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Lindley Distribution and its Generalization in Poisson Mixtures

Research Report in Mathematics, Number four, 2020

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School of Mathematics

Submitted to the School of Mathematics in partial fulfilment for a degree in Master of Science in Statistics

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Abstract

There are many distributions for modeling lifetime data, among the known parametric models, the most popular is Gamma, Beta, Poisson and Lindley. Lindley distribution is a way to describe the lifetime of a process.

The exponential distribution is a close form of the Lindley distribution.Due to the popularity of the exponential distribution in statistics and many applied areas, the Lindley distribution has been overlooked in the literature.

In this project we aim to construct the Lindley distribution using various number of parameters,then examine its properties namely moments,failure rate function and mean residual life function.

The Poisson-Lindley mixture is examined in depth,both construction,properties and estimation of the distribution. Estimation of properties is determined using the method of moments, maximum likelihood method and the expectation maximization algorithm method .The methods are applied on various data sets and the results compared.

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Declaration and Approval

I the undersigned declare that this dissertation is my original work and to the best of my knowledge, it has not been submitted in support of an award of a degree in any other university or institution of learning.

Signature Date

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In my capacity as a supervisor of the candidate's dissertation, I certify that this dissertation has my approval for submission.

Signature Date

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Dedication

I dedicate this work to The Almighty God for giving me the energy,good health and resources to undertake this work.To my father, Joe Gicheru,and my husband, John Kinyati, thank you for the continuous encouragement and the sacrifices you have made in order for me to achieve this.May the Lord God bless you.I also dedicate it to my fellow classmates,thanks you for the various sources for information you provided to me.

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Special thanks to my family for all the sacrifices you made,monetary and emotional support.May the Lord bless you abundantly.

To my fellow classmates,am very grateful for the wonderful memories shared.My hope is that our friendship continues past our academics interactions.

Purity Gicheru

Nairobi, 2020.

Contents

1 INTRODUCTION

1.1 Background Information

The Lindley distribution was first introduced by D.V.Lindley(1958) during the study of the conditions necessary to convert a fiducial distribution into a Bayes' distribution.

The Lindley distribution in particular have found a wide range of applications in mathematical modelling in many real life situations banking industry(Ghitany,2009),computer simulations(S.Nadarajah,2008)and epidemiology(Mishra and Shanker,2015).

The distribution can be constructed from a generalized four parameter distribution which is then modified to derive other distribution namely one parameter(Lindley, 1958), two parameter distribution(Zakerzadeh and Dollati,2010 among others) and three paratemer distribution(Bhati et al,2015).

In this work construction and estimation of the Lindley-Poisson mixture is greatly considered.This study is divided into several chapters.The rest of this chapter gives the notations and terminologies used in this paper, describes research problem, highlights the objectives and examines related work done so far.

Chapter two examines the construction and moments of a generalized four parameter Lindley distribution.At the same,special cases of the distribution are derived and their properties studied.

Chapter three introduces the Poisson-Lindley mixture.The one parameter Possion-Lindley distribution is constructed,estimated using various methods and results compared.

Chapter four describes the various forms of a two parameter Poisson-Lindley distribution, taking into consideration the construction and estimation.

Chapter fives deals with the forms of a three parameter Poisson-Lindley distribution, examining the moments, estimation and application.

Chapter six concludes the study and gives future recommendations.

1.2 Definitions, Notations and Terminologies

Let $f(z)$ be a function of a random variable Z.For a continuous random variable Z the probability distribution function is

$$f(z) \geq 0 \ and \int_{-\infty}^{\infty} f(z)\, d(z) = 1 \tag{1.2.1}$$

While for a discrete random variable the probability mass function is

$$f(z) \geq 0 \ and \sum_{z=-\infty}^{\infty} f(z) = 1 \tag{1.2.2}$$

The probability distribution function of the Possion distribution is

$$f(z) = \frac{e^{-\lambda}\lambda^{z}}{z!},\ z = 0,1,2,...\lambda > 0 \tag{1.2.3}$$

For any Poisson mixture ,the probability distribution function is

$$f(z) = \int_{0}^{\infty} \frac{e^{-\lambda}\lambda^{z}}{z!} g(\lambda)\ d\lambda \tag{1.2.4}$$

Where $g(\lambda)$ are the various forms of the Lindley distributions.

1.3 Research problem

Sarguta(2017) constructed mixed Poisson distribution which are expressed in explicit forms, in terms of modified Bessel functions of third kind and confluent hyper geometric function,recursively form and in expectation forms.

She did not however consider the estimation problem.This paper reconstructs the Poisson -Lindley mixture, then estimates it.

1.4 Objectives

The main objective of this study is to examine various estimation methods of the four parameter generalized Lindley distribution and its special cases.
Specific objectives include

1. To construct the four parameter generalized Lindley distribution and derive the special cases.
2. To obtain the moments of the special cases
3. To construct the various distributions of the Poisson-Lindley mixture
4. To estimate Poisson -Lindley mixture distributions by MOM method then introduce other methods.
5. To compare the different estimation methods.

1.5 Methodology

Methods used in construction of the four parameter generalized distribution and the Poisson-Lindley mixture include direct integration and substitution.

Estimation methods are:

Method of Moments
For a random sample of variable x, equate the first sample moment about the origin $M_1 = \frac{1}{n}\sum_{i=1}^{n} X_i = \bar{X}$ to the first population moments $E(X)$.

Then equate the second sample moment about the mean $M_2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2$ to $E(X-\mu)^2$.

From these equations solve for the parameter.

There are two ways of calculating the moments,namely:

Conditional method:

It is applicable when the sample is incomplete,thus difficult to calculate the sample moments.

Consider $Z = (z1, z2..zn)$ to independent and identically distributed random variables and $m_k(Z)$ to be the sample moments,if Y is the observed sample and $a_k(\theta, Y)$ the conditional expectation of $m_k(Z)$ given the observed Y;

$$a_k(\theta, Y) = E[m_k(Z|Y)]$$

$a_k(\theta, Y)$ is the k-th non central conditional sample moments.If the sample is incomplete, it becomes the optimal estimation for $m_k(Z)$ under MSE error minimization criteria.

Direct method:

Applicable when the sample size is complete.It involves evaluating the experimental moments to the real moments.

Maximum Likelihood Method

The method involves calculating the likelihood of the distribution pdf

$$L = \prod_{i=1}^{n} f(x_i)$$

If L is differentiable, the derivative test to determine the maxima is applied hence solving the parameter value.
The MLE estimator should be consistent(the sequence converges to the value being estimated) and efficient (achieves the Cramer-Rao lower bound when the sample size tends to infinity)

Expectation Maximization Method

The algorithm was derived by Dempster et al(1977) as a way to calculate the MLE for data containing missing values.
There are two main applications of the algorithm.

- Data missing due to problems with or limitations of the observation process.
- When optimizing the likelihood function is analytically intractable but the function can be simplified by assuming the existence of and values for additional but missing parameter.

The EM algorithm consists of two steps: For a Poisson mixture formulation with no covariates, the missing data is the realization of θ_i of the unobserved mixing parameter for each point z_i

1. E-Step : Involves calculating the conditional expectation of some functions of the parameter in order to maximize the likelihood of the complete model which reduces to maximization of the mixing distribution density.

   For mixtures from the exponential family the conditional expectations coincide with the sufficient statistics. If the mixture contains covariate;calculation of the posterior expectation of the sufficient statistics is done.

2. M-step: Maximizes the complete data likelihood and updates the parameters using the conditional expectations obtained in E-step in order to fit a GLM of the simple underlying distribution.

1.6 Applications of Lindley distribution

The Lindley distribution has been applied in several areas such as:

1. Biological sciences:
   Shanker and Fesshaye(2015)used the Poisson -Lindley distribution to analyse the relationship between organisms and their environment in an ecology study.

2. Acturial sciences:
   Sankaran(1970)applied the Poisson-Lindley distribution to errors and accidents while Ghitany (2009) applied the same distribution to determine the service rate(how long a customer waits on queue) at the bank.

2 LITERATURE REVIEW

Lindley distribution

The distribution was derived during the study of fiducial distributions and Bayes' theorem in 1957 by D.V.Lindley.

Let x be one-dimensional random variable whose distribution depends on a single parameter θ. The study aimed to find :

1. The necessary and sufficient condition for θ, given x, to be a Bayes' distribution(existence of transformations of x to μ, and of θ to τ, such that τ is a location parameter for μ.
2. If, for random sized sample from the distribution for x, there exists a single sufficient statistic for θ then the fiducial argument is inconsistent unless condition 1 obtains: and when it does, the fiducial argument is equivalent to a Bayesian argument with uniform prior distribution for τ.

If $F(x|\theta)$ is the distribution of x for values of a real parameter θ in one dimensional sets, then the fiducial distribution for is given by

$$\phi_x(\theta) = -\frac{\partial}{\partial\theta}F(x|\theta)$$

$$fx(\theta) = -\frac{\partial}{\partial\theta}F(x|\theta)$$

Imposing both upper and lower limits on the above equation help to derive the Bayesian argument through prior and posterior distribution of θ which proves that it's a both sufficient and necessary condition.

$$lim_{\theta\to U}F(x|\theta) = 0 \; lim_{\theta\to L}F(x|\theta) = 1$$

For the consistency condition of the fiducial distribution, consider to sufficient statistics x and y for θ which are independent.
If $\psi_{x,y}(\theta)$ is Bayes' posterior distribution for θ using Bayes' theorem for y with prior distribution $\phi_x(\theta)$ and $\phi_{x,y}(\theta)$ is the fiducial distribution for θ given x and y with no prior knowledge of θ ,then the aim is to proof that:

$$\phi_{x,y}(\theta) = \psi_{x,y}(\theta)$$

The results prove that the fiducial argument is consistent if, and only if, it is equivalent to a Bayesian argument under condition 1 with uniform prior distribution for the location parameter.
In one dimensional sets

$$f_x(\theta) = \frac{\theta^2}{\theta+1}(x+1)e^{-\theta x}\,; x>0, \theta>0$$

which is the Lindley one parameter distribution.
Various studies by different people namely M.Ghitany and S.Nadarajah(2007) among others have explored, the properties of this distribution as discussed in this work.

Two parameter Lindley distribution

In this work,the distribution is divided into three namely,Type I,type II and type III.

Type I

The distribution was introduced by Zakerzadeh and Dolati(2009) when they generalized the Lindley distribution properties in order to provide more flexibility which allows analysis of different lifetime data.

The distribution includes special cases the ordinary exponential and gamma distributions.

$$g(\lambda) \sim Gamma\ (\alpha,\theta)\ and\ g(\lambda) \sim Gamma\ (\alpha+1,\theta)$$

$$g(\lambda) = \frac{\theta^{\alpha+1}}{\Gamma\alpha+1}\frac{1}{\lambda+1}\,(\alpha+\lambda)\,e^{-\theta\lambda}\lambda^{\alpha-1}$$

The derivation and properties such as the cdf, moments and survival functions of this distribution have discussed in this work.

Type II

R.Shanker and A.Mishra(2013) discovered the distribution.It comprises of two parameter $\alpha\ and\ \theta$ and a special case of exponential distribution.

$$f(x,\alpha,\theta) = \frac{\theta^2}{\alpha\theta+1}(\alpha+x)e^{-\theta x}\,; x>0, \theta, \alpha>0$$

The equation reduces to Lindley one parameter distribution when$\alpha=1$ and at $\alpha=0$ it reduces to a gamma distribution with parameters $(2,\theta)$.
The derivation and properties such as the cdf, moments and survival functions of this distribution have discussed in this work.

Type III

Following Shanker's work,D.Bhati et al (2015) introduced another two parameter Lindley distribution during the study of a new generalized Poisson-Lindley distribution.

$$f(x,\alpha,\theta) = \frac{\theta^2}{\alpha+\theta}(\alpha x+1)e^{-\theta x}\,; x>0,\theta,\alpha>0$$

The derivation and properties such as the cdf, moments and survival functions of this distribution have discussed in this work.

Three parameter Lindley distribution

Part of Zakerzadeh and Dolati work (2009) includes a three parameter distribution(α,θ,β).For a random variable x,

$$f(x,\alpha,\beta,\theta) = \frac{\theta^{\alpha+1}}{\beta+\theta}\frac{x^{\alpha-1}}{\Gamma\alpha+1}(\alpha+\beta x)\,e^{-\theta x}$$

R.Shanker et al(2017) introduced another three parameter Lindley distribution

$$f(x,\alpha,\beta,\theta) = \frac{\theta^2}{\theta\alpha+\beta}(\alpha+\beta x)\,e^{-\theta x}$$

The derivation and properties such as the cdf, moments and survival functions of this distribution have discussed in this work.

Four parameter Lindley distribution

Consider

$$g(\lambda) \sim Gamma\ (\alpha, \theta)\ and\ g(\lambda) \sim Gamma\ (\alpha+1, \theta)$$

and

$$p_1 = \frac{\beta\theta}{\beta\theta+\delta} \Rightarrow p_2 = \frac{\delta}{\beta\theta+\delta}$$

where $g(\lambda) = p_1 g_1(\lambda) + p_2 g_2(\lambda)$

$$g(\lambda) = \frac{\theta^{\alpha+1}}{\Gamma\alpha+1}\frac{1}{\beta\theta+\delta}(\alpha\beta+\delta\lambda)e^{-\theta\lambda}\lambda^{\alpha-1}$$

$$f(x,\alpha,\beta,\theta,\delta) = \frac{\theta^{\alpha+1}}{\Gamma\alpha+1}\frac{1}{\beta\theta+\delta}(\alpha\beta+\delta\lambda)e^{-\theta\lambda}\lambda^{\alpha-1}$$

3 GENERALIZED FOUR PARAMETER LINDLEY DISTRIBUTION AND ITS SPECIAL CASES

3.1 Introduction

In this chapter a generalized four parameter Lindley (G4L) distribution is constructed using a finite mixture of two gamma distributions.
The rth moment in general is derived and in particular ,the mean and variance have been obtained.
Special cases of the G4L distribution have been deduced and their properties derived.
An extension to a generalized five parameter Lindley (G5L) distribution has been suggested.

3.2 Construction and moments of G4L distribution

Define a finite mixture by

$$g(\lambda) = p_1 g_1(\lambda) + p_2 g_2(\lambda) \tag{1}$$

where

$$p_1 > 0, p_2 > 0 \, and \;\; p_1 + p_2 = 1 \tag{2}$$

Let

$$g_1(\lambda) \sim Gamma\ (\alpha, \theta)$$

$$g_1(\lambda) = \frac{\theta^{\alpha}}{\Gamma\alpha} e^{-\theta\lambda} \lambda^{\alpha-1} \;\; ; \alpha > 0, \theta > 0, \lambda > 0 \;\; g_2(\lambda) \sim Gamma\ (\alpha+1, \theta) \tag{3}$$

$$g_2(\lambda) = \frac{\theta^{\alpha+1}}{\Gamma\alpha+1} e^{-\theta\lambda} \lambda^{(\alpha+1)-1} \;\; ; \alpha > 0, \theta > 0, \lambda > 0 \tag{4}$$

Suppose

$$p_1 = \frac{\beta\theta}{\beta\theta + \delta} \;\Rightarrow\; p_2 = \frac{\delta}{\beta\theta + \delta} \tag{5}$$

Then

$$\begin{aligned} g(\lambda) &= \frac{\beta\theta}{\beta\theta+\delta}\frac{\theta^{\alpha}}{\Gamma\alpha}e^{-\theta\lambda}\lambda^{\alpha-1} + \frac{\delta}{\beta\theta+\delta}\frac{\theta^{\alpha+1}}{\Gamma\alpha+1}e^{-\theta\lambda}\lambda^{\alpha} \\ &= \frac{\theta^{\alpha+1}}{\beta\theta+\delta}\frac{\beta}{\Gamma\alpha}e^{-\theta\lambda}\lambda^{\alpha-1} + \frac{\theta^{\alpha+1}}{\beta\theta+\delta}\frac{\delta}{\Gamma\alpha+1}e^{-\theta\lambda}\lambda^{\alpha} \\ &= \frac{\theta^{\alpha+1}}{\Gamma\alpha+1}\frac{\alpha\beta}{\beta\theta+\delta}e^{-\theta\lambda}\lambda^{\alpha-1} + \frac{\theta^{\alpha+1}}{\Gamma\alpha+1}\frac{\delta}{\beta\theta+\delta}e^{-\theta\lambda}\lambda^{\alpha} \\ &= \frac{\theta^{\alpha+1}}{\Gamma\alpha+1}\frac{1}{\beta\theta+\delta}(\alpha\beta+\delta\lambda)e^{-\theta\lambda}\lambda^{\alpha-1} \\ & for\ \lambda>0;\alpha,\gamma,\delta,\theta>0 \end{aligned} \tag{6}$$

which is a G4L pdf.

The r^{th} moment is

$$\begin{aligned} E(\Lambda^r) &= \int_0^{\infty}\lambda^r\frac{\theta^{\alpha+1}}{\Gamma\alpha+1}\frac{1}{\beta\theta+\delta}(\alpha\beta+\delta\lambda)e^{-\theta\lambda}\lambda^{\alpha-1}\,d\lambda \\ &= \frac{\theta^{\alpha+1}}{\Gamma\alpha+1}\frac{1}{\beta\theta+\delta}\left[\alpha\beta\int_0^{\infty}\lambda^{\alpha+r-1}e^{-\theta\lambda}\,d\lambda+\delta\int_0^{\infty}\lambda^{(\alpha+r+1)-1}e^{-\theta\lambda}\,d\lambda\right] \\ &= \frac{\theta^{\alpha+1}}{\Gamma\alpha+1}\frac{1}{\beta\theta+\delta}\left[\alpha\beta\frac{\Gamma r+\alpha}{\theta^{r+\alpha}}+\delta\frac{\Gamma r+\alpha+1}{\theta^{r+\alpha+1}}\right] \\ &= \frac{\theta^{\alpha+1}}{\Gamma\alpha+1}\frac{1}{\beta\theta+\delta}\left[\frac{\alpha\beta\theta\Gamma r+\alpha}{\theta^{r+\alpha+1}}+\frac{\delta(r+\alpha)\Gamma r+\alpha}{\theta^{r+\alpha+1}}\right] \\ &= \frac{\theta^{\alpha+1}}{\Gamma\alpha+1}\frac{1}{\beta\theta+\delta}\frac{\Gamma r+\alpha}{\theta^{r+\alpha+1}}\left[\alpha\beta\theta+\delta(r+\alpha)\right] \\ &= \frac{\Gamma r+\alpha}{\Gamma\alpha+1(\beta\theta+\delta)\theta^r}\left[\alpha\beta\theta+\delta r+\delta\alpha\right] \end{aligned}$$

When $r=1$

$$E(\Lambda)=\frac{\alpha\beta\theta+\delta+\delta\alpha}{(\beta\theta+\delta)\theta}$$

When $r=2$

$$\begin{aligned} E(\Lambda^2) &= \frac{\Gamma\alpha+2}{\Gamma\alpha+1}\left[\frac{\alpha\beta\theta+\delta+\delta\alpha}{(\beta\theta+\delta)\theta^2}\right] \\ &= \frac{(\alpha+1)(\alpha\beta\theta+2\delta+\delta\alpha)}{(\beta\theta+\delta)\theta^2} \end{aligned}$$

$$\text{Var}\Lambda = E(\Lambda^2) - [E(\Lambda)]^2$$
$$= \frac{(\alpha+1)(\alpha\beta\theta+2\delta+\delta\alpha)}{(\beta\theta+\delta)\theta^2} - \frac{(\alpha\beta\theta+\delta+\delta\alpha)^2}{(\beta\theta+\delta)^2\theta^2}$$
$$= \frac{(\alpha+1)(\beta\theta+\delta)(\alpha\beta\theta+2\delta+\delta\alpha)-(\alpha\beta\theta+\delta+\delta\alpha)^2}{(\beta\theta+\delta)^2\theta^2}$$

3.3 Special cases of the G4L distribution and their properties

3.3.1 Lindley distribution

Let

$$\alpha = \beta = \delta = 1$$

Then

$$g_1(\lambda) \sim Gamma(1,\theta) = exp(\theta)$$
$$g_2(\lambda) \sim Gamma(2,\theta)$$
$$p_1 = \frac{\theta}{\theta+1} \quad p_2 = \frac{1}{\theta+1}$$

Where

$$g(\lambda) = \frac{\theta^2}{\theta+1}(\lambda+1)e^{-\theta\lambda} \; ; \lambda > 0, \theta > 0 \tag{7}$$

As obtained by Lindley(1958) hence the name Lindley one parameter distribution.

Proposition 3.3.1

The cdf of the Lindley distribution is given by

$$G(\lambda) = 1 - \frac{e^{-\theta\lambda}(1+\theta+\theta\lambda)}{\theta+1}$$

The survival function is

$$1 - G(\lambda) = \frac{e^{-\theta\lambda}(1+\theta+\theta\lambda)}{\theta+1}$$

The hazard function is

$$h(\lambda) = \frac{\theta^2(1+\lambda)}{1+\theta+\theta\lambda}$$

Proof:

$$\begin{aligned}
G(\lambda) &= \int_0^\lambda g(t)d(t) \\
&= \int_0^\lambda \frac{\theta^2}{1+\theta}(t+1)e^{-\theta t}dt \\
&= \frac{\theta^2}{1+\theta}\left[\int_0^\lambda e^{-\theta t}dt + \int_0^\lambda te^{-\theta t}dt\right] \\
\frac{\theta^2}{1+\theta}&\left[\frac{1-e^{-\theta\lambda}}{\theta} + \frac{\lambda e^{-\theta\lambda}}{\theta} - \frac{e^{-\theta\lambda}}{\theta^2} + \frac{1}{\theta^2}\right] \\
&= \frac{1}{\theta+1}\left[\theta(1-e^{-\theta\lambda}) - \theta\lambda e^{-\theta\lambda} - e^{-\theta\lambda} + 1\right] \\
&= \frac{1}{\theta+1}\left[(\theta+1) - (1+\theta+\theta\lambda)e^{-\theta\lambda}\right] \\
&= 1 - \frac{e^{-\theta\lambda}(1+\theta+\theta\lambda)}{\theta+1}
\end{aligned}$$

Thus
The Survival function is

$$\begin{aligned}
1-G(\lambda) &= 1 - \left[1 - \frac{e^{-\theta\lambda}(1+\theta+\theta\lambda)}{\theta+1}\right] \\
&= \frac{e^{-\theta\lambda}(1+\theta+\theta\lambda)}{\theta+1}
\end{aligned}$$

The Hazard function is

$$\begin{aligned}
h(\lambda) &= \frac{g(\lambda)}{1-G(\lambda)} \\
&= \frac{\theta^2(1+\lambda)}{1+\theta+\theta\lambda}
\end{aligned}$$

Proposition 3.3.2

The r^{th} moments of a Lindley one parameter distribution is

$$E(\Lambda^r) = \frac{r!}{\theta^r(\theta+1)}(\theta+r+1)$$

When r=1

$$E(\Lambda) = \frac{\theta+2}{\theta(\theta+1)}$$

When r=2

$$E(\Lambda^2) = \frac{2(\theta+3)}{\theta^2(\theta+1)}$$

When r=3

$$E(\Lambda^3) = \frac{6(\theta+4)}{\theta^3(\theta+1)}$$

When r=4

$$E(\Lambda^4) = \frac{24(\theta+5)}{\theta^4(\theta+1)}$$

Proof:

$$E(\Lambda^r) = \int_0^\infty \lambda^r \frac{\theta^2}{\theta+1}(\lambda+1)e^{-\theta\lambda}\,d\lambda$$
$$= \frac{\theta^2}{\theta+1}\left[\int_0^\infty \lambda^r e^{-\theta\lambda}\,d\lambda + \int_0^\infty \lambda^{r+1}e^{-\theta\lambda}\,d\lambda\right]$$
$$= \frac{\theta^2}{\theta+1}\left[\frac{\Gamma r+1}{\theta^{r+1}} + \frac{\Gamma r+2}{\theta^{r+2}}\right]$$
$$= \frac{\theta^2}{\theta+1}\frac{\Gamma r+1}{\theta^{r+2}}[\theta+r+1]$$
$$= \frac{r!}{\theta^r(\theta+1)}(\theta+r+1)$$

Replace r with values 1,2,3,4 in order to get the above equations

Proposition 3.3.3

The r^{th} central moments of the Lindley distribution are:

$$\mu_2(\lambda) = E[\Lambda - E(\Lambda)]^2 = \frac{\theta^2 4\theta + 2}{\theta^2(\theta+1)^2}$$

$$\mu_3(\lambda) = E[\Lambda - E(\Lambda)]^3 = \frac{5\theta^3 + 30\theta^2 + 42\theta + 16}{\theta^3(\theta+1)^3}$$

$$\mu_4(\lambda) = E[\Lambda - E(\Lambda)]^4 = \frac{23\theta^4 + 184\theta^3 + 408\theta^2 + 352\theta + 104}{\theta^4(\theta+1)^4}$$

Proof:

$$\mu_2 = E(\Lambda^2) - (E(\Lambda))^2 = \frac{2(\theta+3)}{\theta^2(\theta+1)} - \frac{(\theta+2)^2}{\theta^2(\theta+1)^2} = \frac{(\theta+1)(2\theta+3) - (\theta+2)(\theta+2)}{\theta^2(\theta+1)^2} = \frac{\theta^2 + 4\theta + 2}{\theta^2(\theta+1)^2}$$

$$\mu_3 = E(\Lambda^3) - (E(\Lambda))^3$$
$$= \frac{6(\theta+4)}{\theta^3(\theta+1)} - \frac{(\theta+2)^3}{\theta^3(\theta+1)^3}$$
$$= \frac{(\theta+1)^2(6\theta+24) - (\theta+2)^3}{\theta^3(\theta+1)^3}$$
$$= \frac{6\theta^3+36\theta^2+54\theta+24-\theta^3-6\theta^2-12\theta-8}{\theta^3(\theta+1)^3}$$
$$= \frac{5\theta^3+30\theta^2+42\theta+16}{\theta^3(\theta+1)^3}$$

$$\mu_4 = E(\Lambda^4) - (E(\Lambda))^4$$
$$= \frac{24(\theta+5)}{\theta^4(\theta+1)} - \frac{(\theta+2)^4}{\theta^4(\theta+1)^4}$$
$$= \frac{(\theta+1)^3(24\theta+12\theta) - (\theta+2)^4}{\theta^4(\theta+1)^4}$$
$$= \frac{23\theta^4+184\theta^3+408\theta^2+352\theta+104}{\theta^4(\theta+1)^4}$$

Proposition 3.3.4

The mode of Lindley one parameter distribution is

$$\lambda = \frac{1-\theta}{\theta} \; for \, 0 < \theta < 1$$

Proof:

$$\frac{d}{d\lambda}g(\lambda) = 0$$

$$\frac{d}{d\lambda}\left[\frac{\theta^2}{\theta+1}(\lambda+1)e^{-\theta\lambda}\right] = 0$$

$$\Rightarrow \frac{d}{d\lambda}(1+\lambda)e^{-\theta\lambda} = 0$$

$$-\theta(1+\lambda)e^{-\theta\lambda} + e^{-\theta\lambda} = 0$$

$$-\theta(1+\lambda)+1 = 0$$

$$\theta(1+\lambda) = 1$$

$$\theta + \theta\lambda = 1 \Rightarrow \lambda = \frac{1-\theta}{\theta} \ ;0,\theta < 1$$

Proposition 3.3.5

The mean residual lifetime is defined as

$$prob(X > z) = 1 - prob(x \leq z) = 1 - F(z)$$

Implying

$$\begin{aligned} 1 - F(z) &= prob(x > z) \\ &= \int_z^\infty f(x)\,dx \\ 1 &= \int_z^\infty \frac{f(x)}{1-F(z)}\,dx \end{aligned}$$

The mean is given by

$$m(z) = E\left[X - z | x > z\right] = \int_z^\infty \frac{(x-z)f(x}{1-F(z)}\,dx$$

which is the expected additional lifetime given that a component has survived until time z.

Using integration by parts;

$$m(z) = \int_z^\infty \frac{1-F(x)}{1-F(z)}\,dx$$

For Lindley distribution;

$$\begin{aligned} m(z) &= \int_z^\infty \frac{1-F(\lambda)}{1-F(z)}\ d\lambda \\ &= \int_z^\infty \frac{(1+\theta+\theta\lambda)e^{-\theta\lambda}}{(1+\theta+\theta z)e^{-\theta z}}\ d\lambda \\ &= \frac{2+\theta+\theta z}{\theta(1+\theta+\theta z)} \end{aligned}$$

3.3.2 Type I generalized two parameter Lindley Distribution

Let

$$\beta = \delta = 1\ \ \alpha > 0\ and\ \theta > 0$$

Then

$$\begin{aligned} g_1(\lambda) &\sim Gamma\ (\alpha, \theta) \\ g_1(\lambda) &= \frac{\theta^\alpha}{\Gamma\alpha} e^{-\theta\lambda} \lambda^{\alpha-1}\ ;\alpha > 0, \theta > 0, \lambda > 0 \\ g_2(\lambda) &\sim Gamma\ (\alpha+1, \theta) \\ g_2(\lambda) &= \frac{\theta^{\alpha+1}}{\Gamma\alpha+1} e^{-\theta\lambda} \lambda^{(\alpha+1)-1}\ ;\alpha > 0, \theta > 0, \lambda > 0 \end{aligned}$$

$$p_1 = \frac{\theta}{\theta+1}\ and\ p_2 = \frac{1}{\theta+1}$$

$$\begin{aligned} g(\lambda) &= p_1 g_1(\lambda) + p_2 g_2(\lambda) \\ &= \frac{\theta}{\theta+1}\frac{\theta^\alpha}{\Gamma\alpha} e^{-\theta\lambda}\lambda^{\alpha-1} + \frac{1}{\theta+1}\frac{\theta^{\alpha+1}}{\Gamma\alpha+1} e^{-\theta\lambda}\lambda^{(\alpha+1)-1} \\ &= \frac{\theta^{\alpha+1}}{\theta+1}\frac{1}{\Gamma\alpha} e^{-\theta\lambda}\lambda^{\alpha-1} + \frac{\theta^{\alpha+1}}{\theta+1}\frac{1}{\Gamma\alpha+1} e^{-\theta\lambda}\lambda^{\alpha} \\ &= \frac{\theta^{\alpha+1}}{\Gamma\alpha+1}\frac{1}{\theta+1}(\alpha+\lambda)\, e^{-\theta\lambda}\lambda^{\alpha-1}\ \ for \lambda > 0, \alpha, \theta > 0 \end{aligned}$$

as obtained by Zakerzadeh and Dolati(2010).This is a type 1 two parameter Lindley distribution.

Proposition 3.3.5

The cdf of the distribution is given by

$$
\begin{aligned}
G(\lambda) &= \int_0^\lambda g(t)dt \\
&= \int_0^\lambda \frac{\theta^{\alpha+1}}{\Gamma\alpha+1}\frac{1}{\theta+1}(\alpha+t)\,e^{-\theta t}t^{\alpha-1}dt \\
&= \frac{\theta^{\alpha+1}}{\Gamma\alpha+1}\frac{1}{\theta+1}\int_0^\lambda(\alpha+t)\,e^{-\theta t}t^{\alpha-1}dt \\
&= \frac{\theta^{\alpha+1}}{\Gamma\alpha+1}\frac{1}{\theta+1}\left[\alpha\int_0^\lambda e^{-\theta t}t^{\alpha-1}dt+\int_0^\lambda te^{-\theta t}dt\right] \\
&= \frac{\theta^{\alpha+1}}{\Gamma\alpha+1}\frac{1}{\theta+1}\left[\frac{\alpha\Gamma\alpha}{\theta^\alpha}+\frac{\lambda e^{-\theta\lambda}}{-\theta}-\frac{e^{-\theta\lambda}}{\theta^2}+\frac{1}{\theta^2}\right] \\
&= \frac{\theta^{\alpha+1}}{\Gamma\alpha+1}\frac{1}{\theta+1}\left[\frac{\theta^2\alpha\Gamma\alpha}{\theta^\alpha}-\theta\lambda e^{-\theta\lambda}-e^{-\theta\lambda}+1\right] \\
&= \frac{\theta^{\alpha+1}}{\Gamma\alpha+1}\frac{1}{\theta+1}\left[\frac{\theta^2\Gamma\alpha+1}{\theta^\alpha}-(\theta+1)e^{-\theta\lambda}+1\right] \\
&= \frac{\theta^{\alpha+1}}{\Gamma\alpha+1}\frac{1}{\theta+1}\frac{\alpha\Gamma\alpha}{\theta^\alpha}\left(-\theta\lambda e^{-\lambda}-e^{-\theta\lambda}+1\right) \\
&= \frac{\theta}{\theta+1}\left(1-(\theta\lambda+1)e^{-\theta\lambda}\right)
\end{aligned}
$$

The survival function can only be given in terms of an incomplete gamma function. The hazard function $h(\lambda)=\frac{g(\lambda)}{1-G(\lambda)}$ can not be expressed in closed form.

Proposition 3.3.6

The rth moments of the distribution is given by

$$
\frac{\Gamma\alpha+r}{\theta^r\Gamma\alpha+1}\left[\frac{\alpha\theta+r+\alpha}{\theta+1}\right]
$$

Proof:

$$E(\Lambda^r) = \int_0^\infty \lambda^r \frac{\theta^{\alpha+1}}{\Gamma\alpha+1}\frac{1}{\theta+1}(\alpha+\lambda)e^{-\theta\lambda}\lambda^{\alpha-1}d\lambda$$
$$= \frac{\theta^{\alpha+1}}{\Gamma\alpha+1}\frac{1}{\theta+1}\int_0^\infty(\alpha+\lambda)e^{-\theta\lambda}\lambda^{\alpha-1+r}d\lambda$$
$$= \frac{\theta^{\alpha+1}}{\Gamma\alpha+1}\frac{1}{\theta+1}\left[\alpha\int_0^\infty e^{-\theta\lambda}\lambda^{\alpha-1+r}d\lambda + \int_0^\infty e^{-\theta\lambda}\lambda^{\alpha+r}d\lambda\right]$$
$$= \frac{\theta^{\alpha+1}}{\Gamma\alpha+1}\frac{1}{\theta+1}\left[\frac{\alpha\Gamma\alpha+r}{\theta^{r+\alpha}} + \frac{\Gamma r+\alpha+1}{\theta^{r+\alpha+1}}\right]$$
$$= \frac{\theta^{\alpha+1}}{\Gamma\alpha+1}\frac{1}{\theta+1}\frac{\Gamma\alpha+r}{\theta^{r+\alpha}}\left(\alpha+\frac{r+\alpha}{\theta}\right)$$
$$= \frac{\theta^{\alpha}+1}{\theta^{r+\alpha+1}}\frac{\Gamma\alpha+r}{\Gamma\alpha+1}\left(\frac{\alpha\theta+r+\alpha}{\theta+1}\right)$$
$$= \frac{\Gamma\alpha+r}{\theta^r\Gamma\alpha+1}\left(\frac{\alpha\theta+r+\alpha}{\theta+1}\right)$$

when r=1

$$E(\Lambda) = \frac{\Gamma\alpha+1}{\theta\Gamma\alpha+1}\left(\frac{\alpha\theta+1+\alpha}{\theta+1}\right) = \frac{\alpha\theta+1+\alpha}{\theta(\theta+1)}$$

when r=2

$$E(\Lambda^2) = \frac{\Gamma\alpha+2}{\theta^2\Gamma\alpha+}\left(\frac{\alpha\theta+2+\alpha}{\theta+1}\right)$$

when r=3

$$E(\Lambda^3) = \frac{\Gamma\alpha+3}{\theta^3\Gamma\alpha+}\left(\frac{\alpha\theta+3+\alpha}{\theta+1}\right)$$

Proposition 3.3.7

The rth central moments of the two parameter Lindley distribution are

$$\mu_2 = E(\Lambda^2) - (E(\Lambda))^2$$
$$= \frac{\alpha\theta^2+2\alpha\theta+2\theta+2\alpha+1}{\theta^2(\theta+1)^2}$$

Proof:

$$\mu_2 = E(\Lambda^2) - (E(\Lambda))^2$$
$$= \frac{\Gamma\alpha+2}{\theta^2\Gamma\alpha+1}\left(\frac{\alpha\theta+2+\alpha}{\theta+1}\right) - \left(\frac{\alpha\theta+1+\alpha}{\theta^2+\theta}\right)^2$$
$$= \frac{(\alpha+1)(\alpha\theta+2+\alpha)}{\theta^2(\theta+1)} - \frac{(\alpha\theta+1+\alpha)(\alpha\theta+1+\alpha)}{(\theta(\theta+1))^2}$$
$$= \frac{\alpha\theta^2+2\alpha\theta+2\theta+2\alpha+1}{\theta^2(\theta+1)^2}$$

3.3.3 Type II generalized two parameter Lindley Distribution

Let

$$\delta = \beta = 1 \ and \alpha > 0, \theta > 0$$

$$f(\lambda,\theta,\alpha) = pf_1(\lambda) + (1-p)f_2(\lambda)$$

Where

$$p = \frac{\alpha\theta}{\theta\alpha+1} \ f_1(\lambda) = \theta e^{-\theta\lambda} \ f_2(\lambda) = \theta^2\lambda e^{-\theta\lambda}$$

$$\begin{aligned} f(\lambda,\theta,\alpha) &= \frac{\alpha\theta}{\theta\alpha+1}\theta e^{-\theta\lambda} + \left(1 - \frac{\alpha\theta}{\theta\alpha+1}\right)\theta^2\lambda e^{-\theta\lambda} \\ &= \frac{\alpha\theta}{\theta\alpha+1}\theta e^{-\theta\lambda} + \left(\frac{\alpha\theta+1-\alpha\theta}{\alpha\theta+1}\right)\theta^2\lambda e^{-\theta\lambda} \\ &= \frac{\alpha\theta}{\theta\alpha+1}\theta e^{-\theta\lambda} + \frac{1}{\alpha\theta+1}\theta^2\lambda e^{-\theta\lambda} \\ &= \frac{\theta^2}{\alpha\theta+1}e^{-\theta\lambda}(\alpha+\lambda) \end{aligned}$$

As obtained by R.Shanker and A.Mishra(2013)

Proposition 3.3.8

The mode of the distribution is given by

$$mode = \left\{\frac{1-\alpha\theta}{\theta} \ \ ,|\alpha\theta| < 0\right\}$$

Proof

$$f'(\lambda) = \frac{\theta^2}{\alpha\theta+1}(1-\alpha\theta-\lambda\theta)e^{-\theta\lambda}$$

$$\Rightarrow f'(\lambda) = 0 \ when \ \ \lambda = \frac{1-\alpha\theta}{\theta}$$

For $|\theta| < 1, \lambda_0 = \frac{1-\alpha\theta}{\theta}$ which is a unique critical point at which $f(\lambda)$ is maximum.
For $\alpha \geq 1$, $f'(\lambda) \geq 0$ thus $f(\lambda)$ decreases in λ

Proposition 3.3.9

The cumulative distribution function is given by

$$G(\lambda) = 1 - \left(\frac{\alpha\theta + \theta\lambda + 1}{\alpha\theta + 1}\right) e^{-\theta\lambda}$$

Proof

$$\begin{aligned}
G(\lambda) &= \int_0^{\lambda} g(t)dt \\
&= \int_0^{\lambda} \frac{\theta^2}{\alpha\theta + 1} (\alpha + t)\, e^{-\theta t} dt \\
&= \frac{\theta^2}{\alpha\theta + 1} \int_0^{z} (\alpha + t)\, e^{-\theta t} dt \\
&= \frac{\theta^2}{\alpha\theta + 1} \left(\alpha \int e^{-\theta t} dt + \int t e^{-\theta t} dt\right) \\
&= \frac{\theta^2}{\alpha\theta + 1} \left(\frac{\alpha(1 - e^{-\theta\lambda})}{\theta} + \frac{\lambda e^{-\theta\lambda}}{\theta} - \frac{e^{-\theta\lambda}}{\theta^2} + \frac{1}{\theta^2}\right) \\
&= 1 - \left(\frac{\alpha\theta + \theta\lambda + 1}{\alpha\theta + 1}\right) e^{-\theta\lambda}
\end{aligned}$$

Proposition 3.3.10

$$\begin{aligned}
E(\Lambda^r) &= \int \lambda^r \frac{\theta^2}{\alpha\theta + 1} (\alpha + \lambda) e^{-\theta\lambda} d\lambda \\
&= \frac{\theta^2}{\alpha\theta + 1} \int \lambda^r (\alpha + \lambda) e^{-\theta\lambda} d\lambda \\
&= \frac{\theta^2}{\alpha\theta + 1} \left[\alpha \int \lambda^r e^{-\theta\lambda} d\lambda + \int \lambda^{r+1} e^{-\theta\lambda} d\lambda\right] \\
&= \frac{\theta^2}{\alpha\theta + 1} \left[\frac{\alpha \Gamma r + 1}{\theta^{r+1}} + \frac{\Gamma r + 2}{\theta^{r+2}}\right] \\
&= \frac{\Gamma r + 1}{\theta^r} \left[\frac{\alpha\theta + 1 + r}{\alpha\theta + 1}\right]
\end{aligned}$$

3.3.4 Type I generalized three parameter Lindley Distribution

Let

$$\delta = 1 \;\; \beta > 0 \;, \alpha > 0 \; and \; \theta > 0$$

Then

$$
\begin{aligned}
g_1(\lambda) &\sim Gamma\ (\alpha, \theta) \\
g_1(\lambda) &= \frac{\theta^{\alpha}}{\Gamma\alpha} e^{-\theta\lambda} \lambda^{\alpha-1}\ ; \alpha > 0, \theta > 0, \lambda > 0 \\
g_2(\lambda) &\sim Gamma\ (\alpha+1, \theta) \\
g_2(\lambda) &= \frac{\theta^{\alpha+1}}{\Gamma\alpha+1} e^{-\theta\lambda} \lambda^{(\alpha+1)-1}\ ; \alpha > 0, \theta > 0, \lambda > 0
\end{aligned}
$$

This implies that eqn 5 will result into

$$
p_1 = \frac{\beta\theta}{\beta\theta+1}\ \ and\ \ p_2 = \frac{1}{\beta\theta+1}
$$

$$
\begin{aligned}
g(\lambda) &= p_1 g_1 + p_2 g_2 \\
&= \frac{\beta\theta}{\beta\theta+1} \frac{\theta^{\alpha}}{\Gamma\alpha} e^{-\theta\lambda} \lambda^{\alpha-1} + \frac{1}{\beta\theta+1} \frac{\theta^{\alpha+1}}{\Gamma\alpha+1} e^{-\theta\lambda} \lambda^{(\alpha+1)-1} \\
&= \frac{\theta^{\alpha+1}}{\beta\theta+1} \frac{\beta}{\Gamma\alpha} e^{-\theta\lambda} \lambda^{\alpha-1} + \frac{\theta^{\alpha+1}}{\beta\theta+1} \frac{1}{\Gamma\alpha+1} e^{-\theta\lambda} \lambda \\
&= \frac{\theta^{\alpha+1}}{\beta\theta+1} \frac{\alpha\beta}{\Gamma\alpha+1} e^{-\theta\lambda} \lambda^{\alpha-1} + \frac{\theta^{\alpha+1}}{\beta\theta+1} \frac{1}{\Gamma\alpha+1} e^{-\theta\lambda} \lambda \\
g(\lambda) &= \frac{\theta^{\alpha+1}}{\beta\theta+1} \frac{1}{\Gamma\alpha+1} (\alpha\beta+\lambda) e^{-\theta\lambda} \lambda^{\alpha-1}
\end{aligned}
$$

which is the pdf for type 1 G3L.

Proposition 3.3.11
The rth moments is given by

$$
\begin{aligned}
E(\Lambda^r) &= \int \lambda^r g(\lambda) d\lambda \\
&= \int \lambda^r \frac{\theta^{\alpha+1}}{\beta\theta+1} \frac{1}{\Gamma\alpha+1} (\alpha\beta+\lambda) e^{-\theta\lambda} \lambda^{\alpha-1} d\lambda \\
&= \frac{\theta^{\alpha+1}}{\beta\theta+1} \frac{1}{\Gamma\alpha+1} \left[(\alpha\beta+\lambda) e^{-\theta\lambda} \lambda^{\alpha-1+r} d\lambda \right] \\
&= \frac{\theta^{\alpha+1}}{\beta\theta+1} \frac{1}{\Gamma\alpha+1} \left[\alpha\beta \int e^{-\theta\lambda} \lambda^{\alpha-1+r} d\lambda + \int e^{-\theta\lambda} \lambda^{\alpha+r} d\lambda \right] \\
&= \frac{\theta^{\alpha+1}}{\beta\theta+1} \frac{1}{\Gamma\alpha+1} \left[\frac{\alpha\beta\Gamma\alpha+r}{\theta^{\alpha}+r} + \frac{\Gamma\alpha+r+1}{\theta^{\alpha}+r+1} \right] \\
&= \frac{\Gamma\alpha+r}{\beta\theta+1} \frac{1}{\theta^r} [\alpha\beta\theta+r+\alpha]
\end{aligned}
$$

When r=1

$$E(\Lambda)=\frac{\Gamma\alpha+1}{\theta(\beta\theta+1)}(\alpha\beta\theta+1+\alpha)$$

When r=2

$$E(\Lambda^2)=\frac{\Gamma\alpha+2}{\theta^2(\beta\theta+1)}(\alpha\beta\theta+2+\alpha)$$

3.3.5 Type II generalized three parameter Lindley Distribution

Let

$$\beta=1\ \ \delta>0\ ,\alpha>0\ and\ \theta>0$$

Then

$$g_1(\lambda)\sim Gamma\ (\alpha,\theta)$$
$$g_1(\lambda)=\frac{\theta^\alpha}{\Gamma\alpha}e^{-\theta\lambda}\lambda^{\alpha-1}\ ;\alpha>0,\theta>0,\lambda>0$$
$$g_2(\lambda)\sim Gamma\ (\alpha+1,\theta)$$
$$g_2(\lambda)=\frac{\theta^{\alpha+1}}{\Gamma\alpha+1}e^{-\theta\lambda}\lambda^{(\alpha+1)-1}\ ;\alpha>0,\theta>0,\lambda>0$$

This implies that eqn 5 will result into

$$p_1=\frac{\theta}{\theta+\delta}\ \ and\ \ p_2=\frac{\delta}{\theta+\delta}$$

$$g(\lambda)=p_1g_1+p_2g_2$$
$$=\frac{\theta}{\theta+\delta}\frac{\theta^\alpha}{\Gamma\alpha}e^{-\theta\lambda}\lambda^{\alpha-1}+\frac{\delta}{\theta+\delta}\frac{\theta^{\alpha+1}}{\Gamma\alpha+1}e^{-\theta\lambda}\lambda^{(\alpha+1)-1}$$
$$=\frac{\theta^{\alpha+1}}{(\theta+\delta)\Gamma\alpha}e^{-\theta\lambda}\lambda^{\alpha-1}+\frac{\delta\theta^{\alpha+1}}{(\theta+\delta)\Gamma\alpha+1}e^{-\theta\lambda}\lambda^{\alpha}$$
$$g(\lambda)=\frac{\theta^{\alpha+1}}{(\theta+\delta)\Gamma\alpha+1}\left[\alpha+\delta\lambda\right]e^{-\theta\lambda}\lambda^{\alpha-1}$$

which is the pdf for type II G3L.

Proposition 3.3.12

The rth moment of the distribution is given by

$$E(\Lambda^r)=\frac{\Gamma\alpha+r}{(\theta+\delta)\theta^r\Gamma\alpha+1}(\theta\alpha+r+\alpha)$$

Proof

$$
\begin{aligned}
E(\Lambda^r) &= \int \lambda^r g(\lambda) d\lambda \\
&= \int \lambda^r \frac{\theta^{\alpha+1}}{(\theta+\delta)\Gamma\alpha+1} [\alpha+\delta\lambda] e^{-\theta\lambda} \lambda^{\alpha-1} d\lambda \\
&= \frac{\theta^{\alpha+1}}{(\theta+\delta)\Gamma\alpha+1} \int [\alpha+\delta\lambda] e^{-\theta\lambda} \lambda^{\alpha-1+r} d\lambda \\
&= \frac{\theta^{\alpha+1}}{(\theta+\delta)\Gamma\alpha+1} \left[\alpha \int e^{-\theta\lambda} \lambda^{\alpha-1+r} d\lambda + \delta \int e^{-\theta\lambda} \lambda^{\alpha+r} d\lambda\right] \\
&= \frac{\theta^{\alpha+1}}{(\theta+\delta)\Gamma\alpha+1} \left[\frac{\alpha\Gamma\alpha+r}{\theta^{\alpha+r}} + \frac{\delta\Gamma\alpha+r+1}{\theta^{\alpha+r+1}}\right] \\
&= \frac{\Gamma\alpha+r}{(\theta+\delta)\theta^r\Gamma\alpha+1} (\theta\alpha+r+\alpha)
\end{aligned}
$$

When r=1

$$E(\Lambda) = \frac{\theta\alpha+1+\alpha}{\theta(\theta+\delta)}$$

When r=2

$$E(\Lambda^2) = \frac{(\alpha+1)(\theta\alpha+2+\alpha)}{\theta^2(\theta+\delta)}$$

4 POISSON MIXTURES:A CASE OF POISSON LINDLEY DISTRIBUTION

4.1 Introduction

Here we will formulate the problem of Poisson mixtures in general and apply it to one parameter Poisson Lindley distribution. We shall specifically construct the Poisson Lindley distribution, analyses some of its properties and then find its estimation through methods of MOM,MLE and EM algorithm.

4.2 Formulation of the mathematical problem

The pdf of Poisson distribution with λ as a parameter is given by

$$f(z|\lambda) = \frac{e^{-\lambda}\lambda^z}{z!} \quad z = 0, 1, 2, ..$$

The marginal distribution of the mixture is

$$\begin{aligned} f(z) &= \int_0^\infty \frac{e^{-\lambda}\lambda^z}{z!} g(\lambda)\, d\lambda && (8) \\ &= \frac{1}{z!}\int_0^\infty \lambda^z \sum_{j=0}^{\infty} \frac{(-\lambda)^j}{j!} g(\lambda)\ d\lambda \\ &= \frac{1}{z!}\sum_{j=0}^{\infty} \frac{(-1)^j}{j!} \int_0^\infty \lambda^{z-j} g(\lambda)\ d\lambda \\ &= \frac{1}{z!}\sum_{j=0}^{\infty} \frac{(-1)^j}{j!} E\left[\Lambda^{z-j}\right] && (9) \end{aligned}$$

From equation 9, Λ can be any distribution e.g beta,pearson or shifted gamma.In this paper ,we are considering Lindley distribution.
The mixing distribution $g(\lambda)$ is a continuous distribution, thus $f(z)$ is a continuous Poisson mixture.
For random variable z with parameter λ;

$$E(Z) = var(Z) = \lambda$$

When $\Lambda = \lambda$ is varying then

$$\begin{aligned} E(Z) &= EE(Z|\lambda) \\ Var(Z) &= E\left(var(Z|\lambda)\right) + var\, E(Z|\lambda) \\ &= E\left(E(Z|\lambda)\right) + var\, E(Z|\lambda) \\ &= E(Z) + var\, E(Z|\lambda) \end{aligned}$$

4.2.1 Properties of a Poisson mixture

There are two methods on how to obtain moments, the direct method and the conditional method.In this paper,we consider the direct method:
The rth raw moments are given by

$$\begin{aligned} E(Z^r) &= EE(Z^r|\Lambda) \\ &= E\left[e^{-\Lambda} \sum_{z=0}^{\infty} \frac{z^r \Lambda^z}{z!}\right] \end{aligned}$$

Proposition 4.2.1
The respective moments are:

- First moment
$$E(Z) = E(\Lambda)$$
- Second moment
$$E(Z^2) = E(\Lambda^2) + E(\Lambda)$$
- Third moment
$$E(Z^3) = E(\Lambda^3) + 3E(\Lambda^2) + E(\Lambda)$$

Proof: The first moment is given by

$$E(Z) = EE(Z|\Lambda) = E(\Lambda)$$

The second moment is given by

$$
\begin{aligned}
E(Z^2) &= E\left[e^{-\Lambda}\sum_{z=0}^{\infty}\frac{z^2\Lambda^z}{z!}\right] \\
&= E\left[e^{-\Lambda}\sum_{z=1}^{\infty}\frac{(z-1+1)\Lambda^z}{(z-1)!}\right] \\
&= E\left[e^{-\Lambda}\Lambda^2\sum_{z=2}^{\infty}\frac{\Lambda^{z-2}}{(z-2)!}+e^{-\Lambda}\Lambda\sum_{z=1}^{\infty}\frac{\Lambda^{z-1}}{(z-1)!}\right] \\
&= E\left[\Lambda^2+\Lambda\right] \\
&= E(\Lambda^2)+E(\Lambda)
\end{aligned}
$$

The third moment is

$$
\begin{aligned}
E(Z^3) &= E\left[e^{-\Lambda}\sum_{z=0}^{\infty}\frac{z^3\Lambda^z}{z!}\right] \\
&= E\left[e^{-\Lambda}\sum_{z=1}^{\infty}\frac{(z-1+1)^2\Lambda^z}{(z-1)!}\right] \\
&= E\left[e^{-\Lambda}\sum_{z=1}^{\infty}\frac{\left[(z-1)^2+2(Z-1)+1\right]\Lambda^z}{(z-1)!}\right] \\
&= E\left[e^{-\Lambda}\left(\sum_{z=2}^{\infty}\frac{(z-2)+3}{(z-2)!}+\sum_{z=1}^{\infty}\frac{1}{(z-1)!}\right)\Lambda^z\right] \\
&= E\left[\Lambda^3+3\Lambda^2+\Lambda\right]
\end{aligned}
$$

Proposition 4.2.2
The central moments of a Poisson mixture are

- Variance
$$\mu_2 = var(\Lambda) + E(\Lambda)$$
- Third moment is
$$\mu_3 = E\left(\Lambda - E(\Lambda)\right)^3 + 3\,var(\Lambda) + E(\Lambda)$$

Proof:
Variance

$$\begin{aligned}\mu_2 &= Var(Z)\\ &= Var\left[E(Z|\Lambda)\right] + E\left[Var(Z|\Lambda)\right]\\ &= Var(\Lambda) + E(\Lambda)\end{aligned}$$

The third central moment is

$$\begin{aligned}\mu_3 &= E\left[Z - E(Z)\right]^3\\ &= E(Z^3) - 3E(Z^2)E(Z) + 2\left[E(Z)\right]^3\\ &= E(\Lambda^3) + 3E(\Lambda^2) + E(\Lambda) - 3E(\Lambda^2)E(\Lambda) - 3\left[E(\Lambda)\right]^2 + \left[E(\Lambda)\right]^3\\ &= E\left(\Lambda - E(\Lambda)\right)^3 + 3\,var(\Lambda) + E(\Lambda)\end{aligned}$$

Proposition 4.2.3
The pgf of the mixture is given by

$$G(s) = \int_0^\infty e^{\lambda(s-1)} g(\lambda) d\lambda$$

The RHS of this equation is $M(s-1)$, the moment generating function of the mixing distribution evaluated at $s-1$ This implies that the pgf of the mixture uniquely determines the mixing distribution through its moment generating function.

4.3 Poisson Lindley distribution

4.3.1 Construction

Proposition 4.3.1
For one parameter distribution, the pdf is

$$f(z) = \frac{\theta^2(z+2+\theta)}{(1+\theta)^{z+3}} \tag{10}$$

Proof:
From the Lindley distribution

$$
\begin{aligned}
g(\lambda) &= \frac{\theta^2}{1+\theta}(1+\lambda)e^{-\theta\lambda} \; \lambda > 0, \theta > 0 \\
f(z) &= \int_0^\infty \frac{e^{-\lambda}\lambda^z}{z!} g(\lambda)\, d\lambda \\
&= \int_0^\infty \frac{e^{-\lambda}\lambda^z}{z!} \frac{\theta^2}{1+\theta}(1+\lambda)e^{-\theta\lambda} d\lambda \\
&= \frac{\theta^2}{1+\theta}\int_0^\infty \left[\frac{\lambda^{z+1}}{z!}e^{-\lambda(1+\theta)} + \frac{\lambda^z}{x!}e^{\lambda}(1+\theta)\right] d\lambda \\
&= \frac{\theta^2(z+2+\theta)}{(1+\theta)^{z+3}}
\end{aligned}
$$

As obtained by Sankaran(1970)

4.3.2 Properties

Proposition 4.3.2

1. The rth moments is

$$E(Z^r) = \frac{r!(r+1+\theta)}{\theta^r(\theta+1)} \tag{11}$$

Proof:

$$
\begin{aligned}
E(Z^r) &= \int_0^\infty z^r f(z,\theta) dz \\
&= \frac{\theta^2}{1+\theta}\int_0^\infty z^r e^{-\theta z}(1+z)\, dz \\
&= \frac{\theta^2}{1+\theta}\left[\int_0^\infty z^r e^{-\theta z}\, dz + \int_0^\infty z^{r+1} e^{-\theta z}\, dz\right] \\
&= \frac{\theta^2}{1+\theta}\left[\frac{\Gamma r+1}{\theta^{r+1}} + \frac{\Gamma r+2}{\theta^{r+2}}\right] \\
&= \frac{r!(r+1+\theta)}{\theta^r(\theta+1)}
\end{aligned}
$$

$$\mu_1' = \frac{\theta+2}{\theta(\theta+1)} \text{ and } \mu_2 = Var = \frac{2+4\theta+\theta^2}{(\theta+1)^2} + \frac{2+\theta}{\theta(\theta+1)}$$

2. The pgf is

$$
\begin{aligned}
G(s) &= \int_0^\infty e^{-\lambda(1-s)} \frac{\theta^2}{1+\theta}(1+\lambda)e^{-\theta\lambda}\, d\lambda \\
&= \frac{\theta^2}{1+\theta}\left[\int_0^\infty \lambda e^{-\lambda(\theta+1-s)}\, d\lambda + \int_0^\infty e^{-\lambda(\theta+1-s)}\, d\lambda\right] \\
&= \frac{\theta^2}{1+\theta}\left[\frac{\Gamma 2}{(\theta+1-s)^2} + \frac{1}{\theta+1-2}\right] \\
&= \frac{\theta^2}{1+\theta}\frac{\theta+2-s}{(\theta+1-s)^2}
\end{aligned}
$$

Estimation:

1. MOM
   From a random sample $x_1, x_2, \ldots\ldots, x_n$

$$
\begin{aligned}
E(X) = \bar{X} &= \mu_1 \\
\bar{X} &= \frac{\theta+2}{\theta^2+\theta} \Rightarrow \bar{X}\theta^2 + \bar{X}\theta = \theta+2 \\
\hat{\theta} &= \frac{-b \pm \sqrt{b^2-4ac}}{2a} \\
&= \frac{-(\bar{X}-1) \pm \sqrt{(\bar{X}-1)^2 - 8\bar{X}}}{2\bar{X}} \qquad (12)
\end{aligned}
$$

However in practice, we use :

$$
\bar{X} = \frac{\sum f_i x_i}{\sum f_i}
$$

2. MLE

$$
\begin{aligned}
L &= \prod_{i=1}^{n} f(x_i) \\
&= \prod_{i=1}^{n}\left(\frac{\theta^2(x+2+\theta)}{(1+\theta)^{x+3}}\right) \\
&= \frac{\theta^{2n}\sum_{i=1}^{n}(x_i+2+\theta)}{(1+\theta)^{\sum_{i=1}^{n}x_i+3n}} \\
&= \frac{\theta^{2n}\sum_{i=1}^{n}(x_i+2+\theta)}{(1+\theta)^{n(\bar{X}+3n)}}
\end{aligned} \tag{13}
$$

$$
\begin{aligned}
logl(\theta) &= 2nlog\theta - n(\bar{X}+3)log(1+\theta) + log\sum_{i=1}^{n}(x_i+2+\theta) \\
\frac{\partial logl(\theta)}{\partial\theta} &= \frac{2n}{\theta} - \frac{n(\bar{X}+3)}{1+\theta} + \sum_{i=1}^{n}\frac{1}{x_i+2+\theta} \\
&= \frac{2n}{\theta} - \frac{n(\bar{X}+3)}{1+\theta} + \frac{1}{x_1+2+\theta} + \frac{1}{x_2+2+\theta} + ... + \frac{1}{x_n+2+\theta} = 0 \\
&= f(\theta) = \frac{2n(1+\theta)\sum_{i=1}^{n}(x_i+2+\theta) - n(\bar{X}+3)\theta\sum_{i=1}^{n}(x_i+2+\theta) + \theta(1+\theta)}{\theta(1+\theta)\sum_{i=1}^{n}(x_i+2+\theta)}
\end{aligned} \tag{14}
$$

This is a polynomial of degree $(n+1)$ in θ which is solved using Newton-Raphson method:

$$\theta_{r+1} = \theta_r - \frac{f(\theta)}{f'(\theta)}$$

3. EM algorithm
From the definition in chapter one;
The pdf of Lindley distribution is given by

$$P(x|p) = \frac{p^2(x+2+p)}{(1+p)^{x+3}}, \;\; x = 0,1,...,p > 0 \tag{15}$$

For a random sample $z1, z2....zn$ the parameter p is given by

$$\frac{p+2}{p(p+1)} = \bar{x}$$

where $\bar{x}$ is the sample mean.This enables us to estimate a new value of p E-step: calculation of the pseudo-values

$$t_i = E(\theta_i|x_i) = \frac{(p_0+x_i+3)(x_i+1)}{(p_0+x_i+2)(p_0+1)}$$

where $\bar{t} = \frac{\sum_{i=1}^{n} t_i}{n}$
M-step:

$$p_{new} = \frac{-(\bar{t}-1)+\sqrt{(\bar{t}^2+6\bar{t}+1)}}{2\bar{t}}$$

Data analysis

The below data is obtained from a study of the Hermite distribution which considers mistakes that occur when copying groups of random digits together with expected frequencies.It incorporates the sum of an ordinary Poisson variable and Poisson doublet variable as studied by Kemp and Kemp;Some properties, of the 'Hermite' distribution(1965)

count	0	1	2	3	4
observed frequency	35	11	8	4	2

1. MOM

$$\begin{aligned}\bar{X} = \mu = \frac{\sum f_i x_i}{\sum f_i} &= \frac{\theta+2}{\theta(\theta+1)} \\ = \frac{47}{60} &= \frac{\theta+2}{\theta(\theta+1)} \\ = 47\theta^2 - 13\theta - 120 &= 0 \\ \hat{\theta} &= \frac{-b \pm \sqrt{(b^2-4ac)}}{2a} \\ \hat{\theta} = \frac{13 \pm 150.7614}{94} &= 1.742\end{aligned}$$

2. MLE
   Newton raphson method:

$$\theta_{r+1} = \theta_r - \frac{f(\theta)}{f'(\theta)}$$

From the above equation

$$\theta_1 = \theta_0 - \frac{1}{\sum_{i=1}^{n}(x_i + 2 + \theta_0) + (1 + \theta_0)}$$

$$\theta_1 = 1.742 - \frac{1}{10 + (2*60) + (60*1.742) + 1 + 1.742} = 1.7378$$

$$\theta_2 = 1.7378 - \frac{1}{10 + (2*60) + (60*1.7378) + 1 + 1.7378} = 1.7336$$

$$\theta_3 = 1.7336 - \frac{1}{10 + (2*60) + (60*1.7336) + 1 + 1.7336} = 1.7294$$

$$\Rightarrow \hat{\theta} = 1.7336$$

3. EM algorithm
   Using $p_0 = 1.742$ and substituting the respective value of x

$$t_i = \frac{(p_0 + x_i + 3)(x_i + 1)}{(p_0 + x_i + 2)(p_0 + 1)}$$

x_i	0	1	2	3	4	total
t_i	0.462	0.8832	1.2846	1.6752	2.059	6.364

$$\Rightarrow \bar{t} = 1.2728$$

$$p_{new} = \frac{-(\bar{t} - 1) + \sqrt{(\bar{t}^2 + 6\bar{t} + 1)}}{2\bar{t}} = 1.15093$$

Remarks

From the above data analysis,the three methods give different correct results as their technics are different.

5 POISSON LINDLEY TWO PARAMETER DISTRIBUTIONS

5.1 Introduction

There are three types of two parameter Poisson Lindley distributions, namely type I, type II and type III . They are obtained by equating certain parameters of the four parameter Lindley distribution to constant values.
For each type,we will construct the distribution then examine its estimation.
Consider the following forms of $g(\lambda)$

5.2 Type I two parameter

The G4 Lindley distribution is given by

$$g(\lambda) = \frac{\theta^{\alpha+1}}{\theta+1}\frac{1}{\Gamma(\alpha+1)}(\alpha+\lambda)\lambda^{\alpha-1}e^{-\theta}$$

5.2.1 Construction

Proposition 5.2.1
The pdf of the Poisson Lindley distribution is given by

$$f(x) = \frac{\Gamma x+\alpha}{\theta^x x!\Gamma\alpha+1}\left(\frac{\alpha\theta+\alpha+x}{\theta+1}\right) \tag{16}$$

Proof

$$
\begin{aligned}
f(x) &= \int_0^\infty \frac{e^{-\lambda}\lambda^x}{x!} g(\lambda)\, d\lambda \\
&= \int_0^\infty \frac{e^{-\lambda}\lambda^x}{x!} \frac{\theta^{\alpha+1}}{\theta+1} \frac{1}{\Gamma(\alpha+1)} (\alpha+\lambda)\lambda^{\alpha-1} e^{-\theta\lambda}\ d\lambda \\
&= \frac{\theta^{\alpha+1}}{\theta+1} \frac{1}{\Gamma(\alpha+1)} \frac{1}{x!} \left(\int_0^\infty e^{-\lambda}\lambda^x (\alpha+\lambda)\lambda^{\alpha-1} e^{-\theta\lambda}\ d\lambda \right) \\
&= \frac{\theta^{\alpha+1}}{\theta+1} \frac{1}{\Gamma(\alpha+1)} \frac{1}{x!} \left(\alpha \int_0^\infty e^{-\lambda(1+\theta)} \lambda^{x+1+\alpha}\ d\lambda + \int_0^\infty e^{-\lambda(1+\theta)} \lambda^{x+\alpha}\ d\lambda \right) \\
&= \frac{\theta^{\alpha+1}}{\theta+1} \frac{1}{\Gamma(\alpha+1)} \frac{1}{x!} \left(\frac{\alpha \Gamma x+\alpha}{\theta^{x+\alpha}} + \frac{\Gamma x+\alpha+1}{\theta^{x+\alpha+1}} \right) \\
&= \frac{\Gamma x+\alpha}{\theta^x x! \Gamma\alpha+1} \left(\frac{\alpha\theta+\alpha+x}{\theta+1} \right)
\end{aligned}
$$

The r^{th} moment is given by

$$
\begin{aligned}
\mu_r' = E\left[E(x^r/\lambda)\right] \\
&= \int_0^\infty \lambda^r \frac{\theta^{\alpha+1}}{\theta+1} \frac{1}{\Gamma(\alpha+1)} (\alpha+\lambda)\lambda^{\alpha-1} e^{-\theta\lambda}\ d\lambda \\
&= \frac{\theta^{\alpha+1}}{\theta+1} \frac{1}{\Gamma(\alpha+1)} \left[\alpha \int_0^\infty \lambda^{r+\alpha+1} e^{-\theta\lambda}\ d\lambda + \int_0^\infty \lambda^{r+\alpha+1} e^{-\theta\lambda}\ d\lambda \right] \\
&= \frac{\theta^{\alpha+1}}{\theta+1} \frac{1}{\Gamma(\alpha+1)} \left[\frac{\alpha\Gamma r+\alpha}{\theta^{r+\alpha}} + \frac{\Gamma r+\alpha+1}{\theta^{r+\alpha+1}} \right] \\
&= \frac{\Gamma\alpha+r}{\theta^r \Gamma r+1} \left[\frac{\alpha r+\alpha+r}{\theta+1} \right]
\end{aligned}
$$

The mean and variance of the distribution are as indicated in proposition 3.3.6 and 3.3.7.

Proposition 5.2.2

Since the hazard function $h(\lambda)$ can not be expressed in closed form,let $h(\lambda)$ be a hazard function of a random variable X distributed according to the type one two parameter Poisson-Lindley distribution,then

1. $h(\lambda)$ is increasing for $\alpha \geq 1$;

2. $h(\lambda)$ is bathtub shaped for$\alpha < 1$ and $\theta > 0$

3. $h(\lambda)$ is decreasing for $\alpha \leq 1$ and $\theta = 0$

Proof
From $g(\lambda)$ we have

$$\rho(\lambda) = \frac{f'(\lambda)}{f(\lambda)} = \frac{1-\alpha}{\lambda} - \frac{\theta}{\alpha+\theta\lambda} + \theta$$

$$It follows\ that$$

$$\rho'(\lambda) = \frac{\alpha-1}{\lambda^2} + \frac{\theta^2}{\theta+\alpha\lambda}^2 \geq 0, for \alpha \geq 1$$

5.2.2 Estimation

1. Method of moments
   Given a random sample $x_1, x_2, ..., x_n$ of size n

$$\mu_1' = \bar{X} = \frac{\alpha\Gamma\alpha(2\alpha+1)}{\theta(\theta+1)}$$

$$var = \mu_2' = \frac{\Gamma\alpha+2(3\alpha+2)}{(\theta+1)\theta^2\Gamma 3}$$

   By substitution, the respective estimates of α and θ are obtained

2. Maximum Likelihood method

$$\begin{aligned} L &= \prod_{i=1}^{n} f(x_i) \\ &= \prod_{i=1}^{n}\left[\frac{\Gamma x_i+\alpha}{\theta^{x_i} x_i! \Gamma\alpha+1}\left(\frac{\alpha\theta+\alpha+x_i}{\theta+1}\right)\right] \\ &= \frac{1}{\theta^{\Sigma x_i}\sum x_i!(\theta+1)^n}\prod_{i=1}^{n}\left[\frac{(\alpha\theta+\alpha+x_i)\Gamma x_i+1}{\Gamma\alpha+1}\right] \end{aligned} \quad (17)$$

   Getting the $\hat{\theta}$and $\hat{\beta}$ from the above equation is not straight forward due to the likelihood of the Gamma part and the EM algorithm is used instead.

3. EM algorithm
   Consider the complete data $Y1, Y2..., Yn$ and the hypothetical random variable $r1, r2, ..., rn$. The joint pdf is such that the marginal density of $Y1, ..., Yn$ is the likelihood of interest.The hypothetical complete-data distribution for each $(Yi, ri) i = 1, 2, ..n$ has a joint probability density function in the form $g(Y, r, \Theta)$ where $\Theta = (\theta, p)$
   E-step:

$$E[r|Y=y] = \frac{1+p(1+\frac{\theta y}{\theta+1})e^{-\theta y}}{1-p(1+\frac{\theta y}{\theta+1})e^{-\theta y}}$$

M-step:
Through iteration

$$y_i^r = \frac{\left(1 + p^r(1 + \frac{\theta^r y_i}{\theta_r + 1)} e^{-\theta r y_i}\right)}{\left(1 + p^r(1 + \frac{\theta^r y_i}{\theta_r + 1)} e^{-\theta r y_i}\right)}$$

5.2.3 Data analysis

Consider the data from Data set 1

count	0	1	2	3	4
observed frequency	35	11	8	4	2

1. MOM

$$\begin{aligned}
\bar{X} = \mu &= \frac{\sum f_i x_i}{\sum f_i} \\
&= \frac{47}{60} = 0.7833 \\
\sigma^2 &= E(x^2) - \mu^2 = 1.85 \\
k = \frac{\sigma^2 - \mu_1}{(\mu_1)^2} &= \frac{1.85 - 0.7833}{0.7833^2} = 1.7385 \\
&\Rightarrow (2-k)b^2 + (8-4k)b + (6-4k) = 0 \\
&\Rightarrow 0.2615b^2 + 1.046b - 0.954 = 0 \\
&\Rightarrow b = 0.7656 \\
&\textit{Substituting accordingly} \\
\hat{\theta} = \frac{b+2}{(b+1)\bar{X}} &= \frac{0.7656+2}{(0.7656+1)0.7833} = 1.9997 \\
\hat{\alpha} = \frac{b(b+1)\bar{X}}{b+2} &= \frac{0.7656(0.7656+1)0.7833}{0.7656+2} = 0.38285
\end{aligned}$$

5.3 Type II two parameter

5.3.1 construction

From the Lindley distribution below:

$$g(\lambda) = \frac{\theta^2}{(\beta\theta+1)}(\beta+\lambda)e^{-\theta lambda}$$

Proposition 4.3.1
The pdf is given by

$$f(x) = \frac{\theta^2}{(\theta+1)^{x+2}}\left(1+\frac{\beta+x}{\beta\theta+1}\right) \quad ,x=0,1,2...,\theta>0,\beta\theta>0 \tag{18}$$

Proof:

$$\begin{aligned}
f(x) &= \int_0^\infty \frac{e^{-\lambda}\lambda^x}{x!}g(\lambda)\,d\lambda \\
&= \int_0^\infty \frac{e^{-\lambda}\lambda^x}{x!}\frac{\theta^2}{\beta\theta+1}(\beta+\lambda)e^{-\theta\lambda}\ d\lambda \\
&= \frac{\theta^2}{\beta\theta+1}\int_0^\infty \frac{1}{x!}(\beta+\lambda)\lambda^x e^{-\theta\lambda}\ d\lambda \\
&= \frac{\theta^2}{\beta\theta+1}\int_0^\infty \left(\frac{\lambda^x}{x!}\beta e^{-\lambda(1+\theta)}+\frac{\lambda^{x+1}}{x!}e^{-\lambda(1+\theta)}\right)\ d\lambda \\
&= \frac{\theta^2}{(\theta+1)^{x+2}}\left(1+\frac{\beta+x}{\beta\theta+1}\right) \quad ,x=0,1,2...,\theta>0,\beta\theta>0
\end{aligned}$$

The rth moments are given by

$$\begin{aligned}
\mu_r' = E\left[E(x^r/\lambda)\right] \\
&= \int_0^\infty \left[\sum_{x=0}^\infty x^r\frac{e^{-\lambda}\lambda^x}{x!}\right]\frac{\theta^2}{\beta\theta+1}(\beta+\lambda)e^{-\theta\lambda}\ d\lambda \\
&= \frac{\theta^2}{\beta\theta+1}\left[\beta\int_0^\infty \lambda^r e^{-\theta\lambda}\,d\lambda+\int_0^\infty \lambda^{r+1}e^{-\theta\lambda}\,d\lambda\right] \\
&= \frac{\theta^2}{\beta\theta+1}\left[\frac{\beta\Gamma r+1}{\theta^{r+1}}+\frac{\Gamma r+2}{\theta^{r+2}}\right] \\
&= \frac{\Gamma r+1}{\theta^r}\left(\frac{\alpha\theta+r+1}{\alpha\theta+1}\right) \quad r=1,2,3...
\end{aligned} \tag{19}$$

Estimation

1. Method of moments
   By use of the moments about the origin for Poisson distribution, the μ_2 for the two parameter PLD is given by

$$\mu_2 = \frac{\theta^2}{\beta\theta+1}\left[\int_0^\infty (\lambda^2+\lambda)(\alpha+\lambda)e^{-\theta\lambda}\,d\lambda\right]$$
$$= \frac{\beta\theta+2}{\theta(\beta\theta+1} + \frac{2(\theta+3)}{\theta^2(\beta\theta+1)}$$

   MOM is given by

$$\frac{\mu_2-\mu_1}{(\mu_1)^2} = \frac{\left[\frac{\beta\theta+2}{\theta(\beta\theta+1)} + \frac{2(\theta+3)}{\theta^2(\beta\theta+1)}\right] - \frac{\beta\theta+2}{\theta(\beta\theta+1)}}{\left(\frac{\beta\theta+2}{\theta(\beta\theta+1)}\right)^2}$$
$$= \frac{2(\beta\theta+3)(\beta\theta+1)}{(\beta\theta+2)^2} \quad \textbf{(20)}$$

   Equation (4.5) is a quadratic in b if $b=\beta\theta$,

$$\frac{2(b+3)(+1)}{(b+2)^2} = k$$

$$(2-k)b^2+(8-4k)b+(6-4k)=0$$

   $\hat{k}$ is obtained by replacing μ_1 and μ_2 by the respective sample moments and sample mean.By substituting $b=\beta\theta$ in the mean $\bar{X}$

$$\hat{\theta} = \frac{b+2}{(b+1)\bar{X}}$$

   and

$$\hat{\beta} = \frac{b}{\hat{\theta}} = \frac{b(b+1)\bar{X}}{b+2}$$

2. Maximum Likelihood method
   Consider a random sample of size n;$x_1,x_2.....x_n$ and let f_x be the observed frequency in the sample corresponding to $X=x$ $(x=1,2,...k)$ such that $\sum_{x=1}^{k} f_x = n$ where k is

the largest observed value having non zero frequency.

$$L = \prod_{i=1}^{n} f(x_i) \tag{21}$$
$$= \prod_{i=1}^{n} \left[\frac{\theta^2}{\beta\theta + 1} \left(\frac{\beta\theta + 1 + \beta + x_i}{(\theta + 1)^{x_i + 2}} \right) \right]$$
$$= \left(\frac{\theta^2}{\beta\theta + 1} \right)^n \frac{1}{(\theta + 1)^{\sum_{i=1}^{n} (x_i + 2) f_x}} \prod_{x=1}^{k} (\beta\theta + 1 + \beta + x)^{f_x}$$
$$log\ l = nlog \left(\frac{\theta^2}{\beta\theta + 1} \right) - \sum_{x=1}^{k} (x_i + 2) f_x log(\theta + 1) + \sum_{x=1}^{k} f_x log(\beta\theta + 1 + \beta + x)$$
$$\frac{\partial logL}{\partial \theta} = \frac{2n}{\theta} - \frac{n\beta}{\beta\theta + 1} - \sum_{x=1}^{k} (x_i + 2) f_x log(\theta + 1) + \sum_{x=1}^{k} \frac{\beta f_x}{\beta\theta + 1 + \beta + x} = 0 \tag{22}$$
$$\frac{\partial logL}{\partial \beta} = -\frac{n\beta}{\beta\theta + 1} + \sum_{x=1}^{k} \frac{(\theta + 1) f_x}{\beta\theta + 1 + \beta + x} = 0 \tag{23}$$

From equations (4.7) and (4.8) using Newton-Raphson method

$$\theta_1 = \theta_0 - \left[n\beta\theta^2(\beta\theta + 3 - \bar{X}\beta\theta^2 - \bar{X} - \bar{X}\theta) + \theta - 2n\theta - n\bar{X}\theta^2 \right]$$

Calculation for the β estimate is complex due to the $\sum_{x=1}^{k} f_x = n$

3. EM algortihm

$$P(x|p,q) = \frac{p^2}{(p+1)^{x+2}} \left(1 + \frac{q + x}{pq + 1} \right), \ \ x = 0, 1, ..., p > 0, q > 0 \tag{24}$$

Data Analysis

Consider the data from Data set 1

count	0	1	2	3	4
observed frequency	35	11	8	4	2

1. MOM

$$
\begin{aligned}
\bar{X} = \mu &= \frac{\sum f_i x_i}{\sum f_i} \\
&= \frac{47}{60} = 0.7833 \\
\sigma^2 &= E(x^2) - \mu^2 = 1.85 \\
k = \frac{\sigma^2 - \mu_1}{(\mu_1)^2} &= \frac{1.85 - 0.7833}{0.7833^2} = 1.7385 \\
&\Rightarrow (2-k)b^2 + (8-4k)b + (6-4k) = 0 \\
&\Rightarrow 0.2615b^2 + 1.046b - 0.954 = 0 \\
&\Rightarrow b = 0.7656 \\
&\textit{Substituting accordingly} \\
\hat{\theta} = \frac{b+2}{(b+1)\bar{X}} &= \frac{0.7656+2}{(0.7656+1)0.7833} = 1.9997 \\
\hat{\alpha} = \frac{b(b+1)\bar{X}}{b+2} &= \frac{0.7656(0.7656+1)0.7833}{0.7656+2} = 0.38285
\end{aligned}
$$

5.4 Type III two parameter

5.4.1 Construction

From the Lindley distribution

$$g(\lambda) = \frac{\theta^2(1+\beta\lambda)e^{-\theta\lambda}}{\beta+\theta}$$

Proposition 4.4.1
The pdf is given by

$$f(x) = \frac{\theta^2}{(\theta+\beta)(1+\theta)^{x+1}}\left[1+\frac{\beta(x+1)}{1+\theta}\right] \; x = 0,1,2... \tag{25}$$

Proof

$$\begin{aligned}
f(x) &= \int_0^\infty \frac{e^{-\lambda}\lambda^x}{x!} g(\lambda)\, d\lambda \\
&= \int_0^\infty \frac{e^{-\lambda}\lambda^x}{x!}\frac{\theta^2(1+\beta\lambda)e^{-\theta\lambda}}{\beta+\theta}\, d\lambda \\
&= \frac{\theta^2}{(\theta+\beta)x!}\left[\int_0^\infty e^{-\lambda(1+\theta)}\lambda^x\, d\lambda + \int_0^\infty e^{-\lambda(1+\theta)}\lambda^{x+1}\, d\lambda\right] \\
&= \frac{\theta^2}{(\theta+\beta)x!}\left[\frac{\Gamma x+1}{(1+\theta)^{x+1}} + \frac{\beta\Gamma x+2}{(1+\theta)^{x+2}}\right] \\
&= \frac{\theta^2}{(\theta+\beta)x!}\frac{\Gamma x+1}{(1+\theta)^{x+1}}\left[1+\frac{\beta(x+1)}{1+\theta}\right] \\
&= \frac{\theta^2}{(\theta+\beta)(1+\theta)^{x+1}}\left[1+\frac{\beta(x+1)}{1+\theta}\right] \; x = 0,1,2...
\end{aligned}$$

The cdf is given by

$$\begin{aligned}
F(x) = \sum_{n=0}^{x}\frac{\theta^2}{\theta+\beta}\frac{1+\theta+n\beta+\beta}{(1+\theta)^{n+2}} \\
= \frac{(\beta+\theta)(\theta+1)^{x+2} - (2\beta\theta+\beta+\theta^2+\theta+\beta\theta x)}{(\theta+1^{x+2}(\beta+\theta)}
\end{aligned} \tag{26}$$

The rth moments about the origin are

$$\begin{aligned}
\mu_r' &= E\left[E(x^r/\lambda)\right] \\
&= \int_0^\infty \lambda^r \frac{\theta^2(1+\beta\lambda)e^{-\theta\lambda}}{\beta+\theta}\, d\lambda \\
&= \frac{\theta^2}{(\theta+\beta)}\left[\int_0^\infty e^{-\lambda\theta}\lambda^r\, d\lambda + \int_0^\infty e^{-\lambda\theta}\lambda^{r+1}\, d\lambda\right] \\
&= \frac{\theta^2}{(\theta+\beta)}\left[\frac{\Gamma r+1}{\theta^{r+1}} + \frac{\beta\Gamma r+2}{\theta^{r+2}}\right] \\
&= \frac{\theta^2}{(\theta+\beta)}\frac{\Gamma r+1}{\theta^{r+1}}\left(1+\frac{\beta(r+1)}{\theta}\right)
\end{aligned} \tag{27}$$

$$\mu_1' = \frac{2\beta+\theta}{\theta(\beta+\theta)}$$

$$\mu_2' = \frac{2\beta(\theta+3)+\theta(\theta+2)}{\theta^2(\beta+\theta)}$$

Estimation

1. Method of moments
   Given a random sample $x_1, x_2, \ldots, x_n$ of size n

$$m_1 = \mu_1' = \bar{X} = \frac{2\beta+\theta}{\theta(\beta+\theta)}$$

$$m_2 = \mu_2' = E(\lambda^2) - (E(\lambda))^2 = \frac{2\beta(\theta+3)+\theta(\theta+2)}{\theta^2(\beta+\theta)}$$

   where m_1 and m_2 are the first and second sample moments

$$\hat{\beta} = \frac{\hat{\theta} - m_1\hat{\theta}^2}{m_1\hat{\theta} - 2}$$

$$\hat{\theta} = \frac{2m_1 + \sqrt{4m_1^2 + 2m_1 - 2m_2}}{m_2 - m_1}$$

2. Maximum Likelihood method

$$
\begin{aligned}
L &= \prod_{i=1}^{n} f(x_i) \qquad (28)\\
&= \prod_{i=1}^{n}\left(\frac{\theta^2}{(\theta+\beta)(1+\theta)^{x+1}}\left[1+\frac{\beta(x+1)}{1+\theta}\right]\right)\\
&= \frac{\theta^{2n}}{(\theta+\beta)^n(1+\theta)^{\sum_{i=1}^{n}(x+1)}}\prod_{i=1}^{n}\left(1+\frac{\beta(x+1)}{1+\theta}\right)\\
&= log\, l = 2nlog\theta - nlog(\theta+\beta) - \sum_{i=1}^{n}(x_i+1)log(1+\theta) - \sum_{i=1}^{n} log\left(1+\frac{\beta(x_i+1)}{1+\theta}\right)
\end{aligned}
$$

$$
\frac{\partial logl}{\partial\theta} = \frac{2n}{\theta} - \frac{n}{\beta+\theta} - \frac{\sum_{i=1}^{n}(x+1)}{1+\theta} - \sum_{i=1}^{n}\frac{\beta(x_i+1)}{1+\theta+\beta(x_i+1)(1+\theta)}
$$

$$
\begin{aligned}
\frac{\partial^2 logl}{\partial\theta^2} &= \frac{2n}{\theta^2} - \frac{n}{(\beta+\theta)^2} - \frac{\sum_{i=1}^{n}(x+1)}{(1+\theta)^2} - \sum_{i=1}^{n}\frac{\beta(x_i+1)(\beta+2\theta+\beta x_i+2)}{(1+\theta)^2+(\beta+\theta\beta(x_i+1)^2}\\
&= \frac{\partial logl}{\partial\beta} = \frac{-n}{\theta+\beta} + \sum_{i=1}^{n}\frac{x_i+1}{1+\theta+\beta(x_i+1)}\\
&= \frac{\partial^2 logl}{\partial\beta^2} = \frac{-n}{(\theta+\beta)^2} - \sum_{i=1}^{n}\frac{(x_i+1)^2}{(1+\theta)^2+(\beta(x_i+1))^2}
\end{aligned}
$$

Getting the $\hat{\theta}$and $\hat{\beta}$ from the above equation is tedious and the EM algorithm is used instead.

3. EM algorithm
The joint probability function is

$$
g(x,\lambda,\beta,\theta) = \frac{\theta^2(1+\beta\lambda)\lambda^x e^{-\lambda(1+\theta)}}{(\theta+\beta)x!} \quad \theta>0,\beta>0 \qquad (29)
$$

E-step:conditional expectation
Consider the below equation

$$
\begin{aligned}
g(\lambda|x) &= \frac{\lambda^x(1+\beta\lambda)e^{\lambda(1+\theta)}(1+\theta)^{x+2}}{x!(1+\theta+\beta(x+1)}\\
E(\lambda|x_i,\theta,\beta) &= \frac{(x_i+1)(1+\theta_0+\beta_0(x_i+2))}{(1+\theta_0(1+\theta_0+\beta_0(x_i+1)} = a \qquad (30)\\
E\left(\frac{\lambda}{1+\beta\lambda}|x_i,\theta,\beta\right) &= \frac{x_i+1}{1+\theta_0+\beta_0(\theta_0+1)} = b \qquad (31)
\end{aligned}
$$

M-steps

$$\theta_1 = \frac{-(\bar{a}\beta_0 - 1) + \sqrt{(\bar{a}\beta_0 - 1)^2 + 8\beta_0\bar{a}}}{2\bar{a}} \quad (32)$$

$$\beta_1 = \frac{1}{\bar{b}} - \theta_0 \quad (33)$$

where $\theta_0$and β_0 are the initial estimates of (θ, β), in this case, from MOM method.

Data Analysis

Using the below table for distribution for epileptic seizure counts

count	0	1	2	3	4	5	6	7	8
observed frequency	126	80	59	42	24	8	5	4	3
expected frequency	122.00	91.00	58.74	35.22	20.52	11.22	6.39	3.25	2.50

1. MOM

$$m_1 = \frac{\sum f_i x_i}{\sum f_i} = \frac{542}{351} = 1.5442$$

$$m_2 = E(x^2) - \mu^2 = \frac{1850}{351} - (1.5442)^2 = 2.8861$$

$$\Rightarrow \hat{\theta} = \frac{2m_1 + \sqrt{4m_1^2 + 2m_1 - 2m_2}}{m_2 - m_1}$$

$$= \frac{(2*1.5442) + \sqrt{(4*1.5442^2) + (2*1.5442) - (2*2.8861)}}{2.8861 - 1.5442}$$

$$\Rightarrow \hat{\theta} = 4.2525$$

$$\hat{\beta} = \frac{\hat{\theta} - m_1\hat{\theta}^2}{m_1\hat{\theta} - 2}$$

$$= \frac{4.2525 - (1.5442*4.2525^2)}{(1.5442*4.2525) - 2}$$

$$\Rightarrow \hat{\beta} = 5.1836$$

2. EM algorithm Using the above values of β and θ as the initial value and substituting the values of x in the below equations:

$$\frac{(x_i+1)(1+\theta_0+\beta_0(x_i+2))}{(1+\theta_0(1+\theta_0+\beta_0(x_i+1)} = a$$

$$= \frac{x_i + 1}{1 + \theta_0 + \beta_0(\theta_0 + 1)} = b$$

x	0	1	2	3	4	5	6	7	8	total
a	1.4967	0.5017	0.7235	0.9134	1.1102	1.3052	1.4962	1.6921	1.8845	11.1169
b	0.0308	0.0616	0.0924	0.1232	0.1539	0.1847	0.2155	0.2463	0.2771	2.2171

$$\Rightarrow \bar{a} = 1.2352 \ and \ \bar{b} = 0.2463$$

$$\theta_1 = \frac{-(\bar{a}\beta_0 - 1) + \sqrt{(\bar{a}\beta_0 - 1)^2 + 8\beta_0\bar{a}}}{2\bar{a}} = 5.8272$$

$$\beta_1 = \frac{1}{\bar{b}} - \theta_0 = 4.0595$$

Remark

For this distribution only the MOM and EM methods were considered.As shown above results for different methods differ but are in close range.

6 POISSON THREE PARAMETER LINDLEY DISTRIBUTION

6.1 Introduction

In this chapter we consider Poisson three parameter Lindley distributions. There are two types of three parameter Poisson Lindley distributions.For each type,we will construct the distribution then examine its estimation
From (3.1) consider the following forms of $g(\lambda)$

6.2 Type I three parameter

6.2.1 Construction

Proposition 7.2.1 The pdf is given by

$$f(x) = \frac{\Gamma x + \alpha}{x!\theta^x \Gamma\alpha + 1}\left(\frac{\beta\alpha\theta + x + \alpha}{\beta\theta + 1}\right) \tag{34}$$

Proof

$$\begin{aligned}
f(x) &= \int_0^\infty \frac{e^{-\lambda}\lambda^x}{x!} g(\lambda)\, d\lambda \\
&= \int_0^\infty \frac{e^{-\lambda}\lambda^x}{x!}\frac{\theta^{\alpha+1}}{\beta\theta+1}\frac{1}{\Gamma\alpha+1}(\beta\alpha+\lambda)\lambda^{\alpha-1}e^{-\theta\lambda}\, d\lambda \\
&= \frac{\theta^{\alpha+1}}{\beta\theta+1}\frac{1}{\Gamma\alpha+1}\frac{1}{x!}\int_0^\infty (\beta\alpha+\lambda)\lambda^{x+\alpha-1}e^{-\lambda(1+\theta}\, d\lambda \\
&= \frac{\theta^{\alpha+1}}{\beta\theta+1}\frac{1}{\Gamma\alpha+1}\frac{1}{x!}\left[\beta\alpha\int_0^\infty \lambda^{x+\alpha-1}e^{-\lambda(1+\theta}\, d\lambda + \int_0^\infty \lambda^{x+\alpha}e^{-\lambda(1+\theta}\, d\lambda\right] \\
&= \frac{\theta^{\alpha+1}}{\beta\theta+1}\frac{1}{\Gamma\alpha+1}\frac{1}{x!}\left[\frac{\beta\alpha\Gamma x+\alpha}{\theta^{x+\alpha}} + \frac{\Gamma x+\alpha+1}{\theta^{\Gamma x+\alpha+1}}\right] \\
&= \frac{\Gamma x + \alpha}{x!\theta^x\Gamma\alpha+1}\left(\frac{\beta\alpha\theta+x+\alpha}{\beta\theta+1}\right)
\end{aligned}$$

The rth moments is given by

$$\mu_r' = E[E(x^r/\lambda)]$$

$$E(\Lambda^r) = \int \lambda^r g(\lambda) d\lambda$$

$$= \int \lambda^r \frac{\theta^{\alpha+1}}{\beta\theta+1} \frac{1}{\Gamma\alpha+1} (\alpha\beta+\lambda) e^{-\theta\lambda} \lambda^{\alpha-1} d\lambda$$

$$= \frac{\theta^{\alpha+1}}{\beta\theta+1} \frac{1}{\Gamma\alpha+1} \left[(\alpha\beta+\lambda) e^{-\theta\lambda} \lambda^{\alpha-1+r} d\lambda \right]$$

$$= \frac{\theta^{\alpha+1}}{\beta\theta+1} \frac{1}{\Gamma\alpha+1} \left[\alpha\beta \int e^{-\theta\lambda} \lambda^{\alpha-1+r} d\lambda + \int e^{-\theta\lambda} \lambda^{\alpha+r} d\lambda \right]$$

$$= \frac{\theta^{\alpha+1}}{\beta\theta+1} \frac{1}{\Gamma\alpha+1} \left[\frac{\alpha\beta\Gamma\alpha+r}{\theta^{\alpha}+r} + \frac{\Gamma\alpha+r+1}{\theta^{\alpha}+r+1} \right]$$

$$= \frac{\Gamma\alpha+r}{\beta\theta+1} \frac{1}{\theta^r} [\alpha\beta\theta+r+\alpha]$$

Let X be a random variable which is continuous;its density function is f (x) and cumulative distribution function F (x). The survival function and hazard rate (failure rate)function of the three parameter generalized Lindley distribution are defined by

$$s(x) = \frac{(1+\lambda\beta+\lambda x^{\alpha}) e^{-\lambda x}}{1+\lambda\beta}$$

$$h(x) = \frac{\alpha\lambda^2(\beta+x^{\alpha}) x^{\alpha-1}}{1+\lambda\beta+\lambda x^{\alpha}}$$

6.3 Type II three parameter

6.3.1 Construction

Proposition 8.2.1 The pdf is given by

$$f(x) = \frac{\Gamma x+\alpha}{x!\theta^x\Gamma\alpha+1}\left(\frac{\beta\alpha\theta+x+\alpha}{\beta\theta+1}\right) \tag{35}$$

Proof

$$\begin{aligned}
f(x) &= \int_0^\infty \frac{e^{-\lambda}\lambda^x}{x!} g(\lambda)\, d\lambda \\
&= \int_0^\infty \frac{e^{-\lambda}\lambda^x}{x!}\frac{\theta^{\alpha+1}}{\theta+\delta}\frac{1}{\Gamma\alpha+1}(\alpha+\delta\lambda)\lambda^{\alpha-1}e^{-\theta\lambda} \\
&= \frac{\theta^{\alpha+1}}{\theta+\delta}\frac{1}{\Gamma\alpha+1}\frac{1}{x!}\left[\alpha\int_0^\infty \lambda^{x+\alpha-1}e^{-\lambda(1+\theta)}\ d\lambda+\delta\int_0^\infty \lambda^{x+\alpha}e^{-\lambda(1+\theta)}\ d\lambda\right] \\
&= \frac{\Gamma x+\alpha}{x!\theta^x\Gamma\alpha+1}\left(\frac{\beta\alpha\theta+x+\alpha}{\beta\theta+1}\right)
\end{aligned}$$

The rth moments is given by

$$\mu_r' = E[E(x^r/\lambda)]$$

$$\begin{aligned}
E(\Lambda^r) &= \int \lambda^r g(\lambda) d\lambda \\
&= \int \lambda^r \frac{\theta^{\alpha+1}}{(\theta+\delta)\Gamma\alpha+1}[\alpha+\delta\lambda]e^{-\theta\lambda}\lambda^{\alpha-1}d\lambda \\
&= \frac{\theta^{\alpha+1}}{(\theta+\delta)\Gamma\alpha+1}\int[\alpha+\delta\lambda]e^{-\theta\lambda}\lambda^{\alpha-1+r}d\lambda \\
&= \frac{\theta^{\alpha+1}}{(\theta+\delta)\Gamma\alpha+1}\left[\alpha\int e^{-\theta\lambda}\lambda^{\alpha-1+r}d\lambda+\delta\int e^{-\theta\lambda}\lambda^{\alpha+r}d\lambda\right] \\
&= \frac{\theta^{\alpha+1}}{(\theta+\delta)\Gamma\alpha+1}\left[\frac{\alpha\Gamma\alpha+r}{\theta^{\alpha+r}}+\frac{\delta\Gamma\alpha+r+1}{\theta^{\alpha+r+1}}\right] \\
&= \frac{\Gamma\alpha+r}{(\theta+\delta)\theta^r\Gamma\alpha+1}(\theta\alpha+r+\alpha)
\end{aligned}$$

7 CONCLUSIONS AND RECOMMENDATION

From the generalized four parameter Lindley distribution, it is possible to generate other distributions by altering the values of the parameters. The G2L and G3L distributions better define the process of a lifetime as compared to the Lindley one parameter distribution.

Estimation
Estimates using the method of moments is easy to obtain as the raw and central moments are easy to derive.
The maximum likelihood estimates for the two parameter Lindley distributions posed a challenge and in some cases was not derived as it required use of technics not discussed in this work. The EM algorithm proved to be of better use where the MLE failed.
Application The Lindley distribution has been applied i several areas such as:

1. Biological sciences:
   Shanker and Fesshaye(2015)used the Poisson -Lindley distribution to analyse the relationship between organisms and their environment in an ecology study.

2. Acturial sciences:
   Sankaran(1970)applied the Poisson-Lindley distribution to errors and accidents while Ghitany (2009) applied the same distribution to determine the service rate(how long a customer waits on queue) at the bank.

3. Computer science:
   Simulation are calculated using the Poisson =Lindley distribution and also help in programming purposes.

Recommendation
The data analysis involved different data set for each parameter.Future studies can involve the same data set for the parameters during estimation in order to determine if the results are similar.

A five parameter generalized Lindley distribution can also be explored then further deriving the distributions in this paper.

References

1. D.V.Lindley (1958).Fiducial distribution and Bayes' Theorem.Journal of Royal Statistical Society,series B,vol 20,pp 102-107

2. M. Sankaran (1970). The Discrete Poisson-Lindley distribution. Biometrics, Vol. 26, No. 1, pp. 145-149.

3. D. Karlis, and E. Xekalaki (2005).Mixed Poisson distributions, International Statistical Review / Revue Internationale de Statistique, 73(1), 35-58.

4. Ghitany, M. E., Atieh, B., Nadarajah, S. (2008). Lindley distribution and its application. Mathematics and Computers in Simulation, 78(4), 493506.

5. Ghitany, M. E. and Al-Mutairi, D. K., (2009). Estimation methods for the discrete Poisson- Lindley distribution, Journal of Statistical Computation and Simulation, Vol. 79 (1), 1-9.

6. Zakerzadeh, H., Dolati, A. (2010). Generalized Lindley distribution. Journal of Mathematical Extension (in press).

7. Mahmoudi,E. and Zakerzadah,H., (2010). Generalized Poisson Lindley distribution, Communications in Statistics-Theory and Methods, 39, 1785-1798.

8. Shanker, R., Sharma, S. and Shanker, R. (2012): A discrete two-parameter Poisson-Lindley distribution, Journal of Ethiopian Statistical Association, 21, 22 – 29.

9. Shanker, R. and Mishra, A. (2013 b): A two-parameter Lindley distribution, Statistics in Transition-new series, 14 (1), 45- 56.

10. Shanker, R., Sharma, S. and Shanker, R. (2013): A two-parameter Lindley distribution for modeling waiting and survival times data, Applied Mathematics, 4, 363 – 368.

11. Shanker, R. and Mishra, A. (2014): A two-parameter Poisson-Lindley distribution, International Journal of Statistics and Systems, 9(1), 79 – 85.

12. Shanker, R. and Hagos, F. (2015): On Poisson-Lindley distribution and its applications to biological Sciences, Biometrics and Biostatistics International Journal, 2(4), 1-5.

13. Shanker, R., Hagos, F. and Sharma, S. (2016 b): On Two-parameter Lindley distribution and its Applications to Model Lifetime data, Biometrics and Biostatistics International Journal, 3(1), 1-8.

14. Shanker, R. and Mishra, A. (2013 a): A quasi Lindley distribution, African Journal of Mathematics and Computer Science Research, 6(4), 64 – 71.

15. Zakerzadeh, H. and Dolati, A. (2009): Generalized Lindley distribution, Journal of Mathematical extension, 3 (2), 13 – 25.

16. M. WANG (2013). A new three-parameter lifetime distribution and associated inference, arXiv:1308.4128v1.

17. Chakraborty, S. and Chakravarty, D. (2012), 'Discrete gamma distributions: Properties and parameters estimations', Communication in Statistics Theory and Methods 41(18), 3301–3324

18. E. G. Deniz and E. C. Ojeda, "The Discrete Lindley distribution-Properties and Applications," Journal of Statis-tical Computation and Simulation, Vol. 81, No. 11, 2011, pp. 1405-1416.

19. Lindley, D. V., (1965). Introduction to Probability and Statistics from Bayesian viewpoint, part II, Inference, Cambridge university press, New York.

20. Mazucheli, J. and Achar, J. A., (2011). The Lindley distribution applied to competing risks lifetime data, Computer Methods and Programs in Biomedicine, Vol. 104 (2), 188-192.

21. Zamani, H. and Ismail, N., (2010). Negative Binomial- Lindley distribution and its Applications, Journal of Mathematics and Statistics, Vol. 6(1), 4-9