University of Nairobi

School of Mathematics

ISSN: 2410-1397

Master Project in Applied Mathematics

Numerical Solutions To Nonlinear Hyperbolic Systems Of Conservation Laws Applied To Traffic Flow Theory

Research Report in Mathematics, Number 29, 2020

Evance Ochieng

December 2020

Submitted to the School of Mathematics in partial fulfilment for a degree in Master of Science in Applied Mathematics

Numerical Solutions To Nonlinear Hyperbolic Systems Of Conservation Laws Applied To Traffic Flow Theory

Research Report in Mathematics, Number 29, 2020

Evance Ochieng

School of Mathematics
College of Biological and Physical sciences
Chiromo, off Riverside Drive
30197-00100 Nairobi, Kenya

Master Thesis

Submitted to the School of Mathematics in partial fulfilment for a degree in Master of Science in Applied Mathematics

Submitted to: The Graduate School, University of Nairobi, Kenya

Abstract

In the recent past, the rapid growing number of vehicles on long crowded roads elicited rigorous scientific research activities in the field of traffic flow modeling. In this thesis we present and discuss some of the macroscopic models of vehicular traffic flow; here we discuss Payne-Whitham(P-W) and the Aw-Rascle models of traffic flow both of which are second order. We study the Riemann problems of these models. The numerical method developed here is the finite volume method (FVM), more specifically the Godunov-type approximation together with the CFL condition for stability test of solutions.

Master Thesis in Mathematics at the University of Nairobi, Kenya.
ISSN 2410-1397: Research Report in Mathematics

DISTRIBUTOR: School of Mathematics, University of Nairobi, Kenya

Declaration and Approval

I the undersigned declare that this dissertation is my original work and to the best of my knowledge, it has not been submitted in support of an award of a degree in any other university or institution of learning.

Signature Date

Evance Ochieng
Reg No. I56/11798/2018

In my capacity as a supervisor of the candidate's dissertation, I certify that this dissertation has my approval for submission.

Signature Date

Dr. Charles Nyandwi
School of Mathematics,
University of Nairobi,
Box 30197, 00100 Nairobi, Kenya.
E-mail: nyandwi@uonbi.ac.ke

Dedication

This thesis is dedicated to my uncle James Opiyo, aunt Rael Adhiambo,cousins Levis, Shantomary and Joshua, my sisters, Belinda and Lencer, and also to my friends, Kennedy and Hyida.

Contents

Acknowledgments

To begin with, I thank God for His protection and care throughout my studies.
I also want to express my true appreciation to the people who have sincerely instructed and extended their help in my favour.
First, I thank my mentor, role model and supervisor , Dr. Charles Nyandwi for the instructions he has been giving me. He gave me useful and helpful suggestions on how to go about the research, he also took a good time to answer my queries and concearns,he properly read my thesis and his suggestions will greatly contunue to influence my future researches. He also introduced me to his colleagues, including Prof. Weke, Prof. Manene, Dr. Katende, Dr. Juma and Dr. Luketero and many others to whom I also owe my appreciation.
I also thank my uncle James for his genuine support during my styudies.
I am also grateful to my former classmate Dr. Juma for sharing his ideas with me.
Finally, I want to thank my friends and family for being a good source of encouragement and support. ....

Evance Ochieng

Nairobi, 2020.

1 Introduction

1.1 Historical Background

Traffic flow is the study of the movements of individual drivers and vehicles between two points and the interactions they make with one another and they play a vital role in the progress of overal social productivity. In the 1950s James Lighthill and Gerard Whitham two experts in fluid dynamics and independently P. Richards modeled the flow of car traffic along a single lane using the same equations describing the flow of water (Lighthill et al, 1955; Richards 1956).
The basic idea is to look at traffic on a large scale so as to consider cars as small particles and to assume the conservation of cars number.
The LWR model is described by a single conservation law, a special partial differential equation the variable car density is a conserved quantity which can neither be created or destroyed.
Traffic engineering usually deals with the analysis of the behaviour of traffic and to design the roadway facilities for a smooth and safe operation of traffic. Traffic flow like fluid flow has several parameters associated with it; the parameters provide information regarding the nature of the traffic flow which helps which helps the analyst in detecting any variation in flow characteristics. Understanding traffic behaviour requires thorough knowledge of traffic stream parameters and their mutual relationship.

Given the continual and dramatic increase in vehicle ownerships and many more people travelling day in day out, at any given time there are millions of vehicles on our roadways these vehicles interact with each other and have great impact on overal movement of traffic or traffic flow. Every driver is therefore faced with with the conflicting objective of reducing travel time and avoiding accidents.
Engineers are given the task of designing controlling and managing the road systems to ensure that drivers travell as efficient as possible on our roadway. For this study we seek to obtain a mathematical model that can provide the understanding necessary for the engineer to undertake his task and to assist in reducing the negative impacts that traffic congestion can have on our lives.
A mathematical model is an inevitable component of scientific and techical progress. The very formulation of the problem of mathematical modellling of traffic flow leads to a better understanding of the traffic phenomena we see around us and a precise plan of action for the engineer.
To better represent the the traffic flow, relationships have been established between the

three main characteristics of traffic flow namely, flow rate density and velocity. THese relationships helps in planning, designing controlling and managing the operations of roadway facilities. Since 1930's scientists in various fields have focused on traffic problems and great progress has been made.The most widely used model is the Greenshields model(1935)which states that the relationship between speed and density is linear.Here we focus our attention on a segment of the roadway where we will analyse traffic phenomena resulting from the complex interaction of many vehicle instead of studying the phenomenon associated with the motion of individual cars.

1.2 Theory of Traffic Flow

Attempts to produce a mathematical theory of traffic flow date back to the 1920s, when Frank Knight first produced an snalysis of traffic equilibrium, which was refined into Wardrop"first and second princilpes of equilibrium in 1952.
Nonetheless, even with the advent of significant computer processing power, to date there has been no satisfactory general theory that can be consistenty applied to real flow conditions. Current traffic models use a mixture of empirical and theoretica techniques. These models are then developped into traffic forecasts and take account of proposed local and major changes, such as increased vehicle use, changes in land use or changes in modes of transport(with people moving from bus to train or car, for example), and to identify the areas of congestion where the network needs to be adjusted.
Traffic behaves in a complex and non linear way, depending on the interactions of a large number of vehicle. Due to the individual reactions of human drivers, vehicles do not interact simply following the laws of mechanics, but rather cluster formation and shock wave propagation both forward and backward, depending on vehicle density. Some mathematical models of traffic flow use a vehicle queue assumption, in which the vehicles along congested link do not spill back along the length of the link.
In a free-flowing network, *traffic flow theory* refers to the traffgic stream variables of speed, flow, and concentration. These relationships are mainly concearned with the uninterrupted traffic flow, primarily found on freeways or expressways. Flow conditions are considered "free" when less than 12 vehiclesper mile per lane are on a road. "Stable" is sometimes described as 12-30 vehicles per mile per lane. As the density reaches maximum mass flow rate (or flux) and exceeds the optimum densiyty (above 30 vehicles per mile per lane), traffic flow becomes unstable, and even a minor incident can result in persistent stop-and-go driving conditions. A "breakdown" condition occurs when traffic becomes unstable and exceeds 67 vehicles per mile per lane. "Jam density" refers to the extreme traffic density when traffic flow stops completely, usually in the range of 185-250 vehicles per mile per lane.

Single vehicle dynamics Motion as a function of time

Let (x)(t) be the vehicle trajectory. Then,

$$\upsilon(t) = x'(t)(speed)$$

$$a(t) = \upsilon'(t) = x''(t)(acceleration)$$
$$j(t) = a'(t) = \upsilon''(t) = x'''(t)(jerk)$$

Traffic stream properties

There are three main variables to visualize a traffic stream: speed (V), density ρ ; the number of vehicles per unit space, and flow q; the number of vehicles per unit of time
Speed Speed is the distance covered per unit time. One cannot track the speed of every vehicle; so in practice, average speed is measured by sampling vevicles in a given area over a period of time. two definitions of average speed are identified; "time mean speed" and "space mean speed"
Time mean speed is measured at reference point on the roadway over a period of time. In practice, it is measured by the use of loop detectors, when spread over a reference area, can identify each vehicle and can track its speed. However, average speed measurements obtained from this method are not accurate because instantaneous speeds averaged over several vehicles do not account for the vehicles that are travelling at different speeds over the same distance.

$$\upsilon_t = \frac{1}{m}\sum_{i=1}^{m}\upsilon_i$$

where m represents the number of vehicles passing the fixed point and υ_i the speed of the ith vehicle.
Space mean speed is measured over the whole roadway segment. Consecutive pictures or video of a roadway segment track the speed of individual vehicles, and then the average speed is calculated. It is considered more accurate than the time mean speed. The data for space calculating space mean speed may be taken from satellite pictures, a camera ,or both.

$$\upsilon_s = \left(\frac{1}{n}\sum_{i=1}^{n}\upsilon_i\right)^{-1}$$

where n represent the number of vehicles passing the roadway segment.
The space mean speed is the harmonic of these speeds. The time mean speeds is never less than the space mean speed:

$$\upsilon_i = \upsilon_s + \frac{\sigma_s^2}{\upsilon_s}$$

where σ_s^2 is the variance of the space mean speed

Density Density ρ is defined as the number of vehicles per unit length of the roadway. In traffic flow, the two most imprtant densities are the critical density ρ_c and the jam density ρ_j. The maximum density achievable under free flow is ρ_c wwhile ρ_j is the maximum density achieved under congestion. in general, jam density is seven times the critical density. Inverse of density spacing (s) which is centre-to-centre distance between two vehicle.

$$\rho = \frac{1}{s}$$

Flow Flow q is the numbers of vehicles passing a reference point per unit time, vehicles per hour. The inverse of flow is the headway (h), which is the time that elapses between the ith vehicle passing a refrence point in space and the (i + 1)th vehicle. In congestion, h remains constant. As a traffic jam forms, h approaches infinity.

$$q = \rho \upsilon$$

$$q = \frac{1}{h}$$

The flow (q) passing a fixed point (x_1) during an interval (T) is equal to the inverse of the average headway of the m vehicles.

Kener's three-phase traffic theory

Kerner's three-phase traffic theory is an alternative theory of traffic flow. Probability the most important result of the three-phase theory is that at any time instance there is arange of highway capacities of free flow bottleneck. The capacity range is between some maximum and minimum capacitie. The range of highway capacities free low bottleneck in three-phase traffic theory contradicts fundamentally classical traffic theories as well as methods for traffic management and traffic control which at any time instant assume the existence of a popular deterministic or stochastic highway capacity of free flow at the bottleneck.

Kerner's network breakdown minimization (BM) principle

Kerner introduced an alternative approach to traffic assignment based based on his network (BM) principle.

Rather than an explicit minimization of travel time that is the objective of system optimum and user equilibrium, the BM principle minimizes the probability of the occurence of congestion in a traffic network. Under sufficient traffic demand, the application of the BM principle should lead to implicit minimization of travel time in the network.

Variable speed limit assignment

This is an upcoming approach of eliminating shocwave and increasing safety on the vehicles. The concept is based on the fact that the risk of accident on a roadway increases with the speed differential between the upstream and downstream vehicles. The two types of crash risk which can be reduced from the VSL imlimentation are the reaer-end crash and the lane change crash. Variable speed limits seeks to homogenize speed, leading to a more constant flow. Different approaches have been imlimented by researchers to build a suitable VSL algorithm.

Variable speed limits ara usually enacter when sensors along the roadway detect that congestion or weather events have exceeded thresholds. The roadway speed limit will then be reduced in 5-mp increaments through the use of signs above the roadway (Dynamic Message Signs) controlled by the department of transportation. The goal of this process is the both increase safety through accident reduction and to avoid or postpone the onset of congestion on the roadway. The resulting traffic flow is slower overal but but less stop-and-go, resulting in fewer instances of rear-end snd lane changes crashes. The use of VSLs also regularly employs shoulder lanes permitted for transportation only under congested states which this process aims to combat.

Through historical data obtained at VSL sites, it has been determined that the implimentation of this practice reduces accident numbers by 20-30In addition to safety and efficiency concearns, VSLs can also garner environmental benefits such as decreased emissions, noise, and fuel consumption. This is due to the fact that the vehicles are more fuel efficient when a constant rate of travel, rather than in a state of constant acceleration and deceleration like that usually found in congested conditions.

Key Background Theory

Fundamental relationships between volume (q), speed (u), and density (ρ) of traffic flow can explain the effectiveness of the VSL. The relationships between these variables is covered in the "Traffic sterem properties".

The Newell's simplified traffic flow theory is also utilized for this model to show the relationship displayed in the flow-density.

VSL Theory

in showing the effectiveness of VSL, several key assumptions are made.

1. No entrance/exit ramps on freeway of analysis.
2. Traffic floe analysis is based upon vehicle trajectory with no acceleration/deceleration.
3. Only passenger vehicles are consiered.

4. Full compliance with VSL from all drivers

5. Focuss on reducing congestion.

Determining VSL Effectiveness

VSL effectiveness can be verified quantitatively through analyzing the shockwaves formed by congestion with and without congestion and without implimentation. In the study cited throught this section, shockwave for an upstream incident were utilized for this comparison. One shockwave was formed through the congestion caused by an upstrem incident, and the other was formed througgh this incident's clearing and recovery to revert to normal flow. It was found that two shockwaves for a system with VSL imlimentation resulted in a much shorte delay and que length due to homogenization of flow through more rapid dissipation of the first shockwave. Through this study, the effectiveness of VSL in reducing congestion is proved, though with the limiting assumptions above.

Limitations of VSL

VSL imlimentation is most ideal under severe congestion states. If a reduced VSL is imlimented in traffic states under critical density, then they will result in reduced flow overall through increased travel times. Thus, the benefits of VSL must be enacted carefully at only threshold states, which depends on the existing traffic data of the roadway. Therefore, sensors must be turned effectively to detect when a congestive state will begin based upon historical data. The VSL must also begin before stop-and-go congested states of traffic are reached in order to be effective.
VSL effectiveness is also nearly complete based upon driver compliance. This can be ensured through enforcement and dynamic signage. Drivers must sense the legitimacy of the VSL for it to be effective; the reasoning for the new speed limit should be explained via signage in order to ensure compliancy. If the VSL is not viwed as mandatory by drivers, then it will work effectively. if the VSL is reduced by a significant amount, compliance will reduce significantly. For this reason, most VSL speeds are above 40 mph on freeways. Several historical example show that the compliance reduces at a much greater rate when the new speed limit falls below this threshold.

Road Junctions

A major consideration in road capacity relates to the design of junctions. By allowing long "weaving sections" on gently curving roads at graded intersections, vehicles can

often move across lanes without causing significant interference to the flow. However this is expensive and takes up a large amount of land, so other patterns are often used, particularly in urban or very rural areas. Most large model use crude simulations for intersections, but computer simulations are avilable to model specific sets of traffic lights, roundabouts and other scenarios where flow is interrupted or shared with other types of road users or pedestrians. A well-designed junction can enable significantly more traffic flow at a range of traffifc densities during the day. By matching such a model to an "Intelligent Transport System" traffic can be sent in uninterrupted "packets" of vehicles at predetermined speeds through a series of phased traffic lights.

Kinematic wave model

The kiinematic wave model was first applied to traffic flow by Lighthill and Whitham in 1955. Their two-part paper was first developed in the theory of kinematic waves using the motion of water as an example. In the second half, they extended the theory of traffic on "crowded arterial roads." This paper was primarily concearned with developping the idea of traffic "humps" (increases in flow) and their effects on speed, especially through bottlenecks.

The authors began by discussing previous approaches to traffic fllow theory. They note that at the time there had been some experimental work, but that "theoretical approaches to the subject were in their infancy." One researcher in particular, John Glen Wardrop, was primarily concearned with the the statistical methods of examination, such as space mean speed, time mean speed and the "effect of increase of flow on overtaking" and the resulting decrease in speed would cause. Other previous research had focused on two separate models: one related traffic speed to flow and another related speed to the headway between vehicles.

The goal of Lighthill and Whitham, on the other hand, was to propose a new method of study "suggested by theories of the flow about supersonic projectiles and of flood movements in rivers." The resulting model would capture both of the aforementioned relationships, speed-flow and speedway, into a single curve "which would sum up all the properties of astretch of road which are relevant to its ability to handle the flow of congested traffic." LWR being a first order approximate model of traffic flow dynamics has a number of serious deficiencies.

1. The model predicts infinite deceleration when a vehicle crosses a shock. This reveals that acceleration or deceleration of a vehilcle stream is proportional to traffic concentration and concentration gradient.

2. The model assumes that the equilibrium speed-concentration holds for non-equilibrium traffic. In reality, traffic flow is hardly in equilibrium and its dynamics is as a result of retarded response of drivers to various frontol stimuli.

3. The model model lacks a mechanism for traffic to accelerate or decelecrate at a finite speed when the concentration gradient is large.

Due to these deficiencies, Payne derived an equation of motion from car-following theory, $response = sensitivity \times stimulus$ The model is called PW model. Payne's model together with its computer imlimentation aoused considerable interest in higher order continum traffic flow models. However application of this model, reported mixed results. Daganzo points out that most of the problems were attributed to the fundamental flaw that the model produces 'wrong way travel'that is, negative travel speed.
Daganzo pointed out that information in the PW model can travel faster than vehicles. There was an enhanced continuum traffic flow theory that removes certain deficiencie of the LWR theory without introducing new flaws. In deriving the nwe theory Zhang neglected higher order terms. This raised concearn by Daganzo as to whether one can neglect higher order terms when the concentration on the road is rapidly changing, as occurs near a shock path. However he showed that these higher order terms can be neglected if the concentration function ρ(x,t) is well behaved and temporal spatial scales are properly treated.
Daganzo argued that the second order models violate the principle that the car is anisotropic particle that responds to frontal stimuli. This motivated Rascle to develop a model which would rectify the above inconsistencies.
The Aw-Rascle respects all natural requirements(frontal stimuli) and the inconsistencies of 'wrong way travel'of second order models disappear.
Klar et al derived a macroscopic traffoic flow model based on Kerner theory using Integral-Differential Equations of kinetic models. The model has the following equations;

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho u)}{\partial x} = 0$$

$$\frac{\partial (\rho u)}{\partial t} + \frac{\partial (\rho u^2)}{\partial x} - a(\rho, u)\frac{\partial u}{\partial x} = \rho R(\rho, u)$$

R(ρ,u), is the relaxation term which describes the tendancy of traffic flow to relax to an equilibrium velocity.
a(ρ, u), is the speed adaptation coefficient given by;

$$a(\rho, u) = c\frac{\rho}{1-\rho}$$

where

$$\frac{a(\rho, u)}{\rho} = \rho p\prime(\rho)$$

Kerner in his work describes traffic in several major cities in the world in three phases;

free folw, wide moving jam and synchronized flow. In our Kenyan roads the three phases exist and knoeledge on how the phase transition takes place can be a great contribution in traffic management. the speed of the vehicles in the synchronized and wide moving jam is influenced by the aggression of the driver. The same driver aggression influences the phase transition from free to congested region.

Finally, initial conditions must be defined to solve a problem using the model. A boundary is defined to be $\rho(t,x)$, representing density as a function of time and position. These boundaries typically take two different forms, resulting in initial value problems (IVPs) and boundary value problems (BVPs). Initial value problems give the traffic density at time $t = 0$, such that $\rho(0,x) = g(x)$, where $g(x)$ is the given density function. Boundary value problems give some function $g(t)$ that represents the density at the $x = 0$ position, such that $\rho(t,0) = g(t)$. The model has many uses in traffic flow. One of the primary uses is in modeling traffic batllenecks.

The inhomogeneous Kinematic Wave Traffic Flow Model as a Resonant Non Linear System

The kinematic wave traffic flow model for an inhomogeneous road is studied as a resonant nonlinear system, where an additional conservation law is introduced to model time invariant road inhomogeneitie such as changes in grades or number of lanes. This resonant system has two families of waves, one which is a standing wave originated originated at the inhomogeneity. The nature of these waves are examined and thgeir time space structures are studied under Riemannian initial conditions and proper entropy conditions.
The kinematic wave traffic flow model was introduced by Lighthill and Whitham (1955) and Richards (1956) (LWR) for modelling dense traffic flow on crowded roads, where the evolution density $\rho(x,t)$ and flow rate $q(x,t)$ over time is described by

$$\rho_t + q_x = 0 \tag{1}$$

This equation follows conservation of traffic that vehicles are neither generated nor destroyed on a road section with no entries and exits.
The conservation equation alone is not sufficient to describe traffic evolution, because it does not capture the unique character of vehicular flow-drivers slow down when their front spacing is reduced to affect safety. The LWR model addresses this issue by assuming that a functional relationship between the local flow rate and density ie $q = f(x,\rho)$. This flow density relation, also known as fundamental diagram of traffic flow is often assumed to be concave in ρ and is a function of the local characteristics of a road location, such as the number of lanes, curvature grades and speed limit, as well as vehicle and driver

composition. When a piece of roadway is homogeneous ie, the eforementioned characteristics of the road are uniform throughout the road sectoin, the fundamental diagram is invariant to location x and the LWR mode is

$$\rho_t + f(\rho)_x = 0 \tag{2}$$

In contrast, if a section of roadway is homogeneous, the LWR model is

$$\rho_t + f(x,\rho)_x = 0 \tag{3}$$

Here we introduce a more explicit notation, an inhomogeneity factor a(x), into the flux function $f(x,\rho)$ and obtain the following equivalent LWR model for an inhomogeneous road:

$$\rho_t + f(a,\rho)_x = 0 \tag{4}$$

This equation is particularly suited for our later analysis of the LWR model for inhomogeneous roads. (we shall hereafter call 2.2 the homogeneous LWR model and 2.4 the inhomogeneous LWR model.

Both the homogeneous and inhomogeneous LWR models have been studied by researchers and applied by practitioners in the transport community. Note that the homogeneous version (2.2) is nothing more than a scalar conservation law. Therefore, its wave solutions exist and are unique under the so called 'Lax entropy condition' (Lax 1972). These solutions are formed by basic solutions to the Riemmanian problem of (2.2) in which the initial conditions jump at a boundary and the constant both upstream and downstream of the jump spot. Nevertheless, because analytic solutions are difficult to obtain for (2.2) with arbitrary initial/boundary conditios, numerical solutions have to be computed in most cases. The most often used approximation of (2.2) is perhaps tha of Godunov. In the Godunov method, a roadway is partitioned into a number of cells, and the change of the number of vehicles in each cell during a time interval is the net inflow of vehicles from its boundaries.

The rate of traffic flowing through a boundary is obtained by solving a Riemannisn problem at this boundary. Besides the Godunov method, there are other types of approximations of the homogeneous LWR model, and some of them are shown to be variants of Godunov's method (Lebacque 1996)

Newel-Daganzo Merge Models

In the condition of traffic flows leaving two branch roadways and merging into a single flow through a single roadway, determining the flows at the pass through the merging process and the state of each branch of roadways becomes an important task for traffic engineers. The Newel- Daganzo merge model is a bbgood approach to solve these problems.

The simple model is the output of the result of both Gordon Newell's description of the

merging process and the Daganzo,s cell transmission model. In order to apply the model to determine the flows which exiting two branch of roadways and the state of each branch of roadways, and the number of lanes of the single roadway. The merge ratio will be calculated in order to determine the proportion of the two input flows when both of branches of roadway are operating in congested conditions.

Incommensurability of Kerner's thrre-phase traffic theory and classical traffic flow theories

The explanation of traffic breakdown at a highway bottleneck by F|S transition in a metastable free flow at the bottleneck is the basic assumption of Kerner's three-phase traffic theory. The tree phase traffic theory is consistent with the fundamental empirical features of traffic breakdown. None of the earlier traffic-flow theories incorporates a F→S in a metastable free flow at the bottleneck. Therefore as above mentioned non of the classical traffic flow theories is consistent with the set of empirical features of real traffic breakdown at a highway bottleneck. The F→S phase transition in metastable free flow at a highway bottleneck does explain the empirical evidence of the induced transition from free flow to synchronized flow together with the flow rate dependence on of the breakdown probability.

Mobility within an urban area is a mojor component of that areas quality of life and an important issue facing many cities as they grow and their transportation faciliteis become cngested. There are no shortage of techniques to improve traffic flow ranging from traffic timing signal optimization(with elaborate computer based routines aswell as simpler manuals, heuristic and methods) to minor physical changes, such as adding a lane by elimination of parking. However the difficulties lies in evaluating the effectiveness of these techniques.
A number of methods currently in use reflecting progress in traffic flow theory and practice in the last thirty years, can effectively evaluate changes in the performance of an arterial. But dilemma is created when these individual components connected to form the traffic network are dealt with collectively.
The need then is for a consistent reliable means to evaluate traffic performance in a network under various traffic and geometric configurations. T the development of such performance models extends traffic flow theory into the network level and provides traffic engineers with means to evaluate system-wide control strategies in urban areas. in addition the quality of service provided to the motorists could be monitored to evaluate a city's ability to manage growth. For instance network performance models could also be used by a state agency to compare traffic conditions between cities in order to more equitably allocate funds for transportation system improvements.
the performance of a traffic system is the response of that system to given travel demand levels. The traffic system consists of the network topology(street width and configuration). the number of trips between origin and destination points along with the desired

arrival/or departure times comprise the travel demand levels. The system response ie, the flow pattern can be measured in terms of the level of the service provided to the motorists.
The taffic flow theory at the intersection and arterial level provides this measurement in terms of the three basic variables of traffic flow, speed, flow and concentration. These three variables approximately defined can also be used to describe traffic at a network level. this description must be one that can overcome the interactabilitie of existing flow theories when network component interactions are taken into account.

Travel Time Models

Travel contour maps provide an overview of how well a street network is operating at a specific time. Vehicles can bbe dispatched away from the specified location in the network, and each vehicle's time and position noted at the desired intervals.
contours of equal travel time can be established, providing information on the average travel times and mean speeds over the network. However the informatin is limited in that the travel times are trelated to a single point, and the study would likely to be repeated for other locations. Also substantial resources are required to establish statistical significance.
Most importantly, though that it is difficult to capture network performance with only one variable (travel time or speed in this case) as the network can be offering quite different levels of service at the same speed. This type of model has been generalised by several authors to estimate average network travel times(per unit distance) or speeds as a function of the distance from the central business district(CBD) of a city unlike travel time contour maps which consider only travel times away from a specific point.

General Network Models

A number of models icorporating performance measures other than speed have been proposed. Early works by Wardrop and Smeed (Wardrop 1952; Smeed 1968) dealt largely with the development of macroscopic models for arterials which were later extended to general network models.

Network Capacity

Smeed(1966) considered the number of vehicles which can "usefully" enter the central area of a city, and N the number of vehicles per unit time that can enter the centre. In general N depends on the general design of the road network, width of roads, type of intersection control, distribution of destinations and vehicle mix. The principal variables for towns with similar networks, shapes, types of control are A, the area of town; f, the

fraction of area devoted to roads; and c the capacity expressed in vehicles per unit time per unit width of road (assumed to be the same for all roads). These are related as follows;

$N = \alpha f c \sqrt{A}$

The Modeling of Automobile Traffic

when one thinks of modeling automobile traffic, it is natural to reason from personal experience and to visualize the car and driver as a coupled system, the driver responding to the sorrounding vehicles and operating the car to make it become a part of the flow of freeway and city traffic. Thus the traffic is not just a mechanical process but one in which human decisions are involved, decisions which we have all experienced and can understand.

In our analysis of traffic we shall however step back from this personal view to take a broader perspective. Think of a traffic helicopter pilot looking on a highway grid. looking at four miles of highway, the pilot will see a line of cars moving with various speeds. On some stretches the traffic may be light and fast on other stretches heavy and slow. To this observer the individual vehicles are not as important as the sense of overal flow of the cars. the reason why cars in the lighter traffic move faster is clear to any driver, but to the observer in the helicopter it seems to be a property of the spacing of the cars. The closer they are they are together, the slower they move.

models of traffic flow try to exploit these observations and use them to formulate a set of assumptions which produce models which can be used to try to understand the peculiar and often frustrating occurences of daily driving. In the picture just suggestd, the cars are viewed in the large, almost as a moving gas or liquid. This kind of picture we will call a continum model of traffic flow. We shall spend much of our time working from such a point of view. there is however another body of traffic theory based upon the point of view of the individual responding to sorrounding traffic just the way we would naturally want to think about driving. This kind of study is called car following theory. Traffic networks consisting of highways, streets and other types of roadways are very important in management of traffic situations; they enable reliable and economical conveyance of people and merchandise. Nowdays, the increasing number of automobles on urban streets and roadways together with the related economical and social implications like prevention of car collisions and polution has motivated me to go into study of traffic model. I believe that at the end of this research, I will be equipped with the needed mathematical tools to go into further studies and practice in the department of traffic.

Traffic flow can be defined as the study of the movement of individual drivers and vehicles between two points. Unfortunately, studying the traffic flow is difficult because driver behaviour is unpredictable with a significant percentage of certainity. Fortunately, drivers tend to behave within a reasonably consistent range, thus traffic streams tend to

have some reasonable consistency and can be roughly represented mathematically.
To better represent traffic flow, mathematical relationships have been established between the three main characteristics: flux/flow, density and mean velocity which are represented by q, ρ, and v, respectively. Vehicular traffic flow can be viewed as an engineering and also a mathematical problem. However, both applied mathematicians and engineers are involved in this field. Mathematicians have been able to come up with suitable methods which describe the evolution in time and space of the flow conditions like vehicle density and velocity. Moreover, the mathematical research also consists in solving mathematical problems generated by the application of models to real traffic flow conditions. The outcome may be useful for engineers involved in traffic control[14].
There are three different types of models in mathematical modeling of traffic flows; microscopic, kinetic and macroscopic models[3],[7],[8],[14]. Microscopic modeling corresponds to modeling the dynamics of each single vehicle under the action of the surrounding vehicles by ordinary differential equations based on Newton's law, ie all vehicles are identified. Position and velocity of each vehicle define the state of the system as dependent variables of time variable, this is referred to as car following models[7],[14]. On the other hand, macroscopic ones model traffic flows such as flow rate q, density of traffic ρ and speed of travel v as a continuum(ref), that is the state is described by locally averaged quantities, q, ρ v are considered as dependent variables of time and space. aaaThe relationship between these variable is $q = \rho v$. Finally, we have the kinetic modeling which as well requires a large number of particles in the system, unlike the macroscopic approach it is based on a microscopic modeling of their mutual interactions [12]
Among the above thre types of model, macroscopic models are more suitable for modeling traffic in large complex networks since less supporting data and computation are needed. In this thesis some of the macroscopic traffic flow models are studied both analytically and numerically; We propose second order models. Traffic flows are classified according to the traffic conditions, roadway conditions and traffic network structure. We say that traffic flows are in equilibrium when their travel speed is uniquely determined as a function of the traffic density, otherwise they are in non-equilibrium The simplest situation in traffic flow theory is the homogeneous equilibrium traffic flow (ref) i.e a uniform flow of vehicles is independent of space and time variables; here the state of the the traffic flow only depends on the vehicle density. Traffic flows are considered inhomogeneous when the roadway has different parameters at different locations.
In This thesis we focus on equilibrium traffic flows.

1.3 Objectives

1. To study and understand the theory of traffic flow.

2. To study and understand scalar conservation laws and how to solve the associated Riemann Problems.

3. To derive some second order traffic flow models.

4. To study an understand numerical methods, specifically the Finite Volume Method for hypebolic conservation laws applied to traffic flow models.

1.4 Statement Of The Problem

We study how to solve hyperbolic systems of first order nonlinear conservation laws numerically. This study is extended to traffic flow models to help in comprehending the dynamics of vehicular flow on highways.

1.5 Literature Review

The most basic continuum traffic flow model was the first order model developed by Lighthill, Whitham (1955) and Richards (1956) (ref), based on the assumption of conservation of mass and density i.e The nummber of vehicles between any two points if there are no entrances or exits is conserved. In this thesis, the model is based on the idea that the classical Euler and Navier-Stokes equations of fluid dynamics describing the flow of fluids could also describe the motion of cars along a road, provided that we consider a large scale point of view so as to consider cars as small particles and their density as the main quantity to be considered [8]. The WLR model is formulated as a scalar hyperbolic conservation law, and is often solved by finite difference methods(LeVeque). The LWR model is given by;

$$\partial_t \rho + \partial_x(\rho v) = 0 \tag{5}$$

for $0 \leq \rho \leq \rho_{max}$ where ρ_{max} is the jam density an because the model is a scalar conservation law for ρ alone, the velocity v in the above equation must be a function of ρ.
LWR used this model to show existence of shockwaves asso765432iated with traffic systems. Lighthill and Whitham introduced a second order model[12], but unfortunately the model was not properly explored until Payne an Whitham developped a second order continuum model for traffiic flow. The model is defined by the conservation law (5) and the acceleration equation [1],[12] in which the average traffic speed satisfies an evolution equation akin to Navier-Stokes equation of the form

$$\partial_t(\rho) + v\partial_x(\rho v) = -\frac{p'(\rho)}{\rho}\partial_x\rho + 1/\tau(V(\rho) - v) + \mu/\rho\frac{\partial^2 v}{\partial x^2}$$

where τ and μ are some positive constants and pressure $p(\rho)$ found in gas dynamics. Last term on the right models viscosity while the second term expresses the tendency of traffic at a given density to relax to some average speed $V(\rho)$(ref). It is defined in such

a way that at low densities it is determined by road conditions and speed limits and it is slightly dependent on ρ while at high densities $V(\rho)$ appraches 0 and again slightly dependent on ρ. V is chosen to be decreasing density function, the first term is the anticipation factor(drivers slow down at the sight of an increase in traffic density ahead of them)

Inspite of this, Daganzo [5] stated that traffic arriving at the end of a densely packed queue of vehicles would result in vehicles backwards in space which is physically absurd. This is due to the isotropic nature of the models since the behaviour of vehicles is influenced by vehicle behind them due to the diffusive effects[5]. Hence the Payne model like other secon order models that are avilable in literature, produced a flawed behaviour for some traffic conditions. The models violate the following principle; a particle of fluid responds to frontal stimuli and from the back, a car for instance is like isotropic particle that mainly responds to the stimuli from the front. To improve this model, Aw and Rascle [1] produced an anisotropic second order model which satisfies this principle and avoided the flaws noted by Daganzo by replacing the p in the equation of momentum by an anticipation factor.
The Aw-Rascle model is given as a coupled system of two equations below

$$\partial_t \rho + \partial_x(\rho v) = 0,$$
$$\partial_t(v + p(\rho)) + u\partial_x(v + p(\rho)) = 0$$

1.6 Outline

The outline of the thesis is as follows:

Chapter 1: Theory and Introduction to traffic flow are discussed in this chapter.

Chapter 2: Hyperbplic conservation laws and Riemann problems are also considered in this chapter.

Chapter 3: Here we derive the two traffic flow models Payne-Whitham(P-W) model and Aw-Rascle(A-R) model.

Chapter 4: Here we discuss the numerical methods for the traffic flow models.

2 Hyperbolic Conservation Laws

2.1 Definition

2.1.1 Systems of First-Order PDEs

System of n first-order PDEs can be expressed in linear algebra form,

$$\mathbf{U}_t + \mathbf{A}\mathbf{U}_x + \mathbf{B} = \mathbf{0} \tag{6}$$

Here $\mathbf{U} = (U_1, ..., U_n)^T$ is a set of independent physical quantities in the system, $\mathbf{A}$ is an $n \times n$ matrix deciding how these quantities interact, and $\mathbf{B}$ is a vector of size n describing the source terms. Subscript t and x means an (element-wise) partial derivative with respect to time or space, $\mathbf{U}_x = \partial\mathbf{U}/\partial x$.
Generally, both $\mathbf{A}$ and $\mathbf{B}$ could depend on any U_i

2.1.2 Conservation Laws

A system describing a set of conserved physical quantities $\mathbf{U}$ (e.g. density, momentum) can in general be written

$$\mathbf{U}_t + F(U)_x = 0, \tag{7}$$

here $\mathbf{F}(\mathbf{U}) = (F_1(\mathbf{U}), ..., F_n(\mathbf{U}))^T$ is called the $flux function$. Each F_i can be a function of all $(U_1, ..., U_n)^T$

2.1.3 Jacobian of the flux function

The matrix of Jacobian is given by

$$\mathbf{A}(\mathbf{U}) = \frac{\partial \mathbf{F}}{\partial \mathbf{U}} \tag{8}$$

A quasi-linear system described by equation (7) can be put on matrix form (6) by using the Jacobian matrix $\mathbf{A}$,

$$\mathbf{U_t} + \mathbf{A}(\mathbf{U})\mathbf{U_x} = \mathbf{0} \tag{9}$$

2.1.4 Hyperbolic, strictly hyperbolic, and elliptic system

A system of the form (6) is *hyperbolic* at point (x,t) if all the n eigenvalues of the matrix $\mathbf{A}$ are real and correspond to n linearly independent eigenvectors $\mathbf{K^{(1)}},...,\mathbf{K^{(n)}}$. The system is *strictly hyprbolic* if all n eigenvalues are distinct. If all the eigenvalues are imaginary, the system is *elliptic*.

2.2 Riemann Problem

A Riemann problem referes to specific (IVP) where initial conditions are piecewise constant, with only one discontinuity (usually taken to be at the origin.). It is given by;

$$\begin{cases} \mathbf{U}_t + F(U)_x = 0 \\ \mathbf{U}(x,0) = \begin{cases} \mathbf{U_L} & forx < 0 \\ \mathbf{U_R} & forx > 0 \end{cases} \end{cases}$$

In the strictly hyperbolic case the n eigenvalues of $\mathbf{A}$ are ordered from lowest to highest, $\lambda_1, \lambda_2, ..., \lambda_n$. Each of them correspond to a characteristic wave travelling at a velocity λ_i from the origin.
The state of the system to the left of λ_1 wave will stay $\mathbf{U_L}$. The wedges between the leftmost and rightmost wave will be called *star regions.* Here, the state $\mathbf{U}^*$ will be piecewise constant with discontinuities at each wave front λ_i. $\mathbf{U}^*$ will be some linear combination of the decomposed right and left states.
To find the general solution, $\mathbf{U}$, let's first write the right and left states as expansions of the n linearly independent eigenvectors of $\mathbf{A}$, which we will denote by $\mathbf{K^{(1)}},...,\mathbf{K^{(n)}}$.

$$\mathbf{U_L} = \sum_{i=1}^{n} \alpha_i \mathbf{K^{(i)}}, \mathbf{U_R} = \sum_{i=1}^{n} \beta_i \mathbf{K^{(i)}} \tag{10}$$

The eigenvectors $\mathbf{K^{(i)}}$ descfribe the portion of the state information $\mathbf{U}$ that is carried along with the wave corresponding to the eigenvalue λ_i. We can combine the expansion (10) and our knowledge of the velocity of each characteristic wave to find a mixed state $\mathbf{U}^*$ in any region. At a point (x,t) where we want to find a solution, we only need to identify which chacteristic lines are on the left and right of that point. Specifically, all $\lambda_i < x/t$ are to the left of (x,t)- and have thus carried information from the *right* state to this point. All $\lambda_i > x/t$ are on the right of point (x,t) and their corresponding information is carried from *left* state. The general solution can in other words be written

$$\mathbf{U}(x,t) = \sum_{i:\lambda_i > x/t} \alpha_i \mathbf{K^{(i)}} + \sum_{i:\lambda_i < x/t} \beta_i \mathbf{K^{(i)}} \tag{11}$$

Note that for a point where x/t is smaller than or lager than *all* λ_i, equation (11) is equivalent to the spectral decomposition $\mathbf{U_L}$ or $\mathbf{U_R}$ respectively. For a 2×2 system, there will only be three solutions at any time: $\mathbf{U} = \mathbf{U_L}$, a region where $\mathbf{U} = \mathbf{U_R}$ and the star region where $\mathbf{U} = \mathbf{U}^*$.

2.3 Introduction and Smooth solution

They are dynamical systems of PDEs, usually nonlinear with a particularly simple structure. In one dimensional space equations assume the form

$$\partial_t u(x,t) + \partial_x f(u(x,t)) = 0 \tag{12}$$

Here, $u : \mathbf{R} \times \mathbf{R} \to \mathbf{R}^m$ is a vector of quantities quantities, or state variables, eg energy, mass and internal energy of a fluid in dynamics problem. More properly, u_i is density for i^{th} conserved quantity with interpratation that $\int_{x^1}^{x^2} u_i(x,t)dx$ is total amount of this conserved quantity in the closed interval $[x^1, x^2]$ at t.

The state variables being conserved implies that $\int_{-\infty}^{\infty} u_i(x,t)dx$ must be constant *w.r.t* t. Hence these functions u_i, representing the spatial distribution of conserved quantities at t, will change as t evolves. The primary postulate underlying (12) is knowing $u(x,t)$ at some given point and time enables us to find the flow rate(flux) of each conserved quantity at (x,t). The flow rate of the i^{th} component is given as $f_i(u(x,t))$. The function $f(u)$ withi^{th} component is known as the **flux function** for the system of conservation laws.
The equation (12) should be augmented with some initial data ie, the boundary conditions and initial conditions on bounded sptial domain. The simplest case is the pure (IVP), or **Cauchy problem**, where (12) holds for $-\infty < x < \infty$ and $t \geq 0$. In such a case we must only specify the initial conditions,

$$u_0(x) = u_0(x), -\infty < x < \infty \tag{13}$$

Let us assume the hyperbolicity of the system (12). Meaning that the $(m \times m)$ Jacobian matrix $f'(u)$ of the flux function has the following property: For each value of u the eigenvalues of $f'(u)$ are real, and the matrix is diagonalizable, i.e, there is a completr set of m linearly independent eigenvectors. The importance of this assumption will be seen later.
In two space dimensions a system of conservation laws takea the form

$$\partial_t u(x,y,t) + \partial_t f(u(x,y,t)) + \partial_t g(u(x,y,t)) = 0 \tag{14}$$

where $u : \mathbf{R}^2 \times \mathbf{R} \to \mathbf{R}^m$ and now we have flow rate functions $f, g : \mathbf{R}^m \to \mathbf{R}^m$. Hyperbolicity now demands that all real linear combinations $\alpha f'(u) + \beta g'(u)$ of the flux Jacobians must be diagonalizable with eigenvalues real.
These flux functions are non-linear functions of u, resulting to non-linear systems of (PDEs). Generally, it is not possible to obtain the analytical solutions to these equations, thus there is a need to devise and study numerical schemes for their approximate solution. This is a fact for any nonlinear PDE, and to some extent the general theory of numerical methods for nonlinear PDEs applies in particular to systems of conservation laws. However there exist a multiple reasons for the study of this type of equations particularly in some reasonable depth.

. There are numerous practical problems particularly in engineering and science involving sets of conserved quantities leading to partial differential equations of type.

. Solving these systems is associated with some special difficulties like shock formation. These must be dealt be carefully dealt with during the development of of numerical scheme. The numerical methods developed may work well if the solutions are smooth and may give poor results in presence of discontinuities.

. Though a few analytical solutions are known, a lot is known about mathematical structure of these PDEs and their solutions. The theory developed here can be used to construct special methods that can deal with exiisting numerical difficulties experienced with a more naive approach.

The equation $\partial_t u + \partial_x f(u) = 0$ is called a conservation law, by integrating over $-\infty < x < \infty$ we get

$$d/dt \int_{-\infty}^{\infty} u(x,t)dx = 0$$

assuming that f(u) vanishes as $|x| \to \infty$. Thus the name derives from the fact that the integral of u is conserved in time. By integrating over $a < x < b$ one gets

$$d/dt \int_a^b u(x,t)dx = f(u(t,a)) - f(u(t,b)) \tag{15}$$

this can be given some interprtation that that the integral of u over a finite interval can change due to in-or outflow at boundaries $x = a$ and $x = b$ If we carry out the differentiation *w.r.t* x we obtain

$$u_t + au_x = 0$$

where $a = f'(u)$

Definition 3.1

The characteristics curves in $x-t$ plane defined by,

$$\frac{dx(t)}{dt} = a(u(t,x(t))) \tag{16}$$

Theorem 2.3.1. *If u(x,t) the solution, is differentiable, then it is not changing8 along the characteristics.*

Proof: The chain rule is used to evaluate the derivative of u along a characteristic curve

$$\frac{du(t,x(t))}{dt} = u_t + \frac{dx(t)}{dt}u_x = u_t + a(u)u_x = 0$$

using definition . The derivative is zero and the solution is constant. The theorem and (1.3) implies that the characteristics are straight lines.

2.4 Non smoothness and Jump Condition

The major difference between the linear and non linear equations is that for the latter, the solution in the class of continous functions may fail to exist after a finite time, no matter how smooth the initial data are. The following two examples show how this failure occurs.

Example 2.4.1. *(Geometric description of smoothness failure)*

$$u_t + (u^2/2)_x = 0, -\infty < x < \infty, 0 > t$$

$$u_0(x) = sinx$$

By differentiation $a(u) = u$ and thus the slope of the characteristics are u. Initially in the point $x = \frac{\pi}{2}$, the slope and the solution are 1 and in the point $x = \frac{3\pi}{2}$ the slope and function are -1. The value 1 is transported to the right and the value -1 to the left, at some time they will meet, thereby causing a failure of smoothness in the solution.

Example 2.4.2. *(Dynamic description of smoothness failure) The problem as in the above example*

$$u_t + (u^2/2)_x = 0, -\infty < x < \infty, t > 0$$

$$u_0(x) = sinx$$

The differential equation can be written

$$u_t + uu_x = 0$$

and u can, in analogy with the linear hyperbolic equation, be interpreted as the speed with which the initial data propagates. For the sine wave below, the maxima travels to the right with speed 1 and the minima to the left with speed -1. This causes a gradual sharpening of the gradients with time, and finally the the waves break into discontinuities.

The above examples show the necessity to extend the solutions into the class of functions with discontinuities. The partial differential equation does not make sense for non differentiable functions. We can however interpret the derivatives in the sense of distributions. There exists an approach resulting into another integra formulation that is easier to deal with. This is a technique that we can generally apply to express a PDE in a format whereby less smoothly require to write a smooth "solution". Basically, we take the partial differential equation and multiply it with a test function which is smooth "test function", integrate severally over a certain domain then apply integration by parts to remove derivatives on u d and onto a "smooth test" function resulting in an an equation with fewer derivatives on u, hence demanding less smoothness.
In the case at hand, we will utilize the test functions $\phi \in C_0^1(\mathbf{R} \times \mathbf{R})$. Here C_0^1 is the function space that are continously differentiable with "compact support". Meaning that that $\phi(x,t)$ is identically zero outside some domain, and so the support of the function lies in a compact set
Multiplication of $\partial_t u + \partial_x f = 0$ by $\phi(x,t)$ and then integrating over space and time, we get

$$\int_0^\infty \int_{-\infty}^\infty [\phi u_t + \phi f(u)_x] dxdt = 0 \tag{17}$$

Now integrate by parts, yielding

$$\int_0^\infty \int_{-\infty}^\infty [\phi_t u + \phi_x f(u)] dxdt = -\int_{-\infty}^\infty \phi(x,0) u(x,0) dx \tag{18}$$

Note that nearly all the boundary terms which normally arise through integration by parts drop out due to the requirement that ϕ have compact support, and hence vanishes at infinity. The remaining boundary term brings in the initial conditions of the PDE, which must still play a role in our weak formulation
The function u(x,t) is called a weak solution of the conservation law if (12) *holds for all functions* $\phi \in C_0^1(\mathbf{R} \times \mathbf{R})$

More specifically this means that the equation is multiplied by a smooth tes function, $\phi \in C_0^\infty(\mathbf{R}^+ \times \mathbf{R})$, and then integrate in time and space. Integration by parts afterwards moves the derivatives to the smooth test functions. Doing this yields

$$\int_0^\infty \int_{-\infty}^\infty [\phi u_t + \phi f(u)_x] dxdt = 0 \tag{19}$$

Now integrate by parts, yielding

The boundary terms at t, $|x| = \infty$ does not contribute, since ϕ is assumed to have compact support

2.5 Weak Solution

A weak solution to (6) is a function $u(t,x)$ satisfying (12) for smooth functions $\phi \in C_0^\infty(\mathbf{R}\times\mathbf{R})$ In specific case of one discontinuity, separating two smooth parts of the solution we can use the conservation property of the solution we can use the conservation property of the original problem to obtain the following theorem.

Theorem 2.5.1. *Rankine- Hugoniot Assume that a discontinuity is moving with speed s and that the value of u to the left of the jump is u_L and to the right u_R. The following data holds*

$$s(u_L - u_R) = g(u_L) - g(u_R)$$

Proof: Use the integrated form (3.4)

$$\frac{d}{dt}\int_a^b u dx = g(u(t,a)) - g(u(t,b)) \tag{20}$$

assume there is one discontinuity moving on the curve at $x(t)$ an that the solution is smooth otherwise. Separate (12) into smooth parts

$$\frac{d}{dt}(\int_a^{x(t)} u dx + \int_{x(t)}^b u dx) = g(u(t,a) - g(u(t,b)))$$

The differentiation can now be carried out, giving

$$\int_a^{x(t)} u_t dx + u(t,x(t)-)x'(t) + \int_{x(t)}^b u_t dx - u(t,x(t)+)x'(t) = g(u(t,a)) - g(u(t,b))$$

Now use $u_t = -f_x$ in the integrals. Performing the integration gives

$$g(u(t,a)) - g(u(t,x(t)-)) + u(t,x(t)-)x'(t)$$

$$+g(u(t,x(t)+)) - f(u(t,b)) - u(t,x(t)+)x'(t) = g(u(t,a)) - g(u(t,b))$$

The desired result is obtained by rearranging this expression, and using the notations

$$u(t,x(t)-) = u_L, u(t,x(t)+) = u_R, x' = s$$

2.6 Case Study

If Cauchy problem is considered for system of n hyperbolic PDE in a conserved form :

$$\partial_t u + \partial_x F(u) = 0, u(x,0) = u_0(x). \tag{21}$$

In this $u = u(x,t)$ in $-\infty < x < \infty, t \geq 0$ and F, u are n-vector valued functions. The solutions of (21) that take values in a region N_0 of u-space in which F is taken as strictly hyperbolic and generally nonlinear Lax's sense. This demands that the matrix of Jacobian $F\prime(u)$ have n real and different eigenvalues $\lambda_{i_{t=1}}^{m}$ for $u's \in N_0$ and also $\nabla\lambda_i R_t$ is not 0 for $i = 1,,,,,n$, where R_i is the corresponding eigenvector to λ_i. We set $R_i(u_i)$ to represent the integral curve which is unique for R_i in N_0 which passes at the point u_i.
Because the solutions for (21) develop discontinuities, the task is to look for the weak solutions which satisfy

$$\int_0^\infty \int_{-\infty}^\infty [\phi_t u + \phi_x F(u)] dx dt = -\int_{-\infty}^\infty \phi(x,0)u(x,0)dx \tag{22}$$

for $\phi \in C_0^1(-\infty,\infty) \times [0,\infty))$ with compact support.
Studies of discontinous solutions are centred on Riemannian problem (15)

$$(3.11a) u_0(x) = \begin{cases} u_L, & x \leq 0 \\ u_R, & x > 0 \end{cases}$$

The Hugoniot locus of u_L is the set of states u_R for which the jump condition $s[u] = [f]$ is is satisfied for some scalar s. Here $[u]$ denote $u_L - u_R$ and $[f] = f(u_L) - f(u_R)$. If u_R lies in the Hugoniot locus of u_L then

$$(3.11b) u(x,t) = u^0(x - st) = \begin{cases} u_L, & x \leq st \\ u_R, & x > st \end{cases}$$

is a weak solution to the Riemannian problem with initial data.
In a neigbourhood of u_L the Hugoniot locus is composed of m curves $S_i(u_L)$ such that $S_i(u_L)$ makes C^2 contact with $R_i(u_L)$ at u_L and λ_i is monotonic on $S_i(u_L)$
the i-shock curve of u_L is defined as

$$S_i^-(u_L) = w \in S_i(u_L) : \lambda_i(w) \leq \lambda_i(u_L) \tag{23}$$

and for $u_R \in S_i^-(u_L)$ we call the solution (3.11b) an i-shock wave.
the i-rarefaction curve associated with u_L is defined as

$$R_i^+(u_L) = w \in S_i(u_L) : \lambda_i(w) \geq \lambda_i(u_L) \tag{24}$$

For $u_R \in R_i^+(u_L)$ we call the solution

$$(3.13a) u(x,t) = u^0(x - st) = \begin{cases} u_L, x \leq \lambda_i(u_L)t \\ u(\xi), x = \xi t, \lambda_i(u_L) < \xi < \lambda_i(u_R) \\ u_R, x \geq \lambda_i(u_R)t \end{cases}$$

an i-rarefuction wave, where u(ξ) denotes the parametrization of $R_i^+(u_L)$ with respect to λ_i. For general systems our assumptions imply that the Riemann problem can be solved uniquely in the class of shock and rarefaction waves in a neigbourhood of any point in N_0 and this is the physically correct solution.

2.7 Uniqueness and Entropy Condition

When we extend the class of admissible solution from the differentiable functions to non differentiable functions, we unfortunately loose uniqueness. The extended class of functions is too large.
We therefore impose an extra condition the so called *entropy condition* which tells us, in case of multiple solutions, which solution is correct one. The name derives from application to gas dynamics, in which case there there is only one solution satisfying the physically correct condition of entropy decrease[4].
As we will see later, entropy conditions are important when we study numerical methods, since some convergent numerical methods do not converge to the solution singled out by the entropy condition. The theory is considerably simplified if the flux function is convex $(f'\prime(u))$

2.7.1 Entropy Condition 1

A discontinuity propagating with speed s given by $s(u_L - u_R) = f(u_L) - f(u_R)$ satisfies the entropy condition if

$$f'(u_L) > s > f'(u_R) \tag{25}$$

2.7.2 Entropy Condition 2

u(x,t) is the entropy solution of all discontinuity has a property that

$$\frac{f(u) - f(u_L)}{u - u_L} \geq s \geq \frac{f(u) - f(u_R)}{u - u_R} \tag{26}$$

for u between u_L and u_R

For f convex, this condition becomes (19)
A separate type of entropy condition is centred on spreading of characteristics in rarefaction plane. If $u(x,t)$ is a monotonic function of x in a region, then the characteristics are spread out if $f'' > 0$.

2.7.3 Entropy Condition (Version 3)

u(x,t) is a solution if there exists constant $G > 0$ *s.t for* $t > 0, a > 0$ *and x*$\in \mathbf{R}$

$$u(x+a,t) - u(x,t)/a < E/t \tag{27}$$

For discontinuities propagating with a constant u_L and u_R, this is satisfied if $u_R - u_L \leq 0$, this is in agreement with (19). The form (21) may be unecessarily convoluted, but it becomes easier to be applied in other contexts. Particularly, this form has merits in the study of numerical schemes. Some numerical schemes are convergent to incorrect *weak solution* sometimes. The application of this criterion(19) is difficult to apply to a discrete solution- a discrete etimation defined only at some grid points is discontinous everywhere. If $U_i < U_{i+1}$ at arbiterary grid point, the problem remains to find out whether determine whetherit is jump inconsistent with the entropy condition, or is just part of smooth estimation of rarefaction wave.

The Entropy Function

Another way to approach entropy condition is to give a definition of entropy function $\xi(u)$ for which additional law of conservation is true for some smooth solutions that are inequality for discontinous solutions. In the study of gas dynamics, there is a quantity known as entropy which is taken to be constant along the path of a particle known to be constant when there is a smooth flow and to rises to a larger value as the gas passes a shock. It cannot rise to lower value, this gives physical entropy criteria that takes the right weak solution.
Let $\xi(u)$ satisfy a law of conservation the form

$$\xi(u)_t + \psi(u)_x = 0 \tag{28}$$

for $\psi(u)$ the entropy flux. The for smooth u,

$$\xi'(u)u_t + \psi'(u)u_x = 0 \tag{29}$$

Remember that (12) can be expressed as $\partial_t u + f'(u)\partial_x u = 0$. Multplyin by $\xi'(u)$ and comparing with (29) gives

$$\psi'(u) = \xi'(u)f'(u) \tag{30}$$

For scalar laws the equation gives many solutions $\xi(u)$, $\psi(u)$. For a system of uquations ξ and ψ are scalar functions, but (24) leads to $\nabla\psi(u) = f'(u)\nabla\xi(u)$, this is system of n equations since the variables ξ and ψ are still scalar functions. If $n > 2$ which may fail to have solutions.

We place an extra condition on entropy function ie, let it be convex, $\xi''(u) > 0$ for reasons below

If the flow are smooth then the entropy is conserved. For discontinous solutions, the manipulations undertaken above are invalid. Because we are in particular interested in the behaviour of the entropy t for the vanishing viscosity weak solution, we consider the related viscous problem and will then set the viscosity to tend to zero. The viscous equation is given by;

$$\partial_t u + \partial_x f(u) = \varepsilon u_{xx} \tag{31}$$

Because we always have a smooth solution of the above equation, we can come up with the corresponding evolution for the entropy copying the mathematical manipulations that was used to get smooth solutions same manipulations we used for smooth solutions for inviscid equation, if multiply (25) by $\xi'(u)$ we obtain

$$\partial_t \xi(u) + \partial_x \psi(u) = \varepsilon \xi'(u) u_{xx} \tag{32}$$

We then can rewrite the RHS to get

$$\xi(u)_t + \psi(u)_x = (\varepsilon \xi'(u) u_x)_x - \varepsilon \xi''(u) u_x^2 \tag{33}$$

If we integrate this equation over $[x_1, x_2] \times [t_1, t_2]$ gives

$$\int_{t_1}^{t_2} \int_{x_1}^{x_2} \xi(u)_t + \psi(u)_x dx dt = \varepsilon \int_{t_1}^{t_2} [\xi'(x_2,t)))u_x(x_2,t) - \xi'(u_1,t))u_x(x_1,t)]dt - \varepsilon \int_{t^1}^{t_2} \int_{x_1}^{x_2} \xi''(u) u_x^2 dx dt$$

As $\varepsilon \to 0$, the first term on the RHS diappears. (This is a fact if u is smooth at x_1 and x_2 and in general can be shown.) The other term, however, involves integrating u_x^2 over the $[x_1, x_2] \times [t_1, t_2]$. If the weak solution is discontinous along a curve in this rectangle, then this term will not vanish in the limit. However, since $\varepsilon > 0, u_x^2 > 0 and \xi''$(by our convexity assumption), we can conclude that the right hand side is nonpositive in the limit and hence the vanishing viscosity weak solution satisfies

$$\int_{t_1}^{t_2} \int_{x_1}^{x_2} \xi(u)_t + \psi(u)_x dx dt \leq 0 \tag{34}$$

for all x_1, x_2, t_1 and t_2

Entropy Condition 4)

u(x,t) is the entropy solution of (12) *if, for convex entropy functions and corresponding entropy fluxes, the inequality*

$$\xi(u)_t + \psi(u)_x \leq 0 \tag{35}$$

This formulation is of good use in analyzing numerical techniques. If we know the discrete form of the entropy enequality to hold for some numerical scheme, it can then be shown that the scheme is convergent to entropy solution.

3 The Payne-Whitham(P-W) and Aw-Rascle(A-R) Models of Traffic Flow

3.1 P-W model

Due to simplicity and low computational complexity,microscopic models are tipically used.The first study of macroscopic traffic flow models was by Lighthill,Withham and Richards who proposedthe LWR model.This is a simple continous traffic model and can be expressed as ;

$$\partial_t \rho + \partial_x(\rho v) = 0 \tag{36}$$

ρ is the density while v is the velocity.

The model can be useful in characterizing traffic during sudden changes in flow of traffic. But it cannot characterise acceleration or inequilibrium flow of traffic accurately.

To do away with the challenges of LWR model, a term known as acceleration term can be incorporated. Approaches to improve the LWR model have taken into account alignment of traffic based on the sorroundings

Payne suggested a second order model of traffic flow based on the car following theory and adjustments of traffic due to response of the driver are due to driver response.This incorporates expectation that describes driver's reaction to traffic situations and conditions. Relaxation term describes adjustments in velocity due to frontal conditions. Whithham came up similar model of traffic flow called the PW model which is based on the postulate that vehicles have same behaviour. In reality, behaviour of vehicles is not similar so the model can bring absurd results.

1. The PW model was improved by Del Castillo by including anticipation and time of reaction for minute changes in velocity and density.
2. Relaxation timeτ was modelled by Philips by assuming that it is a function of density of traffic.
3. It was shown by Daganzo showed that flow of traffic is influenced by frontal stimuli.
4. Papageorgion argued out that the velocities on different lanes are not similar in multilane traffic this difference enables vehicles to move faster than average velocity lanes.

5. A monotonic density function was introduced by Aw and Rascle with an aim of improving the P-W model, so that changes take place below the average velocity, this can lead to larger acceleration when density is high which is unrealistic.

6. The P-W model wa improved by including presumption of the driver which is based on equilibrium speed improved the PW model by incorporating driver presumption which is based on equilibrium speed this was done by Zhang.

3.2 Derivation of P-W model

Vehicles at X(t) move at the velocity ,

$$\dot{x}(t) = v(x(t),t) \tag{37}$$

Its acceleration is

$$\frac{d}{dt}(v(x(t),t)) = v_t + v v_x \tag{38}$$

The model for acceleration is thus given by

$$v_t + v v_x = (v(\rho) - v)/\tau \tag{39}$$

Here $v(\rho)$ is desired velocity and τ is the relaxation time. The full model is

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho v)}{\partial x} = 0 \tag{40}$$

$$v_t + v v_x = -\frac{(p(\rho))_x}{\rho} + \frac{(v(\rho) - v)}{\tau} \tag{41}$$

Where p(ρ)is the traffic pressure (which models preventive driving),without this term vehicles would collide. Equation (41) can be written as;

$$v_t + v v_x + \frac{\partial p}{\partial \rho}\frac{\partial \rho}{\partial x} = \frac{(v(\rho) - v)}{\tau} \tag{42}$$

with

$$c_0^2 = \frac{\partial p}{\partial \rho} \tag{43}$$

The simplified model can be written as[13],[9];

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho v)}{\partial x} = 0 \tag{44}$$

$$v_t + v v_x + \frac{c_0^2}{\rho}\frac{\partial \rho}{\partial x} = \frac{(v(\rho) - v)}{\tau} \tag{45}$$

1. The driver spatial adjustment to frontal situations is characterised by anticipation term $\frac{c_0^2}{\rho}\frac{\partial \rho}{\partial x}$

2. Alignment of traffic occurs during the relaxation time τ
3. Equilibrium speed $v(\rho)$ is reached during alignment of traffic based on distribution of density and is characterised the relaxation term $\frac{(v(\rho)-v)}{\tau}$
4. The constant c_0 is the density parameter.

The P-W model better captures non equilibrium wave phenomena in traffic flow. The P-W model is unstable under certain situations since their are no non analytical solutions to the PW model.Numerical methods are used to solve it.

3.3 Generalities About Roe Decomposition

The Roe approximate Riemann solver, is an approximate Riemann solver based on the Godunov scheme and involves finding an estimate for the intercell numerical flux or Godunov flux $F_{i+\frac{1}{2}}$ at the interface between two computational cells U_i and U_{i+1} on some discretised space-time computational domain.

Quasi-linear hyperbolic system

A non-linear system of hyperbolic partial differential equations representing a set of conservation laws in one spatial dimension can be written in the form

$$\partial_t U + \partial_x F(U) = 0$$

Application of chain rule to second term we obtain a quasilinear hyperbolic system

$$\frac{\partial U}{\partial t} + A(U)\frac{\partial U}{\partial x} = 0$$

where A is matrix of Jacobian of $F(U)$.

Roe matrix

The Roe method includes finding a matrix $\bar{A}(U_i, U_{i+1})$ that is assumed constant between two cells. The Riemann problem can then be solved as a truly linear hyperbolic system at each cell interface. The Roe matrix must obey the following conditions:

1. Diagonalizable with real eigenvalues: ensures that the new linear system is truly hyperbolic.

2. Consistency with the exact Jacobian: when we demand that

$$\vec{A}(U_i, U_{i+1}) = A(U)$$

3. Conserving

$$F_{i+1} - F_i = \vec{A}(U_{i+1} - U_i)$$

Phil Roe introduced a method of parameter vectors to find such a matrix for some systems of conservation laws.

Intercell flux

Once the Roe matrix corresponding to the interface between two cells is found, the intercell flux is given by solving the quasi-linear system as truly linear system.

3.4 Roe Decomposition.

The P-W model is discretized using the Roe Decomposition technique to evaluate their performance. This techniqe can be used to approximate non-linear system of equations,

$$G_t + f(G)_x = S(G) \tag{46}$$

where G denotes the vector of data variables,f(G) denotes the vector of functions of data variables and S(G) is the vector of source terms.Subscripts t and x denotes the partial derivative with respect to time and distance respectively[15]. Equation (46) can be expressed as

$$\frac{\partial G}{\partial t} + \frac{\partial f}{\partial G}\frac{\partial G}{\partial x} = S(G) \tag{47}$$

$\frac{\partial f}{\partial G}$ is the gradient of functions of data variables withn respect to these variables.

Let A(G) be the Jacobian matrix of the system,then (47) can be written as

$$\frac{\partial G}{\partial t} + A(G)\frac{\partial G}{\partial x} = S(G) \tag{48}$$

Setting the source term in (48) to zero gives the quasilinear form;

$$\frac{\partial G}{\partial t} + A(G)\frac{\partial G}{\partial x} = 0 \tag{49}$$

The data variables in the P-W model are ρ and $\rho\upsilon$. Roe technique is used to linearize the Jacobian matrix A(G) by decomposing it to eigenvalues and eigen vectors. It is based

on the concept that the data variables ,eigenvalues and eigenvectors remain conserved for small changes in time and distance, This technique is widely employed because it is able to capture the effects of abrupt changes in the data variables. The eigenvalues are usefull to get approximate solutions and also to analyse the hyperbolicity of traffic system Conserved form of P-W model is found by multyplying (44) by v

$$v\partial_t \rho + v\partial_x(\rho v) = 0 \tag{50}$$

but

$$\partial_t(\rho v) = v\partial_t \rho + \rho \partial_t v \tag{51}$$

so,

$$v\rho_t = (\rho v)_t - \rho v_t \tag{52}$$

substituting,equation (52) into equation (44) we obtain;

$$(\rho v)_t - \rho v_t + v(\rho v)_x = 0$$

or

$$\rho v_t = (\rho v)_t + v(\rho v)_x \tag{53}$$

multiplying (45) by ρ gives ;

$$\rho v_t + \rho v v_x + c_0^2 \rho_x = \rho \frac{v(\rho) - v}{\tau} \tag{54}$$

Now consider

$$(\rho v v)_x = v(\rho v)_x + (\rho v)v_x$$

or

$$\rho v v_x = (\rho v v)_x - v(\rho v)_x \tag{55}$$

substituting (53) and (55) into (45) we obtain,

$$(\rho v)_t + (\rho v)_x + c_0^2 \rho_x = \rho \frac{v(\rho) - v}{\tau} \tag{56}$$

multiplying and dividing $(\rho v v)_x$ by ρ we get

$$\partial_t(\rho v v) = \left(\frac{(\rho v)^2}{\rho}\right)_x \tag{57}$$

so that (56) can be expressed as

$$\partial_t(\rho v) + \partial_x\left(\frac{(\rho v)^2}{\rho} + c_0^2 \rho\right) = \rho \frac{v(\rho) - v}{\tau} \tag{58}$$

which is the conserved form of PW model.

This can be written as;

$$\partial_t G + \partial_G f(G) \partial_x G = S(G)$$

or

$$\partial_t G + A(G) \partial_x G = 0 \tag{59}$$

where

$$f(G) = \begin{bmatrix} \rho \\ \frac{(\rho \upsilon)^2}{\rho} + c_0^2 \rho \end{bmatrix}$$

$$= \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$$

$$G = \begin{bmatrix} \rho \\ \rho \upsilon \end{bmatrix}$$

and

$$S(G) = \begin{bmatrix} 0 \\ \rho \frac{\upsilon(\rho) - \upsilon}{\tau} \end{bmatrix}$$

The matrix of Jacobian ,$A(G) = \frac{\partial f}{\partial G}$

is given by

$$J = \frac{\partial (f_1, f_2)}{\partial (\rho, \rho \upsilon)} = \begin{bmatrix} f_{1\rho} & f_{1\rho\upsilon} \\ f_{2\rho} & f_{2\rho\upsilon} \end{bmatrix}$$

$$A(G) = \begin{bmatrix} 0 & 1 \\ -\upsilon^2 + c_0^2 & 2\upsilon \end{bmatrix} \tag{60}$$

the eigenvalues of the matrix are found by solving

$$|A(G)-\lambda I| = \det \begin{bmatrix} -\lambda & 1 \\ -\upsilon^2 + c_0^2 & 2\upsilon - \lambda \end{bmatrix} = 0 \tag{61}$$

$$= \lambda^2 - 2\lambda\upsilon - c_0^2 + \upsilon^2 = 0 \tag{62}$$

where

$$\lambda_1 = \upsilon + c_0 \tag{63}$$

and

$$\lambda_1 = \upsilon - c_0 \tag{64}$$

are the solutions of (62)

Indeed we a hyperbolic system of conservation laws since the eigenvalues are real and distinct
The eigenvectors are obtained by solving

$$[A(G)-\lambda I]\vec{x} = 0$$

for

$$\lambda_1 = \upsilon + c_0$$

we obtain;

$$x_1 = \begin{bmatrix} 1 \\ v + c_0 \end{bmatrix}$$

and for

$$\lambda_1 = \upsilon - c_0$$

we obtain

$$x_2 = \begin{bmatrix} 1 \\ v - c_1 \end{bmatrix}$$

We have seen that the homogeneous version of the PW model can be expressed as;

$$v_t + f(v)_x = 0 \tag{65}$$

which is system of hyperbolic conservation laws, again the system is genuinely non-linear since

$$\nabla \lambda_i . x_i$$

is non zero.

3.5 Riemann Problem For PW Model

Rieman problem for (65) around x=0 has the following initial conditions

$$u(x.0) = \begin{cases} v_l, & \text{if} x < 0 \\ v_r, & \text{if} x > 0 \end{cases}$$

The solutions to this problem are shock waves and or/rarefaction waves. There are two types of shock waves which are 1-shocks and 2-shocks. Lax(1972) entropy conditions states that 1-shocks satisfy

$$\lambda_1(v_r) < s < \lambda_1(v_l)$$

where $s < \lambda_2(v_r)$
and 2-shocks satisfy

$$\lambda_2(v_r) < s < \lambda_2(v_l)$$

where $s > \lambda_1(v_l)$

The shock wave speed is determined by the Rankine-Hugoniot jump conditions dicussed in chapter 2.
Besides shock wave solutions, the Riemann problem for (65) admits two families of rarefaction waves which are continous in the form $v(x,t) = u(x/t)$ and satisfy $\lambda_i(v(x/t))$

3.6 Aw-Rascle Model of Traffic Flow

Aw and Rascle proposed a new two equation traffic model that seeks to improve the P-W model and adresses the concerns raised by Daganzo

$$\rho_t + (\rho v)_x = 0,$$

$$(\upsilon + p(\rho))_t + u(\upsilon + p(\rho))_x = 0$$

With a suitable choice of the pressure function $p(\rho)$, the model satisfies the following properties

1. The matrix for the system must be diagonalizable.
2. When solving the Riemann problem with arbitrary bounded nonnegative Riemann data (ρ, υ) in a suitable region **R** of the plane, the density and the velocity must remain nonnehative and bounded from above.
3. In solving the same Riemann Problem with arbitrary data U, all waves connecting any state $U = (\rho, \upsilon)$ to its left (i.e. behind it) must have a propagation speed (eigenvalue or shock speed) at most equal to the velocity υ.
4. The solution to the Riemann problem must agree with the qualitative properties that each driver practically observes every day. In particular, braking produces shock waves whose propagation speed can either be negative or nonnegative, whereas accelerating produces rarefaction waves which in any case satisfy principle (3)
5. Near by the vaccum, the solution to the Riemann Problem must be very sensitive to the data. In other words, there must be no continous dependence with respect to the initial data at $\rho = 0$

The property (3), adresses the first point of Daganzo's criticism: a car travellig at a velocity υ recieves no information from the rear.

Derivation of the model

The car-following principle is concerned with the intevehicular dynamics in a single laneof traffic. The mathematical form can be represented as[6]

$$\begin{cases} dx_j/dt = u_j(t) \\ Js_j(t), u_{j+1}(t+J)u_{j+1}(\prime)(t+J) = u_j(t) - u_{j+1}(t) \\ J(u_j(t), s_j(t))u_{j+1}(t+J) + L(u_j(t)), s_j(t) \leq s_j(t) \end{cases}$$

where $J^{-1}(s_i, u_i)$ is the sensitivity, and

$$s_j = x_{j+1} - x_j$$

isthe headway. The third equation of the above system stands for the area. In steady state,

$$u_{j+1}(t+J) = u_{j+1}(t) = U, s_j(t) = \frac{1}{\rho}. \tag{66}$$

then

$$\rho(x_j(t),t) = \frac{1}{x_j(t) - x_{j+1}(t)} \tag{67}$$

finding the derivative of the above equation on both sides, we see that;

$$\rho'(x_j,t) = -\rho'(x_j,t)\partial_x u(x_j(t),t) \tag{68}$$

In addition,

$$\rho'(x_j(t),t) = dx_j/dt\partial_x\rho(x_i,t) + \partial_t\rho(x_{i+1},t) \tag{69}$$

Substituting (68) into (69) we get the equation of continuity

$$\partial_t\rho + \partial_x(\rho\upsilon) = 0 \tag{70}$$

We then turn to derive of equation of conservation of momentum:

$$\upsilon'(x_j,t) = dx_j/dt\partial_x\upsilon(x_j,t) + \partial_t\upsilon(x_{j+1},t) \tag{71}$$

We also

$$dx_j/dt - dx_{j+1}/dt = s_j(t)\partial_t\upsilon(x_{j+1},t) \tag{72}$$

moreover, with (72) and equation of system, we find

$$\partial_t\upsilon(x_j,t) + dx_j/dt\partial_t\upsilon(x_j,t) = \frac{s_i(t)}{J}\partial\_t\upsilon(x_j(t),t) \tag{73}$$

Then recall the equation of conservation of momentum :

$$\upsilon_t + \upsilon\upsilon_x - \frac{1}{\rho J}\upsilon_x = 0 \tag{74}$$

Combining (70) and (74) we the system below for generalising A-R model:

$$\partial_t\rho + \partial_x(\rho\upsilon) = 0$$
$$\upsilon_t + \upsilon\upsilon_x - \frac{1}{\rho J}\upsilon_x = 0$$

The function $J(\rho,\upsilon)$, represents reaction time, in practice should nonincreasing and positive with two variables. Then we have

$$J(\rho,\upsilon) \geq 0, T_u \leq 0, J_\rho \leq 0 \tag{75}$$

To make the equation simpler, we set

$$J(\rho, \upsilon) = h(\rho) f(\upsilon) \tag{76}$$

Setting $p'(\rho) = h^{-1}(\rho)\rho^{-2}, F'(\upsilon)$, we get the conserved form of the model as

$$\begin{cases} \partial_t \rho + \partial_x \rho \upsilon = 0 \\ \partial_t [\rho(F(\upsilon) + p(\rho))] + \partial_x [\rho \upsilon (F(\upsilon) + p(\rho))] = 0 \end{cases}$$

where $p(\rho)$ is pressure. such that

$$p_0 = 0, p(\rho^*) = +\infty \tag{77}$$

From the last equation of the system we get the equation $\upsilon - \rho$ envelope curve,

$$G(\rho, \upsilon) = F(\upsilon) + p(\rho) + K = 0 \tag{78}$$

K constant.
From the theory of traffic flow, the $\upsilon - \rho$ envelope curve passes through $(\rho^*, 0)$ and $(0, \upsilon)$, and from $p(\rho^* = +\infty)$ we obtain

$$C = -F(\upsilon^*), F(0) = -\infty. \tag{79}$$

A-R model doesn't satisfy (79), which is why we have a non-physical invariant region. We then make a conclusion that the model is given by

$$\begin{cases} \partial_t \rho + \partial_x (\rho \upsilon) = 0, \\ \partial_t [\rho(F(\upsilon) + p(\rho))] + \partial_x [\rho \upsilon (F(\upsilon) + p(\rho))] = 0, \end{cases}$$

where
$F_0 = -\infty, F''(\upsilon) \leq 0, p_0 = 0, p(\rho^*) = +\infty, and F(\upsilon) + p(\rho) - F(\upsilon^*) \leq 0$

The model above in conservative form, is given as;

$$\rho_t + (\rho \upsilon)_x = 0,$$

$$(\upsilon + p(\rho))_t + u(\upsilon + p(\rho))_x = 0 \tag{80}$$

where the conserved variables ρ and $y = \rho(\upsilon + p(\rho))$. Note that, there is no obvious physical interpretation of y ("momentum").
Now we present some basic properties of the model. Setting $U = (\rho, \upsilon), Y = (\rho, \rho(\upsilon +$

$p(\rho)))$ and $F = (\rho\upsilon, \rho\upsilon(\upsilon + p(\rho)))$

Now let

$$M = \frac{\partial Y}{\partial U} = \frac{\partial(\rho, \rho(\upsilon + p(\rho)))}{\partial(\rho, \upsilon)} = \begin{bmatrix} 1 & 0 \\ (\upsilon + p(\rho) + \rho p'(\rho)) & \rho \end{bmatrix} \tag{81}$$

and

$$N = \frac{\partial F}{\partial U} = \partial_{(\rho,\upsilon)}(\rho\upsilon, \rho\upsilon(\upsilon + p(\rho))) = \begin{bmatrix} \upsilon & \rho \\ \upsilon(\upsilon + p(\rho) + \rho p'(\rho)) & \rho(2\upsilon + p(\rho)) \end{bmatrix} \tag{82}$$

Since the determinant of M exists $\rho > 0$, there exists a M^{-1} and is given by

$$M^{-1} = \begin{bmatrix} 1 & 0 \\ -\frac{1}{\rho}(\upsilon + p(\rho) + \rho dp/d\rho) & \frac{1}{\rho} \end{bmatrix} \tag{83}$$

hence the model (system) can be expressed in vector form as

$$M\partial_t U + N\partial_x U = 0 \tag{84}$$

$$\implies \partial_t U + M^{-1}N\partial_x U = 0 \tag{85}$$

$$\partial_t U + A(U)\partial_x U = 0 \tag{86}$$

where A(U) is the matrix of Jacobian of the system;

$$A(U) = M^{-1}N = \begin{bmatrix} \upsilon & \rho \\ 0 & \upsilon - \rho p'(\rho) \end{bmatrix} \tag{87}$$

If λ is the eigenvalue the matrix $A(U)$ then it satisfies;

$$|A - \lambda I| = 0$$

which implies that the eigenvalues of the system satisfy

$$(\upsilon - \lambda)(\upsilon - \rho p'(\rho) - \lambda) = 0$$

and therefore we get;

$$\lambda_1 = \upsilon - \rho p'(\rho) and \lambda_2 = \upsilon \tag{88}$$

The right eigenvectors of λ_1 and λ_2 are;

$$r_1 = \begin{bmatrix} 1 \\ -dp/d\rho \end{bmatrix}$$

and

$$r_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

This shows that we have a strictly hyperbolic system except when $\rho = 0$ whereby the two eigenvalues coalesce

3.7 The Riemann Problem of the Aw-Rascle Model

Knowing that,

$$y = \rho(v + p(\rho)). \tag{89}$$

System (80) can be expressed as follows;

$$\begin{aligned} \rho_t + (\rho\upsilon) &= 0 \\ y_t + (y\upsilon)_x &= 0 \end{aligned} \tag{90}$$

Setting $U = (\rho, y)$ the system (90) can be written as;

$$\partial_t U + \partial_x f(U) = 0 \tag{91}$$

here $f(U) = (\rho\upsilon, y\upsilon)$ is flux function.
We now want to solve the Riemann Problem of the Aw-Rascle model associated with the system (90) and with the following piecewise initial condition having a single discontinuity around the space coordinate $x = 0$.

$$U_0(x) = \begin{cases} U_L, & \text{if} x < 0 \\ U_R, & \text{if} x > 0 \end{cases}$$

The initial state components are $U_L = (\rho_L, y_L)$, and $U_R = (\rho_R, y_R)$. The Riemann problem is solved by a combination of two families of waves:

1. 1-waves corresponding to the first eigenvalue λ_1
2. 2-waves corresponding to λ_2

3.7.1 Wave Solutions of The Riemann Problem

Aw and Rascle assumed for analysis of the model the following;

$$p(\rho) \equiv \rho^{\gamma}, near \rho = 0, \gamma > 0 and \forall \rho, \rho d^2/d\rho^2 + 2dp/d\rho > 0 \quad \textbf{(92)}$$

An eigenvalue λ_k is genuinly nonlinear if the function $\nabla\lambda_k(U).r_k(U)$ $k = 1,2$ never vanishes and linearly degenerate if it vanishes for all U, where ∇ is the gradient operator. The waves associated with the linearly degenerate eigenvalues is contact discontinuities and that of genuinly nonlinear is either rarefaction or shock wave depending on the data. Under assumptions (92), λ_1 is genuinely nonlinear and hence admits either a shock wave or rarefaction. On the other hand the λ_2 is linearly degenerate and therefore admit a contact discontinuity.
With the assumption (92), another feature of the model is;

$$\lambda_1 \leq \lambda_2 = v. \quad \textbf{(93)}$$

which means all waves propagate at a velocity not greater than the velocity v which satisfy the property (3).
To solve the Riemann Problem, we first need to calculate the Riemann invariants associated with each eigenvalue λ_k, $k = 1,2$.

Definition

A scalar function z of U is called a 1-Riemann invariant in the sense of Lax if

$$\nabla z.r_1 \equiv 0.$$

and we call $w = w(U)$ a 2-Riemannian invariant in the sense of Lax if

$$\nabla w.r_2 \equiv 0.$$

Now using the Expressions of r_1 and r_2 here we obtain

$$z(U) = v + p(\rho) \tag{94}$$

$$w(U) = v.$$

So, depending on given conditions, waves of the 1st family is either rarefaction or shock and that of the second family is a contact discontinuity. Depending on the data, the Aw-Rascle model may have the following kinds of solutions;

1. A 1-shock wave followed by a 2-contact discontinuity
2. A 1-rarefaction wave followed by a 2-contact discontinuity
3. A 1-rarefaction wave followed by a Vacuum and then a 2-contact discontinuity
4. Isolsted 1-rarefaction wave
5. Isolated 2-contact discontinuity

To discuss all the various wave forms of the model we solve a Riemann problem with initial data.

1-Shock wave

A 1-shock wave of the Riemann problem with initial data is a jump discontinuity with form;

$$u(x,t) = \begin{cases} U_L, & \text{if} x < s_1 t \\ U_R, & \text{if} x < s_1 t \end{cases}$$

The shock waves satisfy the following Lax entropy condition;

$$\lambda_1(U_R) < s_1 < \lambda_1(U_L)$$

where $s_1 < \lambda_2(U_R)$
Where the shock wave velocity is determined by Rankine-Hugoniot jump condition

$$\begin{aligned} s_1[\rho] &= [\rho v] \\ s_1[y] &= [yv] \end{aligned} \tag{95}$$

this is a system of two equations with ρ, y and s_1 as the unknowns. to solve this system we have to express s_1 and y in terms of ρ. Solving s_1 from these two equations gives

$$\frac{[\rho v]}{\rho} = s_1 = \frac{yv}{y}$$

which is equivalent to

$$\frac{\rho_R v_R - \rho_L v_L}{\rho_R - \rho_L} = \frac{y_R v_R - y_L v_L}{y_L - y_L}$$

This leads to

$$y_L \rho_R v_R + y_R \rho_L v_L = y_R \rho_L v_R + y_L \rho_R v_L$$

Rearranging and dividing each term by $\rho_L \rho_R$ we get

$$\frac{y_L}{\rho_L} - \frac{y_R}{\rho_R}(v_R - v_L) = 0$$

If $v_R - v_L \neq 0$, then

$$\frac{y_R}{\rho_R} = \frac{y_L}{\rho_L}$$

from(89) this is equivalent to

$$v_R + \rho_R^\gamma = v_L \rho_L^\gamma$$

or by (94) we get

$$z(U_R) = zU_L$$

This determines the curve of a 1-shock wave connection the state U_R on the left side of discontinuity to the state U_R on the right side.
Generally, a given state U_L to the left can be connected to any other state U to the right by a 1-shock wave if and only if

$$w(U) = w(U_L) \tag{96}$$

with $w(U)$ already defined in (94), this is equevalent to

$$v + \rho^\gamma = v_L + \rho_L^\gamma \tag{97}$$

If $v_R - v_L = 0$ i.e

$$v_R = v_L. \tag{98}$$

then this case corresponds to a contact discontinuity of the second family.

Rarefaction Waves

The RP of the A-R model has the property that the solution U is constant along all rays of the form $x = \beta t$. implying that $\beta = \frac{x}{t}$. As a result the solution is is a function of $\frac{x}{t}$ alone and is called a similarity solution of the partial differential equation. Thus the solution $u()x,t$ is only a function of β and we can comfortably write $u(x,t) = w(\beta)$.
We ned $\beta = x/t$ to increase monotonically as $z(\beta)$ moves from U_L to U_L, along the integral curve in order to have a single-valued rarefaction wave[10]
A given state U_L to the left can be linked to a state U on the right by a 1-rarefaction wave iff

$$\frac{y}{\rho} = \frac{y_L}{\rho_L}$$
$$\lambda_1(U) = \beta$$

which is equivalent to

$$v + \rho^\gamma = v_L + \rho_L^\gamma \tag{99}$$
$$v - \gamma\rho^\gamma = \beta$$

3.8 Analytical Solutions

Here we discuss general cases of the Riemann data we should consider for the A-R model. These cases can be found in[1]. We use the notations $U_L = (\rho_L, y_L)$ to represent the left state, $U_m = (\rho_m, y_m)$ for the intermediate state and $U_R = (\rho_R, y_R)$ for the right state. The aim of this section is to link the given U_L to a determined U_m by a wave of the 1st family and then link U_m to U_R by a contact discontinuity of the 2nd family

Case 1: $\rho_L > 0, \rho_R > 0$ and $0 \leq v_R \leq v_L$:

Here we have a unique solution $U(x,t)$ that consists of a 1-shock wave connecting the state U_L to the intermediate state U_m followed by a 2-contact dicontinuity connecting U_m to the state U_R on the right. Here the 1-shock must satisfy the shock curve defined in (97) and the contact discontinuity satisfies (98). The aim here is to solve for U_m satisfying the following two simultaneous equations

$$v_m + \rho_m^\gamma = v_L + \rho_L^\gamma$$
$$v_m = v_R$$

giving

$$\rho_m = (v_L - v_R + \rho_L^\gamma)^{1/\gamma} \tag{100}$$
$$v_m = v_R$$

the solution $U()x,t$ takes the form

$$U(x,t) = \begin{cases} U_L, & \beta \leq s_1 \\ U_m, & s_1 < \beta \leq v_R \\ U_R, & \beta > v_R \end{cases}$$

where $\beta = x/t$ and s_1 the shock speed

Case 2 $\rho_L > 0, \rho_R > 0$ and $v_L \leq v_R \leq v_L + \rho_L^\gamma$:
In this case the solution $U(x,t)$ consists of a 1-rarefaction wave connecting U_L to U_m followed by a 2-contact discontinuity connecting U_m to U_R Where the 1-rarefaction wave saytisfies the rarefaction curve (99). The solution is therefore given by

$$U(x,t) = \begin{cases} U_L, & \beta \leq \lambda_1(U_L), \\ U^*, & \lambda_1(U_L) < \beta \leq \lambda_1(U_m), \\ U_m, & \lambda_1(U_m) < \beta \leq v_R, \\ U_R, & \beta > v_R \end{cases}$$

Case 3 $\rho_L > 0, \rho_R > 0$ and $v_L + \rho_L^\gamma < v_R$:
Here the intermediate state is the vacuum wave with vacuum states say U_1 and U_2. Hence the solution $U(x,t)$ consists of a shock wave connectin U_L and U_m followed by a vacuum wave which connectU_1 and U_2, followed by a 2-contact discontinuity connecting U_2 to U_R. Once again the 1-shock must satisfy (97) and the contact discontinuity must satisfy (98)

Case 4 $\rho_L > 0, \rho_R = 0$: Here we connect U_L to the origin $U_R = (0,0)$, on the right. This case is similar to the third case but now there is no need to add a contact discontinuity since the state on the right is origin, which is on the rarefaction curve issued from the left state. Therefore, the solution $U(x,t)$ is only a 1-rarefaction wave given by,

$$U(x,t) = \begin{cases} U_L, & x \leq (v_R - \gamma\rho_L^\gamma)t, \\ U^*, & (v_R - \gamma\rho_L^\gamma)t < x < (v_R + \gamma\rho_L^\gamma)t, \\ U_R, & x < (v_R + \gamma\rho_L^\gamma)t \end{cases}$$

Case 5 $\rho_L, \rho_R > 0$
Here the Riemann datum U on the right will be connected to the vacuum state on

the left by a 2-contact discontinuity. The solution $U(x,t)$ is given by a single contact discontinuity moving at the speed of the leading cars and has the form

$$U(x,t) = \begin{cases} U_L, & x \leq v_R t, \\ U_R, & x > v_R t \end{cases}$$

4 Numerical Methods

4.1 Introduction

In practice many interesting solutions to nonlinear hyperbolic laws are not smooth but heve certain discontinuities. These discontinuities can easily develop spontaneously even from smooth initial data and they must be dealt with both mathematically and computationally[11]. At a discontinuity in U, the partial differential equation (91) does not hold in classical sense.
Discontinuities lead to computational difficulties and our main aim is to to obtain accurate approximations of such solutions.
Classical Finite Difference Mthods in which the derivatives are approximated by finite difference can be expected to break down near discontinuities in the solution where the differential equation does not hold. Rather than pointwise approximations at grid points, we break down the domain into grid cells and approximate the total integral of U over each grid cell or actually the cell average of U, which is this integral divided by the volume of the cell. These volumes are modified in each time step by flux through edges of grid cells and the problem is to determine good numerical flux functions that approximate the correct fluxes reasonably well, based on the approximate cell averages. This is Finite Volume Method. Finite Volume Methods have proved to be very effective for computing discontinous solutions as compared to the Finite Differenc Methods.
In this thesis we impliment the finite volume method, Specifically the Godunov's method in the solution of the the Partial differential equations(traffic models).
Other classes of methods have also been applied to hyperbolic equations such as the Finte Element Methods and Spectral Methods both of which are not discussed in this thesis.

4.2 Finite Volume Method (FVM)

In one dimension space, a finite volume method is based on subdividing the spacial domain into finite volumes or grid cells. A fundamental tool in the development of finite volume methods is the Riemann problem-the hyperbolic equation together with the initial data. the data is piecewise constant with a single jump discontinuity at some point say $x = 0$.

The Finite Volume Method (FVM) is a discretization technique for partial differential equations, especially that arise from physical conservation laws. The FVM uses a volume integral formulation of the problem with a finite partioning set of volumes to discretise the equations. Following the FVM's approach, the domain is divided into a number of control volumes or cells where the variable of interest is located at the centroid of the control volume, then the equations are integrated over each control volume. In this way, the discretization of the equations expresses the conservatiopn principle for the variables inside the control volume. the most compelling feature of the FVM is that the resulting solution satisfies the conservation of quantities such as mass, density, momentum, energy etc.
To clearly see this, we begin as follows, we know that,

$$\int_{x_1}^{x_2}\int_{t_1}^{t_2}[U_t + F(U)_x]dxdt = 0$$

or

$$\int_{x_1}^{x_2}\int_{t_1}^{t_2}U_t dxdt + \int_{x_1}^{x_2}\int_{t_1}^{t_2}F(U)_x dxdt = 0$$

using the fundamental law of calculus we have;

$$\int_{x_1}^{x_2}U(x,t_2)dx - \int_{x_1}^{x_2}U(x,t_1)dx + \int_{t_1}^{t_2}F(U(x_2,t))dt - \int_{t_1}^{t_2}F(U(x_1,t))dt = 0 \qquad \textbf{(101)}$$

this is the integral form of our system of conservation laws
Starting from the integral form of our system of conservation laws, we discretise space and time in the following manner;

$x \to j\triangle x$
$t \to n\triangle t$.
and we denote a generic control volume as $[x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}] \times [t_n, t_{n+1}]$
By replacing our generic control volume in (101), we have,

$$\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}}U(x,t_{n+1})dx = \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}}U(x,t_n)dx - \int_{t_n}^{t_{n+1}}F(U(x_{j+\frac{1}{2}},t))dt - \int_{t_n}^{t_{n+1}}F(U(x_{j-\frac{1}{2}},t))dt \qquad \textbf{(102)}$$

From the definition of integral average

$$\bar{w} = \frac{1}{\triangle a}\int_a^{a+\triangle a}w(a)da$$

we can write the first term on the right hand side of equation (102) as

$$\int_{x_{j-1/2}}^{x_{j+1/2}}U(x,t_n)dx = \triangle x\bar{U}$$

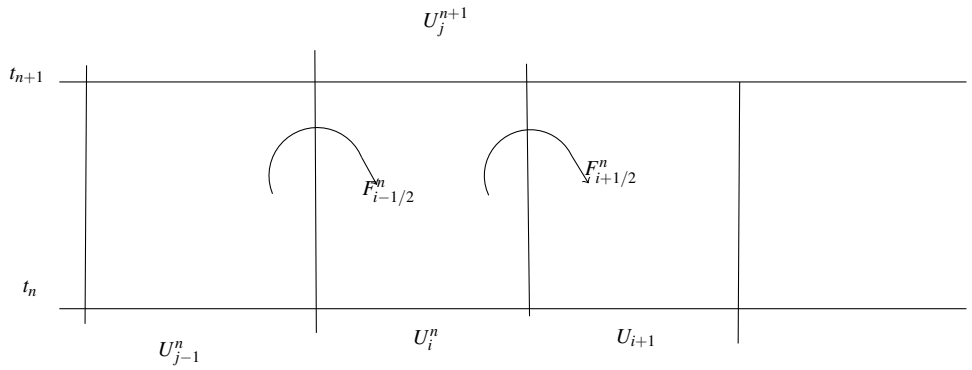

Figure 1. Fluxes Across The Grid Cells

The quantity $\bar{U}$ contains a cell average of each conserved quantity and we assign it to each grid cell, obtaining our dicretised quantities,

$$\bar{U} = U_j^n$$

In the same way the cell averaged vector of conserved quantities at time t_{n+1} is given by the term on the left hand side of (102) as follows.

$$\int_{x_{j-1/2}}^{x_{j+1/2}} U(x,t_{n+1})dx = \triangle x U_j^{n+1}$$

Furthermore, the last two terms on the right hand side of (102) are related to the time averaged fluxes at the boundaries $x_{j\pm1/2}$ of the j^{th} cell. We introduce the following notation for the numerical flux $F_{j\pm1/2}$

$$F_{j\pm1/2} = \frac{1}{\triangle t}\int_{t_n}^{t_{n+1}} F(U(x_{j\pm1/2},t))dt$$

The numerical flux is a (method dependent) numerical approximation of the physical flux. Putting everything together our system of conservation laws reads:

$$\triangle x U_j^{n+1} = \triangle x U_j^n - \triangle t[F^n_{j+1/2} - F^n_{j-1/2}]$$

or equivalently

$$U_j^{n+1} = U_j^n - \frac{\triangle t}{\triangle x}[F^n_{j+1/2} - F^n_{j-1/2}] \tag{103}$$

Now, because the flux entering a given cell is identical to that leaving the adjacent cell, the equation (103) represents a conservative scheme.
For a hyperbolic problem, infoormation propagates with finite speed so it is reasonable to

first suppose that we can find $F^n_{j-1/2}$ based on the values U^n_{j-1} and U^n_j i.e the cell averages on either side of this interface. This might lead us to use a formula of the form;

$$F^n_{j-1/2} = f(U^n_{j-1}, U^n_j) \tag{104}$$

where f is some mumerical flux function thus?? can be written as.

$$U^{n+1}_j = U^n_j - \frac{\triangle t}{\triangle x}[f(U^n_j, U^n_{j+1}) - f(U^n_{j-1}, U^n_j)] \tag{105}$$

The specific method obtained will depend on how we choose f, but in general any method of this type is an explicit method with three point stencil that is, the value of U^{n+1} will depend on the three values U^n_{j-1},U^n_j and U^n_{j+1} at the previous level.
The scheme (105) can be viewed as a direct finite difference approximation to the conservation law (91) because rearranging it gives

$$\frac{U^{n+1} - U^n_j}{\triangle t} + \frac{F^n_{j+1/2} - F^n_{j-1/2}}{\triangle x} = 0 \tag{106}$$

However, nonlinearity introduces a number of difficulties. Stability and convergence are difficult speciffically in that we are mainly interested in discontinous solutions involving shock waves. Again, we must ensure that we are converging to correct weak solution, since weak solutions may not be unique. This necessitates that the numericall method be consistent with a suitable entropy condition[11].
For nonlinear conservation law (91) it is important that the method be in a conserved form (103) so that weak solutions to the conservation law are properly approximated.

Convergence

For a finite volume method we want to compare U^n_j with

$$u^n_j = \frac{1}{\triangle x}\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} u(x, t_n)dx$$

In order to discuss convergence, must pick some finite time T over which we wish to compute. now, as we refine the grid, the number of time steps to reach T will grow like $T/\triangle t$ and go to infinity and in this case we deal with unbounded number of time steps. We introduce N to indicate the time level corresponding to time $T = N\triangle t$, the global errror at this time will be noted by

$$e^N = U^N - u^N$$

To make the notation simple we will generally suppose that $\triangle t$ and $\triangle x$ are related in a

fixed manner as we refine the grid.
For hyperbolic problems it is reasonable to suppose that the ratio $\triangle t/\triangle x$ is fixed. Then we can speak of letting $\triangle t \to 0$ to refine the grid and speak of convergence of order $O(\triangle t^r)$ or as $O(\triangle x^r.)$
To qualify the error, we must choose some norm in which to measure the error at a fixed time. The norm most commonly used is,

$$||e||_p = \left(\triangle x \sum_{j=-\infty}^{\infty} |e_j|^p \right)^{1/p}$$

The factor $\triangle x$ is very important to give correct scaling and order of accuracy as the grid is refined. Particularly the 1-norm (with $p = 1$) is commonly used for conservation laws[11].
We can easily say the method is converges at time T in the norm $\|.\|$ if

$$\lim_{\triangle t \to 0} \|e^N\| = 0$$

The method is said to be accurate of order r if

$$\|e^N\| = O(\triangle t^r)$$

as $\triangle t \to 0$

One-step and Local Truncation Errors

There are two techniques of studying this,
Firstly, we study the error introduced in a single time step, the method is consistent with the differential equation if a small error is introduced in any one step.
Secondly, show that the method is stable so that these local errors do not grow uncontrolably hence a bound on the global error can be obtained in terms of these local errors.
If we can get a bound on the local error in an appropriate sense, then the stability can be used to convert this into a bound on the global error that can be used to prove convergence.
Moreover, we can generally determine the rate of convergence and perhaps even obtain reasonable error bounds.
The exact form of stability needed depends on the type of equation and method.
A general explicit numerical method can be written as

$$U^{n+1} = N(U^n),$$

where $N(.)$ is a numerical operator mapping the approximate solution at one time step to approximate solution at the next,

$$one - steperror = N(u^n) - u^{n+1}$$

Local truncation error is given by

$$\tau^n = \frac{1}{\triangle t}[N(u^n) - u^{n+1}]$$

We say that the metod is consistent with the differential equation if the local truncation error vanishes as $\triangle t \to 0$ for all smooth functions $u(x,t)$ satisfying the differential equation. In this case we expect the method to be convergent provided it is stable[11].

4.3 Godunov's Method

In 1959, Godunov proposed a way to make use of the characteristic information within the framework of a conservative method. Rather than attempting to follow characteristics backwards in time, Godunov suggested solving Riemannian problems forward in time. Solutions to Riemannian problems are relatively easy to compute, give substantial information about the chacteristic structure, and lead to conservative methods since they are the actual solutions of conservation laws hence it is conservative.
In Godunov's scheme, we use U_j^n on the grid cell $x_{j-\frac{1}{2}} < x < x_{j+\frac{1}{2}}$. We use $\bar{u}^n(x,t_n)$ as initial data for the conservation law, which we now solve exactly to obtain $\bar{u}^n(x,t_n)$ for $t_n \leq t \leq t_{n+1}$. The equation can be solved exactly over a short time interval because the initial data $\bar{u}^n(x,t_n)$ is piecewise constant, and hence defines a sequence of Riemann problems. The exact solution, up to the time when waves from neighbouring Riemannian problems start interaction, is found by bringing together those Riemann solutions.[2]
ie; Now consider the following initial value problem

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0, x \in \mathbf{R}, t \in \mathbf{N}, u(x,0) = u_0(x), \forall x \in \mathbf{R}$$

Define

$$x_{j+\frac{1}{2}} = x_j + \frac{h}{2},$$

with h the mesh width. The Godunov scheme will produce approximations $U^n \in \mathbf{R}^m$ to a cell $u(x,t_n)$ given by

$$u_j^n = \frac{1}{h}\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} u(x,t_n)dx \qquad \textbf{(107)}$$

1. Use the given initial data $u_0(x)$ and define the initial dataU^0 for ou numerical method by $U_j^0 = \bar{u}_j^0$.

2. Now construct the estimatiom U^1 from U^0, then U^2 from U^1 repeat this process. In general we construct U^{n+1} from U^n, for $n \in \mathbf{N}$.

3. Use U^n in identifying a constant function $\bar{u}^n(x,t_n)$ such that the value U_j^n is on grid cell $x_{j-\frac{1}{2}} < x < x_{j+\frac{1}{2}}$.

4. Next, make use of $\bar{u}^n(x,t_n)$ as initial value of and consevation law which is solved accurately to find $\bar{u}^n$ for $t_n \leq t \leq t_{n+1}$. Here use a shorter time interval since $\bar{u}^n(x,t_n)$ is constant.

5. Thus define a series of Riemannian problems.

6. Piece these Riemann solutions together to obtain the accurate solution unntil waves from neighbouring Riemann problems begin to interact. Hence we obtain the exact solution over the time interval $[t_n,t_{n+1}]$.

After obtaining this solution over the interval $[t_n,t_{n+1}]$ we define the approximate solution U^{n+1} at time t_{n+1} by averaging this exact solution at time t_{n+1},

$$U_j^{n+1} = \frac{1}{h}\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} \bar{u}^n(x,t_{n+1})dx \tag{108}$$

These values are then used to define a piecewise constant data $\bar{u}^{n+1}(x,t_{n+1})$ and the process repeats. In practice this algorithm is considerably simplified by observing that the cell average (107) can be easily computed using the integral form of the conservation law. Since $\bar{u}^n$ is assumed to be an exact weak solution, we know that

$$\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} \bar{u}^n(x,t_{n+1})dx = \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} \bar{u}^n(x,t_n)dx + \int_{t_n}^{t_{n+1}} f(\bar{u}^n(x_{j-\frac{1}{2}},t))dt - \int_{t_n}^{t_{n+1}} f(\bar{u}^n(x_{j+\frac{1}{2}},t))dt \tag{109}$$

Note that $\bar{u}^n(x,t_n) \equiv U_j^n$ over the cell$(x_{j-\frac{1}{2}},x_{j+\frac{1}{2}})$. Dividing (109) by h and using (108) gives

$$U_j^{n+1} = U_j^n - \frac{k}{h}[F(U_j^n,U_{j+1}) - F(U_{j-1}^n,U_j^n)] \tag{110}$$

with F The numerical flux defined by

$$F(U_j^n,U_{j+1}) = \frac{1}{k}\int_{t_n}^{t_{n+1}} f(\bar{u}^n(x_{j+\frac{1}{2}},t))dt \tag{111}$$

Note that the solution of the Riemann problem at the point $x_{j+\frac{1}{2}}$ is a similarity solution, $(x-x_{j+\frac{1}{2}})/t$ =constant. Therefore $\bar{u}^n$ is constant at $x_{j+\frac{1}{2}}$ over (t_n,t_{n+1}), which simplifies the integral (108).
Denote $u*(U_j^n,U_{j+1}^n)$ value of $\bar{u}^n$ on line $x = x_{j+\frac{1}{2}}$, then the flux defined in (111) becomes

$$F(U_j^n,U_{j+1}) = f(u*(U_j^n,U_{j+1}^n)) \tag{112}$$

We simplify our notation by using

$$F_{j+\frac{1}{2}} = F(U_j^n,U_{j+1}), and F_{j-\frac{1}{2}} = F(U_{j-1}^n,U_j^n) \tag{113}$$

Remark 4.3.1. *The Godunov method can be written in conservative form*

$$U_j^{n+1} = U_j^n - \frac{k}{h}[F^n_{j+\frac{1}{2}} - F^n_{j-\frac{1}{2}}], \tag{114}$$

with intercell numerical flux given by (110)*, if the time step k satisfies*

$$k \leq \frac{h}{\lambda^n_{max}}, \tag{115}$$

where λ^n_{max} denotes the maximum wave velocity at t^n.

4.3.1 Consistency

Approximate $\frac{1}{k}\int_{t_n}^{t_{n+1}} f(u(x_{j+\frac{1}{2},t}))dt \approx F^n_{j+\frac{1}{2}}$ then we have a numerical scheme:

$$U_j^{n+1} = U_j^n - \frac{k}{h}[F^n_{j+\frac{1}{2}} + F^n_{j-\frac{1}{2}}] \tag{116}$$

The method obtained this way is called finite volume method. The schemes of the form (116) is of conservative form, since $\sum_j hu^n_j$ is a constant
F^n_{j+1} is called the numerical flux. In most cases, we construct this numerical flux usins U^n_j and U^n_{j+1}.
we need the approximation to be **consistent**:

$$F_{j+\frac{1}{2}(u,u)} = f(u)$$

.
Usually, we need the Lipschitz continuity:

$$|F^n_{j+\frac{1}{2}} - f(u)| = |F(U^n_j, U^n_{j+1}) - f(u)| \leq Lmax(|U^n_j - u|, |U^n_{j+1} - u|)$$

4.3.2 Convergence and Stability

CFL condition (115) is a condition necessary for stability but it's non sufficient. In this subsection, we look at some convergence results and introduce some stability criteria.

Theorem 4.3.2 (Lax-Wendroff's Theorem)**.** *Suppose a method for the conservation law is conservative and consistent. If the numerical solution U^n_j converges to a function u(x,t) as $h_l, k_l \to 0$, then u(x,t) is a weak solution of the conservation law.*

Remark 4.3.3. *As we know,the solution is non unique. The limit here may not be the entropy solution.*

We now define a stability and then conclude the convergence.
Given an arbitrary function $q(x,t)$, the total variation is defined to be

$$TV(q) = \sup_{\xi} \sum_{j} |q(\xi_j) - q(\xi_{j-1})| = \lim_{\varepsilon \to 0} sup \int \frac{1}{\varepsilon} |q(x) - q(x-\varepsilon)| dx$$

Clearly, if q is a constant function, we have

$$TV(U) = \sum_{j} |U_j - U_{j-1}|$$

.
Now, let us define the (TV) over time interval [0,T] for a piecewise constant solutions:

$$TV_T = \sum_{n=0}^{T/k} kTV(U^n) + \sum_{n=0}^{T/k} ||U^{n+1} - U^n||$$

A conservative method is called **TV-stable** if there exists R independent of h,k but may depend on the initial data such that $TV_T \leq R$ and the numerical solution stays compactly supported. (The compact support condition is always satisfied if the initial data is compactly supported)

Lemma 4.3.4. *For a conservative method, if there exists R independent of h,k but may depend on the initial data such that $TV(U_j^n) \leq R$ for any n such that $nk \leq T$, then the method is TV-stable.*

Theorem 4.3.5. *If a consistent conservative method is TV-stable and the numerical flux is Lipschitz continous, then the numerical solution converges to a weak solution of the conservation law as $k \to 0$*

Here, the weak solution again may not be the entropy solution.

If a **conservative** method for the conservation law can be written as

$$u_j^{n+1} = G(u_{j-m}^n, u_{j-m+1}^n, ..., u_{j+m}^n),$$

such that $\frac{\partial G}{\partial u_{j+p}} \geq 0$ for $|p| \leq m$, then it's called a **monotone scheme**

4.4 Entropy Condition

Finally, let's provide a condition for the solutions to converge to the entropy solution. For conservation law $u_t + f(u)_x = 0$, if we can find ϕ such that $\phi''(u) \geq 0$ and $\phi'(u)f'(u) = \psi'(u)$, then the entropy solution satisfies the entropy inequality:

$$\int\int (\phi(u)_t + \psi(u)_x)dxdt \leq 0$$

The proof follows from the vanishing viscosity. We choose to omit it here.
The dicrete entropy condition says

$$\frac{\phi(u_j^{n+1}) - \phi(u_j^n)}{k} + \frac{\omega^n_{j+\frac{1}{2}} - \omega^n_{j-\frac{1}{2}}}{h} \leq 0$$

where $\omega^n_{j+1/2} = \omega(u^n_{j-m}, ..., u^n_{j+m})$ such that $\omega(u, u, ..., u) = \psi(u)$.

Theorem 4.4.1. *If the numerical solutions to a consistent conservative scheme converges and it satisfies the discrete entropy condition, then the limit is the entropy solution.*

The Godunov's method satisfies entropy condition and hence solution converges to the entropy solution provided k, h are chosen such that it is stable.

4.5 Godunov's Scheme

When we deal with a conservative scheme in the form (4.8), we need to calculate fluxes at the interfaces of every cell in order to advance the solution from time t_n to time t_{n+1}. By exploiting the definition of cell average, we can construct at each step a piece-wise continous constant distribution. In this way we have a set of local Riemann problems at cell boundaries, each one is defined as:

$$\begin{cases} \partial_t U + \partial_x F(U) = 0 \\ U_0(x) = \begin{cases} U_L for x_{j-1} < x < x_{j-\frac{1}{2}} \\ U_R for x_{j-\frac{1}{2}} < x < x_j \end{cases} \end{cases}$$

As we can easily understand, in order to calculate the flues at cell interfaces, we need to solve a pair of Riemann problems for each cell boundaries $x_{j+\frac{1}{2}}$ axis, we do not need a time integral in evaluating the numrtical flux as required by (111), the CFL condition requires

$$max(|\lambda_i|)\frac{\triangle t}{\triangle x} \leq 1$$

where $\max(|\lambda_i|)$ is the absolute value of the highest eigenvalue of the Jacobian matrices (i.e. the highest wave speed of the entire system). If the condition is satisfied, then no waves will interact with the solutions of other Riemann problems.

the idea described above was originally from Godunov. This method is called Godunov's scheme and it is a first order scheme.

4.6 Numerical Solution Method For The Payne-Whitham (PW) Model

The Godunov method is efficient for solving hyperbolic systems of conservation laws. For the homogeneous version of the PW model, therefore we use Godunov-type difference equations to approximate it. The PW model, however, has one relaxation term. Thus, different treatments of the effects of the source term result in different methods.
A generl system (7) can be approximated by the following Godunov-type difference equation,

$$U_j^{n+1} = U_j^n - \frac{\triangle t}{\triangle x}(f(U_{i+\frac{1}{2}}^{*n}) - f(U_{j-\frac{1}{2}}^{*n})) + \triangle t \bar{S}(u), \tag{117}$$

in which $U_j^n = (\rho_j^n, q_j^n)$ is the mean of $u(x,t)$ over cell i at the time $n\triangle t, \bar{S}(u)$ is the average of the source term over $((j-\frac{1}{2})\triangle x, (j+\frac{1}{2})\triangle x) \times (n\triangle t, (n+1)\triangle t)$, and the boundary state $U_{j+\frac{1}{2}}^{*n} = (\rho_{j+\frac{1}{2}}^{*n}, q_{j+\frac{1}{2}}^{*n})$ is computed from the Riemann Problem with initial condition

$$u_{j+\frac{1}{2}}(x,t_n) = \begin{cases} u_L, & x - x_{j+\frac{1}{2}} < 0 \\ u_R, & x - x_{j+\frac{1}{2}} \geq 0 \end{cases}$$

where $u_L = U_j^n$ and $u_R = U_j^{n+1}$.
When the source term is approximated implicitly, we call the numerical solution method as the implicit method. In this method, the PW model can be approximated by.

$$\frac{\rho_j^{n+1} - \rho_j^n}{k} - \frac{q_{j+\frac{1}{2}}^{*n} - q_{j-\frac{1}{2}}^{*n}}{h} = 0, \tag{118}$$

$$\frac{q_j^{n+1} - q_j^n}{k} + \frac{\frac{q_{j+1/2}^{*n^2}}{\rho_j^{*n}} + c_0^2 \rho_{j+1/2}^{*n} - \frac{q_{j-1/2}^{*n^2}}{\rho_{j-1/2}^{*n}} - c_0^2 \rho_{j-1/2}^{*n}}{h} = \frac{f_*(\rho_j^{n+1}) - q_j^{n+1}}{\tau} \tag{119}$$

from which we can write the evolution equations for the PW model as

$$\rho_j^{n+1} = \rho_j^n - \frac{k}{h}(q_{j+1/2}^{*n} - q_{j-1/2}^{*n}), \tag{120}$$

$$q_j^{n+1} = \frac{1}{1+\frac{k}{\tau}} \left[q_j^n - \frac{k}{h} \left(\frac{q_{j+1/2}^{*n^2}}{\rho_j^{*n}} + c_0^2 \rho_{j+1/2}^{*n} - \frac{q_{j-1/2}^{*n^2}}{\rho_{j-1/2}^{*n}} - c_0^2 \rho_{j-1/2}^{*n} \right) + \frac{k}{\tau} f_*(\rho_j^{n+1}) \right] \tag{121}$$

4.7 Numerical Solution Method For The Aw-Rascle (AR) Model

The Aw-Rascle model is a system of two equations and its flux function

$$f(U) = \begin{bmatrix} \rho v \\ yv \end{bmatrix} \tag{122}$$

is a vector of two components. Hence its corresponding numerical Godunov flux is also a vectorof two components, lets say

$$F_G(U) = \begin{bmatrix} H(U) \\ G(U) \end{bmatrix}$$

Here the first component H is associated with the flux of the first conservative variable ρ, and the second component G corresponds to the second variable y. Define the control volume or cells V_j as follows

$$V_j = \left[x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}\right),$$
$$x_{j+\frac{1}{2}} = x_j + \frac{h}{2}$$
$$j = 0, \pm 1, \pm 2, ...$$

Let $\bar{U}(x,t)$ be the solution of the following Riemann problem

$$\bar{U}_t + f(\bar{U})_x = 0,$$

where f is definded by (122) above. Associate the problem with the following initial value

$$\bar{U}(x,t^n) = \begin{cases} U_j^n, & x < x_{j+1/2}, \\ U_{j+1}^n, & x > x_{j+1/2}. \end{cases}$$

The solution of this Riemann Problem is a similarity of the form

$$\bar{U}(x,t) = U_R\left(\beta; U_j^n, U_{j+1}^n\right),$$

$$\beta = \frac{x - x_{j+1/2}}{t - t^n}$$

Clearly, $\beta = 0$ for $x = x_{j+1/2}$, hence the numerical flux is defined by

$$F_G\left(U_j^n, U_{j+1}^n\right) = f\left(U_R\left(0; U_j^n, U_{j+1}^n\right)\right). \tag{123}$$

Defining the numerical flux components

$$H_j^n = H(\rho_j^n, \rho_{j+1}^n),$$

$$H_{j-1}^n = H(\rho_{j-1}^n, \rho_j^n),$$
$$G_j^n = G(y_j^n, y_{j+1}^n),$$

and

$$G_{j-1}^n = G(y_{j-1}^n, y_j^n),$$

Then the Godunov scheme applied to solve the Rascle model is solving the following finite difference equations simultaneously:

$$\rho_j^{n+1} = \rho_j^n - \frac{k}{h}\left(H_j^n - H_{j-1}^n\right),$$

$$y_j^{n+1} = y_j^n - \frac{k}{h}\left(G_j^n - G_{j-1}^n\right)$$

Recall that h stands for the grid mesh size and k represents the time step size.

5 Conclusion

Here we summarize what we have done so far, draw conclusion from the work and present the recommendations for future research work.

5.1 Summary and Conclusion

Chapter one of this thesis is devoted to stuy of the theory of traffic flow. Detailed introduction to various traffic flow models is also captured in the same chapter.
In the second chapter, we presented the sclar consevation laws, specifically, the hyperbolic systems of conservation laws are explained, we also discussed the associated Rieman Problems in the same chapter.
The third chapter was dedicated to the derivation some second order traffic flow models, that is, the Payne-Whitham and Aw-Rascle models of traffic flow.
In chapter four, of this document we studied the finite volume method for the conservation laws, specifically the Godunov's method and how it can be applied in the solution of nonlinear conservation laws which are also the traffic flow models.
As expected, The finite volume method (FVM) specifically of the Godunov's type was found to better approximate the solution of the connservation laws since it takes care of the discontinuities in the solution process.

5.2 Recommendation for Future Research.

Due to time constraints, many are the areas we intended to extend this work to. We point out the following areas this work could be extended to cover in future works.

1. We intended to impliment the finite volume method (FVM) using the existing mathematical packages eg, matlab, python, CLAWPACK etc in order to see the behaviour of the solutions given some conditions.
2. Real life problems in traffic flow can also be studied and managed by implimenting the models and schemes presented in this thesis.
3. The schemes and models could be implimented on a network of roads.

Bibliography

[1] AATM Aw and Michel Rascle. Resurrection of" second order" models of traffic flow. *SIAM journal on applied mathematics*, 60(3):916–938, 2000.

[2] John B Bell, Phillip Colella, and John A Trangenstein. Higher order godunov methods for general systems of hyperbolic conservation laws. *Journal of Computational Physics*, 82(2):362–397, 1989.

[3] Nicola Bellomo, Marcello Delitala, and V Coscia. On the mathematical theory of vehicular traffic flow i: Fluid dynamic and kinetic modelling. *Mathematical Models and Methods in Applied Sciences*, 12(12):1801–1843, 2002.

[4] Gui-Qiang Chen, C David Levermore, and Tai-Ping Liu. Hyperbolic conservation laws with stiff relaxation terms and entropy. *Communications on Pure and Applied Mathematics*, 47(6):787–830, 1994.

[5] Carlos F Daganzo. Requiem for second-order fluid approximations of traffic flow. *Transportation Research Part B: Methodological*, 29(4):277–286, 1995.

[6] DC Gazis and LC Edie. Traffic flow theory. *Proceedings of the IEEE*, 56(4):458–471, 1968.

[7] Axel Klar, Reinhart D Kühne, and Raimund Wegener. Mathematical models for vehicular traffic. 1995.

[8] Axel Klar and Raimund Wegener. Kinetic derivation of macroscopic anticipation models for vehicular traffic. *SIAM Journal on Applied Mathematics*, 60(5):1749–1766, 2000.

[9] Douglas A Kurtze and Daniel C Hong. Traffic jams, granular flow, and soliton selection. *Physical Review E*, 52(1):218, 1995.

[10] Randall J Leveque. Hyperbolic conservation laws and numerical methods. 1990.

[11] Randall J LeVeque. Nonlinear conservation laws and finite volume methods. In *Computational methods for astrophysical fluid flow*, pages 1–159. Springer, 1998.

[12] Christopher Lustri. Continuum modelling of traffic flow. *Web discussion page*, 2010.

[13] HJ Payne. Models of freeway traffic and control, simulation council proc. 28, mathematical models of public systems. *Edited by GA Bekey*, 1, 1971.

[14] Benedetto Piccoli and Andrea Tosin. Vehicular traffic: A review of continuum mathematical models. *Encyclopedia of Complexity and Systems Science*, 22:9727–9749, 2009.

[15] Philip L Roe. Approximate riemann solvers, parameter vectors, and difference schemes. *Journal of computational physics*, 43(2):357–372, 1981.