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Ogola Cynthia Akoth

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Master Thesis

Submitted to the School of Mathematics in partial fulfilment for a degree in Master of Science in Mathematical Statistics

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Abstract

Mixed Poisson distributions have been used in scientific fields for modelling non-homogenous populations, (Karlis and xekalaki, 2005)..
For example, in acturial applications, mixed Poisson distributions are used for modeling total claims in insurance.
In this work, the concentration is mainly on the estimation of the parameters of Poisson-Lindley distribution using EM Algorithm which was first introduced by Dempster (1977).
One-parameter, two-parameter and three-parameter Lindley distributions are compounded by the Poisson distribution to form the Poisson-Lindley distributions and then the parameters estimated.
In order to carry out the EM Algorithm successfully, the posterior distribution is applied and the posterior expectation calculated.
Various properties of each distribution are determined, for example, the r^{th} moment, the cumulative distribution function (cdf), the moment generating function (mgf), the probability generating function (mgf) and the characteristic function.

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Declaration and Approval

I the undersigned declare that this dissertation is my original work and to the best of my knowledge, it has not been submitted in support of an award of a degree in any other university or institution of learning.

C. Akoth — 23/08/2021

Signature — Date

OGOLA CYNTHIA AKOTH
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In my capacity as a supervisor of the candidate's dissertation, I certify that this dissertation has my approval for submission.

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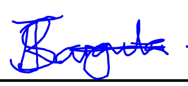

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Dedication

This project is dedicated to my mother who have always believed in my ability to be successful in the academic arena.

Contents

Acknowledgments

First, I thank the Almighty God for giving me strenth, wisdom and knowledge to undertake this project. Second, I am grateful to my supervisors, Prof. J.A.M Ottieno and Dr. Rachel Sarguta for guiding and correcting me brilliantly. May God bless you both abundantly for dedicating your time to make sure that this work was a complete success. For my classmates who helped with some reference materials that were really essential in my work, thank you and my God bless you. To the University, thanks for offering me a scholarship which made my learning and doing my reserch easier without having to worry about the school fees.

Ogola Cynthia Akoth

Nairobi, 2021.

1 General Introduction

1.1 Background Information

Probability distribution can be constructed by combining two or more distributions to obtain a new distribution called called a mixture.
There are three types of mixtures namely; finite mixtures, continous mixtures and discrete mixtures.
Lindley distribution is an example of a finite mixture of two gamma distributions.
It was first introduced by D.V Lindley (1958) in the context of Bayesian analysis as a counter example of fiducial statistics.
The Lindley distribution has been used in the modelling and analyzing of lifetime data that are crucial in many applied sciences including medicine, engineering, insurance and finance.
The introduction of Poisson-Lindley distribution was as a result of comparing the Poisson distribution with the Poisson-Lindley distribution to see the most preferable.
This was achieved by Shanker and Fesshaye, (2015) who fitted some data sets in ecology and genetics to the two distributions and they found that Poisson-Lindley distribution is more flexible for analyzing different types of count data than Poisson distribution.

The EM Algorithm was first introduced by Dempster (1977) to obtain the MLEs of incomplete data.
Mclachlan (2004) states that the EM Algorithm is n applicable approach in the iterative computation of maximum likelihood estimates which is useful in a variety of incomplete data.
Mclachlan (2004) explains that the EM Algorithm is simply a generic method for computing the MLE of an incomplete data by formulating an associated complete data and exploiting the simplicity of the MLE of the latter to compute the MLE of the former.

1.2 Research Problem

Karlis(2005), discussed the EM Algorithm for mixed Poisson and other discrete distributions and he considered the Poisson-Lindley distribution using the posterior expectation which was first introduced by Sapatinas (1995).
However, he did not show how we obtain the posterior expectation of the Poisson-Lindley distribution and how he arrived at the final solution.
The problem is one would like to know how he came up with the final solution.

This project reconstructs the Poisson-Lindley distribution, that is, one parameter, two parameter and three parameter Poisson-Lindley distributions and then estimate the parameters of the distributions using the EM Algorithm method where the posterior distribution and the posterior expectation is applied.

1.3 Objectives

The main objective of this study is to estimate the parameters of Poisson-Lindley distribution with the help of the posterior distribution and the posterior expectation.

The specific objectives are:

1. To estimate the parameter of one-parameter Poisson-Lindley distribution using the EM Algorithm method.
2. Using the EM Algorithm method to estimate the parameters of two-parameter Poisson-Lindley distribution.
3. Estimating the parameters of three-parameter Poisson-Lindley distribution using the EM Algorithm method.

1.4 Methodology

The method used for the estimation of the parameters is the EM Algorithm method which is described below.

Expectation-Maximization Algorithm method

EM Algorithm method was first introduced by Dempster et al. (1977) to obtain the maximum likelihood estimates for an incomplete data.
Assume that the true data are made of an observed part X and unobserved part λ.
This then ensures that the log likelihood function of the complete data (x_i, λ_i) for $i = 1, 2, 3, \ldots, n$ factorizes into two parts (Kostas 2007).

This then implies that the joint density function of X and Λ is given by:

$$f(x,\lambda) = f(x/\lambda)g(\lambda)$$

The likelihood function is:

$$
\begin{aligned}
L &= \prod_{i=1}^{n} f(x_i/\lambda_i) g(\lambda_i) \\
&= \prod_{i=1}^{n} f(x_i/\lambda_i) \prod_{i=1}^{n} g(\lambda_i) \\
\therefore \log\ L &= log \prod_{i=1}^{n} f(x_i/\lambda_i) + log \prod_{i=1}^{n} g(\lambda_i) \\
&= \sum_{i=1}^{n} log\ f(x_i/\lambda_i) + \sum_{i=1}^{n} log\ g(\lambda_i)
\end{aligned}
$$

Let

$$
\begin{aligned}
l_1 &= \sum_{i=1}^{n} log\ f(x_i/\lambda_i) \\
and & \\
l_2 &= \sum_{i=1}^{n} log\ g(\lambda_i)
\end{aligned} \tag{1}
$$

For optimization, l_1 is differentiated with respect to parameters in the function $f(x_i/\lambda_i)$; then equated to zero.
Similarly, l_2 is differentiated with respect to the parameters in $g(\lambda_i)$ and then equated to zero.

1.5 Literature Review

Etimating the parameters of a mixing distribution was first introduced by Pearson (1984) who estimated the parameters of the mixture of two normal densities.
Poisson-Lindley distribution was first introduced by Sankaran (1970) who introduced the one-parameter discrete Poisson-Lindley distribution by mixing Poisson distribution with the Lindley distribution.
Although there are upto five parameters Lindley distribution, Poisson-Lindley distribution have only been introduced upto three-parameter Poisson-Lindley distribution which include Poisson-Lindley (Sankaran 1970), Poisson Quasi-Lindley (Shanker and Mishra, 2013), another two parameter Poisson-Lindley (Shanker and Mishra, 2013), thre-parametr Poisson-Lindley (Kishore et. al, 2018).

Under estimation, EM Algorithm is used to obtain the estimates of the parameters.
EM Algorithm have not been considered in most of the articles as a way of estimating the parameters.
More concentration have been on the method of moments and the maximum likelihood

estimation as a way of estimating the parameters of a distribution.
Karlis (2005) was the first to use EM Algorithm method to estimate the parameters of mixed Poisson and other discrete distributions.
Shanker and Fesshaye (2015) used the method of moments and the maximum likelihood estimation method to estimate the parameter of one-parameter Poisson-Lindley distribution.
Shanker and Mishra (2015) used the method of moments and the maximum likelihood estimation method to estimate the parameters of a Poisson Quasi-Lindley distribution.
Kishore et. al (2018) introduced a new three-parameter Poisson-Lindley distribution and used the method of moments and the maximum likelihood estimation method in estimating the parameters of the distribution.

1.6 Significance of the study

Mixture models have a wide variety of applications in statistics.
The number of applications have increased in the recent years mainly because of the availability of high speed computer resources which have removed any obstacles to apply such methods.

"Thus mixture models have found applications in fields as diverse as data modelling, discriminant analysis, cluster analysis, outlier-robustness studies, ANOVA models, kernel density estimation, latent structure models, empirical Bayes estimation, Bayesian statistics, random variable generation, approximation of the distribution of some statistic and others." (Karlis and Xekalaki, 2005).

Therefore, Poisson mixtures have also a variety of applications in statistical science since its an example of ixture models.
In this work, our main concern is on the applications of Poisson-Lindley distributions.
Poisson-Lindley distribution was first introduced by Sankaran(1970) to model count data.
In Ecology, Poisson-Linldey distribution has been used as an important tool for modelling count data to analyse the relationship between organisms and their environment.(Shanker nad Fesshaye, 2015)
In Genetics, which is a branch of biological science that deals with heredity and variation, Poisson-Lindley distribution has been considered as an important tool for modelling the genetics data.
In acturial science, Sankaran (1970) applied the Poisson-Lindley distribution to errors and accidents.
Ghitany (2009) applied the Poisson-Lindley distribution to determine service rate (how long a customer waits on the queue at the bank).

2 Poisson-One Parameter Lindley Distribution

2.1 Introduction

In this section, we will discuss the one-parameter Poisson-Lindley distribution. We will show the distribution is obtained by constructing the distribution. We will also obtain the posterior distribution and hence the posterior expectation which will aid in the estimation of the parameter using the EM Algorithm. We will then determine the properties which in our case is the r^{th} moment about the origin. Lastly, we will then estimate the parameter of the distribution using the EM Algorithm method.

2.2 Construction

The pdf of a one parameter Lindley distribution is given by:

$$g(\lambda) = \frac{\theta^2}{\theta+1}(\lambda+1)e^{-\lambda\theta}, \lambda>0; \theta>0 \tag{2}$$

The pdf of a poisson distribution is given by:

$$f(x;\lambda) = \frac{e^{-\lambda}\lambda^x}{x!}, x = 1,2,3,... \tag{3}$$

The pdf of the mixed poisson-lindley distribution is given by:

$$\begin{aligned} f(x) &= \int_0^\infty \frac{e^{-\lambda}\lambda^x}{x!} g(\lambda)\, d\lambda \\ f(x) &= \frac{\theta^2(\theta+2+x)}{(1+\theta)^{x+3}}, x = 0,1,2,..., \theta>0 \end{aligned} \tag{4}$$

Proof.

$$f(x) = \int_0^\infty \frac{e^{-\lambda}\lambda^x}{x!}\frac{\theta^2}{\theta+1}(\lambda+1)e^{-\lambda\theta}\,d\lambda$$
$$f(x) = \frac{\theta^2}{x!(\theta+1)}\int_0^\infty (\lambda+1)\lambda^x e^{-\lambda(\theta+1)}\,d\lambda$$
$$f(x) = \frac{\theta^2}{\theta+1}\int_0^\infty [\frac{\lambda^{x+1}}{x!}e^{-\lambda(\theta+1)} + \frac{\lambda^x}{x!}e^{-\lambda(\theta+1)}]\,d\lambda$$
$$Put \phi = \theta+1, \Rightarrow \theta = \phi - 1$$
$$Therefore,$$
$$f(x) = \frac{(\phi-1)^2}{\phi}\int_0^\infty [\frac{\lambda^{x+1}}{x!}e^{-\phi\lambda} + \frac{\lambda^x}{x!}e^{-\phi\lambda}]\,d\lambda$$
$$Put y = \phi\lambda$$
$$\Rightarrow \lambda = \frac{y}{\phi}; d\lambda = \frac{dy}{\phi}$$
$$Therefore,$$
$$f(x) = \frac{(\phi-1)^2}{\phi}\int_0^\infty [\frac{y^{x+1}e^{-y}}{\phi^{x+1}x!} + \frac{y^x e^{-y}}{\phi^x x!}]\frac{dy}{\phi}$$
$$f(x) = \frac{(\phi-1)^2}{\phi}[\frac{\Gamma(x+2)}{\phi^{x+1}x!} + \frac{\Gamma(x+1)}{\phi^x x!}]$$
$$f(x) = \frac{(\phi-1)^2}{(\phi)^2}[\frac{(x+1)\Gamma(x+1)}{\phi^{x+1}x!} + \frac{\Gamma(x+1)}{\phi^x x!}]$$
$$f(x) = \frac{(\phi-1)^2}{\phi^2}[\frac{x+1}{\phi^{x+1}} + \frac{1}{\phi^x}]\frac{\Gamma(x+1)}{x!}$$
$$f(x) = \frac{(\phi-1)^2}{\phi^2}[\frac{x+1+\phi}{\phi^{x+1}}]$$
$$f(x) = \frac{\theta^2}{(1+1)^2}[\frac{x+1+1+\theta}{(1+\theta)^{x+1}}]$$
$$f(x) = \frac{\theta^2(\theta+2+x)}{(1+\theta)^{x+3}}, x = 0,1,2,...; \theta>0$$

(Sankaran 1970)

□

2.3 Posterior Distribution

The pdf of a posterior distribution is defined as follows;

$$f(\lambda/x) = \frac{f(\lambda,x)}{f(x)} = \frac{f(x,\lambda)}{f(x)}\; f(\lambda/x) \quad = \frac{f(x/\lambda)g(\lambda)}{\int_0^\infty f(x/\lambda)g(\lambda)\,d\lambda} \tag{5}$$

The posterior mean is;

$$
\begin{aligned}
E(\Lambda/X) &= \frac{\int_0^\infty \lambda f(x/\lambda)g(\lambda)\,d\lambda}{\int_0^\infty f(x/\lambda)g(\lambda)\,\lambda} \\
E(\Lambda/X) &= \frac{\int_0^\infty \frac{\lambda e^{-\lambda}\lambda^x}{x!}\frac{\theta^2}{\theta+1}(1+\lambda)e^{-\lambda\theta}\,d\lambda}{\int_0^\infty \frac{e^{-\lambda}\lambda^x}{x!}\frac{\theta^2}{\theta+1}(1+\lambda)e^{-\lambda\theta}\,d\lambda} \\
E(\Lambda/X) &= \frac{\int_0^\infty \lambda e^{-\lambda}\lambda^x(1+\lambda)e^{-\lambda\theta}\,d\lambda}{\int_0^\infty e^{-\lambda}\lambda^x(1+\lambda)e^{-\lambda\theta}\,d\lambda} \\
Therefore, & \\
E(\Lambda/X) &= \frac{\int_0^\infty \lambda^{x+1}e^{-\lambda}(1+\lambda)e^{-\lambda\theta}\,d\lambda}{\int_0^\infty e^{-\lambda}\lambda^x(1+\lambda)e^{-\lambda\theta}\,d\lambda} \\
&= \frac{\frac{\Gamma(x+2)}{(\theta+1)^{x+2}}+\frac{\Gamma(x+3)}{(\theta+1)^{x=3}}}{\frac{\Gamma(x+1)}{(\theta+1)^{x+1}}+\frac{\Gamma(x+2)}{(+1)^{x+2}}} \\
&= \frac{\frac{(x+1)\Gamma(x+1)}{(\theta+1)^{x+2}}+\frac{(x+2)(x+1)\Gamma(x+1)}{(\theta+1)^{x+3}}}{\frac{\Gamma(x+1)}{(\theta+1)^{x+1}}+\frac{(x+1)\Gamma(x+1)}{(\theta+1)^{x+2}}} \\
&= \frac{\frac{x+1}{\theta+1}+\frac{(x+2)(x+1)}{\theta^2}}{1+\frac{x+1}{\theta+1}} \\
&= \frac{(x+1)(\theta+1)+(x+2)(x+1)}{(\theta+1)^2+(\theta+1)(x+1)} \\
&= \frac{(x+1)(\theta+1+x+2)}{(\theta+1)(\theta+1+x+1)} \\
&= \frac{(x+1)(x+3+\theta)}{(\theta+1)(x+2+\theta)}
\end{aligned}
\tag{6}
$$

2.4 Properties

2.4.1 The rth moment

The rth moment about of the one-parameter Poisson-Lindley distribution is given by;

$\mu_r' = E[E(x^{(r)}/\lambda)]$

where

x$^r = x(x-1)(x-2)\ldots(x-r+1)$

From
$\int_0^\infty \frac{e^{-\lambda}\lambda^x}{x!}\frac{\theta^2}{\theta+1}(1+\lambda)e^{-\lambda\theta}\,d\lambda$

we have

$$\begin{aligned}\mu_r' &= \int_0^\infty [\sum_{x=0}^\infty x^r \frac{e^{-\lambda}\lambda^x}{x!}]\frac{\theta^2}{\theta+1}(1+\lambda)e^{-\lambda}\,d\lambda \\ &= \int_0^\infty [\lambda^r \frac{e^{-\lambda}\lambda^{x-r}}{(x-r)!}]\frac{\theta^2}{\theta+1}(1+\lambda)e^{-\lambda\theta}\,d\lambda\end{aligned} \tag{7}$$

Subsitituting (x+r) with x, we have;

$$\mu_r' = \int_0^\infty \lambda^r [\sum_{x=0}^\infty \frac{e^{-\lambda}\lambda^x}{x!}]\frac{\theta^2}{\theta+1}(1+\lambda)e^{-\lambda\theta}\,d\lambda$$

Since the expression in the bracket is a unity function, that is, is equivalent to 1, we have,

$$\begin{aligned}\mu_r' &= \frac{\theta^2}{\theta+1}\int_0^\infty \lambda^r(1+\lambda)e^{-\lambda\theta}\,d\lambda \\ &= \frac{\theta^2}{\theta+1}[\int_0^\infty \lambda^r e^{-\lambda\theta}\,d\lambda + \int_0^\infty \lambda^{r+1}e^{-\lambda\theta}\,d\lambda] \\ &= \frac{\theta^2}{\theta+1}[\frac{\Gamma(r+1)}{\theta^{r+1}} + \frac{\Gamma(r+2)}{\theta^{r+2}}] \\ &= \frac{r!(\theta+r+1)}{\theta^r(\theta+1)}\end{aligned} \tag{8}$$

2.5 Estimation

2.5.1 EM Algorithm for One-Parameter Poisson-Lindley Distribution

For a continuous Poisson mixture, the joint distribution is given by;

$$\begin{aligned}f(x,\lambda) &= f(x/\lambda)g(\lambda) \\ &= \frac{e^{-\lambda}\lambda^x}{x!}g(\lambda)\end{aligned} \tag{9}$$

The likelihood function is;

$$
\begin{aligned}
L &= \prod_{i=1}^{n} \frac{e^{-\lambda_i}\lambda_i^{x_i}}{x_i!} g(\lambda_i) \\
log\, L &= log \prod_{i=1}^{n} \frac{e^{-\lambda_i}\lambda_i^{x_i}}{x_i!} + log \prod_{i=1}^{n} g(\lambda_i) \\
&= \sum_{i=1}^{n} log \frac{e^{-\lambda_i}\lambda_i^{x_i}}{x_i!} + \sum_{i=1}^{n} log\ g(\lambda_i)
\end{aligned} \tag{10}
$$

Let

$$
\begin{aligned}
l_1 &= \sum_{i=1}^{n} log \frac{e^{-\lambda_i}\lambda_i^{x}{}_i}{x_i!} \\
l_1 &= \sum_{i=1}^{n} [-\lambda_i + x_i log\lambda_i - log x_i!] \\
&= -\sum_{i=1}^{n} \lambda_i + \sum_{i=1}^{n} x_i log\lambda_i - \sum_{i=1}^{n} log x_i!
\end{aligned} \tag{11}
$$

The log likelihood function of the Poisson distribution remains as it is since there is no parameter to be estimated.

For the Lindley distribution, the pdf of a one parameter Lindley distribution is given by:

$$
g(\lambda) = \frac{\theta^2}{\theta+1}(1+\lambda)e^{-\lambda\theta}; \lambda{>}0; \theta{>}0
$$

Therefore

$$
\begin{aligned}
l_2 &= \sum_{i=1}^{n} log\ g(\lambda_i) \\
&= \sum_{i=1}^{n} log \frac{\theta^2}{\theta+1}(1+\lambda_i)e^{-\lambda_i\theta} \\
&= \sum_{i=1}^{n} 2log\theta + log(1+\lambda_i) - \theta\lambda_i - log(\theta+1) \\
&= 2nlog\theta + \sum_{i=1}^{n}(1+\lambda_i) - \theta\sum_{i=1}^{n}\lambda_i - nlog(\theta+1)
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\frac{\partial}{\partial\theta}l_2 &= \frac{2n}{\theta} - \sum_{i=1}^{n}\lambda_i - \frac{n}{\theta+1} \\
\frac{\partial}{\partial\theta}l_2 &= 0 \\
\Rightarrow \frac{2n}{\theta} - \sum_{i=1}^{n}\lambda_i - \frac{n}{\theta+1} &= 0 \\
\frac{2}{\theta} - \overline{\lambda} - \frac{1}{\theta+1} &= 0 \\
2 - \theta\overline{\lambda} - \frac{\theta}{\theta+1} &= 0 \\
2(\theta+1) - \theta\overline{\lambda}(\theta+1) - \theta &= 0 \\
2\theta + 2 - \overline{\lambda}\theta^2 - \overline{\lambda}\theta - \theta &= 0 \\
\overline{\lambda}\theta^2 + (1 - 2 + \overline{\lambda})\theta - 2 &= 0 \\
\overline{\lambda}\theta^2 + (\overline{\lambda} - 1)\theta - 2 &= 0 \\
\hat{\theta} &= \frac{-(\overline{\lambda}-1) \pm \sqrt{(\overline{\lambda}-1)^2 + 8\overline{\lambda}}}{2\overline{\lambda}} \\
\hat{\theta} &= \frac{(1-\overline{\lambda}) \pm \sqrt{(\overline{\lambda}-1)^2 + 8\overline{\lambda}}}{2\overline{\lambda}}
\end{aligned}
$$

Since λ>0, we have;

$$
\begin{aligned}
\hat{\theta} &= \frac{(1-\overline{\lambda}) + \sqrt{(\overline{\lambda}-1)^2 + 8\overline{\lambda}}}{2\overline{\lambda}} \\
&= \frac{(1-\overline{\lambda}) + \sqrt{1 - 2\overline{\lambda} + \overline{\lambda}^2 + 8\overline{\lambda}}}{2\overline{\lambda}} \\
&= \frac{(1-\overline{\lambda}) + \sqrt{\overline{\lambda}^2 + 6\overline{\lambda} + 1}}{2\overline{\lambda}}
\end{aligned}
$$

Since λ_i is the unobserved part, we estimate using the posterior expectation as obtained in (6) and the iteraton schemes obtained as follows:

Iterative schemes

Iterative scheme 1

Let

$$\begin{aligned} s_i &= E(\Lambda_i/X_i) \\ &= \frac{(x_i+1)(x_i+3+\theta)}{(\theta+1)(x_i+2+\theta)} \end{aligned}$$

The k^{th} iteration is:

$$\begin{aligned} s_i^{(k)} &= E(\Lambda/X_i) \\ &= \frac{(x_i+1)(x_i+3+\theta^{(k)})}{(\theta^{(k)}+1)(x_i+2+\theta^{(k)})} \end{aligned}$$

Based on the explicit solution obtained, we have the following iteration:

$$\begin{aligned} \theta^{(k+1)} &= \frac{1-\overline{s}^{(k)}+\sqrt{\overline{s}^{(k)}+6\overline{s}^{(k)}+1}}{2\overline{s}^{(k)}} \\ &\textit{where} \\ \overline{s}^{(k)} &= \sum_{i=1}^{n} \frac{s_i^{(k)}}{n} \end{aligned}$$

Iterative scheme 2

From the quadratic equation,

$$\overline{\lambda}\theta^2+(\overline{\lambda}-1)\theta-2=0$$

we have

$$\begin{aligned} \overline{\lambda}\theta^2 &= 2-(\overline{\lambda}-1)\theta \\ &= 2+(1-\overline{\lambda})\theta \\ \theta &= \frac{2+(1-\overline{\lambda})\theta}{\overline{\lambda}\theta} \end{aligned}$$

The iteration scheme now becomes:

$$\theta^{(k+1)} = \frac{2 + (1 - \overline{s}^{(k)})\theta}{\overline{s}^{(k)}\theta^{(k)}}$$

3 Poisson-Two Parameter Lindley Distribution

3.1 Introduction

In this section, we will discuss the two parameter Poisson-Lindley. The distribution is obtained by mixing the Poisson distribution with two-parameter Lindley distribution introduced by Shanker and Mishra (2013). This distribution has been found to be a better model than the one parameter Poisson-Lindley distribution for analyzing gruped mortality data, survival time and waiting time. We will discuss in particular, how the parameter is obtained through construction, obtain the posterior distribution which is useful in the EM Algorithm estimation. We will then discuss the properties which include the r^{th} moment, the probability generating function and the moment generating function. We will then estimate the parameters of the distribution using the EM Algorithm method.

3.2 Construction

For a two parameter Lindley distribution, the pdf is given by:

$$g(\lambda) = \frac{\theta^2}{\alpha\theta+1}(\alpha+\lambda)e^{-\lambda\theta};\ \alpha>0,\ \theta>0,\ \lambda>0$$

The pdf of a poisson distribution is as given in (3).
Therefore, the mixed poisson distribution is given by:

$$f(x) = \frac{\theta^2}{(1+\theta)^{x+2}}(1+\frac{\alpha+x}{\alpha\theta});\ x=0,1,2,\ldots;\ \theta>0,\ \alpha\theta>-1$$

Proof.

$$
\begin{aligned}
f(x) &= \int_0^\infty \frac{e^{-\lambda}\lambda^x}{x!}\frac{\theta^2}{\alpha\theta+1}(\alpha+\lambda)e^{-\lambda\theta}\,d\lambda \\
&= \frac{\theta^2}{(\alpha\theta+1)x!}\int_0^\infty (\lambda+\alpha)\lambda^x e^{-\lambda(1+\theta)}\,d\lambda \\
&= \frac{\theta^2}{(\alpha\theta)x!}\{\int_0^\infty \alpha\, e^{-\lambda(1+\theta)}\lambda^x + \lambda^{(x+1)}e^{-\lambda(1+\theta)}\}\,d\lambda \\
&= \frac{\alpha\theta^2}{\alpha\theta+1}\int_0^\infty \frac{\lambda^x}{x!}e^{-\lambda(1+\theta)}\,d\lambda + \frac{\theta^2}{\alpha\theta+1}\int_0^\infty \frac{\lambda^{(x+1)}}{x!}e^{-\lambda(1+\theta)}\,d\lambda \\
Let\ y &= \lambda(1+\theta) \\
\Rightarrow \lambda &= \frac{y}{1+\theta} \\
and\ \ d\lambda &= \frac{dy}{1+\theta} \\
&= \frac{\alpha\theta^2}{\alpha\theta+1}\int_0^\infty \frac{y^x}{(1+\theta)^x}\frac{1}{x!}e^{-y}\frac{dy}{1+\theta} + \frac{\theta^2}{\alpha\theta+1}\int_0^\infty \frac{y^{(x+1)}}{(1+\theta)^{(x+1)}}\frac{1}{x!}e^{-y}\frac{dy}{1+\theta} \\
&= \frac{\alpha\theta^2}{\alpha\theta+1}\int_0^\infty \frac{y^x}{x!}\frac{1}{(1+\theta)^{x+1}}e^{-y}\,dy + \frac{\theta^2}{\alpha\theta+1}\int_0^\infty \frac{y^{x+1}}{x!}\frac{1}{(1+\theta)^{x+2}}e^{-y}\,dy \\
&= \frac{\alpha\theta^2}{\alpha\theta+1}\{\frac{\Gamma(x+1)}{x!}\frac{1}{(1+\theta)^{x+1}}\} + \frac{\theta^2}{\alpha\theta+1}\{\frac{\Gamma(x+2)}{x!}\frac{1}{(1+\theta)^{x+2}}\} \\
&= \frac{\alpha\theta^2}{\alpha\theta+1}\{\frac{1}{(1+\theta)^{x+1}}\} + \frac{\theta^2}{\alpha\theta+1}\{\frac{x+1}{(1+\theta)^{x+2}}\} \\
&= \frac{\theta^2}{\alpha\theta+1}\{\frac{\alpha}{(1+\theta)^{x+1}} + \frac{x+1}{(1+\theta)^{x+2}}\} \\
&= \frac{\theta^2}{\alpha\theta+1}\{\frac{\alpha(1+\theta)+x+1}{(1+\theta)^{x+2}}\} \\
&= \frac{\theta^2}{\alpha\theta}\{\frac{\alpha\theta+\alpha+x+1}{(1+\theta)^{x+2}}\} \\
&= \frac{\theta^2}{(1+\theta)^{x+2}}\{\frac{\alpha\theta+1}{\alpha\theta+1} + \frac{x+\alpha}{\alpha\theta+1}\} \\
&= \frac{\theta^2}{(1+\theta)^{x+2}}(1+\frac{\alpha+x}{\alpha\theta});\ x=0,1,2,\ldots\ \theta{>}0\ \alpha\theta{>}{-1}
\end{aligned}
$$

□

3.3 Posterior Distribution

The pdf of a posterior distribution is as shown in (5).
Therefore,the posterior mean is given by:

$$\begin{aligned}
E(\Lambda/X) &= \frac{\int_0^\infty \lambda\ f(x,\lambda)g(\lambda)\,d\lambda}{\int_0^\infty f(x,\lambda)g(\lambda)\,d\lambda} \\
&= \frac{\int_0^\infty \frac{\lambda\ e^{-\lambda}\lambda^x}{x!}\frac{\theta^2}{\alpha\theta+1}(\alpha+\lambda)e^{-\lambda\theta}\,d\lambda}{\int_0^\infty \frac{e^{-\lambda}\lambda^x}{x!}\frac{\theta^2}{\alpha\theta+1}(\alpha+\lambda)e^{-\lambda\theta}\,d\lambda} \\
&= \frac{\int_0^\infty \lambda\ e^{-\lambda}\lambda^x(\alpha+\lambda)e^{-\lambda\theta}\,d\lambda}{\int_0^\infty e^{-\lambda}\lambda^x(\alpha+\lambda)e^{-\lambda\theta}\,d\lambda} \\
&= \frac{\int_0^\infty \lambda^{x+1}e^{-\lambda}(\alpha+\lambda)e^{-\lambda\theta}\,d\lambda}{\int_0^\infty \lambda^x e^{-\lambda}(\alpha+\lambda)e^{-\lambda\theta}\,d\lambda} \\
&= \frac{\frac{\alpha\Gamma(x+2)}{(1+\theta)^{x+2}}+\frac{\Gamma(x+3)}{(1+\theta)^{x+3}}}{\frac{\alpha\Gamma(x+1)}{(1+\theta)^{x+1}}+\frac{\Gamma(x+2)}{(1+\theta)^{x+2}}} \\
&= \frac{\frac{\alpha(x+1)\Gamma(x+1)}{(1+\theta)^{x+2}}+\frac{(x+2)(x+1)\Gamma(x+1)}{(1+\theta)^{x+3}}}{\frac{\alpha\Gamma(x+1)}{(1+\theta)^{x+1}}+\frac{(x+1)\Gamma(x+1)}{(1+\theta)^{x+2}}} \\
&= \frac{\alpha(\theta+1)(x+1)+(x+2)(x+1)}{(\theta+1)[\alpha(\theta+1)+(x+1)]} \\
&= \frac{(x+1)[\alpha(\theta+1)+(x+2)]}{(\theta+1)[\alpha(\theta+1)+(x+1)]} \\
&= \frac{(x+1)[\alpha\theta+\alpha+x+2]}{(\theta+1)[\alpha\theta+\alpha+x+1]}
\end{aligned} \tag{12}$$

3.4 Properties

3.4.1 The rth moment

The rth moment about origin of the two parameter Poisson-Lindley distribution is obtained as;

$$\mu_r' = E[E(X^{(r)}/\lambda)]$$

From the two parameter poisson lindley distribution, we have;

$$\mu_r' = \int_0^\infty [\sum_{x=0}^\infty \frac{e^{-\lambda}\lambda^x}{x!}]\frac{\theta^2}{\alpha\theta+1}(\alpha+\lambda)e^{-\lambda\theta}\,d\lambda$$

The expression under the bracket is the rth moment about origin of the Poisson distribution.
Therefore, we have:

$$
\begin{aligned}
\mu_r' &= \frac{\theta^2}{\alpha\theta+1}\int_0^\infty \lambda^r(\alpha+\lambda)e^{-\lambda\theta}\,d\lambda \\
&= \frac{\theta^2}{\alpha\theta+1}[\int_0^\infty \alpha\lambda^r e^{-\lambda\theta}\,d\lambda + \int_0^\infty \lambda^{r+1}e^{-\lambda\theta}\,d\lambda] \\
&= \frac{\theta^2}{\alpha\theta+1}[\frac{\alpha\Gamma(r+1)}{\theta^{r+1}} + \frac{\Gamma(r+2)}{\theta^{r+2}}] \\
&= \frac{\theta^2}{\alpha\theta}[\frac{r!(\alpha\theta)+r!(r+1))}{\theta^{r+2}}]
\end{aligned}
$$

Taking r=1, we obtain the mean which is given by;

$$
\begin{aligned}
\mu_1' &= \frac{\theta^2}{\alpha\theta+1}[\frac{\alpha\theta+2}{\theta^3}] \\
&= \frac{\alpha\theta+2}{\theta(\alpha\theta+1)}
\end{aligned}
$$

Taking r=2, we have;

$$
\mu_2' = \frac{\alpha\theta+2}{\theta(\alpha\theta+1)} + \frac{2(\alpha\theta+3)}{\theta^2(\alpha\theta+1)}
$$

3.4.2 Probability Generating Function

The probability generating function of the two-parameter Poisson-Lindley distribution is obtained as;

$$
\begin{aligned}
P_x(t) &= E(t^x) \\
&= \frac{\theta^2}{(\theta+1)^2}\sum_{x=0}^{n}(\frac{t}{\theta+1})^x + \frac{\theta^2}{(\theta+1)^2(\alpha\theta+1)}\sum_{x=0}^{n}(\alpha+x)(\frac{t}{\theta+1})^x \\
&= \frac{\theta^2}{(\theta+1)^2}\frac{\theta+1}{(\theta+1-t)} + \frac{\theta^2}{(\theta+1)^2(\alpha\theta+1)}\frac{(\theta+1)[\alpha(\theta+1-t)+t]}{(\theta+1-t)^2} \\
&= \frac{\alpha\theta^2(\theta+1-t)+\theta^2}{(\alpha\theta+1)(\theta+1-t)^2}
\end{aligned}
$$

3.4.3 Moment Generating Function

For the two-parameter Poisson-Lindley distribution, the mgf is obtained by;

$$\begin{aligned} M_x(t) &= E(e^{tx}) \\ &= \frac{\alpha\theta^2(\theta+1-e^t)+\theta^2}{(\alpha\theta+1)(\theta+1-e^t)} \end{aligned}$$

3.5 Estimation

3.5.1 EM Algorithm for Two-Parameter Poisson-Lindley Distribution

The pdf of a two parameter Lindley distribution is given by:

$$g(\lambda) = \frac{\theta^2}{\alpha\theta+1}(\alpha+\lambda)e^{-\lambda\theta};\ \theta{>}0,\ \lambda{>}0,\ \alpha{>}0$$

The joint distribution is as obtained in equation (9).
The likelihood function is as obtained in equation (10).
l_1 is also as obtained in equation (11).
Therefore l_2 as shown in equation (1) is obtained by:

$$\begin{aligned} l_2 &= \sum_{i=1}^{n} log\ g(\lambda_i) \\ &= \sum_{i=1}^{n} log\frac{\theta^2}{\alpha\theta+1}(\alpha+\lambda_i)e^{-\lambda_i\theta} \\ &= \sum_{i=1}^{n}[2log\theta+log(\alpha+\lambda_i)-\theta\lambda_i-log(\alpha\theta+1)] \\ &= 2nlog\theta+\sum_{i=1}^{n}log(\alpha+\lambda_i)-\theta\sum_{i=1}^{n}\lambda_i-nlog(\alpha\theta+1) \end{aligned}$$

$$\begin{aligned} 2(\alpha\theta+1)-\theta\overline{\lambda}(\alpha\theta+1)-\theta\alpha &= 0 \\ \theta^2\alpha\overline{\lambda}+\theta(\overline{\lambda}-\alpha)-2 &= 0 \end{aligned}$$

Therefore:

$$\frac{\partial}{\partial\theta}l_2 = \frac{2n}{\theta} - \sum_{i=1}^{n}\lambda_i - \frac{n\alpha}{\alpha\theta+1}$$

$$\frac{2n}{\theta} - \sum_{i=1}^{n}\lambda_i - \frac{n\alpha}{\alpha\theta+1} = 0$$

$$\frac{2}{\theta} - \overline{\lambda} - \frac{\alpha}{\alpha\theta+1} = 0$$

$$2 - \theta\overline{\lambda} - \frac{\theta\alpha}{\alpha\theta+1} = 0$$

$$2(\alpha\theta+1) - \theta\overline{\lambda}(\alpha\theta+1) - \alpha\theta = 0$$

$$2\alpha\theta + 2 - \theta^2\alpha\overline{\lambda} - \theta\overline{\lambda} - \alpha\theta = 0$$

$$\alpha\theta + 2 - \theta^2\alpha\overline{\lambda} - \theta\overline{\lambda} = 0$$

$$\theta^2\alpha\overline{\lambda} + \theta(\overline{\lambda} - \alpha) - 2 = 0$$

$$\hat{\theta} = \frac{-(\overline{\lambda}-\alpha) \pm \sqrt{(\overline{\lambda}-\alpha)^2 + 8\alpha\overline{\lambda}}}{2\alpha\overline{\lambda}}$$

Since λ>0, α>0, we take:

$$\hat{\theta} = \frac{(\alpha-\overline{\lambda}) + \sqrt{\overline{\lambda}^2 + \alpha^2 + 6\alpha\overline{\lambda}}}{2\alpha\overline{\lambda}}$$

Next, we differentiate with respect to α:

$$\frac{\partial}{\partial\alpha} = \sum_{i=1}^{n}\frac{1}{\alpha+\lambda_i} - \frac{n\theta}{\alpha\theta+1}$$

$$\sum_{i=1}^{n}\frac{1}{\alpha+\lambda_i} - \frac{n\theta}{\alpha\theta+1} = 0$$

$$\sum_{i=1}^{n}\frac{1}{\alpha+\lambda_i} = \frac{n\theta}{\alpha\theta+1}$$

$$\frac{n\theta}{\sum_{i=1}^{n}(\alpha+\lambda_i)} = \alpha\theta+1$$

$$\frac{n\theta}{\sum_{i=1}^{n}(\alpha+\lambda_i)} - 1 = \alpha\theta$$

$$\frac{n}{\sum_{i=1}^{n}(\alpha+\lambda_i)} - \frac{1}{\theta} = \alpha$$

$$\hat{\alpha} = \frac{n}{\sum_{i=1}^{n}(\alpha+\lambda_i)} - \frac{1}{\theta}$$

Since λ_i is the unobserved part, we estimate using the posterior expectation as obtained in (12) and the obtain the iterative schemes as follows:

Iterative schemes

Iterative scheme 1

Let

$$\begin{aligned} s_i &= E(\Lambda_i/X) \\ &= \frac{(x_i+1)(\alpha\theta+\alpha+x_i+2)}{(\theta+1)(\alpha\theta+\alpha+x_i+1)} \end{aligned}$$

The k^{th} iteration is:

$$\begin{aligned} s_i^{(k)} &= E(\Lambda_i/X) \\ &= \frac{(x_i+1)(\alpha\theta^{(k)}+\alpha+x_i+2)}{(\theta^{(k)}+1)(\alpha\theta^{(k)}+\alpha+x_i+1)} \end{aligned}$$

Based on the explicit solutions obtained, we have the following iterations:

$$\begin{aligned} \theta^{(k+1)} &= \frac{(\alpha-\overline{s}^{(k)})+\sqrt{\overline{s}^{2(k)}+\alpha^2+6\alpha\overline{s}^{(k)}}}{2\alpha\overline{s}^{(k)}} \\ \alpha^{(k+1)} &= \frac{n}{\sum_{i=1}^{n}(\alpha^{(k)}+s_i^{(k)})}-\frac{1}{\theta^{(k+1)}} \end{aligned}$$

Iterative scheme 2

From the quadratic equation:

$$\theta^2\alpha\overline{\lambda}+\theta(\overline{\lambda}-\alpha)-2=0$$

We have:

$$\begin{aligned} \theta^2\alpha\overline{\lambda} &= 2-\theta(\overline{\lambda}-\alpha) \\ &= 2+\theta(\alpha-\overline{\lambda}) \\ \theta &= \frac{2+\theta(\alpha-\overline{\lambda})}{\alpha\overline{\lambda}\theta} \end{aligned}$$

The iteration scheme now becomes;

$$\theta^{(k+1)} = \frac{2 + (\alpha - \bar{s}^{(k)})\theta}{\bar{s}^{(k)}\alpha\theta^{(k)}}$$

4 Poisson-Two Parameter Lindley Distribution

4.1 Introduction

In this section, we will discuss another two parameter Poisson-Lindley distribution which is obtained by mixing a poisson distribution with another two parameter Lindley distribution which was introduced by Shanker (2013). We will do the construction, find the posterior distribution, discuss the various properties which include the cumulative distribution function and the r^{th} moment. We will then estimate the parameters of the distribution using the EM Algorithm method.

4.2 Construction

From the two-parameter Lindley distribution which is given by:

$$g(\lambda) = \frac{\theta^2}{\theta+\beta}(1+\beta\lambda)e^{-\lambda\theta} : \ \theta{>}0, \ \beta{>}0, \ \lambda{>}0$$

The mixed Poisson-Lindley distribution is given by:

$$f(x) = \frac{\theta^2}{(1+\theta)^{x+2}}[1+\frac{\beta(x+1)}{\theta+\beta}]$$

Proof.

$$
\begin{aligned}
f(x) &= \int_0^\infty \frac{e^{-\lambda}\lambda^x}{x!} g(\lambda)\,d\lambda \\
&= \int_0^\infty \frac{e^{-\lambda}\lambda^x}{x!}\frac{\theta^2}{\theta+\beta}(1+\beta\lambda)e^{-\lambda\theta}\,d\lambda \\
&= \frac{\theta^2}{(\theta+\beta)x!}[\int_0^\infty e^{-\lambda(1+\theta)\lambda^x d\lambda} + \beta\int_0^\infty e^{-\lambda(1+\theta)}\lambda^{x+1}\,d\lambda] \\
&= \frac{\theta^2}{(\theta+\beta)x!}\{\frac{\Gamma(x+1)}{(1+\theta)^{x+1}} + \frac{\beta\Gamma(x+2)}{(1+\theta)^{x+2}}\} \\
&= \frac{\theta^2}{(\theta+\beta)x!}\frac{\Gamma(x+1)}{(1+\theta)^{x+1}}\{1+\frac{\beta(x+1)}{1+\theta}\} \\
&= \frac{\theta^2}{(\theta+\beta)(1+\theta)^{x+1}}\{1+\frac{\beta(x+1)}{1+\theta}\} \\
&= \frac{\theta}{(1+\theta)^{x+2}}\{1+\frac{\beta(x+1)}{\theta+\beta}\}, x=0,1,2,\ldots
\end{aligned}
$$

□

4.3 Posterior Distribution

The pdf of a posterior distribution is given by the formula in (5).
Therefore, the posterior mean for two-paremeter Poisson-Lindley distribution is given by:

$$
\begin{aligned}
E(\Lambda/X) &= \frac{\int_0^\infty \lambda\ f(x/\lambda)g(\lambda)\,d\lambda}{\int_0^\infty f(x/\lambda)g(\lambda)\,d\lambda} \\
&= \frac{\int_0^\infty \lambda \frac{e^{-\lambda}\lambda^x}{x!}\frac{\theta^2}{\theta+\beta}(1+\beta\lambda)e^{-\lambda\theta}\,d\lambda}{\int_0^\infty \frac{e^{-\lambda}\lambda^x}{x!}\frac{\theta^2}{\theta+\beta}(1+\beta\lambda)e^{-\lambda\theta}\,d\lambda} \\
&= \frac{\int_0^\infty \lambda\ e^{-\lambda}\lambda^x(1+\beta\lambda)e^{-\lambda\theta}\,d\lambda}{\int_0^\infty e^{-\lambda}\lambda^x(1+\beta\lambda)e^{-\lambda\theta}\,d\lambda}
\end{aligned}
$$

$$
\begin{aligned}
E(\Lambda/X) &= \frac{\int_0^\infty \lambda^{x+1}e^{-\lambda(1+\theta)}(1+\beta\lambda)\,d\lambda}{\int_0^\infty \lambda^x e^{-\lambda(1+\theta)}(1+\beta\lambda)\,d\lambda} \qquad (13) \\
&= \frac{\frac{\Gamma(x+2)}{(\theta+1)^{x+2}} + \frac{\beta\Gamma(x+3)}{(\theta+1)^{x+3}}}{\frac{\Gamma(x+1)}{(\theta+1)^{x+1}} + \frac{\beta\Gamma(x+2)}{(\theta+1)^{x+2}}} \\
&= \frac{(\theta+1)(x+1)+\beta[(x+1)(x+2)]}{(\theta+1)[(\theta+1)+\beta(x+1)]} \\
&= \frac{(x+1)[(\theta+1)+\beta(x+2)]}{(\theta+1)[(\theta+1)+\beta(x+1)]}
\end{aligned}
$$

4.4 Properties

4.4.1 Cumulative Distribution Function

The cdf of the distribution is given by:

$$\begin{aligned} F(x) &= \sum_{n=0}^{x} \frac{\theta^2}{\theta+\beta} \frac{1+\theta+n\beta+\beta}{(1+\theta)^{n+2}} \\ &= \frac{(\beta+\theta)(\theta+1)^{x+2} - (2\beta\theta+\beta+\theta^2+\theta+\beta\theta\, x)}{(\theta+1)^{x+2}(\theta+\beta)} \end{aligned}$$

4.4.2 The r^{th} Moment

The r^{th} moment about the origin of the distribution is given by:

$$\begin{aligned} \mu_r' &= E[E(X^r/\lambda)] \\ &= \int_0^\infty \lambda^r \frac{\theta^2(1+\beta\lambda)e^{-\lambda\theta}}{\theta+\beta} d\lambda \\ &= \frac{\theta^2}{\theta+\beta}[\int_0^\infty e^{-\lambda\theta}\lambda^r d\lambda + \beta \int_0^\infty e^{-\lambda\theta}\lambda^{r+1} d\lambda] \\ &= \frac{\theta^2}{\theta+\beta}[\frac{\Gamma(r+1)}{\theta^{r+1}} + \frac{\beta\Gamma(r+2)}{\theta^{r+2}}] \\ &= \frac{\theta^2}{\theta+\beta}\frac{\Gamma(x+r)}{\theta^{r+1}}[1+\frac{\beta(r+1)}{\theta}] \end{aligned}$$

Taking r=1, the mean is given by:

$$\begin{aligned} \mu_1' &= \frac{2\beta+\theta}{\theta(\beta+\theta)} \\ \mu_2' &= \frac{2\beta(\theta+3)+\theta(\theta+2)}{\theta^2(\theta+\beta)} \end{aligned}$$

4.5 Estimation

4.5.1 EM Algorithm for Two-Parameter Poisson-Linldey Distribution

The joint distribution for a continuous Poisson mixture is as shown in equation (9). Next, we obtain the log likelihood function which is already obtained in equation (10).

From (1), we obtain l_1, which is the log likelihood function of the Poisson distribution and is obtained in (11).
Next, we obtain l_2 which is the log likelihood function of the two-parameter Linldey distribution.
The pdf of two parameter Lindley distribution is given by:

$$g(\lambda) = \frac{\theta^2}{\theta+\beta}(1+\beta\lambda)e^{-\lambda\theta}$$

From the pdf above and from (1), l_2 is obtained by:

$$\begin{aligned}
l_2 &= \sum_{i=1}^{n} log\ g(\lambda_i) \\
&= \sum_{i=1}^{n} log \frac{\theta^2}{\theta+\beta}(1+\beta\lambda_i)e^{-\lambda_i\theta} \\
&= \sum_{i=1}^{n} [2\ log\ \theta + log(1+\beta\lambda_i) - \theta\lambda_i - log(\theta+\beta)] \\
&= 2n\ log\ \theta + \sum_{i=1}^{n} log(1+\beta\lambda_i) - \theta\sum_{i=1}^{n}\lambda_i - n\ log(\theta+\beta)
\end{aligned}$$

Therefore

$$\frac{\partial}{\partial\theta}l_2 = \frac{2n}{\theta} - \sum_{i=1}^{n}\lambda_i - \frac{n}{\theta+\beta}$$

$$\begin{aligned}
\frac{2n}{\theta} - \sum_{i=1}^{n}\lambda_i - \frac{n}{\theta+\beta} &= 0 \\
\frac{2}{\theta} - \overline{\lambda} - \frac{1}{\theta+\beta} &= 0 \\
2 - \theta\overline{\lambda} - \frac{\theta}{\theta+\beta} &= 0 \\
2(\theta+\beta) - \theta\overline{\lambda}(\theta+\beta) - \theta &= 0 \\
2\theta + 2\beta - \theta^2\overline{\lambda} - \theta\overline{\lambda}\beta - \theta &= 0 \\
\theta^2\overline{\lambda} + \theta\beta\overline{\lambda} - \theta - 2\beta &= 0 \\
\theta^2\overline{\lambda} + (\beta\overline{\lambda} - 1)\theta - 2\beta &= 0
\end{aligned}$$

$$\begin{aligned}
\hat{\theta} &= \frac{-(\beta\overline{\lambda}-1) \pm \sqrt{(\beta\overline{\lambda}-1)^2 + 8\beta\overline{\lambda}}}{2\overline{\lambda}} \\
&= \frac{(1-\beta\overline{\lambda}) \pm \sqrt{\beta^2\overline{\lambda}^2 + 6\beta\overline{\lambda} + 1}}{2\overline{\lambda}}
\end{aligned}$$

Since λ>0, β>0, we take;

$$\hat{\theta} = \frac{(1-\beta\overline{\lambda})\sqrt{\beta^2\overline{\lambda}^2 + 6\beta\overline{\lambda} + 1}}{2\overline{\lambda}}$$

We therefore estmate the value of β,

$$\frac{\partial}{\partial\beta}l_2 = \sum_{i=1}^{n}\frac{\lambda_i}{1+\beta\lambda_i} - \frac{n}{\theta+\beta}$$

Therefore,

$$\sum_{i=1}^{n}\frac{\lambda_i}{1+\beta\lambda_i}-\frac{n}{\theta+\beta}=0$$
$$\frac{n}{\theta+\beta}=\sum_{i=1}^{n}\frac{\lambda_i}{1+\beta\lambda_i}$$
$$\theta+\beta=\frac{n}{\sum_{i=1}^{n}\frac{\lambda_i}{1+\beta\lambda_i}}$$
$$\hat{\beta}=\frac{n}{\sum_{i=1}^{n}\frac{\lambda_i}{1+\beta\lambda_i}}-\theta$$

Since $\lambda_i$is the unobserved part, we estimate using the posterior expectation as obtained in (13) and the iterative schemes obtained as follows:

Iterative Schemes

Iterative Scheme 1

Let

$$s_i=E(\Lambda_i/X_i)$$
$$=\frac{(x_i+1)[(\theta+1)+\beta(x_i+2)]}{(\theta+1)[(\theta+1)+\beta(x_i+1)]}$$

The k^{th} iteration is;

$$s_i^{(k)}=E(\Lambda_i/X_i)\qquad=\frac{(x_i+1)[(\theta^{(k)}+1)+\beta(x_i+2)]}{(\theta^{(k)}+1)[(\theta^{(k)}+1)+\beta(x_i+1)]}$$

Based on the explicit solutions obtained:

$$\theta^{(k+1)}=\frac{(1-\beta\overline{s}^{(k)})+\sqrt{\beta^2\overline{s}^{2(k)}+6\beta\overline{s}^{(k)}+1}}{2\overline{s}^{(k)}}$$
$$\beta^{(k+1)}=\frac{n}{\sum_{i=1}^{n}\frac{\overline{s}^{(k)}}{1+\beta^{(k)}\overline{s}^{(k)}}}-\theta^{(k+1)}$$

5 Poisson Quasi-Lindley Distribution

5.1 Introduction

In this chapter, we will discuss a Quasi Poisson-Lindley distribution, of which the Sankaran (1970) Poisson-Lindley distribution is a particular case, which has been introduced by compounding Poisson distribution with the Quasi-Lindley distribution introduced by Shanker and Mishra (2013). The construction on how the distribution is obtained is shown. The posterior distribution is obtained which is useful in estimation of the parameters using the EM Algorithm. The properties have een discussed which include the r^{th} moment, the cumulative distribution function and the moment generating function. We wii then estimate the parameters of the distribution using the EM Algorthm method.

5.2 Construction

The pdf of a poisson Quasi-Lindley distribution is as follows:

$$f(x) = \int_0^\infty \frac{e^{-\lambda}\lambda^x}{x!} g(\lambda)\, d\lambda$$
$$g(\lambda) = \frac{\theta}{\alpha+1}(\alpha+\theta\lambda)e^{-\lambda\theta}\, d\lambda$$
$$f(x) = \frac{\theta}{\alpha+1}[\frac{\alpha+\alpha\theta+\theta\, x+\theta}{(1+\theta)^{x+2}}], x = 0, 1, \ldots,\ \theta>0,\ \alpha>-1$$

Proof.

$$
\begin{aligned}
f(x) &= \int_0^\infty \frac{e^{-\lambda}\lambda^x}{x!} g(\lambda)\,d\lambda \\
&= \int_0^\infty \frac{e^{-\lambda}\lambda^x}{x!} \frac{\theta}{\alpha+1}(\alpha+\theta\lambda)e^{-\lambda\theta}\,d\lambda \\
&= \frac{\theta}{(\alpha+1)x!}[\int_0^\infty \lambda^x(\alpha+\theta\lambda)e^{-\lambda(\theta+1)}] \\
&= \frac{\theta}{(\alpha+1)x!}[\int_0^\infty \alpha\lambda^x e^{-\lambda(\theta+1)}\,d\lambda + \int_0^\infty \theta\lambda^{x+1}e^{-\lambda(\theta+1)}] \\
&= \frac{\theta}{(\alpha+1)x!}[\frac{\alpha\Gamma(x+1)}{(1+\theta)^{x+1}} + \frac{\theta\Gamma(x+2)}{(\theta+1)^{x+2}}] \\
&= \frac{\theta}{\alpha+1}[\frac{\alpha}{(1+\theta)^{x+1}} + \frac{\theta(x+1)}{(1+\theta)^{x+2}}] \\
&= \frac{\theta}{\alpha+1}[\frac{\alpha+\alpha\theta+\theta\,x+\theta}{(1+\theta)^{x+2}}], x = 0,1,2,\ldots,\ \theta>0,\ \alpha>-1
\end{aligned}
$$

□

5.3 Posterior Distribution

The pdf of a posterior distribution is as given in equation (5).
The posterior mean is therefore given by:

$$
\begin{aligned}
E(\Lambda/X) &= \frac{\int_0^\infty \lambda\, f(x/\lambda)g(\lambda)\,d\lambda}{\int_0^\infty f(x/\lambda)g(\lambda)\,d\lambda} \\
&= \frac{\int_0^\infty \lambda \frac{e^{-\lambda}\lambda^x}{x!}\frac{\theta}{\alpha+1}(\alpha+\theta\lambda)e^{-\lambda\theta}\,d\lambda}{\int_0^\infty \frac{e^{-\lambda}\lambda^x}{x!}\frac{\theta}{\alpha+1}(\alpha+\theta\lambda)e^{-\lambda\theta}\,d\lambda} \\
&= \frac{\int_0^\infty \lambda\, e^{-\lambda}\lambda^x(\alpha+\theta\lambda)e^{-\lambda\theta}\,d\lambda}{\int_0^\infty e^{-\lambda}\lambda^x(\alpha+\theta\lambda)e^{-\lambda\theta}\,d\lambda} \\
&= \frac{\int_0^\infty \lambda^{x+1}e^{-\lambda}(\alpha+\theta\lambda)e^{-\lambda\theta}\,d\lambda}{\int_0^\infty \lambda^x e^{-\lambda}(\alpha+\theta\lambda)e^{-\lambda\theta}\,d\lambda} \\
&= \frac{\frac{\alpha\Gamma(x+2)}{(\theta+1)^{x+2}} + \frac{\theta\Gamma(x+3)}{(\theta+1)^{x+3}}}{\frac{\alpha\Gamma(x+1)}{(\theta+1)^{x+1}} + \frac{\theta\Gamma(x+2)}{(\theta+1)^{x+2}}} \\
&= \frac{\alpha[(\theta+1)(x+1)]+\theta[(x+2)(x+1)]}{(\theta+1)[\alpha(\theta+1)+\theta(x+1)]} \\
&= \frac{(x+1)[\alpha(\theta+1)+\theta(x+2)]}{(\theta+1)[\alpha(\theta+1)+\theta(x+1)]}
\end{aligned}
\tag{14}
$$

5.4 Properties

5.4.1 The r^{th} moment

The rth moment about origin of the Quasi-Lindley distribution is given by:

$$\mu_r' = E(X^r)$$
$$= \frac{\theta}{\alpha+1} \frac{r!\theta(\alpha+r+1)}{\theta^{r+2}}$$

Proof.

$$
\begin{aligned}
E(X^r) &= \int_0^\infty \lambda^r f(\lambda,\theta)\, d\lambda \\
&= \int_0^\infty \lambda^r \frac{\theta}{\alpha+1}(\alpha+\theta\lambda)e^{-\lambda\theta}\, d\lambda \\
&= \frac{\theta}{\alpha+1}[\int_0^\infty \alpha\lambda^r e^{-\lambda\theta}\, d\lambda + \int_0^\infty \theta\lambda^{r+1}e^{-\lambda\theta}\, d\lambda] \\
&= \frac{\theta}{\alpha+1}[\frac{\alpha\Gamma(r+1)}{\theta^{r+1}} + \frac{\theta\Gamma(r+2)}{\theta^{r+2}}] \\
&= \frac{\theta}{\alpha+1}[\frac{\theta\alpha\, r! + \theta(r+1)r!}{\theta^{r+2}}] \\
&= \frac{\theta}{\alpha+1}[\frac{r!\theta(\alpha+r+1)}{\theta^{r+2}}]
\end{aligned}
$$

□

5.4.2 Cumulative Distribution Function

The cumulative distribution function (CDF) of the poisson Quasi-Lindley distribution is given by:

$$F(x) = 1 - \frac{\alpha+2\theta+\alpha\theta+x\theta+1}{(1+\alpha)(1+\theta)^{x+2}}$$

5.4.3 Momemnt Generating Function

The moment generating function (mgf) of poisson Quasi-Lindley distribution sis given by;

$$
\begin{aligned}
M_x(t) &= E(e^{tx}) \\
&= \frac{\alpha\theta\ e^t + \theta^2}{(1+\alpha)(e^t + \theta\ e^t - e^{2t})}
\end{aligned}
$$

5.5 Estimation

5.5.1 EM Algorithm for Poisson Quasi-Lindley Distribution

First, we write the joint distribution for a continuous Poisson mixture which is already obtained in (9).
Next, we obtain the log likelihood function of a Poisson-Lindley distribution which is already obtained in (10).
From (1), l_1 is the log likelihood function of the Poisson distribution which is already obtained in (11).
Again, from (1), l_2 is the log likelihood function of the Lindley distribution which in this case is the two parameter Linldey distribution is obtained as follows:

$$
\begin{aligned}
l_2 &= \sum_{i=1}^{n} log\ g(\lambda_i) \\
&= \sum_{i=1}^{n} log \frac{\theta}{\alpha+1}(\alpha+\lambda_i\theta)e^{-\lambda\theta} \\
&= \sum_{i=1}^{n} [log\ \theta + log(\alpha+\theta\lambda_i) - log(\alpha+1) - \theta\lambda_i] \\
&= n\ log\ \theta + \sum_{i=1}^{n} log(\alpha+\theta\lambda_i) - \theta\sum_{i=1}^{n}\lambda_i - n\ log(\alpha+1) \\
\frac{\partial}{\partial\theta} l_2 &= \frac{n}{\theta} + \sum_{i=1}^{n}\frac{\lambda_i}{\alpha+\theta\lambda_i} - \sum_{i=1}^{n}\lambda_i \\
0 &= \frac{n}{\theta} + \sum_{i=1}^{n}\frac{\lambda_i}{\alpha+\theta\lambda_i} - \sum_{i=1}^{n}\lambda_i \\
\frac{n}{\theta} &= \sum_{i=1}^{n}\lambda_i - \sum_{i=1}^{n}\frac{\lambda_i}{\alpha+\theta\lambda_i} \\
\theta &= \frac{n}{\sum_{i=1}^{n}\lambda_i - \sum_{i=1}^{n}\frac{\lambda_i}{\alpha+\theta\lambda_i}} \\
\hat{\theta} &= \frac{1}{\overline{\lambda} - \frac{1}{n}\sum_{i=1}^{n}\frac{1}{\theta+\frac{\alpha}{\lambda_i}}} \\
\frac{\partial}{\partial\alpha} l_2 &= \sum_{i=1}^{n}\frac{1}{\alpha+\theta\lambda_i} - \frac{n}{\alpha+1} \\
\frac{n}{\alpha+1} &= \sum_{i=1}^{n}\frac{1}{\alpha+\theta\lambda_i} \\
\alpha+1 &= \frac{n}{\sum_{i=1}^{n}\frac{1}{\alpha+\theta\lambda_i}} \\
\hat{\alpha} &= \frac{n}{\sum_{i=1}^{n}\frac{1}{\alpha+\theta\lambda_i}} - 1
\end{aligned}
$$

Since λ_i is the unobserved part, we estimate using the posterior expectation as obtained in (14) and the iterative scheme obtained as follows:

Iterative Schemes

Let

$$
\begin{aligned}
s_i &= E(\Lambda_i/X_i) \\
&= \frac{(x_i+1)[\alpha(\theta+1)+\theta(x_i+2)]}{(\theta+1)[\alpha(\theta+1)+\theta(x_i+1)]}
\end{aligned}
$$

The kth iteration is:

$$
\begin{aligned}
s_i^{(k)} &= E(\Lambda/X_i) \\
&= \frac{(x_i+1)[\alpha(\theta^{(k)}+1)+\theta^{(k)}(x_i+2)]}{(\theta^{(k)}+1)[\alpha(\theta^{(k)}+1)+\theta^{(k)}(x_i+1)]}
\end{aligned}
$$

Based on the explicit solution obtained, we have the following iteration:

$$
\begin{aligned}
\theta^{(k+1)} &= \frac{n}{\sum_{i=1}^{n} s_i^{(k)} - \sum_{i=1}^{n} \frac{s_i^{(k)}}{\alpha^{(k)}+\theta^{(K)} s_i^{(k)}}} \\
\alpha^{(k+1)} &= \frac{n}{\sum_{i=1}^{n} \alpha^{(k)} + \theta^{(k+1)} s_i^{(k)}}
\end{aligned}
$$

6 Poisson -Three Parameter Lindley Distribution

6.1 Introduction

In this chapter, we shall discuss a mixed Poisson distribution which is obtained by mixing the Poisson distribution with a three parameter Lindley distribution. We will show how the distribution is obtained by construction. We will also obtain the posterior distribution which is useful in estimation using the EM Algorithm. We will then discuss the various properties which include; the cumulative distribution function, the probability distribution function, the moment generating function, the characteristic function and the r^{th} moment. We will then estimate the parameters using the EM Algorithm method.

6.2 Construction

The pdf of a poisson mixture is as given in (3).

$$g(\lambda) = \frac{\theta^2}{\theta\alpha+\beta}(\alpha+\beta\lambda)e^{-\lambda\theta}$$

Therefore;

$$f(x) = \frac{\theta^2}{(1+\theta)^{x+2}}\{\frac{\alpha+\alpha\theta+\beta\, x+\beta}{\theta\alpha+\beta}\}, \theta>0,\ \alpha>0,\ \beta>0$$

Proof.

$$
\begin{aligned}
f(x) &= \int_0^\infty \frac{e^{-\lambda}\lambda^x}{x!} g(\lambda)\, d\lambda \\
&= \int_0^\infty \frac{e^{-\lambda}\lambda^x}{x!} \frac{\theta^2}{\theta\alpha+\beta} (\alpha+\beta\lambda) e^{-\lambda\theta}\, d\lambda \\
&= \frac{\theta^2}{(\theta\alpha+\beta)x!} \int_0^\infty (\alpha+\beta\lambda)\lambda^x e^{-\lambda(1+\theta)}\, d\lambda \\
&= \frac{\theta^2}{(\theta\alpha+\beta)x!} [\int_0^\infty \alpha\lambda^x e^{-\lambda(1+\theta)}\, d\lambda + \beta\lambda^{x+1} e^{-\lambda(1+\theta)}\, d\lambda] \\
&= \frac{\theta^2}{(\theta\alpha+\beta)x!} [\frac{\alpha\Gamma(x+1)}{(1+\theta)^{x+1}} + \frac{\beta\Gamma(x+2)}{(1+\theta)^{x+2}}] \\
&= \frac{\theta^2}{\theta\alpha+\beta} [\frac{\alpha\Gamma(x+1)}{x!(1+\theta)^{x+1}} + \frac{\beta(x+1)\Gamma(x+1)}{x!(1+\theta)^{x+2}}] \\
&= \frac{\theta^2}{\theta\alpha+\beta} [\frac{\alpha}{(1+\theta)^{x+1}} + \frac{\beta(x+1)}{(1+\theta)^{x+2}}] \\
&= \frac{\theta^2}{\theta\alpha+\beta} \{\frac{\alpha(1+\theta)+\beta(x+1)}{(1+\theta)^{x+2}}\} \\
&= \frac{\theta^2}{(1+\theta)^{x+2}} \{\frac{\alpha+\alpha\theta+\beta\, x+\beta}{\theta\alpha+\beta}\}, x = 0,1,2\ldots\ \theta>0,\ \alpha>0,\ \beta>0
\end{aligned}
$$

$\square$

6.3 Posterior Distribution

The pdf of a posterior distribution is as given 1n (5).
Therefore, the posterior mean is given by;

$$
\begin{aligned}
E(\Lambda/X) &= \frac{\int_0^\infty \lambda\, f(x/\lambda)g(\lambda)\,d\lambda}{\int_0^\infty f(x/\lambda)g(\lambda)\,d\lambda} \\
&= \frac{\int_0^\infty \frac{e^{-\lambda}\lambda^x}{x!}\frac{\theta^2}{\theta\alpha+\beta}(\alpha+\beta\lambda)e^{-\lambda\theta}\,d\lambda}{\int_0^\infty \frac{e^{-\lambda}\lambda^x}{x!}\frac{\theta^2}{\theta\alpha+\beta}(\alpha+\beta\lambda)e^{-\lambda\theta}\,d\lambda} \\
&= \frac{\int_0^\infty \lambda^{x+1}(\alpha+\beta\lambda)e^{-\lambda(1+\theta)}\,d\lambda}{\int_0^\infty \lambda^{x}(\alpha+\beta\lambda)e^{-\lambda(1+\theta)}\,d\lambda} \\
&= \frac{\int_0^\infty \alpha\lambda^{x+1}e^{-\lambda(1+\theta)}\,d\lambda + \int_0^\infty \beta\lambda^{x+2}e^{-\lambda(1+\theta)}\,d\lambda}{\int_0^\infty \alpha\lambda^{x}e^{-\lambda(1+\theta)}\,d\lambda + \int_0^\infty \beta\lambda^{x+1}e^{-\lambda(1+\theta)}\,d\lambda} \\
&= \frac{\frac{\alpha\Gamma(x+2)}{(1+\theta)^{x+2}} + \frac{\beta\Gamma(x+3)}{(1+\theta)^{x+3}}}{\frac{\alpha\Gamma(x+1)}{(1+\theta)^{x+1}} + \frac{\beta\Gamma(x+2)}{(1+\theta)^{x+2}}} \\
&= \frac{\frac{\alpha(x+1)\Gamma(x+1)}{(1+\theta)^{x+2}} + \frac{(x+2)(x+1)\beta\Gamma(x+1)}{(1+\theta)^{x+3}}}{\frac{\alpha\Gamma(x+1)}{(1+\theta)^{x+1}} + \frac{(x+1)\beta\Gamma(x+1)}{(1+\theta)^{x+2}}} \\
&= \frac{(x+1)\alpha(\theta+1)+\beta(x+1)(x+2)}{(\theta+1)[\alpha(\theta+1)+\beta(x+1)]} \\
&= \frac{(x+1)[\alpha(\theta+1)+\beta(x+2)]}{(\theta+1)[\alpha(\theta+1)+\beta(x+1)]}
\end{aligned}
\tag{15}
$$

6.4 Properties

6.4.1 Cumulative Distribution Function

Let X be a random variable which follows a three-parameter Poisson-Lindley distribution; then the cumulative distribution function is given by:

$$
F_X(x;\alpha,\beta,\theta) = \frac{[\theta\{\alpha(\theta+1)+\beta\}\{(\theta+1)^{x+1}-1\}+\beta(\theta+1)\{(\theta+1)^{x}-1\}-x\beta\theta]}{(\theta+1)^{x+2}(\theta\alpha+\beta)}
$$

Proof.

$$\begin{aligned}F_X(x;\alpha,\beta,\theta) &= \sum_{n=0}^{x} \frac{\theta^2}{(\theta+1)^{n+2}(1+\frac{\alpha+\beta\ n}{\theta\alpha+\beta})} \\ &= \frac{[\theta\{\alpha(\theta+1)+\beta\}\{(\theta+1)^{x+1}-1\}+\beta(\theta+1)\{(\theta+1)^x-1\}-x\beta\theta]}{(\theta+1)^{x+2}(\theta\alpha+\beta)}\end{aligned}$$

□

6.4.2 Probability Generating Function

Let X be a three-parameter Poisson-Lindley variable with parameters α, β and θ; then the probability generating function of X denoted by $P_x(t)$ is given by:

$$P_x(t) = \frac{\alpha(\theta+1-t)\theta^2+\theta^2\beta}{(\alpha\theta+\beta)(\theta+1-t)^2}$$

Proof.

$$\begin{aligned}P_x(t) &= E(t^x) \\ &= \sum_{x=0}^{\infty} t^x \frac{\theta^2}{(\theta+1)^{x+2}}[1+\frac{\alpha+x\beta}{\theta\alpha+\beta}] \\ &= \frac{\theta^2}{(\theta+1)^2}\sum_{x=0}^{\infty}(\frac{t}{\theta+1})^x + \frac{\alpha\theta^2}{(\theta+1)^2(\theta\alpha+\beta)}\sum_{x=0}^{\infty}(\frac{t}{\theta+1})^x + \frac{\beta\theta^2}{(\theta+1)^2(\theta\alpha+\beta)}\sum_{x=0}^{\infty}x(\frac{t}{\theta+1})^x \\ &= \frac{\alpha(\theta+1-t)\theta^2+\theta^2\beta}{(\alpha\theta+\beta)(\theta+1-t)^2}\end{aligned}$$

□

6.4.3 Moment Generating Function

The moment generating function is obtained by setting $t = e^t$ in the expression for the probability generating function and is therefore given by:

$$M_x(t) = \frac{\alpha(\theta+1-e^t)\theta^2+\theta^2\beta}{(\alpha\theta+\beta)(\theta+1-e^t)^2}$$

6.4.4 Characteristic Function

The characteristic function is obtained by setting $t = e^{it}$ in the expression for the probability generating function and is therefore given by:

$$\phi_x(t) = \frac{\alpha(\theta+1-e^{it})\theta^2+\theta^2\beta}{(\alpha\theta+\beta)(\theta+1-e^{it})^2}$$

6.4.5 The rth Moment

The rth moment of the three parameter Lindley distribution is given by;

$$\begin{aligned}
\mu_r' &= E(X^r) \\
&= \int_0^\infty \lambda^r f(\theta,\alpha,\beta)\,d\lambda \\
&= \int_0^\infty \lambda^r \frac{\theta^2}{\theta\alpha+\beta}(\alpha+\beta\lambda)e^{-\lambda\theta}\,d\lambda \\
&= \frac{\theta^2}{\theta\alpha+\beta}[\int_o^\infty \alpha\lambda^r e^{-\lambda\theta}\,d\lambda + \int_0^\infty \beta\lambda^{r+1}e^{-\lambda\theta}\,d\lambda] \\
&= \frac{\theta^2}{\theta\alpha+\beta}[\frac{\alpha\Gamma(r+1)}{\theta^{r+1}} + \frac{\beta\Gamma(r+2)}{\theta^{r+2}}] \\
&= \frac{\theta^2}{\theta\alpha+\beta}[\frac{r!\alpha}{\theta^{r+1}} + \frac{\beta(r+1)r!}{\theta^{r+2}}] \\
&= \frac{\theta^2}{\theta\alpha+\beta}[\frac{r!\theta\alpha+\beta(r+1)r!}{\theta^{r+2}}] \\
&= \frac{\theta^2}{\theta\alpha+\beta}[\frac{r!\{\theta\alpha+\beta(r+1)\}}{\theta^{r+2}}]
\end{aligned}$$

Taking $r = 1$, we have:

$$\begin{aligned}
\mu_1' &= \frac{\theta\alpha+2\beta}{\theta(\theta\alpha+\beta)} \\
\mu_2' &= \frac{\theta\alpha+2\beta}{\theta(\theta\alpha+\beta)} + \frac{2(\theta\alpha+3\beta)}{\theta^2(\theta\alpha+\beta)} \\
\mu_3' &= \frac{\theta\alpha+2\beta}{\theta(\theta\alpha+\beta)} + \frac{6(\theta\alpha+3\beta)}{\theta^2(\theta\alpha+\beta)} + \frac{6(\theta\alpha+4\beta)}{\theta^3(\theta\alpha+\beta)}
\end{aligned}$$

6.5 Estimation

6.5.1 EM Algorithm for Three Parameter Poisson-Lindley Distribution

First, we obtain the joint distribution for a continuous Poisson mixture which has been obtained in equation (9).
Next, we obtain the log likelihood function of the Poisson-Lindley distribution which is already obtained in equation (10).
From the log likelihood function, l_1 and l_2 is obtained where l_1 is the log likelihood of the Poisson distribution and l_2 is the log likelihood of the Linldey distribution as shown in (1).
l_1 is already in (11).
To calculate l_2, we write the pdf of the three-parameter Lindley distribution and find its log likelihood function.
Given the pdf of a three parameter Lindley distribution as:

$$g(\lambda) = \frac{\theta^2}{\theta\alpha+\beta}(\alpha+\beta\lambda)e^{-\lambda\theta}$$

Therefore;

$$\begin{aligned}
l_2 &= \sum_{i=1}^{n} log\ g(\lambda_i) \\
&= \sum_{i=1}^{n} log \frac{\theta^2}{\theta\alpha+\beta}(\alpha+\beta\lambda_i)e^{-\lambda_i\theta} \\
&= \sum_{i=1}^{n} [2\ log\ \theta + log(\alpha+\beta\lambda_i) - \theta\lambda_i - log(\theta\alpha+\beta)] \\
&= 2n\ log\ \theta + \sum_{i=1}^{n} log(\alpha+\beta\lambda_i) - \theta\sum_{i=1}^{n}\lambda_i - n\ log(\theta\alpha+\beta)
\end{aligned}$$

Therefore;

$$\frac{\partial}{\partial\theta}l_2 = \frac{2n}{\theta} - \sum_{i=1}^{n}\lambda_i - \frac{n\alpha}{\theta\alpha+\beta}$$

$$\frac{\partial}{\partial\theta}l_2 = 0 \Rightarrow \frac{2n}{\theta} - \sum_{i=1}^{n}\lambda_i - \frac{n\alpha}{\theta\alpha+\beta} = 0$$

$$\frac{2}{\theta} - \overline{\lambda} - \frac{\alpha}{\theta\alpha+\beta} = 0$$

$$2 - \theta\overline{\lambda} - \frac{\theta\alpha}{\theta\alpha+\beta} = 0$$

$$2(\theta\alpha+\beta) - \theta\overline{\lambda}(\theta\alpha+\beta) - \theta\alpha = 0$$

$$2\theta\alpha + 2\beta - \theta^2\overline{\lambda}\alpha - \theta\overline{\lambda}\beta - \theta\alpha = 0$$

$$\theta\alpha + 2\beta - \theta^2\alpha\overline{\lambda} - \theta\beta\overline{\lambda} = 0$$

$$\theta^2\alpha\overline{\lambda} + \theta\beta\overline{\lambda} - \theta\alpha - 2\beta = 0$$

$$\theta^2\alpha\overline{\lambda} + \theta(\beta\overline{\lambda} - \alpha) - 2\beta = 0$$

$$\hat{\theta} = \frac{-(\beta\overline{\lambda}-\alpha) \pm \sqrt{(\beta\overline{\lambda}-\alpha)^2 + 8\alpha\beta\overline{\lambda}}}{2\alpha\overline{\lambda}}$$

$$= \frac{(\alpha-\beta\overline{\lambda}) \pm \sqrt{\beta^2\overline{\lambda}^2 + \alpha^2 + 6\alpha\beta\overline{\lambda}}}{2\alpha\overline{\lambda}}$$

Since λ>0, α>0, β>0, we have:

$$\hat{\theta} = \frac{(\alpha-\beta\overline{\lambda}) + \sqrt{\beta^2\overline{\lambda}^2 + \alpha^2 + 6\alpha\beta\overline{\lambda}}}{2\alpha\overline{\lambda}}$$

Next, we differentiate with respect to α,

$$\frac{\partial}{\partial\alpha}l_2 = \sum_{i=1}^{n}\frac{1}{\alpha+\beta\lambda_i} - \frac{n\theta}{\theta\alpha+\beta} = 0$$

$$\frac{n\theta}{\theta\alpha+\beta} = \sum_{i=1}^{n}\frac{1}{\alpha+\beta\lambda_i}$$

$$\theta\alpha+\beta = \frac{n\theta}{\sum_{i=1}^{n}\frac{1}{\alpha+\beta\lambda_i}}$$

$$\theta\alpha = \frac{n\theta}{\sum_{i=1}^{n}\frac{1}{\alpha+\beta\lambda_i}} - \beta$$

$$\hat{\alpha} = \frac{n}{\sum_{i=1}^{n}\frac{1}{\alpha+\beta\lambda_i}} - \frac{\beta}{\theta}$$

Next, we differentiate with respect to β,

$$\frac{\partial}{\partial\beta}l_2 = \sum_{i=1}^{n}\frac{\lambda_i}{\alpha+\beta\lambda_i} - \frac{n}{\theta\alpha+\beta}$$

$$\frac{n}{\theta\alpha+\beta} = \sum_{i=1}^{n}\frac{\lambda_i}{\alpha+\beta\lambda_i}$$

$$\theta\alpha+\beta = \frac{n}{\sum_{i=1}^{n}\frac{\lambda_i}{\alpha+\beta\lambda_i}}$$

$$\hat{\beta} = \frac{n}{\sum_{i=1}^{n}\frac{\lambda_i}{\alpha+\beta\lambda_i}} - \theta\alpha$$

Since λ_i is the unobserved part, we estimate using the posterior expectation as obtained in (15) and then obtain the iterative schemes as follows:

Iterative Schemes

Iterative Scheme 1

Let

$$s_i = E(\Lambda_i/X_i) = \frac{(x_i+1)[\alpha(\theta+1)+\beta(x_i+2)]}{(\theta+1)[\alpha(\theta+1)+\beta(x_i+1)]}$$

The kth iteration is;

$$
\begin{aligned}
s_i^{(k)} &= E(\Lambda_i/X_i) \\
&= \frac{(x_i+1)[\alpha(\theta^{(k)}+1)+\beta(x_i+2)]}{(\theta^{(k)}+1)[\alpha(\theta^{(k)}+1)+\beta(x_i+1)]}
\end{aligned}
$$

Based on the explicit solutions obtained, we have the following iterations;

$$
\begin{aligned}
\theta^{(k+1)} &= \frac{\alpha - s_i^{(k)}\beta + \sqrt{\beta^2 s_i^{(2k)} + \alpha^2 + 6\alpha\beta\ s_i^{(k)}}}{2\alpha\ s_i^{(k)}} \\
\alpha^{(k+1)} &= \frac{n}{\sum_{i=1}^{n} \frac{1}{\alpha^{(K)}+s_i^{(k)}\beta}} - \frac{\beta}{\theta^{(k+1)}} \\
\beta^{(k+1)} &= \frac{n}{\sum_{i=1}^{n} \frac{s_i^{(k)}}{\alpha^{(k+1)}+\beta^{(K)}s_i^{(k)}}} - \theta^{(k+1)}\alpha^{(k+1)}
\end{aligned}
$$

Iterative Scheme 2

From the quadratic equation;
$\theta^2\alpha\overline{\lambda} + \theta(\beta\overline{\lambda} - \alpha) - 2\beta = 0,$

we have:

$$
\begin{aligned}
\theta^2\alpha\overline{\lambda} &= 2\beta - \theta(\beta\overline{\lambda} - \alpha) \\
&= 2\beta + \theta(\alpha - \beta\overline{\lambda}) \\
\theta &= \frac{2\beta + \theta(\alpha - \beta\overline{\lambda})}{\theta\alpha\overline{\lambda}}
\end{aligned}
$$

The iteration scheme now becomes:

$$
\theta^{(k+1)} = \frac{2\beta + \theta^{(k)}(\alpha - \beta\overline{s}^{(k)})}{\overline{s}^{(k)}\theta^{(k)}\alpha}
$$

7 Conclusion and Recomendations

7.1 Conclusion

In this paper, Poisson-Lindley distributions have been obtained right from the one-parameter Poisson-Lindley distribution all the way upto the three-parameter Poisson-Lindley distribution by mixing the Poisson distribution with one parameter, two parameter and three parameter Lindley distributions respectively.
Under two parameters, three types of two parameters Poisson-Lindley distributions have been considered.
Various properties have been discussed including the r^{th} moment, the cumulative distribution function, the probability generating function, the moment generating function and the characteristic function of the distributions
The posterior distribution and the posterior expectations have also been obtained to aid in the estimation using the Expectation-Maximization Algorithm method.
The estimation of the parameters of the distributions have been discussed using the Expectation-Maximization Algorithm method.

7.2 Recommendations

In this work, we have considered te Poisson-lindley distribution and estimated the parameters of the distribution using the EM Algorithm method. One can extend the study to estimate the parameters of the distributions using the method of moments and the maximum likelihood method together with the EM Algorithm method and then determine which method is the most preferable among the three.

Again, we have only considered the Linldley distribution which ia an example of two componenets finite Gamma mixing distribution. One can extend the study by considering other Poisson mixtures based on two components based on two components Gamma mixing distribution like Akash and Shanker distributions and then estimate the parameters using the EM Algorithm method.

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