TWO-TYPE STEP-WISE GROUP SCREENING DESIGNS WITH ERRORS IN OBSERVATIONS

By

MECHA OCHARO HENRY

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This project is my original work and has not been presented for a degree in any other University.

Signature.....

H.O. MECHA

This project has been submitted for examination with my approval as the University Supervisor.

Signature.....

PROF. M.M. MANENE

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DEDICATION

I dedicate this project to my parents, Kwamboka Mellen and Mecha Francis and Monica Kwamboka (girlfriend).

Abstract

In part one of this project, we discuss the problem of two-type step-wise group screening designs with errors in observations and equal prior probability of factor being defective; wherein f factors are subdivided into groups of k_1 factors each, forming g_1 group-factors called first order group-factors. The first order group-factors are then studied using fractional factorial designs of type given by Placket and Burman (1946) in g_1+h runs. The two versions of the first order group-factors are formed by maintaining all component factors at their upper and lower levels respectively. All the first order group-factors found to be defective are subdivided into g_2 second order group-factors of sizes k_2 factors each. In type-one search steps of the experiments, the second order group-factors are tested for their effects using fractional factorial designs. Then the effects of individual factors from the second order group-factors declared defective are studied in type-two search steps of the experiments using non-orthogonal fractional factorial designs. The expression for the expected number of runs for two-type step-wise group screening designs is obtained and used to generate tables by numerical approximation. In part two of this project, we discuss the problem of two-type step-wise group screening designs with errors in observations and unequal prior probability of a factor being defective, wherein f factors are subdivided into g_1 first order group-factors of sizes k_{1i} factors each. Then the group-factors and individual factors are tested for their effects as in part one, the expression for expected number of runs is obtained for two-type step-wise group screening designs with errors in observations and unequal prior probability of factor being defective.

CHAPTER 1

1.1: Concept of group screening designs

The problem of detecting defective factors in a large population consisting of defective and non-defective factors has been tackled in various ways. Designs used in this kind of investigation have been called screening designs. One such class of designs is the group screening designs. In group screening designs, the factors or members of the population are divided into groups called group-factors. The group-factors are then tested for their significance and classified as either defective or non-defective. If a group-factor is classified as non-defective, then it is dropped from further investigation since it is assumed that all the factors within that group-factor are non-defective. If however a group-factor is classified as defective, individual factors from the group-factor are investigated further.

Group screening experiments can be carried out in several stages. In a two stage group screening design the group-factors formed are tested in the first stage and factors from defective group-factors are only tested in the second stage. In a three stage group screening design, the first stage consists of dividing the factors into group-factors known as first order group-factors which are then tested and classified as defective or non-defective. In the second stage of the experiment, each first order group-factor classified as defective in the first stage is further divided into smaller group-factors called second order group-factors which are then tested and classified as defective or non-defective. Finally in the third stage all the factors belonging to the second order group-factors found to be defective in the second stage are tested individually and classified as either defective or non-defective. This can be extended to n-stage group screening designs ($n \geq 3$)

In a step-wise group-screening design the analysis is carried out as follows; in the initial step, the factors are divided into first order group-factors. Then the first order group-factors are

tested for their significance. Those that are found to be non-defective are set aside. In step two, we start with any defective first order group-factor and test the factors within it one by one till we find a defective factor. We set aside the factors which are found to be non-defective, keeping the defective factor separate. The remaining factors are then pooled into a group. In step three we test the group-factor obtained after step two is performed. If the group-factor is a non-defective, we terminate the test procedure. If the group-factor is defective we continue with step four. In step four, factors within a group-factor found to be defective in step three are tested one by one till a defective factor is found. Factors which are found to be non-defective are again set aside keeping the defective factor separate. The remaining factors are grouped into a group-factor. In step five the group-factor obtained in step four is tested. The test procedure is repeated until the analysis terminates with a test on a non-defective group-factor. Steps two onwards are carried out of all the first order group-factors found to be defective on step one. Two-type stepwise group screening design is carried in three steps; initial step, factors are grouped into first order group-factors and tested for their significance, in step two the first order group-factors declared defective are further divided into second order group-factors which again are tested for their significance using type-one search steps. In step three the individual factors from the second order group-factors declared defective are tested for their significance using type-two search steps.

The main objective of group screening is to reduce the number of tests or observations by eliminating a large number of non-defective factors in a bunch thus reducing the cost of experiment.

1.2: Literature review

The method of group testing was first introduced by Dorfman (1943), who proposed that instead of testing each blood sample individually for the presence of rare disease, blood samples be pooled and analyzed together.

Sterrett (1957) proposed that individual items from defective pooled sample be tested one at a time until a defective item is found. The remaining items from the defective pooled sample

were again tested in pool. If the result was negative the work was complete for the pooled sample. Otherwise testing items individually was continued with until another defective item was found. The remaining items were again tested in a pool. The process was continued until all the defective factors from a pooled sample were weeded out.

Watson (1961) introduced the two stage group screening procedure. This method was generalized to more than two stages by Li (1962) and Patel (1962). In particular, Patel discussed multi-stage group screening designs in which all the factors had the same prior probability of being defective.

Ottieno and Patel (1984) extended the idea of the two stage group screening with unequal prior probabilities to include situations when no prior information is available so that no natural partitioning can be assumed.

Odhiambo and Patel (1986) extended the work done by Ottieno and Patel (1984) to multi-stage group screening designs.

The group testing procedure first introduced by Sterrett (1957) was extended by Manene (1985). Patel and Manene (1987) worked along the line adopted by Sterrett and called their designs step-wise group screening designs. They restricted themselves to what we shall call one type step wise group screening designs.

Manene et.al (2002) extended step-wise group screening designs to multi-type step-wise group screening designs. They considered the case when all factors have the same prior probability of being defective.

Later Manene (2005) extended multi-type step-wise group screening designs to a case where factors have different prior probabilities of being defective.

Achia (2004) re-examined Dorfman- Sterrett procedures with and without errors. He derived an expression for the expected number of runs in multi-step Dorfman-Sterrett procedure and he compared the results with expected number of runs in Dorfman procedure and the (unrestricted procedure) proposed by Sterrett

Manene and Odhiambo (1987) studied the one-type stepwise group screening designs with errors in observations.

Manene (2003) extended the work done by Manene and Odhiambo (1987) to the case when we have unequal group sizes.

The problem of two-type stepwise group screening designs with errors in decisions has not been addressed so far. In my project I decided to address and discuss this problem of two-type step-wise group screening designs.

1.3: Design, Structure and Assumptions

We shall assume that there is a single response variable y, which is related to a set of f factors through the first order linear regression model.

$$Y_{ij} = \beta_0 + \sum_{i=1}^{f} \beta_j X_{ij} + \varepsilon_{ij} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots 1.3.1$$

where Y_{ij} is the i^{th} response β_0 is a constant term common to every response, β_j ($j \geq 1$) is the linear effect on the j^{th} factor, $X_{ij} = \pm 1$ is the level of the j^{th} factor in the i^{th} run and ε_{ij} is the i^{th} error term.

In addition to model (1.3.1) .We shall make the following assumptions;

i) All factors have independently the same a priori- probability 'p' of being effective(defective)

ii) Defective factors have the same effect $\Delta > 0$ i.e.

$$\beta_j = \begin{cases} 1, & if\ the\ factor\ is\ defective \\ 0, & if\ the\ factor\ is\ non-defective \end{cases}$$

iii) None of the factors interact

iv) The required designs exist.

v) The directions of possible effects are known

vi) The errors of all observations are independently normal with a constant known variance σ^2

vii) The total number of factors is $f=k_1g_1$, where g_1 is the number of first order group-factors and k_1 is the number of factors in first order group-factors.

In testing the significance of the first order group-factors and second order group-factors, we shall use orthogonal fractional factorial designs of type given by Plackett and Burmann (1946). In testing the significance of individual factors and group-factors in subsequent steps, we shall use non-orthogonal designs to simplify computations. To test the significant of group-factors, we shall use usual tests based on the normal distributions, since σ^2 is assumed to be known. In addition to above in testing the significance of the first order group-factors we shall use α_1 as level of significance and in testing second order group-factors we shall use α_2 as the level of significance.

For later development with unequal a-priori probabilities we shall assume that it is possible to partition the f-factors into a fixed number g_1 of first order group-factors such that the i^{th} first order group-factor contains k_{1i} factors. The factors will be partitioned into first order group-factors of unequal sizes by selecting a set of numbers ($p_1 \leq p_2 \leq \ldots\ldots\ldots\ldots\ldots \leq p_{g_1}$, $0<p_i<1$) and identifying p_i as the probability that a factor belonging to i^{th} first order group-factor is defective. Thus p_i and k_{1i} will be variables. This is generalization of natural partitioning when p_i's are actual probabilities.

The following modified assumptions are made:

i) The total number of factors $f = \sum_{i=1}^{g_1} k_{1i}$ where k_{1i} is the number of factors in the i^{th} first order group-factor.

ii) $p_i > 0, i=1,2,\ldots\ldots g_1$ is the a-priori probability that a factor in the i^{th} first order group-factor is defective

iii) $\Delta_i > 0$, $i=1, 2\ldots\ldots g_1$, is the effect of a factor within the i^{th} first order group-factor.

iv) None of the factors interact

v) The directions of possible effects are known

vi) The required design exist

vii) The errors of all observations are independently normal with a constant known variance σ^2

viii) α_{1i} the level of significance for testing the i^{th} first order group-factor in the initial step and α_{2i} is the level of significance for testing the second order group-factors within the i^{th} first order group-factor which has been declared defective in the initial step.

1.4: The objective of the study

There are investigations where a large number of factors need to be examined. In such a situation we have to run an experiment to identify the influential factors. Once these have been isolated, future experimentation can study them in greater detail.

By reducing the size of the experiment at the screening stage, one can conserve resources and more efficiently study the important factors. To study this we have to derive an expression for the expected number of runs required to analyze the whole procedure in two-type step-wise group screening designs.

1.5 Methodology

The goal of the group screening procedure is to minimize the number of tests (runs) required to isolate all the defective factors. There have arisen various methods obtaining designs that minimize the expected number of runs. Some of these methods are;

i) Computer simulation:

We use computer surge to generate the best combination of group-factor sizes and the probability of factor being defective for which the number of runs for the design

is minimum. These values are then used to compare the efficiency of the design with other designs.

ii) Numerical approximation:

We are going to use testing of hypothesis as our approach. The hypothesis will be used in the following steps;

a) H_0 : 1^{st} order group-factor is declared non-defective.

H_1 : 1^{st} order group-factor is declared defective.

b) H_0 : 2^{nd} order group-factor is declared non-defective

H_1 : 2^{nd} order group-factor is declared defective

c) H_0 : A factor is declared non-defective

H_1 : A factor is declared defective

Using the distribution functions of the 1^{st} and 2^{nd} order group-factors declared defective we derive an expression for the expected number of runs required for two-type stepwise group screening designs with errors in observations for two cases.

1.6: List of symbols and their definitions

Symbol	Definition
f	Number of factors under investigations
g_1	Number of first order group-factors in the initial step
g_2	Number of second order group-factors in the type-one search steps

k_1	The size of the first order group-factor in the initial step
k_2	The size of the second order group-factor in the type-one search steps
p	The a-priori probability that a factor is defective (q=1-p)
π_1^*	The probability that a first order group-factor in the initial step is defective
π_2^*	The probability that a second order group-factor in type-one search steps is defective
n_1	Number of defective first order group-factors in the initial step.
n_2	Number of defective second order group-factors in the type-one search steps
$f(n_1)$	Probability distribution of n_1
$f(n_2 \mid n_1)$	Conditional probability distribution of n_2 given n_1
$\pi_2^{*'}$	Probability that the second order group-factor is defective given that it is within defective first order group-factor.
π^*	The probability that a second order group-factor chosen at random from the first order group-factor in the initial step that has been declared defective is defective.
α_{11}	The probability of declaring a non- defective second order group-factor defective.
γ_{11}	The probability of declaring a defective second order group-factor defective

$\bar{\beta}$	The probability that second order group-factor chosen at random from the first order group-factor that has been declared defective in initial step is declared defective in subsequent steps of type-one search steps
α_1^*	The proportion of first order group-factors that are declared defective but second order group-factors within them are declared non defective.
$P_{g_2}^*(j_1)$	The probability that exactly j_1 second order group-factors that are declared defective in initial step are declared defective in type-one search steps.
$E_{g_2}^*(R_{j_1})$	The expected number of runs required to declare as defective or non-defective all the g_2 second order group-factors within the defective first order group-factors
p'	The probability that a factor chosen at random from second order group-factor in type-one search steps containing s defective factors that have been declared defective is defective.
α_s	The probability of declaring a non defective factor defective
γ_s	The probability of declaring a defective factor defective in the type-two search steps

$\bar{\beta}^*$	The probability that a factor chosen at random from second order group-factor that has been declared defective in type-one search steps is declared defective in

	subsequent steps of type-two search steps
α_2^*	Proportion of the second order group-factors that are declared defective but the factors within them are declared non-defective
$P_{k_2}(j_2)$	Probability that defective second order group-factor contains exactly j_2 defective factors
$E_{k_2}(j_2)$	Expected numbers of runs required to declared as defective or non-defective all factors within a defective second order group-factor
$R_{t_1}^0$	Number of runs required to declare as defective or non-defective all g_2 second order group-factors within a defective first order group-factor
R_{t_1}	Number of runs required to declare as defective or non-defective all n_1g_2 second order group-factors
$R_{t_2}^0$	Number of runs required to declare as defective or non-defective all the k_2 factors within a defective second order group-factor
R_{t_2}	Number of runs required to declare as defective or non-defective all n_2k_2 factors within the defective second order group-factors
R	Total number of runs required to investigate the f factors
R_I	Number of runs required to declare the g_1 first order group-factors as defective or non-defective in the initial step

α_1	The level of significance of tests in the initial step
α_2	The level of significance of tests in type-one search steps
Δ	The effect of a factor
σ^2	The error variance
$\pi_1(\delta\Phi_1,\alpha_1)$	The power of the test in the initial step
$\pi_2(\delta\Phi_2,\alpha_2)$	The power of the test in type-one search steps

When screening with unequal group sizes the above symbols have been slightly modified, subscripting them appropriately.

CHAPTER 2

Two-type step-wise Group Screening Designs With Equal prior probabilities and Errors in observations.

2.1: Introduction.

When screening with two-types of search steps, we first divide the f-factors into g_1 first order group-factors each of size k_1 ($f=k_1g_1$). In the initial step, the first order group-factors are tested for their effects. Let n_1 be number of first order group-factors found to be defective in the initial step. Each defective first order group-factor is further divided into g_2 second order group-factors each containing k_2 factors ($k_1=k_2g_2$).

In the first of type-one search steps, we start with any of n_1 defective first order group-factors. We test the second order group-factors within it one by one till we declare second order group-factor defective. The second order group-factor declared defective is kept separate. In second of the type-one search steps we test the remaining second order group-factors in pooled group. If the pooled group-test is declared non-defective, then the test procedure is terminated. Otherwise in the third type-one search steps, we continue testing the remaining second order group-factors one by one till another second order group-factor is declared defective

The second and the third type-one search steps are repeated successively in the subsequent type-one search steps till the analysis terminates with a pooled group-factor declared non-defective or a single second order group-factor declared non-defective. This test procedure is performed on all the n_1 first order group-factors declared defective in the initial step.

Finally factors within each second order group-factors found to be defective in type-one search steps are declared as either defective or non-defective using type-two search steps.

2.2: Expected number of runs

Suppose f-factors are divided into g_1 first order group-factors of k_1 factors each in initial step. Each first order group-factor is tested at two levels. Assuming that all interactions effects are negligible, we shall require,

$$R_I = g_1 + 4 - g_1 \pmod 4$$

$$= g_1 + h, \qquad h = 1, 2, 3, 4 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots 2.2.1$$

runs to estimate the main effect of g_1 first order group-factors orthogonally.

Let $\hat{G}_1$ be the estimate of the main effect of any first order group-factor in the initial step with δ (=1, 2, ..., k_1) defective factors each with effect $\Delta > 0$. Then,

$$E(\hat{G}_1) = \delta\Delta \text{ and } var(\hat{G}_1) = \frac{\sigma^2}{g_1 + h} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots 2.2.2$$

where σ^2 is the error in observation. Now define

$$z_1 = \frac{\hat{G}_1 - \delta\Delta}{\sqrt{\frac{\sigma^2}{g_1 + h}}} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots 2.2.3$$

$$= y_1 - \delta\Phi_1.$$

where $y_1 = \frac{\hat{G}_1}{\sqrt{\frac{\sigma^2}{g_1 + h}}}$ and $\Phi_1 = \frac{\Delta}{\sqrt{\frac{\sigma^2}{g_1 + h}}}$

Assuming that the observations are normally distributed, z_1 is a standardized normal variate. We shall say that a first order group-factor is non-defective if $\delta = 0$, which implies that $\delta\Phi_1 = 0$. On the other hand a first order group-factor will be defective if $\delta\Phi_1 \neq 0$. Therefore our hypothesis will be expressed as,

$H_0 : \delta\Phi_1=0$

.....2.2.4

$H_1 : \delta\Phi_1\neq0$

In testing the above hypothesis we shall use normal deviate test if σ^2 is know, otherwise we shall use the t-test if σ^2 is estimated from the experiment.

Let $\pi_1(\delta\Phi_1, \alpha_1)$ denote the power of test in the initial step. Then,

$$\pi_1(\delta\Phi_1, \alpha_1) = \int_{z_1(\alpha_1)-\delta\Phi_1}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-z_1^2}{2}\right) \partial z_1 \dots\dots\dots\dots\dots\dots\dots\dots 2.2.5$$

where $z_1(\alpha_1)$ is given by

$$\alpha_1 = \int_{z_1(\alpha_1)}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-z_1^2}{2}\right) \partial z_1 \dots\dots\dots\dots\dots\dots\dots\dots 2.2.6$$

When $\delta=0$ or $\frac{\Delta}{\sigma} = 0$, we have $\pi_1(0, \alpha_1) =\alpha_1$ and when $\delta\neq0$ and $\frac{\Delta}{\sigma}$ is large, then we have

$\pi_1(\delta\Phi_1, \alpha_1) =1.$

Let π_1^* denote the probability that first order group-factor is declared defective in the initial step. Then,

$$\pi_1^* = \sum_{\delta=0}^{k_1} \binom{k_1}{\delta} p^{\delta}(1-p)^{k_1-\delta}\pi_1(\delta\Phi_1, \alpha_1) \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots 2.2.7$$

Further let n_1 be the number of the first order group-factors declared defective in the initial step. Then the probability function of n_1 is given by,

$$f(n_1^0) = p(n_1 = n_1^0) = \begin{cases} \binom{g_1}{n_1^0} \pi_1^{*n_1^0} (1-\pi_1^*)^{g_1 - n_1^0}, & n_1^0 = 1,2,\dots,g_1 \\ 0, & otherwise \end{cases} \dots\dots\dots\dots\dots 2.2.8$$

Thus

$$E(n_1^0) = g_1 \pi_1^* = \frac{f\pi_1^*}{k_1}$$

Now suppose n_1 first order group-factors are declared defective and each of these first order group-factors are divided into g_2 second order group-factors of size k_2. Then in total we have $n_1 g_2$ second order group-factors to test.

Let $\hat{G}_2$ be the estimate of the main effect of any second order group-factor with s (=0, 1, 2...k_2) defective factors each with $\Delta>0$. Then,

$$E(\hat{G}_2) = s\Delta \text{ and } var(\hat{G}_2) = \frac{\sigma^2}{n_1 g_2 + h} \dots\dots\dots\dots\dots\dots 2.2.9$$

where σ^2 is the error in observation. Define,

$$z_2 = \frac{\hat{G}_2 - s\Delta}{\sqrt{\frac{\sigma^2}{n_1 g_1 + h}}} \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots 2.2.10$$

$$= y_2 - s\Phi_2.$$

where $y_2 = \frac{\hat{G}_2}{\sqrt{\frac{\sigma^2}{n_1 g_1 + h}}}$ and $\Phi_2 = \frac{\Delta}{\sqrt{\frac{\sigma^2}{n_1 g_1 + h}}}$

Again in type-one search steps we assume that the observations are normally distributed, then z_2 is a standardized normal variate. We shall say that a second order group-factor in type-one search steps is non-defective if s=0, which implies that $s\Phi_2$=0. On the other hand a second order group-factor will be defective if $s\Phi_2 \neq 0$. Therefore our hypothesis in type-one search steps is expressed as,

$H_0 : s\Phi_2 = 0$

.....2.2.11

$H_1 : s\Phi_2 \neq 0$

To test above hypothesis we shall use normal deviate test if σ^2 is known. Otherwise we shall use the t-test if σ^2 is estimated from the experiment.

Let $\pi_2(s\Phi_2, \alpha_2)$ denote the power of test in the type-one search steps. Then,

$$\pi_2(s\Phi_2, \alpha_2) = \int_{z_2(\alpha_2)-s\Phi_2}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-z_2^2}{2}\right) \partial z_2 \dots\dots\dots\dots\dots\dots\dots\dots 2.2.12$$

where $z_2(\alpha_2)$ is given by

$$\alpha_2 = \int_{z_2(\alpha_2)}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-z_2^2}{2}\right) \partial z_2 \dots\dots\dots\dots\dots\dots\dots\dots 2.2.13$$

When s=0 or $\frac{\Delta}{\sigma} = 0$, we have $\pi_2(0, \alpha_2) = \alpha_2$ and when $s \neq 0$ and $\frac{\Delta}{\sigma}$ is large, then we have

$\pi_2(s\Phi_2, \alpha_2) = 1$.

Let π_2^* denote the probability that a second order group-factor is declared defective in type-one search steps. Then,

$$\pi_2^* = \sum_{s=0}^{k_2} \binom{k_2}{s} p^s (1-p)^{k_1 - s} \pi_2(s\Phi_2, \alpha_2) \dots\dots\dots\dots\dots\dots\dots\dots 2.2.14$$

Also let $\pi_2^{*'}$ denote the probability that the second order group-factor is declared defective given that it is within a first order group-factor that was declared defective in the initial step. Then,

$$\pi_2^{*'} = \frac{\pi_2^*}{\pi_1^*}$$

Then the conditional probability of n_2 the number of second order group-factors declared defective at the end of type-one search steps given n_1 is given by,

$$f(n_2^0/n_1^0) = \begin{cases} \binom{n_1^0 g_2}{n_2^0} \pi_2^{*'^{n_2^0}} (1-\pi_2^{*'})^{g_1 - n_2^0}, & n_2^0 = 1,2,\dots,g_2 \\ 0, & otherwise \end{cases} \quad \dots\dots\dots\dots\dots\dots 2.2.15$$

Then

$$E(n_2^0/n_1^0) = n_1^0 g_2 \pi_2^{*'}$$

$$E(n_2^0) = E\{E(n_2^0/n_1^0)\} = E(g_2 \pi_2^{*'} n_1)$$

$$= g_1 g_2 \pi_1^* \pi_2^{*'}$$

$$= \frac{f}{k_2} \pi_2^*$$

Let π^* be the probability that a second order group-factor chosen at random from the first order group-factor containing r defective second order group-factors that have been declared defective is defective. Then,

$$\pi^* = \frac{\pi_2^*}{\pi_1^*} \sum_{r=1}^{g_2} \binom{g_2 - 1}{r-1} \pi_2^{*r-1} (1-\pi_2^*)^{g_2 - r} \pi_1(\delta\Phi_1, \alpha_1)$$

$$= \frac{\pi_2^* \pi^+}{\pi_1^*} \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots 2.2.16$$

Where $\pi^+ = \sum_{r=1}^{g_2} \binom{g_2-1}{r-1} \pi_2^{*r-1} (1-\pi_2^*)^{g_2-r} \pi_1(\delta\Phi_1, \alpha_1)$ is the probability that the first order group-factor containing at least one defective second order group-factor is declared defective in the initial step.

Let α_{11} be the probability of declaring a non-defective second order group-factor defective and γ_{11} be the probability of declaring a defective second order group-factor defective in type-one search steps. Further let $\bar{\beta}$ be the probability that second order group-factor chosen at random from a first order group-factor that has been declared defective in initial step is declared defective in type-one search steps. Then,

$$\begin{aligned}\bar{\beta} &= \pi^+\gamma_{11} + \alpha_{11}(1-\pi^+) \\ &= \gamma_{11}\frac{\pi_2^*}{\pi_1^*}\pi^+ + \alpha_{11}\left(1 - \frac{\pi_2^*}{\pi_1^*}\pi^+\right) \\ &= \frac{\bar{\beta}^+}{\pi_1^*}\end{aligned}$$

Where $\bar{\beta}^+ = \pi_2^*(\gamma_{11} - \alpha_{11})\pi^+ + \pi_1^*\alpha_{11}$

Let α_1^* be the proportion of the first order group-factors that are declared defective but second order group-factors within them are declared non-defective. Obvious α_1^* is different in all steps of type-one search steps but because of simplicity in algebra we shall assume it is constant.

Let $P_{g_2}^*(j_1)$, j_1=1, 2,...,g_2 be the probability that exactly j_1 second order group-factors from the first order group-factor that has been declared defective in the initial step are declared defective in the type-one search steps. Then,

$$P_{g_2}^*(j_1) = \begin{cases} 1 - \frac{1}{\pi_1^*}\{1 - (1-\bar{\beta}^+)^{g_2}, j_1 = 0, \ldots\ldots\ldots\ldots .2.2.17 \\ \frac{1}{\pi_1^*}\binom{g_2}{j_1}\bar{\beta}^{+j_1}(1-\bar{\beta}^+)^{g_2-j_1}, \qquad j_1 = 1,2,\ldots,g_2 \end{cases}$$

Let $E^*_{g_2}(R_{j_1})$, $j_1=1,2,\ldots,g_2$ be the expected number of runs required to declare as defective or non-defective all the g_2 second order group-factors within a first order group-factor which has been declared defective if exactly j_1 second order group-factors are declared defective. Then following Manene and Odhiambo (1987) we have,

$$E^*_{g_2}(R_{j_1}) = \begin{cases} g_2 & j_1 = 0 \\ \frac{j_1 g_2}{j_1+1} + j_1 + \frac{j_1}{j_1+1} - \frac{j_1(g_2+j_1-2)}{g_2(g_2-1)} & \\ +\alpha_1^*\left(\frac{g_2}{j_1+1} - \frac{j_1}{j_1+1} - \frac{j_1}{g_{2i}-1} + \frac{j_1^2}{g_2(g_2-1)}\right) & \\ -\frac{j_1(1-\xi_1)(g_2-j_1)}{g_2(g_2-1)}, & for\ j_1 = 1,2,\ldots,g_2 \end{cases}$$

.....2.2.18

Where

$$\xi_1 = \begin{cases} 0, & if\ \alpha_1^* = 0 \\ 1, & otherwise \end{cases}$$

In type-two search steps we shall use non-orthogonal designs. We use one of the h runs in the initial step as the control run. Let p' be the probability that a factor chosen at random from the second order group-factor containing s defective factors which has been declared defective is defective. Then,

$$p' = \frac{p}{\pi_2^*}\sum_{s=1}^{k_2}\binom{k_2-1}{s-1}\mathrm{p}^{s-1}(1-\mathrm{p})^{k_2-s}\pi_2(s\Phi_2,\alpha_2)$$

$$= \frac{p\pi_1^+}{\pi_2^*} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots 2.2.19$$

Where $\pi_1^+ = \sum_{s=1}^{k_2} \binom{k_2-1}{s-1} p^{s-1}(1-p)^{k_2-s}\pi_2(s\Phi_2, \alpha_2)$

Let α_s be the probability of declaring a non-defective factor defective and γ_s be the probability of declaring a defective factor defective in the type-two search steps. Further let $\bar{\beta}^*$ be the probability that a factor chosen at random from second order group-factor which has been declared defective in type-one search steps is declared defective in type-two search steps. Then,

$$\begin{aligned}\bar{\beta}^* &= p'\gamma_s + \alpha_s(1-p') \\ &= \gamma_s \frac{p\pi_1^+}{\pi_2^*} + \alpha_s\left(1 - \frac{p\pi_1^+}{\pi_2^*}\right) \\ &= \frac{\bar{\beta}_1^*}{\pi_2^*}\end{aligned}$$

Where $\bar{\beta}_1^* = p(\gamma_s - \alpha_s)\pi_1^+ + \pi_2^*\alpha_s$

Let α_2^* be the proportion of second order group-factors which are declared defective but individual factors within them are declared non-defective. Obvious α_2^* is different at every stage but because of simplicity in algebra we shall assume it is constant.

Let $P_{k_2}(j_2)$ j_2=1, 2... K_2 be the probability that exactly j_2 factors from the second order group-factor that has been declared defective in type-one search steps are declared defective in the type-two search steps. Then,

,

$$P_{k_2}(j_2) = \begin{cases} 1 - \frac{1}{\pi_2^*}\{1 - (1-\bar{\beta}_1^*)^{k_2}, j_2 = 0, \ldots\ldots\ldots\ldots.2.2.20 \\ \frac{1}{\pi_2^*}\binom{k_2}{j_2}\bar{\beta}_1^{*j_2}(1-\bar{\beta}_1^*)^{k_2} \qquad j_2 = 1,2,\ldots,k_2 \end{cases}$$

Further let $E_{k_2}(R_{j_2})$ denote the expected number of runs required to declare as defective or non-defective all the factors within a second order group-factor that has been declared defective in type-one search steps if exactly j_2 factors are declared defective. Then,

$$E_{k_2}(R_{j_2}) = \begin{cases} k_2, & j_2 = 0 \\ \frac{j_2 k_2}{j_2+1} + j_2 + \frac{j_2}{j_2+1} - \frac{j_2(k_2+j_2-2)}{k_2(k_2-1)} & \\ +\alpha_2^*\left(\frac{k_2}{j_2+1} - \frac{j_2}{j_2+1} - \frac{j_2}{k_2-1} + \frac{j_2^2}{k_2(k_2-1)}\right) & \\ -\frac{j_2(1-\xi_2)(k_2-j_2)}{k_2(k_2-1)}, & for\ j_2 = 1,2,\dots,k_2 \end{cases}$$

.....2.2.21

Where

$$\xi_2 = \begin{cases} 0, & if\ \alpha_2^* = 0 \\ 1, & otherwise \end{cases}$$

Denote $R_{t_1}^0$ the number of runs required to declare as defective or non-defective all the g_2 second order group-factors within a defective first order group-factor using type-one search steps. Then,

$$E(R_{t_1}^0) = \sum_{j_1=0}^{g_2} E_{g_2}^*(R_{j_1})\, P_{g_2}^*(j_1)$$

$$= g_2 - \frac{g_2}{\pi_1^*}\left\{1-\left(1-\bar{\beta}^+\right)^{g_2}\right\} - \frac{1}{\pi_1^*}\left\{(2-\xi_1)\bar{\beta}^+ + \xi_1\bar{\beta}^{+2}\right\}$$
$$+ \frac{1}{\pi_1^*}\left\{g_2 + 1 + g_2\bar{\beta}^+ - \frac{1}{\bar{\beta}^+}\left[1-\left(1-\bar{\beta}^+\right)^{g_2+1}\right]\right\}$$
$$+ \frac{\alpha_1^*}{\pi_1^*}\{\frac{1}{\bar{\beta}^+}\left[1-\left(1-\bar{\beta}^+\right)^{g_2+1} - g_2\bar{\beta}^+\left(1-\bar{\beta}^+\right)^{g_2}\right] - 1 - \bar{\beta}^+ + \bar{\beta}^{+2}\}$$

.....2.2.22

Let R_{t_1} be the number of runs required to declare as defective or non-defective all n_1g_2 second order group-factors within n_1 defective first order group-factors using type-one search steps. Then,

$$R_{t_1} = n_1 E\left(R_{t_1}^0\right)$$

Thus

$$E\left(R_{t_1}\right) = E\{n_1 E\left(R_{t_1}^0\right)\}$$

$$= \frac{f}{k_1} + \frac{f}{k_2} + \frac{f\pi_1^*}{k_2} - \frac{f}{k_1\bar{\beta}^+}\left\{1-\left(1-\bar{\beta}^+\right)^{k_1+k_2}\right\}(1-\alpha_1^*)$$
$$+ f\bar{\beta}^+\left[\frac{1}{k_2} - \frac{2-\xi_1}{k_1} - \frac{\alpha_1^*}{k_1}\right] - \frac{f}{k_2}\left\{1-\left(1-\bar{\beta}^+\right)^{k_1}\right\} - \frac{f\bar{\beta}^{+2}}{k_1}(\xi_1 - \alpha_1^*)$$
$$- \frac{f\alpha_1^*}{k_1} - \frac{f\alpha_1^*}{k_1} - \frac{f\alpha_1^*}{k_2}((1-\bar{\beta}^+)^{k_1})$$

.....2.2.23

Denote $R_{t_2}^0$ the number of runs required to declare as defective or non-defective all the k_2 factors within a defective second order group-factor using type-two search steps. Then,

$$E\left(R_{t_2}^0\right) = \sum_{j_2=0}^{k_2} E_{k_2}\left(R_{j_2}\right)P_{k_2}(j_2)$$

$$= k_2 - \frac{k_2}{\pi_2^*}\left\{1-\left(1-\bar{\beta}_1^*\right)^{k_2}\right\} - \frac{1}{\pi_2^*}\{(2-\xi_2)\bar{\beta}_1^* + \xi_2\bar{\beta}_1^{*2}\}$$
$$+ \frac{1}{\pi_2^*}\left\{k_2 + 1 + k_2\bar{\beta}_1^* - \frac{1}{\bar{\beta}_1^*}\left[1-\left(1-\bar{\beta}^+\right)^{k_2+1}\right]\right\}$$
$$+ \frac{\alpha_2^*}{\pi_2^*}\{\frac{1}{\bar{\beta}_1^*}\left[1-\left(1-\bar{\beta}_1^*\right)^{k_2+1} - k_2\bar{\beta}^+\left(1-\bar{\beta}_1^*\right)^{k_2}\right] - 1 - \bar{\beta}_1^* + \bar{\beta}_1^{*2}\}$$

.....2.2.24

Let R_{t_2} be the number of runs required to declare as defective or non-defective all n_2k_2 factors within n_2 second order group-factors found to be defective at end of type-two search steps. Then,

$$R_{t_2} = n_2 E\left(R_{t_2}^0\right)$$

Thus

$$E\left(R_{t_2}\right) = E\{n_2 E\left(R_{t_2}^0\right)\}$$

$$= f + \frac{f}{k_2} + f\pi_2^* + f\bar{\beta}_1^*\left[1 - \frac{2-\xi_2}{k_2} - \frac{\alpha_2^*}{k_2}\right] - f\left\{1-\left(1-\bar{\beta}_1^*\right)^{k_2}\right\}$$
$$- \frac{f}{k_2\bar{\beta}_1^*}\left\{1-\left(1-\bar{\beta}_1^*\right)^{k_2+1}\right\}(1-\alpha_2^*) - \frac{f\bar{\beta}_1^{*2}}{k_2}(\xi_2 - \alpha_2^*) - \frac{f\alpha_2^*}{k_2}$$
$$- f\alpha_2^*\left(1-\bar{\beta}_1^*\right)^{k_2}$$

.....2.2.25

Theorem.

The expected total number of runs in a two-type step-wise group screening designs with errors in observations , in which k_1 is the size of each first order group-factor and k_2 is the size of second order group-factor where all symbols are as stated earlier is given by

$$E(R) = h + \frac{f}{k_2} + \frac{2f}{k_1} + f\pi_2^* + \frac{f\pi_1^*}{k_2} - f\left[\frac{\alpha_1^*}{k_1} + \frac{\alpha_2^*}{k_2}\right] + f\left\{\frac{\bar{\beta}_1^{*2}}{k_2}(\xi_2 - \alpha_2^*) + \frac{\bar{\beta}^{+2}}{k_1}(\xi_1 - \alpha_1^*)\right\} + f\bar{\beta}_1^*\left[1 - \frac{2-\xi_2}{k_2} - \frac{\alpha_2^*}{k_2}\right] + f\bar{\beta}^+\left[\frac{1}{k_2} - \frac{2-\xi_1}{k_1} - \frac{\alpha_1^*}{k_1}\right] - f((1 - \alpha_2^*)\left\{\left(1 - \bar{\beta}_1^*\right)^{k_2} - \frac{1}{k_2\bar{\beta}_1^*}\left\{1 - \left(1 - \bar{\beta}_1^*\right)^{k_2+1}\right\}\right\} - f(1 - \alpha_1^*)\{\frac{\left(1-\bar{\beta}^+\right)^{k_1}}{k_2} - \frac{f}{k_1\bar{\beta}^+}\left\{1 - \left(1 - \bar{\beta}^+\right)^{k_1+k_2}\right\}\}$$

Proof

The number of runs required in initial step is R_I, the number of runs required in type-one search steps is $E(R_{t_1})$ and the number of runs required in type-two of search steps is $E(R_{t_2})$. Thus

$$E(R) = R_I + E(R_{t_1}) + E(R_{t_2}) \text{ and}$$

$$E(R) = R_I + E\{n_2 E(R_{t_2}^0)\} + E\{n_1 E(R_{t_1}^0)\}$$

The expression for expected number of runs required reduces to that in the theorem on substituting the expressions in equations (2.2.1), (2.2.23) and (2.2.25) and simplifying it. This completes the proof.

Special case

If there are no errors in observations, then $\alpha_{11} = \alpha_s = \xi_1 = \xi_2 = \alpha_1^* = \alpha_2^* = 0$, where all symbols have usual meanings. Also $\bar{\beta}_1^* = p$ and $\bar{\beta}^+ = p$, $\gamma_{11}=\gamma_s=1$ and $\pi_2^* = 1 - q^{k_2}$ and $\pi_1^* = 1 - q^{k_1}$ implying $\pi_1=\pi_2=1$.

Therefore the expected total number of runs required to analyze all f factors using two-type step-wise group screening design is

$$E(R) = h + \frac{f}{k_2} + \frac{2f}{k_1} + f(1 - q^{k_2}) + \frac{f(1 - q^{k_1})}{k_2} fp\left(1 - \frac{2}{k_2}\right) + fp\left(\frac{1}{k_2} - \frac{2}{k_1}\right) + fq^{k_2} - \frac{f}{k_2 p}(1 - q^{k_2+1}) + \frac{f}{k_2} q^{k_2} - \frac{f}{k_1 p}(1 - q^{k_2+k_1})$$

$$= h + \frac{f}{k_2} + \frac{2f}{k_1} + f - fq^{k_2} + \frac{f}{k_2} - \frac{f}{k_2} q^{k_1} + fp - \frac{2pf}{k_2} + \frac{f}{k_2} - \frac{f}{k_2} q^{k_2} - \frac{2f}{k_1} + \frac{2f}{k_1} q^{k_2} + fq^{k_2} - \frac{f}{k_2 p}(1 - q^{k_2+1}) + \frac{f}{k_2} q^{k_1} - \frac{f(1 - q^{k_2})^{-1}}{k_1}(1 - q^{k_2+k_1})$$

.....2.2.26

Simplifying the second equation (2.2.26) we get

$$E(R) = h + f + \frac{3f}{k_2} + fp - \frac{2pf}{k_2} - \frac{f}{k_2} q^{k_2} + \frac{f}{k_1} q^{k_2} - \frac{f}{k_2 p}(1 - q^{k_2+1}) - \frac{f(1 - q^{k_2})^{-1}}{k_1}(1 - q^{k_2+k_1})$$

which the expression obtained by Manene (1987).

Corollary 2.1

For large values of $\frac{\Delta}{\sigma}$ and arbitrary values of p, the expected total number of runs required in two-type step-wise group screening designs is approximately equal to

$$h+\frac{f}{k_2}+\frac{2f}{k_1}+f\{1-(1-\alpha_2)q^{k_2}\}+\frac{f}{k_2}\{f\{1-(1-\alpha_1)q^{k_1}\}-f\left[\frac{\alpha_1^*}{k_1}+\frac{\alpha_2^*}{k_2}\right]$$
$$+f\left\{\frac{(1-A)^2}{k_2}(\xi_2-\alpha_2^*)+\frac{(1-B)^2}{k_1}(\xi_1-\alpha_1^*)\right\}$$
$$+f\{(1-A)\left[1-\frac{2-\xi_2}{k_2}-\frac{\alpha_2^*}{k_2}\right]+(1-B)\left[\frac{1}{k_2}-\frac{2-\xi_1}{k_1}-\frac{\alpha_1^*}{k_1}\right]$$
$$-f(1-\alpha_2^*)\left\{A^{k_2}-\frac{(1-A^{k_2+1})}{k_2(1-A)}\right\}-f(1-\alpha_1^*)\{\frac{B^{k_1}}{k_2}-\frac{\left(1-B^{k_2+k_1}\right)}{k_1(1-B)}\}$$

where

$$A=1-\{p(1-\alpha_s)+\alpha_s(1-\alpha_2)q^{k_2}\}$$

$$B=1-\{p(1-\alpha_{11})+\alpha_{11}(1-\alpha_1)q^{k_1}\}$$

Proof

If $\frac{\Delta}{\sigma}$ is large, we have the following approximations,

$$\pi_1^*=1-(1-\alpha_1)q^{k_1} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots 2.2.27$$

$$\pi_2^*=1-(1-\alpha_2)q^{k_2} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots 2.2.28$$

$$\gamma_{11}=\gamma_s=1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots 2.2.29$$

$$\pi^+=\pi_1^+=1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots 2.2.30$$

$$\bar{\beta}^+=p(1-\alpha_{11})+\alpha_{11}(1-(1-\alpha_1)q^{k_1} \ldots\ldots\ldots\ldots\ldots\ldots 2.2.31$$

$$\bar{\beta}_1^*=p(1-\alpha_s)+\alpha_s(1-(1-\alpha_2)q^{k_2} \ldots\ldots\ldots\ldots\ldots\ldots 2.2.32$$

The corollary follows immediately on substituting the above approximations on the theorem above. This completes the proof.

Corollary 2.2

For large values of $\frac{\Delta}{\sigma}$ and small values of p, the expected total number of runs in two-type step-wise group screening designs is approximately equal to

$$h+\frac{2f}{k_1}+f\{(1-\alpha_2)pk_2+\alpha_2\}+\frac{f}{k_2}\{(1-\alpha_1)pk_1+\alpha_1\}$$
$$+fp(1-\alpha_s)\left\{1-\frac{2-\xi_2}{k_2}-k_2+\frac{(k_2+1)(1-\alpha_2^*)}{2}\right\}$$
$$+fp(1-\alpha_{11})\left\{\frac{1}{k_2}-\frac{2-\xi_1}{k_1}-\frac{k_1}{k_2}\right.$$
$$\left.+\frac{(k_1+k_2)(k_1+k_2-1)(1-\alpha_1^*)}{2k_1}\right\}-f(1-\alpha_1^*)\{\frac{k_1+k_2}{k_1}\}$$

Proof

If $\frac{\Delta}{\sigma}$ is large then α_1^* , α_2^*,α_s, α_{11},α_1, and $\alpha_s$are relatively small, we have

$$1-(1-\alpha_1)q^{k_1}=k_1p+\alpha_1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots 2.2.33$$

$$1-(1-\alpha_2)q^{k_2}=(1-\alpha_2)k_2p+\alpha_2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots 2.2.34$$

The corollary follows immediately on substituting the approximate values given above in the expression in corollary (2.1) and approximating the resulting expression to terms of order p. This completes the proof.

CHAPTER 3

TWO-TYPE STEP-WISE GROUP SCREENING DESIGNS WITH UNEQUAL PRIOR PROBABILITIES AND ERRORS IN OBSERVATIONS

3.1: Expected number of runs.

Suppose f-factors are divided into g_1 first order group factors of k_{1i} factors each in initial step. Each first order group factor is tested at two levels. Assuming that all interaction effects are negligible. We shall require

$$R_I = g_1 + 4 - g_1 \pmod 4$$

$$= g_1 + h,\ h = 1, 2, 3, 4 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots 3.1.1$$

runs to estimate the main effects of g_1 first order group factors orthogonally.

Let $\hat{G}_{1i}$ be the estimate of the main effect of the i^{th} first order group factor in the initial step with δ_i ($i=1, 2 \ldots k_{1i}$) defective factors each with effect $\Delta_i > 0$. Then

$$E(\hat{G}_{1i}) = \delta_i \Delta_i \text{ and } Var(\hat{G}_{1i}) = \frac{\sigma^2}{\sqrt{g_{1+h}}}$$

Next define

$$z_{1i} = \frac{\hat{G}_{1i} - \delta_i \Delta_i}{\sqrt{\frac{\sigma^2}{g_1 + h}}} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots 3.1.2$$

$$= y_{1i} - \delta_i \phi_{1i}$$

where

$$y_{1i} = \frac{\hat{G}_{1i}}{\sqrt{\frac{\sigma^2}{g_1 + h}}}$$

$$\Phi_{1i} = \frac{\Delta_i}{\sqrt{\frac{\sigma^2}{g_1 + h}}}$$

Assuming that the observations are normally distributed. Z_{1i} is a standardized normal variate. We shall say the i^{th} first order group-factor is non-defective if $\delta_i=0$, which implies that $\delta_i\phi_{1i}=0$ and it is defective if $\delta_i\neq0$ or $\delta_i\phi_{1i}\neq0$. Thus we wish to test the hypothesis,

$H_0 : \delta_i\phi_{1i}=0$

.....3.1.3

$H_1 : \delta_i\phi_{1i}\neq0$

Assuming σ is known, we shall use the normal deviate test, and otherwise we would use a corresponding t-test. The power of the test for the i^{th} first order group factor is

$$\pi_{1i}(\delta_i\Phi_{1i}, \alpha_{1i}) = \int_{z_{1i}(\alpha_{1i})-\delta_{i\Phi_{1i}}}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-z_{1i}^2}{2}\right)\partial z_{1i} \dots\dots\dots\dots 3.1.4$$

where $z_{1i}(\alpha_{1i})$ is given by

$$\alpha_{1i} = \int_{z_{1i}(\alpha_{1i})}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-z_{1i}^2}{2}\right)\partial z_{1i}$$

which is the size of the critical region for testing significance of the i^{th} first order group-factor.

When $\delta_i=0$ or $\frac{\Delta_i}{\sigma}=0$ we have,

$$\pi_{1i}(0, \alpha_{1i}) = \alpha_{1i}$$

When $\delta_i\neq0$ and $\frac{\Delta_i}{\sigma}$ is large, then we have

$$\pi_{1i}(\delta_{i\Phi_{1i}}, \alpha_{1i}) \approx 1$$

Let π_{1i}^{*} denotes the probability that the i[th] first order group-factor is declared defective in initial step. Then,

$$\pi_{1i}^{*} = \sum_{\delta_i=0}^{k_{1i}} \binom{k_{1i}}{\delta_i} \mathrm{p_i}^{\delta_i}(1-\mathrm{p_i})^{k_{1i}-\delta_i}\pi_{1i}(\delta_i\Phi_{1i}, \alpha_{1i}) \ldots\ldots\ldots\ldots\ldots\ldots 3.1.5$$

where p_i is the probability that a factor in the i[th] first order group factor in initial is step defective.

Define a random variable U_{1i} as

$$U_{1i} = \begin{cases} 1 & if\ the\ i^{th}\ first\ order\ group\ factor\ is\ defective \\ 0 & if\ the\ i^{th}\ first\ order\ group\ factor\ is\ non-defective \end{cases}$$

Then

$$E(U_{1i}) = \pi_{1i}^{*}$$

Now let n_1 be the number of first order group- factors declared defective in initial step. Thus

$$n_1 = \sum_{i=1}^{g_1} U_{1i} \text{ and } E(n_1) = \sum_{i=1}^{g_1} \pi_{1i}^{*}$$

If n_1 is the number of first order group-factors found to be defective in the initial step. Then the i[th] first order group-factor found to be defective is further divided into g_{2i} second order group-factors each containing k_{2i} factors such that.

$$k_{1i}=k_{2i}g_{2i}$$

In total we have $\sum_{i=1}^{g_1} U_{1i}g_{2i}$ second order group-factors to test.

Let $\hat{G}_{2i}$ be the estimate of the main effect of the i[th] second order group-factor in type-one search steps with s_i (=1, 2.....k_{2i}) defective factors each with $\Delta_i>0$. Then,

$$E(\hat{G}_{2i}) = s_i\Delta_i \text{ and } Var(\hat{G}_{2i}) = \frac{\sigma^2}{\sqrt{\sum U_{1i}g_{2i}+h}}$$

Now define,

$$z_{2i} = \frac{\hat{G}_{2i} - s_i \Delta_i}{\sqrt{\frac{\sigma^2}{\sum_{i=1}^{g_1} U_{1i} g_{2i} + h}}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots 3.1.6$$

$=y_{2i}-s_i\phi_{2i}$

Where

$$y_{2i} = \frac{\hat{G}_{2i}}{\sqrt{\frac{\sigma^2}{\sum_{i=1}^{g_1} U_{1i} g_{2i} + h}}}$$

$$\Phi_{2i} = \frac{\Delta_i}{\sqrt{\frac{\sigma^2}{\sum_{i=1}^{g_1} U_{1i} g_{2i} + h}}}$$

Assuming that the observations are normally distributed, z_{2i} is standardized normal variate. Then the i^{th} second order group-factor is non-defective if $s_i = 0$ or $s_i\Phi_{2i}=0$ and its defective if $s_i \neq 0$ or $s_i\Phi_{2i}\neq 0$.Thus we test the following hypothesis,

$H_0 : s_i\phi_{2i}=0$

.....3.1.7

$H_1 : s_i\phi_{2i}\neq 0$

If σ is known we shall use normal deviate test or otherwise we would use the corresponding t-test. The power of the test is given by

$$\pi_{2i}(s_i\Phi_{2i}, \alpha_{2i}) = \int_{z_{2i}(\alpha_{2i})-s_{i\Phi_{2i}}}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-z_{2i}^2}{2}\right) \partial z_{2i} \ldots\ldots\ldots\ldots\ldots\ldots 3.1.8$$

where $z_{2i}(\alpha_{2i})$ is given by

$$\alpha_{2i} = \int_{z_{2i}(\alpha_{2i})}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-z_{2i}^2}{2}\right) \partial z_{2i}$$

which is the size of critical region for testing the significance of the i^{th} second order group-factor. Also when $s_i=0$ or $\frac{\Delta_i}{\sigma} = 0$ we have $\pi_{2i}(0, \alpha_{2i}) = \alpha_{2i}$ and if $s_i \neq 0$ and is $\frac{\Delta_i}{\sigma}$ large then,

$$\pi_{2i}(s_i \Phi_{2i}) \approx 1$$

Let π_{2i}^* be the probability that a second order group-factor is declared defective in type-one search steps. Then,

$$\pi_{2i}^* = \sum_{s_i=0}^{k_{2i}} \binom{k_{2i}}{s_i} p_i^{s_i} (1 - p_i)^{k_{2i} - s_i} \pi_{2i}(s_i \Phi_{2i}, \alpha_{2i}) \ldots\ldots\ldots\ldots\ldots\ldots 3.1.9$$

Also let $\pi_{2i}^{*\prime}$ denote the probability that the second order group-factor is defective given that it is within defective i^{th} first order group-factor

$$\pi_{2i}^{*\prime} = \frac{\pi_{2i}^*}{\pi_{1i}^*}$$

Let n_{2i} be the number of second order group-factors declared defective among g_{2i} second order group-factors within the i^{th} first order group-factor which was declared defective in the initial step, then for $n_{2i}=1,2,3,\ldots,g_{2i}$ and $i=1,2,\ldots\ldots,g_1$.

$$E(n_{2i}/U_{1i}) = g_{2i}\pi_{2i}^{*\prime} \text{ and } E(n_{2i}) = g_{2i}\pi_{2i}^* \ldots\ldots\ldots\ldots\ldots\ldots 3.1.10$$

Let π_i^* be the probability that a second order group-factor chosen at random from the i^{th} first order group-factor containing r_i defective second order group-factors that have been declared defective is defective. Then

$$\pi_i^* = \frac{\pi_{2i}^*}{\pi_{1i}^*} \sum_{r_i=1}^{g_{2i}} \binom{g_{2i}-1}{r_i-1} \pi_{2i}^{*r_i-1} (1-\pi_{2i}^*)^{g_{2i}-r_i} \pi_{1i}(\delta_i \Phi_{1i}, \alpha_{1i})$$

$$= \frac{\pi_{2i}^* \pi_i^+}{\pi_{1i}^*} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots 3.1.11$$

where $\pi_i^+ = \sum_{r_i=1}^{g_{2i}} \binom{g_{2i}-1}{r_i-1} \pi_{2i}^{*r_i-1} (1-\pi_{2i}^*)^{g_{2i}-r_i} \pi_{1i}(\delta_i \Phi_{1i}, \alpha_{1i})$

Let α_{11i} be the probability of declaring a non-defective second order group-factor defective from the i^{th} first order group-factor and let γ_{11i} be the probability of declaring a defective second order group-factor defective in type-one search steps. Further let $\bar{\beta}_i$ be the probability that a second order group-factor chosen at random from the i^{th} first order group-factor that has been declared defective in initial step is declared defective in type-one search steps. Then

$$\bar{\beta}_i = \pi_i^* \gamma_{11i} + \alpha_{11i}(1-\pi_i^*)$$

$$= \gamma_{11i} \frac{\pi_{2i}^* \pi_i^+}{\pi_{1i}^*} + \alpha_{11i}\left(1 - \frac{\pi_{2i}^* \pi_i^+}{\pi_{1i}^*}\right)$$

$$= \frac{\bar{\beta}_i^+}{\pi_{1i}^*}$$

where $\bar{\beta}_i^+ = \pi_{2i}^*(\gamma_{11i} - \alpha_{11i})\pi_i^+ + \pi_{1i}^* \alpha_{11i}$

Let α_{1i}^* be the proportion of first order group-factors that are declared defective but second order group-factors within them are declared non-defective. We shall assume α_{1i}^* is constant for all steps for simplicity in algebra.

Denote $P_{g_{2i}}^*(j_1)$, $j_1=1,2,\ldots,g_{2i}$ be the probability that exactly j_1 second order group-factors from the i^{th} first order group-factor in the initial step that has been declared defective in type-one search steps are declared defective

$$P^*_{g_{2i}}(j_1) = \begin{cases} 1 - \dfrac{1}{\pi^*_{1i}}\left\{1 - (1 - \bar{\beta}^+_i)^{g_{2i}}\right\}, & j_1 = 0 \\ \dfrac{1}{\pi^*_{1i}}\binom{g_{2i}}{j_1}\bar{\beta}^{+j_1}_i(1 - \bar{\beta}^+_i)^{g_{2i}-j_1}, & j_1 0 = 1,2,..g_{2i} \end{cases}$$

.....3.1.12

Let $E_{g_{2i}}(R_{j_1})$ j_i=1,2,...,g_{2i} be the expected number of runs required to declare exactly j_1 second order group-factors defective from the i^{th} initial step first order group-factor which has been declared defective in the initial step. Then,

$$E_{g_{2i}}(R_{j_1}) = \begin{cases} g_{2i}, & j_1 = 0 \\ \dfrac{j_1 g_{2i}}{j_1 + 1} + j_1 + \dfrac{j_1}{j_1 + 1} - \dfrac{j_1(g_{2i} + j_1 - 2)}{g_{2i}(g_{2i} - 1)} \\ +\alpha^*_{1i}\left(\dfrac{g_{2i}}{j_1 + 1} - \dfrac{j_1}{j_1 + 1} - \dfrac{j_1}{g_{2i} - 1} + \dfrac{j_1^2}{g_{2i}(g_{2i} - 1)}\right) \\ -\dfrac{j_1(1 - \xi_{1i})(g_{2i} - j_1)}{g_{2i}(g_{2i} - 1)}, & for\ j_1 = 1,2,\dots,g_{2i} \end{cases}$$

.....3.1.13

Where

$$\xi_{1i} = \begin{cases} 0, & if\ \alpha^*_{1i} = 0 \\ 1, & otherwise \end{cases}$$

In type-two search steps we shall use non-orthogonal designs. Let p'_i be the probability that a factor chosen at random from the i^{th} second order group-factor containing s_i defective factors that has been declared defective in type-one search steps is defective in type-two search steps. Then,

$$p_i' = \frac{p_i}{\pi_{1i}^*} \sum_{s_i=1}^{k_{2i}} \binom{k_{2i}-1}{s_i-1} p_i^{s_i-1}(1-p_i)^{k_{2i}-s_i} \pi_{2i}(s_i \Phi_{2i}, \alpha_{2i})$$

$$= \frac{p_i \pi_{1i}^+}{\pi_{2i}^*} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots 3.1.14$$

where $\pi_{1i}^+ = \sum_{s_i=1}^{k_{2i}} \binom{k_{2i}-1}{s_i-1} p_i^{s_i-1}(1-p_i)^{k_{2i}-s_i} \pi_{2i}(s_i \Phi_{2i}, \alpha_{2i})$

Let α_{si} be the probability of declaring a non-defective factor from the i^{th} second order group-factor in type-two search steps as defective and γ_{si} be the probability of declaring a defective factor as defective in type-two search steps. Further let $\bar{\beta}_i^*$ be the probability that a factor chosen at random from a second order group-factor that has been declared defective in type-one search steps is declared defective in type-two search steps. Then,

$$\bar{\beta}_i^* = p_i' \gamma_{si} + \alpha_{si}(1-p_i')$$

$$= \gamma_{si} \frac{p_i \pi_{1i}^+}{\pi_{2i}^*} + \alpha_{si}\left(1 - \frac{p_i \pi_{1i}^+}{\pi_{2i}^*}\right)$$

$$= \frac{\bar{\beta}_{1i}^*}{\pi_{2i}^*}$$

Where $\bar{\beta}_{1i}^* = p_i(\gamma_{si} - \alpha_{si})\pi_{1i}^+ + \pi_{2i}^* \alpha_{si}$

Let α_{2i}^* be the proportion of second order group-factors which are declared defective but the factors within them are declared non-defective. Obvious α_{2i}^* is different in all steps of type-one search steps but we shall assume it is constant in all steps of type-one search steps for simplicity in algebra.

Denote$P_{k_{2i}}(j_2)$, (j_2=1,2,...,k_{2i}) the expected number of runs required to declare exactly j_2 factors defective from the i^{th} second order group-factor which has been declared defective in type-one search steps. Then,

$$P_{k_{2i}}(j_2) = \begin{cases} 1 - \frac{1}{\pi_{2i}^*}\left\{1 - \left(1 - \bar{\beta}_{1i}^*\right)^{k_{2i}}\right\}, & j_2 = 0 \\ \frac{1}{\pi_{2i}^*}\binom{k_{2i}}{j_2}\bar{\beta}_{1i}^{*j_2}(1 - \bar{\beta}_i^*)^{k_{2i}-j_2}, & j_2 = 1,2,..k_{2i} \end{cases}$$

.....3.1.15

Further let $E_{k_{2i}}(R_{j_2})$ denote the expected number of runs required to declare as defective or non-defective all factors within a second order group-factor if exactly j_2 factors are declared defective. Then,

$$E_{k_{2i}}(R_{j_2}) = \begin{cases} k_{2i}, & j_2 = 0 \\ \frac{j_2 k_{2i}}{j_2+1} + j_2 + \frac{j_2}{j_2+1} - \frac{j_2(k_{2i}+j_2-2)}{k_{2i}(k_{2i}-1)} \\ +\alpha_{2i}^*\left(\frac{k_{2i}}{j_2+1} - \frac{j_2}{j_2+1} - \frac{j_2}{k_{2i}-1} + \frac{j_2^2}{k_{2i}(k_{2i}-1)}\right) \\ -\frac{j_2(1-\xi_{2i})(k_{2i}-j_2)}{k_{2i}(k_{2i}-1)}, & for\ j_2 = 1,2,\ldots,k_{2i} \end{cases}$$

.....3.1.16

Where

$$\xi_{2i} = \begin{cases} 0, & if\ \alpha_{2i}^* = 0 \\ 1, & otherwise \end{cases}$$

Let $R_{t_1}^0$ be the number of runs required to analyze the i^{th} first order group-factor once it has been declared defective in the initial step. Then,

$$E(R_{t_1}^0) = \sum_{j_1=1}^{g_{2i}} E_{g_{2i}}(R_{j_1}) P_{g_{2i}}(j_1)$$

$$= g_{2i} - \frac{g_{2i}}{\pi_{1i}^*}\left\{1 - \left(1 - \bar{\beta}_i^+\right)^{g_{2i}}\right\} - \frac{1}{\pi_{1i}^*}\left\{(2 - \xi_{1i})\bar{\beta}_i^+ + \xi_{1i}\bar{\beta}_i^{+^2}\right\}$$

$$+ \frac{1}{\pi_{1i}^*}\{g_{2i} + 1 + g_{2i}\bar{\beta}_i^+ - \frac{1}{\bar{\beta}_i^+}\left\{1 - \left(1 - \bar{\beta}_i^+\right)^{g_{2i}+1}\right\}\}$$

$$+ \frac{\alpha_{1i}^*}{\pi_{1i}^*}\{\frac{1}{\bar{\beta}_i^+}\{\left(1 - \left(1 - \bar{\beta}_i^+\right)^{g_{2i}+1} - g_{2i}\bar{\beta}_i^+\left(1 - \bar{\beta}_i^+\right)^{g_{2i}}\right\} - 1 - \bar{\beta}_i^+ + \bar{\beta}_i^{+^2}\}$$

.....3.1.17

Denote R_{t_1} the number of runs required to analyze all the first order group factors declared defective in initial step. Then,

$$R_{t_1} = \sum_{i=1}^{g_1} U_{1i}\, E(R_{t_1}^0)$$

$$E(R_{t_1}) = \sum_{i=1}^{g_1}\left\{\pi_{1i}^* g_{2i} - g_{2i}\left\{1 - \left(1 - \bar{\beta}_i^+\right)^{g_{2i}}\right\} - (2 - \xi_{1i})\bar{\beta}_i^+ + \xi_{1i}\bar{\beta}_i^{+^2}\right\}$$

$$+ \sum_{i=1}^{g_1}\{g_{2i} + 1 + g_{2i}\bar{\beta}_i^+ - \frac{1}{\bar{\beta}_i^+}\left\{1 - \left(1 - \bar{\beta}_i^+\right)^{g_{2i}+1}\right\}\} +$$

$$\sum_{i=1}^{g_1}\alpha_{1i}^*\{\frac{1}{\bar{\beta}_i^+}\left\{\left\{1 - \left(1 - \bar{\beta}_i^+\right)^{g_{2i}+1}\right\} - g_{2i}\bar{\beta}_i^+\left(1 - \bar{\beta}_i^+\right)^{g_{2i}}\right\} - 1 - \bar{\beta}_i^+ + \bar{\beta}_i^{+^2}\}$$

.....3.1.18

Let $R_{t_2}^0$ be the number of runs required to analyze the ith second order group-factor once it has been declared defective in type-one search steps. Then,

$$E(R_{t_2}^0) = \sum_{j_2=1}^{k_{2i}} E_{k_{2i}}(R_{j_2}) P_{k_{2i}}(j_2)$$

$$= k_{2i} - \frac{k_{2i}}{\pi_{2i}^*}\left\{1 - \left(1 - \bar{\beta}_{1i}^*\right)^{k_{2i}}\right\} - \frac{1}{\pi_{2i}^*}\{(2 - \xi_{2i})\bar{\beta}_{1i}^* + \xi_{2i}\bar{\beta}_{1i}^{*^2}\}$$

$$+ \frac{1}{\pi_{2i}^*}\{k_{2i} + 1 + k_{2i}\bar{\beta}_{1i}^* - \frac{1}{\bar{\beta}_{1i}^*}\left\{1 - \left(1 - \bar{\beta}_{1i}^*\right)^{k_{2i}+1}\right\}\}$$

$$+ \frac{\alpha_{2i}^*}{\pi_{2i}^*}\{\frac{1}{\bar{\beta}_{1i}^*}\{\left(1 - \left(1 - \bar{\beta}_{1i}^*\right)^{k_{2i}+1} - k_{2i}\bar{\beta}_{1i}^*\left(1 - \bar{\beta}_{1i}^*\right)^{k_{2i}}\right\} - 1 - \bar{\beta}_{1i}^* + \bar{\beta}_{1i}^{*^2}\}$$

.....3.1.19

Denote by R_{t_2} the total number of runs required to analyze all the second order group-factors declared defective in the type-one search steps. Then,

$$R_{t_2} = \sum_{i=1}^{g_1} U_{1i} n_{2i} E(R_{t_2}^0)$$

$$E(R_{t_2}) = \sum_{i=1}^{g_1} \frac{k_{1i}}{k_{2i}}\{k_{2i}\pi_{2i}^* - k_{2i}\left\{1 - \left(1 - \bar{\beta}_{1i}^*\right)^{k_{2i}}\right\} - (2 - \xi_{2i})\bar{\beta}_{1i}^* + \xi_{2i}\bar{\beta}_{1i}^{*^2}\}$$

$$+ \sum_{i=1}^{g_1} \frac{k_{1i}}{k_{2i}}\{k_{2i} + 1 + k_{2i}\bar{\beta}_{1i}^* - \frac{1}{\bar{\beta}_{1i}^*}\left\{1 - \left(1 - \bar{\beta}_{1i}^*\right)^{k_{2i}+1}\right\}\} +$$

$$\sum_{i=1}^{g_1} \frac{k_{1i}\alpha_{2i}^*}{k_{2i}}\{\frac{1}{\bar{\beta}_{1i}^*}\left\{1 - \left(1 - \bar{\beta}_{1i}^*\right)^{k_{2i}+1}\right\} - k_{2i}(1 - \bar{\beta}_{1i}^*)^{k_{2i}} - 1 - \bar{\beta}_{1i}^* + \bar{\beta}_{1i}^{*^2}\}$$

.....3.1.20

Theorem

The expected total number of runs required to analyze all the factors in two-type step-wise group screening designs with unequal a prior probability is given by,

$$E(R) = h + 2g_1 + \sum_{i=1}^{g_1} \frac{k_{1i}}{k_{2i}} \{\pi_{1i}^* + \left(1 - \bar{\beta}_i^+\right)^{k_{1i}} (1 - \alpha_{1i}^*) +$$

$$\sum_{i=1}^{g_1} \{k_{1i}\bar{\beta}_i^+ (\frac{1}{k_{2i}} - \frac{2 - \xi_{1i}}{k_{1i}} - \frac{\alpha_{1i}^*}{k_{1i}}) - \bar{\beta}_i^+ (\xi_{1i} - \alpha_{1i}^*)\} -$$

$$\sum_{i=1}^{g_1} \{\frac{\left(1 - \left(1 - \bar{\beta}_i^+\right)^{k_{1i}+k_{2i}}\right)}{\bar{\beta}_i^+} (1 - \alpha_{1i}^*) - \alpha_{1i}^*\} +$$

$$\sum_{i=1}^{g_1} \frac{k_{1i}}{k_{2i}} \{1 + k_{2i}\pi_{2i}^* - \alpha_{2i}^* + k_{2i}\bar{\beta}_{1i}^* \left(1 - \frac{2 - \xi_{2i}}{k_{2i}} - \frac{\alpha_{2i}^*}{k_{2i}}\right) - \bar{\beta}_{1i}^* (\xi_{2i} - \alpha_{2i}^*)\} +$$

$$\sum_{i=1}^{g_1} \frac{k_{1i}}{k_{2i}} \{ k_{2i}\left(1 - \bar{\beta}_{1i}^*\right)^{k_{2i}} (1 - \alpha_{2i}^*) - \frac{\left(1 - \left(1 - \bar{\beta}_{1i}^*\right)^{k_{2i}+1}\right)}{\bar{\beta}_{1i}^*} (1 - \alpha_{2i}^*)\}$$

Proof:

We note that

$$E(R) = R_I + E\left(R_{t_1}\right) + E(R_{t_2})$$

Then,

$$E(R) = R_I + E\left[\sum_{i=1}^{g_1} U_{1i}\, E\left(R_{t_1}^0\right)\right] + E\left[\sum_{i=1}^{g_1} U_{1i} n_{2i}\, E\left(R_{t_2}^0\right)\right]$$

The expression for the expected number of runs reduces to that in the theorem on substituting the expressions in the equations (3.1.1), (3.1.18) and (3.1.20) and simplifying

Special case

If there are no errors in observations then $\alpha_{11i} = \alpha_{si} = \xi_{1i} = \xi_{2i} = \alpha_{1i}^* = \alpha_{2i}^* = 0$,where all symbols have their usual meanings. Also $\bar{\beta}_{1i}^* \approx p_i$ and $\bar{\beta}_i^* \approx p_i$ with γ_{11i}=1 and γ_{si}=1. $\pi_{1i}^* = 1 - q_i^{k_{1i}}$ and $\pi_{2i}^* = 1 - q_i^{k_{2i}}$ with π_{1i}=1 and π_{2i}=1

Then the expected total number of runs required to analyze two-type stepwise group screening design without errors in decision is,

$$\begin{aligned} E(R) = h + 2g_1 &+ \sum_{i=1}^{g_1} \frac{k_{1i}}{k_{2i}}\left(1 - q_i^{k_{1i}} + q_i^{k_{1i}}\right) \\ &+ \sum_{i=1}^{g_1} \left\{k_{1i} p_i \left(\frac{1}{k_{2i}} - \frac{2}{k_{1i}}\right) - \frac{1}{p_i}\left(1 - q_i^{k_{1i}+k_{2i}}\right)\right\} \\ &+ \sum_{i=1}^{g_1} \frac{k_{1i}}{k_{2i}} \left\{1 + k_{2i}\left(1 - q_i^{k_{2i}}\right) + k_{2i} p_i \left(1 - \frac{2}{k_{2i}}\right) + k_{2i} q^{k_{2i}} \right. \\ &\left. - \frac{1}{p_i}\left(1 - q^{k_{2i}+1}\right)\right\} \end{aligned}$$

Simplifying the equation above reduces to

$$E(R) = h + f + \sum_{i=1}^{g_1}\{2q_i^{k_{2i}} + k_{1i}p_1 - \left(1 - q_i^{k_{2i}}\right)^{-1}(1 - q_i^{k_{1i}+k_{2i}})\}$$
$$+ \sum_{i=1}^{g_1}\frac{k_{1i}}{k_{2i}}\{3 - q_i^{k_{2i}} - 2p_i - \frac{1}{p_i}(1 - q^{k_{2i}+1})\}$$

which is the expression obtained by Manene (2005).

Corollary 3.1

For large values of $\frac{\Delta_i}{\sigma}$ and arbitrary values of p_i the expected total number of runs in two-type step-wise group screening design with the i^{th} first order group-factor of size k_{1i} i=1,2,...,g_1 is approximately equal to

$$h + 2g_1 + \sum_{i=1}^{g_1}\frac{k_{1i}}{k_{2i}}\{1 - (1 - \alpha_{1i})q_i^{k_{1i}} + B_i(1 - \alpha_{1i}^*)\}$$
$$+ \sum_{i=1}^{g_1}\{k_{1i}(1 - B_i)\left[\frac{1}{k_{2i}} - \frac{2 - \xi_{1i}}{k_{1i}} - \frac{\alpha_{1i}^*}{k_{1i}}\right] - (1 - B_i)^2[\xi_{1i} - \alpha_{1i}^*]$$
$$+ \sum_{i=1}^{g_1}\frac{(1 - B_i^{k_{1i}+k_{2i}})}{1 - B_i}(1 - \alpha_{1i}^*) - \alpha_{1i}^*\} + \sum_{i=1}^{g_1}\frac{k_{1i}}{k_{2i}}\{1$$
$$+ k_{2i}\left(1 - (1 - \alpha_{2i})q_i^{k_{2i}}\right) + \alpha_{2i}^* + k_{2i}(1 - A_i)\left[1 - \frac{2 - \xi_{2i}}{k_{2i}} - \frac{\alpha_{2i}^*}{k_{2i}}\right]$$
$$- (1 - A_i)^2[\xi_{2i} - \alpha_{2i}^*] + k_{2i}A_i^{k_{2i}}(1 - \alpha_{2i}^*)$$
$$- \frac{\left(1 - A_i^{k_{2i}+1}\right)}{1 - A_i}(1 - \alpha_{2i}^*)\}$$

where $A_i = 1 - [p_i(1 - \alpha_{si}) + \alpha_{si}(1 - \alpha_{2i})q_i^{k_{2i}}]$

$$B_i = 1 - [p_i(1 - \alpha_{11i}) + \alpha_{11i}(1 - \alpha_{1i})q_i^{k_{1i}}].$$

Proof

If $\frac{\Delta_i}{\sigma}$ are large, then we have the following approximations,

$$\pi_{1i}^{*} = 1 - (1 - \alpha_{1i})q_i^{k_{1i}} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots 3.1.21$$

$$\pi_{2i}^{*} = 1 - (1 - \alpha_{2i})q_i^{k_{2i}} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots 3.1.22$$

$$\pi_i^{+} = \pi_{1i}^{+} = 1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots 3.1.23$$

$$\gamma_{11i} = \gamma_{si} = 1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots 3.1.24$$

$$\bar{\beta}_i^{+} = p_i(1 - \alpha_{11i}) + \alpha_{11i}(1 - \alpha_{1i})q_i^{k_{1i}} \ldots\ldots\ldots\ldots\ldots\ldots 3.1.25$$

$$\bar{\beta}_{1i}^{*} = p_i(1 - \alpha_{si}) + \alpha_{si}(1 - \alpha_{2i})q_i^{k_{2i}} \ldots\ldots\ldots\ldots\ldots\ldots 3.1.26$$

The corollary follows immediately on substituting the above approximate values on expression (3.1.20). This completes the proof.

Corollary 3.2

For large values of $\frac{\Delta_i}{\sigma}$ and small values of p_i the expected total number of runs in two-type step-wise group screening designs with g_1 first order group-factors in the initial step, the i^{th} first order group-factor being of size k_{1i}, i=1,2,...,g_1 is approximately equal to

$$h + 2g_1 + \sum_{i=1}^{g_1} \frac{k_{1i}}{k_{2i}} \{1 + (1 - \alpha_{1i}) p_i k_{1i} + \alpha_{1i} - (1 - \alpha_{11i}) p_i k_{1i} - \alpha_{1i}^*\}$$

$$+ \sum_{i=1}^{g_1} \{ p_i k_{1i} (1 - \alpha_{11i}) \left[\frac{1}{k_{2i}} - \frac{2 - \xi_{1i}}{k_{1i}} \right]$$

$$+ \sum_{i=1}^{g_1} \{ (k_{1i} + k_{2i}) \left[1 - \frac{(k_{1i} + k_{2i} - 1) p_i (1 - \alpha_{11i})}{2} - \alpha_{1i}^* \right] - \alpha_{1i}^* \}$$

$$+ \sum_{i=1}^{g_1} \frac{k_{1i}}{k_{2i}} \{1 + \alpha_{2i}^* + (1 - \alpha_{2i}) p_i k_{2i}^2 + k_{2i} p_i (1 - \alpha_{si}) \left[1 - \frac{2 - \xi_{2i}}{k_{2i}} \right]$$

$$+ k_{2i} [1 - k_{2i} p_i (1 - \alpha_{si}) - \alpha_{2i}^*]$$

$$- (k_{2i} + 1) \left[1 - \frac{k_{2i} p_i (1 - \alpha_{si})}{2} - \alpha_{2i}^* \right] \}$$

Proof

If $\frac{\Delta_i}{\sigma}$ are large, then $\alpha_{1i}, \alpha_{2i}, \alpha_{1i}^*, \alpha_{2i}^*, \alpha_{11i}$ and α_{si} are relatively small. Hence if p_i's are small we have,

$$(1 - \alpha_{1i}) q_i^{k_{1i}} = (1 - \alpha_{1i}) k_{1i} p_i + \alpha_{1i} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots 3.1.29$$

$$(1 - \alpha_{2i}) q_i^{k_{2i}} = (1 - \alpha_{2i}) k_{2i} p_i + \alpha_{2i} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots 3.1.30$$

The corollary follows immediately on substituting the approximate values given above in corollary (3.1) and approximating the resulting expression to terms of order p_i. This completes the proof.

CHAPTER 4

4.1: Concluding Remarks.

The usual sampling inspection plan consists of drawing samples from the population. All the items in the sample(s) are then examined. If the proportion of defective items in the sample(s) is small, then they are replaced by good ones and all items in the population are accepted. In such cases some items are passed without being inspected. In group screening designs however every item is subject to inspection either in groups or individually. Group screening designs are thus some kind of 100% sampling inspection plans. Thus screening efficiency of two-type stepwise group screening design can be measured in terms of expected number of runs E(R). The small value of E(R) indicates the better performance on average.

When screening with errors in observations the value of the expected number of runs is higher than the value for the corresponding case when screening without errors in observations in two-type step-wise group screening designs as seen in tables in appendix.

The result presented in tables 1, 2, 3 and 4 are only for illustration and are intended to indicate that it is possible to use corollary (2.1) to show how two-type stepwise group screening design works for certain values of p, α_1, α_2, α_1^*, α_2^*, α_s, α_{11}, k_1, k_1 and $\frac{\Delta}{\sigma}$ such that E(R) is minimized.

Group screening techniques can be used in industries in sorting out defective items from non-defective ones with substantial saving in cost of inspection and time. In chemical industry, the technique has been used for example in;

i. classifying an unknown chemical element,

ii. selecting the best catalyst for chemical reaction from a large number of compounds which was are possible candidates.

Group screening techniques have also been applied in biological experiments.

4.2: Recommendations for Further Research.

The value of a factor being defective should always be small for this design to work.

This design, one can extend to three-type stepwise group screening designs and generalize to multi-type step-wise group screening designs with errors in observations considering the two cases that is equal and unequal prior probability of factors being defective.

APPENDIX

Simulation of Tables.

Using the corollary (2.1) for expected number of runs required to analyze the f-factors in the two-type step-wise group screening designs which is

$$h+\frac{f}{k_2}+\frac{2f}{k_1}+f\{1-(1-\alpha_2)q^{k_2}\}+\frac{f}{k_2}\{f\{1-(1-\alpha_1)q^{k_1}\}-f\left[\frac{\alpha_1^*}{k_1}+\frac{\alpha_2^*}{k_2}\right]$$
$$+f\left\{\frac{(1-A)^2}{k_2}(\xi_2-\alpha_2^*)+\frac{(1-B)^2}{k_1}(\xi_1-\alpha_1^*)\right\}$$
$$+f\{(1-A)\left[1-\frac{2-\xi_2}{k_2}-\frac{\alpha_2^*}{k_2}\right]+(1-B)\left[\frac{1}{k_2}-\frac{2-\xi_1}{k_1}-\frac{\alpha_1^*}{k_1}\right]$$
$$-f(1-\alpha_2^*)\left\{A^{k_2}-\frac{(1-A^{k_2+1})}{k_2(1-A)}\right\}-f(1-\alpha_1^*)\{\frac{B^{k_1}}{k_2}-\frac{\left(1-B^{k_2+k_1}\right)}{k_1(1-B)}\}$$

where

$$A=1-\{p(1-\alpha_s)+\alpha_s(1-\alpha_2)q^{k_2}\}$$
$$B=1-\{p(1-\alpha_{11})+\alpha_{11}(1-\alpha_1)q^{k_1}\}$$

where the symbols above have the usual meanings as stated before.

We then generate some tables for specified values of p with the best combination of k_1 and k_2 which gives the minimum number of runs required to analyze two-type stepwise group screening design with errors in observations and equal prior probability of factor being defective, we then compare with the case when we have no errors in observations.

Table 4.1

$f = 100, h = 3, \alpha_1 = \alpha_2 = 0.005, \alpha_1^* = \alpha_2^* = 0.005, and\ \alpha_s = \alpha_{11} = 0.002$

p	k_1	k_2	E(R) with errors in observations	E(R) without errors in observations
0.001	50	23	19.6581	17.0182
0.002	45	18	23.2323	21.0894
0.003	35	13	27.5916	25.8969
0.010	25	10	43.2445	41.9011
0.020	15	8	62.8761	61.6136
0.030	15	7	73.3567	72.0303

Table 4.2

$f = 100, h = 3, \alpha_1 = \alpha_2 = 0.01, \alpha_1^* = \alpha_2^* = 0.01\ and\ \alpha_s = \alpha_{11} = 0.01$

p	k_1	k_2	E(R) with errors in observations	E(R) without errors in observations
0.01	35	9	44.1574	39.3938
0.02	25	7	59.3295	55.5443
0.03	15	6	75.0831	71.6544
0.04	15	6	84.2162	80.9174
0.05	15	5	92.5327	89.1929

Table 4.3

$f = 200, h = 3, \alpha_1 = \alpha_2 = 0.005, \alpha_1^* = \alpha_2^* = 0.005$ *and* $\alpha_s = \alpha_{11} = 0.002$

P	k_1	k_2	E(R) with errors in observations	E(R) without errors in observations
0.001	50	21	36.2646	31.3087
0.002	35	17	47.0039	42.8924
0.003	20	14	64.1534	60.6594
0.010	15	9	97.3544	94.7718
0.020	15	7	122.1501	119.62954
0.030	15	6	143.0944	140.3087

Table 4.4

$f = 200, h = 3, \alpha_1 = \alpha_2 = 0.01, \alpha_1^* = \alpha_2^* = 0.01 \; and \; \alpha_s = \alpha_{11} = 0.01$

P	k_1	k_2	E(R) with errors in observations	E(R) without errors in observations
0.010	45	9	83.1829	73.6100
0.020	25	7	115.6590	108.0886
0.030	15	6	147.1663	140.3087
0.040	15	6	165.4324	158.8349
0.050	15	5	182.0655	175.3858

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