On Unitary Invariance of Some Classes of Operators

in Hilbert Spaces

Linety N. Muhati [1] and J. M. Khalagai [2]

[1] Department of Mathematics and Computer Science
University of Eldoret, Kenya

[2] School of Mathematics, University of Nairobi, Kenya

Abstract

It is a known fact in operator theory that two similar operators have equal spectra but they do not necessarily have to belong to the same class of operators. However, under the stronger relation of unitary equivalence it can be shown that two unitarily equivalent operators may belong to the same class of operators. In this paper we endeavor to exhibit some results on some pairs of operators which may belong to the same class under not only unitary equivalence but also isometric and co-isometric equivalence.

1. Introduction

Let H be a complex Hilbert space and $B(H)$ denote the Banach algebra of all bounded linear operators on H. An operator X is said to be a quasi-affinity if X is both one to one and has a dense range. Two operators A and B are said to be similar if there exists an invertible operator S such that $A = S^{-1}BS$. If there exists a unitary operator U such that $A = U^*BU$ then A and B are said to be unitarily equivalent. A and B are said to be quasi-similar if there exist quasi-affinities X and Y such that $AX = XB$ and $BY = YA$. These properties of similarity unitary equivalence and quasi-similarity, have been studied by a number of authors who by a large extent relate them to equality of spectra. For example, see W.S Clary [1]. However, adequate investigations of unitarily equi-

valent operators belonging to the same class has not been addressed. G. Messaond and N. Mostefa [6] showed that if $S, T \in B(H)$ are unitarily equivalent operators and T is n-power-hyponormal, then so is S. Also K. Rasimi et al [7], showed that if $S, T \in B(H)$ are unitarily equivalent operators and T is skew-n-binormal, so is S. In this case we can assert that all the classes of operators mentioned above are invariant under unitary equivalence. S.W. Luketero and J.M. Khalagai [4] considered some classes of operators under both isometric and co-isometric equivalences. To this end they were able to show that if $A, B \in B(H)$ are such that either $A = UBU^*$ with U an isometry or $A = U^*BU$ with U co-isometry and A is binormal, then so is B. The same result holds true for the classes of hypornormal operators and partial isometries. Recently S.W. Luketero [5], extended the results above to cover the classes of n-power hyponormal and n-binormal. Also S.K. Karani et al [3], showed that the results above hold true for classes of θ-operators and posinormal operators. In this paper we show that the classes of skew n-binormal and quasi n-normal which are independent also yield similar results.

2. Notations, definitions and terminologies

Given a complex Hilbert space H and operators $A, B \in B(H)$ the commutator of A and B is given by $[A, B] = AB - BA$.

The spectrum of A is denoted by $\sigma(A) = \{\lambda \in \mathbb{C} : A - \lambda I \text{ is not invertible}\}$, where I is the identity operator and $\mathbb{C}$ is the complex field. We denote the range of A and the kernel of A by ran$A$ and ker$A$ respectively.

Recall that an operator $A \in B(H)$ is said to be:

- Normal if $[A, A^*] = 0$ where A^* is the adjoint of A
- θ-operator if $[A^*A, A^* + A] = 0$
- Posinormal if $AA^* = A^*PA$ where P is a positive operatorType equation here.
- Quasi-normal if $[A^*A, A] = 0$
- Quasi n-normal if $A(A^nA^*) = (A^*A^n)A$
- Binormal if $[A^*A, AA^*] = 0$
- n-binormal if $A^*A^nA^nA^* = A^nA^*A^*A^n$
- Skew binormal if $(A^*AAA^*)A = A(AA^*A^*A)$
- Skew n-binormal if $(A^*A^nA^nA^*)A = A(A^nA^*A^*A^n)$
- Hyponormal if $A^*A \geq AA^*$
- n-power hyponormal if $A^*A^n \geq A^nA^*$
- partial isometry if $A = AA^*A$
- isometry if $A^*A = I$
- coisometry if $AA^* = I$

-      unitary if $A^*A = AA^* = I$

The class of skew n-binormal is denoted by $[snBN]$.

We also have the following inclusions of the classes of operators as defined above:
{Normal} $\subseteq$ {Posinormal}
{Normal} $\subseteq$ {θ-operator}
{Normal} $\subseteq$ {quasi n-normal}
{Normal} $\subseteq$ {quasi-normal} $\subseteq$ {Binormal} $\subseteq$ {n-binormal}
{Normal} $\subseteq$ {Hyponormal} $\subseteq$ {n-power-hyponormal}
{Normal} $\subseteq$ {Skew binormal} $\subseteq$ {Skew n-binormal}
{Unitary} $\subseteq$ {Isometry} $\subseteq$ {Partial isometry}
{Unitary} $\subseteq$ {Coisometry} $\subseteq$ {Partial isometry}

3. Main results

We first note that in our main results we will refer to the following results.

Theorem A [7, Theorem 2.4 (ii)]
Let $T \in [snBN]$. If S is an operator which is unitarily equivalent to T then S is also skew n- binormal.

Theorem B [7, Theorem 2.9]
Let $S, T \in [snBN]$ and be double commuting operators. Then ST is also skew n-binormal.

Theorem C [5, Theorem 3.7]
Let $A, B \in B(H)$ be such that A is skew-binormal and either $B = UAU^*$ with U isometry or $B = U^*AU$ with U coisometry, then B is also skew-binormal.

Theorem 3.1
Let $A \in [snBN]$ and B be another operator such that:
(i) $B = UAU^*$ with U isometry then B is also skew n-binormal.
(ii) $B = U^*AU$ with U coisometry, then B is also skew n-binormal.

Proof
(i)   Let $= UAU^*$. Then we have $B^* = UA^*U^*$.

Thus $B^n = (UAU^*)^n$
$$= \underbrace{UAU^*UAU^*UAU^* \dots UAU^*}_{n\ times}$$
$$= UAAA \dots AU^*$$

$$= UA^n U^*$$

Therefore $(B^* B^n B^n B^*)B = (UA^* U^* UA^n U^* UA^n U^* UA^* U^*)UAU^*$

$$= UA^* A^n A^n A^* A U^*$$

$$= U(A^* A^n A^n A^* A)U^* \dots \dots (1)$$

Also $B(B^n B^* B^* B^n) = UAU^*(UA^n U^* UA^* U^* UA^* U^* UA^n U^*)$

$$= UAU^*(UA^n A^* A^* A^n U^*)$$

$$= UAU^* UA^n A^* A^* A^n U^*$$

$$= UA(A^n A^* A^* A^n)U^* \dots \dots (2)$$

Since $A \in [snBN], (A^* A^n A^n A^*)A = A(A^n A^* A^* A^n)$. From (1) and (2) we have $(B^* B^n B^n B^*)B = B(B^n B^* B^* B^n)$. Hence $B \in [snBN]$.

(ii)   Similarly let $B = U^* AU$. Then we have $B^* = U^* A^* U$.

Thus $B^n = (U^* AU)^n$

$$= \underbrace{U^* AUU^* AUU^* \dots U^* AU}_{n \text{ times}}$$

$$= U^* AAA \dots AU$$

$$= U^* A^n U$$

Therefore $(B^* B^n B^n B^*)B = (U^* A^* UU^* A^n UU^* A^n UU^* A^* U)U^* AU$

$$= U^* A^* A^n A^n A^* AU$$

$$= U^*(A^* A^n A^n A^*)AU \dots \dots (1)$$

Also $B(B^n B^* B^* B^n) = U^* AU(U^* A^n UU^* A^* UU^* A^* UU^* A^n U)$

$$= U^* A(A^n A^* A^* A^n)U \dots \dots (2)$$

But $A \in [snBN], (A^* A^n A^n A^*)A = A(A^n A^* A^* A^n)$. From (1) and (2) we have $(B^* B^n B^n B^*)B = B(B^n B^* B^* B^n)$. Hence $B \in [snBN]$.

Remark 3.2

We note that theorems A and C above are mere corollaries to theorem 3.1 as shown below;

Corollary 3.3 [7, Theorem 2.4 (ii)]

Let $A \in [snBN]$. If B is an operator which is unitarily equivalent to A then B is also skew n- binormal.

Proof: This is immediate since the class of unitary operators is contained in the classes of isometry and coisometry operators. Thus U is unitary implies U is both isometry and coisometry.

Corollary 3.4 [5, Theorem 3.7]
Let $A, B \in B(H)$ be such that A is skew-binormal and either $B = UAU^*$ with U isometry or $B = U^*AU$ with U coisometry, then B is also skew-binormal.

Proof: This is also immediate since every skew binormal operator is skew n-binormal.

Theorem 3.5
Let A be a quasi n-normal operator and B be another operator such that:
(i) $B = UAU^*$ with U isometry then B is also quasi n-normal.
(ii) $B = U^*AU$ with U coisometry, then B is also quasi n-normal.

Proof
(i)   Let $= UAU^*$. Then we have $B^* = UA^*U^*$.

Thus $B^n = (UAU^*)^n$
$$= \underbrace{UAU^*UAU^*UAU^* \dots UAU^*}_{n\ times}$$
$$= UAAA \dots AU^*$$
$$= UA^nU^*$$

Therefore $B(B^nB^*) = UAU^*(UA^nU^*UA^*U^*)$
$$= UAA^nA^*U^* \dots \dots \dots (1)$$

Also $(B^*B^n)B = (UA^*U^*UA^nU^*)UAU^*$
$$= UA^*A^nAU^* \dots \dots \dots (2)$$

But $A(A^nA^*) = (A^*A^n)A$ since A is quasi n-normal. Now from (1) and (2) we have $(B^nB^*)B = B(B^*B^n)$. Hence B is also quasi n-normal.
(ii)   Similarly let $B = U^*AU$. Then we have $B^* = U^*A^*U$.

Thus $B^n = (U^*AU)^n$
$$= \underbrace{U^*AUU^*AUU^* \dots U^*AU}_{n\ times}$$
$$= U^*AAA \dots AU$$
$$= U^*A^nU$$

Therefore $B(B^nB^*) = U^*AU(U^*A^nUU^*A^*U) = U^*AA^nA^*U$
$$= U^*(AA^nA^*)U \dots \dots \dots (1)$$

Also $(B^*B^n)B = (U^*A^*UU^*A^nU)U^*AU$
$$= U^*(A^*A^nA)U \dots \dots \dots (2)$$

But $A(A^n A^*) = (A^* A^n)A$. Therefore, from (1) and (2) we have $B(B^n B^*) = (B^* B^n)B$. Hence B is also quasi n-normal.
The following corollary is also immediate.

Corollary 3.6
Let A be a quasi n-normal operator and B be another operator which is unitarily equivalent to A, then B is also quasi n-normal.

Remark 3.7
We note that the results above show that the two classes of operators namely skew n-binormal and quasi n-normal are not only unitarily invariant but also isometrically and coisometrically invariant.

We now show that we have similar results to theorem B above for classes of quasi n-normal and n-binormal operators.

Theorem 3.8
Let $A, B \in B(H)$ be quasi n-normal operators such that $[A, B] = 0$ and $[A, B^*] = 0$. Then AB is also quasi n-normal.

Proof:
We have $A(A^n A^*) = (A^* A^n)A$ and $B(B^n B^*) = (B^* B^n)B$. Also, $AB = BA$ implies $B^* A^* = A^* B^*$ and $AB^* = B^* A$ implies $A^* B = BA^*$, we have to show that
$AB((AB)^n (AB)^*) = ((AB)^* (AB)^n)AB$.

Now
$$
\begin{aligned}
AB((AB)^n (AB)^*) &= AB(A^n B^n B^* A^*) \\
&= ABA^n B^n B^* A^* \\
&= AA^n BB^n B^* A^* \\
&= AA^n B^* B^n BA^* \quad (B \text{ is qusi n-normal}) \\
&= AA^n A^* B^* B^n B \quad ([A^*, B^* B^n B] = 0) \\
&= A^* A^n AB^* B^n B \quad (A \text{ is qusi n-normal}) \\
&= B^* A^* A^n AB^n B \quad ([B^*, A^* A^n A] = 0) \\
&= B^* A^* A^n B^n AB \quad ([A, B^n] = 0) \\
&= (AB)^* (AB)^n AB
\end{aligned}
$$

Hence AB is also quasi n-normal.

Corollary 3.9
Let A be quasi n-normal operator. If B is a normal operator such that $[A, B] = 0$, then AB is also quasi n-normal.

Proof:
We first note that B is normal implies B is quasi n-normal. Also $AB = BA$ implies $BA^* = A^*B$ by Putman Fuglede theorem. Hence by the theorem above the proof comes through.

Theorem 3.10
Let $A, B \in B(H)$ be n-binormal operators such that $[A, B] = 0$ and $[A, B^*] = 0$. Then AB is also n-binormal.

Proof:
Since A and B are n-binormal, we have $A^*A^nA^nA^* = A^nA^*A^*A^n$ and $B^*B^nB^nB^* = B^nB^*B^*B^n$. Now, $AB = BA$ and $AB^* = B^*A$ implies $B^*A^* = A^*B^*$ and $BA^* = A^*B$. Thus, we have to show that $(AB)^*(AB)^n(AB)^n(AB)^* = (AB)^n(AB)^*(AB)^*(AB)^n$.

Now $(AB)^*(AB)^n(AB)^n(AB)^* = B^*A^*A^nB^nA^nB^nB^*A^*$
$= B^*B^nB^nA^*A^nA^nB^*A^*$ $(BA = AB, AB^* = B^*A)$
$\qquad\qquad = B^*B^nB^nB^*A^*A^nA^nA^*$
$\qquad\qquad = B^nB^*B^*B^nA^nA^*A^*A^n$ (A and B are n-binormal)
$\qquad\qquad = B^nA^nB^*B^*A^*A^*A^nB^n$ $(BA^n = A^nB)$
$\qquad\qquad = A^nB^nB^*A^*B^*A^*A^nB^n$
$\qquad\qquad = (AB)^n(AB)^*(AB)^*(AB)^n$

Hence AB is also n-binormal.

Corollary 3.11
Let A be n-binormal and B be a normal operator such that $AB = BA$, then AB is also n-binormal.

Proof:
We first note that B is normal implies B is n-binormal. We also have that $AB = BA$ implies $BA^* = A^*B$ by Putnam Fuglede theorem. Hence by the theorem above, the result follows.

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