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European Option Pricing Using Truncated Normal Distribution

Research Report in Mathematics, Number 01, 2020

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June 2020

Submitted to the School of Mathematics in partial fulfilment for a degree in Master of Science in Actuarial Science

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Submitted to the School of Mathematics in partial fulfilment for a degree in Master of Science in Actuarial Science

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Abstract

Option trading is one of the activities that take place in the financial market. Pricing these option is key for investor to ensure that the position they take offers good returns. The Black & Scholes model is widely used in pricing option although its underlying assumptions are inconsistent with the market dynamics. Some studies have been done aimed at improving the Black & Scholes model and in general the pricing of option.

In this paper, we take the same motive but now use the truncated normal distribution instead of the normal distribution that as been used in previous studies. Under the truncated normal distribution, denoted by TND in this paper, the underlying asset's log-return of is assumed to be bounded below and above. The boundary values are determined by the investor's perceived realistic price ranges of the underlying asset. The basic statistics of the proposed model are derive. The martingale restriction and closed formulas for option pricing as well as the pricing error are presented. The put - call parity and duality and some of the Greeks are also formulated. From the numerical result of the study, the proposed model performs better than the classical Black & Scholes at different price ranges for European options.

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Declaration and Approval

I the undersigned declare that this dissertation is my original work and to the best of my knowledge, it has not been submitted in support of an award of a degree in any other university or institution of learning.

Signature   Date

ONSOTI ALEX NYONG'A
Reg No. I56/12462/2018

In my capacity as a supervisor of the candidate's dissertation, I certify that this dissertation has my approval for submission.

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Dedication

This project is dedicated to my Mum, wife, my daughters, family, friends and my late father. The immeasurable support, love and motivation you all offered me when doing this project is indisputable. Because you this thesis has been accomplished. May God bless you abundantly.

Figures and Tables

Figures

Tables

Acknowledgments

First, I thank the Almighty God for the accomplishment of this work. My sincere gratitude goes to my Supervisor Professor Ivivi Joseph Mwaniki for his enthusiastic support , fervent guidance and encouragement throughout this research work. His constructive suggestion from the planning and development of this project as well as the sacrifices he made to schedule zoom meetings regularly in spite of the COVID-19 pandemic are highly appreciated. Many great thanks to may friends who we shared a lot during the progress of this work. I salute all that have made this work a success.

Onsoti Alex Nyong'a

Nairobi, 2020.

Contents

1 Chapter 1: Introduction

1.1 Background of the Study

The derivative market is a lucrative market where investor hedge their investments against unforeseeable risk. Financial instruments such as options, warrants, swaps and futures are widely used in financial market. Of these instruments, options are considered as the mostly used and traded tools. These financial instruments (options) which derive their prices from an underlying asset(s) have attracted the attention of investor and researcher on the optimal way of pricing them. Options have their origins since the 1600s - where they was used by the Dutch for the harvesting of olives, Romans and Phoenicians for the transportation of cargo [Str03]. They have been known to be the most preferred tools used in the market by investor to hedge against unforeseeable risks. Before the advent of the world economic crash in 1987, two famous American Scholars developed an well known and celebrated option pricing model - commonly referred to as Black - Scholes Model.

1.1.1 Black - Scholes - Merton Option Pricing Model

The Black - Scholes (B-S) model [BS73] has been used as the basis of pricing these options. This model was coined by Fisher Black and Myron Scholes in 1973 and later improved by Robert C. Merton [Mer76] when he assumed that the stock returns of the underlying are discontinuous. This model provides a closed formula for evaluating the call and put option prices of European options. With the following key assumptions, the closed form formula for the Black - Scholes model for a non-dividend paying European call and put option is given by;

$$
\begin{aligned}
C_t &= S_t\Phi(d_1) - Ke^{-r\tau}\Phi(d_2) \\
P_t &= Ke^{-r\tau}\Phi(-d_2) - S_t\Phi(-d_1) && \text{where,} \\
d_1 &= \frac{\log_e\left(\frac{S_t}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}} && \text{and,} \\
d_2 &= d_1 - \sigma\sqrt{\tau} \\
\tau &= T - t
\end{aligned}
$$

where
C_t is the value of a European Call option
P_t is the value of a European Put option
S_t is the price of the underlying stock/asset at time t

K is the option's strike price
r is the risk - free interest rate
σ is the volatility of the underlying stock/asset
τ is the time to maturity of the option
$\Phi(*)$ is the standard normal cumulative density function (CDF) defined by;

$$\Phi(u) = \frac{1}{2\pi} \int_{-\infty}^{u} e^{-\frac{1}{2}x^2} dx$$

The key assumptions of the Black & Scholes model are:

1. The model assumes that the market is "Frictionless" which implies that there are no transaction cost, taxes nor brokerage fee in the market. There are no arbitrage opportunities in the market implying that there are equal returns for all portfolios in the market and no restriction for short-selling in the market and that the stock can be subdidived into smaller units

2. The underlying asset or stock is assumed to be following a Geometric Brownian motion. The underlying asset/stock returns are log-normally distribution with mean μ and constant volatility $\sigma > 0$

3. The option is only excised on maturity date, i.e the model is assumed to price European options as opposed to American options which can be exercised before the maturity date

4. The underlying stock/asset does not pay dividends or any other distribution for the entire option contract period

5. The risk - free interest rate (borrowing and lending rate) is assumed to be constant and known during the option contract period

6. The market is assumed to be operating continuously

The above assumptions have been found to be unrealistic and thus more and more scholars have been attracted to the modification of the B-S model to "improve" the prices obtained by the Black & Scholes model. For instance, various parameters used in the Black & Scholes have been improved by adding some terms into the model such as jump as proposed by and Merton [Mer76] who introduced the discontinuity of the underlying stock returns. Ingersoll [IJ76] considered the transactions costs which are incurred during the buying and selling of the options while Whaley [Wha82] considered the dividend paying stock, i.e American options, in the B-S model. More studies have been proposed to improve the Black & Scholes model are presented in the literature review section such as the assumptions of constant volatility has been found to be at odds with so called volatility smile as proposed by Dumas [DFW98], the observed returns have been found to be skewed and fat-tailed as indicated by Piero and Rachev [Pei99, RMF05], Mwaniki [Mwa19] found that the log-returns of the underlying asset are heavy-tailed and follows Generalized hyperbolic Distribution as opposed to the Geometric Brownian motion.

Few approaches has been proposed to correct and modify the B-S model so as to obtain more "improved" option prices that are in-line with the market dynamics. "Improved" option prices is therefore been an interesting area by various financial analysts and hence development and modification of existing model to improve on the pricing process of options is necessary. In this study, the focus is the assumption that the underlying stock/asset prices are normally distribution.

1.2 Statement of the Problem

Some studies that have been done in option pricing indicate that the returns of the underlying assets are fat-tailed and skewed. The changes in the stock prices of the underlying assets from the start date of the option significantly affects the returns which investors get from their investment. Whenever the stock price is above the strike price, the option call holder makes a profit and thus exercise the call at the expiration date. On the other hand, if the stock price is below the strike price, the call option holder suffers a loss and thus fails to exercise the call option.

There are two key modifications that has been done as far as option pricing is concerned, i.e. structural and non-structural models. Structural models provide the definition of the dynamics of the underlying asset prices at every moment of time between now and the maturity period. Examples of structural models are the celebrated Black-Scholes, the Variance Gamma, Jump diffusion, Carr, German, Madan and Yor (CGMY) and Stochastic volatility models that were proposed to capture the various features that are shown by the real market data. On the other hand, non-structural models only defines the probability density function (p.d.f) of the underlying asset price at the time of maturity conditioned on the current time, i.e $\mathbb{E}^{\mathbb{Q}}\left[e^{-rt}S_t|\mathscr{F}_0\right]$ without necessarily defining the stochastic nature of the process at every moment in time for the entire duration of the option contract [ZH18]. The truncated normal distribution (TND) is an example of the non-structural model. With the more flexible distributions, different characteristics of the underlying asset returns and the volatility term structure can be captured in modeling.

To protect themselves from the extreme losses, investors have adopted the usage of market order and trading orders. Trade orders are orders which are placed by investors to either buy or sell a stock when a specified price is reached or a stock takes a specific direction as specified by the investor. For instance, stop limit orders are exercised when a specified price is reached by the stock while market orders are exercised whenever they are placed at current market price. Investors may specify a percentage for which if the price fluctuates beyond, their orders are exercised. The choice of the prices for which the orders are set and which type of order to choose posses a dilemma to investors are indicated by Bae [BJP03]. The use of trade orders has resulted in the truncation of the stock prices thereby providing the bounded prices. A proper method is therefore necessary to ensure that the interests and profits of the investors are protected while setting the prices for which the orders are to be exercised.

Unfortunately, all the pricing models that have so far been proposed in the existing literature assume that the underlying prices and returns are unbounded, i.e. the price ranges from zero to infinity, $[0,\infty)$. In this study, we adopt the assumption that the log-returns of the underlying asset follows the truncated normal distribution.

1.3 Research Objectives

1.3.1 General Objectives

To price European Options when underlying asset prices and log-returns are assumed to be following the truncated normal distribution.

1.3.2 Specific Objectives

1. To derive the basic statistics of the proposed truncated distribution model
2. To determine the value European option using the truncated normal distribution
3. To compare the option prices from the truncated model with those obtained under the Black & Scholes formula
4. To formulate the put-call parity for the truncated normal distribution
5. To formulate the put-call duality for the truncated normal distribution
6. To formulate the some Greeks of the proposed truncated normal distribution model

The rest of the dissertation is outlined as follows: Chapter 2 present the existing literature in option pricing and use of truncated normal distribution. Chapter 3 presents the methodologies including the properties of the truncated normal distribution such as MGF, Mean, Variance, Skewness and Kurtosis. The closed formulas for pricing European options are also presented in this section together with the parity relations and option Greeks. Chapter 4 presents data description, analysis and empirical results. Lastly, Chapter 5 presents the conclusions of the study and future research.

2 Chapter 2: Literature Review

This sections presents the existing literature for the pricing of options. Option pricing has undergone through a lot of revolution and transformation since its inception in 1973 by famous American scholars Fisher Black & Myron Scholes. The model for pricing option was first proposed by Black and Scholes [BS73] which is a well celebrated and applied in the derivative market. The assumption behind the model has frequently attracted critics on their validity in the pricing of option. For instance Merton [Mer73] extended the Black and Scholes model to include the pricing of underlying assets when they are not continuous. The assumption of constant volatility is among the areas which has received many criticism by academic researchers. The volatility smile investigated by Dumas [DFW98] to check how volatility relate with time. Also, [Sco87] applied the varying volatility in the pricing of option. Brechmann [BCA12] introduced the truncated Lévy process in the pricing of option.

The distribution of the underlying process which is assumed by Black and Scholes as normally distributed and following Geometric Brownian motion has been investigated by various academic researchers. While investigating the underlying asset's return, Piero and Rachev [Pei99, RMF05] found out that the returns were actually skewed and fat-tailed respectively. Similar results were obtained by Mwaniki [Mwa19] that the log-returns of the underlying asset are heavy-tailed which is not in line with the normality assumption proposed by Black & Scholes. The constant volatility assumption and normal distribution of the underlying asset among other parameter assumptions have been investigated by academic researchers. In many of these studies, the return of the underlying asset is assumed to be normally distributed with no bounds. However, Zhu [ZH18] proposed a bounded normal distribution (truncated normal distribution) for modeling the log-return of the underlying asset. A closed formula is proposed for the pricing of European options which takes into account the investor's expectation on the range of the returns of the underlying asset. This is the center of this paper as it tries to investigate the application and comparability of the truncated normal distribution model to the existing Black & Scholes model.

The truncated normal distribution has been applied across various disciplines. For instance, Norgaard and Killer [NK80] applied the truncated normal distribution to analyze investment. In this study, Norgaad found out that the truncated normal distribution serves as good alternative in when investors are not contended that the probability at the tail end of normal distribution is not in line with their investment decisions. Also, Hasan [HKMB12] and Dey [DC12] used the truncated normal distribution in measuring the efficiency of the stock market and modeling of the single period inventory model respectively. Hasan found out that the truncated normal distribution was in a better position for measuring the technical inefficiency in Bangladesh Stock Market as opposed to half-normal distributions. Similarly, Dey reported that the truncated normal distribution provided higher profits than the non-truncated case. Other studies where the truncated normal distribution has been applied is the queuing process of the impatient customer and in the investigation of portfolio insurance by Pender [Pen15] and Hocquard [HPR15] respectively. Pender concluded that the truncated normal distributions approximate the mean, variance and kurtosis of the queue process better than the normal distribution although the skewness was overestimated. On the hand, Hocquard reported that the left truncated normal distribution included into the Payoff Distribution Model reduces the downside risk for the investor significantly with no requirement of the fund manager. The left-truncated model model minimizes the downward risk inherent in the investment portfolio when the normal distribution is used.

Similarly, Gatti [GGGP03] introduced the truncated normal distribution in analyzing the pattern of entry and exit of firms in industrial market based on the financial bankruptcy - more specifically the firm's accumulation of capital. Gatti found that the entry and exit of firms follows the truncated normal distribution and that firms that remain in the market posses a higher equity ratio and low volatility (right-truncation) while firms that exit the market experiences weak equity ratio or have higher fragility in bankruptcy (left-truncation). Del [dCD09] used the mixture of left-right truncated normal distribution in the pricing of the bid and ask price of the exchange rates between the Euros and dollars. From the findings of the study, Del found that the truncated normal distribution (left and right) performed better than other distribution such as Laplace, inverse Gaussian, normal and the mixture of normal distribution is measuring the exchange rates between the US dollar and European euros. Additionally, Cox [Cox09] incorporated the truncated normal distribution in the development of the statistical process control - commonly referred to as control charts and their application in finance. In his/her findings, Cox reported that truncated financial data provides reliable charts that show better fit with predicable behaviors that are quite reliable. Zhu & He [ZH18] proposed the TND model in option pricing when they modified the B-S model in the pricing of S&P 500 index and found that the model performs better than the B-S model. Having evaluated the existing literature on how the truncated normal distribution has been applied in various fields, it is therefore profound to evaluate its applicability and performance in options market.

3 Chapter 3: Methodology

In this section, the probability distribution of the truncated normal, its properties including, MGF, Mean, Variance, Skewness and Kurtosis are presented. The closed formulas for the European call and put options as well as the put-call parity and the put-call duality are presented. Later in the chapter, the option Greeks, i.e. Delta, Gamma and Rho as well as the sensitivity of the stock price with respect to the strike prices are presented.

3.1 Truncated Normal Distribution (TND) and its Properties

A random variable X is assumed to follow a truncated normal distribution with mean, μ, and variance, σ^2, ($\sigma > 0$), and $x \in [a,b]$ if its probability density function (p.d.f) is given by

$$f(x;\sigma,\mu,a,b) = \begin{cases} \frac{1}{\sigma}\frac{\phi\left(\frac{x-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}, & a \leq x \leq b \\ 0, & \text{otherwise} \end{cases}$$

where $\phi(*)$ is the standard normal density function and $\Phi(*)$ is the standard normal distribution function.

3.1.1 Moment Generating Function (MGF)

For a random variable z, its MGF is given by;

$$\begin{aligned} M_Z(t) &= E\left(e^{tZ}\right) \\ &= \int e^{tz} f_Z(z)dz \end{aligned}$$

Therefore, the MGF of the truncated normal distribution is given as;

$$\begin{aligned} M_X(t) &= E\left(e^{tX}\right) \\ &= \int_a^b e^{tx} f_X(x)dx \\ &= \int_a^b e^{tx} \frac{\frac{1}{\sigma}\phi\left(\frac{x-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)} dx \\ &= \frac{1}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)} \int_a^b e^{tx}\frac{1}{\sigma}\phi\left(\frac{x-\mu}{\sigma}\right)dx \end{aligned}$$

Let $M = \Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)$ and since

$$\phi\left(\frac{x-\mu}{\sigma}\right) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

Then we have that;

$$\begin{aligned}
M_X(t) &= \frac{1}{M}\int_a^b e^{tx}\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx \\
&= \frac{1}{M}\int_a^b \frac{1}{\sigma\sqrt{2\pi}} e^{tx} e^{-\frac{1}{2\sigma^2}[(x-\mu)(x-\mu)]} dx \\
&= \frac{1}{M}\int_a^b \frac{1}{\sigma\sqrt{2\pi}} e^{tx} e^{-\frac{1}{2\sigma^2}(x^2-2x\mu+\mu^2)} dx \\
&= \frac{1}{M}\int_a^b \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}(x^2-2x\mu+\mu^2-2\sigma^2 tx)} dx \\
&= \frac{1}{M}\int_a^b \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}(x^2-2x(\mu+\sigma^2 t)+\mu^2)} dx
\end{aligned}$$

Let $k = (\mu + \sigma^2 t)$ and $m = \mu^2$ then;

$$\begin{aligned}
x^2 - 2x(\mu+\sigma^2 t) + \mu^2 &= x^2 - 2xk + m \\
&= x^2 - 2xk + k^2 - k^2 + m, \qquad \text{add zero to complete the square} \\
&= (x-k)^2 - k^2 + m \\
&= (x-(\mu+\sigma^2 t))^2 - (\mu+\sigma^2 t)^2 + \mu^2
\end{aligned}$$

Hence;

$$\begin{aligned}
M_X(t) &= \frac{1}{M}\int_a^b \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}(x-(\mu+\sigma^2 t))^2-(\mu+\sigma^2 t)^2+\mu^2)} dx \\
&= \frac{e^{-\frac{1}{2\sigma^2}\left[-(\mu+\sigma^2 t)^2+\mu^2)\right]}}{M}\int_a^b \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}(x-(\mu+\sigma^2 t))^2} dx \\
&= \frac{e^{-\frac{1}{2\sigma^2}\left[\mu^2-(\mu^2+2\mu\sigma^2 t+\sigma^4 t^2)\right]}}{M}\int_a^b \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}(x-\mu^*)^2} dx, \quad \text{where } \mu^* = \mu + \sigma^2 t \\
&= \frac{e^{-\frac{1}{2\sigma^2}\left[-(2\mu\sigma^2 t+\sigma^4 t^2)\right]}}{M}\int_a^b \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}(x-\mu^*)^2} dx
\end{aligned}$$

By the definition of normal distribution:

$$\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}; \qquad x \sim N(0,1)$$
$$\phi(\frac{x-\mu}{\sigma}) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(\frac{x-\mu}{\sigma})^2}; \qquad x \sim N(\mu, \sigma)$$

Consequently;

$$\phi(\frac{x-\mu^*}{\sigma}) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu^*}{\sigma}\right)^2}; \qquad x \sim N(\mu^*, \sigma)$$

Thus;

$$M_X(t) = \frac{e^{-\frac{1}{2\sigma^2}\left[-(2\mu\sigma^2 t + \sigma^4 t^2)\right]}}{M} \int_a^b \frac{1}{\sigma}\phi(\frac{x-\mu^*}{\sigma})dx$$
$$= \frac{e^{(\mu t + \frac{1}{2}\sigma^2 t^2)}}{M} \int_a^b \frac{1}{\sigma}\phi(\frac{x-\mu^*}{\sigma})dx$$

Let $u = \frac{x-\mu^*}{\sigma}$, then;

$$\frac{du}{dx} = \frac{d}{dx}\left(\frac{x-\mu^*}{\sigma}\right) = \frac{1}{\sigma} \qquad \Rightarrow dx = \sigma du$$

The lower and upper limits are $(\frac{a-\mu^*}{\sigma})$ and $(\frac{b-\mu^*}{\sigma})$ respectively. Thus, the MGF becomes;

$$M_X(t) = \frac{e^{(\mu t + \frac{1}{2}\sigma^2 t^2)}}{M} \int_{\frac{a-\mu^*}{\sigma}}^{\frac{b-\mu^*}{\sigma}} \frac{1}{\sigma}\phi(u)\sigma du$$
$$= \frac{e^{(\mu t + \frac{1}{2}\sigma^2 t^2)}}{M} \int_{\frac{a-\mu^*}{\sigma}}^{\frac{b-\mu^*}{\sigma}} \phi(u)du$$

By definition;

$$\int_a^b \phi(u)du = \Phi(b) - \Phi(a)$$

Thus;

$$\begin{aligned}
M_X(t) &= \frac{e^{(\mu t+\frac{1}{2}\sigma^2 t^2)}}{M}\left[\Phi\left(\frac{b-\mu^*}{\sigma}\right)-\Phi\left(\frac{a-\mu^*}{\sigma}\right)\right]; && \text{but } \mu^*=\mu+\sigma^2 t\\
&= \frac{e^{(\mu t+\frac{1}{2}\sigma^2 t^2)}}{M}\left[\Phi\left(\frac{b-(\mu+\sigma^2 t)}{\sigma}\right)-\Phi\left(\frac{a-(\mu+\sigma^2 t)}{\sigma}\right)\right]\\
&= \frac{e^{(\mu t+\frac{1}{2}\sigma^2 t^2)}}{M}\left[\Phi\left(\frac{b-\mu}{\sigma}-\sigma t\right)-\Phi\left(\frac{a-\mu}{\sigma}-\sigma t\right)\right], && M=\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)\\
M_X(t) &= e^{(\mu t+\frac{1}{2}\sigma^2 t^2)}\left[\frac{\Phi\left(\frac{b-\mu}{\sigma}-\sigma t\right)-\Phi\left(\frac{a-\mu}{\sigma}-\sigma t\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right]
\end{aligned}$$

Therefore, the moment generating function of the truncated normal distribution is given by;

$$M_X(t) = e^{(\mu t+\frac{1}{2}\sigma^2 t^2)}\left[\frac{\Phi\left(\frac{b-\mu}{\sigma}-\sigma t\right)-\Phi\left(\frac{a-\mu}{\sigma}-\sigma t\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right] \tag{1}$$

3.1.2 Hermite Polynomials

In finding the mean, variance, skewness and kurtosis of the truncated normal distribution, the first four moments of the MGF are involved. The Hermite polynomial are considered useful and hence they are reviewed in this section. The Hermite polynomials is defined as,

$$P_k(z) = \frac{1}{\phi(z)}\left(-\frac{d}{dz}\right)^k \phi(z)$$

where $P_k(z)$ is called the Hermite Polynomial [Nua06]. For instance, the first four polynomials are;

$$\begin{aligned}
P_0(z) &= \frac{1}{\phi(z)}\left(-\frac{d}{dz}\right)^0 \phi(z) = 1\\
P_1(z) &= \frac{1}{\phi(z)}\left(-\frac{d}{dz}\right)^1 \phi(z)\\
&= \frac{1}{\phi(z)}\left(-\frac{d}{dz}\right) \phi(z)\\
&= \frac{1}{\phi(z)}(-)(-z\phi(z))\\
\Rightarrow P_1(z) &= z
\end{aligned}$$

$$
\begin{aligned}
P_2(z) &= \frac{1}{\phi(z)}\left(-\frac{d}{dz}\right)^2 \phi(z) \\
&= \frac{1}{\phi(z)}\frac{d}{dz}\left(\frac{d}{dz}\phi(z)\right) \\
&= \frac{1}{\phi(z)}\frac{d}{dz}\left((-z)\phi(z)\right) \\
&= \frac{1}{\phi(z)}\left(-\phi(z)+(-z)(-z)\phi(z)\right) \\
&= \frac{1}{\phi(z)}\left(z^2-1\right)\phi(z) \\
\Rightarrow P_2(z) &= z^2-1 \\
P_3(z) &= \frac{1}{\phi(z)}\left(-\frac{d}{dz}\right)^3 \phi(z) \\
&= \frac{1}{\phi(z)}\left(-\frac{d}{dz}\right)\left[\left(\frac{d}{dz}\right)^2 \phi(z)\right] \\
&= \frac{1}{\phi(z)}\left(-\frac{d}{dz}\right)\left[(z^2-1)\phi(z)\right] \\
&= \frac{1}{\phi(z)}(-)\left[2z\phi(z)+(z^2-1)(-z)\phi(z)\right] \\
&= \frac{1}{\phi(z)}(-)\left[2z\phi(z)-z^3\phi(z)+z\phi(z)\right] \\
&= \frac{1}{\phi(z)}\left[z^3-3z\right]\phi(z) \\
\Rightarrow P_3(z) &= z^3-3z
\end{aligned}
$$

These Hermite polynomials are used while finding the mean, variance, skewness and kurtosis of the distribution in this paper. By defining

$$
\frac{d^k}{dz^k}\Phi(z) = \phi(z)P_k(z)
$$

Then

$$
\begin{aligned}
\frac{d^0}{dz^0}\Phi(z) &= \phi(z)P_0(z) = \phi(z) \\
\frac{d^1}{dz^1}\Phi(z) &= \phi(z)P_1(z) = \phi(z)z \\
\frac{d^2}{dz^2}\Phi(z) &= \phi(z)P_2(z) = \phi(z)(z^2-1) \\
\frac{d^3}{dz^3}\Phi(z) &= \phi(z)P_3(z) = \phi(z)\left(3z-z^3\right)
\end{aligned}
$$

3.1.3 Mean

Using equation (1), the mean of the truncated normal distribution is determined as follows;

$$\mathbb{E}[X] = M_X^1(t)|_{t=0} \qquad \text{where}$$

$$M_X(t) = e^{(\mu t + \frac{1}{2}\sigma^2 t^2)} \left[\frac{\Phi\left(\frac{b-\mu}{\sigma} - \sigma t\right) - \Phi\left(\frac{a-\mu}{\sigma} - \sigma t\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)} \right]$$

Let $y = \left(\frac{b-\mu}{\sigma} - \sigma t\right), z = \left(\frac{a-\mu}{\sigma} - \sigma t\right)$ and M as define before, then;

$$M_X(t) = e^{(\mu t + \frac{1}{2}\sigma^2 t^2)} \left[\frac{\Phi(y) - \Phi(z)}{M} \right]$$

Then, applying the chain rule and Hermite polynomial and given that;

$$\frac{dy}{dt} = \frac{d}{dt}\left(\frac{b-\mu}{\sigma} - \sigma t\right) = -\sigma \qquad \frac{dz}{dt} = \frac{d}{dt}\left(\frac{a-\mu}{\sigma} - \sigma t\right) = -\sigma$$

Hence;

$$M_X^1(t) = (\mu + \sigma^2 t)e^{(\mu t + \frac{1}{2}\sigma^2 t^2)} \left[\frac{\Phi(y) - \Phi(z)}{M} \right] + e^{(\mu t + \frac{1}{2}\sigma^2 t^2)} \left[\frac{\phi(z)P_0(z) - \phi(y)P_0(y)}{M} \right] \sigma$$

$$M_X^1(t) = (\mu + \sigma^2 t)e^{(\mu t + \frac{1}{2}\sigma^2 t^2)} \left[\frac{\Phi(y) - \Phi(z)}{M} \right] + e^{(\mu t + \frac{1}{2}\sigma^2 t^2)} \left[\frac{\phi(z) - \phi(y)}{M} \right] \sigma$$

$$= (\mu + \sigma^2 t)e^{(\mu t + \frac{1}{2}\sigma^2 t^2)} \left[\frac{\Phi\left(\frac{b-\mu}{\sigma} - \sigma t\right) - \Phi\left(\frac{a-\mu}{\sigma} - \sigma t\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)} \right] + e^{(\mu t + \frac{1}{2}\sigma^2 t^2)} \left[\frac{\phi\left(\frac{a-\mu}{\sigma} - \sigma t\right) - \phi\left(\frac{b-\mu}{\sigma} - \sigma t\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)} \right] \sigma$$

Evaluating at $t = 0$, we have

$$M_X^1(0) = \mu \left[\frac{\Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)} \right] + \left[\frac{\phi\left(\frac{a-\mu}{\sigma}\right) - \phi\left(\frac{b-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)} \right] \sigma$$

$$= \mu + \left[\frac{\phi\left(\frac{a-\mu}{\sigma}\right) - \phi\left(\frac{b-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)} \right] \sigma$$

Therefore, the mean, E[X], of the truncated normal distribution is given as;

$$\mathbb{E}[X] = M_X^1(0) = \mu + \left[\frac{\phi\left(\frac{a-\mu}{\sigma}\right) - \phi\left(\frac{b-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)} \right] \sigma$$

3.1.4 Variance

The variance is given by

$$\mathbb{V}(X) = M_X^2(0) - [M_X^1(0)]^2$$

Hence;

$$M_X^1(t) = (\mu + \sigma^2 t)e^{(\mu t + \frac{1}{2}\sigma^2 t^2)} \left[\frac{\Phi(y) - \Phi(z)}{M}\right] + e^{(\mu t + \frac{1}{2}\sigma^2 t^2)} \left[\frac{\phi(z) - \phi(y)}{M}\right] \sigma$$

Then;

$$\begin{aligned}
M_X^2(t) &= \frac{d^2}{dt^2}(M_X(t)) = \frac{d}{dt}\left(M_X'(t)\right) \\
&= \frac{d}{dt}\left((\mu + \sigma^2 t)e^{(\mu t + \frac{1}{2}\sigma^2 t^2)} \left[\frac{\Phi(y) - \Phi(z)}{M}\right] + e^{(\mu t + \frac{1}{2}\sigma^2 t^2)} \left[\frac{\phi(z) - \phi(y)}{M}\right] \sigma\right) \\
&= \sigma^2 e^{(\mu t + \frac{1}{2}\sigma^2 t^2)} \left[\frac{\Phi(y) - \Phi(z)}{M}\right] + (\mu + \sigma^2 t)^2 e^{(\mu t + \frac{1}{2}\sigma^2 t^2)} \left[\frac{\Phi(y) - \Phi(z)}{M}\right] \\
&+ (\mu + \sigma^2 t)e^{(\mu t + \frac{1}{2}\sigma^2 t^2)} \left[\frac{\phi(z)P_0(z) - \phi(y)P_0(y)}{M}\right] \sigma + (\mu + \sigma^2 t)e^{(\mu t + \frac{1}{2}\sigma^2 t^2)} \left[\frac{\phi(z) - \phi(y)}{M}\right] \sigma \\
&+ e^{(\mu t + \frac{1}{2}\sigma^2 t^2)} \left[\frac{\phi(z)P_1(z) - \phi(y)P_1(y)}{M}\right] \sigma^2 \\
&= \sigma^2 e^{(\mu t + \frac{1}{2}\sigma^2 t^2)} \left[\frac{\Phi(y) - \Phi(z)}{M}\right] + (\mu + \sigma^2 t)^2 e^{(\mu t + \frac{1}{2}\sigma^2 t^2)} \left[\frac{\Phi(y) - \Phi(z)}{M}\right] \\
&+ (\mu + \sigma^2 t)e^{(\mu t + \frac{1}{2}\sigma^2 t^2)} \left[\frac{\phi(z) - \phi(y)}{M}\right] \sigma + (\mu + \sigma^2 t)e^{(\mu t + \frac{1}{2}\sigma^2 t^2)} \left[\frac{\phi(z) - \phi(y)}{M}\right] \sigma \\
&+ e^{(\mu t + \frac{1}{2}\sigma^2 t^2)} \left[\frac{\phi(z)z - \phi(y)y}{M}\right] \sigma^2
\end{aligned}$$

Substituting back the value of y, z and M,we have;

$$\begin{aligned}
&M_X^2(t) \\
&= \sigma^2 e^{(\mu t + \frac{1}{2}\sigma^2 t^2)} \left[\frac{\Phi\left(\frac{b-\mu}{\sigma} - \sigma t\right) - \Phi\left(\frac{a-\mu}{\sigma} - \sigma t\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)}\right] + (\mu + \sigma^2 t)^2 e^{(\mu t + \frac{1}{2}\sigma^2 t^2)} \left[\frac{\Phi\left(\frac{b-\mu}{\sigma} - \sigma t\right) - \Phi\left(\frac{a-\mu}{\sigma} - \sigma t\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)}\right] \\
&+ (\mu + \sigma^2 t)e^{(\mu t + \frac{1}{2}\sigma^2 t^2)} \left[\frac{\phi\left(\frac{a-\mu}{\sigma} - \sigma t\right) - \phi\left(\frac{b-\mu}{\sigma} - \sigma t\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)}\right] \sigma \\
&+ (\mu + \sigma^2 t)e^{(\mu t + \frac{1}{2}\sigma^2 t^2)} \left[\frac{\phi\left(\frac{a-\mu}{\sigma} - \sigma t\right) - \phi\left(\frac{b-\mu}{\sigma} - \sigma t\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)}\right] \sigma \\
&+ e^{(\mu t + \frac{1}{2}\sigma^2 t^2)} \left[\frac{\phi\left(\frac{a-\mu}{\sigma} - \sigma t\right)\left(\frac{a-\mu}{\sigma} - \sigma t\right) - \phi\left(\frac{a-\mu}{\sigma} - \sigma t\right)\left(\frac{b-\mu}{\sigma} - \sigma t\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)}\right] \sigma^2
\end{aligned}$$

At time $t=0$, we have;

$$M_X^2(0)=\sigma^2\left[\frac{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right]+\mu^2\left[\frac{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right]+\mu\left[\frac{\phi\left(\frac{a-\mu}{\sigma}\right)-\phi\left(\frac{b-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right]\sigma$$
$$+\mu\left[\frac{\phi\left(\frac{a-\mu}{\sigma}\right)-\phi\left(\frac{b-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right]\sigma+\left[\frac{\phi\left(\frac{a-\mu}{\sigma}\right)\left(\frac{a-\mu}{\sigma}\right)-\phi\left(\frac{a-\mu}{\sigma}\right)\left(\frac{b-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right]\sigma^2$$
$$=\sigma^2+\mu^2+2\sigma\mu\left[\frac{\phi\left(\frac{a-\mu}{\sigma}\right)-\phi\left(\frac{b-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right]+\sigma^2\left[\frac{\left(\frac{a-\mu}{\sigma}\right)\phi\left(\frac{a-\mu}{\sigma}\right)-\left(\frac{b-\mu}{\sigma}\right)\phi\left(\frac{b-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right]$$

Therefore, the variance is given as;

$$\mathbb{V}(X)=M_X^2(0)-[M_X^1(0)]^2$$
$$(M_X^1(0))^2=\left(\mu+\left[\frac{\phi\left(\frac{a-\mu}{\sigma}\right)-\phi\left(\frac{b-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right]\sigma\right)^2$$
$$=\mu^2+2\sigma\mu\left[\frac{\phi\left(\frac{a-\mu}{\sigma}\right)-\phi\left(\frac{b-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right]+\sigma^2\left[\frac{\phi\left(\frac{a-\mu}{\sigma}\right)-\phi\left(\frac{b-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right]^2$$

$$\mathbb{V}(X)=\sigma^2+\mu^2+2\sigma\mu\left[\frac{\phi\left(\frac{a-\mu}{\sigma}\right)-\phi\left(\frac{b-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right]+\sigma^2\left[\frac{\left(\frac{a-\mu}{\sigma}\right)\phi\left(\frac{a-\mu}{\sigma}\right)-\left(\frac{b-\mu}{\sigma}\right)\phi\left(\frac{b-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right]$$
$$-\left(\mu^2+2\sigma\mu\left[\frac{\phi\left(\frac{a-\mu}{\sigma}\right)-\phi\left(\frac{b-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right]+\sigma^2\left[\frac{\phi\left(\frac{a-\mu}{\sigma}\right)-\phi\left(\frac{b-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right]^2\right)$$
$$=\sigma^2+\mu^2+2\sigma\mu\left[\frac{\phi\left(\frac{a-\mu}{\sigma}\right)-\phi\left(\frac{b-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right]+\sigma^2\left[\frac{\left(\frac{a-\mu}{\sigma}\right)\phi\left(\frac{a-\mu}{\sigma}\right)-\left(\frac{b-\mu}{\sigma}\right)\phi\left(\frac{b-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right]$$
$$-\mu^2-2\sigma\mu\left[\frac{\phi\left(\frac{a-\mu}{\sigma}\right)-\phi\left(\frac{b-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right]-\sigma^2\left[\frac{\phi\left(\frac{a-\mu}{\sigma}\right)-\phi\left(\frac{b-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right]^2$$
$$=\sigma^2+\sigma^2\left[\frac{\left(\frac{a-\mu}{\sigma}\right)\phi\left(\frac{a-\mu}{\sigma}\right)-\left(\frac{b-\mu}{\sigma}\right)\phi\left(\frac{b-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right]-\sigma^2\left[\frac{\phi\left(\frac{a-\mu}{\sigma}\right)-\phi\left(\frac{b-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right]^2$$

Therefore, the variance of the truncated normal distribution is given by;

$$\mathbb{V}(X) = \sigma^2 \left[1 + \frac{\left(\frac{a-\mu}{\sigma}\right)\phi\left(\frac{a-\mu}{\sigma}\right) - \left(\frac{b-\mu}{\sigma}\right)\phi\left(\frac{b-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)} - \left(\frac{\phi\left(\frac{a-\mu}{\sigma}\right) - \phi\left(\frac{b-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)} \right)^2 \right]$$

3.1.5 Skewness

Skewness is given by

$$\mathbb{S}(X) = \frac{M_X^3(0) - 3M_X^2(0)M_X^1(0) + 2[M_X^1(0)]^3}{[\mathbb{V}(X)]^{\frac{3}{2}}}$$

Given that the second moment is

$$\begin{aligned}
M_X^2(t) = &\left(\sigma^2 e^{(\mu t + \frac{1}{2}\sigma^2 t^2)} + (\mu + \sigma^2 t)^2 e^{(\mu t + \frac{1}{2}\sigma^2 t^2)}\right) \left[\frac{\Phi\left(\frac{b-\mu}{\sigma} - \sigma t\right) - \Phi\left(\frac{a-\mu}{\sigma} - \sigma t\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)} \right] \\
&+ 2\sigma(\mu + \sigma^2 t) e^{(\mu t + \frac{1}{2}\sigma^2 t^2)} \left[\frac{\phi\left(\frac{b-\mu}{\sigma} - \sigma t\right) - \phi\left(\frac{a-\mu}{\sigma} - \sigma t\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)} \right] \\
&+ \sigma^2 e^{(\mu t + \frac{1}{2}\sigma^2 t^2)} \left[\frac{\left(\frac{a-\mu}{\sigma} - \sigma t\right)\phi\left(\frac{a-\mu}{\sigma} - \sigma t\right) - \left(\frac{b-\mu}{\sigma} - \sigma t\right)\phi\left(\frac{a-\mu}{\sigma} - \sigma t\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)} \right]
\end{aligned}$$

The third moment is given by

$$\begin{aligned}
M_X^3(t) &= \frac{d^3}{dt^3}(M_X(t)) \\
&= \frac{d^3}{dt^3}\left(e^{(\mu t + \frac{1}{2}\sigma^2 t^2)} \left[\frac{\Phi(y) - \Phi(z)}{M} \right] \right) \\
M_X^2(t) &= \sigma^2 e^{(\mu t + \frac{1}{2}\sigma^2 t^2)} \left[\frac{\Phi(y) - \Phi(z)}{M} \right] + 2(\mu + \sigma^2 t)^2 e^{(\mu t + \frac{1}{2}\sigma^2 t^2)} \left[\frac{\Phi(y) - \Phi(z)}{M} \right] \\
&\quad + (\mu + \sigma^2 t) e^{(\mu t + \frac{1}{2}\sigma^2 t^2)} \left[\frac{\phi(z) - \phi(y)}{M} \right] \sigma + e^{(\mu t + \frac{1}{2}\sigma^2 t^2)} \left[\frac{\phi(z)z - \phi(y)y}{M} \right] \sigma^2 \\
M_X^3(t) &= \sigma^2 \left(\mu + \sigma^2 t\right) e^{(\mu t + \frac{1}{2}\sigma^2 t^2)} \left[\frac{\Phi(y) - \Phi(z)}{M} \right] + \sigma^2 e^{(\mu t + \frac{1}{2}\sigma^2 t^2)} \left[\frac{\phi(z)P_0(z) - \phi(y)P_0(y)}{M} \right] \sigma \\
&\quad + 2\sigma^2(\mu + \sigma^2 t) e^{(\mu t + \frac{1}{2}\sigma^2 t^2)} \left[\frac{\Phi(y) - \Phi(z)}{M} \right] + (\mu + \sigma^2 t)^3 e^{(\mu t + \frac{1}{2}\sigma^2 t^2)} \left[\frac{\Phi(y) - \Phi(z)}{M} \right] \\
&\quad + (\mu + \sigma^2 t)^2 e^{(\mu t + \frac{1}{2}\sigma^2 t^2)} \left[\frac{\phi(z)P_0(z) - \phi(y)P_0(y)}{M} \right] \sigma + 2\sigma^2 e^{(\mu t + \frac{1}{2}\sigma^2 t^2)} \left[\frac{\phi(z)P_0(z) - \phi(y)P_0(y)}{M} \right] \sigma \\
&\quad + 2(\mu + \sigma^2 t)^2 e^{(\mu t + \frac{1}{2}\sigma^2 t^2)} \left[\frac{\phi(z)P_0(z) - \phi(y)P_0(y)}{M} \right] \sigma + 2(\mu + \sigma^2 t) e^{(\mu t + \frac{1}{2}\sigma^2 t^2)} \left[\frac{\phi(z)P_1(z) - \phi(y)P_1(y)}{M} \right] \sigma^2 \\
&\quad + (\mu + \sigma^2 t) e^{(\mu t + \frac{1}{2}\sigma^2 t^2)} \left[\frac{\phi(z)z - \phi(y)y}{M} \right] \sigma^2 + e^{(\mu t + \frac{1}{2}\sigma^2 t^2)} \left[\frac{\phi(z)P_2(z) - \phi(y)(P_2(y)}{M} \right] \sigma^3
\end{aligned}$$

$$
\begin{aligned}
M_X^3(t) &= \sigma^2\left(\mu+\sigma^2 t\right)e^{(\mu t+\frac{1}{2}\sigma^2 t^2)}\left[\frac{\Phi(y)-\Phi(z)}{M}\right]+\sigma^2 e^{(\mu t+\frac{1}{2}\sigma^2 t^2)}\left[\frac{\phi(z)-\phi(y)}{M}\right]\sigma \\
&+2\sigma^2(\mu+\sigma^2 t)e^{(\mu t+\frac{1}{2}\sigma^2 t^2)}\left[\frac{\Phi(y)-\Phi(z)}{M}\right]+(\mu+\sigma^2 t)^3 e^{(\mu t+\frac{1}{2}\sigma^2 t^2)}\left[\frac{\Phi(y)-\Phi(z)}{M}\right] \\
&+(\mu+\sigma^2 t)^2 e^{(\mu t+\frac{1}{2}\sigma^2 t^2)}\left[\frac{\phi(z)-\phi(y)}{M}\right]\sigma+2\sigma^2 e^{(\mu t+\frac{1}{2}\sigma^2 t^2)}\left[\frac{\phi(z)-\phi(y)}{M}\right]\sigma \\
&+2(\mu+\sigma^2 t)^2 e^{(\mu t+\frac{1}{2}\sigma^2 t^2)}\left[\frac{\phi(z)-\phi(y)}{M}\right]\sigma+2(\mu+\sigma^2 t)e^{(\mu t+\frac{1}{2}\sigma^2 t^2)}\left[\frac{\phi(z)z-\phi(y)y}{M}\right]\sigma^2 \\
&+(\mu+\sigma^2 t)e^{(\mu t+\frac{1}{2}\sigma^2 t^2)}\left[\frac{\phi(z)z-\phi(y)y}{M}\right]\sigma^2+e^{(\mu t+\frac{1}{2}\sigma^2 t^2)}\left[\frac{\phi(z)(z^2-1)-\phi(y)(y^2-1)}{M}\right]\sigma^3 \\
&= \sigma^2\left(\mu+\sigma^2 t\right)e^{(\mu t+\frac{1}{2}\sigma^2 t^2)}\left[\frac{\Phi\left(\frac{b-\mu}{\sigma}-\sigma t\right)-\Phi\left(\frac{a-\mu}{\sigma}-\sigma t\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right] \\
&+\sigma^2 e^{(\mu t+\frac{1}{2}\sigma^2 t^2)}\left[\frac{\phi\left(\frac{a-\mu}{\sigma}-\sigma t\right)-\phi\left(\frac{b-\mu}{\sigma}-\sigma t\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right]\sigma \\
&+2\sigma^2(\mu+\sigma^2 t)e^{(\mu t+\frac{1}{2}\sigma^2 t^2)}\left[\frac{\Phi\left[\frac{a-\mu}{\sigma}-\sigma t\right]-\Phi\left[\frac{b-\mu}{\sigma}-\sigma t\right]}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right] \\
&+(\mu+\sigma^2 t)^3 e^{(\mu t+\frac{1}{2}\sigma^2 t^2)}\left[\frac{\Phi\left[\frac{b-\mu}{\sigma}-\sigma t\right]-\Phi\left[\frac{b-\mu}{\sigma}-\sigma t\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right]-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right] \\
&+(\mu+\sigma^2 t)^2 e^{(\mu t+\frac{1}{2}\sigma^2 t^2)}\left[\frac{\phi\left(\frac{a-\mu}{\sigma}-\sigma t\right)-\phi\left(\frac{b-\mu}{\sigma}-\sigma t\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right]\sigma \\
&+2\sigma^2 e^{(\mu t+\frac{1}{2}\sigma^2 t^2)}\left[\frac{\phi\left(\frac{a-\mu}{\sigma}-\sigma t\right)-\phi\left(\frac{b-\mu}{\sigma}-\sigma t\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right]\sigma \\
&+2(\mu+\sigma^2 t)^2 e^{(\mu t+\frac{1}{2}\sigma^2 t^2)}\left[\frac{\phi\left(\frac{a-\mu}{\sigma}-\sigma t\right)-\phi\left(\frac{b-\mu}{\sigma}-\sigma t\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right]\sigma \\
&+2(\mu+\sigma^2 t)e^{(\mu t+\frac{1}{2}\sigma^2 t^2)}\left[\frac{\phi\left(\frac{a-\mu}{\sigma}-\sigma t\right)\left(\frac{a-\mu}{\sigma}-\sigma t\right)-\phi\left(\frac{b-\mu}{\sigma}-\sigma t\right)\left(\frac{b-\mu}{\sigma}-\sigma t\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right]\sigma^2 \\
&+(\mu+\sigma^2 t)e^{(\mu t+\frac{1}{2}\sigma^2 t^2)}\left[\frac{\phi\left(\frac{a-\mu}{\sigma}-\sigma t\right)\left(\frac{a-\mu}{\sigma}-\sigma t\right)-\phi\left(\frac{b-\mu}{\sigma}-\sigma t\right)\left(\frac{b-\mu}{\sigma}-\sigma t\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right]\sigma^2 \\
&+e^{(\mu t+\frac{1}{2}\sigma^2 t^2)}\left[\frac{\phi\left(\frac{a-\mu}{\sigma}-\sigma t\right)\left[\left(\frac{a-\mu}{\sigma}-\sigma t\right)^2-1\right]-\phi\left(\frac{b-\mu}{\sigma}-\sigma t\right)\left[\left(\frac{b-\mu}{\sigma}-\sigma t\right)^2-1\right]}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right]\sigma^3
\end{aligned}
$$

Substituting back the values of y, z and M in the last equality and evaluating at $t=0$, we have

$$
\begin{aligned}
M_X^3(0) &= \sigma^2\mu\left[\frac{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right]+\sigma^2\left[\frac{\phi\left(\frac{a-\mu}{\sigma}\right)-\phi\left(\frac{b-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right]\sigma+2\sigma^2\mu\left[\frac{\Phi\left[\frac{a-\mu}{\sigma}\right]-\Phi\left[\frac{b-\mu}{\sigma}\right]}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right]\\
&+\mu^3\left[\frac{\Phi\left[\frac{b-\mu}{\sigma}\right]-\Phi\left[\frac{b-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right]-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right]+\mu^2\left[\frac{\phi\left(\frac{a-\mu}{\sigma}\right)-\phi\left(\frac{b-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right]\sigma+2\sigma^2\left[\frac{\phi\left(\frac{a-\mu}{\sigma}\right)-\phi\left(\frac{b-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right]\sigma\\
&+2\mu^2\left[\frac{\phi\left(\frac{a-\mu}{\sigma}\right)-\phi\left(\frac{b-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right]\sigma+2\mu\left[\frac{\phi\left(\frac{a-\mu}{\sigma}\right)\left(\frac{a-\mu}{\sigma}\right)-\phi\left(\frac{b-\mu}{\sigma}\right)\left(\frac{b-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right]\sigma^2\\
&+\mu\left[\frac{\phi\left(\frac{a-\mu}{\sigma}\right)\left(\frac{a-\mu}{\sigma}\right)-\phi\left(\frac{b-\mu}{\sigma}\right)\left(\frac{b-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right]\sigma^2+\left[\frac{\phi\left(\frac{a-\mu}{\sigma}\right)\left[\left(\frac{a-\mu}{\sigma}\right)^2-1\right]-\phi\left(\frac{b-\mu}{\sigma}\right)\left[\left(\frac{b-\mu}{\sigma}\right)^2-1\right]}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right]\sigma^3\\
&=\sigma^2\mu+\sigma^3\left[\frac{\phi\left(\frac{a-\mu}{\sigma}\right)-\phi\left(\frac{b-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right]+2\sigma^2\mu+\mu^3+\sigma\mu^2\left[\frac{\phi\left(\frac{a-\mu}{\sigma}\right)-\phi\left(\frac{b-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right]\\
&+2\sigma^3\left[\frac{\phi\left(\frac{a-\mu}{\sigma}\right)-\phi\left(\frac{b-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right]+2\sigma\mu^2\left[\frac{\phi\left(\frac{a-\mu}{\sigma}\right)-\phi\left(\frac{b-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right]\\
&+2\sigma^2\mu\left[\frac{\phi\left(\frac{a-\mu}{\sigma}\right)\left(\frac{a-\mu}{\sigma}\right)-\phi\left(\frac{b-\mu}{\sigma}\right)\left(\frac{b-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right]\\
&+\sigma^2\mu\left[\frac{\phi\left(\frac{a-\mu}{\sigma}\right)\left(\frac{a-\mu}{\sigma}\right)-\phi\left(\frac{b-\mu}{\sigma}\right)\left(\frac{b-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right]+\sigma^3\left[\frac{\phi\left(\frac{a-\mu}{\sigma}\right)\left[\left(\frac{a-\mu}{\sigma}\right)^2-1\right]-\phi\left(\frac{b-\mu}{\sigma}\right)\left[\left(\frac{b-\mu}{\sigma}\right)^2-1\right]}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right]\\
&=3\sigma^2\mu+\mu^3+3\sigma^3\left[\frac{\phi\left(\frac{a-\mu}{\sigma}\right)-\phi\left(\frac{b-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right]+3\sigma\mu^2\left[\frac{\phi\left(\frac{a-\mu}{\sigma}\right)-\phi\left(\frac{b-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right]\\
&+3\sigma^2\mu\left[\frac{\phi\left(\frac{a-\mu}{\sigma}\right)\left(\frac{a-\mu}{\sigma}\right)-\phi\left(\frac{b-\mu}{\sigma}\right)\left(\frac{b-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right]+\sigma^3\left[\frac{\phi\left(\frac{a-\mu}{\sigma}\right)\left[\left(\frac{a-\mu}{\sigma}\right)^2-1\right]-\phi\left(\frac{b-\mu}{\sigma}\right)\left[\left(\frac{b-\mu}{\sigma}\right)^2-1\right]}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right]
\end{aligned}
$$

$$3M_X^2(0)M_X^1(0) = 3\sigma^2\mu + 3\mu^3 + 6\sigma\mu^2\left[\frac{\phi\left(\frac{a-\mu}{\sigma}\right)-\phi\left(\frac{b-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right] + 3\sigma^2\mu\left[\frac{\left(\frac{a-\mu}{\sigma}\right)\phi\left(\frac{a-\mu}{\sigma}\right)-\left(\frac{b-\mu}{\sigma}\right)\phi\left(\frac{b-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right]$$

$$+3\sigma^3\left[\frac{\phi\left(\frac{a-\mu}{\sigma}\right)-\phi\left(\frac{b-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right] + 3\sigma\mu^2\left[\frac{\phi\left(\frac{a-\mu}{\sigma}\right)-\phi\left(\frac{b-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right] + 6\sigma^2\mu\left[\frac{\phi\left(\frac{a-\mu}{\sigma}\right)-\phi\left(\frac{b-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right]^2$$

$$+3\sigma^3\left\{\left[\frac{\left(\frac{a-\mu}{\sigma}\right)\phi\left(\frac{a-\mu}{\sigma}\right)-\left(\frac{b-\mu}{\sigma}\right)\phi\left(\frac{b-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right]\left[\frac{\phi\left(\frac{a-\mu}{\sigma}\right)-\phi\left(\frac{b-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right]\right\}$$

$$= 3\sigma^2\mu + 3\mu^3 + 9\sigma\mu^2\left[\frac{\phi\left(\frac{a-\mu}{\sigma}\right)-\phi\left(\frac{b-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right] + 3\sigma^2\mu\left[\frac{\left(\frac{a-\mu}{\sigma}\right)\phi\left(\frac{a-\mu}{\sigma}\right)-\left(\frac{b-\mu}{\sigma}\right)\phi\left(\frac{b-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right]$$

$$+3\sigma^3\left[\frac{\phi\left(\frac{a-\mu}{\sigma}\right)-\phi\left(\frac{b-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right] + 6\sigma^2\mu\left[\frac{\phi\left(\frac{a-\mu}{\sigma}\right)-\phi\left(\frac{b-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right]^2$$

$$+3\sigma^3\left\{\left[\frac{\left(\frac{a-\mu}{\sigma}\right)\phi\left(\frac{a-\mu}{\sigma}\right)-\left(\frac{b-\mu}{\sigma}\right)\phi\left(\frac{b-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right]\left[\frac{\phi\left(\frac{a-\mu}{\sigma}\right)-\phi\left(\frac{b-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right]\right\}$$

Therefore;

$$M_X^3(0) - 3M_X^2(0)M_X^1(0) = -2\mu^3 - 6\sigma\mu^2\left[\frac{\phi\left(\frac{a-\mu}{\sigma}\right)-\phi\left(\frac{b-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right] - 6\sigma^2\mu\left[\frac{\phi\left(\frac{a-\mu}{\sigma}\right)-\phi\left(\frac{b-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right]^2$$

$$+\sigma^3\left[\frac{\phi\left(\frac{a-\mu}{\sigma}\right)\left[\left(\frac{a-\mu}{\sigma}\right)^2-1\right]-\phi\left(\frac{b-\mu}{\sigma}\right)\left[\left(\frac{b-\mu}{\sigma}\right)^2-1\right]}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right]$$

$$-3\sigma^3\left\{\left[\frac{\left(\frac{a-\mu}{\sigma}\right)\phi\left(\frac{a-\mu}{\sigma}\right)-\left(\frac{b-\mu}{\sigma}\right)\phi\left(\frac{b-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right]\left[\frac{\phi\left(\frac{a-\mu}{\sigma}\right)-\phi\left(\frac{b-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right]\right\}$$

$$
\begin{aligned}
2[M_X^1(0)]^3 &= 2[M_X^1(0)]^2.[M_X^1(0)] \\
&= 2\mu^3 + 2\sigma\mu^2\left[\frac{\phi\left(\frac{a-\mu}{\sigma}\right)-\phi\left(\frac{b-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right] + 4\sigma\mu^2\left[\frac{\phi\left(\frac{a-\mu}{\sigma}\right)-\phi\left(\frac{b-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right] \\
&+ 4\sigma^2\mu\left[\frac{\phi\left(\frac{a-\mu}{\sigma}\right)-\phi\left(\frac{b-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right]^2 + 2\sigma^2\mu\left[\frac{\phi\left(\frac{a-\mu}{\sigma}\right)-\phi\left(\frac{b-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right]^2 + 2\sigma^3\left[\frac{\phi\left(\frac{a-\mu}{\sigma}\right)-\phi\left(\frac{b-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right]^3 \\
&= 2\mu^3 + 6\sigma\mu^2\left[\frac{\phi\left(\frac{a-\mu}{\sigma}\right)-\phi\left(\frac{b-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right] + 6\sigma^2\mu\left[\frac{\phi\left(\frac{a-\mu}{\sigma}\right)-\phi\left(\frac{b-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right]^2 \\
&+ 2\sigma^3\left[\frac{\phi\left(\frac{a-\mu}{\sigma}\right)-\phi\left(\frac{b-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right]^3
\end{aligned}
$$

$$
\begin{aligned}
M_X^3(0) - 3M_X^2(0)M_X^1(0) + 2[M_X^{(1)}(0)]^3 &= 2\sigma^3\left[\frac{\phi\left(\frac{a-\mu}{\sigma}\right)-\phi\left(\frac{b-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right]^3 \\
&+ \sigma^3\left[\frac{\phi\left(\frac{a-\mu}{\sigma}\right)\left[\left(\frac{a-\mu}{\sigma}\right)^2-1\right]-\phi\left(\frac{b-\mu}{\sigma}\right)\left[\left(\frac{b-\mu}{\sigma}\right)^2-1\right]}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right] \\
&- 3\sigma^3\left\{\left[\frac{\left[\frac{a-\mu}{\sigma}\right]\phi\left[\frac{a-\mu}{\sigma}\right]-\left[\frac{b-\mu}{\sigma}\right]\phi\left[\frac{b-\mu}{\sigma}\right]}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right]\left[\frac{\phi\left[\frac{a-\mu}{\sigma}\right]-\phi\left[\frac{b-\mu}{\sigma}\right]}{\Phi\left[\frac{b-\mu}{\sigma}\right]-\Phi\left[\frac{a-\mu}{\sigma}\right]}\right]\right\}
\end{aligned}
$$

Define $V = \left[\frac{\phi\left(\frac{a-\mu}{\sigma}\right)-\phi\left(\frac{b-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right]$, $W = \left[\frac{\phi\left(\frac{a-\mu}{\sigma}\right)\left[\left(\frac{a-\mu}{\sigma}\right)^2-1\right]-\phi\left(\frac{b-\mu}{\sigma}\right)\left[\left(\frac{b-\mu}{\sigma}\right)^2-1\right]}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right]$ and $Z = \left[\frac{\left(\frac{a-\mu}{\sigma}\right)\phi\left(\frac{a-\mu}{\sigma}\right)-\left(\frac{b-\mu}{\sigma}\right)\phi\left(\frac{b-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right]$, then;

$$
\begin{aligned}
M_X^3(0) - 3M_X^2(0)M_X^1(0) + 2[M_X^1(0)]^3 &= 2\sigma^3V^3 + \sigma^3W - 3\sigma^3VZ \\
&= \sigma^3[2V^3 + W - 3VZ]
\end{aligned}
$$

$$\mathbb{V}(X) = \sigma^2 + \sigma^2 \left[\frac{\left(\frac{a-\mu}{\sigma}\right)\phi\left(\frac{a-\mu}{\sigma}\right) - \left(\frac{b-\mu}{\sigma}\right)\phi\left(\frac{b-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)} \right] - \sigma^2 \left[\frac{\phi\left(\frac{a-\mu}{\sigma}\right) - \phi\left(\frac{b-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)} \right]^2$$
$$= \sigma^2 + \sigma^2 Z - \sigma^2 V^2$$
$$= \sigma^2\left(1 + Z - V^2\right)$$
$$\mathbb{S}(X) = \frac{M_X^3(0) - 3M_X^2(0)M_X^1(0) + 2[M_X^1(0)]^3}{[\mathbb{V}(X)]^{\frac{3}{2}}}$$
$$= \frac{\sigma^3[2V^3 + W - 3VZ]}{[\sigma^2(1 + Z - V^2)]^{\frac{3}{2}}}$$
$$= \frac{2V^3 + W - 3VZ}{(1 + Z - V^2)^{\frac{3}{2}}}$$

3.1.6 Kurtosis

The Kurtosis of the truncated normal distribution using the MGF technique is determined as

$$\mathbb{K}(X) = \frac{M_X^{(4)}(0) - 4[M_X^{(1)}(0)M_X^{(3)}(0)] + 6[M_X^{(1)}(0)]^2 M_X^{(2)}(0) + 3[M_X^{(1)}(0)]^4}{\mathbb{V}(X)}$$

The fourth moment is determined as follows

$$M_X^{(4)}(t) = \frac{d^4}{dt^4}(M_X(t))$$
$$= \frac{d^4}{dt^4}\left(e^{(\mu t + \frac{1}{2}\sigma^2 t^2)}\left[\frac{\Phi(y) - \Phi(z)}{M}\right]\right)$$
$$M_X^{(3)}(t) = 3\sigma^2\left(\mu + \sigma^2 t\right)e^{(\mu t + \frac{1}{2}\sigma^2 t^2)}\left[\frac{\Phi(y) - \Phi(z)}{M}\right] + 3\sigma^2 e^{(\mu t + \frac{1}{2}\sigma^2 t^2)}\left[\frac{\phi(z)P_0(z) - \phi(y)P_0(y)}{M}\right]\sigma$$
$$+ (\mu + \sigma^2 t)^3 e^{(\mu t + \frac{1}{2}\sigma^2 t^2)}\left[\frac{\Phi(y) - \Phi(z)}{M}\right] + 3(\mu + \sigma^2 t)^2 e^{(\mu t + \frac{1}{2}\sigma^2 t^2)}\left[\frac{\phi(z)P_0(z) - \phi(y)P_0(y)}{M}\right]\sigma$$
$$+ 3(\mu + \sigma^2 t)e^{(\mu t + \frac{1}{2}\sigma^2 t^2)}\left[\frac{\phi(z)P_1(z) - \phi(y)P_1(y)}{M}\right]\sigma^2 + e^{(\mu t + \frac{1}{2}\sigma^2 t^2)}\left[\frac{\phi(z)P_2(z) - \phi(y)(P_2(y)}{M}\right]\sigma^3$$

Thus, we have

$$
\begin{aligned}
&M_X^{(4)}(t)\\
&= 3\sigma^4 e^{(\mu t+\frac{1}{2}\sigma^2t^2)}\left[\frac{\Phi(y)-\Phi(z)}{M}\right] + 3\sigma^2\left(\mu+\sigma^2t\right)^2 e^{(\mu t+\frac{1}{2}\sigma^2t^2)}\left[\frac{\Phi(y)-\Phi(z)}{M}\right]\\
&+3\sigma^2\left(\mu+\sigma^2t\right)e^{(\mu t+\frac{1}{2}\sigma^2t^2)}\left[\frac{\phi(z)P_0(z)-\phi(y)P_0(y)}{M}\right]\sigma + 3\sigma^2(\mu+\sigma^2t)e^{(\mu t+\frac{1}{2}\sigma^2t^2)}\left[\frac{\phi(z)P_0(z)-\phi(y)P_0(y)}{M}\right]\sigma\\
&+3\sigma^2 e^{(\mu t+\frac{1}{2}\sigma^2t^2)}\left[\frac{\phi(z)P_1(z)-\phi(y)P_1(y)}{M}\right]\sigma^2 + 3\sigma^2(\mu+\sigma^2t)^2 e^{(\mu t+\frac{1}{2}\sigma^2t^2)}\left[\frac{\Phi(y)-\Phi(z)}{M}\right]\\
&+(\mu+\sigma^2t)^4 e^{(\mu t+\frac{1}{2}\sigma^2t^2)}\left[\frac{\Phi(y)-\Phi(z)}{M}\right] + (\mu+\sigma^2t)^3 e^{(\mu t+\frac{1}{2}\sigma^2t^2)}\left[\frac{\phi(z)P_0(z)-\phi(y)P_0(y)}{M}\right]\sigma\\
&+6\sigma^2(\mu+\sigma^2t)e^{(\mu t+\frac{1}{2}\sigma^2t^2)}\left[\frac{\phi(z)P_0(z)-\phi(y)P_0(y)}{M}\right]\sigma + 3(\mu+\sigma^2t)^3 e^{(\mu t+\frac{1}{2}\sigma^2t^2)}\left[\frac{\phi(z)P_0(z)-\phi(y)P_0(y)}{M}\right]\sigma\\
&+3(\mu+\sigma^2t)^2 e^{(\mu t+\frac{1}{2}\sigma^2t^2)}\left[\frac{\phi(z)P_1(z)-\phi(y)P_1(y)}{M}\right]\sigma^2 + 3\sigma^2 e^{(\mu t+\frac{1}{2}\sigma^2t^2)}\left[\frac{\phi(z)P_1(z)-\phi(y)P_1(y)}{M}\right]\sigma^2\\
&+3(\mu+\sigma^2t)^2 e^{(\mu t+\frac{1}{2}\sigma^2t^2)}\left[\frac{\phi(z)P_1(z)-\phi(y)P_1(y)}{M}\right]\sigma^2 + 3(\mu+\sigma^2t)e^{(\mu t+\frac{1}{2}\sigma^2t^2)}\left[\frac{\phi(z)P_2(z)-\phi(y)P_2(y)}{M}\right]\sigma^3\\
&+(\mu+\sigma^2t)e^{(\mu t+\frac{1}{2}\sigma^2t^2)}\left[\frac{\phi(z)P_2(z)-\phi(y)(P_2(y)}{M}\right]\sigma^3 + e^{(\mu t+\frac{1}{2}\sigma^2t^2)}\left[\frac{\phi(z)P_3(z)-\phi(y)(P_3(y)}{M}\right]\sigma^4\\
&= 3\sigma^4 e^{(\mu t+\frac{1}{2}\sigma^2t^2)}\left[\frac{\Phi(y)-\Phi(z)}{M}\right] + 3\sigma^2\left(\mu+\sigma^2t\right)^2 e^{(\mu t+\frac{1}{2}\sigma^2t^2)}\left[\frac{\Phi(y)-\Phi(z)}{M}\right]\\
&+3\sigma^2\left(\mu+\sigma^2t\right)e^{(\mu t+\frac{1}{2}\sigma^2t^2)}\left[\frac{\phi(z)-\phi(y)}{M}\right]\sigma + 3\sigma^2(\mu+\sigma^2t)e^{(\mu t+\frac{1}{2}\sigma^2t^2)}\left[\frac{\phi(z)-\phi(y)}{M}\right]\sigma\\
&+3\sigma^2 e^{(\mu t+\frac{1}{2}\sigma^2t^2)}\left[\frac{\phi(z)z-\phi(y)y}{M}\right]\sigma^2 + 3\sigma^2(\mu+\sigma^2t)^2 e^{(\mu t+\frac{1}{2}\sigma^2t^2)}\left[\frac{\Phi(y)-\Phi(z)}{M}\right]\\
&+(\mu+\sigma^2t)^4 e^{(\mu t+\frac{1}{2}\sigma^2t^2)}\left[\frac{\Phi(y)-\Phi(z)}{M}\right] + (\mu+\sigma^2t)^3 e^{(\mu t+\frac{1}{2}\sigma^2t^2)}\left[\frac{\phi(z)-\phi(y)}{M}\right]\sigma\\
&+6\sigma^2(\mu+\sigma^2t)e^{(\mu t+\frac{1}{2}\sigma^2t^2)}\left[\frac{\phi(z)-\phi(y)}{M}\right]\sigma + 3(\mu+\sigma^2t)^3 e^{(\mu t+\frac{1}{2}\sigma^2t^2)}\left[\frac{\phi(z)-\phi(y)}{M}\right]\sigma\\
&+3(\mu+\sigma^2t)^2 e^{(\mu t+\frac{1}{2}\sigma^2t^2)}\left[\frac{\phi(z)z-\phi(y)y}{M}\right]\sigma^2 + 3\sigma^2 e^{(\mu t+\frac{1}{2}\sigma^2t^2)}\left[\frac{\phi(z)z-\phi(y)y}{M}\right]\sigma^2\\
&+3(\mu+\sigma^2t)^2 e^{(\mu t+\frac{1}{2}\sigma^2t^2)}\left[\frac{\phi(z)z-\phi(y)y}{M}\right]\sigma^2 + 3(\mu+\sigma^2t)e^{(\mu t+\frac{1}{2}\sigma^2t^2)}\left[\frac{\phi(z)\left(z^2-1\right)-\phi(y)\left(y^2-1\right)}{M}\right]\sigma^3\\
&+(\mu+\sigma^2t)e^{(\mu t+\frac{1}{2}\sigma^2t^2)}\left[\frac{\phi(z)\left(z^2-1\right)-\phi(y)\left(y^2-1\right)}{M}\right]\sigma^3 + e^{(\mu t+\frac{1}{2}\sigma^2t^2)}\left[\frac{\phi(z)\left(z^3-3z\right)-\phi(y)\left(y^3-3y\right)}{M}\right]\sigma^4
\end{aligned}
$$

Substituting back the values of y, z and M, we have

$$
\begin{aligned}
&M_X^4(t)\\
&=3\sigma^2 e^{(\mu t+\frac{1}{2}\sigma^2t^2)}\left\{\sigma^2\left[\frac{\Phi\left(\frac{b-\mu}{\sigma}-\sigma t\right)-\Phi\left(\frac{a-\mu}{\sigma}-\sigma t\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right]+\left(\mu+\sigma^2t\right)^2\left[\frac{\Phi\left(\frac{b-\mu}{\sigma}-\sigma t\right)-\Phi\left(\frac{a-\mu}{\sigma}-\sigma t\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right]\right\}\\
&+3\sigma^2\left(\mu+\sigma^2t\right)e^{(\mu t+\frac{1}{2}\sigma^2t^2)}\left\{\left[\frac{\phi\left(\frac{a-\mu}{\sigma}-\sigma t\right)-\phi\left(\frac{b-\mu}{\sigma}-\sigma t\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right]\sigma+\left[\frac{\phi\left(\frac{a-\mu}{\sigma}-\sigma t\right)-\phi\left(\frac{b-\mu}{\sigma}-\sigma t\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right]\sigma\right\}\\
&+3\sigma^2 e^{(\mu t+\frac{1}{2}\sigma^2t^2)}\left[\frac{\phi\left[\frac{a-\mu}{\sigma}-\sigma t\right]\left[\frac{a-\mu}{\sigma}-\sigma t\right]-\phi\left[\frac{b-\mu}{\sigma}-\sigma t\right]\left[\frac{b-\mu}{\sigma}-\sigma t\right]}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right]\sigma^2\\
&+\left(3\sigma^2+(\mu+\sigma^2t)^2\right)\left\{(\mu+\sigma^2t)^2e^{(\mu t+\frac{1}{2}\sigma^2t^2)}\left[\frac{\Phi\left[\frac{b-\mu}{\sigma}-\sigma t\right]-\Phi\left[\frac{a-\mu}{\sigma}-\sigma t\right]}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right]\right\}\\
&+\left(6\sigma^2+(\mu+\sigma^2t)^2\right)\left\{(\mu+\sigma^2t)e^{(\mu t+\frac{1}{2}\sigma^2t^2)}\left[\frac{\phi\left(\frac{a-\mu}{\sigma}-\sigma t\right)-\phi\left(\frac{b-\mu}{\sigma}-\sigma t\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right]\sigma\right\}\\
&+3(\mu+\sigma^2t)^3e^{(\mu t+\frac{1}{2}\sigma^2t^2)}\left[\frac{\phi\left(\frac{a-\mu}{\sigma}-\sigma t\right)-\phi\left(\frac{b-\mu}{\sigma}-\sigma t\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right]\sigma\\
&+3(\mu+\sigma^2t)^2e^{(\mu t+\frac{1}{2}\sigma^2t^2)}\left[\frac{\phi\left(\frac{a-\mu}{\sigma}-\sigma t\right)\left(\frac{a-\mu}{\sigma}-\sigma t\right)-\phi\left(\frac{b-\mu}{\sigma}-\sigma t\right)\left(\frac{b-\mu}{\sigma}-\sigma t\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right]\sigma^2\\
&+3\sigma^2e^{(\mu t+\frac{1}{2}\sigma^2t^2)}\left[\frac{\phi\left(\frac{a-\mu}{\sigma}-\sigma t\right)\left(\frac{a-\mu}{\sigma}-\sigma t\right)-\phi\left(\frac{b-\mu}{\sigma}-\sigma t\right)\left(\frac{b-\mu}{\sigma}-\sigma t\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right]\sigma^2\\
&+3(\mu+\sigma^2t)^2e^{(\mu t+\frac{1}{2}\sigma^2t^2)}\left[\frac{\phi\left(\frac{a-\mu}{\sigma}-\sigma t\right)\left(\frac{a-\mu}{\sigma}-\sigma t\right)-\phi\left(\frac{b-\mu}{\sigma}-\sigma t\right)\left(\frac{b-\mu}{\sigma}-\sigma t\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right]\sigma^2\\
&+3(\mu+\sigma^2t)e^{(\mu t+\frac{1}{2}\sigma^2t^2)}\left[\frac{\phi\left(\frac{a-\mu}{\sigma}-\sigma t\right)\left(\left(\frac{a-\mu}{\sigma}-\sigma t\right)^2-1\right)-\phi\left(\frac{b-\mu}{\sigma}-\sigma t\right)\left(\left(\frac{b-\mu}{\sigma}-\sigma t\right)^2-1\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right]\sigma^3\\
&+(\mu+\sigma^2t)e^{(\mu t+\frac{1}{2}\sigma^2t^2)}\left[\frac{\phi\left(\frac{a-\mu}{\sigma}-\sigma t\right)\left(\left(\frac{a-\mu}{\sigma}-\sigma t\right)^2-1\right)-\phi\left(\frac{b-\mu}{\sigma}-\sigma t\right)\left(\left(\frac{b-\mu}{\sigma}-\sigma t\right)^2-1\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right]\sigma^3\\
&+e^{(\mu t+\frac{1}{2}\sigma^2t^2)}\left[\frac{\phi\left[\frac{a-\mu}{\sigma}-\sigma t\right]\left[\left[\frac{a-\mu}{\sigma}-\sigma t\right]^3-3\left[\frac{a-\mu}{\sigma}-\sigma t\right]\right]-\phi\left[\frac{b-\mu}{\sigma}-\sigma t\right]\left[\left[\frac{b-\mu}{\sigma}-\sigma t\right]^3-3\left[\frac{b-\mu}{\sigma}-\sigma t\right]\right]}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right]\sigma^4
\end{aligned}
$$

Evaluating at $t=0$, we have
$M_X^{(4)}(0)$

$$= 3\sigma^4\left[\frac{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right]+3\sigma^2\mu^2\left[\frac{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right]+3\sigma^2\mu\left[\frac{\phi\left(\frac{a-\mu}{\sigma}\right)-\phi\left(\frac{b-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right]\sigma$$

$$+3\sigma^2\mu\left[\frac{\phi\left(\frac{a-\mu}{\sigma}\right)-\phi\left(\frac{b-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right]\sigma+3\sigma^2\left[\frac{\phi\left[\frac{a-\mu}{\sigma}\right]\left[\frac{a-\mu}{\sigma}\right]-\phi\left[\frac{b-\mu}{\sigma}\right]\left[\frac{b-\mu}{\sigma}\right]}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right]\sigma^2$$

$$+3\sigma^2\mu^2\left[\frac{\Phi\left[\frac{b-\mu}{\sigma}\right]-\Phi\left[\frac{b-\mu}{\sigma}\right]}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right]+\mu^4\left[\frac{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right]+\mu^3\left[\frac{\phi\left(\frac{a-\mu}{\sigma}\right)-\phi\left(\frac{b-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right]\sigma$$

$$+6\sigma^2\mu\left[\frac{\phi\left(\frac{a-\mu}{\sigma}\right)-\phi\left(\frac{b-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right]\sigma+3\mu^3\left[\frac{\phi\left(\frac{a-\mu}{\sigma}\right)-\phi\left(\frac{b-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right]\sigma$$

$$+3\mu^2\left[\frac{\phi\left(\frac{a-\mu}{\sigma}\right)\left(\frac{a-\mu}{\sigma}\right)-\phi\left(\frac{b-\mu}{\sigma}\right)\left(\frac{b-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right]\sigma^2+3\sigma^2\left[\frac{\phi\left(\frac{a-\mu}{\sigma}\right)\left(\frac{a-\mu}{\sigma}\right)-\phi\left(\frac{b-\mu}{\sigma}\right)\left(\frac{b-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right]\sigma^2$$

$$+3\mu^2\left[\frac{\phi\left(\frac{a-\mu}{\sigma}\right)\left(\frac{a-\mu}{\sigma}\right)-\phi\left(\frac{b-\mu}{\sigma}\right)\left(\frac{b-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right]\sigma^2+3\mu\left[\frac{\phi\left(\frac{a-\mu}{\sigma}\right)\left[\left(\frac{a-\mu}{\sigma}\right)^2-1\right]-\phi\left(\frac{b-\mu}{\sigma}\right)\left[\left(\frac{b-\mu}{\sigma}\right)^2-1\right]}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right]\sigma^3$$

$$+\mu\left[\frac{\phi\left[\frac{a-\mu}{\sigma}\right]\left[\left[\frac{a-\mu}{\sigma}\right]^2-1\right]-\phi\left(\frac{b-\mu}{\sigma}\right)\left[\left[\frac{b-\mu}{\sigma}\right]^2-1\right]}{\Phi\left[\frac{b-\mu}{\sigma}\right]-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right]\sigma^3$$

$$+\left[\frac{\phi\left[\frac{a-\mu}{\sigma}\right]\left[\left[\frac{a-\mu}{\sigma}\right]^3-3\left[\frac{a-\mu}{\sigma}-\sigma t\right]\right]-\phi\left[\frac{b-\mu}{\sigma}\right]\left[\left[\frac{b-\mu}{\sigma}\right]^3-3\left[\frac{b-\mu}{\sigma}\right]\right]}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right]\sigma^4$$

Simplifying further, we have

$$
\begin{aligned}
M_X^{(4)}(0) &= 3\sigma^4 + 3\sigma^2\mu^2 + 3\sigma^3\mu\left[\frac{\phi\left(\frac{a-\mu}{\sigma}\right)-\phi\left(\frac{b-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right] + 3\sigma^3\mu\left[\frac{\phi\left(\frac{a-\mu}{\sigma}\right)-\phi\left(\frac{b-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right] \\
&\quad + 3\sigma^4\left[\frac{\phi\left[\frac{a-\mu}{\sigma}\right]\left[\frac{a-\mu}{\sigma}\right]-\phi\left[\frac{b-\mu}{\sigma}\right]\left[\frac{b-\mu}{\sigma}\right]}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right] + 3\sigma^2\mu^2 + \mu^4 + \sigma\mu^3\left[\frac{\phi\left(\frac{a-\mu}{\sigma}\right)-\phi\left(\frac{b-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right] \\
&\quad + 6\sigma^3\mu\left[\frac{\phi\left(\frac{a-\mu}{\sigma}\right)-\phi\left(\frac{b-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right] + 3\sigma\mu^3\left[\frac{\phi\left(\frac{a-\mu}{\sigma}\right)-\phi\left(\frac{b-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right] \\
&\quad + 3\sigma^2\mu^2\left[\frac{\phi\left(\frac{a-\mu}{\sigma}\right)\left(\frac{a-\mu}{\sigma}\right)-\phi\left(\frac{b-\mu}{\sigma}\right)\left(\frac{b-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right] + 3\sigma^4\left[\frac{\phi\left(\frac{a-\mu}{\sigma}\right)\left(\frac{a-\mu}{\sigma}\right)-\phi\left(\frac{b-\mu}{\sigma}\right)\left(\frac{b-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right] \\
&\quad + 3\sigma^2\mu^2\left[\frac{\phi\left(\frac{a-\mu}{\sigma}\right)\left(\frac{a-\mu}{\sigma}\right)-\phi\left(\frac{b-\mu}{\sigma}\right)\left(\frac{b-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right] + 3\sigma^3\mu\left[\frac{\phi\left[\frac{a-\mu}{\sigma}\right]\left[\left[\frac{a-\mu}{\sigma}\right]^2-1\right]-\phi\left[\frac{b-\mu}{\sigma}\right]\left[\left[\frac{b-\mu}{\sigma}\right]^2-1\right]}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right] \\
&\quad + \sigma^3\mu\left[\frac{\phi\left[\frac{a-\mu}{\sigma}\right]\left[\left[\frac{a-\mu}{\sigma}\right]^2-1\right]-\phi\left(\frac{b-\mu}{\sigma}\right)\left[\left[\frac{b-\mu}{\sigma}\right]^2-1\right]}{\Phi\left[\frac{b-\mu}{\sigma}\right]-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right] \\
&\quad + \sigma^4\left[\frac{\phi\left[\frac{a-\mu}{\sigma}\right]\left[\left[\frac{a-\mu}{\sigma}\right]^3-3\left[\frac{a-\mu}{\sigma}-\sigma t\right]\right]-\phi\left[\frac{b-\mu}{\sigma}\right]\left[\left[\frac{b-\mu}{\sigma}\right]^3-3\left[\frac{b-\mu}{\sigma}\right]\right]}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right] \\
&= 3\sigma^4 + 6\sigma^2\mu^2 + 12\sigma^3\mu\left[\frac{\phi\left(\frac{a-\mu}{\sigma}\right)-\phi\left(\frac{b-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right] + 6\sigma^4\left[\frac{\phi\left[\frac{a-\mu}{\sigma}\right]\left[\frac{a-\mu}{\sigma}\right]-\phi\left[\frac{b-\mu}{\sigma}\right]\left[\frac{b-\mu}{\sigma}\right]}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right] \\
&\quad + \mu^4 + 4\sigma\mu^3\left[\frac{\phi\left(\frac{a-\mu}{\sigma}\right)-\phi\left(\frac{b-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right] + 6\sigma^2\mu^2\left[\frac{\phi\left(\frac{a-\mu}{\sigma}\right)\left(\frac{a-\mu}{\sigma}\right)-\phi\left(\frac{b-\mu}{\sigma}\right)\left(\frac{b-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right] \\
&\quad + 4\sigma^3\mu\left[\frac{\phi\left(\frac{a-\mu}{\sigma}\right)\left[\left(\frac{a-\mu}{\sigma}\right)^2-1\right]-\phi\left(\frac{b-\mu}{\sigma}\right)\left[\left(\frac{b-\mu}{\sigma}\right)^2-1\right]}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right] \\
&\quad + \sigma^4\left[\frac{\phi\left[\frac{a-\mu}{\sigma}\right]\left[\left[\frac{a-\mu}{\sigma}\right]^3-3\left[\frac{a-\mu}{\sigma}-\sigma t\right]\right]-\phi\left[\frac{b-\mu}{\sigma}\right]\left[\left[\frac{b-\mu}{\sigma}\right]^3-3\left[\frac{b-\mu}{\sigma}\right]\right]}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right]
\end{aligned}
$$

$$
\begin{aligned}
&4M_X^1(0)M_X^3(0)\\
&= 12\sigma^2\mu^2 + 4\mu^4 + 12\sigma^3\mu\left[\frac{\phi\left(\frac{a-\mu}{\sigma}\right)-\phi\left(\frac{b-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right] + 12\sigma\mu^3\left[\frac{\phi\left(\frac{a-\mu}{\sigma}\right)-\phi\left(\frac{b-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right]\\
&+ 12\sigma^2\mu^2\left[\frac{\phi\left[\frac{a-\mu}{\sigma}\right]\left[\frac{a-\mu}{\sigma}\right]-\phi\left[\frac{b-\mu}{\sigma}\right]\left[\frac{b-\mu}{\sigma}\right]}{\Phi\left[\frac{b-\mu}{\sigma}\right]-\Phi\left[\frac{a-\mu}{\sigma}\right]}\right] + 4\sigma^3\mu\left[\frac{\phi\left[\frac{a-\mu}{\sigma}\right]\left[\left[\frac{a-\mu}{\sigma}\right]^2-1\right]-\phi\left[\frac{b-\mu}{\sigma}\right]\left[\left[\frac{b-\mu}{\sigma}\right]^2-1\right]}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right]\\
&+ 12\sigma^3\mu\left[\frac{\phi\left(\frac{a-\mu}{\sigma}\right)-\phi\left(\frac{b-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right] + 4\sigma\mu^3\left[\frac{\phi\left(\frac{a-\mu}{\sigma}\right)-\phi\left(\frac{b-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right] + 12\sigma^4\left[\frac{\phi\left(\frac{a-\mu}{\sigma}\right)-\phi\left(\frac{b-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right]^2\\
&+ 12\sigma^2\mu^2\left[\frac{\phi\left[\frac{a-\mu}{\sigma}\right]-\phi\left[\frac{b-\mu}{\sigma}\right]}{\Phi\left[\frac{b-\mu}{\sigma}\right]-\Phi\left[\frac{a-\mu}{\sigma}\right]}\right]^2 + 12\sigma^3\mu\left[\frac{\phi\left[\frac{a-\mu}{\sigma}\right]\left[\frac{a-\mu}{\sigma}\right]-\phi\left[\frac{b-\mu}{\sigma}\right]\left(\frac{b-\mu}{\sigma}\right)}{\Phi\left[\frac{b-\mu}{\sigma}\right]-\Phi\left[\frac{a-\mu}{\sigma}\right]}\right]\left[\frac{\phi\left[\frac{a-\mu}{\sigma}\right]-\phi\left[\frac{b-\mu}{\sigma}\right]}{\Phi\left[\frac{b-\mu}{\sigma}\right]-\Phi\left[\frac{a-\mu}{\sigma}\right]}\right]\\
&+ 4\sigma^4\left\{\left[\frac{\phi\left(\frac{a-\mu}{\sigma}\right)\left[\left(\frac{a-\mu}{\sigma}\right)^2-1\right]-\phi\left(\frac{b-\mu}{\sigma}\right)\left[\left(\frac{b-\mu}{\sigma}\right)^2-1\right]}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right]\left[\frac{\phi\left(\frac{a-\mu}{\sigma}\right)-\phi\left(\frac{b-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right]\right\}\\
&= 12\sigma^2\mu^2 + 4\mu^4 + 24\sigma^3\mu\left[\frac{\phi\left(\frac{a-\mu}{\sigma}\right)-\phi\left(\frac{b-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right] + 16\sigma\mu^3\left[\frac{\phi\left(\frac{a-\mu}{\sigma}\right)-\phi\left(\frac{b-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right]\\
&+ 12\sigma^2\mu^2\left[\frac{\phi\left[\frac{a-\mu}{\sigma}\right]\left[\frac{a-\mu}{\sigma}\right]-\phi\left[\frac{b-\mu}{\sigma}\right]\left[\frac{b-\mu}{\sigma}\right]}{\Phi\left[\frac{b-\mu}{\sigma}\right]-\Phi\left[\frac{a-\mu}{\sigma}\right]}\right] + 4\sigma^3\mu\left[\frac{\phi\left[\frac{a-\mu}{\sigma}\right]\left[\left[\frac{a-\mu}{\sigma}\right]^2-1\right]-\phi\left[\frac{b-\mu}{\sigma}\right]\left[\left[\frac{b-\mu}{\sigma}\right]^2-1\right]}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right]\\
&+ 12\sigma^4\left[\frac{\phi\left(\frac{a-\mu}{\sigma}\right)-\phi\left(\frac{b-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right]^2 + 12\sigma^2\mu^2\left[\frac{\phi\left(\frac{a-\mu}{\sigma}\right)-\phi\left(\frac{b-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right]^2\\
&+ 12\sigma^3\mu\left\{\left[\frac{\phi\left(\frac{a-\mu}{\sigma}\right)\left(\frac{a-\mu}{\sigma}\right)-\phi\left(\frac{b-\mu}{\sigma}\right)\left(\frac{b-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right]\left[\frac{\phi\left(\frac{a-\mu}{\sigma}\right)-\phi\left(\frac{b-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right]\right\}\\
&+ 4\sigma^4\left\{\left[\frac{\phi\left(\frac{a-\mu}{\sigma}\right)\left[\left(\frac{a-\mu}{\sigma}\right)^2-1\right]-\phi\left(\frac{b-\mu}{\sigma}\right)\left[\left(\frac{b-\mu}{\sigma}\right)^2-1\right]}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right]\left[\frac{\phi\left(\frac{a-\mu}{\sigma}\right)-\phi\left(\frac{b-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right]\right\}
\end{aligned}
$$

Thus

$$M_X^4(0) - 4M_X^1(0)M_X^3(0)$$

$$= 3\sigma^4 - 6\sigma^2\mu^2 - 3\mu^4 - 12\sigma^3\mu \left[\frac{\phi\left(\frac{a-\mu}{\sigma}\right) - \phi\left(\frac{b-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)}\right] + 6\sigma^4 \left[\frac{\phi\left(\frac{a-\mu}{\sigma}\right)\left(\frac{a-\mu}{\sigma}\right) - \phi\left(\frac{b-\mu}{\sigma}\right)\left(\frac{b-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)}\right]$$

$$- 12\sigma\mu^3 \left[\frac{\phi\left(\frac{a-\mu}{\sigma}\right) - \phi\left(\frac{b-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)}\right] - 6\sigma^2\mu^2 \left[\frac{\phi\left(\frac{a-\mu}{\sigma}\right)\left(\frac{a-\mu}{\sigma}\right) - \phi\left(\frac{b-\mu}{\sigma}\right)\left(\frac{b-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)}\right]$$

$$+ \sigma^4 \left[\frac{\phi\left[\frac{a-\mu}{\sigma}\right]\left[\left[\frac{a-\mu}{\sigma}\right]^3 - 3\left[\frac{a-\mu}{\sigma}\right]\right] - \phi\left[\frac{b-\mu}{\sigma}\right]\left[\left[\frac{b-\mu}{\sigma}\right]^3 - 3\left[\frac{b-\mu}{\sigma}\right]\right]}{\Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)}\right]$$

$$- 12\sigma^4 \left[\frac{\phi\left(\frac{a-\mu}{\sigma}\right) - \phi\left(\frac{b-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)}\right]^2 - 12\sigma^2\mu^2 \left[\frac{\phi\left(\frac{a-\mu}{\sigma}\right) - \phi\left(\frac{b-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)}\right]^2$$

$$- 12\sigma^3\mu \left\{\left[\frac{\phi\left(\frac{a-\mu}{\sigma}\right)\left(\frac{a-\mu}{\sigma}\right) - \phi\left(\frac{b-\mu}{\sigma}\right)\left(\frac{b-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)}\right]\left[\frac{\phi\left(\frac{a-\mu}{\sigma}\right) - \phi\left(\frac{b-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)}\right]\right\}$$

$$- 4\sigma^4 \left\{\left[\frac{\phi\left(\frac{a-\mu}{\sigma}\right)\left[\left(\frac{a-\mu}{\sigma}\right)^2 - 1\right] - \phi\left(\frac{b-\mu}{\sigma}\right)\left[\left(\frac{b-\mu}{\sigma}\right)^2 - 1\right]}{\Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)}\right]\left[\frac{\phi\left(\frac{a-\mu}{\sigma}\right) - \phi\left(\frac{b-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)}\right]\right\}$$

Also

$$6(M_X^1(0))^2 M_X^2(0)$$

$$= 6\sigma^2\mu^2 + 6\mu^4 + 12\sigma\mu^3 \left[\frac{\phi\left(\frac{a-\mu}{\sigma}\right) - \phi\left(\frac{b-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)}\right] + 6\sigma^2\mu^2 \left[\frac{\left(\frac{a-\mu}{\sigma}\right)\phi\left(\frac{a-\mu}{\sigma}\right) - \left(\frac{b-\mu}{\sigma}\right)\phi\left(\frac{b-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)}\right]$$

$$+ 12\sigma^3\mu \left[\frac{\phi\left(\frac{a-\mu}{\sigma}\right) - \phi\left(\frac{b-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)}\right] + 12\sigma\mu^3 \left[\frac{\phi\left(\frac{a-\mu}{\sigma}\right) - \phi\left(\frac{b-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)}\right] + 24\sigma^2\mu^2 \left[\frac{\phi\left(\frac{a-\mu}{\sigma}\right) - \phi\left(\frac{b-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)}\right]^2$$

$$+ 12\sigma^3\mu \left\{\left[\frac{\left(\frac{a-\mu}{\sigma}\right)\phi\left(\frac{a-\mu}{\sigma}\right) - \left(\frac{b-\mu}{\sigma}\right)\phi\left(\frac{b-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)}\right]\left[\frac{\phi\left(\frac{a-\mu}{\sigma}\right) - \phi\left(\frac{b-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)}\right]\right\}$$

$$+ 6\sigma^4 \left[\frac{\phi\left(\frac{a-\mu}{\sigma}\right) - \phi\left(\frac{b-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)}\right]^2 + 6\sigma^2\mu^2 \left[\frac{\phi\left(\frac{a-\mu}{\sigma}\right) - \phi\left(\frac{b-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)}\right]^2 + 12\sigma^3\mu \left[\frac{\phi\left(\frac{a-\mu}{\sigma}\right) - \phi\left(\frac{b-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)}\right]^3$$

$$+ 6\sigma^4 \left\{\left[\frac{\left(\frac{a-\mu}{\sigma}\right)\phi\left(\frac{a-\mu}{\sigma}\right) - \left(\frac{b-\mu}{\sigma}\right)\phi\left(\frac{b-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)}\right]\left[\frac{\phi\left(\frac{a-\mu}{\sigma}\right) - \phi\left(\frac{b-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)}\right]\right\}$$

Correct like-terms we have

$$6(M_X^1(0))^2M_X^2(0)$$

$$= 6\sigma^2\mu^2 + 6\mu^4 + 24\sigma\mu^3\left[\frac{\phi\left(\frac{a-\mu}{\sigma}\right)-\phi\left(\frac{b-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right] + 6\sigma^2\mu^2\left[\frac{\left(\frac{a-\mu}{\sigma}\right)\phi\left(\frac{a-\mu}{\sigma}\right)-\left(\frac{b-\mu}{\sigma}\right)\phi\left(\frac{b-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right]$$

$$+ 12\sigma^3\mu\left[\frac{\phi\left(\frac{a-\mu}{\sigma}\right)-\phi\left(\frac{b-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right] + 30\sigma^2\mu^2\left[\frac{\phi\left(\frac{a-\mu}{\sigma}\right)-\phi\left(\frac{b-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right]^2$$

$$+ 12\sigma^3\mu\left\{\left[\frac{\left(\frac{a-\mu}{\sigma}\right)\phi\left(\frac{a-\mu}{\sigma}\right)-\left(\frac{b-\mu}{\sigma}\right)\phi\left(\frac{b-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right]\left[\frac{\phi\left(\frac{a-\mu}{\sigma}\right)-\phi\left(\frac{b-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right]\right\}$$

$$+ 6\sigma^4\left[\frac{\phi\left(\frac{a-\mu}{\sigma}\right)-\phi\left(\frac{b-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right]^2 + 12\sigma^3\mu\left[\frac{\phi\left(\frac{a-\mu}{\sigma}\right)-\phi\left(\frac{b-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right]^3$$

$$+ 6\sigma^4\left\{\left[\frac{\left(\frac{a-\mu}{\sigma}\right)\phi\left(\frac{a-\mu}{\sigma}\right)-\left(\frac{b-\mu}{\sigma}\right)\phi\left(\frac{b-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right]\left[\frac{\phi\left(\frac{a-\mu}{\sigma}\right)-\phi\left(\frac{b-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right]\right\}$$

Hence

$$M_X^4(0) - 4M_X^1(0)M_X^3(0) + 6(M_X^1(0))^2M_X^2(0)$$

$$= 3\sigma^4 + 3\mu^4 + 6\sigma^4\left[\frac{\phi\left(\frac{a-\mu}{\sigma}\right)\left(\frac{a-\mu}{\sigma}\right)-\phi\left(\frac{b-\mu}{\sigma}\right)\left(\frac{b-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right] + 12\sigma\mu^3\left[\frac{\phi\left(\frac{a-\mu}{\sigma}\right)-\phi\left(\frac{b-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right]$$

$$+ \sigma^4\left[\frac{\phi\left[\frac{a-\mu}{\sigma}\right]\left[\left[\frac{a-\mu}{\sigma}\right]^3-3\left[\frac{a-\mu}{\sigma}\right]\right]-\phi\left[\frac{b-\mu}{\sigma}\right]\left[\left[\frac{b-\mu}{\sigma}\right]^3-3\left[\frac{b-\mu}{\sigma}\right]\right]}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right] - 6\sigma^4\left[\frac{\phi\left(\frac{a-\mu}{\sigma}\right)-\phi\left(\frac{b-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right]^2$$

$$+ 18\sigma^2\mu^2\left[\frac{\phi\left(\frac{a-\mu}{\sigma}\right)-\phi\left(\frac{b-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right]^2 + 12\sigma^3\mu\left[\frac{\phi\left(\frac{a-\mu}{\sigma}\right)-\phi\left(\frac{b-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right]^3$$

$$- 4\sigma^4\left\{\left[\frac{\phi\left(\frac{a-\mu}{\sigma}\right)\left[\left(\frac{a-\mu}{\sigma}\right)^2-1\right]-\phi\left(\frac{b-\mu}{\sigma}\right)\left[\left(\frac{b-\mu}{\sigma}\right)^2-1\right]}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right]\left[\frac{\phi\left(\frac{a-\mu}{\sigma}\right)-\phi\left(\frac{b-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right]\right\}$$

$$+ 6\sigma^4\left\{\left[\frac{\left(\frac{a-\mu}{\sigma}\right)\phi\left(\frac{a-\mu}{\sigma}\right)-\left(\frac{b-\mu}{\sigma}\right)\phi\left(\frac{b-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right]\left[\frac{\phi\left(\frac{a-\mu}{\sigma}\right)-\phi\left(\frac{b-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right]^2\right\}$$

Lastly,

$$3(M_X^1(0))^4 = 3\left[\mu+\sigma\left(\frac{\phi\left(\frac{a-\mu}{\sigma}\right)-\phi\left(\frac{b-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right)\right]^4$$

$$= 3\mu^4+12\sigma\mu^3\left[\frac{\phi\left(\frac{a-\mu}{\sigma}\right)-\phi\left(\frac{b-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right]+18\sigma^2\mu^2\left[\frac{\phi\left(\frac{a-\mu}{\sigma}\right)-\phi\left(\frac{b-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right]^2$$

$$+12\sigma^3\mu\left[\frac{\phi\left(\frac{a-\mu}{\sigma}\right)-\phi\left(\frac{b-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right]^3+3\sigma^4\left[\frac{\phi\left(\frac{a-\mu}{\sigma}\right)-\phi\left(\frac{b-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right]^4$$

Hence

$$M_X^4(0)-4M_X^1(0)M_X^3(0)+6(M_X^1(0))^2M_X^2(0)+3(M_X^1(0))^4$$

$$= 3\sigma^4+3\mu^4+6\sigma^4\left[\frac{\phi\left(\frac{a-\mu}{\sigma}\right)\left(\frac{a-\mu}{\sigma}\right)-\phi\left(\frac{b-\mu}{\sigma}\right)\left(\frac{b-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right]+12\sigma\mu^3\left[\frac{\phi\left(\frac{a-\mu}{\sigma}\right)-\phi\left(\frac{b-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right]$$

$$+\sigma^4\left[\frac{\phi\left[\frac{a-\mu}{\sigma}\right]\left[\left[\frac{a-\mu}{\sigma}\right]^3-3\left[\frac{a-\mu}{\sigma}\right]\right]-\phi\left[\frac{b-\mu}{\sigma}\right]\left[\left[\frac{b-\mu}{\sigma}\right]^3-3\left[\frac{b-\mu}{\sigma}\right]\right]}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right]-6\sigma^4\left[\frac{\phi\left(\frac{a-\mu}{\sigma}\right)-\phi\left(\frac{b-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right]^2$$

$$+18\sigma^2\mu^2\left[\frac{\phi\left(\frac{a-\mu}{\sigma}\right)-\phi\left(\frac{b-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right]^2+12\sigma^3\mu\left[\frac{\phi\left(\frac{a-\mu}{\sigma}\right)-\phi\left(\frac{b-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right]^3$$

$$-4\sigma^4\left\{\left[\frac{\phi\left(\frac{a-\mu}{\sigma}\right)\left[\left(\frac{a-\mu}{\sigma}\right)^2-1\right]-\phi\left(\frac{b-\mu}{\sigma}\right)\left[\left(\frac{b-\mu}{\sigma}\right)^2-1\right]}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right]\left[\frac{\phi\left(\frac{a-\mu}{\sigma}\right)-\phi\left(\frac{b-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right]\right\}$$

$$+6\sigma^4\left[\frac{\left(\frac{a-\mu}{\sigma}\right)\phi\left(\frac{a-\mu}{\sigma}\right)-\left(\frac{b-\mu}{\sigma}\right)\phi\left(\frac{b-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right]\left[\frac{\phi\left(\frac{a-\mu}{\sigma}\right)-\phi\left(\frac{b-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right]^2+3\mu^4+12\sigma\mu^3\left[\frac{\phi\left(\frac{a-\mu}{\sigma}\right)-\phi\left(\frac{b-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right]$$

$$+18\sigma^2\mu^2\left[\frac{\phi\left(\frac{a-\mu}{\sigma}\right)-\phi\left(\frac{b-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right]^2+12\sigma^3\mu\left[\frac{\phi\left(\frac{a-\mu}{\sigma}\right)-\phi\left(\frac{b-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right]^3+3\sigma^4\left[\frac{\phi\left(\frac{a-\mu}{\sigma}\right)-\phi\left(\frac{b-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right]^4$$

Correcting like terms, we have

$$M_X^4(0)-4M_X^1(0)M_X^3(0)+6(M_X^1(0))^2M_X^2(0)+3(M_X^1(0))^4$$

$$=3\sigma^4\left[\frac{\phi\left(\frac{a-\mu}{\sigma}\right)-\phi\left(\frac{b-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right]^4-6\sigma^4\left[\frac{\phi\left(\frac{a-\mu}{\sigma}\right)-\phi\left(\frac{b-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right]^2+6\sigma^4\left[\frac{\phi\left(\frac{a-\mu}{\sigma}\right)\left(\frac{a-\mu}{\sigma}\right)-\phi\left(\frac{b-\mu}{\sigma}\right)\left(\frac{b-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right]$$

$$+6\sigma^4\left\{\left[\frac{\phi\left(\frac{a-\mu}{\sigma}\right)\left(\frac{a-\mu}{\sigma}\right)-\phi\left(\frac{b-\mu}{\sigma}\right)\left(\frac{b-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right]\left[\frac{\phi\left(\frac{a-\mu}{\sigma}\right)-\phi\left(\frac{b-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right]^2\right\}$$

$$-4\sigma^4\left\{\left[\frac{\phi\left(\frac{a-\mu}{\sigma}\right)\left[\left(\frac{a-\mu}{\sigma}\right)^2-1\right]-\phi\left(\frac{b-\mu}{\sigma}\right)\left[\left(\frac{b-\mu}{\sigma}\right)^2-1\right]}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right]\left[\frac{\phi\left(\frac{a-\mu}{\sigma}\right)-\phi\left(\frac{b-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right]\right\}$$

$$\sigma^4\left[\frac{\phi\left[\frac{a-\mu}{\sigma}\right]\left[\left[\frac{a-\mu}{\sigma}\right]^3-3\left[\frac{a-\mu}{\sigma}\right]\right]-\phi\left[\frac{b-\mu}{\sigma}\right]\left[\left[\frac{b-\mu}{\sigma}\right]^3-3\left[\frac{b-\mu}{\sigma}\right]\right]}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right]+3\sigma^4+24\sigma^3\mu\left[\frac{\phi\left(\frac{a-\mu}{\sigma}\right)-\phi\left(\frac{b-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right]^3$$

$$+36\sigma^2\mu^2\left[\frac{\phi\left(\frac{a-\mu}{\sigma}\right)-\phi\left(\frac{b-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right]^2+24\sigma\mu^3\left[\frac{\phi\left(\frac{a-\mu}{\sigma}\right)-\phi\left(\frac{b-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right]+6\mu^4$$

Define $S=\left[\frac{\phi\left[\frac{a-\mu}{\sigma}\right]\left[\left[\frac{a-\mu}{\sigma}\right]^3-3\left[\frac{a-\mu}{\sigma}\right]\right]-\phi\left[\frac{b-\mu}{\sigma}\right]\left[\left[\frac{b-\mu}{\sigma}\right]^3-3\left[\frac{b-\mu}{\sigma}\right]\right]}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right]$ and V,W,Z as before, we have

$$M_X^4(0)-4M_X^1(0)M_X^3(0)+6(M_X^1(0))^2M_X^2(0)+3(M_X^1(0))^4$$

$$\begin{aligned}&=3\sigma^4V^4-6\sigma^4V^2+6\sigma^4Z+6\sigma^4V^2Z-4\sigma^4VW+\sigma^4S+3\sigma^4\\&+24\sigma^3\mu V^3+36\sigma^2\mu^2V^2+24\sigma\mu^3V+6\mu^4\\&=\sigma^4\left[3V^4-6V^2+6Z+6V^2Z-4VW+S+3\right]\\&+24\sigma^3\mu V^3+36\sigma^2\mu^2V^2+24\sigma\mu^3V+6\mu^4\end{aligned}$$

Therefore, kurtosis is given as

$$\begin{aligned}\mathbb{K}(X)&=\frac{M_X^4(0)-4[M_X^1(0)M_X^3(0)]+6[M_X^1(0)]^2M_X^2(0)+3[M_X^1(0)]^4}{\mathbb{V}(X)}\\&=\frac{\sigma^4\left[3V^4-6V^2+6Z+6V^2Z-4VW+S+3\right]+24\sigma^3\mu V^3+36\sigma^2\mu^2V^2+24\sigma\mu^3V+6\mu^4}{\sigma^2\left(1+Z-V^2\right)}\end{aligned}$$

3.2 Martingale Restriction

Option pricing models require an imposition of a restriction known as the martingale restriction to ensure that there are no arbitrage opportunities in the market. As stated by Black and Scholes, Cox and Ross and Merton, [BS73, CR76, Mer73], the implied market price of the underlying asset must be equal to its current market value, otherwise, there could exist an arbitrage opportunity in the market. The martingale restriction of the structural model - which defines the dynamics of the underlying process at every moment of the contract period, define $t \in [0,T]$, then, to avoid arbitrage opportunities in the market, the following martingale restriction should hold.

$$\mathbb{E}^{\mathbb{Q}}\left[e^{-r(T-t)}S_T \middle| \mathscr{F}_t\right] = S_t, \qquad \text{where } t \in [0,T]$$

Therefore, this condition should also hold for the case of non-structural models - which defines the dynamics of the underlying asset at the beginning of the contract period and at maturity denoted as 0 and t as proposed by Longstaff [Lon95], i.e.

$$\mathbb{E}^{\mathbb{Q}}\left[e^{-rt}S_t \middle| \mathscr{F}_0\right] = S_0$$

The violation of this condition results in the existence of arbitrage opportunities as noted by Harrisson et al. [HK79]. The proof of this condition is presented in the next subsection.

3.2.1 Non-Structural Pricing Models - Black & Scholes Model

Under the assumption of Black & Scholes, the underlying stock price, S_t, follows a Geometric Brownian Motion (GBM) given by;

$$dS_t = rS_t dt + \sigma S_t dW_t$$

where r and σ are the expected return and volatility respectively, and W_t is a standard Brownian Motion under the risk - neutral probability measure $\mathbb{Q}$. The continuously compounded risk - free rate r is also assumed to be constant. To find the distribution of S_t, we apply the Itô Lemma. The solution of the GBM is determined as follows;

Let
$f(S_t,t) = \log_e S_t, \Rightarrow f_t'(t) = 0$, $f_s'(t) = \frac{1}{S_t}$ and $f_s''(t) = -\frac{1}{S_t^2}$. Therefore, applying the Itô lemma, we have;

$$
\begin{aligned}
df(S_t,t) &= f_t'(t)dt + f_s'(t)dS_t + \frac{1}{2}f_s''(t)(dS_t)^2 \\
&= 0dt + \frac{1}{S_t}dS_t + \frac{1}{2}\left(-\frac{1}{S_t^2}\right)(dS_t)^2 \\
&= \frac{1}{S_t}\left(rS_tdt + \sigma S_tdW_t\right) + \frac{1}{2}\left(-\frac{1}{S_t^2}\right)\left(rS_tdt + \sigma S_tdW_t\right)^2 \\
&= \frac{1}{S_t}\left(rS_tdt + \sigma S_tdW_t\right) + \frac{1}{2}\left(-\frac{1}{S_t^2}\right)\left[\left(rS_tdt + \sigma S_tdW_t\right)\left(rS_tdt + \sigma S_tdW_t\right)\right] \\
&= \frac{1}{S_t}\left(rS_tdt + \sigma S_tdW_t\right) + \frac{1}{2}\left(-\frac{1}{S_t^2}\right)\left[\left((rS_tdt)^2 + rS_tdt.\sigma S_tdW_t + \sigma S_tdW_t.S_tdt + (\sigma S_tdW_t)^2\right)\right]
\end{aligned}
$$

Using Itô's multiplication table, i.e

$$
\begin{cases}
(dt)^2 = 0 \\
dt(dW_t) = 0 \\
(dW_t)^2 = dt
\end{cases}
$$

Therefore;

$$
\begin{aligned}
df(S_t,t) &= \frac{1}{S_t}\left(rS_tdt + \sigma S_tdW_t\right) + \frac{1}{2}\left(-\frac{1}{S_t^2}\right)(\sigma^2 S_t^2)dt \\
&= rdt + \sigma dW_t - \frac{1}{2}\sigma^2 dt \\
&= \left(r - \frac{1}{2}\sigma^2\right)dt + \sigma dW_t \\
\Rightarrow d\left(\log_e S_t\right) &= \left(r - \frac{1}{2}\sigma^2\right)dt + \sigma dW_t \\
\int_0^t d\left(\log_e S_s\right) &= \int_0^t \left(r - \frac{1}{2}\sigma^2\right)ds + \sigma\int_0^t dW_s \\
\log_e S_t - \log_e S_0 &= \left(r - \frac{1}{2}\sigma^2\right)t + \sigma(W_t - W_0) \\
\log_e\left(\frac{S_t}{S_0}\right) &= \left(r - \frac{1}{2}\sigma^2\right)t + \sigma(W_t - W_0) \\
\frac{S_t}{S_0} &= e^{\left\{\left(r-\frac{1}{2}\sigma^2\right)t + \sigma(W_t - W_0)\right\}} \\
S_t &= S_0 e^{\left\{\left(r-\frac{1}{2}\sigma^2\right)t + \sigma(W_t - W_0)\right\}}
\end{aligned}
$$

Therefore,

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}}\left[e^{-rt}S_t\middle|\mathscr{F}_0\right] &= e^{-rt}\mathbb{E}^{\mathbb{Q}}\left[S_t\middle|\mathscr{F}_0\right] \\
&= e^{-rt}\mathbb{E}^{\mathbb{Q}}\left[S_0e^{\left\{\left(r-\frac{1}{2}\sigma^2\right)t+\sigma(W_t-W_0)\right\}}\middle|\mathscr{F}_0\right] \\
&= e^{-rt}S_0\mathbb{E}^{\mathbb{Q}}\left[e^{\left\{\left(r-\frac{1}{2}\sigma^2\right)t+\sigma(W_t-W_0)\right\}}\middle|\mathscr{F}_0\right] \\
&= e^{-rt}S_0\mathbb{E}^{\mathbb{Q}}\left[e^{\left(r-\frac{1}{2}\sigma^2\right)t}e^{\sigma(W_t-W_0)}\middle|\mathscr{F}_0\right] \\
&= e^{-rt}S_0e^{\left(r-\frac{1}{2}\sigma^2\right)t}\mathbb{E}^{\mathbb{Q}}\left[e^{\sigma(W_t-W_0)}\middle|\mathscr{F}_0\right]
\end{aligned}$$

The MGF of the standard normal distribution, where $X \sim N(0,1)$ is given by;

$$\begin{aligned}
M_X(t) &= \mathbb{E}\left[e^{tX}\right] \\
&= e^{\frac{1}{2}t^2}
\end{aligned}$$

Therefore, using the concept of MGF where $(W_t - W_0) \sim N(0,t)$ under the probability measure $\mathbb{Q}$, we have then that;

$$\begin{aligned}
M_{[W_t-W_0]}(\sigma) &= \mathbb{E}^{\mathbb{Q}}\left[e^{\sigma(W_t-W_0)}\right] \\
&= e^{\frac{1}{2}\sigma^2 t}
\end{aligned}$$

Therefore;

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}}\left[e^{-rt}S_t\middle|\mathscr{F}_0\right] &= e^{-rt}S_0e^{\left(r-\frac{1}{2}\sigma^2\right)t}\mathbb{E}^{\mathbb{Q}}\left[e^{\sigma(W_t-W_0)}\middle|\mathscr{F}_0\right] \\
&= e^{-rt}S_0e^{\left(r-\frac{1}{2}\sigma^2\right)t}e^{\frac{1}{2}\sigma^2 t} \\
&= e^{-rt}S_0e^{\left(r-\frac{1}{2}\sigma^2+\frac{1}{2}\sigma^2\right)t} \\
&= e^{-rt}S_0e^{rt}
\end{aligned}$$

$$\Rightarrow \mathbb{E}^{\mathbb{Q}}\left[e^{-rt}S_t\middle|\mathscr{F}_0\right] = S_0 \tag{2}$$

3.2.2 Structural Pricing Models - Truncated Normal Distribution (TND) Model

The underlying log-price is now assumed to follow truncated normal distribution under the martingale measure $\mathbb{Q}$ which is defined as;

$$X_t = \log_e\left(\frac{S_t}{S_0}\right) = f(x;\mu t,\sigma\sqrt{t},a,b)$$

where S_0 is the current underlying asset price. Define

$$Y_t = \left(\frac{S_t}{S_0}\right)$$

Then

$$\log_e(Y_t) = X_t$$
$$\Rightarrow Y_t = e^{X_t}$$

The probability density function of the log-returns of the underlying process is determine as follows;

$$\begin{aligned}
F_Y(y) &= Pr[Y \le y] \\
&= Pr\left[e^X \le y\right] \\
&= Pr[X \le \log_e(y)] \\
&= F_X[\log_e(y)] \\
f_Y(y) &= \frac{d}{dy}F_Y(y) \\
&= \frac{d}{dy}F_X[\log_e(y)] \\
&= \frac{d}{dx}F_X[\log_e(y)]\frac{dx}{dy} \\
&= f_X[\log_e(y))\frac{1}{y}, \text{ where } x = \log_e(y), \Rightarrow \frac{dx}{dy} = \frac{1}{y} \\
f_Y(y) &= f((\log_e(y); \mu t, \sigma\sqrt{t}, a, b)
\end{aligned}$$

When $x = a, y = e^a$ and when $x = b, y = e^b$

Hence, the p.d.f of the of the returns of the underlying asset is given by;

$$f_Y(y) = \begin{cases} \frac{1}{\sigma\sqrt{t}y}\frac{\phi\left(\frac{\log_e(y)-\mu t}{\sigma\sqrt{t}}\right)}{\Phi\left(\frac{b-\mu t}{\sigma\sqrt{t}}\right)-\Phi\left(\frac{a-\mu t}{\sigma}\right)}, & e^a \le y \le e^b \\ 0, & \text{otherwise} \end{cases} \quad (3)$$

where $\phi(*)$ and $\Phi(*)$ are as defined before.

Since the truncated normal distribution is among the non-structural models, then the martingale restriction condition is necessary when determining the expectation of the underlying process. Hence, applying the martingale restriction from equation (2) and expectation to equation (3), we have;

$$
\begin{aligned}
E\left[Y_t \middle| \mathscr{F}_0\right] &= \int_{e^a}^{e^b} y(f_Y(y))dy \\
&= \int_{e^a}^{e^b} y\left(\frac{1}{\sigma\sqrt{t}y}\frac{\phi\left(\frac{\log_e(y)-\mu t}{\sigma\sqrt{t}}\right)}{\Phi\left(\frac{b-\mu t}{\sigma\sqrt{t}}\right)-\Phi\left(\frac{a-\mu t}{\sigma}\right)}\right)dy \\
&= \int_{e^a}^{e^b} \frac{1}{\sigma\sqrt{t}}\frac{\phi\left(\frac{\log_e(y)-\mu t}{\sigma\sqrt{t}}\right)}{\Phi\left(\frac{b-\mu t}{\sigma\sqrt{t}}\right)-\Phi\left(\frac{a-\mu t}{\sigma}\right)}dy
\end{aligned}
$$

Let:

$$
\begin{aligned}
u &= \frac{\log_e(y)-\mu t}{\sigma\sqrt{t}}, &\Rightarrow \frac{du}{dy} = \frac{1}{\sigma\sqrt{t}y}, \Rightarrow dy = \sigma\sqrt{t}ydu \\
\log_e(y) &= \mu t + \sigma\sqrt{t}u &\Rightarrow y = e^{(\mu t+\sigma\sqrt{t}u)}
\end{aligned}
$$

When

$$
y = e^a, u = \frac{\log_e(e^a)-\mu t}{\sigma\sqrt{t}} \Rightarrow u = \frac{a-\mu t}{\sigma\sqrt{t}}
$$

Similarly, when

$$
y = e^b, u = \frac{\log_e(e^b)-\mu t}{\sigma\sqrt{t}} \Rightarrow u = \frac{b-\mu t}{\sigma\sqrt{t}}
$$

Hence;

$$
\begin{aligned}
E\left[Y_t \middle| \mathscr{F}_0\right] &= \int_{\frac{a-\mu t}{\sigma\sqrt{t}}}^{\frac{b-\mu t}{\sigma\sqrt{t}}} \frac{1}{\sigma\sqrt{t}} \frac{\phi(u)}{\Phi\left(\frac{b-\mu t}{\sigma\sqrt{t}}\right) - \Phi\left(\frac{a-\mu t}{\sigma}\right)} \sigma\sqrt{t} y du \\
&= \int_{\frac{a-\mu t}{\sigma\sqrt{t}}}^{\frac{b-\mu t}{\sigma\sqrt{t}}} \frac{\phi(u)}{\Phi\left(\frac{b-\mu t}{\sigma\sqrt{t}}\right) - \Phi\left(\frac{a-\mu t}{\sigma}\right)} y du \\
&= \int_{\frac{a-\mu t}{\sigma\sqrt{t}}}^{\frac{b-\mu t}{\sigma\sqrt{t}}} \frac{1}{\sqrt{2\pi}} \frac{e^{-\frac{u^2}{2}}}{M} e^{(\mu t + \sigma\sqrt{t} u)} du, \quad \text{where } M = \Phi\left(\frac{b-\mu t}{\sigma\sqrt{t}}\right) - \Phi\left(\frac{a-\mu t}{\sigma}\right) \\
&= \frac{e^{\mu t}}{M} \int_{\frac{a-\mu t}{\sigma\sqrt{t}}}^{\frac{b-\mu t}{\sigma\sqrt{t}}} e^{-\frac{1}{2}(u^2 - 2\sigma\sqrt{t} u)} du
\end{aligned}
$$

Considering the exponent and completing the square, we have;

$$
\begin{aligned}
-\frac{1}{2}(u^2 - 2\sigma\sqrt{t}u) &= -\frac{1}{2}(u^2 - 2\sigma\sqrt{t}u + \sigma^2 t - \sigma^2 t) \\
&= -\frac{1}{2}(u - \sigma\sqrt{t})^2 + \frac{1}{2}\sigma^2 t
\end{aligned}
$$

Thus;

$$
\begin{aligned}
E\left[Y_t \middle| \mathscr{F}_0\right] &= \frac{e^{\left(\mu + \frac{1}{2}\sigma^2\right)t}}{M} \int_{\frac{a-\mu t}{\sigma\sqrt{t}}}^{\frac{b-\mu t}{\sigma\sqrt{t}}} e^{-\frac{1}{2}(u - \sigma\sqrt{t})^2} du \\
&= \frac{e^{\left(\mu + \frac{1}{2}\sigma^2\right)t}}{M} \left[\Phi\left(\frac{b-\mu t}{\sigma\sqrt{t}} - \sigma\sqrt{t}\right) - \Phi\left(\frac{a-\mu t}{\sigma\sqrt{t}} - \sigma\sqrt{t}\right)\right] \\
&= e^{\left(\mu + \frac{1}{2}\sigma^2\right)t} \left[\frac{\Phi\left(\frac{b-\mu t}{\sigma\sqrt{t}} - \sigma\sqrt{t}\right) - \Phi\left(\frac{a-\mu t}{\sigma\sqrt{t}} - \sigma\sqrt{t}\right)}{\Phi\left(\frac{b-\mu t}{\sigma\sqrt{t}}\right) - \Phi\left(\frac{a-\mu t}{\sigma}\right)}\right]
\end{aligned}
$$

Rearranging equation (2), we have that

$$
\begin{aligned}
E\left[e^{-rt} S_t \middle| \mathscr{F}_0\right] &= S_0 \\
E\left[\left(\frac{S_t}{S_0}\right) \middle| \mathscr{F}_0\right] &= e^{rt} \\
E\left[Y_t \middle| \mathscr{F}_0\right] &= e^{rt}
\end{aligned}
$$

Therefore, incorporating this condition, we have;

$$E\left[Y_t \middle| \mathscr{F}_0\right] = e^{rt}$$
$$e^{\left(\mu+\frac{1}{2}\sigma^2\right)t}\left[\frac{\Phi\left(\frac{b-\mu t}{\sigma\sqrt{t}}-\sigma\sqrt{t}\right)-\Phi\left(\frac{a-\mu t}{\sigma\sqrt{t}}-\sigma\sqrt{t}\right)}{\Phi\left(\frac{b-\mu t}{\sigma\sqrt{t}}\right)-\Phi\left(\frac{a-\mu t}{\sigma}\right)}\right] = e^{rt}$$

Therefore, the expected value of the option given the current information is given as;

$$e^{\left(r-\mu-\frac{1}{2}\sigma^2\right)t} = \frac{\Phi\left(\frac{b-\mu t}{\sigma\sqrt{t}}-\sigma\sqrt{t}\right)-\Phi\left(\frac{a-\mu t}{\sigma\sqrt{t}}-\sigma\sqrt{t}\right)}{\Phi\left(\frac{b-\mu t}{\sigma\sqrt{t}}\right)-\Phi\left(\frac{a-\mu t}{\sigma}\right)} \tag{4}$$

Thus, if the values of a, b, t, σ and r are provided, then the value of μ can be determined using the root finding process such as uniroot or uniroot.all in R - software which is an open source. However, any other iteration method such as the Newton Raphson Method can be used in finding the root. In this paper, we adopt the uniroot code in R software.

3.3 Closed Formulas for Pricing Options

3.3.1 Call Option

The value of a European Call Option (V_c) is determine as;

$$V_c = max\left[(S_t - K, 0) \middle| \mathscr{F}_0\right]$$

Consider two functions f and g and let a be any constant. The using the properties of expectation, we have that;

$$\mathbb{E}\left[a\left(f\left[\frac{X}{a}\right]+g\left[\frac{X}{a}\right]\right)\right] = a\mathbb{E}\left[f\left[\frac{X}{a}\right]+g\left[\frac{X}{a}\right]\right]$$

According to the Fayman-Kac stochastic representation as noted by Bj$\ddot{o}$rk [Bjö09], which is the solution of the Black & Scholes equation, consider the SDE

$$dS_t = rS_t dt + \sigma S_t dW_t^{\mathbb{Q}}$$

where t is option's time to maturity.

Then applying the above property of expectation, the value of a European Call option is given as;

$$\begin{aligned} V_c &= e^{-rt}\mathbb{E}^{\mathbb{Q}}\left[max\left(S_t - K, 0\right)\Big|\mathscr{F}_0\right] \\ &= S_0 e^{-rt}\mathbb{E}^{\mathbb{Q}}\left[max\left(\frac{S_t}{S_0} - \frac{K}{S_0}, 0\right)\Bigg|\mathscr{F}_0\right] \\ &= S_0 e^{-rt}\mathbb{E}^{\mathbb{Q}}\left[max\left(y - \frac{K}{S_0}, 0\right)\Bigg|\mathscr{F}_0\right] \end{aligned}$$

In determining the closed form of the European call option, three situation are considered, i.e when $\frac{K}{S_0} < e^a$, $\frac{K}{S_0} > e^b$ and when $e^a \leq \frac{K}{S_0} \leq e^b$ with respect to the underlying price and the strike price.

Case I, $\frac{K}{S_0} < e^a$

The returns of the underlying asset is assumed to be higher than e^a but lower than e^b, i.e $e^a < \frac{S_t}{S_0} < e^b$.

Therefore, since $\frac{S_t}{S_0} > e^a$ and $\frac{K}{S_0} < e^a$, then $\frac{S_t}{S_0} - \frac{K}{S_0} > 0$, hence the option has value. Thus, the option price is determined as follows;

$$\begin{aligned} V_c &= S_0 e^{-rt}\mathbb{E}^{\mathbb{Q}}\left[max\left(y - \frac{K}{S_0}, 0\right)\Bigg|\mathscr{F}_0\right] \\ &= S_0 e^{-rt}\int_{e^a}^{e^b} max\left(y - \frac{K}{S_0}, 0\right) f_Y(y)dy \\ &= S_0 e^{-rt}\int_{e^a}^{e^b} \left(y - \frac{K}{S_0}\right) f_Y(y)dy, \quad \text{Let } M = \Phi\left(\frac{b-\mu t}{\sigma\sqrt{t}}\right) - \Phi\left(\frac{a-\mu t}{\sigma}\right), \text{ then} \\ &= \frac{S_0 e^{-rt}}{M}\int_{e^a}^{e^b} \left(y - \frac{K}{S_0}\right)\frac{1}{\sigma\sqrt{t}y}\phi\left(\frac{\log_e(y) - \mu t}{\sigma\sqrt{t}}\right)dy, \\ &= \frac{S_0 e^{-rt}}{M}\int_{e^a}^{e^b} \frac{1}{\sigma\sqrt{t}}\phi\left(\frac{\log_e(y) - \mu t}{\sigma\sqrt{t}}\right)dy - \frac{Ke^{-rt}}{M}\int_{e^a}^{e^b} \frac{1}{\sigma\sqrt{t}y}\phi\left(\frac{\log_e(y) - \mu t}{\sigma\sqrt{t}}\right)dy \end{aligned}$$

Let

$$A_1 = \frac{S_0 e^{-rt}}{M} \int_{e^a}^{e^b} \frac{1}{\sigma\sqrt{t}} \phi\left(\frac{\log_e(y) - \mu t}{\sigma\sqrt{t}}\right) dy$$
$$A_2 = \frac{K e^{-rt}}{M} \int_{e^a}^{e^b} \frac{1}{\sigma\sqrt{t}y} \phi\left(\frac{\log_e(y) - \mu t}{\sigma\sqrt{t}}\right) dy$$

Consider A_2, i.e

$$A_2 = \frac{K e^{-rt}}{M} \int_{e^a}^{e^b} \frac{1}{\sigma\sqrt{t}y} \phi\left(\frac{\log_e(y) - \mu t}{\sigma\sqrt{t}}\right) dy$$

Define

$$z = \frac{\log_e(y) - \mu t}{\sigma\sqrt{t}}, \Rightarrow \frac{dz}{dy} = \frac{d}{dy}\left(\frac{\log_e(y) - \mu t}{\sigma\sqrt{t}}\right) = \frac{1}{\sigma\sqrt{t}y}, \Rightarrow dy = \sigma\sqrt{t}y dz$$
$$y = e^b, z = \frac{\log_e(e^b) - \mu t}{\sigma\sqrt{t}} = \frac{b - \mu t}{\sigma\sqrt{t}}$$
$$y = e^a, z = \frac{\log_e(e^a) - \mu t}{\sigma\sqrt{t}} = \frac{a - \mu t}{\sigma\sqrt{t}}$$

Therefore,

$$\begin{aligned}
A_2 &= \frac{K e^{-rt}}{M} \int_{\frac{a-\mu t}{\sigma\sqrt{t}}}^{\frac{b-\mu t}{\sigma\sqrt{t}}} \frac{1}{\sigma\sqrt{t}y} \phi(z) \sigma\sqrt{t}y dz \\
&= \frac{K e^{-rt}}{M} \int_{\frac{a-\mu t}{\sigma\sqrt{t}}}^{\frac{b-\mu t}{\sigma\sqrt{t}}} \phi(z) dz \\
&= \frac{K e^{-rt}}{M} \left[\Phi\left(\frac{b-\mu t}{\sigma\sqrt{t}}\right) - \Phi\left(\frac{a-\mu t}{\sigma\sqrt{t}}\right)\right] \\
&= K e^{-rt} \left[\frac{\Phi\left(\frac{b-\mu t}{\sigma\sqrt{t}}\right) - \Phi\left(\frac{a-\mu t}{\sigma\sqrt{t}}\right)}{\Phi\left(\frac{b-\mu t}{\sigma\sqrt{t}}\right) - \Phi\left(\frac{a-\mu t}{\sigma\sqrt{t}}\right)}\right]
\end{aligned}$$
$$\Rightarrow A_2 = K e^{-rt}$$

Similarly, considering A_1,i.e.

$$A_1 = \frac{S_0 e^{-rt}}{M} \int_{e^a}^{e^b} \frac{1}{\sigma\sqrt{t}} \phi\left(\frac{\log_e(y) - \mu t}{\sigma\sqrt{t}}\right) dy$$

And defining

$$\begin{aligned}
v = \frac{\log_e(y) - \mu t}{\sigma\sqrt{t}}, &\Rightarrow \frac{dv}{dy} = \frac{d}{dy}\left(\frac{\log_e(y) - \mu t}{\sigma\sqrt{t}}\right) = \frac{1}{\sigma\sqrt{t}y}, \Rightarrow dy = \sigma\sqrt{t}ydv \\
y = e^b, v &= \frac{\log_e(e^b) - \mu t}{\sigma\sqrt{t}} = \frac{b - \mu t}{\sigma\sqrt{t}} \\
y = e^a, v &= \frac{\log_e(e^a) - \mu t}{\sigma\sqrt{t}} = \frac{a - \mu t}{\sigma\sqrt{t}} \\
\log_e(y) = \mu t + \sigma\sqrt{t}v, &\Rightarrow y = e^{(\mu t + \sigma\sqrt{t}v)}
\end{aligned}$$

Then

$$\begin{aligned}
A_1 &= \frac{S_0 e^{-rt}}{M} \int_{\frac{a-\mu t}{\sigma\sqrt{t}}}^{\frac{b-\mu t}{\sigma\sqrt{t}}} \frac{1}{\sigma\sqrt{t}} \phi(v) \sigma\sqrt{t}ydv \\
&= \frac{S_0 e^{-rt}}{M} \int_{\frac{a-\mu t}{\sigma\sqrt{t}}}^{\frac{b-\mu t}{\sigma\sqrt{t}}} \phi(v)ydv \\
&= \frac{S_0 e^{-rt}}{M} \int_{\frac{a-\mu t}{\sigma\sqrt{t}}}^{\frac{b-\mu t}{\sigma\sqrt{t}}} \frac{1}{\sqrt{2\pi}} e^{-\frac{v^2}{2}} e^{(\mu t + \sigma\sqrt{t}v)} dv \\
&= \frac{S_0 e^{-(r-\mu)t}}{M} \int_{\frac{a-\mu t}{\sigma\sqrt{t}}}^{\frac{b-\mu t}{\sigma\sqrt{t}}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(v^2 - 2\sigma\sqrt{t}v)} dv
\end{aligned}$$

Completing the square in the exponent, we have;

$$\begin{aligned}
-\frac{1}{2}(v^2 - 2\sigma\sqrt{t}v) &= -\frac{1}{2}(v^2 - 2\sigma\sqrt{t}v + \sigma^2 t - \sigma^2 t) \\
&= -\frac{1}{2}(v - \sigma\sqrt{t})^2 + \frac{1}{2}\sigma^2 t
\end{aligned}$$

$$
\begin{aligned}
A_1 &= \frac{S_0 e^{-(r-\mu-\frac{1}{2}\sigma^2)t}}{M} \int_{\frac{a-\mu t}{\sigma\sqrt{t}}}^{\frac{b-\mu t}{\sigma\sqrt{t}}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(v-\sigma\sqrt{t})^2} dv \\
&= \frac{S_0 e^{-(r-\mu-\frac{1}{2}\sigma^2)t}}{M} \left[\Phi\left(\frac{b-\mu t}{\sigma\sqrt{t}} - \sigma\sqrt{t}\right) - \Phi\left(\frac{a-\mu t}{\sigma\sqrt{t}}\right) - \sigma\sqrt{t} \right]
\end{aligned}
$$

Combining the martingale restriction from (4), we have

$$
\begin{aligned}
e^{-(r-\mu-\frac{1}{2}\sigma^2)t} &= \frac{1}{e^{(r-\mu-\frac{1}{2}\sigma^2)t}} \\
&= \frac{\Phi\left(\frac{b-\mu t}{\sigma\sqrt{t}}\right) - \Phi\left(\frac{a-\mu t}{\sigma\sqrt{t}}\right)}{\Phi\left(\frac{b-\mu t}{\sigma\sqrt{t}} - \sigma\sqrt{t}\right) - \Phi\left(\frac{a-\mu t}{\sigma\sqrt{t}} - \sigma\sqrt{t}\right)}
\end{aligned}
$$

Hence;

$$
A_1 = S_0 \frac{\Phi\left(\frac{b-\mu t}{\sigma\sqrt{t}}\right) - \Phi\left(\frac{a-\mu t}{\sigma\sqrt{t}}\right)}{\Phi\left(\frac{b-\mu t}{\sigma\sqrt{t}} - \sigma\sqrt{t}\right) - \Phi\left(\frac{a-\mu t}{\sigma\sqrt{t}} - \sigma\sqrt{t}\right)} \left[\frac{\Phi\left(\frac{b-\mu t}{\sigma\sqrt{t}} - \sigma\sqrt{t}\right) - \Phi\left(\frac{a-\mu t}{\sigma\sqrt{t}} - \sigma\sqrt{t}\right)}{\Phi\left(\frac{b-\mu t}{\sigma\sqrt{t}}\right) - \Phi\left(\frac{a-\mu t}{\sigma\sqrt{t}}\right)} \right]
$$

$$
\Rightarrow A_1 = S_0
$$

Therefore

$$
\begin{aligned}
V_c &= A_1 - A_2 \\
&= S_0 - K e^{-rt}
\end{aligned}
$$

Case II, $\frac{K}{S_0} > e^b$

Given that;

$$
\frac{S_t}{S_0} < e^b, \quad \text{and } \frac{K}{S_0} > e^b
$$

Then

$$
\begin{aligned}
&\frac{S_t}{S_0} - \frac{K}{S_0} < 0 \\
&\qquad \Rightarrow V_c = S_0 e^{-rt} \mathbb{E}^{\mathbb{Q}} \left[max(\frac{S_t}{S_0} - \frac{K}{S_0}, 0) \middle| \mathscr{F}_0 \right] \\
&\qquad = 0
\end{aligned}
$$

hence, the option is worthless, i.e has no value.

Case III, $e^a \leq \frac{K}{S_0} \leq e^b$

The value of the European Call Option is given as

$$
\begin{aligned}
V_c &= S_0 e^{-rt}\mathbb{E}^{\mathbb{Q}}\left[max\left(y-\frac{K}{S_0},0\right)\middle|\mathscr{F}_0\right] \\
&= S_0 e^{-rt}\int_{e^{\frac{K}{S_0}}}^{e^b} max\left(y-\frac{K}{S_0},0\right) f_Y(y)dy \\
&= S_0 e^{-rt}\int_{e^{\frac{K}{S_0}}}^{e^b} \left(y-\frac{K}{S_0}\right) f_Y(y)dy, \quad \text{Let } M = \Phi\left(\frac{b-\mu t}{\sigma\sqrt{t}}\right) - \Phi\left(\frac{a-\mu t}{\sigma}\right) \\
&= \frac{S_0 e^{-rt}}{M}\int_{e^{\frac{K}{S_0}}}^{e^b} \left(y-\frac{K}{S_0}\right) \frac{1}{\sigma\sqrt{t}y}\phi\left(\frac{\log_e(y)-\mu t}{\sigma\sqrt{t}}\right)dy \\
&= \frac{S_0 e^{-rt}}{M}\int_{e^{\frac{K}{S_0}}}^{e^b} \frac{1}{\sigma\sqrt{t}}\phi\left(\frac{\log_e(y)-\mu t}{\sigma\sqrt{t}}\right)dy - \frac{Ke^{-rt}}{M}\int_{e^{\frac{K}{S_0}}}^{e^b} \frac{1}{\sigma\sqrt{t}y}\phi\left(\frac{\log_e(y)-\mu t}{\sigma\sqrt{t}}\right)dy
\end{aligned}
$$

Define

$$
\begin{aligned}
B_1 &= \frac{S_0 e^{-rt}}{M}\int_{e^{\frac{K}{S_0}}}^{e^b} \frac{1}{\sigma\sqrt{t}}\phi\left(\frac{\log_e(y)-\mu t}{\sigma\sqrt{t}}\right)dy \\
B_2 &= \frac{Ke^{-rt}}{M}\int_{e^{\frac{K}{S_0}}}^{e^b} \frac{1}{\sigma\sqrt{t}y}\phi\left(\frac{\log_e(y)-\mu t}{\sigma\sqrt{t}}\right)dy
\end{aligned}
$$

Consider

$$
B_2 = \frac{Ke^{-rt}}{M}\int_{e^{\frac{K}{S_0}}}^{e^b} \frac{1}{\sigma\sqrt{t}y}\phi\left(\frac{\log_e(y)-\mu t}{\sigma\sqrt{t}}\right)dy
$$

And define

$$
\begin{aligned}
& v = \frac{\log_e(y)-\mu t}{\sigma\sqrt{t}}, \Rightarrow \frac{dv}{dy} = \frac{d}{dy}\left(\frac{\log_e(y)-\mu t}{\sigma\sqrt{t}}\right) = \frac{1}{\sigma\sqrt{t}y}, \Rightarrow dy = \sigma\sqrt{t}ydv \\
& y = e^b, v = \frac{\log_e(e^b)-\mu t}{\sigma\sqrt{t}} = \frac{b-\mu t}{\sigma\sqrt{t}} \\
& y = e^{\frac{K}{S_0}}, v = \frac{\log_e e^{\frac{K}{S_0}} - \mu t}{\sigma\sqrt{t}} = \frac{\log_e\left(\frac{K}{S_0}\right)-\mu t}{\sigma\sqrt{t}}
\end{aligned}
$$

Hence

$$\begin{aligned}
B_2 &= \frac{Ke^{-rt}}{M}\int_{\frac{\log_e(\frac{K}{S_0})-\mu t}{\sigma\sqrt{t}}}^{\frac{b-\mu t}{\sigma\sqrt{t}}} \frac{1}{\sigma\sqrt{t}y}\phi(v)\sigma\sqrt{t}ydv \\
&= \frac{Ke^{-rt}}{M}\int_{\frac{\log_e(\frac{K}{S_0})-\mu t}{\sigma\sqrt{t}}}^{\frac{b-\mu t}{\sigma\sqrt{t}}} \phi(v)dv \\
&= \frac{Ke^{-rt}}{M}\left[\Phi\left(\frac{b-\mu t}{\sigma\sqrt{t}}\right)-\Phi\left(\frac{\log_e(\frac{K}{S_0})-\mu t}{\sigma\sqrt{t}}\right)\right] \\
\Rightarrow B_2 &= Ke^{-rt}\left[\frac{\Phi\left(\frac{b-\mu t}{\sigma\sqrt{t}}\right)-\Phi\left(\frac{\log_e(\frac{K}{S_0})-\mu t}{\sigma\sqrt{t}}\right)}{\Phi\left(\frac{b-\mu t}{\sigma\sqrt{t}}\right)-\Phi\left(\frac{a-\mu t}{\sigma\sqrt{t}}\right)}\right]
\end{aligned}$$

Similarly,

$$B_1 = \frac{S_0e^{-rt}}{M}\int_{e^{\frac{K}{S_0}}}^{e^b} \frac{1}{\sigma\sqrt{t}}\phi\left(\frac{\log_e(y)-\mu t}{\sigma\sqrt{t}}\right)dy$$

Letting

$$\begin{aligned}
&z = \frac{\log_e(y)-\mu t}{\sigma\sqrt{t}}, \Rightarrow \frac{dz}{dy} = \frac{d}{dy}\left(\frac{\log_e(y)-\mu t}{\sigma\sqrt{t}}\right) = \frac{1}{\sigma\sqrt{t}y}, \Rightarrow dy = \sigma\sqrt{t}ydz \\
&y = e^b, z = \frac{\log_e(e^b)-\mu t}{\sigma\sqrt{t}} = \frac{b-\mu t}{\sigma\sqrt{t}} \\
&y = e^{\frac{K}{S_0}}, z = \frac{\log_e e^{\frac{K}{S_0}}-\mu t}{\sigma\sqrt{t}} = \frac{\log_e\left(\frac{K}{S_0}\right)-\mu t}{\sigma\sqrt{t}} \\
\log_e(y) = \mu t + \sigma\sqrt{t}z, &\Rightarrow y = e^{(\mu t+\sigma\sqrt{t}z)}
\end{aligned}$$

Therefore

$$
\begin{aligned}
B_1 &= \frac{S_0 e^{-rt}}{M} \int_{\frac{\log_e\left(\frac{K}{S_0}\right)-\mu t}{\sigma\sqrt{t}}}^{\frac{b-\mu t}{\sigma\sqrt{t}}} \frac{1}{\sigma\sqrt{t}} \phi(z) \sigma\sqrt{t} y dz \\
&= \frac{S_0 e^{-rt}}{M} \int_{\frac{\log_e\left(\frac{K}{S_0}\right)-\mu t}{\sigma\sqrt{t}}}^{\frac{b-\mu t}{\sigma\sqrt{t}}} \phi(z) y dz \\
&= \frac{S_0 e^{-rt}}{M} \int_{\frac{\log_e\left(\frac{K}{S_0}\right)-\mu t}{\sigma\sqrt{t}}}^{\frac{b-\mu t}{\sigma\sqrt{t}}} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} e^{(\mu t + \sigma\sqrt{t}z)} dz \\
&= \frac{S_0 e^{-(r-\mu)t}}{M} \int_{\frac{\log_e\left(\frac{K}{S_0}\right)-\mu t}{\sigma\sqrt{t}}}^{\frac{b-\mu t}{\sigma\sqrt{t}}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(z^2 - 2\sigma\sqrt{t}z)} dz
\end{aligned}
$$

Complete the square to the exponent;

$$
\begin{aligned}
-\frac{1}{2}(z^2 - 2\sigma\sqrt{t}z) &= -\frac{1}{2}(z^2 - 2\sigma\sqrt{t}z + \sigma^2 t - \sigma^2 t) \\
&= -\frac{1}{2}(z - \sigma\sqrt{t})^2 + \frac{1}{2}\sigma^2 t
\end{aligned}
$$

Thus

$$
\begin{aligned}
B_1 &= \frac{S_0 e^{-(r-\mu-\frac{1}{2}\sigma^2)t}}{M} \int_{\frac{\log_e(\frac{K}{S_0})-\mu t}{\sigma\sqrt{t}}}^{\frac{b-\mu t}{\sigma\sqrt{t}}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(z-\sigma\sqrt{t})^2} dz \\
&= \frac{S_0 e^{-(r-\mu-\frac{1}{2}\sigma^2)t}}{M} \left[\Phi\left(\frac{b-\mu t}{\sigma\sqrt{t}} - \sigma\sqrt{t} \right) - \Phi\left(\frac{\log_e(\frac{K}{S_0}) - \mu t}{\sigma\sqrt{t}} - \sigma\sqrt{t} \right) \right]
\end{aligned}
$$

Combining the martingale restriction from (4), we have

$$
B_1 = \frac{S_0 \left(\Phi\left[\frac{b-\mu t}{\sigma\sqrt{t}}\right] - \Phi\left[\frac{a-\mu t}{\sigma\sqrt{t}}\right] \right)}{\Phi\left[\frac{b-\mu t}{\sigma\sqrt{t}} - \sigma\sqrt{t}\right] - \Phi\left[\frac{a-\mu t}{\sigma} - \sigma\sqrt{t}\right]} \left[\frac{\Phi\left[\frac{b-\mu t}{\sigma\sqrt{t}} - \sigma\sqrt{t}\right] - \Phi\left[\frac{\log_e(\frac{K}{S_0})-\mu t}{\sigma\sqrt{t}} - \sigma\sqrt{t}\right]}{\Phi\left[\frac{b-\mu t}{\sigma\sqrt{t}}\right] - \Phi\left[\frac{a-\mu t}{\sigma\sqrt{t}}\right]} \right]
$$

$$
\Rightarrow B_1 = S_0 \left[\frac{\Phi\left(\frac{b-\mu t}{\sigma\sqrt{t}} - \sigma\sqrt{t}\right) - \Phi\left(\frac{\log_e(\frac{K}{S_0})-\mu t}{\sigma\sqrt{t}} - \sigma\sqrt{t}\right)}{\Phi\left(\frac{b-\mu t}{\sigma\sqrt{t}} - \sigma\sqrt{t}\right) - \Phi\left(\frac{a-\mu t}{\sigma\sqrt{t}} - \sigma\sqrt{t}\right)} \right]
$$

Therefore

$V_c = B_1 - B_2$

$$V_c = S_0 \left[\frac{\Phi\left[\frac{b-\mu t}{\sigma\sqrt{t}} - \sigma\sqrt{t}\right] - \Phi\left[\frac{\log_e(\frac{K}{S_0})-\mu t}{\sigma\sqrt{t}} - \sigma\sqrt{t}\right]}{\Phi\left[\frac{b-\mu t}{\sigma\sqrt{t}} - \sigma\sqrt{t}\right] - \Phi\left[\frac{a-\mu t}{\sigma\sqrt{t}} - \sigma\sqrt{t}\right]} \right] - Ke^{-rt} \left[\frac{\Phi\left[\frac{b-\mu t}{\sigma\sqrt{t}}\right] - \Phi\left[\frac{\log_e(\frac{K}{S_0})-\mu t}{\sigma\sqrt{t}}\right]}{\Phi\left[\frac{b-\mu t}{\sigma\sqrt{t}}\right] - \Phi\left[\frac{a-\mu t}{\sigma\sqrt{t}}\right]} \right] \tag{5}$$

3.3.2 Put Option

Considering the case when $e^a \leq \frac{K}{S_0} \leq e^b$, the value of a European Put Option is given by;

$$\begin{aligned} V_p &= e^{-rt}\mathbb{E}^{\mathbb{Q}}\left[max(K - S_t, 0)\middle|\mathscr{F}_0\right] \\ &= S_0 e^{-rt}\mathbb{E}^{\mathbb{Q}}\left[max\left(\frac{K}{S_0} - \frac{S_t}{S_0}, 0\right)\middle|\mathscr{F}_0\right] \\ &= S_0 e^{-rt}\mathbb{E}^{\mathbb{Q}}\left[max\left(\frac{K}{S_0} - y, 0\right)\middle|\mathscr{F}_0\right] \end{aligned}$$

Hence;

$$\begin{aligned} V_p &= S_0 e^{-rt}\int_{e^a}^{e^{\frac{K}{S_0}}} max\left(\frac{K}{S_0} - y, 0\right) f_Y(y)dy \\ &= S_0 e^{-rt}\int_{e^a}^{e^{\frac{K}{S_0}}} \left(\frac{K}{S_0} - y\right) f_Y(y)dy, \quad \text{Let } M = \Phi\left(\frac{b-\mu t}{\sigma\sqrt{t}}\right) - \Phi\left(\frac{a-\mu t}{\sigma}\right) \\ &= \frac{S_0 e^{-rt}}{M}\int_{e^a}^{e^{\frac{K}{S_0}}} \left(\frac{K}{S_0} - y\right) \frac{1}{\sigma\sqrt{t}y}\phi\left(\frac{\log_e(y) - \mu t}{\sigma\sqrt{t}}\right) dy \\ &= \frac{Ke^{-rt}}{M}\int_{e^a}^{e^{\frac{K}{S_0}}} \frac{1}{\sigma\sqrt{t}y}\phi\left(\frac{\log_e(y) - \mu t}{\sigma\sqrt{t}}\right) dy - \frac{S_0 e^{-rt}}{M}\int_{e^{\frac{K}{S_0}}}^{e^b} \frac{1}{\sigma\sqrt{t}}\phi\left(\frac{\log_e(y) - \mu t}{\sigma\sqrt{t}}\right) dy \end{aligned}$$

Let

$$\begin{aligned} I_1 &= \frac{Ke^{-rt}}{M}\int_{e^a}^{e^{\frac{K}{S_0}}} \frac{1}{\sigma\sqrt{t}y}\phi\left(\frac{\log_e(y) - \mu t}{\sigma\sqrt{t}}\right) dy \\ I_2 &= \frac{S_0 e^{-rt}}{M}\int_{e^a}^{e^{\frac{K}{S_0}}} \frac{1}{\sigma\sqrt{t}}\phi\left(\frac{\log_e(y) - \mu t}{\sigma\sqrt{t}}\right) dy \end{aligned}$$

Let us consider

$$I_1 = \frac{Ke^{-rt}}{M}\int_{e^a}^{e^{\frac{K}{S_0}}} \frac{1}{\sigma\sqrt{t}y}\phi\left(\frac{\log_e(y)-\mu t}{\sigma\sqrt{t}}\right)dy$$

Define

$$u = \frac{\log_e(y)-\mu t}{\sigma\sqrt{t}}, \Rightarrow \frac{du}{dy} = \left(\frac{\log_e(y)-\mu t}{\sigma\sqrt{t}}\right) = \frac{1}{\sigma\sqrt{t}y}, \Rightarrow dy = \sigma\sqrt{t}ydu$$
$$y = e^a, u = \frac{\log_e e^a - \mu t}{\sigma\sqrt{t}} = \frac{a-\mu t}{\sigma\sqrt{t}}$$
$$y = e^{\frac{K}{S_0}}, u = \frac{\log_e e^{(\frac{K}{S_0})} - \mu t}{\sigma\sqrt{t}} = \frac{\log_e(\frac{K}{S_0})-\mu t}{\sigma\sqrt{t}}$$

Thus

$$I_1 = \frac{Ke^{-rt}}{M}\int_{\frac{a-\mu t}{\sigma\sqrt{t}}}^{\frac{\log_e\left(\frac{K}{S_0}\right)-\mu t}{\sigma\sqrt{t}}} \frac{1}{\sigma\sqrt{t}y}\phi(u)\sigma\sqrt{t}ydu$$
$$= \frac{Ke^{-rt}}{M}\int_{\frac{a-\mu t}{\sigma\sqrt{t}}}^{\frac{\log_e\left(\frac{K}{S_0}\right)-\mu t}{\sigma\sqrt{t}}} \phi(u)dz$$
$$= \frac{Ke^{-rt}}{M}\left[\Phi\left(\frac{\log_e\left(\frac{K}{S_0}\right)-\mu t}{\sigma\sqrt{t}}\right) - \Phi\left(\frac{a-\mu t}{\sigma\sqrt{t}}\right)\right]$$
$$\Rightarrow I_1 = Ke^{-rt}\left[\frac{\Phi\left(\frac{\log_e(\frac{K}{S_0})-\mu t}{\sigma\sqrt{t}}\right) - \Phi\left(\frac{a-\mu t}{\sigma\sqrt{t}}\right)}{\Phi\left(\frac{b-\mu t}{\sigma\sqrt{t}}\right) - \Phi\left(\frac{a-\mu t}{\sigma\sqrt{t}}\right)}\right]$$

Considering

$$I_2 = \frac{S_0 e^{-rt}}{M}\int_{e^a}^{e^{\frac{K}{S_0}}} \frac{1}{\sigma\sqrt{t}}\phi\left(\frac{\log_e(y)-\mu t}{\sigma\sqrt{t}}\right)dy$$

Define

$$v = \frac{\log_e(y) - \mu t}{\sigma\sqrt{t}}, \Rightarrow \frac{dv}{dy} = \left(\frac{\log_e(y) - \mu t}{\sigma\sqrt{t}}\right) = \frac{1}{\sigma\sqrt{t}y}, \Rightarrow dy = \sigma\sqrt{t}ydv$$

$$y = e^a, v = \frac{\log_e(e^a) - \mu t}{\sigma\sqrt{t}} = \frac{a - \mu t}{\sigma\sqrt{t}}$$

$$y = e^{\frac{K}{S_0}}, v = \frac{\log_e e^{\frac{K}{S_0}} - \mu t}{\sigma\sqrt{t}} = \frac{\log_e\left(\frac{K}{S_0}\right) - \mu t}{\sigma\sqrt{t}}$$

$$\log_e(y) = \mu t + \sigma\sqrt{t}v, \Rightarrow y = e^{(\mu t + \sigma\sqrt{t}v)}$$

Then

$$\begin{aligned}
I_2 &= \frac{S_0 e^{-rt}}{M}\int_{\frac{a-\mu t}{\sigma\sqrt{t}}}^{\frac{\log_e\left(\frac{K}{S_0}\right)-\mu t}{\sigma\sqrt{t}}} \frac{1}{\sigma\sqrt{t}}\phi(v)\sigma\sqrt{t}ydv \\
&= \frac{S_0 e^{-rt}}{M}\int_{\frac{a-\mu t}{\sigma\sqrt{t}}}^{\frac{\log_e\left(\frac{K}{S_0}\right)-\mu t}{\sigma\sqrt{t}}} \phi(z)ydv \\
&= \frac{S_0 e^{-rt}}{M}\int_{\frac{a-\mu t}{\sigma\sqrt{t}}}^{\frac{\log_e\left(\frac{K}{S_0}\right)-\mu t}{\sigma\sqrt{t}}} \frac{1}{\sqrt{2\pi}}e^{-\frac{v^2}{2}}e^{(\mu t + \sigma\sqrt{t}v)}dv \\
&= \frac{S_0 e^{-(r-\mu)t}}{M}\int_{\frac{a-\mu t}{\sigma\sqrt{t}}}^{\frac{\log_e\left(\frac{K}{S_0}\right)-\mu t}{\sigma\sqrt{t}}} \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(v^2 - 2\sigma\sqrt{t}v)}dv
\end{aligned}$$

Completing the square to the exponent;

$$\begin{aligned}
-\frac{1}{2}(v^2 - 2\sigma\sqrt{t}v) &= -\frac{1}{2}(v^2 - 2\sigma\sqrt{t}v + \sigma^2 t - \sigma^2 t) \\
&= -\frac{1}{2}(v - \sigma\sqrt{t})^2 + \frac{1}{2}\sigma^2 t
\end{aligned}$$

Thus

$$I_2 = \frac{S_0 e^{-(r-\mu-\frac{1}{2}\sigma^2)t}}{M} \int_{\frac{a-\mu t}{\sigma\sqrt{t}}}^{\frac{\log_e\left(\frac{K}{S_0}\right)-\mu t}{\sigma\sqrt{t}}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(v-\sigma\sqrt{t})^2} dv$$
$$= \frac{S_0 e^{-(r-\mu-\frac{1}{2}\sigma^2)t}}{M} \left[\Phi\left(\frac{\log_e\left(\frac{K}{S_0}\right)-\mu t}{\sigma\sqrt{t}} - \sigma\sqrt{t}\right) - \Phi\left(\frac{a-\mu t}{\sigma\sqrt{t}} - \sigma\sqrt{t}\right)\right]$$

Combining the martingale restriction from (4), we have

$$I_2 = S_0 \frac{\Phi\left(\frac{b-\mu t}{\sigma\sqrt{t}}\right) - \Phi\left(\frac{a-\mu t}{\sigma\sqrt{t}}\right)}{\Phi\left(\frac{b-\mu t}{\sigma\sqrt{t}} - \sigma\sqrt{t}\right) - \Phi\left(\frac{a-\mu t}{\sigma} - \sigma\sqrt{t}\right)} \left[\frac{\Phi\left(\frac{\log_e\left(\frac{K}{S_0}\right)-\mu t}{\sigma\sqrt{t}} - \sigma\sqrt{t}\right) - \Phi\left(\frac{a-\mu t}{\sigma\sqrt{t}} - \sigma\sqrt{t}\right)}{\Phi\left(\frac{b-\mu t}{\sigma\sqrt{t}}\right) - \Phi\left(\frac{a-\mu t}{\sigma\sqrt{t}}\right)}\right]$$
$$\Rightarrow I_2 = S_0 \left[\frac{\Phi\left(\frac{\log_e\left(\frac{K}{S_0}\right)-\mu t}{\sigma\sqrt{t}} - \sigma\sqrt{t}\right) - \Phi\left(\frac{a-\mu t}{\sigma\sqrt{t}} - \sigma\sqrt{t}\right)}{\Phi\left(\frac{b-\mu t}{\sigma\sqrt{t}} - \sigma\sqrt{t}\right) - \Phi\left(\frac{a-\mu t}{\sigma\sqrt{t}} - \sigma\sqrt{t}\right)}\right]$$

The value of the European Put is thus given by

$$V_p = I_1 - I_2$$

$$V_p = Ke^{-rt} \left[\frac{\Phi\left(\frac{\log_e\left(\frac{K}{S_0}\right)-\mu t}{\sigma\sqrt{t}}\right) - \Phi\left(\frac{a-\mu t}{\sigma\sqrt{t}}\right)}{\Phi\left(\frac{b-\mu t}{\sigma\sqrt{t}}\right) - \Phi\left(\frac{a-\mu t}{\sigma\sqrt{t}}\right)}\right] - S_0 \left[\frac{\Phi\left(\frac{\log_e\left(\frac{K}{S_0}\right)-\mu t}{\sigma\sqrt{t}} - \sigma\sqrt{t}\right) - \Phi\left(\frac{a-\mu t}{\sigma\sqrt{t}} - \sigma\sqrt{t}\right)}{\Phi\left(\frac{b-\mu t}{\sigma\sqrt{t}} - \sigma\sqrt{t}\right) - \Phi\left(\frac{a-\mu t}{\sigma\sqrt{t}} - \sigma\sqrt{t}\right)}\right] \quad \textbf{(6)}$$

3.4 Parity Relations

3.4.1 Put - Call Parity

The Put-Call Parity relation for the B -S model is defined as

$$C_t = S_0 + P_t - Ke^{-rt}$$
$$\Rightarrow C_t - P_t = S_0 - Ke^{-rt}$$

This relation should also hold for the proposed truncated normal distribution model. Hence, from equations (5) and (6), we have that;

$$V_c = S_0 \left[\frac{\Phi\left(\frac{b-\mu t}{\sigma\sqrt{t}} - \sigma\sqrt{t}\right) - \Phi\left(\frac{\log_e\left(\frac{K}{S_0}\right)-\mu t}{\sigma\sqrt{t}} - \sigma\sqrt{t}\right)}{\Phi\left(\frac{b-\mu t}{\sigma\sqrt{t}} - \sigma\sqrt{t}\right) - \Phi\left(\frac{a-\mu t}{\sigma\sqrt{t}} - \sigma\sqrt{t}\right)} \right] - Ke^{-rt} \left[\frac{\Phi\left(\frac{b-\mu t}{\sigma\sqrt{t}}\right) - \Phi\left(\frac{\log_e\left(\frac{K}{S_0}\right)-\mu t}{\sigma\sqrt{t}}\right)}{\Phi\left(\frac{b-\mu t}{\sigma\sqrt{t}}\right) - \Phi\left(\frac{a-\mu t}{\sigma\sqrt{t}}\right)} \right]$$

$$V_p = Ke^{-rt} \left[\frac{\Phi\left(\frac{\log_e\left(\frac{K}{S_0}\right)-\mu t}{\sigma\sqrt{t}}\right) - \Phi\left(\frac{a-\mu t}{\sigma\sqrt{t}}\right)}{\Phi\left(\frac{b-\mu t}{\sigma\sqrt{t}}\right) - \Phi\left(\frac{a-\mu t}{\sigma\sqrt{t}}\right)} \right] - S_0 \left[\frac{\Phi\left(\frac{\log_e\left(\frac{K}{S_0}\right)-\mu t}{\sigma\sqrt{t}} - \sigma\sqrt{t}\right) - \Phi\left(\frac{a-\mu t}{\sigma\sqrt{t}} - \sigma\sqrt{t}\right)}{\Phi\left(\frac{b-\mu t}{\sigma\sqrt{t}} - \sigma\sqrt{t}\right) - \Phi\left(\frac{a-\mu t}{\sigma\sqrt{t}} - \sigma\sqrt{t}\right)} \right]$$

Thus

$$\begin{aligned}
V_c - V_p &= S_0 \left[\frac{\Phi\left[\frac{b-\mu t}{\sigma\sqrt{t}} - \sigma\sqrt{t}\right] - \Phi\left[\frac{\log_e(\frac{K}{S_0})-\mu t}{\sigma\sqrt{t}} - \sigma\sqrt{t}\right]}{\Phi\left(\frac{b-\mu t}{\sigma\sqrt{t}} - \sigma\sqrt{t}\right) - \Phi\left[\frac{a-\mu t}{\sigma\sqrt{t}} - \sigma\sqrt{t}\right]} + \frac{\Phi\left[\frac{\log_e(\frac{K}{S_0})-\mu t}{\sigma\sqrt{t}} - \sigma\sqrt{t}\right] - \Phi\left[\frac{a-\mu t}{\sigma\sqrt{t}} - \sigma\sqrt{t}\right]}{\Phi\left[\frac{b-\mu t}{\sigma\sqrt{t}} - \sigma\sqrt{t}\right] - \Phi\left[\frac{a-\mu t}{\sigma\sqrt{t}} - \sigma\sqrt{t}\right]} \right] \\
&\quad - Ke^{-rt} \left[\frac{\Phi\left(\frac{b-\mu t}{\sigma\sqrt{t}}\right) - \Phi\left(\frac{\log_e\left(\frac{K}{S_0}\right)-\mu t}{\sigma\sqrt{t}}\right)}{\Phi\left(\frac{b-\mu t}{\sigma\sqrt{t}}\right) - \Phi\left(\frac{a-\mu t}{\sigma\sqrt{t}}\right)} + \frac{\Phi\left(\frac{\log_e\left(\frac{K}{S_0}\right)-\mu t}{\sigma\sqrt{t}}\right) - \Phi\left(\frac{a-\mu t}{\sigma\sqrt{t}}\right)}{\Phi\left(\frac{b-\mu t}{\sigma\sqrt{t}}\right) - \Phi\left(\frac{a-\mu t}{\sigma\sqrt{t}}\right)} \right] \\
&= S_0 \left[\frac{\Phi\left[\frac{b-\mu t}{\sigma\sqrt{t}} - \sigma\sqrt{t}\right] - \Phi\left[\frac{\log_e\left(\frac{K}{S_0}\right)-\mu t}{\sigma\sqrt{t}} - \sigma\sqrt{t}\right] + \Phi\left[\frac{\log_e\left(\frac{K}{S_0}\right)-\mu t}{\sigma\sqrt{t}} - \sigma\sqrt{t}\right] - \Phi\left[\frac{a-\mu t}{\sigma\sqrt{t}} - \sigma\sqrt{t}\right]}{\Phi\left[\frac{b-\mu t}{\sigma\sqrt{t}} - \sigma\sqrt{t}\right] - \Phi\left[\frac{a-\mu t}{\sigma\sqrt{t}} - \sigma\sqrt{t}\right]} \right] \\
&\quad - Ke^{-rt} \left[\frac{\Phi\left(\frac{b-\mu t}{\sigma\sqrt{t}}\right) - \Phi\left(\frac{\log_e(\frac{K}{S_0})-\mu t}{\sigma\sqrt{t}}\right) + \Phi\left(\frac{\log_e(\frac{K}{S_0})-\mu t}{\sigma\sqrt{t}}\right) - \Phi\left(\frac{a-\mu t}{\sigma\sqrt{t}}\right)}{\Phi\left(\frac{b-\mu t}{\sigma\sqrt{t}}\right) - \Phi\left(\frac{a-\mu t}{\sigma\sqrt{t}}\right)} \right] \\
&= S_0 \left[\frac{\Phi\left(\frac{b-\mu t}{\sigma\sqrt{t}} - \sigma\sqrt{t}\right) - \Phi\left(\frac{a-\mu t}{\sigma\sqrt{t}} - \sigma\sqrt{t}\right)}{\Phi\left(\frac{b-\mu t}{\sigma\sqrt{t}} - \sigma\sqrt{t}\right) - \Phi\left(\frac{a-\mu t}{\sigma\sqrt{t}} - \sigma\sqrt{t}\right)} \right] - Ke^{-rt} \left[\frac{\Phi\left(\frac{b-\mu t}{\sigma\sqrt{t}}\right) - \Phi\left(\frac{a-\mu t}{\sigma\sqrt{t}}\right)}{\Phi\left(\frac{b-\mu t}{\sigma\sqrt{t}}\right) - \Phi\left(\frac{a-\mu t}{\sigma\sqrt{t}}\right)} \right] \\
&= S_0 - Ke^{-rt}
\end{aligned}$$

$$V_c - V_p = S_0 - Ke^{-rt} \tag{7}$$

Therefore, the put-call parity holds for the truncated normal distribution as it does with the B-S model.

3.4.2 Put - Call Duality

The Classical Black & Scholes Model (B-S)

Theorem 3.4.1. *The put - call duality holds for the Black & Scholes model*

Proof: The European Call under the B-S model assumption is given by;

$$C(t,S_t) = S_t\Phi(d_1) - Ke^{-rt}\Phi(d_2) \qquad \text{where,}$$
$$d_1 = \frac{\log_e\left(\frac{S_t}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)t}{\sigma\sqrt{t}} \qquad \text{and,}$$
$$d_2 = d_1 - \sigma\sqrt{t}$$

Define $\tilde{S}_t = -S_t, \tilde{K} = -K$ and $\tilde{\sigma} = -\sigma$, the put - call duality for the B-S model is given by;

$$C(t,\tilde{S}_t) = \tilde{S}_t\Phi\left[\frac{\log_e\left(\frac{\tilde{S}_t}{\tilde{K}}\right) + \left(r + \frac{\tilde{\sigma}^2}{2}\right)t}{\tilde{\sigma}\sqrt{t}}\right] - \tilde{K}e^{-rt}\Phi\left[\frac{\log_e\left(\frac{\tilde{S}_t}{\tilde{K}}\right) + \left(r + \frac{\tilde{\sigma}^2}{2}\right)t}{\tilde{\sigma}\sqrt{t}} - \tilde{\sigma}\sqrt{t}\right]$$

Define

$$\tilde{d_1} = \frac{\log_e\left(\frac{\tilde{S}_t}{\tilde{K}}\right) + \left(r + \frac{\tilde{\sigma}^2}{2}\right)t}{\sigma\sqrt{t}} \qquad \text{and,}$$
$$\tilde{d_2} = \frac{\log_e\left(\frac{\tilde{S}_t}{\tilde{K}}\right) + \left(r + \frac{\tilde{\sigma}^2}{2}\right)t}{\tilde{\sigma}\sqrt{t}} - \tilde{\sigma}\sqrt{t}$$
$$= \tilde{d_1} - \tilde{\sigma}\sqrt{t}$$

Then, the call value is given by

$$C(t,\tilde{S}_t) = \tilde{S}_t\Phi(\tilde{d_1}) + \tilde{K}e^{-rt}\Phi(\tilde{d_2})$$

Since $\tilde{S}_t = -S_t, \tilde{K} = -K$ and $\tilde{\sigma} = -\sigma$, replacing them in the equation we have

$$\tilde{d_1} = \frac{\log_e\left(\frac{-S_t}{-K}\right) + \left(r + \frac{(-\sigma)^2}{2}\right)t}{-\sigma\sqrt{t}}$$
$$= -\frac{\log_e\left(\frac{S_t}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)t}{\sigma\sqrt{t}} = -d_1$$
$$\tilde{d_2} = -d_1 - (-\sigma)\sqrt{t}$$
$$= -d_1 + \sigma\sqrt{t}$$
$$= -\left(d_1 - \sigma\sqrt{t}\right) = -d_2$$
$$\Rightarrow C(t,-S_t) = -S_t\Phi(-d_1) + Ke^{-rt}\Phi(-d_2)$$

Therefore, the put - call duality under the B - S model is given as;

$$\begin{aligned} C(t,-S_t) &= Ke^{-rt}\Phi(-d_2) - S_t\Phi(-d_1) \\ &= Ke^{-rt}[1-\Phi(d_2)] - S_t[1-\Phi(d_1)] \\ &= S_t\Phi(d_1) - Ke^{-rt}\Phi(d_2) + \left[Ke^{-rt} - S_t\right] \\ &= P(t,S_t) \end{aligned}$$

Hence, the duality relation holds for the classical Black & Scholes model.

Truncated Normal Distribution

The European Call Option price under the truncated normal distribution is given by;

$$\begin{aligned} V_c = S_0 &\left[\frac{\Phi\left(\frac{b-\mu t}{\sigma\sqrt{t}} - \sigma\sqrt{t}\right) - \Phi\left(\frac{\log_e(\frac{K}{S_0})-\mu t}{\sigma\sqrt{t}} - \sigma\sqrt{t}\right)}{\Phi\left(\frac{b-\mu t}{\sigma\sqrt{t}} - \sigma\sqrt{t}\right) - \Phi\left(\frac{a-\mu t}{\sigma\sqrt{t}} - \sigma\sqrt{t}\right)}\right] \\ - Ke^{-rt} &\left[\frac{\Phi\left(\frac{b-\mu t}{\sigma\sqrt{t}}\right) - \Phi\left(\frac{\log_e(\frac{K}{S_0})-\mu t}{\sigma\sqrt{t}}\right)}{\Phi\left(\frac{b-\mu t}{\sigma\sqrt{t}}\right) - \Phi\left(\frac{a-\mu t}{\sigma\sqrt{t}}\right)}\right] \end{aligned}$$

Using the same approach as was with the B - S model above and by substituting S_0, K, b, a and σ by $-S_0, -K, -b, -a$ and $-\sigma$ respectively, the put - call duality under the TND model is given by;

$$\begin{aligned}
\tilde{V}_c &= -S_0 \left[\frac{\Phi\left[\frac{-b-\mu t}{(-\sigma)\sqrt{t}} - (-\sigma)\sqrt{t}\right] - \Phi\left[\frac{\log_e\left(\frac{-K}{-S_0}\right)-\mu t}{(-\sigma)\sqrt{t}} - (-\sigma)\sqrt{t}\right]}{\Phi\left(\frac{-b-\mu t}{(-\sigma)\sqrt{t}} - (-\sigma)\sqrt{t}\right) - \Phi\left(\frac{-a-\mu t}{(-\sigma)\sqrt{t}} - (-\sigma)\sqrt{t}\right)} \right] - (-K)e^{-rt} \left[\frac{\Phi\left[\frac{-b-\mu t}{(-\sigma)\sqrt{t}}\right] - \Phi\left[\frac{\log_e\left(\frac{-K}{-S_0}\right)-\mu t}{(-\sigma)\sqrt{t}}\right]}{\Phi\left(\frac{-b-\mu t}{(-\sigma)\sqrt{t}}\right) - \Phi\left(\frac{-a-\mu t}{(-\sigma)\sqrt{t}}\right)} \right] \\
&= -S_0 \left[\frac{\Phi\left[\frac{-b-\mu t}{(-\sigma)\sqrt{t}} + \sigma\sqrt{t}\right] - \Phi\left[\frac{\log_e\left(\frac{K}{S_0}\right)-\mu t}{(-\sigma)\sqrt{t}} + \sigma\sqrt{t}\right]}{\Phi\left[\frac{-b-\mu t}{(-\sigma)\sqrt{t}} + \sigma\sqrt{t}\right] - \Phi\left[\frac{-a-\mu t}{(-\sigma)\sqrt{t}} + \sigma\sqrt{t}\right]} \right] + Ke^{-rt} \left[\frac{\Phi\left(\frac{-b-\mu t}{(-\sigma)\sqrt{t}}\right) - \Phi\left(\frac{\log_e\left(\frac{K}{S_0}\right)-\mu t}{(-\sigma)\sqrt{t}}\right)}{\Phi\left(\frac{-b-\mu t}{(-\sigma)\sqrt{t}}\right) - \Phi\left(\frac{-a-\mu t}{(-\sigma)\sqrt{t}}\right)} \right] \\
&= -S_0 \left[\frac{\Phi\left(-\left[\frac{-b-\mu t}{\sigma\sqrt{t}} - \sigma\sqrt{t}\right]\right) - \Phi\left(-\left[\frac{\log_e\left(\frac{K}{S_0}\right)-\mu t}{\sigma\sqrt{t}} - \sigma\sqrt{t}\right]\right)}{\Phi\left(-\left[\frac{-b-\mu t}{\sigma\sqrt{t}} - \sigma\sqrt{t}\right]\right) - \Phi\left(-\left[\frac{-a-\mu t}{\sigma\sqrt{t}} - \sigma\sqrt{t}\right]\right)} \right] \\
&\quad + Ke^{-rt} \left[\frac{\Phi\left(-\left[\frac{-b-\mu t}{\sigma\sqrt{t}}\right]\right) - \Phi\left(-\left[\frac{\log_e\left(\frac{K}{S_0}\right)-\mu t}{\sigma\sqrt{t}}\right]\right)}{\Phi\left(-\left[\frac{-b-\mu t}{\sigma\sqrt{t}}\right]\right) - \Phi\left(-\left[\frac{-a-\mu t}{\sigma\sqrt{t}}\right]\right)} \right]
\end{aligned}$$

Considering that $\Phi(-x) = 1 - \Phi(x)$ and;

$$\begin{aligned}
\Phi(-u) &= \int_u^\infty \phi(v)dv \\
&= \int_{-\infty}^\infty \phi(v)dv - \int_{-\infty}^u \phi(v)dv \\
&= 1 - \Phi(u), \qquad \text{since}
\end{aligned}$$

$$\begin{aligned}
\int_{-\infty}^\infty \phi(v)dv &= 1 \\
\int_{-\infty}^u \phi(v)dv &= \Phi(u) \\
\int_u^\infty \phi(v)dv &= \int_{-\infty}^{-u} \phi(v)dv = \Phi(-u) \qquad \text{and} \\
\phi(v) &= \phi(-v)
\end{aligned}$$

Then

$$\begin{aligned}
\tilde{V}_c &= -S_0 \left[\frac{1 - \Phi\left[\frac{-b-\mu t}{\sigma\sqrt{t}} - \sigma\sqrt{t}\right] - 1 + \Phi\left[\frac{\log_e(\frac{K}{S_0})-\mu t}{\sigma\sqrt{t}} - \sigma\sqrt{t}\right]}{1 - \Phi\left[\frac{-b-\mu t}{\sigma\sqrt{t}} - \sigma\sqrt{t}\right] - 1 + \Phi\left[\frac{-a-\mu t}{\sigma\sqrt{t}} - \sigma\sqrt{t}\right]} \right] + Ke^{-rt} \left[\frac{1 - \Phi\left[\frac{-b-\mu t}{\sigma\sqrt{t}}\right] - 1 + \Phi\left[\frac{\log_e(\frac{K}{S_0})-\mu t}{\sigma\sqrt{t}}\right]}{1 - \Phi\left[\frac{-b-\mu t}{\sigma\sqrt{t}}\right] - 1 + \Phi\left[\frac{-a-\mu t}{\sigma\sqrt{t}}\right]} \right] \\
&= -S_0 \left[\frac{\Phi\left(\frac{\log_e(\frac{K}{S_0})-\mu t}{\sigma\sqrt{t}} - \sigma\sqrt{t}\right) - \Phi\left(\frac{-b-\mu t}{\sigma\sqrt{t}} - \sigma\sqrt{t}\right)}{\Phi\left(\frac{-a-\mu t}{\sigma\sqrt{t}} - \sigma\sqrt{t}\right) - \Phi\left(\frac{-b-\mu t}{\sigma\sqrt{t}} - \sigma\sqrt{t}\right)} \right] + Ke^{-rt} \left[\frac{\Phi\left(\frac{\log_e(\frac{K}{S_0})-\mu t}{\sigma\sqrt{t}}\right) - \Phi\left(\frac{-b-\mu t}{\sigma\sqrt{t}}\right)}{\Phi\left(\frac{-a-\mu t}{\sigma\sqrt{t}}\right) - \Phi\left(\frac{-b-\mu t}{\sigma\sqrt{t}}\right)} \right]
\end{aligned}$$

Assuming that $-a = b$, implying that $-b = a$, we have then that;

$$\tilde{V}_c = -S_0 \left[\frac{\Phi\left(\frac{\log_e\left(\frac{K}{S_0}\right)-\mu t}{\sigma\sqrt{t}} - \sigma\sqrt{t}\right) - \Phi\left(\frac{a-\mu t}{\sigma\sqrt{t}} - \sigma\sqrt{t}\right)}{\Phi\left(\frac{b-\mu t}{\sigma\sqrt{t}} - \sigma\sqrt{t}\right) - \Phi\left(\frac{a-\mu t}{\sigma\sqrt{t}} - \sigma\sqrt{t}\right)} \right] + Ke^{-rt} \left[\frac{\Phi\left(\frac{\log_e\left(\frac{K}{S_0}\right)-\mu t}{\sigma\sqrt{t}}\right) - \Phi\left(\frac{a-\mu t}{\sigma\sqrt{t}}\right)}{\Phi\left(\frac{b-\mu t}{\sigma\sqrt{t}}\right) - \Phi\left(\frac{a-\mu t}{\sigma\sqrt{t}}\right)} \right]$$

Thus, the duality relation for the truncated normal distribution is;

$$\begin{aligned}\tilde{V}_c &= Ke^{-rt} \left[\frac{\Phi\left(\frac{\log_e\left(\frac{K}{S_0}\right)-\mu t}{\sigma\sqrt{t}}\right) - \Phi\left(\frac{a-\mu t}{\sigma\sqrt{t}}\right)}{\Phi\left(\frac{b-\mu t}{\sigma\sqrt{t}}\right) - \Phi\left(\frac{a-\mu t}{\sigma\sqrt{t}}\right)} \right] - S_0 \left[\frac{\Phi\left(\frac{\log_e\left(\frac{K}{S_0}\right)-\mu t}{\sigma\sqrt{t}} - \sigma\sqrt{t}\right) - \Phi\left(\frac{a-\mu t}{\sigma\sqrt{t}} - \sigma\sqrt{t}\right)}{\Phi\left(\frac{b-\mu t}{\sigma\sqrt{t}} - \sigma\sqrt{t}\right) - \Phi\left(\frac{a-\mu t}{\sigma\sqrt{t}} - \sigma\sqrt{t}\right)} \right] \\ &= V_p\end{aligned}$$

Therefore, the put-call duality relation hold for the truncated normal distribution as it does with the B-S model with the assumption that $-a = b$ and $-b = a$.

3.5 Truncated Normal Distribution Model Greeks

Greeks can be defined as the statistical measures that presents the sensitivity of the option price to either a change in value of a state variable or parameter. They are used to showcase the various dimensions of the risk of an option. In this subsection, we will derive three Greeks with respect to our model (TND). They include:

- Delta, Δ
- Gamma, Γ
- Rho, ρ

The Sensitivity of the option's strike price K with respect to the underlying asset prices is also determined.

3.5.1 Delta, Δ

Delta, Δ is the sensitivity of the option price to changes in the stock price of the underlying asset. It is determined as $\frac{\partial V_c}{\partial S_0}$, Hence, given that

$$V_c = S_0 \left[\frac{\Phi\left[\frac{b-\mu t}{\sigma\sqrt{t}} - \sigma\sqrt{t}\right] - \Phi\left[\frac{\log_e(\frac{K}{S_0})-\mu t}{\sigma\sqrt{t}} - \sigma\sqrt{t}\right]}{\Phi\left[\frac{b-\mu t}{\sigma\sqrt{t}} - \sigma\sqrt{t}\right] - \Phi\left[\frac{a-\mu t}{\sigma\sqrt{t}} - \sigma\sqrt{t}\right]} \right] - Ke^{-rt} \left[\frac{\Phi\left[\frac{b-\mu t}{\sigma\sqrt{t}}\right] - \Phi\left[\frac{\log_e(\frac{K}{S_0})-\mu t}{\sigma\sqrt{t}}\right]}{\Phi\left[\frac{b-\mu t}{\sigma\sqrt{t}}\right] - \Phi\left[\frac{a-\mu t}{\sigma\sqrt{t}}\right]} \right]$$

Let $u = \left(\frac{\log_e(\frac{K}{S_0})-\mu t}{\sigma\sqrt{t}} - \sigma\sqrt{t}\right), V = \frac{K}{S_0} \Rightarrow u = \left(\frac{\log_e(V)-\mu t}{\sigma\sqrt{t}} - \sigma\sqrt{t}\right)$ Considering that $\frac{\partial}{\partial u}\Phi(u) = \phi(u)$, we have

$$\frac{\partial V}{\partial S_0} = \frac{\partial}{\partial S_0}\left(\frac{K}{S_0}\right) = -\frac{K}{S_0^2}; \quad \frac{\partial u}{\partial V} = \frac{\partial}{\partial V}\left(\frac{\log_e(V)-\mu t}{\sigma\sqrt{t}} - \sigma\sqrt{t}\right) = \frac{1}{V\sigma\sqrt{t}}$$

$$\begin{aligned}
\frac{\partial}{\partial S_0} &= \frac{\partial}{\partial u}\frac{\partial u}{\partial V}\frac{\partial V}{\partial S_0} \\
&= \phi(u)\frac{1}{V\sigma\sqrt{t}}\left(-\frac{K}{S_0^2}\right) \\
&= -\phi\left(\frac{\log_e(\frac{K}{S_0})-\mu t}{\sigma\sqrt{t}} - \sigma\sqrt{t}\right)\frac{S_0}{K\sigma\sqrt{t}}\frac{K}{S_0^2} \\
&= -\frac{1}{S_0\sigma\sqrt{t}}\phi\left(\frac{\log_e(\frac{K}{S_0})-\mu t}{\sigma\sqrt{t}} - \sigma\sqrt{t}\right)
\end{aligned}$$

Thus,the Delta is determine as

$$\begin{aligned}
\frac{\partial V_c}{\partial S_0} = &\left[\frac{\Phi\left[\frac{b-\mu t}{\sigma\sqrt{t}} - \sigma\sqrt{t}\right] - \Phi\left[\frac{\log_e(\frac{K}{S_0})-\mu t}{\sigma\sqrt{t}} - \sigma\sqrt{t}\right]}{\Phi\left[\frac{b-\mu t}{\sigma\sqrt{t}} - \sigma\sqrt{t}\right] - \Phi\left[\frac{a-\mu t}{\sigma\sqrt{t}} - \sigma\sqrt{t}\right]} \right] + \frac{S_0}{S_0\sigma\sqrt{t}} \left[\frac{\phi\left[\frac{\log_e(\frac{K}{S_0})-\mu t}{\sigma\sqrt{t}} - \sigma\sqrt{t}\right]}{\Phi\left[\frac{b-\mu t}{\sigma\sqrt{t}} - \sigma\sqrt{t}\right] - \Phi\left[\frac{a-\mu t}{\sigma\sqrt{t}} - \sigma\sqrt{t}\right]} \right] \\
&- \frac{Ke^{-rt}}{S_0\sigma\sqrt{t}} \left[\frac{\phi\left[\frac{\log_e(\frac{K}{S_0})-\mu t}{\sigma\sqrt{t}}\right]}{\Phi\left[\frac{b-\mu t}{\sigma\sqrt{t}}\right] - \Phi\left[\frac{a-\mu t}{\sigma\sqrt{t}}\right]} \right]
\end{aligned}$$

Therefore, the Delta of under the Truncated normal distribution is;

$$\Delta = \frac{\partial V_c}{\partial S_0} = \left[\frac{\Phi\left[\frac{b-\mu t}{\sigma\sqrt{t}} - \sigma\sqrt{t}\right] - \Phi\left[\frac{\log_e(\frac{K}{S_0})-\mu t}{\sigma\sqrt{t}} - \sigma\sqrt{t}\right]}{\Phi\left[\frac{b-\mu t}{\sigma\sqrt{t}} - \sigma\sqrt{t}\right] - \Phi\left[\frac{a-\mu t}{\sigma\sqrt{t}} - \sigma\sqrt{t}\right]}\right] + \frac{1}{\sigma\sqrt{t}}\left[\frac{\phi\left[\frac{\log_e(\frac{K}{S_0})-\mu t}{\sigma\sqrt{t}} - \sigma\sqrt{t}\right]}{\Phi\left[\frac{b-\mu t}{\sigma\sqrt{t}} - \sigma\sqrt{t}\right] - \Phi\left[\frac{a-\mu t}{\sigma\sqrt{t}} - \sigma\sqrt{t}\right]}\right]$$
$$- \frac{Ke^{-rt}}{S_0\sigma\sqrt{t}}\left[\frac{\phi\left[\frac{\log_e(\frac{K}{S_0})-\mu t}{\sigma\sqrt{t}}\right]}{\Phi\left[\frac{b-\mu t}{\sigma\sqrt{t}}\right] - \Phi\left[\frac{a-\mu t}{\sigma\sqrt{t}}\right]}\right]$$

3.5.2 Gamma, Γ

The Gamma is a measure of sensitivity of the delta to the changes of the underlying stock price. Considering that $\frac{\partial}{\partial u}\phi(u) = -u\phi(u)$, the Gamma, Γ, is given as $\frac{\partial^2 V_c}{\partial S_0^2}$, i.e;

$$\Gamma = \frac{1}{S_0\sigma\sqrt{t}}\left[\frac{\phi\left[\frac{\log_e(\frac{K}{S_0})-\mu t}{\sigma\sqrt{t}} - \sigma\sqrt{t}\right]}{\Phi\left[\frac{b-\mu t}{\sigma\sqrt{t}} - \sigma\sqrt{t}\right] - \Phi\left[\frac{a-\mu t}{\sigma\sqrt{t}} - \sigma\sqrt{t}\right]}\right] + \frac{1}{S_0\sigma\sqrt{t}}\frac{1}{\sigma\sqrt{t}}\left[\frac{\left(\frac{\log_e(\frac{K}{S_0})-\mu t}{\sigma\sqrt{t}} - \sigma\sqrt{t}\right)\phi\left[\frac{log_e(\frac{K}{S_0})-\mu t}{\sigma\sqrt{t}} - \sigma\sqrt{t}\right]}{\Phi\left[\frac{b-\mu t}{\sigma\sqrt{t}} - \sigma\sqrt{t}\right] - \Phi\left[\frac{a-\mu t}{\sigma\sqrt{t}} - \sigma\sqrt{t}\right]}\right]$$
$$+ \frac{Ke^{-rt}}{S_0^2\sigma\sqrt{t}}\left[\frac{\phi\left[\frac{\log_e(\frac{K}{S_0})-\mu t}{\sigma\sqrt{t}}\right]}{\Phi\left[\frac{b-\mu t}{\sigma\sqrt{t}}\right] - \Phi\left[\frac{a-\mu t}{\sigma\sqrt{t}}\right]}\right] - \frac{Ke^{-rt}}{S_0^2(\sigma\sqrt{t})^2}\left[\frac{\left(\frac{\log_e(\frac{K}{S_0})-\mu t}{\sigma\sqrt{t}}\right)\phi\left[\frac{log_e(\frac{K}{S_0})-\mu t}{\sigma\sqrt{t}}\right]}{\Phi\left[\frac{b-\mu t}{\sigma\sqrt{t}}\right] - \Phi\left[\frac{a-\mu t}{\sigma\sqrt{t}}\right]}\right]$$

$$\Gamma = \frac{1}{S_0\sigma\sqrt{t}}\left[\frac{\phi\left[\frac{\log_e(\frac{K}{S_0})-\mu t}{\sigma\sqrt{t}} - \sigma\sqrt{t}\right]}{\Phi\left[\frac{b-\mu t}{\sigma\sqrt{t}} - \sigma\sqrt{t}\right] - \Phi\left[\frac{a-\mu t}{\sigma\sqrt{t}} - \sigma\sqrt{t}\right]} + \frac{1}{\sigma\sqrt{t}}\frac{\left(\frac{\log_e(\frac{K}{S_0})-\mu t}{\sigma\sqrt{t}} - \sigma\sqrt{t}\right)\phi\left[\frac{\log_e(\frac{K}{S_0})-\mu t}{\sigma\sqrt{t}} - \sigma\sqrt{t}\right]}{\Phi\left[\frac{b-\mu t}{\sigma\sqrt{t}} - \sigma\sqrt{t}\right] - \Phi\left[\frac{a-\mu t}{\sigma\sqrt{t}} - \sigma\sqrt{t}\right]}\right]$$
$$+ \frac{Ke^{-rt}}{S_0^2\sigma\sqrt{t}}\left[\frac{\phi\left[\frac{\log_e(\frac{K}{S_0})-\mu t}{\sigma\sqrt{t}}\right]}{\Phi\left[\frac{b-\mu t}{\sigma\sqrt{t}}\right] - \Phi\left[\frac{a-\mu t}{\sigma\sqrt{t}}\right]} - \frac{1}{\sigma\sqrt{t}}\frac{\left(\frac{\log_e(\frac{K}{S_0})-\mu t}{\sigma\sqrt{t}}\right)\phi\left[\frac{\log_e(\frac{K}{S_0})-\mu t}{\sigma\sqrt{t}}\right]}{\Phi\left[\frac{b-\mu t}{\sigma\sqrt{t}}\right] - \Phi\left[\frac{a-\mu t}{\sigma\sqrt{t}}\right]}\right]$$
$$= \frac{\phi\left[\frac{\log_e(\frac{K}{S_0})-\mu t}{\sigma\sqrt{t}} - \sigma\sqrt{t}\right]}{S_0\sigma\sqrt{t}}\left[\frac{1}{\Phi\left[\frac{b-\mu t}{\sigma\sqrt{t}} - \sigma\sqrt{t}\right] - \Phi\left[\frac{a-\mu t}{\sigma\sqrt{t}} - \sigma\sqrt{t}\right]} + \frac{1}{\sigma\sqrt{t}}\frac{\left(\frac{\log_e(\frac{K}{S_0})-\mu t}{\sigma\sqrt{t}} - \sigma\sqrt{t}\right)}{\Phi\left[\frac{b-\mu t}{\sigma\sqrt{t}} - \sigma\sqrt{t}\right] - \Phi\left[\frac{a-\mu t}{\sigma\sqrt{t}} - \sigma\sqrt{t}\right]}\right]$$
$$+ \frac{\phi\left[\frac{\log_e(\frac{K}{S_0})-\mu t}{\sigma\sqrt{t}}\right]Ke^{-rt}}{S_0^2\sigma\sqrt{t}}\left[\frac{1}{\Phi\left[\frac{b-\mu t}{\sigma\sqrt{t}}\right] - \Phi\left[\frac{a-\mu t}{\sigma\sqrt{t}}\right]} - \frac{1}{\sigma\sqrt{t}}\frac{\left(\frac{\log_e(\frac{K}{S_0})-\mu t}{\sigma\sqrt{t}}\right)}{\Phi\left[\frac{b-\mu t}{\sigma\sqrt{t}}\right] - \Phi\left[\frac{a-\mu t}{\sigma\sqrt{t}}\right]}\right]$$

Thus, the Gamma is given by

$$\Gamma = \frac{\phi\left[\frac{\log_e(\frac{K}{S_0})-\mu t}{\sigma\sqrt{t}}-\sigma\sqrt{t}\right]}{S_0\sigma\sqrt{t}}\left[\frac{\sigma\sqrt{t}+\left(\frac{\log_e(\frac{K}{S_0})-\mu t}{\sigma\sqrt{t}}-\sigma\sqrt{t}\right)}{\sigma\sqrt{t}\left(\Phi\left[\frac{b-\mu t}{\sigma\sqrt{t}}-\sigma\sqrt{t}\right]-\Phi\left[\frac{a-\mu t}{\sigma\sqrt{t}}-\sigma\sqrt{t}\right]\right)}\right]$$
$$+\frac{\phi\left[\frac{\log_e(\frac{K}{S_0})-\mu t}{\sigma\sqrt{t}}\right]Ke^{-rt}}{S_0^2\sigma\sqrt{t}}\left[\frac{\sigma\sqrt{t}-\left(\frac{\log_e(\frac{K}{S_0})-\mu t}{\sigma\sqrt{t}}\right)}{\sigma\sqrt{t}\left(\Phi\left[\frac{b-\mu t}{\sigma\sqrt{t}}\right]-\Phi\left[\frac{a-\mu t}{\sigma\sqrt{t}}\right]\right)}\right]$$

3.5.3 Rho, ρ

The Rho which is the sensitivity of the option price with respect to risk-free interest rate is given as

$$\rho = \frac{\partial V_c}{\partial r} = -(-t)Ke^{-rt}\left[\frac{\Phi\left[\frac{b-\mu t}{\sigma\sqrt{t}}\right]-\Phi\left[\frac{\log_e(\frac{K}{S_0})-\mu t}{\sigma\sqrt{t}}\right]}{\Phi\left[\frac{b-\mu t}{\sigma\sqrt{t}}\right]-\Phi\left[\frac{a-\mu t}{\sigma\sqrt{t}}\right]}\right]$$
$$= tKe^{-rt}\left[\frac{\Phi\left[\frac{b-\mu t}{\sigma\sqrt{t}}\right]-\Phi\left[\frac{\log_e(\frac{K}{S_0})-\mu t}{\sigma\sqrt{t}}\right]}{\Phi\left[\frac{b-\mu t}{\sigma\sqrt{t}}\right]-\Phi\left[\frac{a-\mu t}{\sigma\sqrt{t}}\right]}\right]$$

With Respect K

Consider the sensitivity of the option price with respect to the strike price K. Consider $\Phi\left[\frac{\log_e(\frac{K}{S_0})-\mu t}{\sigma\sqrt{t}}-\sigma\sqrt{t}\right]$ and let $v=\left(\frac{\log_e(\frac{K}{S_0})-\mu t}{\sigma\sqrt{t}}-\sigma\sqrt{t}\right)$, $u=\frac{K}{S_0}$ implying that $v=\left(\frac{\log_e(u)-\mu t}{\sigma\sqrt{t}}-\sigma\sqrt{t}\right)$ then

$$\frac{d}{dv}\Phi(v)=\phi(v);\quad \frac{dv}{du}\left(\frac{log_e(u)-\mu t}{\sigma\sqrt{t}}-\sigma\sqrt{t}\right)=\frac{1}{\sigma\sqrt{t}u};\quad \frac{du}{dK}\left(\frac{K}{S_0}\right)=\frac{1}{S_0}$$
$$\frac{d}{dK}=\frac{d}{dv}\frac{dv}{du}\frac{du}{dK}$$
$$=\phi\left(\frac{\log_e(\frac{K}{S_0})-\mu t}{\sigma\sqrt{t}}-\sigma\sqrt{t}\right)\frac{S_0}{\sigma\sqrt{t}K}\frac{1}{S_0}$$
$$=\frac{1}{\sigma\sqrt{t}K}\phi\left(\frac{\log_e(\frac{K}{S_0})-\mu t}{\sigma\sqrt{t}}-\sigma\sqrt{t}\right)$$

$$\frac{\partial V_c}{\partial K} = S_0\left[\frac{-\frac{1}{\sigma\sqrt{t}K}\phi\left(\frac{\log_e(\frac{K}{S_0})-\mu t}{\sigma\sqrt{t}}-\sigma\sqrt{t}\right)}{\Phi\left(\frac{b-\mu t}{\sigma\sqrt{t}}-\sigma\sqrt{t}\right)-\Phi\left(\frac{a-\mu t}{\sigma\sqrt{t}}-\sigma\sqrt{t}\right)}\right] - e^{-rt}\left[\frac{\Phi\left(\frac{b-\mu t}{\sigma\sqrt{t}}\right)-\Phi\left(\frac{\log_e(\frac{K}{S_0})-\mu t}{\sigma\sqrt{t}}\right)}{\Phi\left(\frac{b-\mu t}{\sigma\sqrt{t}}\right)-\Phi\left(\frac{a-\mu t}{\sigma\sqrt{t}}\right)}\right]$$

$$+Ke^{-rt}\left[\frac{-\frac{1}{\sigma\sqrt{t}K}\phi\left(\frac{\log_e(\frac{K}{S_0})-\mu t}{\sigma\sqrt{t}}\right)}{\Phi\left(\frac{b-\mu t}{\sigma\sqrt{t}}\right)-\Phi\left(\frac{a-\mu t}{\sigma\sqrt{t}}\right)}\right]$$

$$= -S_0\left[\frac{\frac{1}{\sigma\sqrt{t}K}\phi\left(\frac{\log_e(\frac{K}{S_0})-\mu t}{\sigma\sqrt{t}}-\sigma\sqrt{t}\right)}{\Phi\left(\frac{b-\mu t}{\sigma\sqrt{t}}-\sigma\sqrt{t}\right)-\Phi\left(\frac{a-\mu t}{\sigma\sqrt{t}}-\sigma\sqrt{t}\right)}\right] - e^{-rt}\left[\frac{\Phi\left(\frac{b-\mu t}{\sigma\sqrt{t}}\right)-\Phi\left(\frac{\log_e(\frac{K}{S_0})-\mu t}{\sigma\sqrt{t}}\right)}{\Phi\left(\frac{b-\mu t}{\sigma\sqrt{t}}\right)-\Phi\left(\frac{a-\mu t}{\sigma\sqrt{t}}\right)}\right]$$

$$+Ke^{-rt}\left[\frac{\frac{1}{\sigma\sqrt{t}K}\phi\left(\frac{\log_e(\frac{K}{S_0})-\mu t}{\sigma\sqrt{t}}\right)}{\Phi\left(\frac{b-\mu t}{\sigma\sqrt{t}}\right)-\Phi\left(\frac{a-\mu t}{\sigma\sqrt{t}}\right)}\right]$$

Therefore, the sensitivity of the price to strike price, K is

$$\frac{\partial V_c}{\partial K} = Ke^{-rt}\left(\frac{\frac{1}{\sigma\sqrt{t}K}\phi\left[\frac{\log_e(\frac{K}{S_0})-\mu t}{\sigma\sqrt{t}}\right]}{\Phi\left[\frac{b-\mu t}{\sigma\sqrt{t}}\right]-\Phi\left[\frac{a-\mu t}{\sigma\sqrt{t}}\right]}\right) - e^{-rt}\left(\frac{\Phi\left[\frac{b-\mu t}{\sigma\sqrt{t}}\right]-\Phi\left[\frac{\log_e(\frac{K}{S_0})-\mu t}{\sigma\sqrt{t}}\right]}{\Phi\left[\frac{b-\mu t}{\sigma\sqrt{t}}\right]-\Phi\left[\frac{a-\mu t}{\sigma\sqrt{t}}\right]}\right)$$

$$-S_0\left(\frac{\frac{1}{\sigma\sqrt{t}K}\phi\left[\frac{\log_e(\frac{K}{S_0})-\mu t}{\sigma\sqrt{t}}-\sigma\sqrt{t}\right]}{\Phi\left[\frac{b-\mu t}{\sigma\sqrt{t}}-\sigma\sqrt{t}\right]-\Phi\left[\frac{a-\mu t}{\sigma\sqrt{t}}-\sigma\sqrt{t}\right]}\right)$$

$$= \frac{Ke^{-rt}}{\sigma\sqrt{t}K}\left(\frac{\phi\left[\frac{\log_e(\frac{K}{S_0})-\mu t}{\sigma\sqrt{t}}\right]}{\Phi\left[\frac{b-\mu t}{\sigma\sqrt{t}}\right]-\Phi\left[\frac{a-\mu t}{\sigma\sqrt{t}}\right]}\right) - e^{-rt}\left(\frac{\Phi\left[\frac{b-\mu t}{\sigma\sqrt{t}}\right]-\Phi\left[\frac{\log_e(\frac{K}{S_0})-\mu t}{\sigma\sqrt{t}}\right]}{\Phi\left[\frac{b-\mu t}{\sigma\sqrt{t}}\right]-\Phi\left[\frac{a-\mu t}{\sigma\sqrt{t}}\right]}\right)$$

$$-\frac{S_0}{\sigma\sqrt{t}K}\left(\frac{\phi\left[\frac{\log_e(\frac{K}{S_0})-\mu t}{\sigma\sqrt{t}}-\sigma\sqrt{t}\right]}{\Phi\left[\frac{b-\mu t}{\sigma\sqrt{t}}-\sigma\sqrt{t}\right]-\Phi\left[\frac{a-\mu t}{\sigma\sqrt{t}}-\sigma\sqrt{t}\right]}\right)$$

4 Chapter 4: Data Description and Analysis

4.1 Data Description

The data used in the analysis is for three different companies, i.e. Russell 2000 index, Facebook and Apple Corporations with maturity periods of ($t \leq 30$ days and $t > 30$ days for each company. The historical data for the underlying assets consists of 2,516 observations retrieved on February 14, 2020 from yahoo finance (Russell 2000 Historical Data) for the Russell 2000 index. The data is for the past 10-year period ranging from February 16, 2010 to February 12, 2020. The adjusted closing prices were used as the prices of the underlying asset in obtaining the stock's log-returns. The options data for Russell 2000 Index Option (RUT) with 30 days and 83 days to expiry was obtained from Market-Watch (Russell 2000 Options Data) on the same date as the historical data for the underlying asset (i.e. on February 14, 2020). The days to expiry for options were determined using Options Expiration Calender. The expiry dates for the options is March 31, 2020 and June 19, 2020 and consists of 158 and 41 option prices respectively. The risk free interest rate used in the analysis is the one-month and three-month Treasury Bill Rate obtained from the U.S. Dept. of the Treasury (Daily Treasury Yield Curve Rates) on February 14, 2020 respectively. The 1-month and 3-month T-Bills rates are 1.60% and 1.53% - which are used as the risk-free rate in the analysis for the 30-days and 83-days to maturity respectively because it coincides with the expiry date for the options as proposed by Shu [SZ04].

The historical data for Facebook Inc. consists of 1,948 observations for obtained from yahoo finance for 7-year period (May 18, 2012 to February 14,2020). Option data for data for 14 days and 64 days to maturity were obtained from Market-Watch with 36 and 41 observations respectively. The two-month risk-free rate for the 63-day to maturity contract is 1.60%. Apple historical data and option price data were similarly obtained from the yahoo finance and Market-Watch respectively. The 10-year historical data (from October 19, 2009 to October 17, 2020). The option data for Apple Corporation consists of 30-days and 95-days to maturity with 42 observations for all of them. The data was obtained on October 17, 2019 from yahoo finance for historical data and Market-Watch for options data. The risk-free rate is 1.74% and 1.66% for the 30-days and 95-days to maturity options respectively. In both cases,the adjusted closing prices were used while the market call and put were obtained by average the bid and ask prices accordingly. Options data for Russell 2000 Index and Facebook Corporation were retrieved during the COVID-19 Pandemic while that for Apple Inclusion were retrieved before the COVID-19 pandemic. Therefore, it worth interesting to study how the models work for the two scenarios (i.e. before the pandemic and during the pandemic).

4.2 Data Analysis

4.2.1 Estimation of Parameters

The parameters that require to be estimated before the pricing is done are a,b and μ. The upper and lower bounds are determined by the investor's perceived realistic price range of the underlying asset. The value of μ is estimated using the root-finding method (particularly uniroot function) in R-program once the value of a and b have been determined. The volatility (σ) used in the model is the same as that used in Black & Scholes model which is derived from the historical data. The risk-free rate (r) is the 3-month T-Bill rate while t is the option's number of days to maturity. Assuming that the investor estimates the price range of the underlying asset as 10%, 14%, 15% and 20% for the 83-days options and 8%, 10%, 14% and 15% for the 30-day options for Russell 2000 index, then the estimated parameters are as given in Table 1 below.

Table 1. Estimated Parameters: Russell 2000 Index Options

Range	a	b	t	R	Lower	Upper	σ	μ
10%	-0.1053605	0.09531018	83	0.0001903614	1520.442	1858.318	0.01020331	0.0006647598
14%	-0.1508229	0.1310283	83	0.0001903614	1452.867	1925.893	0.01020331	0.0003933144
15%	-0.1625189	0.1397619	83	0.0001903614	1435.973	1942.787	0.01020331	0.0003547773
20%	-0.2231436	0.1823216	83	0.0001903614	1351.504	2027.256	0.01020331	0.0002341286
8%	-0.08338161	0.07696104	30	0.0005333333	1554.23	1824.53	0.01020331	0.001122183
10%	-0.1053605	0.09531018	30	0.0005333333	1520.442	1858.318	0.01020331	0.0008201757
14%	-0.1508229	0.1310283	30	0.0005333333	1452.867	1925.893	0.01020331	0.0005750999
15%	-0.1625189	0.1397619	30	0.0005333333	1435.973	1942.787	0.01020331	0.0005474799

The parameters for Facebook Corporation and Apple Corporation Options are as presented in Table 2 and 3 below.

Table 2. Estimated Parameters: Facebook Corporation Options

Range	a	b	t	R	Lower	Upper	σ	μ
10%	-0.1053605	0.09531018	14	0.001142857	192.762	235.598	0.02249525	0.003341362
15%	-0.1625189	0.1397619	14	0.001142857	182.053	246.307	0.02249525	0.001815905
20%	-0.2231436	0.1823216	14	0.001142857	171.344	257.016	0.02249525	0.001247113
15%	-0.1625189	0.1397619	63	0.0002539683	182.053	246.307	0.02249525	0.001612246
20%	-0.2231436	0.1823216	63	0.0002539683	171.344	257.016	0.02249525	0.001058277
25%	-0.2876821	0.2231436	63	0.0002539683	160.635	267.725	0.02249525	0.0007538312

Table 3. Estimated Parameters: Apple Corporation Options

Range	a	b	t	R	Lower	Upper	σ	μ
10%	-0.1053605	0.09531018	30	0.0005866667	211.752	258.808	0.01636316	0.001887374
15%	-0.1625189	0.1397619	30	0.0005866667	199.988	270.572	0.01636316	0.00101256
20%	-0.2231436	0.1823216	30	0.0005866667	188.224	282.336	0.01636316	0.0006829072
30%	-0.3566749	0.2623643	95	0.0001747368	164.696	305.864	0.01636316	0.0002851128
35%	-0.4307829	0.3001046	95	0.0001747368	152.932	317.628	0.01636316	0.0002018363
40%	-0.5108256	0.3364722	95	0.0001747368	141.168	329.392	0.01636316	0.0001425555

4.2.2 Pricing and Model Comparison

After estimating the parameters that are relevant to the model, the next step is price the put and call options and comparing them to the classical Black & Scholes model to check the performance of the model. The three price deviation scenarios are evaluated separately to access the impact of the different deviations on the model. First, we compute and compare the prices and later estimate the Mean Squared Errors ($MSEs$) of each model and compare them. When the price range is estimated to be 10% the option price payoffs are as given in Figure 1 below. Clearly, it is evident that the model underestimate the market prices call a little lower whereas the B-S model overestimates the market prices far much higher. On the other hand, the put for the model is far much lower while the put for B-S model is slightly lower than the market observed put values for options with 83-days to maturity.

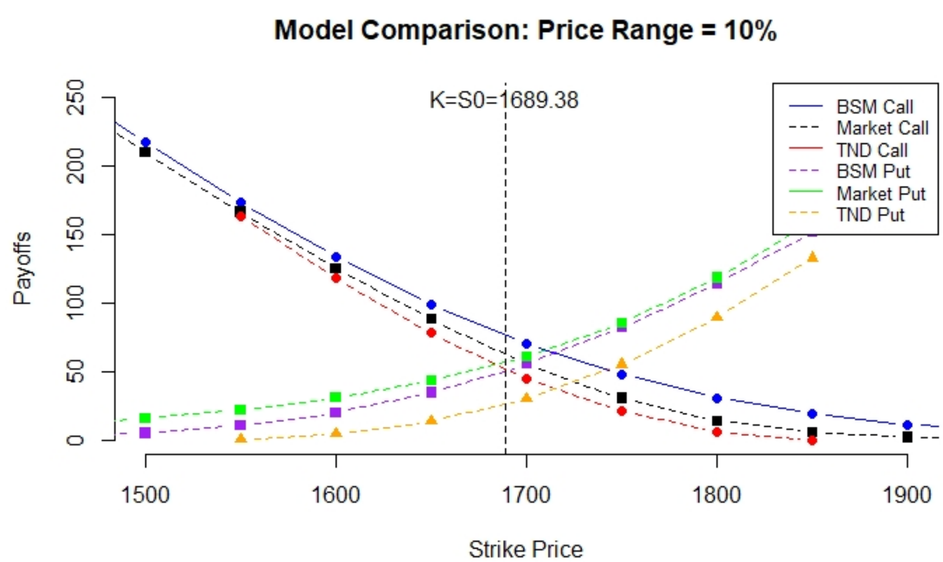

Figure 1. Russell 2000: a = -0.1053605, b = 0.09531018, t = 83 days

Similarly, Figure 2 presents the option prices when the investor's perceived price range is 15%.

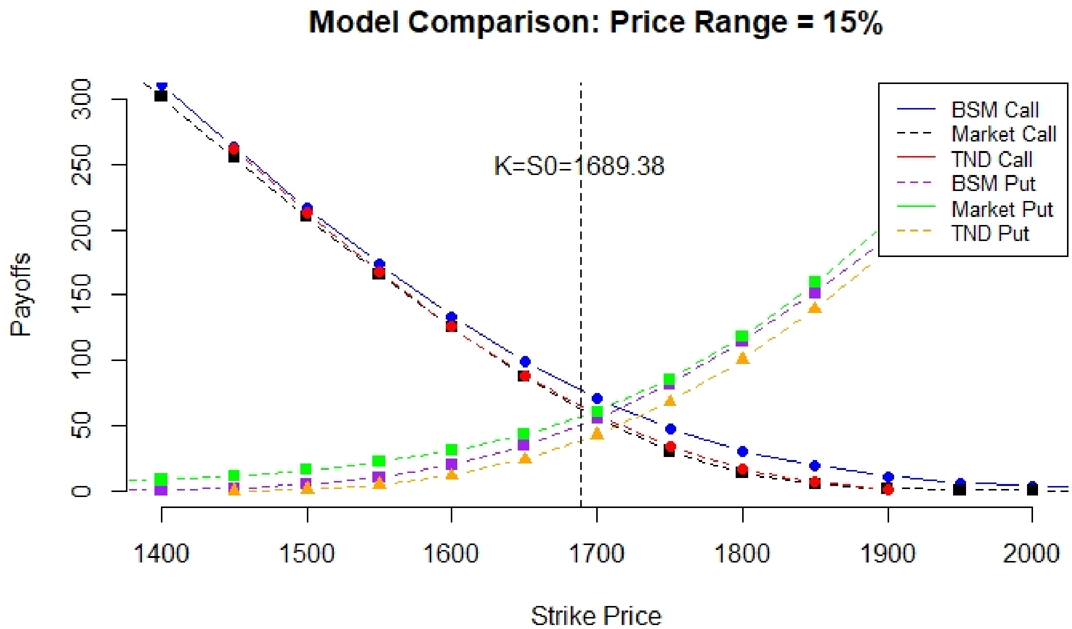

Figure 2. Russell 2000: a = -0.1625189, b = 0.1397619, t = 83 days

As Figure 2 above depicts, the call payoff for the model lies between the B-S model and the observed market values but very close to the observed market call values. The put payoff for the model moves nearer to the B-S put payoff at the extreme ends all of which lies below the observed market put values but lies closer to the model at the point where $S_0 = K = \$1689.38$ where the B-S model payoff is above the market observed value.

Also, from Figure 3 below, the model call payoffs lies between the B-S model payoffs and the observed market values but this time closer to the B-S model payoffs than the observed market values. At the point where $S_0 = K = \$1689.38$, the model tends to the observed market call values than at the lower strike values.

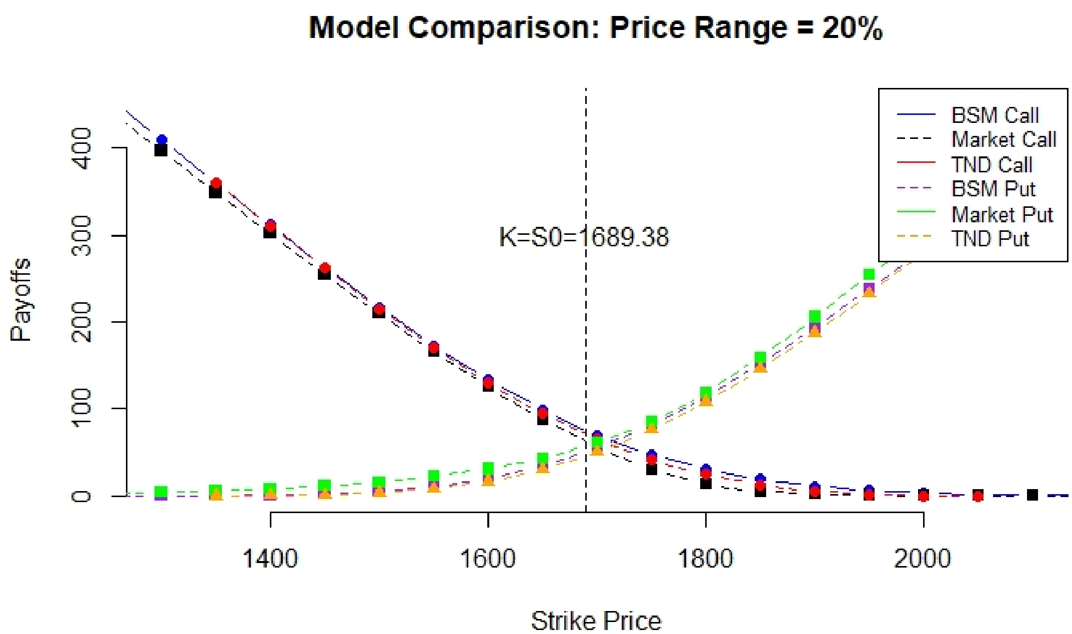

Figure 3. Russell 2000: a = -0.2231436, b = 0.1823216, t = 83 days

Options that have time to maturity of 30 days for Russell 2000 contracts behave in the same manner as those for with different price range than those used for the 83-days to expiry. Using 8% price as the minimum price movement, the TND model offers a better estimate than the B-S model as shown in Figure 4 below.

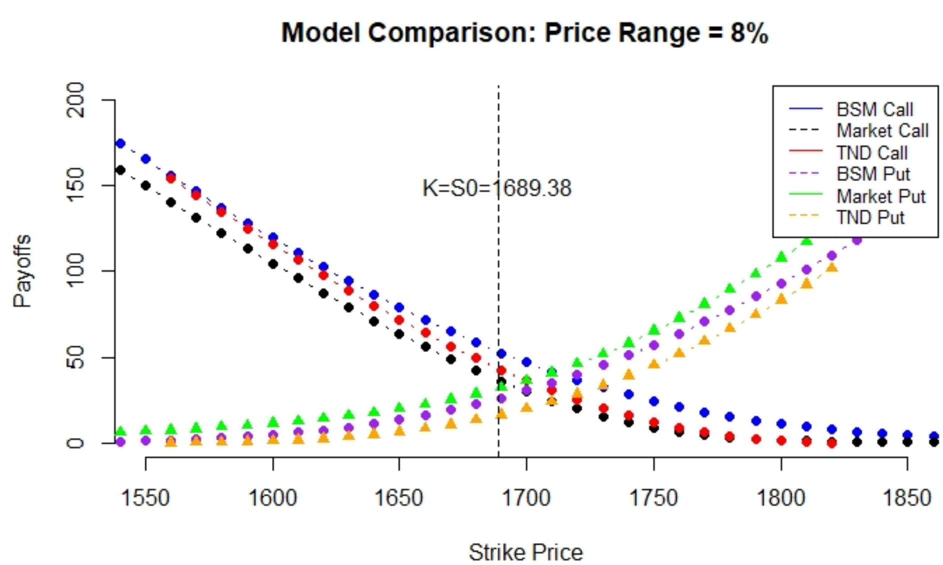

Figure 4. Russell 2000: a = -0.08338161, b = 0.07696104, t = 30 days

Increasing the range to 10%, the TND models slightly deviates upwards from market calls. However, the put prices improves as the models converges to the B-S which also underestimates the put as shown in Figure 5.

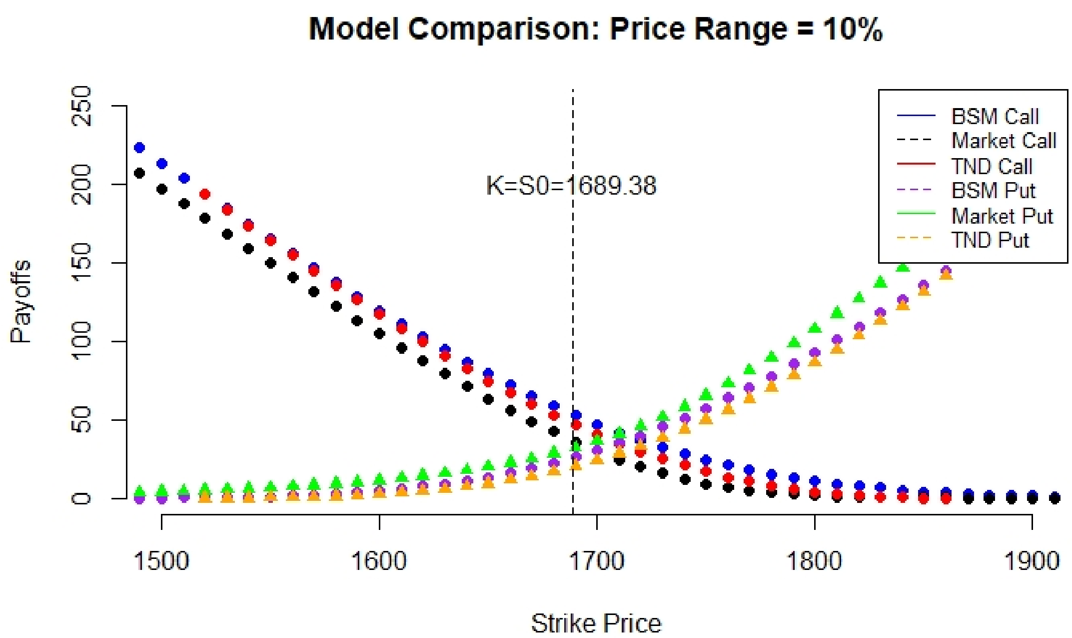

Figure 5. Russell 2000: a = -0.1053605, b = 0.09531018, t = 30 days

When the price range the 30-days to maturity option is set to be 15%, the truncated model estimates shifts towards the Black - Scholes estimates as shown in Figure 6. This phenomenon is similar to the observations for the options with maturity time of 83 days. Therefore, enlarging the price range of the option prices from the current prices results to the TND model mapping the B-S model and decreasing the price range, the truncated model estimates tends towards the observed call values and may lead to underestimations for both the calls and puts.

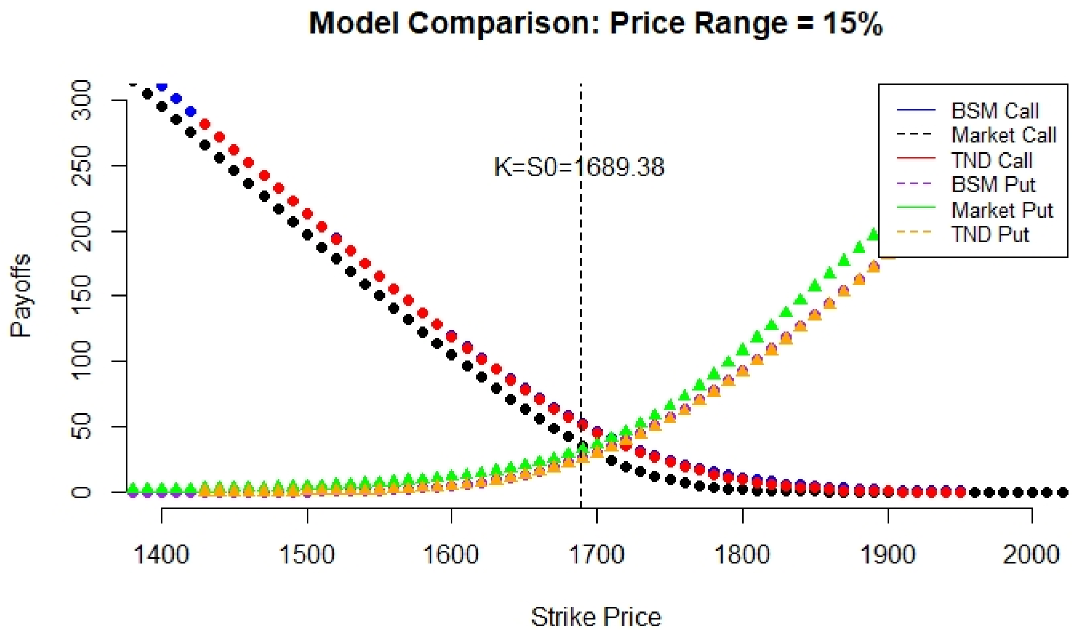

Figure 6. Russell 2000: a = -0.1625189, b = 0.1397619, t = 30 days

The following figure, Figure 7 ,8 and 9, shows the price movement for the Facebook option prices with maturity period of 63 days under the three price range scenarios, i.e 15%, 20% and 25%. A price range below 10% undervalues the market, hence the need to use 15% as the minimum price range. At the price range of 15%, the TND model is better than B-S model. As the range rises to 25%, the model converges to the B-S model.

Figure 7. Facebook: a = -0.1625189, b = 0.1397619, t = 63 days

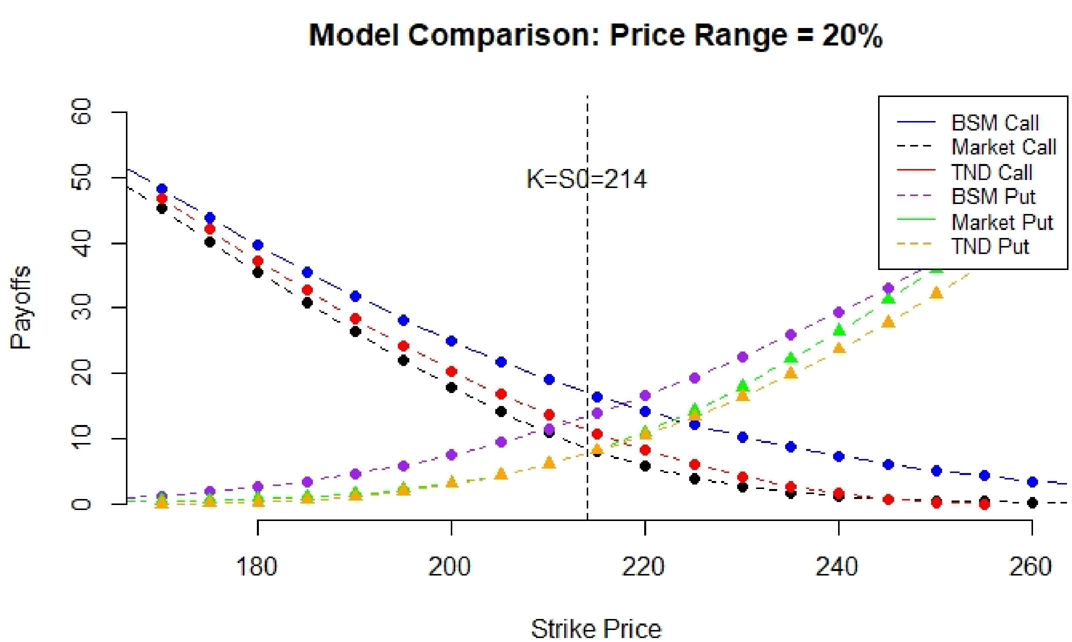

Figure 8. Facebook:a = -0.2231436, b = 0.1823216, t = 63 days

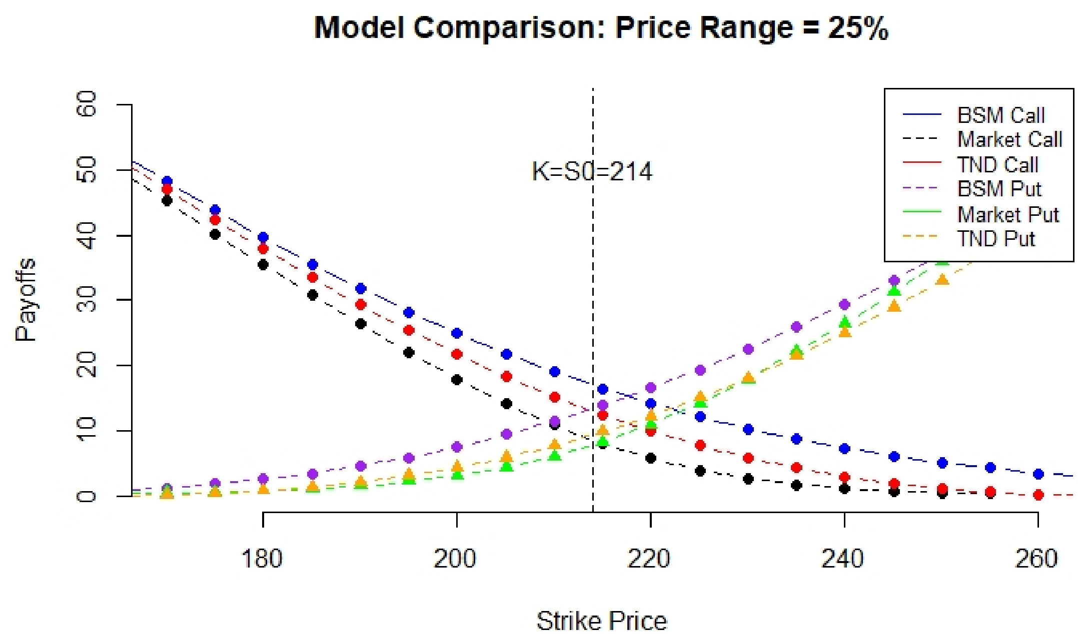

Figure 9. Facebook: a = -0.2876821, b = 0.2231436, t = 63 days

Using the option data that have a maturity period of 14 days, the price of 20% and above is the similar to the B-S model as shown by Figure 12. However, the TND model performs better than the B-S at 15% and 10% below which it underestimates the market as presented in Figures 11 and 10 respectively.

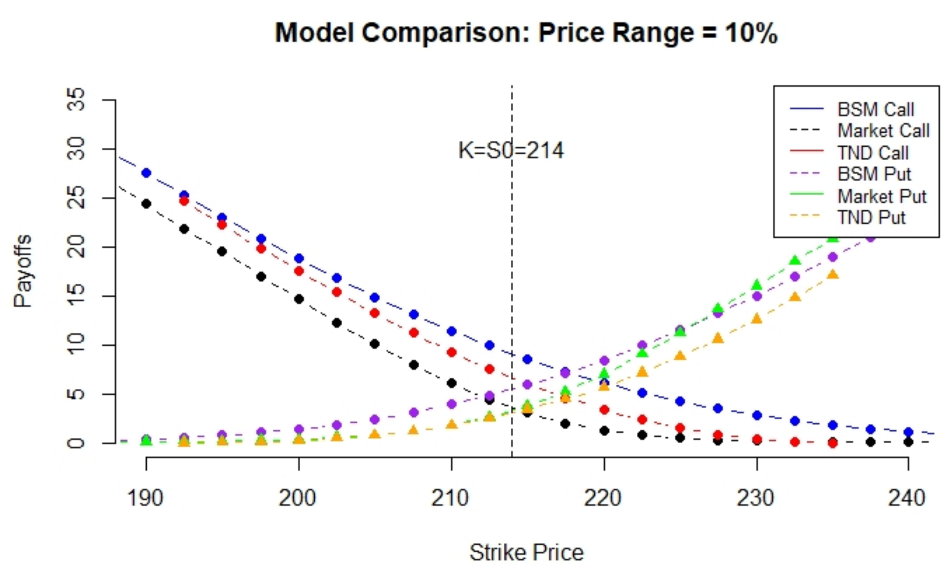

Figure 10. Facebook: a = -0.1053605, b = 0.09531018, t = 14 days

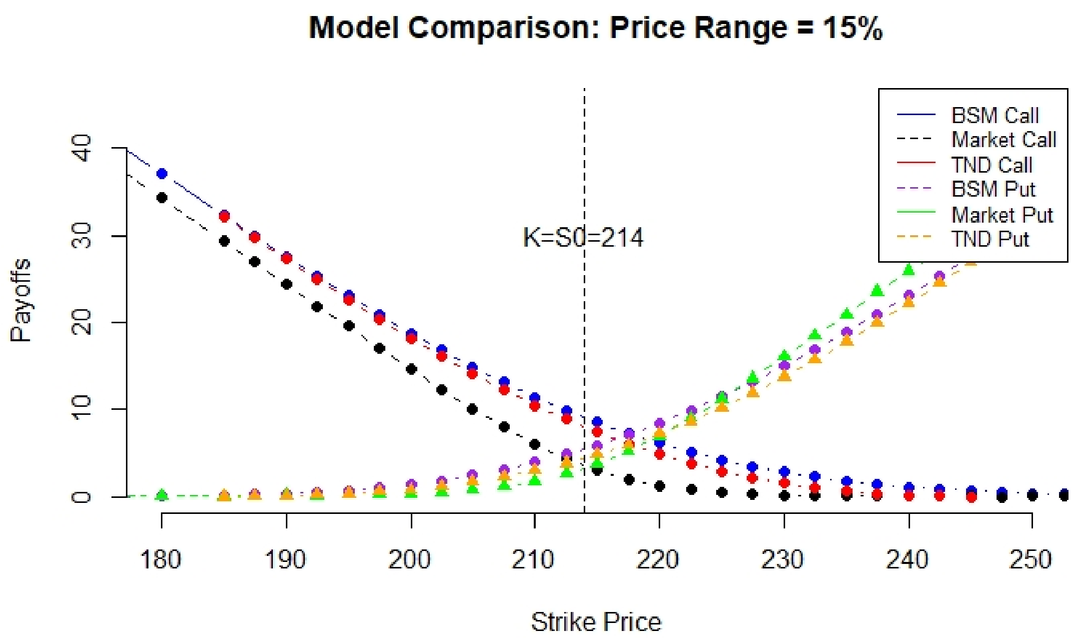

Figure 11. Facebook:a = -0.1625189, b = .1397619, t = 14 days

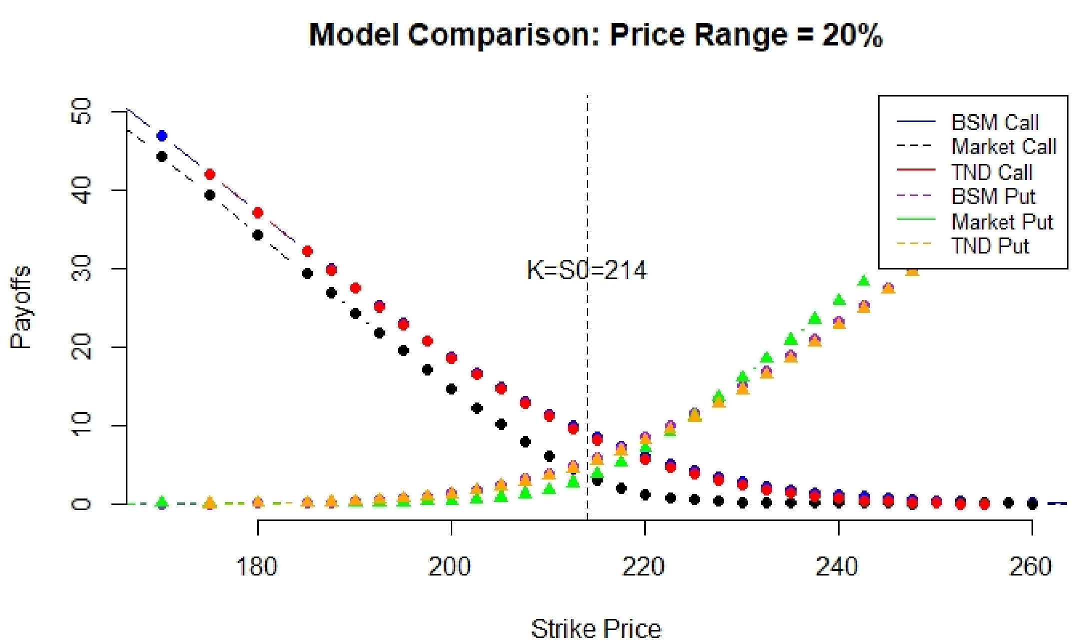

Figure 12. Facebook: a = -0.2231436, b = 0.1823216, t = 14 days

Similarly, the estimated prices for Apple Corporation option data depicts the same trend as it was for Russell 2000 index and Facebook Company. The estimated prices for options with a maturity period of 30 days when the price range is assumed to be 10%, 15% and 20% are shown in Figures 13, 14 and 15 respectively.

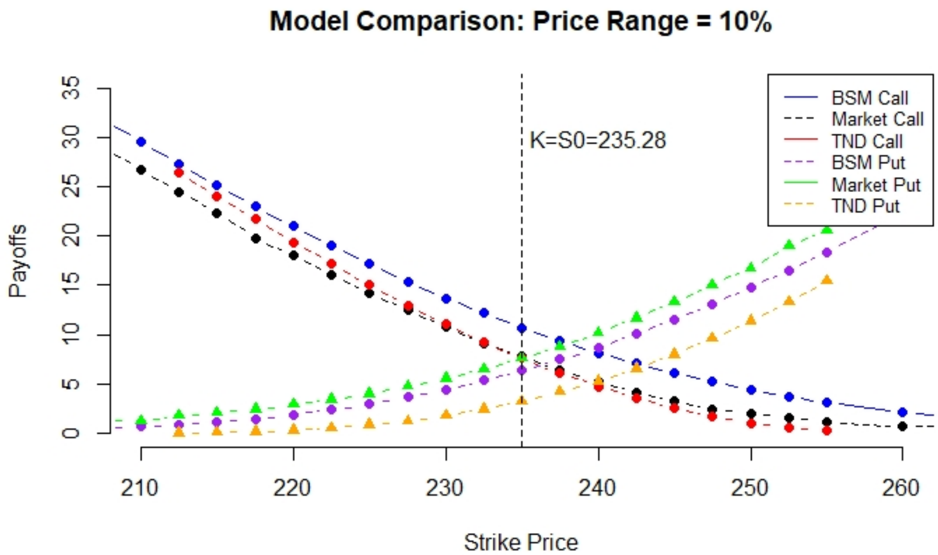

Figure 13. Apple Inc.: a = -0.1053605, b = 0.09531018, t = 30 days

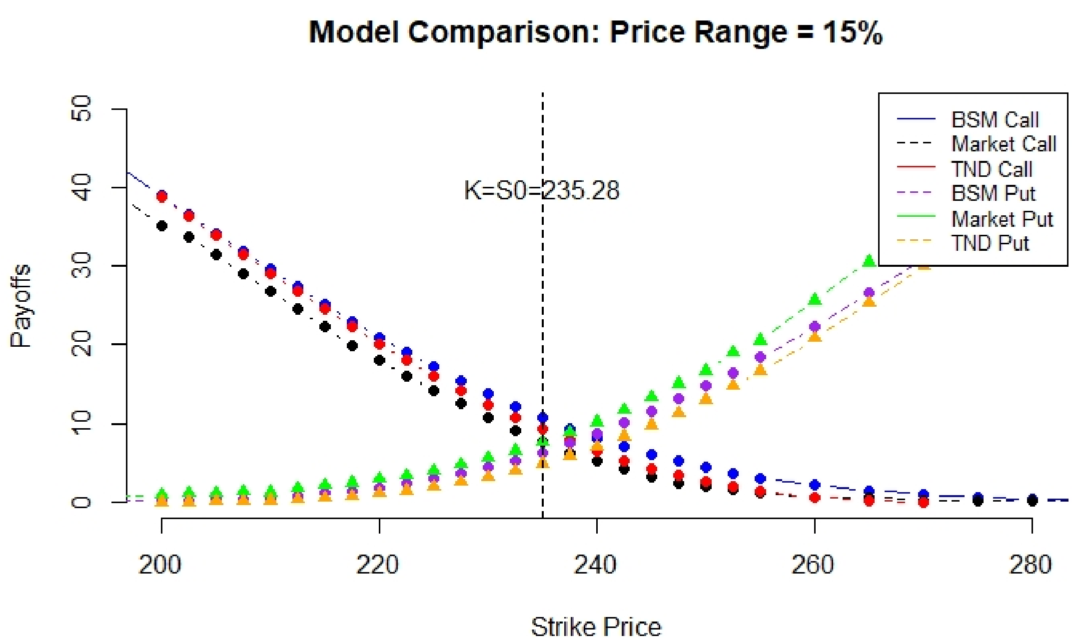

Figure 14. Apple Inc.:a = -0.1625189, b = .1397619, t = 30 days

Figure 15. Apple Inc.: a = -0.2231436, b = 0.1823216, t = 30 days

From the above figures, it is evident the the TND model estimates the market call prices better than the B-S models at 10% and 15% but at 20% price range, the models appears to be similar which is the same scenarios for the other companies.

Let us now consider options with a maturity period of 95 days with 30%, 35% and 40% price ranges. The behavior of the estimated option prices is as shown in Figures 16, 17 and 18 respectively.

Figure 16. Apple Inc.: a = -0.3566749, b = 0.2623643, t = 95 days

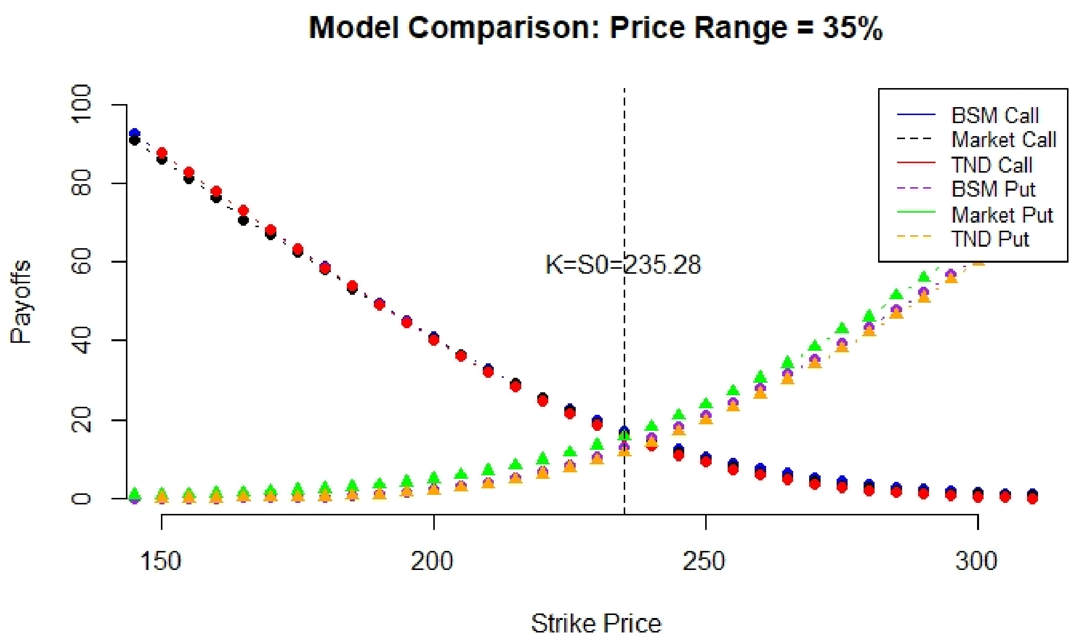

Figure 17. Apple Inc.: a = -0.4307829, b = 0.3001046, t = 95 days

Figure 18. Apple Inc.: a = -0.5108256, b = 0.3364722, t = 95 days

The TND model show same trend in all the three companies involved in this study. It is worth noting that as the value of a and b tends to negative and positive infinity respectively, the TND model converges towards the B-S model. Also, when the time to maturity is 30 days and below, small values of a and b offers a best estimates than the B-S for calls while when the time to maturity of the options is more than 30 days, small values of a and b tends to underestimate the market call values. Market puts are underestimated both by the TND and B-S models for Russell 2000 index and Apple Corporation while TND model does better for Facebook options than B-S model.

4.3 Comparison on the model Greeks

The price sensitivity measures (Greeks) that are discussed in this paper are the call Delta, Gamma and Rho. These Greeks are compared to the those calculated using the B-S model.

4.3.1 Call Delta

The Call Delta is a sensitivity measure of the option price with respect to the changes of the stock (underlying asset) price, S_0 [Cha94]. When the price range is considered to be 10%, the Call Delta for the B-S model is 0.5560 while for the TND is 0.6111 at the current underlying asset price $1,700 (used as the close proximity to the current underlying asset price $1,689.38). This comparison is shown in Figure 19 below

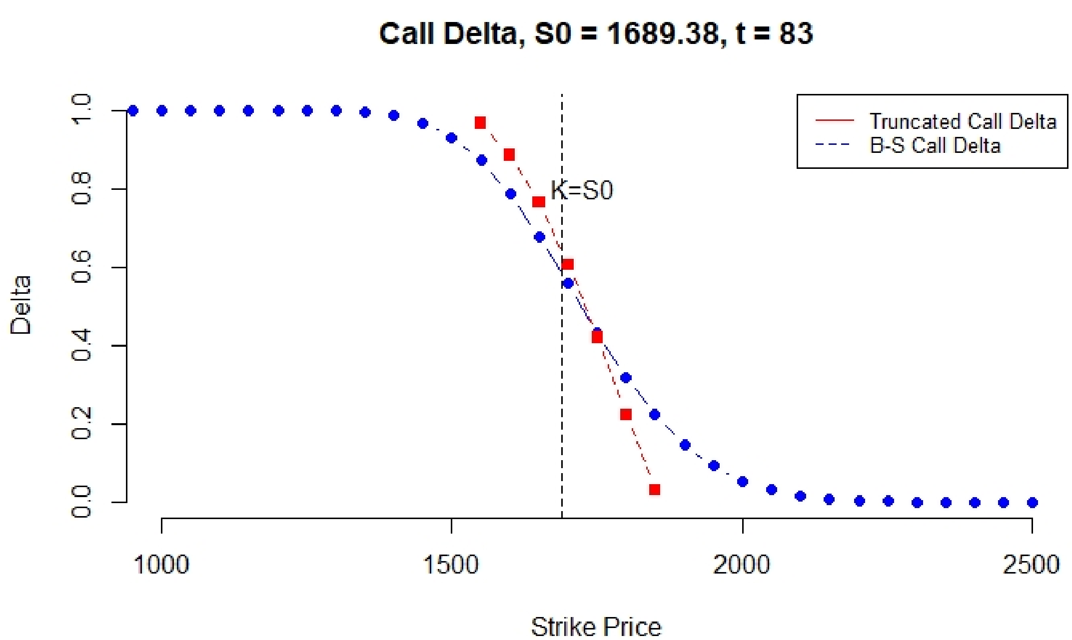

Figure 19. Russell 2000: Range = 10%, a = -0.1053605,b = 0.09531018

With a price range of 15%, the TND Call Delta becomes 0.5893 while that of the B-S remains the same (i.e. 0.5560). Figure 20 shows the graph of the Call Delta for both models against the strike price.

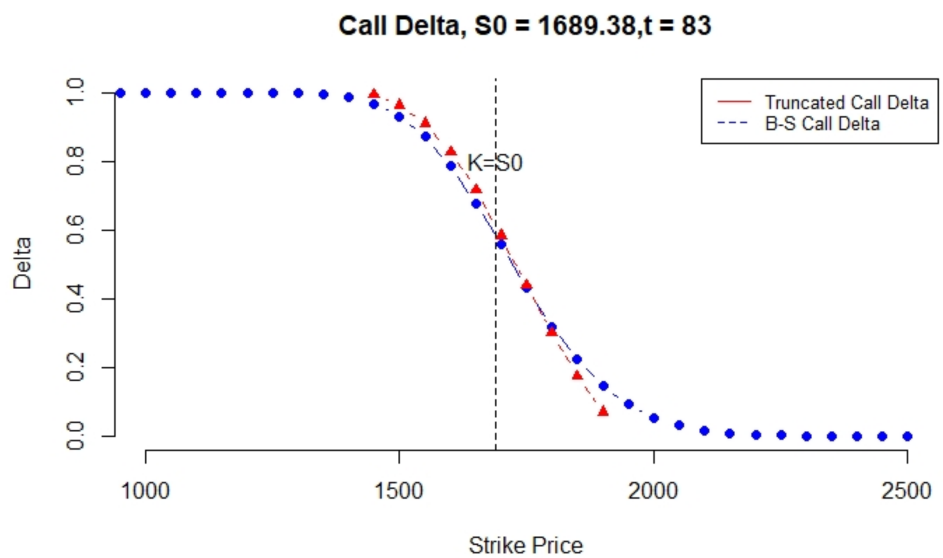

Figure 20. Russell 2000: Range = 15%, a = -0.1625189,b = 0.1397619

When the price range is 20% and the underlying asset price is $1,700, the option delta is 0.5560 under the B-S model while it is 0.5786 under the TND model. Figure 21 below shows the comparison of the Call Deltas for both models and there relation to the option's strike price. It is evident that the Call Delta for the TND model converges to the B-S model as the range of the underlying asset's increases. The upper limit depicts a significant deviation from the B-S model which diminishes as the price range changes from 10% to 20%.

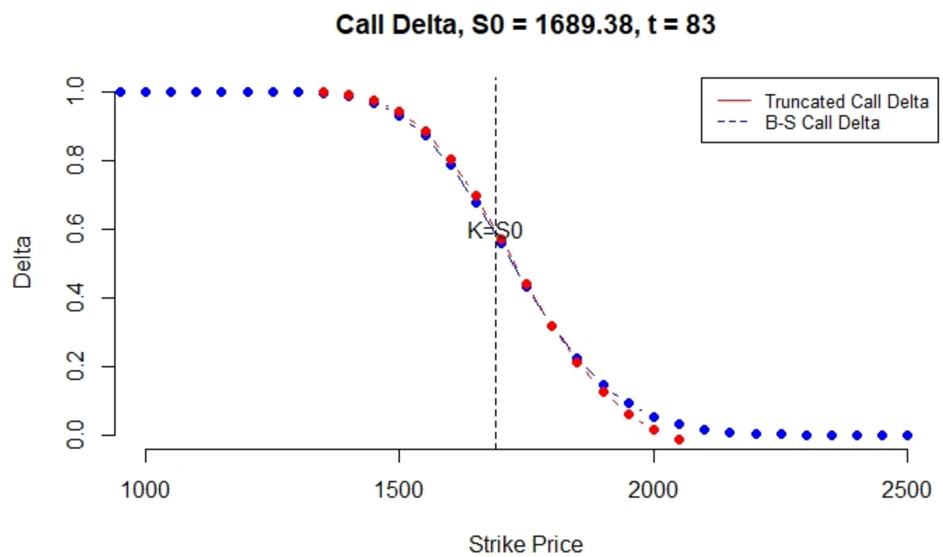

Figure 21. Russell 2000: Range = 20%, a = -0.2231436,b = 0.1823216

With the maturity period of 30 days, the call delta for Russell 2000 options are as shown in Figures 22, 23 and 24 with the price range of 8%, 10% and 15% respectively.

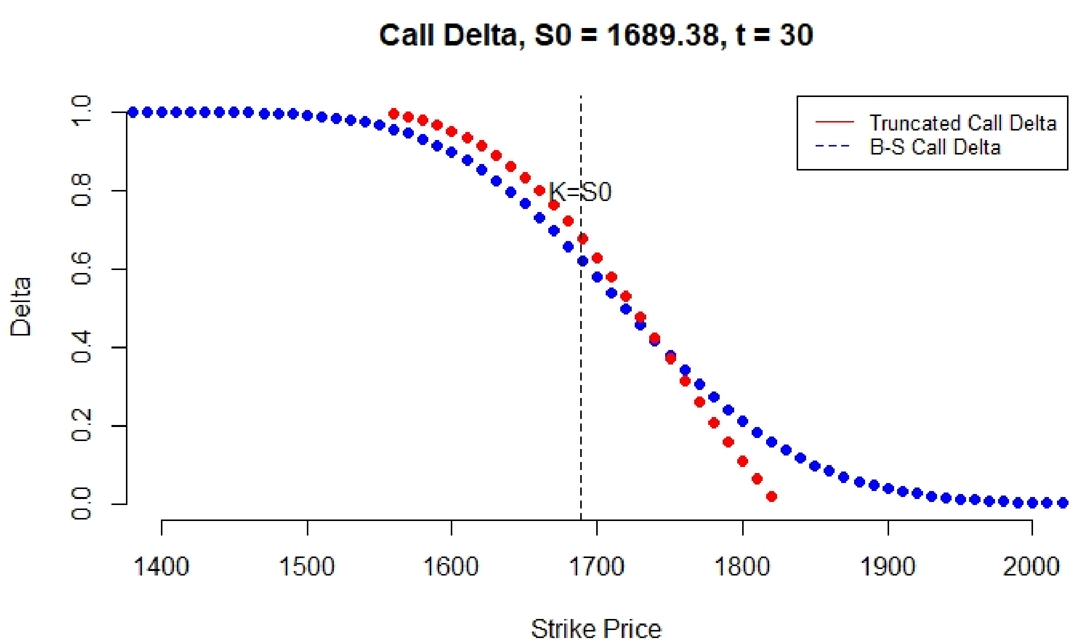

Figure 22. Russell 2000: Range = 8%, a = -0.08338161,b = 0.07696104

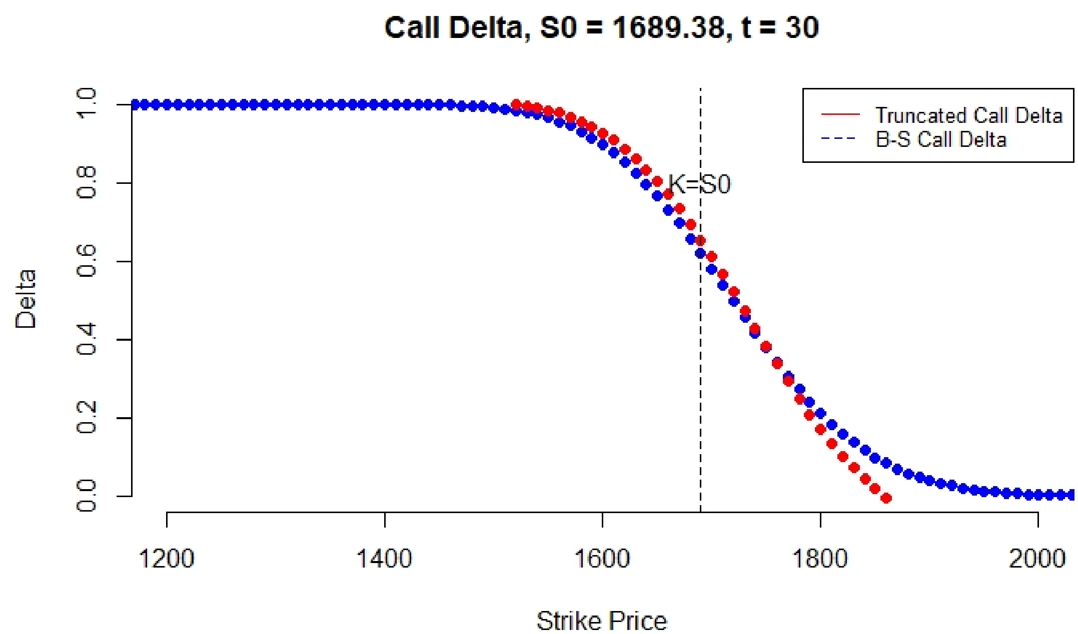

Figure 23. Russell 2000: Range = 10%, a = -0.1053605, b = 0.09531018

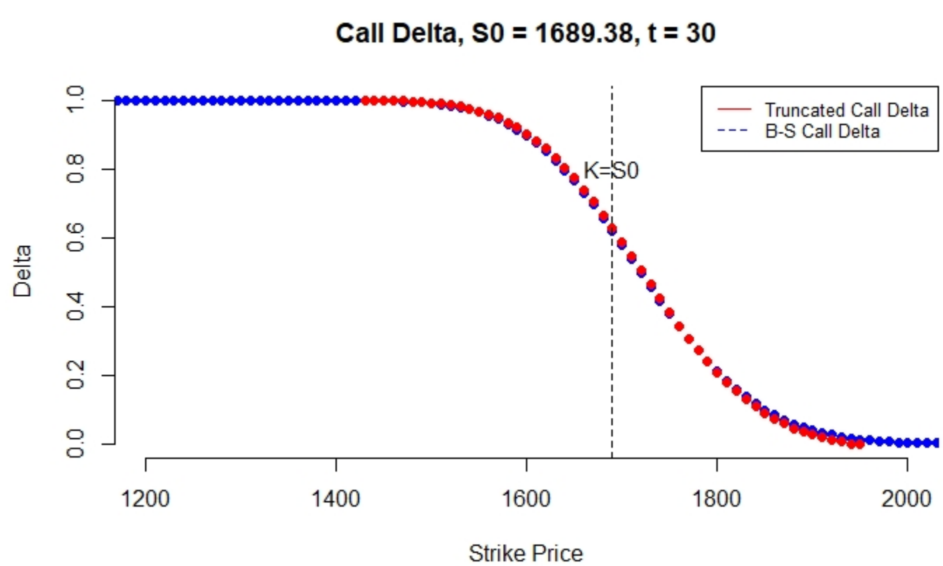

Figure 24. Russell 2000: Range = 15%, a = -0.1625189, b = 0.1397619

The same trend was observed for Facebook Options with time to maturity of 63 days and 14 days as indicated in Figures 25 and 25 respectively.

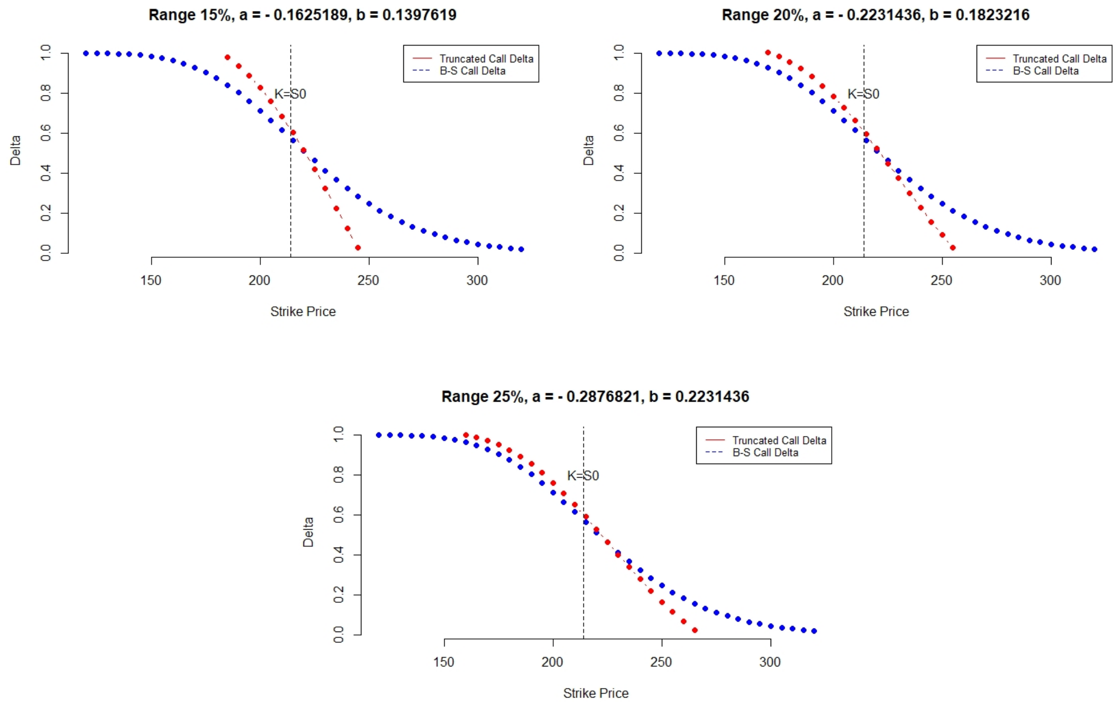

Figure 25. Facebook Options Call Deltas: S0 = 214.18, t = 63 days

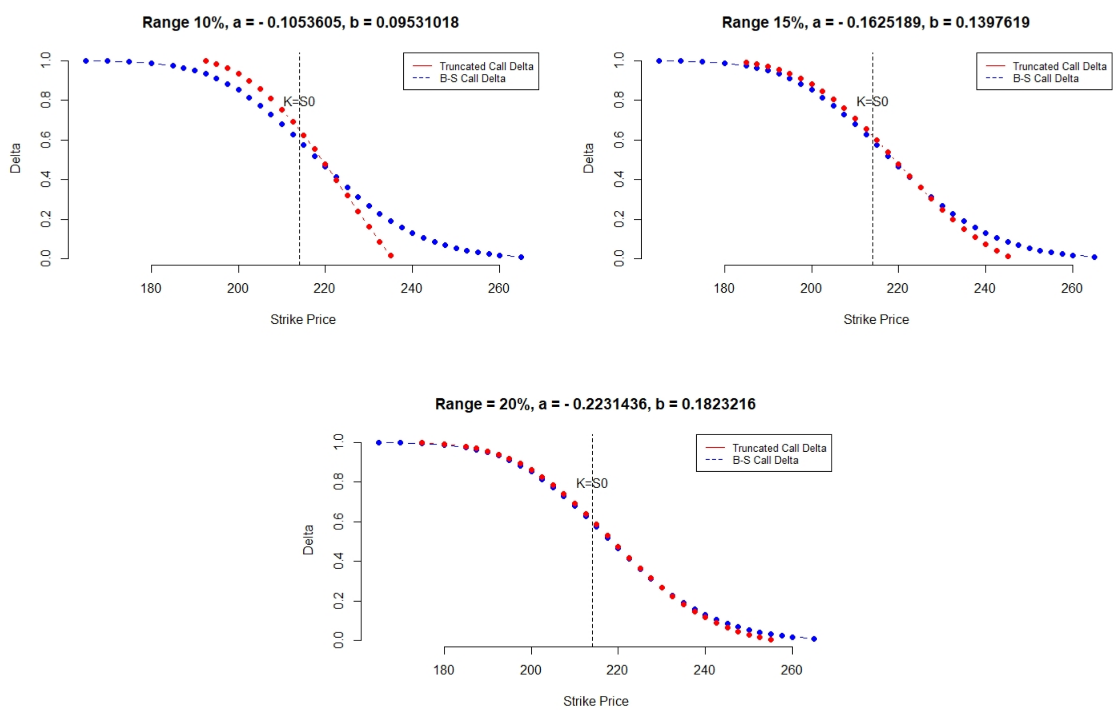

Figure 26. Facebook Options Call Deltas: S0 = 214.18, t = 14 days

Similarly, as Figure 4 and 6 below shows, the Call Deltas for Apple Inc. Options behaves in a similar manners as Russell 2000 index and Facebook Options.

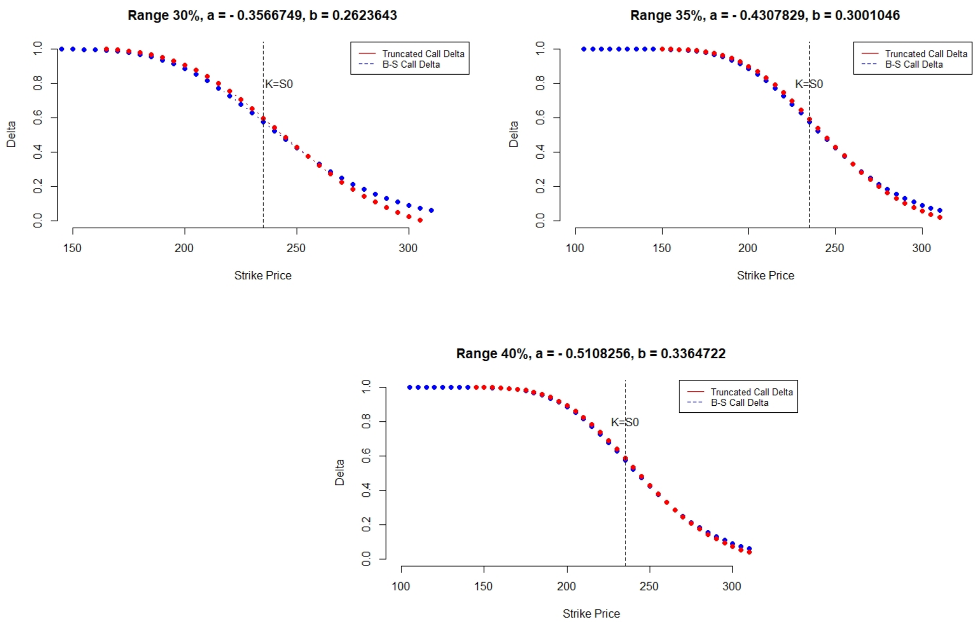

Figure 27. Apple Inc. Options Call Delta: S0 = 235.28, t = 95 days

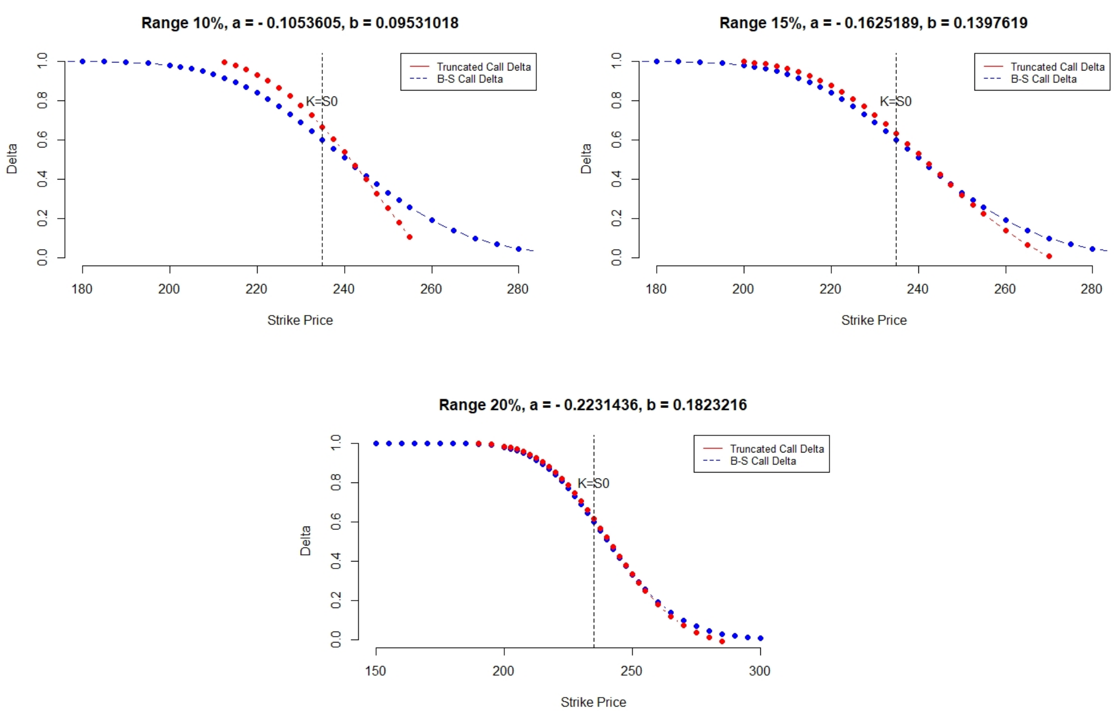

Figure 28. Apple Inc. Options Call Delta: S0 = 235.28, t = 30 days

4.3.2 Call Gamma

The Call Gamma measures the rate at which the Call Delta changes with respect to the changes of the underlying asset price, S_0. Figure 29 below shows the Option's Call Gamma plot when the price range is 10%. Clearly, the sensitiveness of the underlying price is profound for the TND at 10% price range when compared to the B-S model.

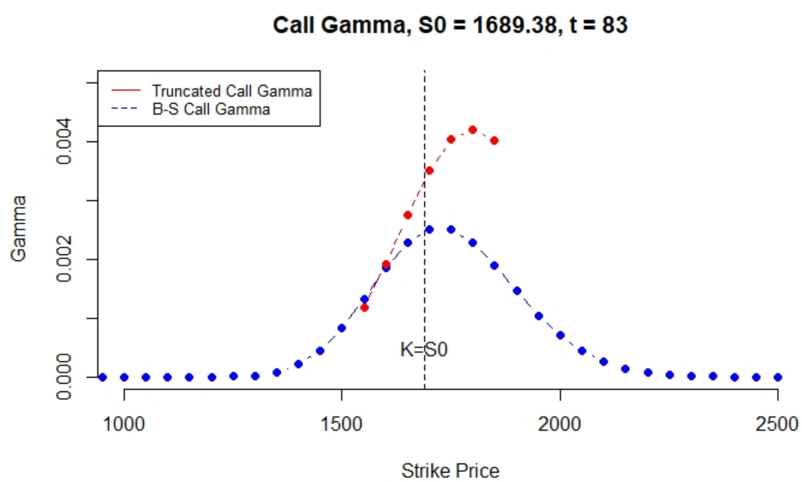

Figure 29. Russell 2000: Range = 10%, a = -0.1053605,b = 0.09531018

When the price range of the underlying asset increases to 15%, the Gamma plot for the European Call option is as shown in Figure 30 below. When the option is out-of-the-money, the TND and B-S depicts a significant difference that when the option is in-the-money.

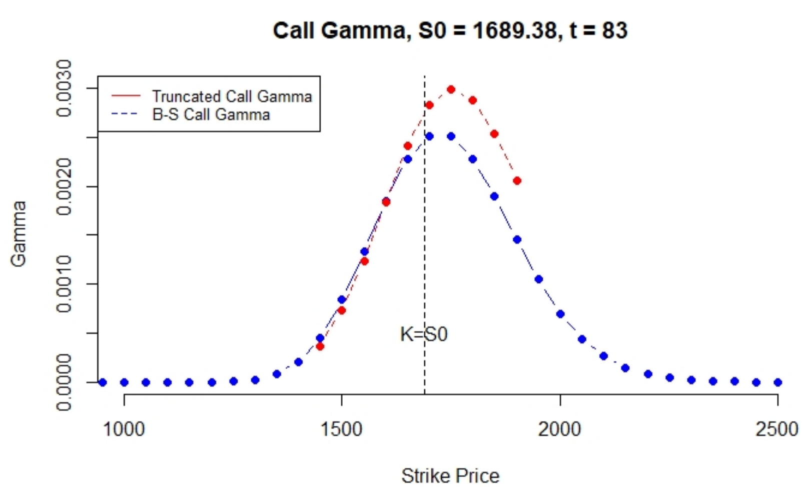

Figure 30. Call Gamma, Range = 15%, a = -0.1625189,b = 0.1397619

Similarly, with a price range of 20%, the European Call Option Gamma against the strike price is as shown below in Figure 31. There is less difference between the B-S and TND when the price is below the current price of the underlying asset than at the above. Nevertheless, the TND model converges to the B-S as the price range rises from 10% to 20% as expected.

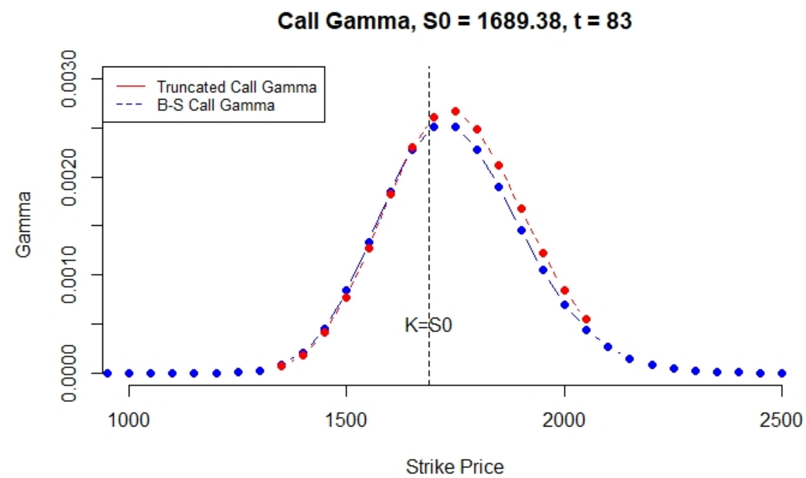

Figure 31. Russell 2000: Range = 20%, a = -0.2231436,b = 0.1823216

Considering the Russell 2000 index options with maturity period of 30 days, the Call Gamma are more sensitive with a price range of 8% than 15% compared to B-S model Call Gamma. Nevertheless, as the sensitivity decreases as the price range increases as shown in Figures 32, 33 and 34 below.

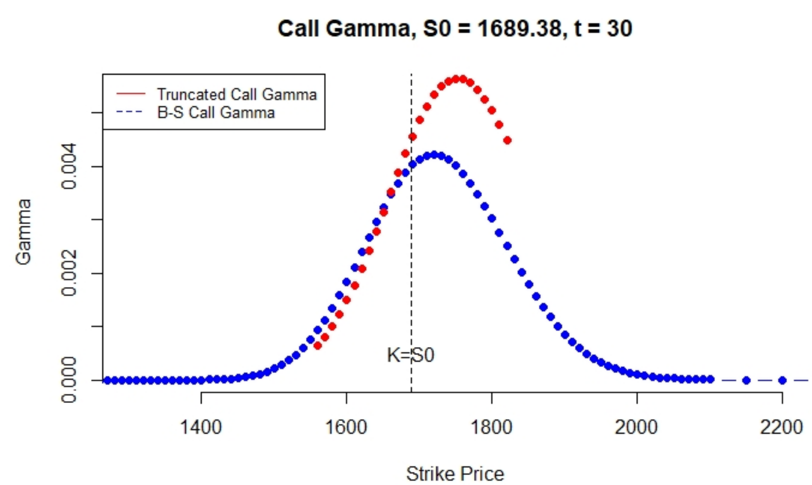

Figure 32. Russell 200: Range = 8%, a = -0.08338161, b = 0.07696104

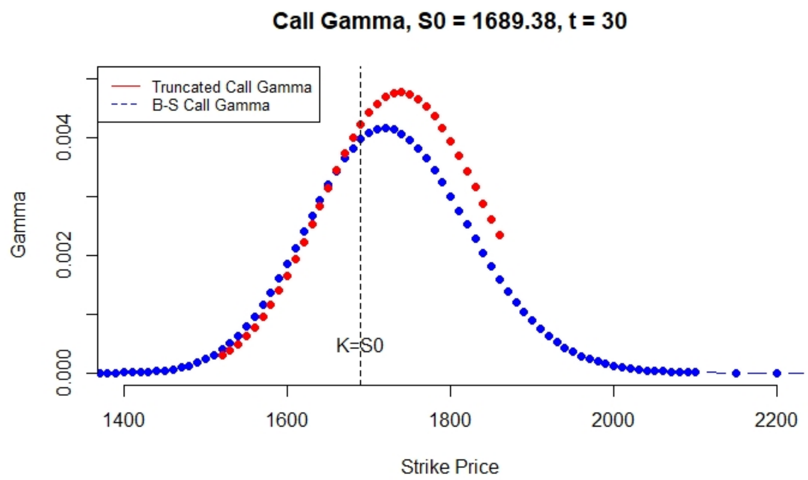

Figure 33. Russell 2000: Range = 10%, a = -0.1053605, b = 0.09531018

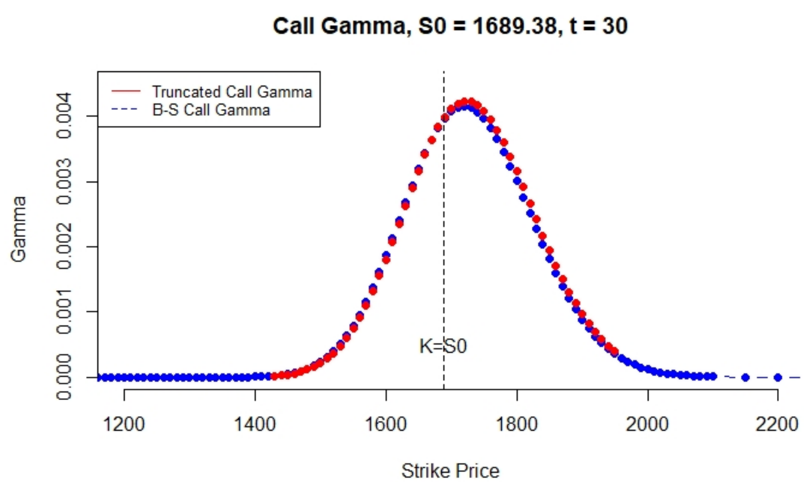

Figure 34. Russell 2000: Range = 15%, a = -0.1625189, b = 0.1397619

The options of Facebook Corporation's sensitivity of Deltas with respect to the underlying stock prices shows similar results as with Russell 2000 index options as can be observed in the Figures 35 and 36 below for 63 and 14 days to maturity options respectively. The Call Gamma for Apple Inc. Options similarly shows the same results as Figures 37 and 38 indicates for 95 and 30 days to maturity respectively.

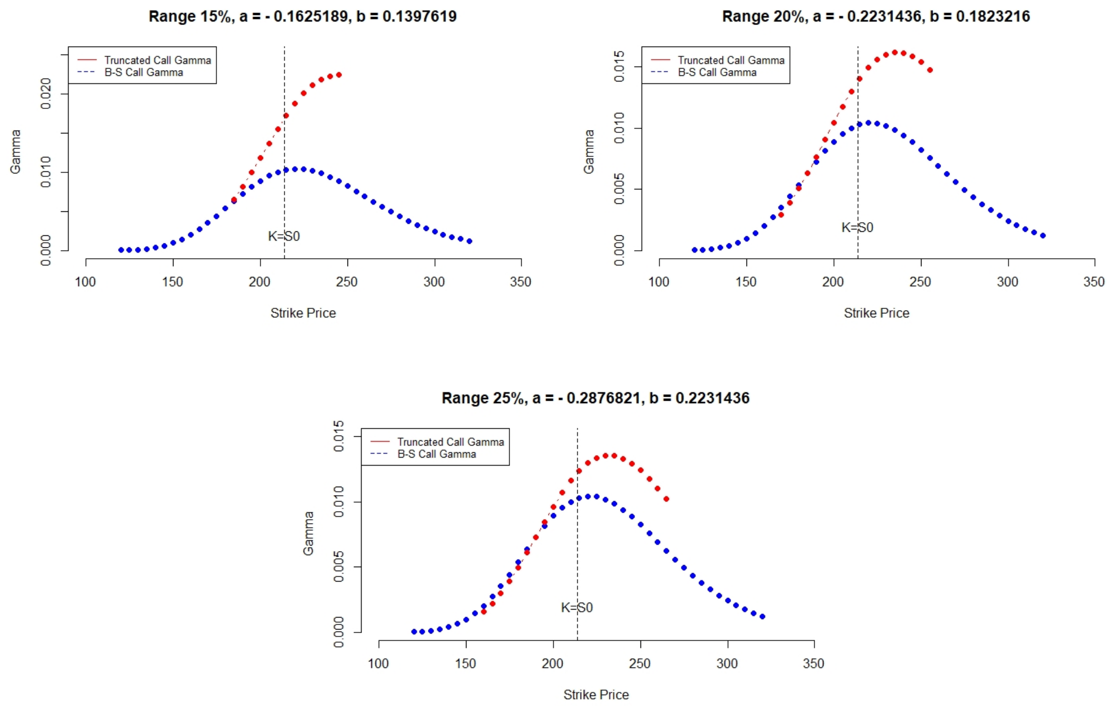

Figure 35. Facebook Options Call Gamma: S0 = 214.18, t = 63 days

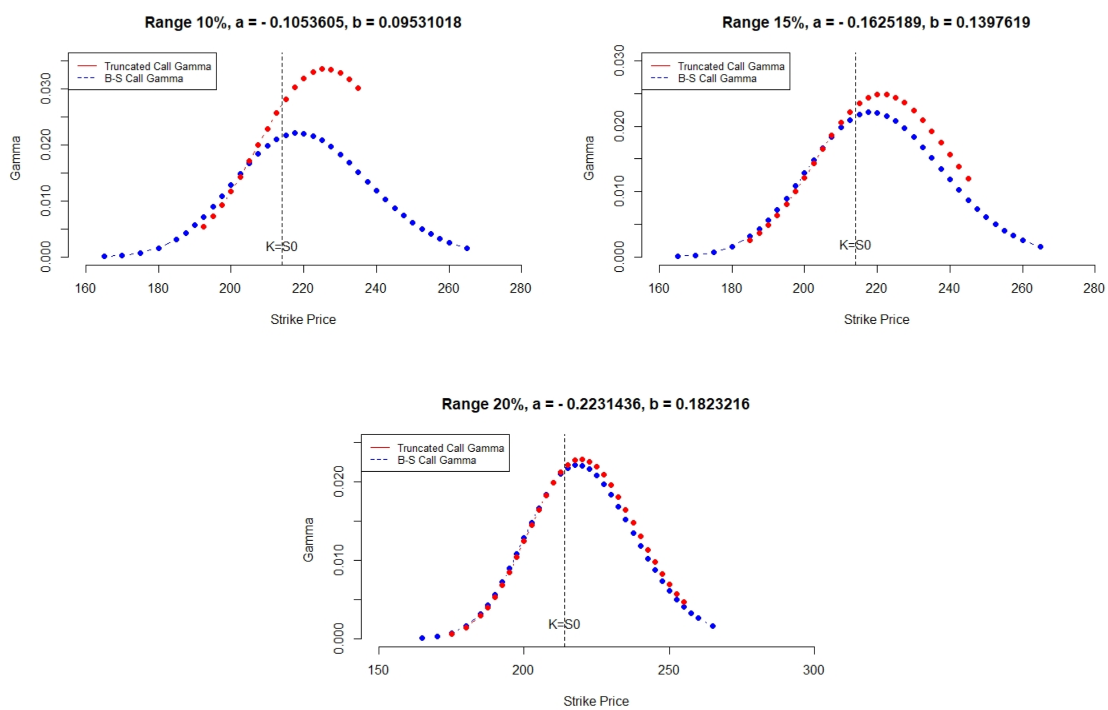

Figure 36. Facebook Options Call Gamma: S0 = 214.18, t = 14 days

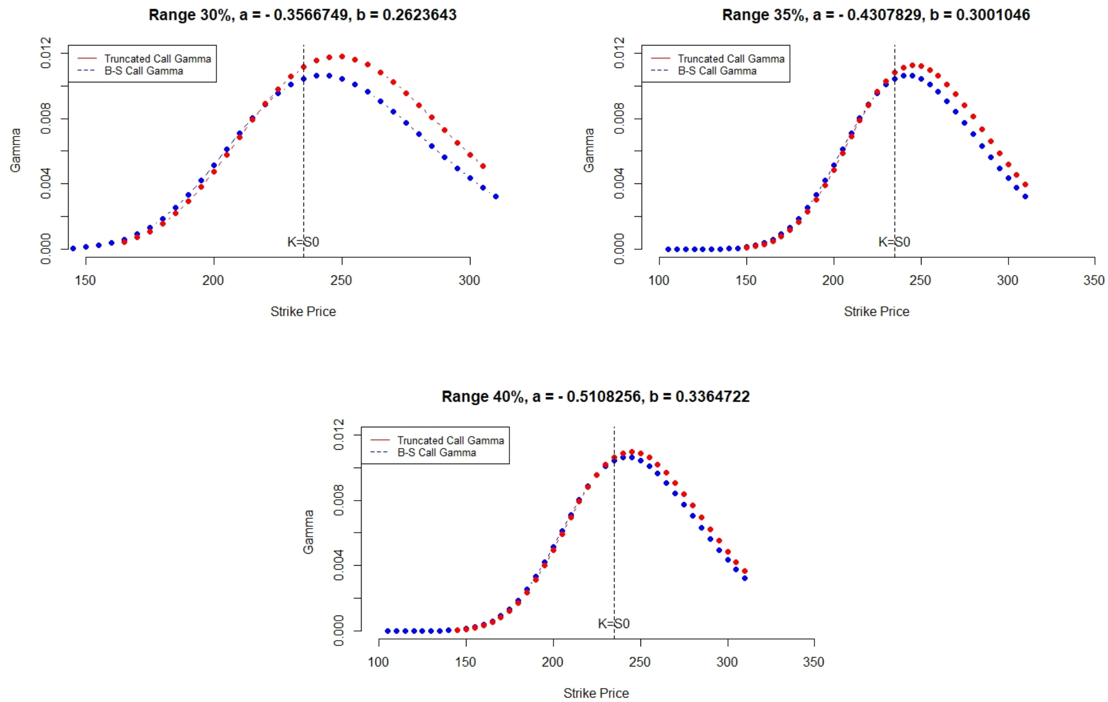

Figure 37. Apple Inc. Options Call Gamma: S0 = 235.28, t = 95 days

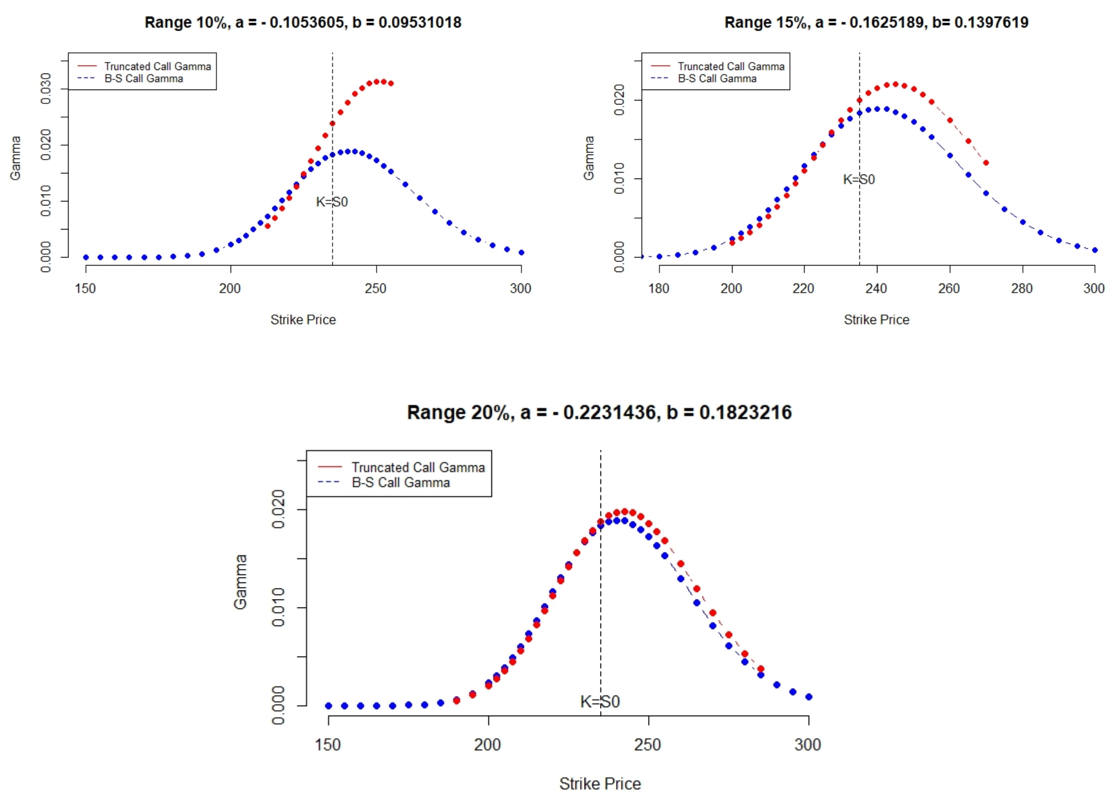

Figure 38. Apple Inc. Options Call Gamma: S0 = 235.28, t = 30 days

4.3.3 Call Rho

The option Greek Rho is a measure of how sensitive of the price of the option with changes in the risk-free rate, r. The call option's rho should be positive ($\rho > 0$) because when the interest rate is higher, the present value of the strike price, K, is reduced and thereby increasing the option's call value. At the 10% price range the call option rho is as shown in Figure 39 below which appears to be very sensitive for TND than B-S as the price moves away from the current price (i.e, call rho increases in sensitivity in-and-out-of the money) when an investor restricts to price range to 10%.

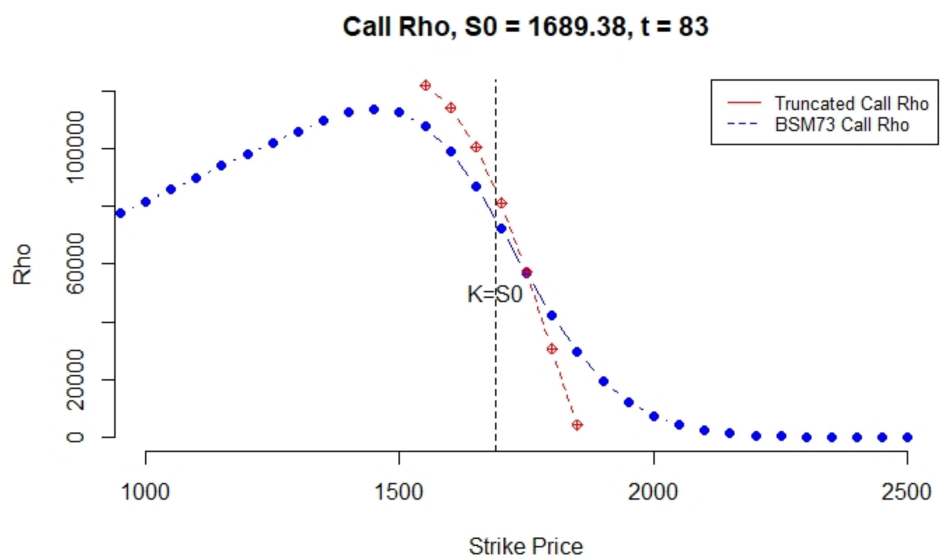

Figure 39. Russell 2000: Range = 10%, a = -0.1053605,b = 0.09531018

With a price range of 15%, the robust deviation of the call rho for TND decreases towards B-S when the range of the option price is increased to 15% as shown in Figures 40 below both in-the-money and out-of-the money options.

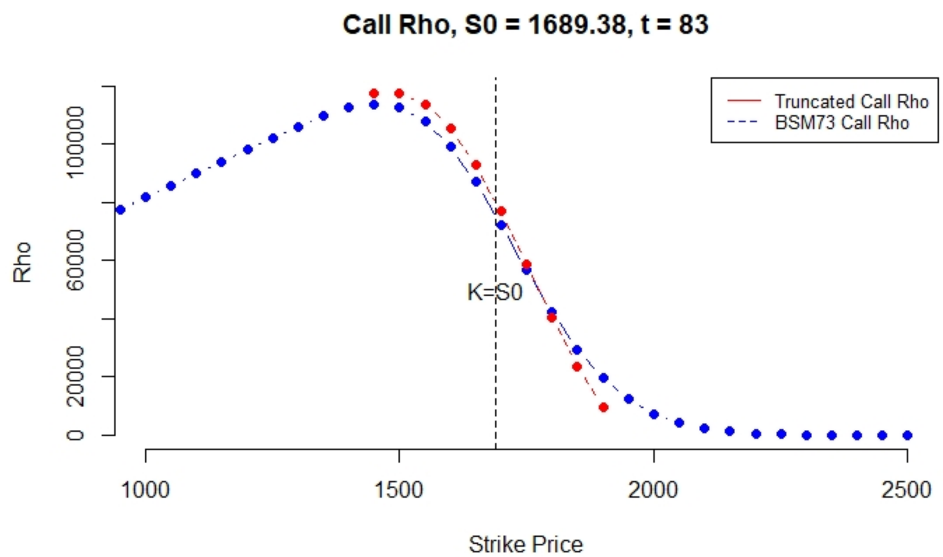

Figure 40. Russell 2000: Range = 15%, a = -0.1625189,b = 0.1397619

Similarly, as Figure 41 depicts, the European call option's rho for TND when the price range is 20% from the current price almost perfectly fits the B-S in-the-money than out-of-the money options. Nevertheless, the sensitive is not as robust as it appears with 15% price range.

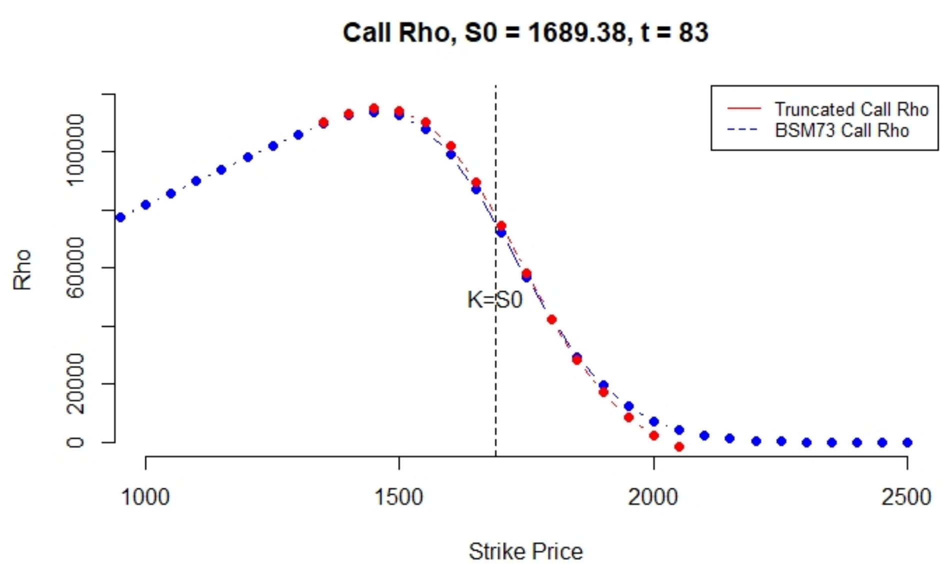

Figure 41. Russell 2000: Range = 20%, a = -0.2231436,b = 0.1823216

When options with 30 days to maturity are used and the price ranges of 8%, 10% and 15%, the Call Rho for Russell 2000 index are as shown in Figures 42, 43 and 44.

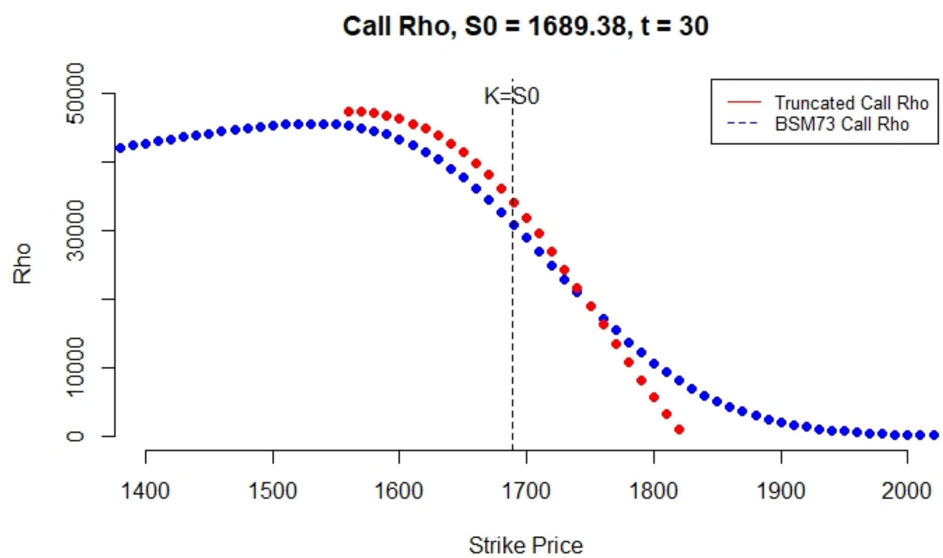

Figure 42. Russell 2000: Range = 8%, a = -0.08338161, b = 0.07696104

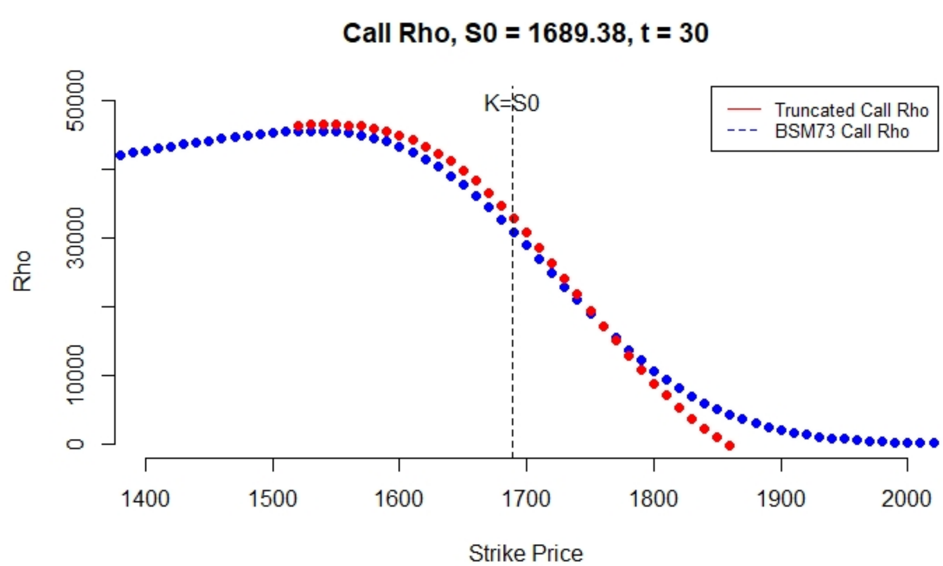

Figure 43. Russell 2000: Range = 10%, a = -0.1053605, b = 0.09531018

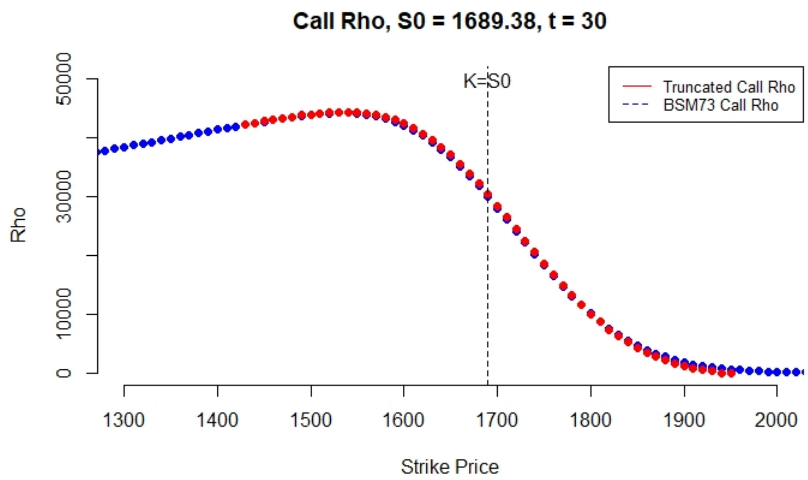

Figure 44. Russell 2000: Range = 15%, a = -0.1625189, b = 0.1397619

Clearly, the call rho for TND model converges to the B-S models as the price range increases from 8% to 15% with deviations from each model decreasing significantly.

The same results are experienced for Facebook and Apple Inc. Options with the different maturity periods and price ranges as Figures 45, 46, 47 and 48 indicates.

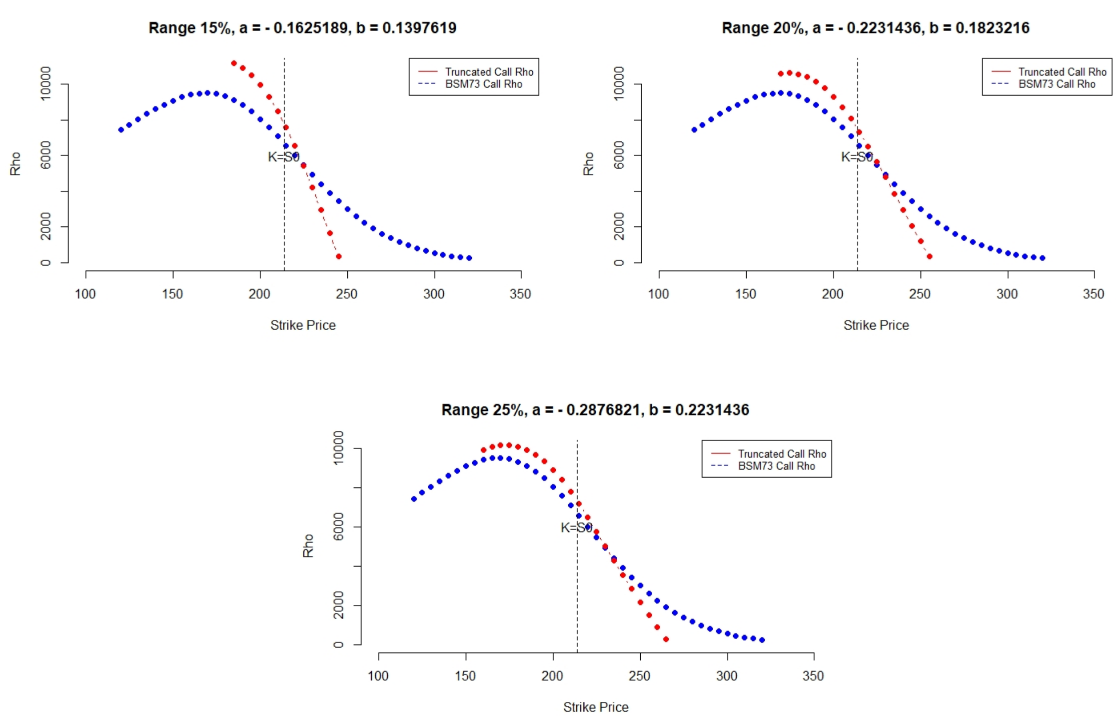

Figure 45. Facebook Options Call Rho: S0 = 214.18, t = 63 days

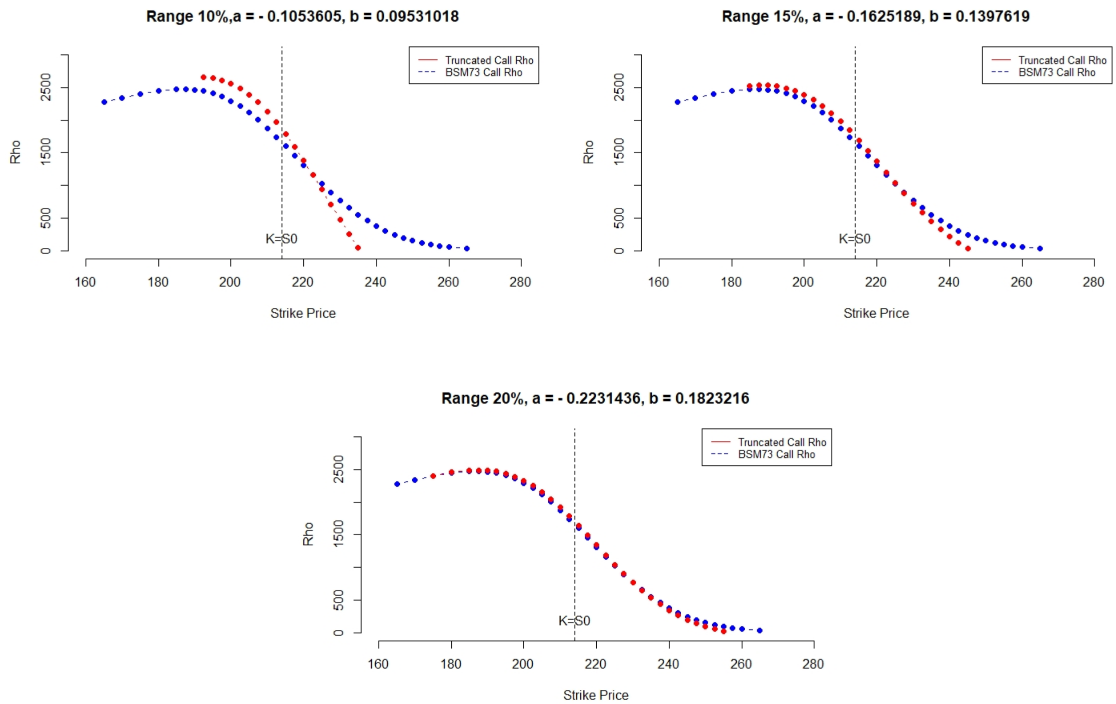

Figure 46. Facebook Options Call Rho: S0 = 214.18, t = 14 days

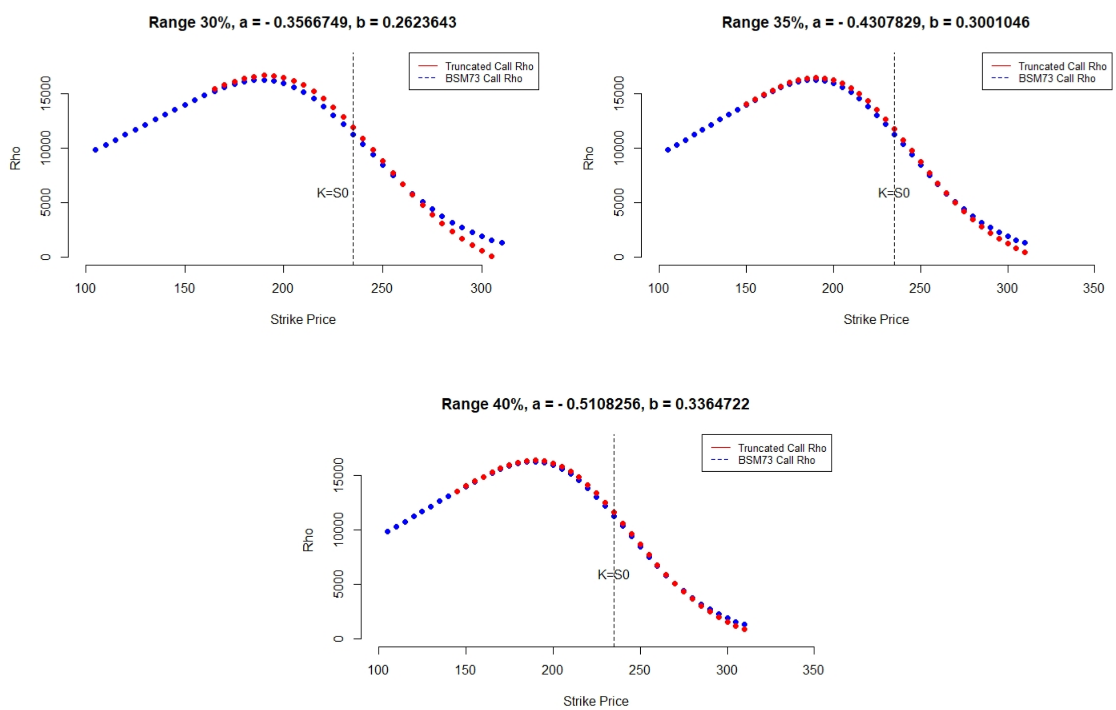

Figure 47. Apple Inc. Options Call Rho: S0 = 235.28, t = 95 days

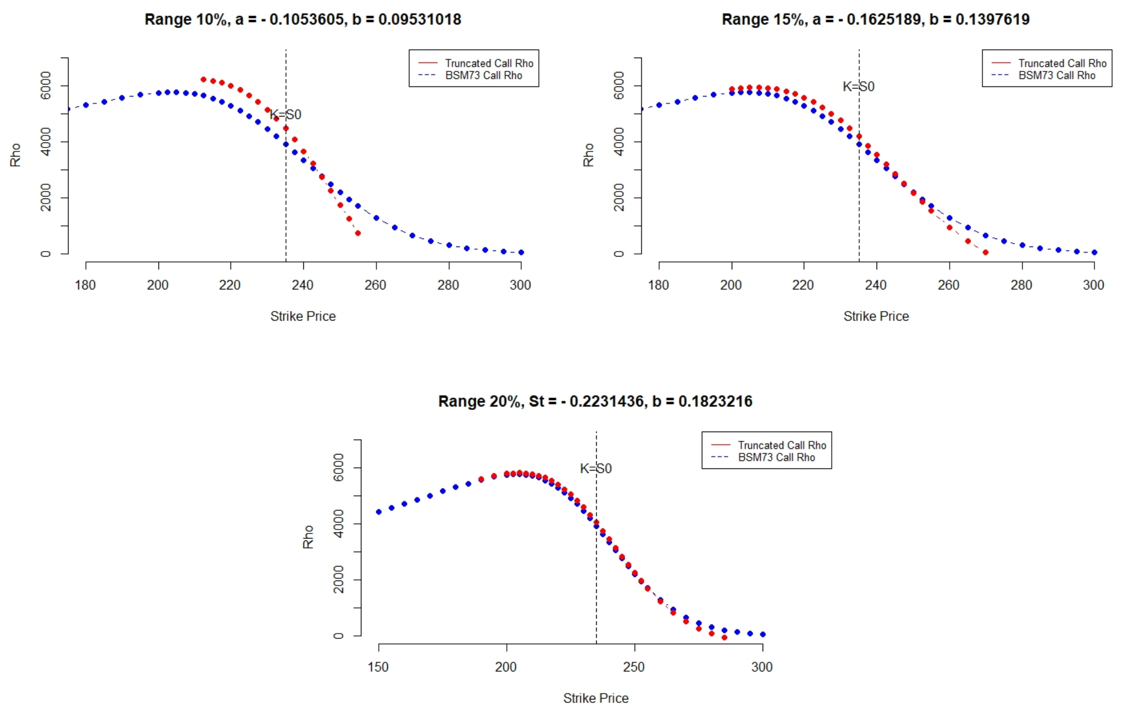

Figure 48. Apple Inc. Options Call Rho: S0 = 235.28, t = 30 days

The call rho appears to be more sensitive with a smaller price range than larger price range with respect to the B-S model calls. As expected, the call rho for TND model converges to the B-S model rho as the value a and b tends to negative infinity and positive infinity (i.e. by enlarging the price range from the current stock price).

4.4 Statistical Properties

The statistical properties such as mean, standard deviation, skewness and kurtosis for the truncated normal distribution as compared to those used in the B-S model are as given in Table 4 below. As it can be noted from Table 4, the volatility is the same for the B-S and the Truncated normal distribution (TND) Models. Similarly, the mean for the TND is the same as the mean obtained in Table 1 for μ. The skewness and kurtosis of the model are near zero. However, the TND model has higher expected mean return than the B-S for shorter price ranges with longer time to maturity (83 days) which decreases as price range increases. The mean returns is higher than B-S for options with shorter maturity periods (30 days) for prices ranges between 10% and 15% whereby the mean return is highest at 8%. A dash (-) in table indicates that the statistical parameters were not determine at the corresponding price range or model.

Table 4. Statistical Properties: Russell 2000 Index

Statistic	t	8%	10%	15%	20%	B - S
Mean	83	-	0.0006647598	0.0003547773	0.0002341286	0.0003429912
	30	0.001122183	0.0008201757	0.0005474799	-	0.0003429912
Volatility	83	-	0.01020331	0.01020331	0.01020331	0.01020331
	30	0.01020331	0.01020331	0.01020331	-	0.01020331
Skewness	83	-	- 6.900774e-18	- 2.125947e-39	- 8.720507e-68	-
	30	- 2.178018e-11	- 8.060204e-18	- 2.781708e-39	-	-
Kurtosis	83	-	0.0003123326	0.0003123234	0.0003123228	-
	30	0.000312414	0.0003123487	0.0003123278	-	-

Table 5 and 6 below shows the statistical properties for Facebook and Apple Options respectively.

Table 5. Statistical Properties: Facebook Corporation Options

Statistic	t	10%	15%	20%	25%	B - S
Mean	63	-	0.001612246	0.001058277	0.0007538312	0.000885052
	14	0.003339332	0.001815905	0.001247113	-	0.000885052
Volatility	63	-	0.02249525	0.02249525	0.02249525	0.02249525
	14	0.02249076	0.02249525	0.02249525	-	0.02249525
Skewness	63	-	-9.456047e-08	-2.029918e-13	2.310593e-20	-
	14	-0.001396318	-9.966692e-08	-2.167305e-13	-	-
Kurtosis	63	-	0.001518189	0.001518124	0.001518113	-
	14	0.001516151	0.001518238	0.001518138	-	-

Table 6. Statistical Properties: Apple Corporation Options

Statistic	t	10%	15%	20%	30%	35%	B - S
Mean	95	-	-	0.0005717515	.0002851128	0.0002018363	0.0009127773
	30	0.001887374	0.00101256	0.0006829072	0.0004753061	-	0.0009127773
Volatility	95	-	-	0.01636316	0.01636316	0.01636316	0.01636316
	30	0.01636316	0.01636316	0.01636316	0.01636316	-	0.01636316
Skewness	95	-	-	-7.92324e-26	-2.016344e-54	-1.525261e-71	-
	30	-1.04484e-06	-6.897906e-15	-8.533466e-26	-2.425223e-54	-	-
Kurtosis	95	-	-	0.0008032619	0.0008032596	0.0008032595	-
	30	0.0008035421	0.000803283	0.0008032643	0.0008032606	-	-

4.5 Pricing Errors

To test the performance of the Truncated Normal Distribution model in comparison to the Black & Scholes Model, the following three key indicators are used as noted by both Badescu et al. and Christofferson et al. [BK08, CDJW10], i.e:

1. The Dollar Root Mean Squared Error (RMSE($))
2. Average Absolute Error (APE(%)) and
3. Average Relative Pricing Error (ARPE(%))

The formulas for calculating these indicators are given as below;

$$RMSE(\$) = \sqrt{\sum_{i=1}^{N} \frac{(V_i^{market} - V_i^{model})^2}{N}},$$
$$ARPE(\%) = \frac{1}{N}\sum_{i=1}^{N} \frac{|V_i^{market} - V_i^{model}|}{V_i^{market}} \times 100,$$
$$APE(\%) = \frac{1}{N}\sum_{i=1}^{N} \frac{|V_i^{market} - V_i^{model}|}{\overline{V^{market}}} \times 100,$$

Where N is the total number of observations and $\overline{V^{market}}$ is the average value of the market option prices. Tables 7, 9 and 11 below presents the pricing errors for Calls while Tables 8, 10 and 12 presents the pricing errors for Puts for European Options for the Russell 2000 index, Facebook and Apple Options respectively. The last column indicates the errors for the classical Black & Scholes model for the entire data while the rows indicates the pricing errors for Truncated distribution model and Black & Scholes model within the indicated price ranges at the top row.

Table 7. Pricing Errors: Russell 2000 Index Calls

Indicator	Model	t	8%	10%	14%	15%	20%	B-S
RMSE ($)	TND	83	13.94013	7.662783	2.6350834	3.36482	7.70738	10.68518
	BS	83	14.53229	13.70699	12.02255	12.47751	11.10818	10.68518
	TND	30	7.581714	9.931964	12.2096	12.41819	12.74649	12.29002
	BS	30	14.19545	13.59167	13.12258	12.99859	12.81082	12.29002
APE (%)	TND	83	21.83558	10.38505	1.961139	2.88922	6.266521	2.472057
	BS	83	22.61075	19.12351	13.1601	12.566552	9.50008	2.472057
	TND	30	12.97236	15.35278	15.57118	14.25076	11.82957	2.877815
	BS	30	28.40206	22.92887	17.26819	15.27383	11.93539	2.877815
ARPE (%)	TND	83	39.02956	31.56237	12.95375	10.90533	50.53603	115.3791
	BS	83	46.26234	72.06162	162.5999	102.1757	226.9627	115.3791
	TND	30	22.83928	47.55402	124.218	138.0701	193.6492	103.4894
	BS	30	147.3372	214.2946	268.6344	264.0103	219.826	103.4894

Table 8. Pricing Errors: Russell 2000 Index Puts

Indicator	Model	t	8%	10%	14%	15%	20%	B-S
RMSE ($)	TND	83	34.87868	28.41343	20.19615	18.46177	15.36447	14.45224
	BS	83	7.442571	8.331137	10.38711	9.36576	12.16299	14.45224
	TND	30	17.10751	15.15378	13.90963	14.23379	14.82768	12.29002
	BS	30	9.146001	10.76677	12.69713	13.42754	14.71313	12.29002
APE (%)	TND	83	51.38183	38.05795	21.72625	24.18124	12.84401	5.878943
	BS	83	10.25534	10.54901	10.56265	11.18124	9.721858	5.878943
	TND	30	35.79208	26.0826	17.88761	16.23098	13.25609	8.753727
	BS	30	18.72418	17.88285	16.08498	15.23098	13.13801	8.753727
ARPE (%)	TND	83	62.15531	54.79584	45.93129	47.12577	39.37461	54.63921
	BS	83	15.05448	19.13364	27.24688	29.2792	33.38869	54.63921
	TND	30	55.23094	48.97446	43.57534	44.00637	44.78718	66.40577
	BS	30	31.46501	35.71957	40.36786	41.99834	44.59605	66.40577

Table 9. Pricing Errors: Facebook Corporation Call Options

Indicator	Option	t	10%	15%	20%	22%	B-S
RMSE ($)	TND	63	1.542654	0.83606	1.869731	2.266653	4.389487
	BS	63	7.649193	7.051997	6.325461	6.201712	4.389487
	TND	14	2.427216	3.013156	3.198696	3.171051	3.258968
	BS	14	4.230824	3.765212	3.451933	3.325339	3.258968
APE (%)	TND	63	14.0689	6.798324	11.12156	14.0007	14.58617
	BS	63	78.50975	61.72287	40.75917	41.93123	14.58617
	TND	14	31.28327	32.19757	30.20187	27.80384	26.52495
	BS	14	59.98022	42.51555	33.56838	29.78812	26.52495
ARPE (%)	TND	63	34.44174	20.59825	28.92845	40.11283	491.1579
	BS	63	147.9184	200.6228	254.9602	309.9449	491.1579
	TND	14	81.01143	190.1262	294.7563	316.7216	428.7623
	BS	14	425.9404	475.9404	473.6976	444.0691	428.7623

Table 10. Pricing Errors: Facebook Corporation Put Options

Indicator	Option	t	10%	15%	20%	22%	B-S
RMSE ($)	TND	63	4.232306	2.687948	1.848225	1.856292	3.162934
	BS	63	4.861191	4.29261	3.695804	3.600639	3.162934
	TND	14	1.843663	1.893007	2.132766	2.245237	2.210517
	BS	14	1.466081	1.692321	2.011619	2.157727	2.210517
APE (%)	TND	63	41.51365	21.41189	9.861883	8.802932	8.451392
	BS	63	48.51365	35.77338	25.60792	21.5711	8.451392
	TND	14	19.13184	15.4864	13.83743	12.93457	12.26512
	BS	14	20.91577	15.99764	13.57598	12.69922	12.26512
ARPE (%)	TND	63	57.20631	38.2455	24.17814	19.25856	61.99839
	BS	63	78.67937	86.52122	96.55473	91.55123	61.99839
	TND	14	27.25601	44.37338	66.92025	70.36194	76.95254
	BS	14	112.4185	97.57811	82.56682	78.40648	76.95254

Table 11. Pricing Errors: Apple Corporation Call Options

Indicator	Option	t	10%	15%	20%	25%	B-S
RMSE ($)	TND	95	0.6270006	0.6542114	0.6623861	0.6572748	1.096466
	BS	95	7.801481	5.186203	3.380691	2.03054	1.096466
	TND	30	1.010708	1.776693	2.245984	2.449755	2.621543
	BS	30	2.801821	2.699299	2.605883	2.570601	2.621543
APE (%)	TND	95	3.237152	3.025258	2.925913	2.747874	2.03043
	BS	95	44.56207	26.72137	15.94271	8.558026	2.03043
	TND	30	8.484477	11.93961	14.6746	14.24831	10.35188
	BS	30	27.7231	20.10689	17.68761	15.15683	10.35188
ARPE (%)	TND	95	4.581283	6.575091	8.373743	10.24012	10.36712
	BS	95	54.98859	45.02938	36.02404	28.14518	10.36712
	TND	30	18.53643	22.83029	39.66943	53.70966	61.40541
	BS	30	58.15171	68.91884	75.75342	74.34065	61.40541

Table 12. Pricing Errors: Apple Corporation Put Options

Indicator	Option	t	10%	15%	20%	25%	B-S
RMSE ($)	TND	95	11.10845	8.462173	6.649547	5.321244	2.506095
	BS	95	2.761073	2.747789	2.736375	2.77163	2.506095
	TND	30	4.1658	3.025908	2.800111	2.621586	2.517993
	BS	30	1.457339	1.907905	2.387088	2.465113	2.517993
APE (%)	TND	95	66.97649	46.45825	33.87507	24.9431	11.98028
	BS	95	16.76648	15.32363	14.18461	13.19975	11.98028
	TND	30	45.9062	27.59942	18.20355	15.45815	13.28366
	BS	30	15.98662	16.17398	14.75583	14.29324	13.28366
ARPE (%)	TND	95	74.13637	61.6553	52.70013	46.00327	50.74115
	BS	95	19.6582	22.84793	25.4667	28.30286	50.74115
	TND	30	61.95972	49.99816	39.59798	37.78803	43.55854
	BS	30	22.65149	30.29713	32.46275	35.66489	43.55854

The truncated model is well applicable when the B-S model over-estimates the market option prices. The model shows significant improvement in the pricing of European call options as compared to the B-S model. The pricing errors are extremely reduced when TND is used in the pricing of call options. The choice of the value of a and b is critical when using the TND model. The errors are minimized when an appropriate price range is selected for the different call options with different maturity periods.

The puts options are under-priced when the TND is used thereby resulting in larger pricing errors in some measures when compared to the B-S model. For the data used, observed market puts are highly priced for Russell 2000 index and Apple Corporation options. Facebook Corporation market puts are under-priced by the B-S resulting to a better performance of the truncated model in pricing them for the price ranges chosen. The B-S model under-prices the market put, hence, the truncated model under-prices them further since the prices of the truncated model are always below the B-S model.

5 Chapter Five: Conclusion and Future Research

5.1 Conclusion

The Truncated Normal Distribution (TND) Model offers a more flexible closed form formula for pricing options. The martingale restriction for the TND that was obtained ensured that arbitrage opportunities does not exit in the market. The uniroot function for R-Software was used to solve for μ that is incorporated in the development of the closed formulas for the call and put options. The Moment Generating Function, Mean, Variance, Skewness and Kurtosis were obtained for the truncated model. The introduction of the lower and upper bounds to the distribution of the underlying asset prices and assuming that the log-returns of the process are truncated, the model offers an alternative method of pricing European Options. A key observation from the TND model is that the prices of the model converges to the B-S model when the upper and lower bounds increases and decreases respectively. The prices of the model is always below the Black - Scholes formula.

The option Greeks obtained shows that they approach the B-S Greeks when the lower and upper bounds are decreased and increased respectively as it was the case with the option prices. The statistical properties obtained indicates that the historical mean returns of the underlying process is the same as the solution of μ from the martingale restriction. The standard deviation (volatility) is the same for the truncated normal distribution model and that obtained from the historical data of the underlying asset. The model performs well in the pricing of Facebook Options - where market puts where observed to be lower than that obtained using the B-S model than other options data used in the analysis. The pricing of calls were significantly better for all the data used in this study. Based on the numerical results of the data used in the analysis, the Truncated Normal Distribution model outperforms the Black & Scholes model at least for the case of European Call Options for the data used in the analysis. Therefore, the truncated model can be used as an alternative options pricing model.

5.2 Future Research

The following are the recommendations for this study;

1. The investigation of the other Greeks, i.e Theta (θ) and Vega (ν)
2. The investigation of the volatility smile for the proposed truncated normal distribution model

3. Application of other distributions such as the truncated normal inverse Gaussian instead of the truncated normal distribution

4. Application of the model on American Options

5. Further Empirical Studies are recommended to check the consistency of the results and determine when does the model over-prices or under-prices the market data

Bibliography

[BCA12] Eike C Brechmann, Claudia Czado, and Kjersti Aas. Truncated regular vines in high dimensions with application to financial data. *Canadian Journal of Statistics*, 40(1):68–85, 2012.

[Bjö09] Tomas Björk. *Arbitrage theory in continuous time*. Oxford university press, 2009.

[BJP03] Kee-Hong Bae, Hasung Jang, and Kyung Suh Park. Traders' choice between limit and market orders: evidence from nyse stocks. *Journal of Financial Markets*, 6(4):517–538, 2003.

[BK08] Alexandru M Badescu and Reg J Kulperger. Garch option pricing: A semi-parametric approach. *Insurance: Mathematics and Economics*, 43(1):69–84, 2008.

[BS73] Fischer Black and Myron Scholes. The pricing of options and corporate liabilities. *Journal of political economy*, 81(3):637–654, 1973.

[CDJW10] Peter Christoffersen, Christian Dorion, Kris Jacobs, and Yintian Wang. Volatility components, affine restrictions, and nonnormal innovations. *Journal of Business & Economic Statistics*, 28(4):483–502, 2010.

[Cha94] Don M Chance. Translating the greek: the real meaning of call option derivatives. *Financial Analysts Journal*, 50(4):43–49, 1994.

[Cox09] MAA Cox. Control charts for monitoring observations from a truncated normal distribution. *The Journal of Risk Finance*, 2009.

[CR76] John C Cox and Stephen A Ross. The valuation of options for alternative stochastic processes. *Journal of financial economics*, 3(1-2):145–166, 1976.

[DC12] Oshmita Dey and Debjani Chakraborty. A single period inventory model with a truncated normally distributed fuzzy random variable demand. *International Journal of Systems Science*, 43(3):518–525, 2012.

[dCD09] Joan del Castillo and Jalila Daoudi. The mixture of left–right truncated normal distributions. *Journal of statistical planning and inference*, 139(10):3543–3551, 2009.

[DFW98] Bernard Dumas, Jeff Fleming, and Robert E Whaley. Implied volatility functions: Empirical tests. *The Journal of Finance*, 53(6):2059–2106, 1998.

[GGGP03] Domenico Delli Gatti, Mauro Gallegati, Gianfranco Giulioni, and Antonio Palestrini. Financial fragility, patterns of firms' entry and exit and aggregate dynamics. *Journal of economic behavior & organization*, 51(1):79–97, 2003.

[HK79] J Michael Harrison and David M Kreps. Martingales and arbitrage in multiperiod securities markets. *Journal of Economic theory*, 20(3):381–408, 1979.

[HKMB12] Md Zobaer Hasan, Anton Abdulbasah Kamil, Adli Mustafa, and Md Azizul Baten. Stochastic frontier model approach for measuring stock market efficiency with different distributions. *PloS one*, 7(5), 2012.

[HPR15] Alexandre Hocquard, Nicolas Papageorgiou, and Bruno Remillard. The payoff distribution model: an application to dynamic portfolio insurance. *Quantitative Finance*, 15(2):299–312, 2015.

[IJ76] Jonathan E Ingersoll Jr. A theoretical and empirical investigation of the dual purpose funds: An application of contingent-claims analysis. *Journal of Financial Economics*, 3(1-2):83–123, 1976.

[Lon95] Francis A Longstaff. Option pricing and the martingale restriction. *The Review of Financial Studies*, 8(4):1091–1124, 1995.

[Mer73] RC Merton. Theory of rational option pricing. the bell journal of econometrics and management science, 4, 141-183, 1973.

[Mer76] Robert C Merton. Option pricing when underlying stock returns are discontinuous. *Journal of financial economics*, 3(1-2):125–144, 1976.

[Mwa19] Ivivi J Mwaniki. Geometric brownian motion assumption and generalized hyperbolic distribution on modeling returns. *Journal of Advances in Applied Mathematics*, 4(3), 2019.

[NK80] R Norgaard and T Killeen. Expected utility and the truncated normal distribution. *Management Science*, 26(9):901–909, 1980.

[Nua06] David Nualart. *The Malliavin calculus and related topics*, volume 1995. Springer, 2006.

[Pei99] Amado Peiro. Skewness in financial returns. *Journal of Banking & Finance*, 23(6):847–862, 1999.

[Pen15] Jamol Pender. The truncated normal distribution: Applications to queues with impatient customers. *Operations Research Letters*, 43(1):40–45, 2015.

[RMF05] Svetlozar T Rachev, Christian Menn, and Frank J Fabozzi. *Fat-tailed and skewed asset return distributions: implications for risk management, portfolio selection, and option pricing*, volume 139. John Wiley & Sons, 2005.

[Sco87] Louis O Scott. Option pricing when the variance changes randomly: Theory, estimation, and an application. *Journal of Financial and Quantitative analysis*, 22(4):419–438, 1987.

[Str03] Edward Stringham. The extralegal development of securities trading in seventeenth-century amsterdam. *The Quarterly Review of Economics and Finance*, 43(2):321–344, 2003.

[SZ04] Jinghong Shu and Jin E Zhang. Pricing s&p 500 index options under stochastic volatility with the indirect inference method. *Journal of derivatives accounting*, 1(02):171–186, 2004.

[Wha82] Robert E Whaley. Valuation of american call options on dividend-paying stocks: Empirical tests. *Journal of Financial Economics*, 10(1):29–58, 1982.

[ZH18] Song-Ping Zhu and Xin-Jiang He. A modified black–scholes pricing formula for european options with bounded underlying prices. *Computers & Mathematics with Applications*, 75(5):1635–1647, 2018.

Appendices

Appendix A: Root Finding Problem R-Codes

```
### Root finding to determine the value of mu at 20% price range

BSM.MUT<-function(mu){
  #Onsoti Alex Nyong'a class of 2020
  # University of Nairobi
  da<-diff(log(stock_data[,6]))
  sigmat<-sd(da)
  R<-r*0.01/tau
  a<-log(0.80)
  b<-log(1.20)
  A<-(a-mu*tau)/(sigmat*sqrt(tau))
  B<-(b-mu*tau)/(sigmat*sqrt(tau))
  num<-pnorm(B-sigmat*sqrt(tau))-pnorm(A-sigmat*sqrt(tau))
  den<-pnorm(B)-pnorm(A)
  exp1<-exp((R-0.5*sigmat^2-mu)*tau)
  fn.mu<-(num/den)-exp1
  return(fn.mu)
}
library(rootSolve)

Mu<-uniroot.all(BSM.MUT, c(-0.4,0.4),tol = 1e-25,maxiter = 1000)

### Alternative code

Mu1<-uniroot(BSM.MUT, c(-0.04,0.04),tol = 1e-25,maxiter = 1000)
Mu<-Mu1$root;Mu
```

Appendix B: Black & Scholes Option Prices R-codes

```
BSM73<-function(stock_data,r,so,option_data,tau)
{#Onsoti Alex Nyong'a class of 2020
# University of Nairobi
  da<-diff(log(stock_data[,6]))
  sigmab<-sd(da)
  k<-option_data[,4]
  R<-r*0.01/tau
  d3<-(log(so/k)+(R+.5*sigmab^2)*tau)
  d1<-(1/(sigmab*sqrt(tau)))*d3
  d2<-d1-sigmab*sqrt(tau)
  Ecal<-so*pnorm(d1)-exp(-R*tau)*k*pnorm(d2)
  Eput<--so*pnorm(-d1)+exp(-R*tau)*k*pnorm(-d2)
  Edelta<-pnorm(d1)
  Egamma<-dnorm(d1)/(so*sigmab*sqrt(tau))
  Evega<-so*sqrt(tau)*dnorm(d1)
  Etheta<--so*sigmab/(2*sqrt(tau))*dnorm(d1)-R*k*exp(-R*tau)*pnorm(d2)
  Erho<-tau*k*exp(-R*tau)*pnorm(d2)
  price<-cbind(tau,r,so,k,Ecal,Eput,Edelta,Egamma,Evega,Etheta,Erho)
  return(price)
}
BSM73_Payoffs<-BSM73(stock_data,r,so,option_data,tau)
```

Appendix C: Truncated Normal Disbribution Option Prices R-Codes

```
###### Price of a Truncated Normal Distribution

option<-function(a,b,tau,Mu,R){
 #Onsoti Alex Nyong'a class of 2020
 # University of Nairobi
  da<-diff(log(stock_data[,6]))
  sigmat<-sd(da)
  R<-r*0.01/tau
  k<-option_t_data[,4]
  D1<-(b-Mu*tau)/(sigmat*sqrt(tau))-(sigmat*sqrt(tau))
  N1<-(a-Mu*tau)/(sigmat*sqrt(tau))-(sigmat*sqrt(tau))
  P1<-((log(k/so)-Mu*tau)/(sigmat*sqrt(tau)))-(sigmat*sqrt(tau))
  D2<-(b-Mu*tau)/(sigmat*sqrt(tau))
  N2<-(a-Mu*tau)/(sigmat*sqrt(tau))
  P2<-(log(k/so)-(Mu*tau))/(sigmat*sqrt(tau))
  N1.Call<-pnorm(D1)-pnorm(P1)
  D1.Call<-pnorm(D1)-pnorm(N1)
  N2.Call<-pnorm(D2)-pnorm(P2)
  D2.Call<-pnorm(D2)-pnorm(N2)
  N1.Put<-pnorm(P2)-pnorm(N2)
  D1.Put<-pnorm(D2)-pnorm(N2)
  N2.Put<-pnorm(P1)-pnorm(N1)
  D2.Put<-pnorm(D1)-pnorm(N1)
  vc<-so*(N1.Call/D1.Call)-k*exp(-R*tau)*(N2.Call/D2.Call)
  vp<-k*exp(-R*tau)*(N1.Put/D1.Put)-so*(N2.Put/D2.Put)
  price<-cbind(vc,vp,k,a,b,so,tau,sigmat,R)
  return(price)
}
TND_Payoffs<-option(a,b,tau,Mu,R);TND_Payoffs
```

Appendix D: Pricing Errors R-Codes

D.1 Call Errors

```
####################################################
BCALL<-BSM73_Payoffs[,5];MCALL<-option_data[,3]
TCALL<-TND_Payoffs[,1];TMCALL<-option_t_data[,3]

###### Dollar Root Mean Squared Absolute Error
RMSE.BCALL<-sqrt(sum((MCALL-BCALL)^2)/length(MCALL))
RMSE.TCALL<-sqrt(sum((TMCALL-TCALL)^2)/length(TMCALL))

####### Average Absolute Error
APE.BCALL<-1/length(MCALL)*sum(abs(MCALL-BCALL)/mean(MCALL))*100
APE.TCALL<-1/length(TMCALL)*sum(abs(TMCALL-TCALL)/mean(TMCALL))*100

###### Average Relative Pricing Error
ARPE.BCALL<-1/length(MCALL)*sum(abs(MCALL-BCALL)/MCALL)*100
ARPE.TCALL<-1/length(TMCALL)*sum(abs(TMCALL-TCALL)/TMCALL)*100
```

D.2 Put Errors

```
####################################################
BPUT<-BSM73_Payoffs[,6];MPUT<-option_data[,7]
TPUT<-TND_Payoffs[,2];TMPUT<-option_t_data[,7]

###### Dollar Root Mean Squared Absolute Error
RMSE.BPUT<-sqrt(sum((MPUT-BPUT)^2)/length(MPUT))
RMSE.TPUT<-sqrt(sum((TMPUT-TPUT)^2)/length(TMPUT))

####### Average Absolute Error
APE.BPUT<-1/length(MPUT)*sum(abs(MPUT-BPUT)/mean(MPUT))*100
APE.TPUT<-1/length(TMPUT)*sum(abs(TMPUT-TPUT)/mean(TMPUT))*100

###### Average Relative Pricing Error
ARPE.BPUT<-1/length(MPUT)*sum(abs(MPUT-BPUT)/MPUT)*100
ARPE.TPUT<-1/length(TMPUT)*sum(abs(TMPUT-TPUT)/TMPUT)*100
```