frac{\beta}{\beta+1}\right)^{2\alpha}\right]$$

4.2.4 Binomial-Generalized exponential with 1 parameter distribution

4.2.4.1 Generalized exponential with 1 parameter distribution

The general form of the Generalized exponential with 1 parameter is

$$g(t) = \begin{cases} \alpha(1-e^{-t})^{\alpha-1}e^{-t}, & t > 0; \alpha > 0 \\ 0\ , & elsewhere \end{cases}$$

Its Laplace transform $L_t(s)$ is

$$E(e^{-ts}) = \alpha\int_0^{\infty}(1-e^{-t})^{\alpha-1}e^{-t(s+1)}\,dt$$

But $B(\alpha,\beta) = \int_0^{\infty} e^{-t\alpha}(1-e^{-t})^{\beta-1}\,dt$

where $B(\alpha,\beta)$ is the Beta function.

Thus

$$L_t(s) = \alpha B(s+1, \alpha) \quad (4.14)$$

4.2.4.2 Generalized Exponential with 1 parameter mixing distribution

The Binomial-Generalized Exponential (I) distribution is given by

$$f(x) = \binom{n}{x} \sum_{k=0}^{n-x} \binom{n-x}{k} (-1)^k L_t(x+k)$$

$L_t(x+k) = \alpha B(x+k+1, \alpha)$from(4.14).

Thus

$$f(x) = \begin{cases} \binom{n}{x} \alpha \sum_{k=0}^{n-x} \binom{n-x}{k} (-1)^k B(x+k+1, \alpha), & x = 0,1,\ldots,n;\ \alpha > 0 \\ 0, & elsewhere \end{cases} \quad (4.15)$$

Properties of the Binomial-Generalized Exponential with 1 parameter distribution

The mean is

$$E(X) = nL_t(1) = n\alpha B(2, \alpha)$$

$$= \frac{n}{\alpha+1}$$

And the Variance is

$$Var(X) = nL_t(1) - nL_t(2) + n^2[L_t(2) - [L_t(1)]^2]$$

$$= n\alpha B(2, \alpha) - n\alpha B(3, \alpha) + n^2\left[\alpha B(3, \alpha) - (\alpha B(2, \alpha))^2\right]$$

$$= \frac{n}{\alpha+1} - \frac{2n}{(\alpha+2)(\alpha+1)} + n^2\left[\frac{2}{(\alpha+2)(\alpha+1)} - \left(\frac{1}{\alpha+1}\right)^2\right]$$

$$= \frac{n\alpha}{(\alpha+2)(\alpha+1)} + \frac{n^2\alpha}{(\alpha+2)(\alpha+1)^2}$$

$$= \frac{n\alpha(\alpha+n+1)}{(\alpha+2)(\alpha+1)^2}$$

4.2.5 Binomial-Generalized Exponential with 2 parameters distribution

4.2.5.1 Generalized Exponential with 2 parameters distribution

The general form of a Generalized Exponential (I) distribution is

$$g(\lambda) = \begin{cases} \alpha\left(1-e^{-\lambda}\right)^{\alpha-1} e^{-\lambda} & \lambda > 0; \alpha > 0 \\ 0 \ , & elsewhere \end{cases}$$

Putting $\lambda = t\beta$, $t > 0$

Then $g(\lambda)$becomes

$$g_1(t) = \begin{cases} \alpha\beta\left(1-e^{-t\beta}\right)^{\alpha-1} e^{-t\beta}, & t > 0; \alpha, \beta > 0 \\ 0 \ , & elsewhere \end{cases} \quad (4.16)$$

This is the p.d.f. of the Generalized Exponential distribution, with 2 parameters α and β . We denote it as the Generalized Exponential (II) distribution.

Its Laplace transform$L_t(s)$is

$$E(e^{-ts}) = \alpha\beta \int_0^\infty \left(1-e^{-t\beta}\right)^{\alpha-1} e^{-t(s+\beta)}\, dt$$

Letting $a = t\beta$, then

$$E(e^{-ts}) = \alpha \int_0^\infty (1-e^{-a})^{\alpha-1} e^{-a\left(\frac{s+\beta}{\beta}\right)}\, da$$

But $B(\alpha,\beta) = \int_0^\infty e^{-t\alpha}(1-e^{-t})^{\beta-1}\, dt$

where $B(\alpha,\beta)$is the Beta function.

Thus

$$L_t(s) = \alpha\, B\left(\frac{s+\beta}{\beta}, \alpha\right) \quad (4.17)$$

4.2.5.2 Generalized Exponential with 2 parameters mixing distribution

The Binomial-Generalized Exponential (II) distribution is given by

$$f(x) = \binom{n}{x}\sum_{k=0}^{n-x}\binom{n-x}{k}(-1)^k L_t(x+k)$$

$L_t(x+k) = \alpha\, B\left(\frac{x+k+\beta}{\beta}, \alpha\right)$from(4.17).

Thus

$$f(x) = \begin{cases} \binom{n}{x}\alpha\sum_{k=0}^{n-x}\binom{n-x}{k}(-1)^k B\left(\frac{x+k+\beta}{\beta}, \alpha\right), & x = 0,1,\dots,n;\ \alpha,\beta > 0 \\ 0\ , & elsewhere \end{cases} \quad (4.18)$$

Properties of the Binomial-Generalized Exponential with 2 parameters distribution

The mean is

$$E(X) = nL_t(1) = n\alpha\, B\left(\frac{1+\beta}{\beta}, \alpha\right)$$

And the Variance is

$$Var(X) = nL_t(1) - nL_t(2) + n^2[L_t(2) - [L_t(1)]^2]$$

$$= n\alpha B\left(\frac{1+\beta}{\beta}, \alpha\right) - n\alpha B\left(\frac{2+\beta}{\beta}, \alpha\right) + n^2\left[\alpha B\left(\frac{2+\beta}{\beta}, \alpha\right) - \left(\alpha\, B\left(\frac{1+\beta}{\beta}, \alpha\right)\right)^2\right]$$

$$= \frac{n}{\alpha+1} - \frac{2n}{(\alpha+2)(\alpha+1)} + n^2\left[\frac{2}{(\alpha+2)(\alpha+1)} - \left(\frac{1}{\alpha+1}\right)^2\right]$$

4.2.6 Binomial-Variated Exponential distribution

4.2.6.1 Variated Exponential distribution

Consider an exponential distribution given by

$$g(t) = \begin{cases} \alpha e^{-\alpha t} & t > 0; \alpha > 0 \\ 0\ , & elsewhere \end{cases}$$

where α is randomized, and takes on the distribution$\frac{1}{\alpha \ln\left(\frac{b}{a}\right)}$

with $0 < a \leq \alpha < b$

Then the Variated Exponential distribution$g_1(t)$becomes

$$\frac{1}{\ln\left(\frac{b}{a}\right)}\int_a^b e^{-\alpha t}\, d\alpha$$

$$= \frac{1}{\ln\left(\frac{b}{a}\right)}\left[\frac{e^{-\alpha t}}{-t}\right]_a^b$$

$$g_1(t) = \frac{e^{-at} - e^{-bt}}{t\ln\left(\frac{b}{a}\right)}, \quad t > 0; 0 < a < b \tag{4.19}$$

This is the p.d.f. of the Variated Exponential distribution, with parameters a and b .

Its Laplace transform$L_t(s)$is

$$E(e^{-ts}) = \frac{1}{\ln\left(\frac{b}{a}\right)}\int_0^\infty e^{-ts}\frac{e^{-at} - e^{-bt}}{t}\, dt$$

$$= \frac{1}{\ln\left(\frac{b}{a}\right)}\int_0^\infty \frac{e^{-t(s+a)} - e^{-t(s+b)}}{t}\, dt$$

But,

$$\int_0^\infty \frac{f(at) - f(bt)\, dt}{t} = [f(0) - f(\infty)]\ln\left(\frac{b}{a}\right)$$

which is a frullani integral

Thus

$$L_t(s) = \frac{\ln\left(\frac{s+b}{s+a}\right)}{\ln\left(\frac{b}{a}\right)} \tag{4.20}$$

asobtained by Bowman et al (1992).

4.2.6.2 Variated Exponential mixing distribution

The Binomial-Variated Exponential distribution is given by

$$f(x) = \binom{n}{x}\sum_{k=0}^{n-x}\binom{n-x}{k}(-1)^k L_t(x+k)$$

$L_t(x+k) = \dfrac{\ln\left(\frac{x+k+b}{x+k+a}\right)}{\ln\left(\frac{b}{a}\right)}$ from(4.20)

Thus

$$f(x) = \begin{cases} \binom{n}{x}\alpha\sum_{k=0}^{n-x}\binom{n-x}{k}(-1)^k \dfrac{\ln\left(\frac{x+k+b}{x+k+a}\right)}{\ln\left(\frac{b}{a}\right)}, & x = 0,1,\dots,n;\ 0 < a < b \\ 0\ , & elsewhere \end{cases} \qquad (4.21)$$

4.2.7 Binomial-Variated Gamma$(2, \alpha)$distribution

4.2.7.1 Variated Gamma$(2, \alpha)$distribution

Let

$$g(t) = \begin{cases} \alpha^2 t\, e^{-\alpha t} & t > 0; \alpha > 0 \\ 0\ , & elsewhere \end{cases}$$

where α is randomized, and takes on the distribution $\frac{1}{\alpha \ln\left(\frac{b}{a}\right)}$

with $0 < a \leq \alpha < b$

Then the Variated Gamma$(2, \alpha)$distribution $g_1(t)$becomes

$$\frac{t}{\ln\left(\frac{b}{a}\right)}\int_a^b \alpha\, e^{-\alpha t}\, d\alpha$$

Using integration by parts, $\int u dv = uv - \int v du$

Let $u = \alpha$ then $\frac{du}{d\alpha} = 1$

$$du = d\alpha$$

and $dv = e^{-\alpha t}$then $v = \int e^{-\alpha t}\, d\alpha$

$$v = \frac{-e^{-\alpha t}}{t}$$

Therefore

$$\int_a^b \alpha t\, e^{-\alpha t}\, d\alpha = \frac{-\alpha e^{-\alpha t}}{t} - \int \frac{-e^{-\alpha t}}{t}\, d\alpha$$

$$\left[\frac{-\alpha e^{-\alpha t}}{t} - \frac{e^{-\alpha t}}{t^2}\right]_a^b$$

$$= \frac{t}{\ln\left(\frac{b}{a}\right)}\left[\frac{ae^{-at}}{t} + \frac{e^{-at}}{t^2} - \frac{be^{-bt}}{t} - \frac{e^{-bt}}{t^2}\right]$$

And

$$g_1(t) = \frac{1}{t\ln\left(\frac{b}{a}\right)}[e^{-at}(at+1) - e^{-bt}(bt+1)] \tag{4.22}$$

with $t > 0; 0 < a < b$

Its Laplace transform$L_t(s)$is

$$E(e^{-ts}) = \frac{1}{\ln\left(\frac{b}{a}\right)}\int_0^\infty e^{-ts}\frac{1}{t}[e^{-at}(at+1) - e^{-bt}(bt+1)]\, dt$$

$$= \frac{1}{\ln\left(\frac{b}{a}\right)}\int_0^\infty \frac{e^{-t(s+a)}(at+1) - e^{-t(s+b)}(bt+1)}{t}\, dt$$

$$= \frac{1}{\ln\left(\frac{b}{a}\right)}\left[\int_0^\infty \frac{e^{-t(s+a)} - e^{-t(s+b)}}{t} dt + \int_0^\infty ae^{-t(s+a)}dt - \int_0^\infty be^{-t(s+b)}dt\right]$$

$$= \frac{1}{\ln\left(\frac{b}{a}\right)}\left[\ln\left(\frac{s+b}{s+a}\right) + \frac{a}{s+a} - \frac{b}{s+b}\right]$$

Thus

$$L_t(s) = \frac{1}{\ln\left(\frac{b}{a}\right)}\left[\ln\left(\frac{s+b}{s+a}\right) + \frac{a}{s+a} - \frac{b}{s+b}\right] \tag{4.23}$$

as obtained by Bowman et al (1992).

4.2.7.2 Variated Exponential mixing distribution

The Binomial-Variated Exponential distribution is given by

$$f(x) = \binom{n}{x} \sum_{k=0}^{n-x} \binom{n-x}{k} (-1)^k L_t(x+k)$$

$L_t(x+k) = \frac{1}{\ln\left(\frac{b}{a}\right)}\left[\ln\left(\frac{x+k+b}{x+k+a}\right) + \frac{a}{x+k+a} - \frac{b}{x+k+b}\right]$from(4.23).

Thus

$$f(x) = \binom{n}{x} \alpha \sum_{k=0}^{n-x} \binom{n-x}{k} (-1)^k \frac{1}{\ln\left(\frac{b}{a}\right)}\left[\ln\left(\frac{x+k+b}{x+k+a}\right) + \frac{a}{x+k+a} - \frac{b}{x+k+b}\right] \qquad (4.24)$$

with $x = 0,1,\dots,n; 0 < a, b$

4.2.8 Binomial-Inverse Gaussian distribution

4.2.8.1 Inverse Gaussian distribution

Construction

The pdf of the Inverse Gaussian distribution is given by

$$g(\lambda) = \left(\frac{\Phi}{2\pi\lambda^3}\right)^{\frac{1}{2}} exp\left\{\frac{-\Phi(\lambda-\mu)^2}{2\mu^2\lambda}\right\} \quad ; \quad \lambda > 0, \mu > 0, \phi > 1 \qquad (4.25)$$

Put $\mu = (2\alpha)^{-\frac{1}{2}} \Rightarrow \mu^2 = (2\alpha)^{-1}$

Then

$$g(\lambda) = \left(\frac{\Phi}{2\pi\lambda^3}\right)^{\frac{1}{2}} exp\left\{\frac{-\Phi\left(\lambda-(2\alpha)^{-\frac{1}{2}}\right)^2}{2(2\alpha)^{-1}\lambda}\right\}$$

$$= \left(\frac{\Phi}{2\pi\lambda^3}\right)^{\frac{1}{2}} exp\left\{\frac{-\Phi(2\alpha)\left(\lambda-(2\alpha)^{-\frac{1}{2}}\right)^2}{2\lambda}\right\}$$

$$= \left(\frac{\Phi}{2\pi\lambda^3}\right)^{\frac{1}{2}} exp\left\{\frac{-\Phi(2\alpha)}{2\lambda}\left[\lambda^2 - 2\lambda(2\alpha)^{-\frac{1}{2}} + (2\alpha)^{-1}\right]\right\}$$

$$= \left(\frac{\Phi}{2\pi\lambda^3}\right)^{\frac{1}{2}} exp\left\{\left[-\lambda\alpha\phi + \phi(2\alpha)^{\frac{1}{2}} - \frac{\phi}{2\lambda}\right]\right\} \qquad (4.26)$$

The Laplace transform is given by

$$L_\lambda(s) = \int_0^\infty e^{-s\lambda}\, g(\lambda)d\lambda$$

$$= \int_0^\infty \left(\frac{\Phi}{2\pi\lambda^3}\right)^{\frac{1}{2}} exp\left\{\left[-s\lambda - \lambda\alpha\phi + \phi(2\alpha)^{\frac{1}{2}} - \frac{\phi}{2\lambda}\right]\right\} d\lambda$$

$$= exp\left(\phi(2\alpha)^{\frac{1}{2}}\right)\int_0^\infty \left(\frac{\Phi}{2\pi\lambda^3}\right)^{\frac{1}{2}} exp\left\{\left[-s\lambda - \lambda\alpha\phi - \frac{\phi}{2\lambda}\right]\right\} d\lambda$$

Thus

$$L_\lambda(s) = exp\left(\phi(2\alpha)^{\frac{1}{2}}\right)\int_0^\infty \left(\frac{\Phi}{2\pi\lambda^3}\right)^{\frac{1}{2}} exp\left\{-\left(\phi\alpha + \frac{\phi s}{\phi}\right)\lambda - \frac{\phi}{2\lambda}\right\} d\lambda$$

$$L_\lambda(s) = exp\left(\phi(2\alpha)^{\frac{1}{2}}\right)\int_0^\infty \left(\frac{\Phi}{2\pi\lambda^3}\right)^{\frac{1}{2}} exp\left\{-\left(\alpha + \frac{s}{\phi}\right)\phi\lambda - \frac{\phi}{2\lambda}\right\} d\lambda \qquad (4.27)$$

The exponent in formula (4.26) is

$$-\lambda\alpha\phi + \phi(2\alpha)^{\frac{1}{2}} - \frac{\phi}{2\lambda}$$

Substitute α by $\alpha + \frac{s}{\varphi}$ to become

$$-\lambda\phi\left(\alpha+\frac{s}{\phi}\right)+\phi\left(2\left(\alpha+\frac{s}{\phi}\right)\right)^{\frac{1}{2}}-\frac{\phi}{2\lambda}$$

which is the exponent of formula (4.27), without the middle term. To include it, the exponent of formula (4.27) becomes

$$-\lambda\phi\left(\alpha+\frac{s}{\phi}\right)+\phi\left(2\left(\alpha+\frac{s}{\phi}\right)\right)^{\frac{1}{2}}-\frac{\phi}{2\lambda}-\phi\left(2\left(\alpha+\frac{s}{\phi}\right)\right)^{\frac{1}{2}}$$

Therefore

$$L_\lambda(s)= exp\left(\phi(2\alpha)^{\frac{1}{2}}\right)exp\left(-\phi\left(2\left(\alpha+\frac{s}{\phi}\right)\right)^{\frac{1}{2}}\right)\int_0^\infty\left(\frac{\Phi}{2\pi\lambda^3}\right)^{\frac{1}{2}} exp\left\{-\lambda\phi\left(\alpha+\frac{s}{\phi}\right)\right.$$

$$\left.+\phi\left(2\left(\alpha+\frac{s}{\phi}\right)\right)^{\frac{1}{2}}-\frac{\phi}{2\lambda}\right\}d\lambda$$

$$= exp\left(\phi(2\alpha)^{\frac{1}{2}}\right)exp\left(-\phi\left(2\left(\alpha+\frac{s}{\phi}\right)\right)^{\frac{1}{2}}\right)1$$

$$= exp\left(\phi(2\alpha)^{\frac{1}{2}}-\phi\left(2\left(\alpha+\frac{s}{\phi}\right)\right)^{\frac{1}{2}}\right) \qquad (4.28)$$

But$(2\alpha)^{-\frac{1}{2}}=\mu \Rightarrow (2\alpha)^{-1}=\mu^2$

Therefore

$$L_\lambda(s)= exp\left(\frac{\phi}{(2\alpha)^{-\frac{1}{2}}}-\phi\left(2\alpha+\frac{2s}{\phi}\right)^{\frac{1}{2}}\right)$$

$$= exp\left(\frac{\phi}{(2\alpha)^{-\frac{1}{2}}}-\phi\left(\frac{1}{(2\alpha)^{-1}}+\frac{2s}{\phi}\right)^{\frac{1}{2}}\right)$$

$$= exp\left(\frac{\phi}{\mu}-\phi\left(\frac{1}{\mu^2}+\frac{2s}{\phi}\right)^{\frac{1}{2}}\right)$$

$$= exp\left(\frac{\phi}{\mu} - \phi\left(\frac{\phi + 2s\mu^2}{\mu^2\phi}\right)^{\frac{1}{2}}\right)$$

$$= exp\left(\frac{\phi}{\mu}\left(1 - \left(\frac{\phi + 2s\mu^2}{\phi}\right)^{\frac{1}{2}}\right)\right) \tag{4.29}$$

Further, let $\alpha = 0$ in (4.26) and (4.28).

Then

$$g(\lambda) = \left(\frac{\Phi}{2\pi\lambda^3}\right)^{\frac{1}{2}} exp\left\{\left[-\frac{\phi}{2\lambda}\right]\right\}$$

and

$$L_\lambda(s) = exp\left(-\phi\left(2\left(\frac{s}{\phi}\right)\right)^{\frac{1}{2}}\right)$$

$$= exp\left(-(2(s\phi))^{\frac{1}{2}}\right)$$

Putting $\phi = \frac{k^2}{2}$we have

$$g(\lambda) = \left(\frac{k^2}{4\pi\lambda^3}\right)^{\frac{1}{2}} exp\left\{\left[-\frac{k^2}{4\lambda}\right]\right\} \tag{4.30}$$

and

$$L_\lambda(s) = exp\left\{-\left(2\left(s\frac{k^2}{2}\right)\right)^{\frac{1}{2}}\right\}$$

$$L_\lambda(s) = exp(-k\sqrt{s}) \tag{4.31}$$

as obtained by Bowman et al (1992).

4.2.8.2 Inverse Gaussian mixing distribution

The Binomial-Inverse Gaussian distribution is given by

$$f(x) = \binom{n}{x} \sum_{j=0}^{n-x} \binom{n-x}{j} (-1)^k L_\lambda(x+j)$$

$$L_\lambda(x+j) = exp\left(-k\sqrt{(x+j)}\right)$$

Thus

$$f(x) = \binom{n}{x} \sum_{j=0}^{n-x} \binom{n-x}{j} (-1)^k exp\left(-k\sqrt{(x+j)}\right)$$

for $x = 0,1, \dots, n;\ k > 0$

4.3 Using $p = 1 - \exp(-t)$

4.3.1 Binomial-Exponential distribution

The Binomial-Exponential distribution is given by

$$f(x) = \binom{n}{x} \sum_{k=0}^{x} \binom{x}{k} (-1)^k L_t(n-x+k)$$

$L_t(n-x+k) = \frac{\beta}{(n-x+k+\beta)}$ from(4.7)

Thus

$$f(x) = \beta \binom{n}{x} \sum_{k=0}^{x} \binom{x}{k} \frac{(-1)^k}{(n-x+k+\beta)} \qquad (4.32)$$

with $x = 0,1, \dots, n\ ;\ \beta > 0$

Properties of the Binomial-Exponential distribution

The mean is

$$E(X) = n(1 - L_t(1)) = n\left(1 - \frac{\beta}{(1+\beta)}\right)$$

4.3.2 Binomial-Gamma with 1 parameter distribution

The Binomial- Gamma (I) distribution is given by

$$f(x) = \binom{n}{x} \sum_{k=0}^{x} \binom{x}{k} (-1)^k L_t(n - x + k)$$

$L_t(x + k) = \frac{1}{(n-x+k+1)^\alpha}$from(4.9).

Thus

$$f(x) = \binom{n}{x} \sum_{k=0}^{x} \binom{x}{k} \frac{(-1)^k}{(n - x + k + 1)^\alpha} \quad (4.33)$$

with $x = 0,1, \dots, n\ ;\ \alpha > 0$

Properties of the Binomial- Gamma (I) distribution

The mean is

$$E(X) = n(1 - L_t(1)) = n\left(1 - \frac{1}{2^\alpha}\right)$$

4.3.3 Binomial- Gamma with 2 parameters distribution

The Binomial- Gamma (II) distribution is given by

$$f(x) = \binom{n}{x} \sum_{k=0}^{x} \binom{x}{k} (-1)^k L_t(n - x + k)$$

$L_t(n - x + k) = \left(\frac{\beta}{n-x+k+\beta}\right)^\alpha$ from(4.12).

Thus

$$f(x) = \begin{cases} \binom{n}{x} \beta^\alpha \sum_{k=0}^{x} \binom{x}{k} (-1)^k \left(\frac{1}{n - x + k + \beta}\right)^\alpha, & x = 0,1, \dots, n;\ \alpha, \beta > 0 \\ 0\ , & elsewhere \end{cases} \quad (4.34)$$

asobtained by Alanko and Duffy (1996).

Properties of the Binomial- Gamma (II) distribution

The mean is

$$E(X) = n(1 - L_t(1)) = n\left(1 - \left(\frac{\beta}{1+\beta}\right)^{\alpha}\right)$$

4.3.4 Binomial-Generalized Exponential with 1 parameter distribution

The Binomial- Generalized Exponential (I) distribution is given by

$$f(x) = \binom{n}{x} \sum_{k=0}^{x} \binom{x}{k} (-1)^k L_t(n - x + k)$$

$L_t(n - x + k) = \alpha B(n - x + k + 1, \alpha)$from(4.14).

Thus

$$f(x) = \begin{cases} \binom{n}{x} \alpha \sum_{k=0}^{x} \binom{x}{k} (-1)^k B(n - x + k + 1, \alpha), & x = 0,1,\dots,n;\ \alpha > 0 \\ 0, & elsewhere \end{cases} \quad (4.35)$$

Properties of the Binomial-Generalized Exponential (I) distribution

The mean is

$$E(X) = n(1 - L_t(1)) = n(1 - \alpha B(2, \alpha))$$

$$= n - \frac{n}{\alpha + 1}$$

4.3.5 Binomial-Generalized Exponential with 2 parameters distribution

The Binomial- Generalized Exponential (II) distribution is given by

$$f(x) = \binom{n}{x} \sum_{k=0}^{x} \binom{x}{k} (-1)^k L_t(n - x + k)$$

$L_t(n - x + k) = \alpha B\left(\frac{n-x+k+\beta}{\beta}, \alpha\right)$from(4.17).

Thus

$$f(x) = \begin{cases} \binom{n}{x} \alpha \sum_{k=0}^{x} \binom{x}{k} (-1)^k B\left(\frac{n-x+k+\beta}{\beta}, \alpha\right), & x = 0,1,\dots,n;\ \alpha > 0 \\ 0\ , & elsewhere \end{cases} \tag{4.36}$$

Properties of the Binomial-Generalized Exponential (II) distribution

The mean is

$$E(X) = n\big(1 - L_t(1)\big) = n\left(1 - \alpha\, B\left(\frac{1+\beta}{\beta}, \alpha\right)\right)$$

4.4 Using $p = cy$

4.4.1 Binomial-Generalized Beta with 4 parameters distribution

4.4.1.1 Generalized Beta with 4 parameters distribution

Consider a Beta distribution with p.d.f.

$$g(p) = \frac{p^{\left(\frac{r}{a}-1\right)}(1-p)^{w-1}}{B\left(\frac{r}{a}, w\right)}$$

where $0 < p < 1; r, a, w > 0$ and $0, elsewhere$.

Letting $P = \frac{y^a}{bw}$

We have

$$g(y) = \frac{\left(\frac{y^a}{bw}\right)^{\left(\frac{r}{a}-1\right)} \left(1 - \frac{y^a}{bw}\right)^{w-1} |J|}{B\left(\frac{r}{a}, w\right)}$$

where $|J| = \left|\frac{dp}{dy}\right| = \frac{a}{bw} y^{a-1}$

Thus

$$g(y) = \frac{y^{(r-1)} \left(1 - \frac{y^a}{bw}\right)^{w-1} \frac{a}{(bw)^{\frac{r}{a}}}}{B\left(\frac{r}{a}, w\right)} \tag{4.37}$$

for $0 < y < (bw)^{\frac{1}{a}}$; $a, w, r, b > 0$ and zero elsewhere.

$g(y)$is the p.d.f. of a Generalized Beta distribution, with 4 parameters a, w, r and b .

We denote this distribution as the Generalized Beta (4) distribution.

4.4.1.2 Generalized Beta (4) mixing distribution

The Binomial-Generalized Beta (4) distribution is given by

$$f(x) = \int_0^t (cy)^x(1-cy)^{n-x}\, g(y)dy \quad , \qquad 0 < y < t$$

where $g(y)$is the p.d.f. of the GB4 distribution.

Thus

$$f(x) = \binom{n}{x} \frac{c^x a}{B\left(\frac{r}{a}, w\right)} \sum_{k=0}^{n-x} \binom{n-x}{k} (-c)^k \int_0^{(bw)^{\frac{1}{a}}} \frac{y^{(x+k+r-1)}\left(1-\frac{y^a}{bw}\right)^{w-1}}{(bw)^{\frac{r}{a}}} dy$$

Putting $t = \frac{y^a}{bw}$we have

$$f(x) = \binom{n}{x} \frac{c^x a}{B\left(\frac{r}{a}, w\right)} \sum_{k=0}^{n-x} \binom{n-x}{k} (-c)^k \int_0^1 \frac{(tbw)^{\frac{1}{a}(x+k+r-1)}(1-t)^{w-1}}{(bw)^{\frac{r}{a}}} dy$$

where $dy = \frac{(bw)^{\frac{1}{a}}}{a} t^{\frac{1}{a}-1} dt$

$$f(x) = \binom{n}{x} \frac{c^x}{B\left(\frac{r}{a}, w\right)} \sum_{k=0}^{n-x} \binom{n-x}{k} (-c)^k \int_0^1 \frac{t^{\frac{x+k+r}{a}-1}(1-t)^{w-1}(bw)^{\frac{x+k}{a}}}{(bw)^{\frac{r}{a}}} dt$$

Therefore

$$f(x) = \binom{n}{x} \frac{c^x (bw)^{\frac{x}{a}}}{B\left(\frac{r}{a}, w\right)} \sum_{k=0}^{n-x} \binom{n-x}{k} (-c)^k (bw)^{\frac{k}{a}} B\left(\frac{x+k+r}{a}, w\right) \tag{4.38}$$

with $x = 0,1, \ldots, n; a, r, b, w > 0$

as obtained by Gerstenkorn (2004).

CHAPTER 5

BINOMIAL MIXTURES BASED ON LOG-INVERSE DISTRIBUTIONS IN THE [0, 1] DOMAIN

5.1 Introduction

Let $f(x)$be the pdf of X such that $0 < x < \infty$. If

$$Y = e^{-X} \implies X = -\log Y$$

Then the pdf of Y is given by

$$g(y) = f(x)\left|\frac{dx}{dy}\right|$$

$$= f(-\log y)\left|-\frac{1}{y}\right|$$

$$= \frac{1}{y}f(-\log y) \ , \ 0 < y < 1 \tag{5.1}$$

Next, let $Z = 1 - Y$. Then $y = 1 - z$ and $dy = -dz$

Therefore the pdf of Z is given by

$$h(z) = g(y)\left|\frac{dz}{dy}\right| = g(y) = g(1-z)$$

$$= \frac{1}{1-z} f(-\log(1-z)) \ , \ 0 < z < 1 \tag{5.2}$$

In using the derived pdf $h(z)$as a prior distribution for the Binomial, we make use of the methods used in chapters 2 and 3:

- Moments
- Direct substitution and integration.

5.2 Binomial-Type 1 Log inverse Exponential distribution

5.2.1 Type 1 Log inverse Exponential distribution

Construction

The general form of the exponential distribution is

$$g(y;\beta) = \begin{cases} \beta e^{-\beta y}, & y > 0;\ \beta > 0 \\ 0, & elsewhere \end{cases}$$

Putting $Y = \ln\left(\frac{1}{p}\right)$ yields the distribution

$$g(p;\beta) = \beta e^{\beta \ln p}|J|$$

where $|J| = \left|\frac{dy}{dp}\right| = \frac{1}{p}$

for $0 < p < 1$; $\beta > 0$ and $zero\ elsewhere$.

Thus

$$g(p;\beta) = \begin{cases} \beta p^{\beta-1}, & 0 < p < 1;\ \beta > 0 \\ 0, & elsewhere \end{cases} \tag{5.3}$$

This is the p.d.f. of the Type 1 Log inverse exponential distribution with parameter β . We denote this distribution as Type 1 LIE distribution. It can also be referred to as the power function distribution.

For properties, and details on the corresponding Binomial mixture, refer to (2.8).

5.3 Binomial-Type 2 Log inverse Exponential distribution

5.3.1 Type 2 Log inverse Exponential distribution

Construction

Given a Type 1 LIE distribution with pdf

$$g(p) = \begin{cases} \beta p^{\beta-1}, & 0 < p < 1;\ \beta > 0 \\ 0, & elsewhere \end{cases}$$

Let $P = 1 - U$, $0 < u < 1$.

Then $g(p)$ becomes

$$g(u) = \beta(1-u)^{\beta-1}|J|$$

where $|J| = \left|\frac{dp}{du}\right| = 1$

Thus

$$g(u) = \begin{cases} \beta(1-u)^{\beta-1}, & 0 < u < 1\,;\ \beta > 0 \\ 0\,, & elsewhere \end{cases} \qquad (5.4)$$

This is the p.d.f. of the Type 2 LIE distribution with parameter β . It is also called the Kumaraswamy (I) distribution.

Properties

The moment of order j about the origin of the Type 2 LIE distribution is given by

$$\begin{aligned} E(U^j) &= \int_0^1 u^j \beta(1-u)^{\beta-1} du \\ &= \beta \int_0^1 u^j (1-u)^{\beta-1} du \\ &= \beta\, B(j+1, \beta) \end{aligned} \qquad (5.5)$$

Thus the mean is

$$E(U) = \beta\, B(2,\beta) = \frac{1}{\beta+1}$$

and the variance is

$$\begin{aligned} Var(U) = \{E(U^2) - (E(U))^2\} &= \frac{2}{(\beta+2)(\beta+1)} - \frac{1}{(\beta+1)^2} \\ &= \frac{\beta}{(\beta+2)(\beta+1)^2} \end{aligned}$$

5.3.2 Type 2 LIE mixing distribution

The Binomial-Type 2 LIE distribution is given by

$$f(x) = \binom{n}{x} \int_0^1 p^x (1-p)^{n-x} g(p) dp$$

Where $g(p)$ is the pdf of the Type 2LIE distribution

Thus

$$f(x) = \binom{n}{x} \beta \int_0^1 p^x (1-p)^{n-x} (1-p)^{\beta-1}\, dp$$

$$= \binom{n}{x} \beta \int_0^1 p^x (1-p)^{n-x+\beta-1} dp$$

$$= \binom{n}{x} \beta B(x+1, n-x+\beta)$$

And

$$f(x) = \begin{cases} \binom{n}{x} \beta B(x+1, n-x+\beta), & x = 0,1,\dots,n\,; \beta > 0 \\ 0\ , & elsewhere \end{cases} \quad (5.6)$$

From the moments method this probability distribution can also be written as,

$$f(x) = \sum_{j=x}^{n} \frac{n!\ (-1)^{j-x}}{x!\ (j-x)!\,(n-j)!} E(P^j) \quad , \qquad x = 0,1,\dots,n$$

for $j \geq x$ and 0 if $j < x$

The Jth moment, $E(P^j)$ of the Type 2 LIE distribution is $\beta\ B(j+1, \beta) = \frac{j!\,\beta!}{(\beta+j)!}$ from (5.5)

Thus,

$$f(x) = \begin{cases} \beta! \sum_{j=x}^{n} \frac{n!\ (-1)^{j-x}}{x!\ (j-x)!\,(n-j)!} \frac{j!}{(\beta+j)!} , x = 0,1,\dots,n;\ \beta > 0 \\ 0\ , \qquad elsewhere \end{cases} \quad (5.7)$$

Properties of the Binomial-Type 2 LIE distribution

The mean is

$$E(X) = nE(P) = \frac{n}{\beta + 1}$$

and the variance, is

$$Var(X) = nE(P) - nE(P^2) + n^2Var(P)$$

$$= \frac{n}{\beta+1} - \frac{2n}{(\beta+2)(\beta+1)} + \frac{n^2\beta}{(\beta+2)(\beta+1)^2}$$

$$= \frac{n\beta(\beta+n+1)}{(\beta+2)(\beta+1)^2}$$

5.4 Binomial-Type 1 Log inverse Gamma with 1 parameter distribution

5.4.1 Type 1 Log inverse Gamma with 1 parameter distribution

Construction

The general form of the gamma distribution is

$$g(y;\alpha) = \begin{cases} \frac{e^{-y}y^{\alpha-1}}{\Gamma(\alpha)}, & y > 0; \alpha > 0 \\ 0, & elsewhere \end{cases}$$

Using the Type 1 transform $Y = \ln\left(\frac{1}{P}\right)$ yields the distribution

$$g(p;\alpha) = \frac{e^{\ln p}}{\Gamma(\alpha)}\left(\ln\left(\frac{1}{p}\right)\right)^{\alpha-1} |J|$$

where $|J| = \left|\frac{dy}{dp}\right| = \frac{1}{p}$

for $0 < p < 1 ; \alpha > 0$

Thus

$$g(p;\alpha) = \begin{cases} \left(\ln\left(\frac{1}{p}\right)\right)^{\alpha-1} \frac{1}{\Gamma(\alpha)}, 0 < p < 1 ; \alpha > 0 \\ 0, \quad elsewhere \end{cases} \quad (5.8)$$

This is the p.d.f. of the Type 1 Log inverse Gamma with the parameter α distribution. We denote this distribution as Type 1 LIG (I) distribution.

It is also referred to as the Standard unit gamma distribution

Properties

The moment of order j about the origin of this distribution is given by

$$E(P^j) = \int_0^1 p^j \left(\ln\left(\frac{1}{p}\right)\right)^{\alpha-1} \frac{1}{\Gamma(\alpha)}\, dp$$

Letting $\ln\left(\frac{1}{p}\right) = a \; ; \; a > 0$

We have

$$E(P^j) = \int_0^\infty e^{-a(j+1)} a^{\alpha-1} \frac{1}{\Gamma(\alpha)}\, da$$

$$= \frac{1}{(j+1)^\alpha} \qquad (5.9)$$

Thus the mean is

$$E(P) = \frac{1}{2^\alpha}$$

and the variance is

$$Var(P) = \{E(P^2) - (E(P))^2\} = \frac{1}{3^\alpha} - \left(\frac{1}{2^\alpha}\right)^2$$

5.4.2 Type 1 LIG (I) mixing distribution

The Binomial-Type 1 LIG distribution with (I) is given by

$$f(x) = \binom{n}{x} \int_0^1 p^x (1-p)^{n-x} g(p)\, dp$$

Where $g(p)$ is the pdf of the Type 1 LIG (I) distribution.

Thus

$$f(x) = \binom{n}{x} \int_0^1 p^x (1-p)^{n-x} \left(\ln\left(\frac{1}{p}\right)\right)^{\alpha-1} \frac{1}{\Gamma(\alpha)}\, dp$$

Putting $a = \ln\left(\frac{1}{p}\right)$, $a > 0$

We have,

$$f(x) = \binom{n}{x}\int_0^{\infty} e^{-a(x+1)}(1-e^{-a})^{n-x} a^{\alpha-1}\frac{1}{\Gamma(\alpha)}\, da$$

$$= \binom{n}{x}\sum_{k=0}^{n-x}\binom{n-x}{k}(-1)^k\int_0^{\infty}\frac{a^{\alpha-1}}{\Gamma(\alpha)}e^{-a(x+k+1)}\, da$$

$$= \binom{n}{x}\sum_{k=0}^{n-x}\binom{n-x}{k}(-1)^k\left(\frac{1}{x+k+1}\right)^{\alpha}$$

Thus,

$$f(x) = \begin{cases}\binom{n}{x}\sum_{k=0}^{n-x}\binom{n-x}{k}(-1)^k\left(\frac{1}{x+k+1}\right)^{\alpha}, & x = 0,1,\dots,n\,;\alpha > 0\\ 0\ , & elsewhere\end{cases} \qquad (5.10)$$

From the Moments method, this probability distribution can also be written as,

$$f(x) = \sum_{j=x}^{n}\frac{n!\,(-1)^{j-x}}{x!\ (j-x)!\,(n-j)!}E(P^j) \quad , \quad x = 0,1,\dots,n$$

for $j \geq x$ and 0 if $j < x$

The Jth moment, $E(P^J)$ of the Type 1 LIG (I) distribution is $\frac{1}{(j+1)^{\alpha}}$, from (5.9).

Thus,

$$f(x) = \sum_{j=x}^{n}\frac{n!\ (-1)^{j-x}}{x!\ (j-x)!\,(n-j)!}\frac{1}{(j+1)^{\alpha}} \qquad (5.11)$$

with $x = 0,1,\dots,n\,;\alpha > 0$

5.5 Binomial-Type 2 Log Inverse Gamma with 1 parameter distribution

5.5.1 Type 2 Log inverse Gamma with 1 parameter distribution

Construction

Given a Type 1 LIG (I) distribution, withdensity function

$$g(p) = \begin{cases} \left(\ln\left(\frac{1}{p}\right)\right)^{\alpha-1} \frac{1}{\Gamma(\alpha)}, & 0 < p < 1 ; \alpha > 0 \\ 0, & elsewhere \end{cases}$$

Let $p = 1 - u$, $0 < u < 1$.

Then $g(p)$becomes

$$g(u) = \left[\ln\left(\frac{1}{1-u}\right)\right]^{\alpha-1} \frac{1}{\Gamma(\alpha)} |J|$$

where$|J| = \left|\frac{dp}{du}\right| = 1$

Thus

$$g(u) = \begin{cases} \left[\ln\left(\frac{1}{1-u}\right)\right]^{\alpha-1} \frac{1}{\Gamma(\alpha)}, & 0 < u < 1 ; \alpha > 0 \\ 0, & elsewhere \end{cases} \tag{5.12}$$

This is the p.d.f. of the Type 2 LIG (I) distribution with parameter α .

Properties

The moment of order j about the origin of the Type 2 LIG (I) distribution is given by

$$E(U^j) = \int_0^1 u^j \left(\ln \frac{1}{1-u}\right)^{\alpha-1} \frac{1}{\Gamma(\alpha)} du$$

Letting$\ln \frac{1}{1-u} = a \; ; \; a > 0$

We have

$$E(U^j) = \int_0^\infty (1 - e^{-a})^j e^{-a} a^{\alpha-1} \frac{1}{\Gamma(\alpha)} da$$

$$= \sum_{k=0}^{j} \binom{j}{k} (-1)^k \int_0^\infty \frac{1}{\Gamma(\alpha)} e^{-a(k+1)} a^{\alpha-1} \, da$$

$$= \sum_{k=0}^{j} \binom{j}{k} (-1)^k \frac{1}{(k+1)^\alpha} \tag{5.13}$$

Thus the mean is

$$E(U) = 1 - \left(\frac{1}{2}\right)^\alpha$$

and the variance is

$$Var(U) = \{E(U^2) - (E(U))^2\} = \frac{1}{3^\alpha} - \frac{1}{2^{2\alpha}}$$

5.5.2 Type 2 LIG (I) mixing distribution

The Binomial-Type 2 LIG (I) distribution is given by

$$f(x) = \binom{n}{x} \int_0^1 p^x (1-p)^{n-x} g(p) dp$$

Where $g(p)$ is the pdf of the Type 2 LIG (I) distribution,

Thus

$$f(x) = \binom{n}{x} \int_0^1 p^x (1-p)^{n-x} \left[\ln\left(\frac{1}{1-p}\right)\right]^{\alpha-1} \frac{1}{\Gamma(\alpha)} \, dp$$

Putting $a = \ln\left(\frac{1}{1-p}\right)$, $a > 0$

We have,

$$f(x) = \binom{n}{x} \int_0^\infty (1 - e^{-a})^x (e^{-a})^{n-x+1} a^{\alpha-1} \frac{1}{\Gamma(\alpha)} \, da$$

$$= \binom{n}{x} \sum_{k=0}^{x} \binom{x}{k} (-1)^k \int_0^\infty (e)^{-a(n-x+k+1)} a^{\alpha-1} \frac{1}{\Gamma(\alpha)} \, da$$

$$= \binom{n}{x}\sum_{k=0}^{x}\binom{x}{k}(-1)^k\left(\frac{1}{n-x+k+1}\right)^{\alpha}$$

Therefore,

$$f(x) = \begin{cases} \binom{n}{x}\sum_{k=0}^{x}\binom{x}{k}(-1)^k\left(\frac{1}{n-x+k+1}\right)^{\alpha}, & x = 0,1,\dots,n;\ \alpha > 0 \\ 0 \quad , & elsewhere \end{cases} \tag{5.14}$$

From the Moments method, this probability distribution can also be written as,

$$f(x) = \sum_{j=x}^{n}\frac{n!\ (-1)^{j-x}}{x!\ (j-x)!\,(n-j)!}E(P^j) \quad , \qquad x = 0,1,\dots,n$$

for $j \geq x$ and 0 if $j < x$

The Jth moment, $E(P^j)$ of this distribution is $\sum_{k=0}^{j}\binom{j}{k}(-1)^k\frac{1}{(k+1)^{\alpha}}$ from (5.13).

Thus,

$$f(x) = \sum_{j=x}^{n}\frac{n!\ (-1)^{j-x}}{x!\ (j-x)!\,(n-j)!}\frac{1}{(j+1)^{\alpha}}\sum_{k=0}^{j}\binom{j}{k}(-1)^k\frac{1}{(k+1)^{\alpha}} \tag{5.15}$$

with $x = 0,1,\dots,n\,;\alpha > 0\,;$

5.6 Binomial-Type 1 Log inverse Gamma distribution with 2 parameters

5.6.1 Type 1 Log inverse Gamma distribution with 2 parameters

Construction

Consider a 2 parameter Gamma distribution

$$g(y) = \begin{cases} \frac{\beta^{\alpha}}{\Gamma(\alpha)}y^{\alpha-1}e^{-y\beta} \quad , & y > 0;\ \alpha,\beta > 0 \\ 0 \quad , & elsewhere \end{cases}$$

Using the Type 1 transform $Y = \ln\left(\frac{1}{P}\right)$ yields the distribution

$$g(p) \;=\; \frac{e^{\beta \ln p}}{\Gamma(\alpha)}\left(\ln\left(\frac{1}{p}\right)\right)^{\alpha-1} \beta^{\alpha}|J|$$

where$|J| \;=\; \left|\frac{dy}{dp}\right| \;=\; \frac{1}{p}$

for $0 < p < 1\,;\, \alpha, \beta > 0$ and $zero\ elsewhere\,.$

Thus

$$g(p) \;=\; \begin{cases} \left(\ln\left(\frac{1}{p}\right)\right)^{\alpha-1} \frac{\beta^{\alpha} p^{\beta-1}}{\Gamma(\alpha)}\,, & 0 < p < 1\,;\, \alpha, \beta > 0 \\ 0\,, \;\; elsewhere & \end{cases} \qquad (5.16)$$

asobtained by Grassia (1977).

This is the p.d.f. of the Type 1 Log inverse Gamma distribution with parameters α and β .We denote this distribution as the Type 1 LIG (II) distribution.

It is also called the Unit Gamma (Log-gamma) or the Grassia 1 distribution (see McDonald & Yexiao, (1995))

Properties

The moment of order j about the origin of this distribution is given by

$$E(P^j) = \int_0^1 p^j \left(\ln\left(\frac{1}{p}\right)\right)^{\alpha-1} \frac{\beta^{\alpha} p^{\beta-1}}{\Gamma(\alpha)}\, dp$$

Letting$\ln\left(\frac{1}{p}\right) \;=\; a\,,\;\; a > 0$

We have

$$E(P^j) = \frac{\beta^{\alpha}}{\Gamma(\alpha)}\int_0^{\infty} e^{-a(j)} e^{-a(\beta-1)} a^{\alpha-1}\, e^{-a}\, da$$

$$= \frac{\beta^{\alpha}}{\Gamma(\alpha)}\int_0^{\infty} e^{-a(j+\beta)} a^{\alpha-1}\; da$$

Therefore,

$$E(P^j) \;=\; \left(\frac{\beta}{j+\beta}\right)^{\alpha} \qquad (5.17)$$

Thus the mean is

$$E(P) = \left(\frac{\beta}{1+\beta}\right)^{\alpha}$$

and the variance is

$$Var(P) = \{E(P^2) - (E(P))^2\} = \left(\frac{\beta}{\beta+2}\right)^{\alpha} - \left(\frac{\beta^2}{\beta^2+2\beta+1}\right)^{\alpha}$$

5.6.2 Type 1 LIG (II) mixing distribution

The Binomial-Type 1 LIG (II) distribution is given by

$$f(x) = \binom{n}{x}\int_0^1 p^x(1-p)^{n-x}g(p)dp$$

Where $g(p)$ is the pdf of the Type 1 LIG (II) distribution.

Thus

$$f(x) = \binom{n}{x}\int_0^1 p^x(1-p)^{n-x}\left(\ln\left(\frac{1}{p}\right)\right)^{\alpha-1}\frac{\beta^{\alpha}p^{\beta-1}}{\Gamma(\alpha)}dp$$

$$= \binom{n}{x}\sum_{k=0}^{n-x}\binom{n-x}{k}(-1)^k\int_0^1 p^{x+k}\left(\ln\left(\frac{1}{p}\right)\right)^{\alpha-1}\frac{\beta^{\alpha}p^{\beta-1}}{\Gamma(\alpha)}dp$$

But,

$$\int_0^1 p^{x+k}\left(\ln\frac{1}{p}\right)^{\alpha-1}\frac{\beta^{\alpha}p^{\beta-1}}{\Gamma(\alpha)}dp = E(p^{x+k}) = \left(\frac{\beta}{x+k+\beta}\right)^{\alpha}$$

Therefore,

$$f(x) = \binom{n}{x}\sum_{k=0}^{n-x}\binom{n-x}{k}(-1)^k\left(\frac{\beta}{x+k+\beta}\right)^{\alpha} \quad (5.18)$$

with $x = 0,1,\ldots,n$; $\alpha,\beta > 0$ and zero elsewhere.

From the Moments method, this probability distribution can also be written as,

$$f(x) = \sum_{j=x}^{n} \frac{n!\ (-1)^{j-x}}{x!\ (j-x)!\,(n-j)!} E(P^j) \quad , \quad x = 0,1,\dots,n$$

for $j \geq x$ and 0 if $j < x$

The Jth moment, $E(P^j)$ of the Type 1 LIG (II) distribution is $\left(\frac{\beta}{j+\beta}\right)^{\alpha}$ from (5.17).

Thus,

$$f(x) = \sum_{j=x}^{n} \frac{n!\ (-1)^{j-x}}{x!\ (j-x)!\,(n-j)!} \left(\frac{\beta}{j+\beta}\right)^{\alpha} \tag{5.19}$$

with $x = 0,1,\dots,n\ ;\ \alpha,\beta > 0$; and zero elsewhere.

5.7 Binomial-Type 2 LIG (II) distribution

5.7.1 Type 2 LIG (II) distribution

Construction

Given a Type 1 LIG (II) distribution, with density function

$$g(p) = \begin{cases} \left(\ln\left(\frac{1}{p}\right)\right)^{\alpha-1} \frac{\beta^{\alpha} p^{\beta-1}}{\Gamma(\alpha)} , & 0 < p < 1\ ;\ \alpha,\beta > 0 \\ 0\ , \quad elsewhere \end{cases}$$

Let $P = 1 - U\ ,\ \ 0 < u < 1$.

Then $g(p)$becomes

$$g(u) = \left[\ln\left(\frac{1}{1-u}\right)\right]^{\alpha-1} \frac{\beta^{\alpha}(1-u)^{\beta-1}}{\Gamma(\alpha)} |J|$$

where $|J| = \left|\frac{dp}{du}\right| = 1$

Therefore,

$$g(u) = \begin{cases} \left[\ln\left(\frac{1}{1-u}\right)\right]^{\alpha-1} \frac{\beta^{\alpha}(1-u)^{\beta-1}}{\Gamma(\alpha)} , & 0 < u < 1;\ \alpha,\beta > 0 \\ 0\ , \quad elsewhere \end{cases} \tag{5.20}$$

This is the p.d.f. of the Type 2 LIG (II)/Grassia 2 distribution with parameters α, β .

asobtained by Grassia (1977).

Properties

The moment of order j about the origin of this distribution is given by

$$E(U^j) = \int_0^1 u^j \left[\ln\left(\frac{1}{1-u}\right)\right]^{\alpha-1} \frac{\beta^\alpha (1-u)^{\beta-1}}{\Gamma(\alpha)} \, dp$$

Lettingln $\left(\frac{1}{1-u}\right) = a , \; a > 0$

We have

$$E(U^j) = \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^\infty (1-e^{-a})^j e^{-a(\beta-1)} a^{\alpha-1} e^{-a} \, da$$

$$= \frac{\beta^\alpha}{\Gamma(\alpha)} \sum_{k=0}^{j} \binom{j}{k} (-1)^k \int_0^\infty e^{-a(k+\beta)} a^{\alpha-1} \, da$$

$$= \beta^\alpha \sum_{k=0}^{j} \binom{j}{k} (-1)^k \left(\frac{1}{k+\beta}\right)^\alpha \tag{5.21}$$

Thus the mean is

$$E(U) = 1 - \left(\frac{\beta}{1+\beta}\right)^\alpha$$

and the variance is

$$Var(U) = \{E(u^2) - (E(u))^2\} = \left(\frac{\beta}{\beta+2}\right)^\alpha - \left(\frac{\beta}{\beta+1}\right)^{2\alpha}$$

5.7.2 Type 2 LIG (II) mixing distribution

The Binomial-Type 2 LIG (II) distribution is given by

$$f(x) = \binom{n}{x}\int_0^1 p^x(1-p)^{n-x} g(p)dp$$

Where $g(p)$ is the pdf of the Type 2 LIG (II) distribution.

Thus

$$f(x) = \binom{n}{x}\int_0^1 p^x(1-p)^{n-x}\left[\ln\left(\frac{1}{1-p}\right)\right]^{\alpha-1}\frac{\beta^\alpha(1-p)^{\beta-1}}{\Gamma(\alpha)}\,dp$$

Puttingln $\left(\frac{1}{1-p}\right) = a\ ,\ a > 0$

We have,

$$f(x) = \binom{n}{x}\frac{\beta^\alpha}{\Gamma(\alpha)}\int_0^\infty e^{-a(n-x+\beta-1)}(1-e^{-a})^x a^{\alpha-1}e^{-a}\,dp$$

$$= \binom{n}{x}\sum_{k=0}^{x}\binom{x}{k}(-1)^k\frac{\beta^\alpha}{\Gamma(\alpha)}\int_0^\infty e^{-a(n-x+k+\beta)}a^{\alpha-1}\,dp$$

$$= \binom{n}{x}\beta^\alpha\sum_{k=0}^{x}\binom{x}{k}(-1)^k\left(\frac{1}{n-x+k+\beta}\right)^\alpha$$

Thus,

$$f(x) = \begin{cases}\binom{n}{x}\beta^\alpha\sum_{k=0}^{x}\binom{x}{k}(-1)^k\left(\frac{1}{n-x+k+\beta}\right)^\alpha, & x = 0,1,\dots,n;\ \alpha,\beta > 0\\ 0\ , & elsewhere\end{cases} \qquad (5.22)$$

From the Moments method, this probability distribution can also be written as,

$$f(x) = \sum_{j=x}^{n}\frac{n!\ (-1)^{j-x}}{x!\ (j-x)!\,(n-j)!}E(P^j)\ ,\quad x = 0,1,\dots,n$$

for $j \geq x$ and 0 if $j < x$

The Jth moment, $E(P^j)$ of the Type 2 LIG (II) distribution is$\beta^\alpha\sum_{k=0}^{j}\binom{j}{k}(-1)^k\left(\frac{1}{k+\beta}\right)^\alpha$ from(5.21).

Thus,

$$f(x) = \sum_{j=x}^{n} \frac{n!\,(-1)^{j-x}}{x!\;(j-x)!\,(n-j)!} \beta^{\alpha} \sum_{k=0}^{j} \binom{j}{k} (-1)^{k} \left(\frac{1}{k+\beta}\right)^{\alpha} \qquad (5.23)$$

with $x = 0,1,\dots,n\,;\alpha,\beta > 0$; and $zero\;\; elsewhere$.

5.8 Binomial-Type 1 Log inverse Gamma distribution with 3 parameters

5.8.1 Type 1 Log Inverse Gamma distribution with 3 parameters

Construction

Consider a Type 1 LIG (II) distribution, with p.d.f.

$$g(p) = \begin{cases} \left(\ln\left(\frac{1}{p}\right)\right)^{\alpha-1} \dfrac{\beta^{\alpha} p^{\beta-1}}{\Gamma(\alpha)}, & 0 < p < 1\,;\, \alpha,\beta > 0 \\ 0\,, \;\; elsewhere \end{cases}$$

Putting $P = \frac{Y}{s}$, $s > 0$

We have

$$g(y) = \left[-\ln\left(\frac{y}{s}\right)\right]^{\alpha-1} \frac{\beta^{\alpha} \left(\frac{y}{s}\right)^{\beta-1}}{\Gamma(\alpha)} |J|$$

where $|J| = \left|\frac{dp}{dy}\right| = \frac{1}{s}$

Therefore,

$$g(y) = \begin{cases} \left[-\ln\left(\frac{y}{s}\right)\right]^{\alpha-1} \dfrac{\beta^{\alpha} \left(\frac{y}{s}\right)^{\beta-1}}{\Gamma(\alpha) s}, & 0 < y < s;\; \alpha,\beta > 0 \\ 0\,, \quad elsewhere \end{cases} \qquad (5.24)$$

This is the **p.d.f.** of the Type 1 Log inverse Gamma distribution with 3 parameters α, s and β . We denote this distribution as the Type 1 LIG (III) distribution.

Properties

The moment of order j about the origin of this distribution is given by

$$E(Y^j) = \int_0^s y^j \left[-\ln\left(\frac{y}{s}\right)\right]^{\alpha-1} \frac{\beta^\alpha \left(\frac{y}{s}\right)^{\beta-1}}{\Gamma(\alpha)s}\, dy$$

Letting $y = as$

We have

$$E(Y^j) = \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^1 \frac{(as)^{j+\beta-1}}{s^\beta} \left[-\ln\left(\frac{as}{s}\right)\right]^{\alpha-1} da$$

$$= \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^1 a^{j+\beta-1} s^j (-\ln a)^{\alpha-1}\, da$$

Let $-\ln a = t$

Then,

$$E(Y^j) = -\frac{\beta^\alpha}{\Gamma(\alpha)} \int_\infty^0 e^{-t(j+\beta-1)} s^j t^{\alpha-1} e^{-t}\, dt$$

$$= \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^\infty e^{-t(j+\beta-1)} s^j t^{\alpha-1} e^{-t}\, dt$$

Thus,

$$E(Y^j) = \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^\infty e^{-t(j+\beta)} s^j t^{\alpha-1}\, dt$$

$$= \frac{s^j \beta^\alpha}{(j+\beta)^\alpha} \qquad (5.25)$$

Thus the mean is

$$E(Y) = \frac{s\,\beta^\alpha}{(1+\beta)^\alpha}$$

and the variance is

$$Var(Y) = \{E(Y^2) - (E(Y))^2\} = \frac{s^2\beta^\alpha}{(2+\beta)^\alpha} - \left(\frac{s\,\beta^\alpha}{(1+\beta)^\alpha}\right)^2$$

5.8.2 Type 1 LIG (III) mixing distribution

The Binomial-Type 1 LIG (III) distribution is given by

$$f(x) = \binom{n}{x}\int_0^s (cy)^x(1-cy)^{n-x}g(y)dy$$

Where $g(y)$ is the pdf of the Type 1 LIG (III) distribution

Thus

$$f(x) = \binom{n}{x}\int_0^s (cy)^x(1-cy)^{n-x}\left[-\ln\left(\frac{y}{s}\right)\right]^{\alpha-1}\frac{\beta^\alpha\left(\frac{y}{s}\right)^{\beta-1}}{\Gamma(\alpha)s}\,dy$$

Putting $-\ln\left(\frac{y}{s}\right) = a\,,\qquad 0 < a < \infty$

We have,

$$f(x) = \binom{n}{x}\frac{\beta^\alpha c^x}{\Gamma(\alpha)s^\beta}\sum_{k=0}^{n-x}\binom{n-x}{k}(-c)^k\int_0^\infty (se^{-a})^{x+k+\beta-1}a^{\alpha-1}se^{-a}$$

Therefore,

$$f(x) = \binom{n}{x}\frac{\beta^\alpha c^x}{\Gamma(\alpha)s^\beta}\sum_{k=0}^{n-x}\binom{n-x}{k}(-c)^k s^{x+k+\beta}\int_0^\infty (e^{-a})^{x+k+\beta}a^{\alpha-1}\,da$$

$$= \binom{n}{x}\beta^\alpha c^x\sum_{k=0}^{n-x}\binom{n-x}{k}(-c)^k\frac{s^{x+k}}{(x+k+\beta)^\alpha}$$

$$f(x) = \begin{cases}\binom{n}{x}\beta^\alpha c^x\sum_{k=0}^{n-x}\binom{n-x}{k}(-c)^k\frac{s^{x+k}}{(x+k+\beta)^\alpha}, & x = 0,1,\dots,n;\ \alpha,\beta > 0\\ 0\ , & elsewhere\end{cases} \quad (5.26)$$

5.9 Binomial-Type 1 Log inverse Generalized Exponential distribution with 1 parameter

5.9.1 Type 1 Log inverse Generalized Exponential distribution with 1 parameter

Construction

The general form of the 1 parameter Generalized Exponential distribution is

$$g(t) = \begin{cases} \alpha(1-e^{-t})^{\alpha-1}e^{-t}, & t>0; \alpha>0 \\ 0, & elsewhere \end{cases}$$

Putting $T = \ln\left(\frac{1}{p}\right)$yields the distribution,

$$g(p) = \alpha\left(1-e^{\ln p}\right)^{\alpha-1} e^{\ln p}|J|$$

where$|J| = \left|\frac{dt}{dp}\right| = \frac{1}{p}$

for $0<p<1$; $\alpha>0$ and zero elsewhere.

Thus

$$g(p) = \begin{cases} \alpha(1-p)^{\alpha-1}, 0<p<1; \alpha>0 \\ 0, \quad elsewhere \end{cases} \tag{5.27}$$

This is the p.d.f. of the Type 1 Log inverse Generalized Exponential distribution with parameter α .We denote this distribution as Type 1 LIGE (I)

It is also referred to as the Kumaraswamy (I) distribution.

For properties and the corresponding Binomial mixture, refer to (3.4).

5.10 Binomial-Type 2 LIGE (I) distribution

5.10.1 Type 2 LIGE (I) distribution

Construction

Given a Type 1 LIGE (I) distribution, with density function

$$g(p) = \begin{cases} \alpha(1-p)^{\alpha-1}, 0<p<1; \alpha>0 \\ 0, \quad elsewhere \end{cases}$$

Let $P=1-U$, $0<u<1$.

Then $g(p;\alpha)$becomes

$$g(u) = \alpha u^{\alpha-1}|J|$$

where$|J| = \left|\frac{dp}{du}\right| = 1$

Therefore,

$$g(u) = \begin{cases} \alpha u^{\alpha-1}, & 0 < u < 1;\ \alpha, > 0 \\ 0\ , & elsewhere \end{cases} \tag{5.28}$$

This is the p.d.f. of the Type 2 LIGE (I) distribution, with parameter α . It is also called the power function distribution.

For properties, and details on the corresponding Binomial mixture, refer to (2.8).

5.11 Binomial-Type 1 Log inverse Generalized Exponential distribution with 2 parameters

5.11.1 Type 1 Log inverse Generalized Exponential distribution with 2 parameters Construction

The general form of a 2 parameter generalized exponential distribution is,

$$g(t) = \begin{cases} \alpha\beta\left(1 - e^{-t\beta}\right)^{\alpha-1} e^{-t\beta}, & t > 0; \alpha, \beta > 0 \\ 0\ , & elsewhere \end{cases}$$

Putting $T = \ln\left(\frac{1}{P}\right)$gives us the distribution,

$$g(p) = \alpha\beta\left(1 - e^{\beta\ln p}\right)^{\alpha-1} e^{\beta\ln p}|J|$$

where$|J| = \left|\frac{dt}{dp}\right| = \frac{1}{p}$

for $0 < p < 1\ ;\ \alpha, \beta > 0$ and zero elsewhere.

Thus

$$g(p) = \begin{cases} \alpha\beta\left(1 - p^{\beta}\right)^{\alpha-1} p^{\beta-1}, 0 < p < 1\ ;\ \alpha, \beta > 0 \\ 0\ , \quad elsewhere \end{cases} \tag{5.29}$$

This is the p.d.f. of the Type 1 Log inverse Generalized Exponential distribution with parameters α and β . We denote this distribution as Type 1 LIGE (II)

It is also referred to as the Kumaraswamy (II) distribution.

For details on properties and the corresponding Binomial mixture, refer to (3.3).

5.12 Binomial-Type 2 Log inverse Generalized Exponential distribution with 2 parameters

5.12.1 Type 2 Log Inverse Generalized Exponential distribution with 2 parameters

Construction

Given a Type 1 LIGE (II) distribution, with density function

$$g(p) = \begin{cases} \alpha\beta\left(1-p^{\beta}\right)^{\alpha-1}p^{\beta-1}, 0 < p < 1 ; \alpha,\beta > 0 \\ 0, \quad elsewhere \end{cases}$$

Let $P = 1 - U$, $0 < u < 1$.

Then $g(p)$becomes

$$g(u) = \alpha\beta\left(1-(1-u)^{\beta}\right)^{\alpha-1}(1-u)^{\beta-1}|J|$$

where$|J| = \left|\frac{dp}{du}\right| = 1$

Therefore,

$$g(u) = \begin{cases} \alpha\beta\left(1-(1-u)^{\beta}\right)^{\alpha-1}(1-u)^{\beta-1}, \quad 0 < u < 1;\ \alpha,> 0 \\ 0, \quad elsewhere \end{cases} \tag{5.30}$$

This is the pdf of the Type 2 LIGE (II) distribution, with parameters α , β .

Properties

The moment of order j about the origin of this distribution is given by

$$E(U^j) = \alpha\beta\int_0^1 u^j\left(1-(1-u)^{\beta}\right)^{\alpha-1}(1-u)^{\beta-1}\,du$$

Letting $t = (1-u)^{\beta}$, for $0 < t < 1$, we have,

$$E(U^j) = \alpha \int_0^1 \left(1 - t^{\frac{1}{\beta}}\right)^j (1-t)^{\alpha-1}\, dt$$

$$= \alpha \sum_{k=0}^{j} \binom{j}{k} (-1)^k \int_0^1 t^{\frac{k}{\beta}} (1-t)^{\alpha-1} dt$$

$$= \alpha \sum_{k=0}^{j} \binom{j}{k} (-1)^k B\left(\frac{k}{\beta} + 1, \alpha\right) \tag{5.31}$$

5.12.2 Type 2 LIGE (II) mixing distribution

The Binomial-Type 2 LIGE (II) distribution is given by

$$f(x) = \binom{n}{x} \int_0^1 p^x (1-p)^{n-x} g(p) dp$$

Where $g(p)$ is the pdf of the Type 2 LIGE (II) distribution.

Thus

$$f(x) = \alpha\beta \binom{n}{x} \int_0^1 p^x (1-p)^{n-x} \left(1-(1-p)^{\beta}\right)^{\alpha-1} (1-p)^{\beta-1} dp$$

$$= \alpha\beta \binom{n}{x} \sum_{k=0}^{\alpha-1} (-1)^k \binom{\alpha-1}{k} \int_0^1 p^x (1-p)^{n-x+k\beta+\beta-1} dp$$

$$= \alpha\beta \binom{n}{x} \sum_{k=0}^{\alpha-1} \binom{\alpha-1}{k} (-1)^k B(x+1, n-x+k\beta+\beta)$$

$$f(x) = \begin{cases} \alpha\beta \binom{n}{x} \sum_{k=0}^{\alpha-1} \binom{\alpha-1}{k} (-1)^k B(x+1, n-x+k\beta+\beta), & x = 0,1,\dots,n;\ \alpha,\beta > 0 \\ 0\ , & elsewhere \end{cases} \tag{5.32}$$

From the Moments method, this probability distribution can also be written as,

$$f(x) = \sum_{j=x}^{n} \frac{n!\ (-1)^{j-x}}{x!\ (j-x)!\,(n-j)!} E(P^j) \quad , \quad x = 0,1,\dots,n$$

for $j \geq x$ and 0 if $j < x$

The Jth moment, $E(P^j)$ of the Type 2 LIGE (II) distribution is $\alpha \sum_{k=0}^{j} \binom{j}{k} (-1)^k B\left(\frac{k}{\beta} + 1, \alpha\right)$, from (5.31)

Thus,

$$f(x) = \alpha \sum_{j=x}^{n} \frac{n!\ (-1)^{j-x}}{x!\ (j-x)!\,(n-j)!} \sum_{k=0}^{j} \binom{j}{k} (-1)^k B\left(\frac{k}{\beta} + 1, \alpha\right) \tag{5.33}$$

with $x = 0,1,\dots,n\ ;\ \alpha, \beta > 0$; and zero elsewhere.

CHAPTER 6

BINOMIAL MIXTURES BASED ON DISTRIBUTIONS GENERATED FROM CUMULATIVE DISTRIBUTION FUNCTIONS

6.1 Introduction

In this chapter, we derive mixing distributions using the generator method, as pioneered by Eugene et al (2002).

6.1.1 The generator method

Theorem 7.1.1

Let$G(x)$be the cdf of a random variable. A method to generalize distributions, consists of defining a new cdf$F(x)$from the baseline$G(x)$by

$$F\big(G(x)\big) = \int_0^{G(x)} f(t)dt$$

Proof

Consider

$$f(x) = \int_{-\infty}^{x} f(t)dt \qquad , \quad -\infty < x < \infty$$

Then the cdf of the random variable$X$is

$F(-\infty) = 0$ and$F(\infty) = 1$

So that

$$0 \leq F(x) \leq 1$$

Concentrating on $0 \leq x \leq 1$ we have,

$$F(x) = \int_0^{x} f(t)dt \quad , \qquad 0 \leq x \leq 1$$

Thus

$$F(G(y)) = \int_0^{G(y)} f(t)dt \qquad (6.1)$$

Where $G(y)$ is the cdf of another random variable Y ,

$-\infty < y < \infty$.

Note that since $G(y)$ is a cdf, then $0 \leq G(y) \leq 1$.

Put

$$H(y) = F[G(y)]$$

Then

$$H(y) = \int_0^{G(y)} f(t)dt$$

$$H(-\infty) = F[G(-\infty)] = F(0) = 0$$

And

$$H(\infty) = F[G(\infty)] = F(1) = 1$$

Therefore

$$0 \leq H(y) \leq 1$$

Hence $H(y)$ is also a cdf.

Thus,

$$h(y) = \frac{dH(y)}{dy} = \frac{d}{dy}\int_0^{G(y)} f(t)dt$$

$$= f[G(y)]\frac{dG(y)}{dy}$$

Using Leibnitz technique of differentiation, then

$$h(y) = f[G(y)]g(y) \quad , \quad -\infty < y < \infty \qquad (6.2)$$

where $h(y)$ is the p.d.f. of the new distribution.

We identify 2 classes of these distributions:

- Beta-generated
- Kumaraswamy-generated distributions.

6.2 Beta-generated distributions (Beta-G distributions)

The cumulative distribution function (cdf) of the class of generalized Beta (Beta-G) distributions, is defined by

$$F(t) = \frac{1}{B(a,b)} \int_0^{G(t)} x^{a-1}(1-x)^{b-1}dx \quad , \quad a>0, b>0$$

where $G(t)$ is the cdf of the parent random variable, and

$B(a,b)$ is the Beta function.

The Beta-G distributions generalize the distribution G of a random variable, with cdf $G(t)$.

To find $f(t)$ the p.d.f. of the Beta-G distribution, we proceed as follows

Since,

$$F(t) = \frac{1}{B(a,b)} \int_0^{G(t)} x^{a-1}(1-x)^{b-1}dx$$

$$= \frac{1}{B(a,b)} \sum_{k=0}^{b-1} \binom{b-1}{k} (-1)^k \int_0^{G(t)} x^{a+k-1}\, dx$$

$$= \frac{1}{B(a,b)} \sum_{k=0}^{b-1} \binom{b-1}{k} \frac{(-1)^k}{a+k} [G(t)]^{a+k}$$

Thus

$$f(t) = \frac{dF(t)}{dt} = \frac{1}{B(a,b)} \sum_{k=0}^{b-1} \binom{b-1}{k} (-1)^k [G(t)]^{a+k-1} g(t)$$

$$= \frac{g(t)}{B(a,b)} G(t)^{a-1}[1-G(t)]^{b-1} \tag{6.3}$$

We note that, $f(t)$ will be most tractable when the cdf $G(t)$ and the p.d.f. $g(t)$ have simple analytic expressions.

6.2.1 Binomial-Beta Exponential distribution

6.2.1.1 Beta Exponential distribution

Construction

The p.d.f. of the Beta Exponential distribution is given by the formula (6.3) as

$$g(t) = \frac{f(t)}{B(a,b)}F(t)^{a-1}[1-F(t)]^{b-1}\ , \qquad a,b>0$$

Where

$f(t)$ is the p.d.f. of the exponential distribution and,

$F(t)$ is the cdf of the exponential distribution, found as $1-e^{-t\beta}$

Thus

$$g(t) = \frac{\beta e^{-t\beta}}{B(a,b)}\left(1-e^{-t\beta}\right)^{a-1}\left[1-\left(1-e^{-t\beta}\right)\right]^{b-1}$$

$$= \frac{\beta e^{-t(\beta+\beta b-\beta)}}{B(a,b)}\left(1-e^{-t\beta}\right)^{a-1}$$

$$= \frac{\beta e^{-t\beta b}}{B(a,b)}\left(1-e^{-t\beta}\right)^{a-1}\ , \quad a,b,\beta>0; t>0 \qquad (6.4)$$

This is the p.d.f. of the Beta Exponential distribution as obtained by Nadarajah & Kotz, (2006).

The Laplace Transform of the Beta Exponential distribution is

$$L_t(s) = \frac{\beta}{B(a,b)}\int_0^\infty e^{-ts}e^{-t\beta b}\left(1-e^{-t\beta}\right)^{a-1}dt$$

$$= \frac{\beta}{B(a,b)}\int_0^\infty e^{-t(s+\beta b)}\left(1-e^{-t\beta}\right)^{a-1}dt$$

Letting $t = \frac{p}{\beta}$

We have

$$L_t(s) = \frac{\beta}{B(a,b)} \int_0^{\infty} e^{-p\left(\frac{s+\beta b}{\beta}\right)} (1-e^{-p})^{a-1} \frac{1}{\beta}\, dp$$

$$= \frac{1}{B(a,b)} \int_0^{\infty} e^{-p\left(\frac{s+\beta b}{\beta}\right)} (1-e^{-p})^{a-1} dp$$

But $\int_0^{\infty} e^{-ta}(1-e^{-t})^{b-1}dt = B(a,b)$

Thus

$$L_t(s) = \frac{B\left(\frac{s+\beta b}{\beta}, a\right)}{B(a,b)} \tag{6.5}$$

6.2.1.2 Beta Exponential mixing distribution

The Binomial-Beta Exponential distribution is given by

$$f(x) = \binom{n}{x} \sum_{k=0}^{n-x} \binom{n-x}{k} (-1)^k L_t(x+k)$$

From (4.1).

But $L_t(x+k) = \frac{B\left(\frac{x+k+\beta b}{\beta}, a\right)}{B(a,b)}$ from (6.5),

Therefore

$$f(x) = \binom{n}{x} \sum_{k=0}^{n-x} \binom{n-x}{k} (-1)^k \frac{B\left(\frac{x+k+\beta b}{\beta}, a\right)}{B(a,b)} \tag{6.6}$$

with $x = 0,1,\dots,n; a, b, \beta > 0$

6.2.2 Binomial-Beta Generalized Exponential distribution

6.2.2.1 Beta Generalized Exponential (BGE) distribution

Construction

The p.d.f. of the BGE distribution is given by the formula

$$g(t) = \frac{f(t)}{B(a,b)} F(t)^{a-1}[1-F(t)]^{b-1}\ , \qquad a,b>0$$

Where

$f(t)$is the p.d.f. of the Generalized Exponential distribution and,

$F(t)$is the cdf of the Generalized exponential distribution, found as follows

$$F(t) = \alpha\beta \int_0^t \left(1-e^{-\lambda\beta}\right)^{\alpha-1} e^{-\lambda\beta}\, d\lambda$$

Letting $1-e^{-\lambda\beta} = u$ we have

$$F(t) = \alpha\beta \int_0^{1-e^{-t\beta}} (u)^{\alpha-1}(1-u)\frac{1}{\beta(1-u)}\, du$$

Thus

$$F(t) = \alpha \int_0^{1-e^{-t\beta}} (u)^{\alpha-1}\, du$$

$$= [u^{\alpha}]_0^{1-e^{-t\beta}}$$

$$= \left(1-e^{-t\beta}\right)^{\alpha}$$

Therefore, pdf of the BGE distribution is

$$g(t) = \frac{\alpha\beta\left(1-e^{-t\beta}\right)^{\alpha-1} e^{-t\beta}}{B(a,b)} \left[\left(1-e^{-t\beta}\right)^{\alpha}\right]^{a-1} \left[1-\left(1-e^{-t\beta}\right)^{\alpha}\right]^{b-1}$$

$$= \frac{\alpha\beta\left(1-e^{-t\beta}\right)^{a\alpha-1} e^{-t\beta}}{B(a,b)} \left[1-\left(1-e^{-t\beta}\right)^{\alpha}\right]^{b-1}, \qquad \alpha, a, b, \beta > 0; t>0$$

asobtained by Barreto-Souza et al(2009).

The Laplace Transform of the BGE distribution is

$$L_t(s) = \frac{\alpha\beta}{B(a,b)}\int_0^\infty e^{-ts}\left(1-e^{-t\beta}\right)^{a\alpha-1}e^{-t\beta}\left[1-\left(1-e^{-t\beta}\right)^{\alpha}\right]^{b-1}dt$$

$$= \frac{\alpha\beta}{B(a,b)}\int_0^\infty e^{-t(s+\beta)}\left(1-e^{-t\beta}\right)^{a\alpha-1}\left[1-\left(1-e^{-t\beta}\right)^{\alpha}\right]^{b-1}dt$$

$$= \frac{\alpha\beta}{B(a,b)}\sum_{k=0}^{b-1}\binom{b-1}{k}(-1)^k\int_0^\infty e^{-t(s+\beta)}\left(1-e^{-t\beta}\right)^{\alpha k+a\alpha-1}dt$$

Letting $z = t\beta$ we have

$$L_t(s) = \frac{\alpha\beta}{B(a,b)}\sum_{k=0}^{b-1}\binom{b-1}{k}(-1)^k\int_0^\infty e^{-\frac{z}{\beta}(s+\beta)}(1-e^{-z})^{\alpha k+a\alpha-1}\frac{1}{\beta}\,dz$$

Thus

$$L_t(s) = \frac{\alpha}{B(a,b)}\sum_{k=0}^{b-1}\binom{b-1}{k}(-1)^k\int_0^\infty e^{-\frac{z}{\beta}(s+\beta)}(1-e^{-z})^{\alpha k+a\alpha-1}dz$$

But $\int_0^\infty e^{-ta}(1-e^{-t})^{b-1}dt \;=\; B(a,b)$

Thus

$$L_t(s) = \frac{\alpha}{B(a,b)}\sum_{j=0}^{b-1}\binom{b-1}{j}(-1)^j B\left(\frac{s+\beta}{\beta}, \alpha j + a\alpha\right) \qquad (6.7)$$

6.2.2.2 Beta Generalized Exponential mixing distribution

The Binomial-Beta Generalized Exponential distribution is given by

$$f(x) = \binom{n}{x}\sum_{k=0}^{n-x}\binom{n-x}{k}(-1)^k L_t(x+k)$$

Butfrom (6.7)

$$L_t(x+k) = \frac{\alpha}{B(a,b)}\sum_{k=0}^{b-1}\binom{b-1}{k}(-1)^k B\left(\frac{x+k+\beta}{\beta}, \alpha k + a\alpha\right)$$

Therefore

$$f(x) = \frac{\alpha}{B(a,b)}\binom{n}{x}\sum_{k=0}^{n-x}\binom{n-x}{k}(-1)^k\sum_{j=0}^{b-1}\binom{b-1}{j}(-1)^j B\left(\frac{x+k+\beta}{\beta}, \alpha j + a\alpha\right) \quad (6.8)$$

with $x = 0,1,\dots,n;\ \alpha, a, b, \beta > 0$

6.2.3 Binomial-Beta Power distribution

6.2.3.1 Beta Power distribution

Construction

The p.d.f. of the Beta Power distribution is given by the formula

$$g(t) = \frac{f(t)}{B(a,b)}F(t)^{a-1}[1-F(t)]^{b-1}, \qquad a,b > 0$$

Where

$f(t)$is the p.d.f. of the Power distribution and,

$F(t)$is the cdf of the Power distribution, found as follows

$$F(t) = \alpha\int_0^t p^{\alpha-1}\,dp$$

Thus

$$F(t) = [p^\alpha]_0^t$$

$$t^\alpha$$

$$g(t) = \frac{\alpha t^{\alpha-1}}{B(a,b)}[t^\alpha]^{a-1}[1-t^\alpha]^{b-1}$$

Therefore, p.d.f. of the Beta Power distribution is

$$= \frac{\alpha t^{\alpha a-1}(1-t^{\alpha})^{b-1}}{B(a,b)}, \qquad \alpha, a, b > 0; 0 < t < 1 \tag{6.9}$$

The jth moment of the Beta Power distribution is

$$E(T^j) = \frac{\alpha}{B(a,b)} \int_0^{\infty} t^{\alpha a+j-1}(1-t^{\alpha})^{b-1}\, dt$$

Let$t^{\alpha} = z$

Then

$$E(T^j) = \frac{\alpha}{B(a,b)} \int_0^{\infty} z^{\frac{\alpha a+j-1}{\alpha}}(1-z)^{b-1} \frac{Z^{\frac{1}{\alpha}-1}}{\alpha} dz$$

$$\frac{1}{B(a,b)} \int_0^{\infty} z^{\frac{\alpha a+j}{\alpha}-1}(1-z)^{b-1}\, dz$$

$$E(T^j) = \frac{B\left(\frac{\alpha a+j}{\alpha}, b\right)}{B(a,b)} \tag{6.10}$$

6.2.3.2 Beta Power mixing distribution

The Binomial-Beta Power distribution is given by

$$f(x) = \binom{n}{x} \int_0^1 t^x(1-t)^{n-x}\, g(t)dt$$

where $g(t)$is the pdf of the Beta Power distribution.

Therefore

$$f(x) = \binom{n}{x} \frac{\alpha}{B(a,b)} \int_0^1 t^x(1-t)^{n-x} t^{\alpha a-1}(1-t^{\alpha})^{b-1}\, dt$$

$$= \binom{n}{x}\frac{\alpha}{B(a,b)}\int_0^1 t^{x+\alpha a-1}(1-t)^{n-x}(1-t^{\alpha})^{b-1}\,dt$$

$$= \binom{n}{x}\frac{\alpha}{B(a,b)}\sum_{k=0}^{b-1}\binom{b-1}{k}(-1)^k\int_0^1 t^{x+\alpha a+\alpha k-1}(1-t)^{n-x}\,dt$$

$$f(x) = \binom{n}{x}\frac{\alpha}{B(a,b)}\sum_{k=0}^{b-1}\binom{b-1}{k}(-1)^k B(x+\alpha a+\alpha k, n-x+1) \qquad (6.11)$$

with $x = 0,1,\dots,n;\ \alpha, a, b > 0$

Using the Moments method, the Binomial-Beta Power distribution can also be expressed as

$$f(x) = \sum_{j=x}^{n}\frac{n!}{x!\,(n-j)!\,(j-x)!}E(T^j)$$

for $j \geq x$ and 0 if $j < x$

The jthmoment of the Beta Power distribution from (6.10) is

$$E(T^j) = \frac{B\left(\frac{\alpha a+j}{\alpha}, b\right)}{B(a,b)}$$

Thus

$$f(x) = \sum_{j=x}^{n}\frac{n!\,B\left(\frac{\alpha a+j}{\alpha}, b\right)}{x!\,(n-j)!\,(j-x)!\,B(a,b)} \qquad (6.12)$$

with $x = 0,1,\dots,n\ ; a, b, \alpha > 0$

6.3 Kumaraswamy-generated distributions (Kw-G distributions)

The cdf of the class of generalized Kw distributions is defined by

$$F(t) = \int_0^{G(t)} \alpha\beta\, x^{\alpha-1}(1-x^{\alpha})^{\beta-1}\,dt$$

where $G(t)$is the cdf of the parent random variable.

To find $f(t)$, the p.d.f. of the Kw-G distribution, we proceed as follows:

$$F(t) = \int_0^{G(t)} \alpha\beta\, x^{\alpha-1}(1-x^\alpha)^{\beta-1}\, dt$$

Thus

$$F(t) = \alpha\beta \sum_{k=0}^{\beta-1} \binom{\beta-1}{k} (-1)^k \int_0^{G(t)} x^{\alpha+\alpha k-1}\, dt$$

$$= \alpha\beta \sum_{k=0}^{\beta-1} \binom{\beta-1}{k} (-1)^k \left[\frac{x^{\alpha+\alpha k}}{\alpha+\alpha k}\right]_0^{G(t)}$$

And,

$$f(t) = \frac{dF(t)}{dt} = \alpha\beta \sum_{k=0}^{\beta-1} \binom{\beta-1}{k} (-1)^k [G(t)]^{\alpha+\alpha k-1} g(t)$$

$$f(t) = \alpha\beta G(t)^{\alpha-1}\, g(t)[1-G(t)^\alpha]^{\beta-1} \tag{6.13}$$

6.3.1 Binomial-Kumaraswamy Exponential distribution

6.3.1.1 Kumaraswamy (Kw) Exponential distribution

Construction

The p.d.f. of the Kw Exponential distribution is given by the formula

$$g(t) = \alpha\beta f(t)F(t)^{\alpha-1}\left[1-\left(F(t)\right)^\alpha\right]^{\beta-1}, \qquad \alpha,\beta > 0$$

Where

$f(t)$is the p.d.f. of the exponential distribution and,

$F(t)$is the cdf of the exponential distribution, found as$1-e^{-t\lambda}$

Thus

$$g(t) = \lambda\alpha\beta e^{-t\lambda}\left(1-e^{-t\lambda}\right)^{\alpha-1}\left[1-\left(1-e^{-t\lambda}\right)^{\alpha}\right]^{\beta-1} \quad , \quad \lambda,\alpha,\beta > 0\,;\ t > 0 \qquad (6.14)$$

This is the p.d.f. of the Kw Exponential distribution.

The Laplace Transform of the Kw Exponential distribution is

$$L_t(s) = \alpha\beta\lambda \int_0^\infty e^{-ts}e^{-t\lambda}\left(1-e^{-t\lambda}\right)^{\alpha-1}\left[1-\left(1-e^{-t\lambda}\right)^{\alpha}\right]^{\beta-1} dt$$

$$= \alpha\beta\lambda \int_0^\infty e^{-t(s+\lambda)}\left(1-e^{-t\lambda}\right)^{\alpha-1}\left[1-\left(1-e^{-t\lambda}\right)^{\alpha}\right]^{\beta-1} dt$$

$$= \alpha\beta\lambda \sum_{j=0}^{\beta-1}\binom{\beta-1}{j}(-1)^j \int_0^\infty e^{-t(s+\lambda)}\left(1-e^{-t\lambda}\right)^{\alpha+\alpha j-1} dt$$

Let $a = t\lambda$

Then

$$L_t(s) = \alpha\beta\lambda \sum_{j=0}^{\beta-1}\binom{\beta-1}{j}(-1)^j \int_0^\infty e^{-\frac{a}{\lambda}(s+\lambda)}(1-e^{-a})^{\alpha+\alpha j-1}\frac{1}{\lambda}\, da$$

But $\int_0^\infty e^{-ta}(1-e^{-t})^{b-1}dt \quad = \quad B(a,b)$

Thus

$$L_t(s) = \alpha\beta \sum_{j=0}^{\beta-1}\binom{\beta-1}{j}(-1)^j B\left(\frac{s+\lambda}{\lambda}, \alpha+\alpha j\right) \qquad (6.15)$$

6.3.1.2 Kumaraswamy-Exponential mixing distribution

The Binomial-Kw Exponential distribution is given by

$$f(x) = \binom{n}{x}\sum_{k=0}^{n-x}\binom{n-x}{k}(-1)^k L_t(x+k)$$

Butfrom (6.15)

$$L_t(x+k) = \alpha\beta\sum_{j=0}^{\beta-1}\binom{\beta-1}{j}(-1)^j B\left(\frac{x+k+\lambda}{\lambda}, \alpha+\alpha j\right)$$

Therefore

$$f(x) = \binom{n}{x}\alpha\beta\sum_{k=0}^{n-x}\binom{n-x}{k}(-1)^k\sum_{j=0}^{\beta-1}\binom{\beta-1}{j}(-1)^j B\left(\frac{x+k+\lambda}{\lambda}, \alpha+\alpha j\right) \qquad (6.16)$$

with $x = 0,1,\dots,n;\ \alpha,\beta,\lambda > 0$

6.3.2 Binomial-Kumaraswamy Power distribution

6.3.2.1 Kumaraswamy (Kw)-Power distribution

Construction

The p.d.f. of the Kw-Power distribution is given by the formula

$$g(t) = \alpha\beta f(t)F(t)^{\alpha-1}\left[1-\left(F(t)\right)^{\alpha}\right]^{\beta-1}, \qquad \alpha,\beta > 0$$

Where

$f(t)$is the p.d.f. of the Power distribution and,

$F(t)$is the cdf of the Power distribution, found to be t^k

Therefore, the p.d.f. of the Kw-Power distribution is

$$g(t) = \alpha\beta k t^{k-1}[t^k]^{\alpha-1}[1-t^{k\alpha}]^{\beta-1}$$

$$= \alpha\beta k t^{k\alpha-1}[1-t^{k\alpha}]^{\beta-1}, \qquad \alpha,k,\beta > 0; 0 < t < 1 \qquad (6.17)$$

The jth moment of the Kw-Power distribution is

$$E(T^j) = \alpha\beta k\int_0^1 t^{j+k\alpha-1}[1-t^{k\alpha}]^{\beta-1}\,dt$$

Let$t^{\alpha k} = a$

Then

$$E(T^j) = \alpha\beta k \int_0^1 a^{\frac{1}{\alpha k}(j+k\alpha-1)}(1-a)^{\beta-1}\frac{a^{\frac{1}{k\alpha}-1}}{k\alpha}da$$

$$\beta\int_0^\infty a^{\frac{1}{\alpha k}(j)}(1-a)^{\beta-1}\,da$$

$$= \beta\ B\left(\frac{j}{k\alpha}+1,\beta\right) \tag{6.18}$$

6.3.2.2 Kumaraswamy Power mixing distribution

The Binomial-Kw Power distribution is given by

$$f(x) = \binom{n}{x}\int_0^1 t^x(1-t)^{n-x}\ g(t)dt$$

where $g(t)$is the p.d.f. of the Kw Power distribution.

Therefore

$$f(x) = \binom{n}{x}\alpha\beta k\int_0^1 t^{x+k\alpha-1}(1-t)^{n-x}(1-t^{k\alpha})^{\beta-1}\,dt$$

$$= \binom{n}{x}\alpha\beta k\sum_{m=0}^{\beta-1}\binom{\beta-1}{m}(-1)^m\int_0^1 t^{x+\alpha k+k\alpha m-1}(1-t)^{n-x}\,dt$$

$$f(x) = \binom{n}{x}\alpha\beta k\sum_{m=0}^{\beta-1}\binom{\beta-1}{m}(-1)^m B(x+\alpha k+km\alpha, n-x+1) \tag{6.19}$$

With $x = 0,1,\ldots,n;\ \alpha,\beta,k > 0$

Using the Moments method, the Binomial-Kw Power distribution can also be expressed as

$$f(x) = \sum_{j=x}^{n}\frac{n!}{x!\ (n-j)!\ (j-x)!}\ E(T^j)$$

for $j \geq x$ and 0 if $j < x$

The jthmoment of theKw Power distribution, from (6.18)is

$$E(T^j) = \beta\, B\left(\frac{j}{k\alpha} + 1, \beta\right)$$

Thus

$$f(x) = \sum_{j=x}^{n} \frac{n!\ \beta\, B\left(\frac{j}{k\alpha} + 1, \beta\right)}{x!\ (n-j)!\ (j-x)!}$$

with $x = 0,1,\dots,n;\ \alpha,\beta,k > 0$

CHAPTER 7

SUMMARY AND CONCLUSIONS

7.1 Summary

This literature was dedicated to the construction of Binomial mixture distributions based on continuous mixing distributions for the success parameter p .

Since$p$is restricted to the range$[0,1]$our main problem was in finding mixing distributions beyond those naturally available in this domain.

In the first Chapter, we introduced the Binomial distribution upon which these mixtures are based.

In the second and third Chapters, we explored mixing distributions in the unit interval$[0,1]$, in which the Beta remained the premier distribution of choice. Chapter two was thus dedicated to the Beta distribution, and its generalizations, while Chapter three investigated alternative mixing distributions to the Beta, hence the title: Beyond the Beta.

Chapter four focused on Binomial mixtures based on a transformed success parameter p . This Chapter was motivated by the works ofBowman et al(1992) and (Alanko & Duffy, (1996) who used the transformations $p = e^{-t}$and $p = 1 - e^{-t}$to extend p into the interval$[0, \infty]$through the random variable t for $t > 0$. From this transformation it was evident that $0 < e^{-t} < 1$. We were therefore able to shift our attention to mixing distributions beyond the domain$[0,1]$. A major limitation to our work with this class of distributions was that this formulation only provided closed forms for the marginal probabilities in the compound distribution if the Laplace transform of the mixing distribution could be written in a closed form. To improve this situation, we then took recourse to methods of generating distributions with tractable Laplace transforms in this domain, hence Chapters six and seven.

Chapter five extended the work of Grassia (1977) in deriving mixing distributions in the interval$[0,1]$by applying the log-inverse transform directly to distributions defined in the interval$[0, \infty]$. Binomial mixtures in this chapter were found to be similar to those derived in chapter 4, where a transformed parameter p had been used. The table below provides a summary of these results.

Comparing results when the transformations $p = exp(-t)$ and $Y = -\ln U$ are used.

$p = exp(-t)$ [g(t) is the pdf of the mixing distribution]	$T = -\ln P$ $t > 0;\ 0 < p < 1$ [g(p) is the pdf of the mixing distribution]	Mixed Binomial f(x)
$g(t) = \beta e^{-t\beta}$ $t > 0;\ \beta > 0$	$g(p) = \beta p^{\beta-1}$ $0 < p < 1;\ \beta > 0$	$f(x) = \beta\binom{n}{x}\sum_{k=0}^{n-x}\binom{n-x}{k}\frac{(-1)^k}{(x+k+\beta)}$ $x = 0,1,\dots,n;\ \beta > 0$
$g(t) = \frac{e^{-t}t^{\alpha-1}}{\Gamma(\alpha)}$ $t > 0;\ \alpha > 0$	$g(p) = \left(\ln\left(\frac{1}{p}\right)\right)^{\alpha-1}\frac{1}{\Gamma(\alpha)}$ $0 < p < 1;\ \alpha > 0$	$f(x) = \binom{n}{x}\sum_{k=0}^{n-x}\binom{n-x}{k}(-1)^k\frac{1}{(x+k+1)^\alpha}$ $x = 0,1,\dots,n;\ \alpha > 0$
$g(t) = \frac{e^{-t\beta}t^{\alpha-1}\beta^\alpha}{\Gamma(\alpha)}$ $t > 0; \alpha,\beta > 0$	$g(p) = \left(\ln\left(\frac{1}{p}\right)\right)^{\alpha-1}\frac{\beta^\alpha p^{\beta-1}}{\Gamma(\alpha)}$ $0 < p < 1;\ \alpha,\beta > 0$	$f(x) = \binom{n}{x}\sum_{k=0}^{n-x}\binom{n-x}{k}(-1)^k\left(\frac{\beta}{x+k+\beta}\right)^\alpha$ $x = 0,1,\dots,n;\ \alpha,\beta > 0$
$g(t) = \alpha(1-e^{-t})^{\alpha-1}e^{-t}$ $t > 0; \alpha > 0$	$g(p) = \alpha(1-p)^{\alpha-1}$ $0 < p < 1;\ \alpha > 0$	$f(x) = \binom{n}{x}\alpha\sum_{k=0}^{n-x}\binom{n-x}{k}(-1)^k B(x+k+1,\alpha)$ $x = 0,1,\dots,n;\ \alpha > 0$
$g(t) = \alpha\beta\left(1-e^{-t\beta}\right)^{\alpha-1}e^{-t\beta}$ $t > 0; \alpha,\beta > 0$	$g(p) = \alpha\beta\left(1-p^\beta\right)^{\alpha-1}p^{\beta-1}$ $0 < p < 1;\ \alpha,\beta > 0$	$f(x) = \binom{n}{x}\alpha\sum_{k=0}^{n-x}\binom{n-x}{k}(-1)^k B\left(\frac{x+k+\beta}{\beta},\alpha\right)$ $x = 0,1,\dots,n;\ \alpha,\beta > 0$

Comparing results when the transformations $p = 1 - exp(-t)$ and $Y = -\ln(1-U)$are used.

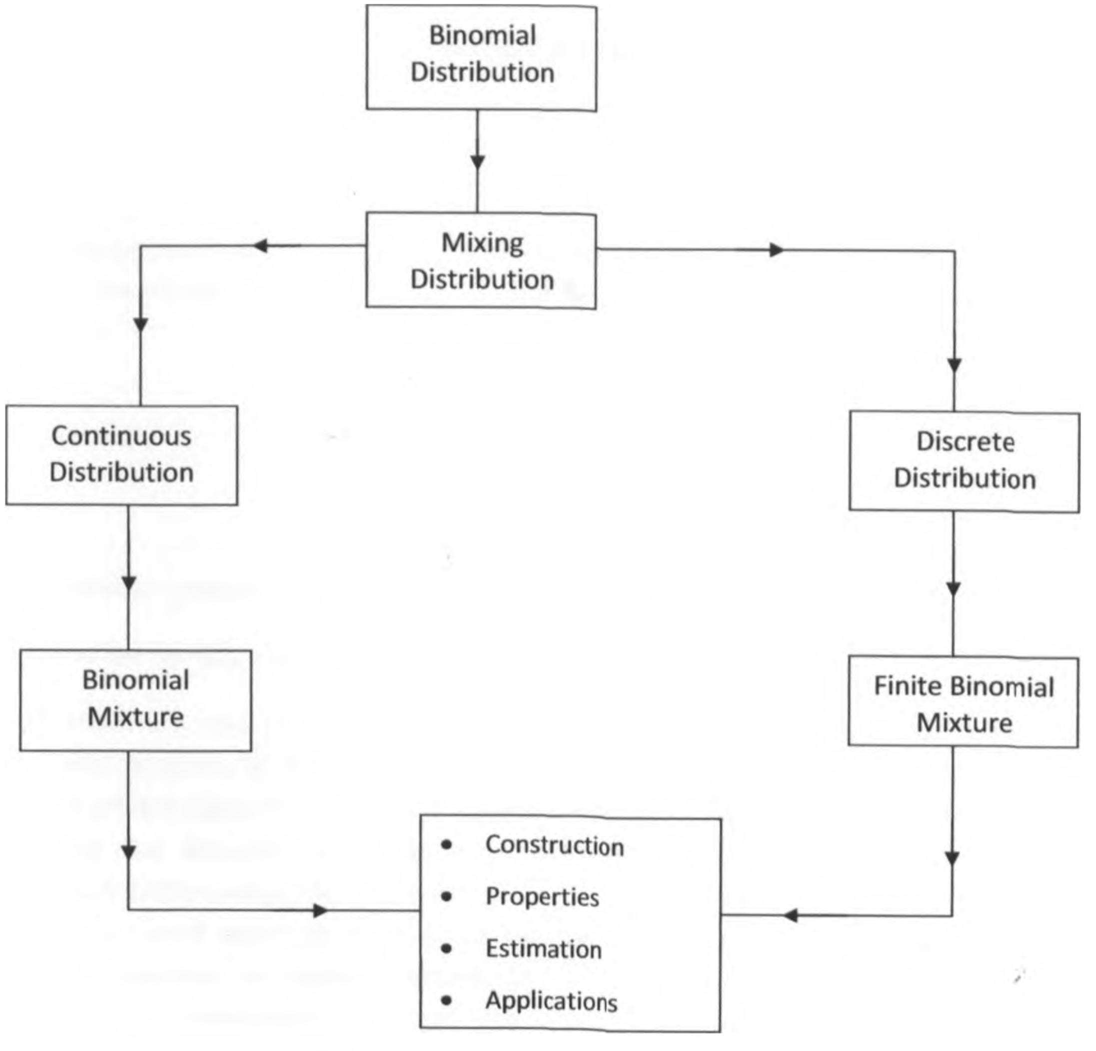
$p = 1 - exp(-t)$ [g(t) is the pdf of the mixing distribution]	$T = -\ln(1-U)$ $t > 0;\ 0 < u < 1$ [g(u) is the pdf of the mixing distribution]	Mixed Binomial $f(x)$
$g(t) = \beta e^{-t\beta}$ $t > 0;\ \beta > 0$	$g(u) = \beta(1-u)^{\beta-1}$ $0 < u < 1;\ \beta > 0$	$f(x) = \beta\binom{n}{x}\sum_{k=0}^{x}\binom{x}{k}\frac{(-1)^k}{(n-x+k+\beta)}$ $x = 0,1,\dots,n;\ \beta > 0$
$g(t) = \frac{e^{-t}t^{\alpha-1}}{\Gamma(\alpha)}$ $t > 0;\ \alpha > 0$	$g(u) = \left[\ln\left(\frac{1}{1-u}\right)\right]^{\alpha-1}\frac{1}{\Gamma(\alpha)}$ $0 < u < 1;\ \alpha > 0$	$f(x) = \binom{n}{x}\sum_{k=0}^{x}\binom{x}{k}\frac{(-1)^k}{(n-x+k+1)^\alpha}$ $x = 0,1,\dots,n;\ \alpha > 0$
$g(t) = \frac{e^{-t\beta}t^{\alpha-1}\beta^\alpha}{\Gamma(\alpha)}$ $t > 0;\ \alpha,\beta > 0$	$g(u) = \left[\ln\left(\frac{1}{1-u}\right)\right]^{\alpha-1}\frac{\beta^\alpha(1-u)^{\beta-1}}{\Gamma(\alpha)}$ $0 < u < 1;\ \alpha,\beta > 0$	$f(x) = \binom{n}{x}\beta^\alpha \sum_{k=0}^{x}\binom{x}{k}(-1)^k\left(\frac{1}{n-x+k+\beta}\right)^\alpha$ $x = 0,1,\dots,n;\ \alpha,\beta > 0$

Other observations that were made in this chapter, involved the newly constructed distributions. It was found that:

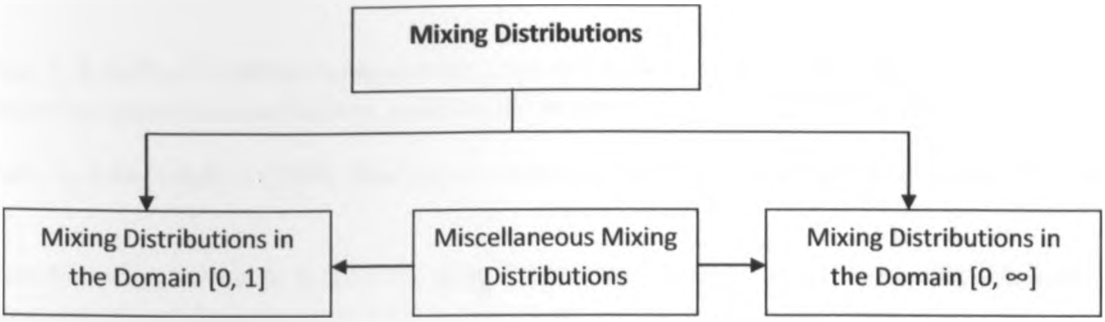
1. The Generalized exponential distribution is a result of applying the Log-inverse transform on the Kumaraswamy distribution
2. The exponential distribution is a result of applying the Log-inverse transform on the power distribution.

Finally, Chapter six explored mixtures based on generalized Beta and Kumaraswamy distributions obtained through the generator method pioneered by Eugene et al (2002). The work in this project can be summarized using the following frameworks as a guideline:

A FRAMEWORK FOR BINOMIAL MIXTURES



A FRAMEWORK FOR THE MIXING DISTRIBUTIONS



7.1.2 Further research areas

We identified the following areas of further research:

1) From the work of Bowman et al (1992) it was discovered that a large number of new distributions could be derived based on frullani integrals. These new distributions have tractable Laplace transforms mostly expressible in closed forms, hence possible mixers for the Binomial parameter p in the interval$[0, \infty]$. Further study of these new distributions would be of import.
2) More work needs to be done on the use of mixture/variated distributions as mixing distributions. In Chapter 4, sections (4.2.7) and (4.2.8) we look at two such instances.
3) The transformation $p = exp(-t)$or $p = 1 - exp(-t)$could be extended to other distributions that have a parameter p lying in the domain$[0,1]$particularly with regards to mixing procedures.
4) Methods of expressing Laplace transforms of various distributions in closed forms, where such forms do not exist.
5) The use of the Beta-generated and Kumaraswamy-generated distributions in mixing procedures.

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