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Dissertation in Pure Mathematics

Foundations to the theory of schemes

Research Report in Mathematics, Number 05, 2021

Peter Omondi Oluoch

June 2021

Submitted to the Department of Mathematics in partial fulfillment for a degree in Master of Science in Pure Mathematics

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Peter Omondi Oluoch

Department of Mathematics
Faculty of science and Technology
Chiromo, off Riverside Drive
30197-00100 Nairobi, Kenya

Master Thesis

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Abstract

Grothendieck's magnificent theory of schemes pervades the spectrum of modern algebraic geometry and underpins its wide applications in the field of Number theory,Medicine, Physics , Applied Mathematics,image encryption and finger printing. This report which is a simple account of the foundations to the theory of schemes underscores and demonstrates the common geometric concepts that form the basis of the definitions. The report begins these foundations with Some local algebra where we make a mention of Noether's Normalization Lemma, Going-up theorem of Cohen-Seidenberg and the Weak Nullstellensatz result before giving some properties of Cohen-Macaulay rings. The report then introduces the language of categories and functors which then leads to a discussion on the sheaf theory. We then introduce the spectrum of rings and the Zariski topology before defining an affine scheme and scheme in general. This is then followed by a number of examples of schemes and some of the properties of affine schemes. The report discusses dimension of a scheme and ends by exhibiting on the concept of gluing construction. In this dissertation all the results are well-known and therefore our contribution is only at the level of presentation.

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Declaration and Approval

I the undersigned declare that this dissertation is my original work and to the best of my knowledge, it has not been submitted in support of an award of a degree in any other university or institution of learning.

10/9/2021

Signature Date

PETER OMONDI OLUOCH
Reg No. I56/13765/2018

In my capacity as a supervisor of the candidate's dissertation, I certify that this dissertation has my approval for submission.

10/09/2021

Signature Date

Dr Jared Ongaro
Department of Mathematics,
University of Nairobi,
Box 30197, 00100 Nairobi, Kenya.
E-mail: ongaro@uonbi.ac.ke

10/09/2021

Signature Date

Dr Benjamin KIKWAI
Department of Mathematics
University of Nairobi,
Box 30197, 00100 Nairobi, Kenya.
E-mail: kikwaib@gmail.com

Dedication

This project is dedicated to my lovely wife Fenny.

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Acknowledgments

I would like to express my very sincere appreciation to Jared Ongaro for his valuable and constructive guidance in helping me choose the dissertation topic and during the entire development of this project report. His willingness to set aside time is much appreciated. Similarly I salute my second supervisor Benjamin Kikwai for his constructive and valuable input in improving the quality of this dissertation. Thank you abundantly.

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PETER OMONDI OLUOCH

Nairobi, 2021.

1 Introduction

1.1 Historical perspective

The theory of schemes is the foundation for Algebraic Geometry as formulated by Alexander Grothendieck and his many coworkers. It is viewed as the basis for a grand unification of Number theory and Algebraic Geometry, a situation that many number theorists and geometers yearned and wished for over a century. By allowing flexible geometric arguments about infinitesimals and limits in away that could not be achieved via the classical theory, the theory of schemes has strengthened and enriched classical algebraic geometry[EH00]. It is in view of this that the report gives the foundations to the theory of schemes.

1.1.1 Motivation (Why schemes?)

The goal of this dissertation is to introduce the reader to an accessible Foundations to the Theory of Schemes, that is widely used in modern geometry.

Unlike in classical Algebraic Geometry, Schemes:

1. Make **intrinsic** sense of X without referring to the ambient space.
2. Take **any commutative ring R with** 1 (cr1ng) and consider its corresponding variety $\mathbf{Spec}(R)$ instead of just the polynomial ring.
3. Take care of **nilpotents** in a variety without ignoring them. In classical AG, such information is lost under continuous deformation e.g with $a \neq b \in \mathbb{R}^*$

$$\mathbb{R}^2 \cong \mathbb{R}[\mathbb{V}(x^2-(a+b)x+ab)] \neq \mathbb{R}[\mathbb{V}(x^2)] \cong \mathbb{R}.$$

4. **Focus on sheaf of functions** $\mathscr{O}_{\mathbf{Spec}(R)}$ on $\mathbf{Spec}(R)$ than just the space.

1.2 The aesthetic nature of the theory of schemes.

The beautiful nature of the theory of schemes is evidenced by the pictures that follow which are but a few of the many interesting representations that we use to reduce the abstractness of the theory if indeed it exists. By this, we put the foundations to the theory schemes in a natural environmental setting with the full alluring and soothing ambiance.

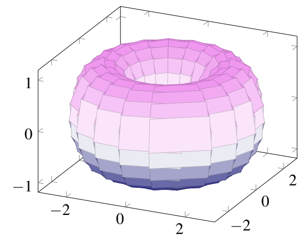

Figure 1. Torus

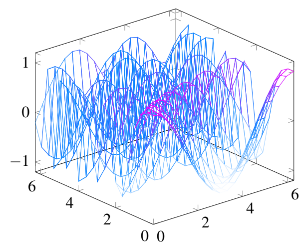

Figure 2. a projective surface for $y = cos\,(\deg\,(x - y^2))$ **in 3 D**

1.3 Outline of the dissertation

The outline of the dissertation report is as follows:

Chapter 1:
The report focuses on the historical background of the theory of schemes, some motivation, the aesthetic nature of the theory and the basic structure of the report.

Chapter 2
We begin the chapter by a discussion on some basic facts on Algebraic Geometry, some local algebra and consider the basic tool in studying the loci V of roots of a finite set of polynomials $f_i(x_1,...,x_n)$ in k^n, k algebraically closed field. We present some of the most important results which include Noethers' normalization Lemma, the Going-up theorem of Cohen-Seidenberg and some results on depth and Cohen-Macaulay rings.

The report discusses Categories, Functors and the theory on sheaves. In particular we present the data that constitutes the categories, functors and sheaves together with fundamental examples in each case.

Chapter 3:
We discuss Schemes as follows:

Section 3.1: **The spectrum of a ring and the Zariski topology.** In this section we construct the space **Spec** R, define a topology on it, give some of the basic properties of the $\mathcal{O}$ on **Spec** R and give mophisms of ringed spaces and locally ringed spaces.

Section 3.2: **First definition and examples of schemes.** We define the affine scheme and a scheme. The report then provides some examples of schemes. A presentation on Proj S then follows as we also describe the projective **n**-space of a ring.

Section 3.3: **First properties of schemes.** Here, we deal with connectedness of schemes, irreducible and reduced schemes, the notion of integral schemes, locally notherian schemes. We then explain open, closed subschemes and the products of schemes.

Section 3.4: **Dimension and morphism of a scheme.** In this section we give the definition and first properties of dimension of a scheme, discuss dimension and schemes of finite type over a field and conclude the section with a discussion on morphisms and dimension.

Section 3.5: **Constructions.** In this section we define a few useful constructions on schemes (the gluing construction).

Chapter 4:
We conclude our report by a recap of the main ideas in the dissertation and finally indicating the future research direction.

2 Preliminaries, Categories, Functors and Sheaf theory

2.1 Preliminaries

2.1.1 Some basic results of Algebraic Geometry

Transition from classical Algebraic Geometry to schemes

The proofs of results in this section can be found in [Ewal96] and [Fult93] for example.

Let $\mathbb{C}[x] = \mathbb{C}[x_1, \ldots, x_{\mathbf{n}}]$ be the ring of polynomials in $\mathbf{n}$ variables over $\mathbb{C}$.

Definition 2.1.1. *If $E = (f_1, \ldots, f_r) \subset \mathbb{C}[x]$, then $V(E) = \{x \in \mathbb{C}^{\mathbf{n}} : f_1(x) = \ldots = f_{\mathbf{n}}(x) = 0\}$ is called the affine algebraic set defined by E.*

Let $X \subset \mathbb{C}^{\mathbf{n}}$; then

$$I(X) = \{f \in \mathbb{C}[x] : f|_X = 0\},$$

is an ideal called the vanishing ideal of X.

Example 2.1.2. *For $a = (a_1, \ldots, a_{\mathbf{n}}) \in \mathbb{C}^{\mathbf{n}}$, Consider $E = \{x_1 - a_1, \ldots x_{\mathbf{n}} - a_{\mathbf{n}}\}$. Then $V(E) = \{a\}$ and $I(\{a\}) = \mathbb{C}[x](x_1 - a_1) + \cdots + \mathbb{C}[x](x_{\mathbf{n}} - a_{\mathbf{n}})$. It is a maximal ideal denoted by $\mathfrak{m}_a$.*

Definition 2.1.3. *An $\mathbf{n}$-dimensional affine space is defined by $\mathbb{A}^n := \mathbb{C}^n$ to be the set of all $\mathbf{n}$-tuples of elements of $\mathbb{C}$.*

Definition 2.1.4. *For a positive integer $\mathbf{n}$, Let $R = \mathbb{C}[x_1, \ldots, x_n]$ be a polynomial ring. An affine variety defined on $S \subset R = \mathbb{C}[x_1, \ldots, x_n]$ is the varnishing set*

$$(S) := \{p \in \mathbb{C}^n | f(p) = 0, \textit{for } f \in S\}$$

Definition 2.1.5. *Let $V \subset \mathbb{C}^n$ be affine variety. The varnishing ideal of V*

$$\mathbb{I}(V) = \{f \in \mathbb{C}[x_0, \ldots, x_n] : f(p) = 0, \textit{for all } p \in V\}$$

is the ideal of all homogeneous polynomials on V. The coordinate ring $\mathbb{C}[V]$ of V is the quotient ring

$$\mathbb{C}[V] = \frac{\mathbb{C}[x_0, \ldots, x_n]}{\mathbb{I}(V)}$$

Furthermore, we call a subset $V \subset \mathbb{P}^n$ an algebraic set if there exists a homogeneous ideal I for which $V = \mathbb{V}(I)$. We say V is reducible if $V = \mathbb{V}(I) = \mathbb{V}(I_1) \cup \mathbb{V}(I_2)$ for some proper ideals $I_1, I_2 \in \mathbb{C}[x_0, \ldots, x_n]$, otherwise it is irreducible.

Theorem 2.1.6 (Hilbert Nullstellensatz.). *Let I be an ideal of R. Then*

(i) $I \subsetneq R$ if and only if $\mathbb{V}(I) \neq \emptyset$

(ii) $\mathbb{I}(V(I)) = \sqrt{I} = \{f \in R = \mathbb{C}[x_0, \ldots, x_n] : f^m \in I \text{ for some } \mathfrak{m}\}$. In particular, if I is radical (i.e $\sqrt{I} = I$) or equivalently R/I is a reduced ring (has no nilpotent elements) e.g $I = (x) \subset \mathbb{C}[x]$, then $\mathbb{I}(\mathbb{V}(I)) = I$

Corollary 2.1.7 (The $\mathbb{I} - \mathbb{V}$ Correspondence in classical AG)**.** *There are order-reversing bijections between ideals $I \lhd R = k[x_1, \ldots, x_n]$ where k an algebraically closed field with algebraic varieties $X \subset \mathbb{A}^n_k$. That is*

$$\begin{aligned}
\{\textit{varieties }\} &\longleftrightarrow \{\textit{radical ideals }\} \\
\{\textit{irreducible varieties}\} &\longleftrightarrow \{\textit{prime ideals } \mathfrak{p} \lhd k[X]\} = \mathbf{Spec}\ (R) \\
\{\textit{points}\} &\longleftrightarrow \{\textit{maximal ideals } \mathfrak{m}_a \lhd k[X]\} = \mathbf{Spec}\ m(R) \\
X &\longleftrightarrow \mathbb{I}(X) \\
\mathbb{V}(I) &\longleftrightarrow I
\end{aligned}$$

Remark 2.1.8. *An algebraic set is irreducible if and only if its ideal is prime.*

The examples that follow offer some explanation the Corollary 2.1.7

Example 2.1.9. *$\mathbb{A}^{\mathbf{n}}_k$ is irreducible, since it corresponds to the zero ideal in R, which is prime.*

Example 2.1.10. *Let f be an irreducible polynomial in $R = k[x,y]$. Then f generates a prime ideal in R, since R a unique factorization domain, so the zero set $X' = V(f)$ is irreducible. We call it the affine curve defined by $f(x,y) = 0$. If f has degree d, then X is a curve of degree d*

Example 2.1.11. *More generally, if f is an irreducible polynomial in $R = k[x_1, \ldots, x_{\mathbf{n}}]$, we obtain an affine variety $X = V(f)$, which is called a surface if $\mathbf{n} = 3$ or a hypersurface if $\mathbf{n} \geq 3$.*

Example 2.1.12. *A maximal ideal $\mathfrak{m}$ of $R = k[x_1, \ldots, x_{\mathbf{n}}]$ corresponds to a minimal irreducible closed subset of $\mathbb{A}^{\mathbf{n}}_k$ which must be a point, say $P = (a_1, \ldots, a_{\mathbf{n}})$. This shows that every maximal ideal of R is of the form $m = (x_1 - a_1, \ldots, x_{\mathbf{n}} - a_{\mathbf{n}})$ for some $a_1, \ldots, a_{\mathbf{n}} \in k$*

Example 2.1.13. *If k is not algebraically closed, these results do not hold, for example if $k = \mathbb{R}$, the curve $x^2 + y^2 + 1 = 0$ in $\mathbb{A}^2_{\mathbb{R}}$ has no points.*

Definition 2.1.14. *If $X \subseteq \mathbb{A}^{\mathbf{n}}_k$ is an affine algebraic set, we say the affine coordinate ring $R(X)$ of X, is $R/I(X)$.*

Lemma 2.1.15. *A collection of* ***algebraic sets*** *has the following properties.*

(i) Empty space and the whole projective space are algebraic sets.

(ii) Arbitrary intersection of algebraic sets is an algebraic set.

(iii) Finite union of algebraic sets is an algebraic set.

Definition 2.1.16 (Zariski topology)**.** *Zariski topology on $\mathbb{P}^n$ is the topology on algebraic varieties where the open sets are complements of algebraic sets which satisfy Lemma 2.1.15*

2.1.2 Some local Algebra

In this section, we will discuss some local algebra. We consider the basic tool in studying the locus V of roots of a finite set of polynomials $f_i(x_1,\ldots,x_\mathbf{n})$ in $k^\mathbf{n}$, (k being an algebraically closed field). This is the ring of functions from V to k obtained by restricting polynomials from $k^\mathbf{n}$ to V. On the other hand we assume known to the reader the following topics in algebra.

(i) The essentials of field theory (Galois theory, separability, transcendence degree);

(ii) Ring localization, the behavior of ideals in localization, local ring concept;

(iii) Noetherian rings, and the decomposition theorem of ideals in these rings;

(iv) The integral dependence concept.

Theorem 2.1.17 (Noethers' Normalization Lemma.). *If A is an integral domain, finitely generated over a field k and if R has transcendence degree n over k, then we have elements $x_1,\ldots,x_n \in A$, algebraically independent on the sub-ring $k(x_1,\ldots,x_n)$ generated by $x's$.*

For the proof, see Mumford [Mum99].

Theorem 2.1.18 (Going-up theorem of Cohen-Seidenberg.). *Let R be a commutative ring and $S \subset R$ a sub-ring such that R is integrally dependent on S. For all prime ideals $P \subset S$, there exists prime ideals $P' \subset R$ such that $P' \cap S = P$.*

Again for the proof check [Mum99].

Lemma 2.1.19. *If A is a field, and $S \subset A$ a subring such that A is integrally dependent on B, then B is a field.*

We refer to [Mum99] for the proof of Lemma 2.1.3.

In the remaining portion of this section, we present some results majorly on **depth** and **Cohen-Macaulay rings**, which play a pivotal role in **Algebraic Geometry.**

We refer to Matsumura [Mat89] for proofs.

Let R be a ring, and M be an R-module. Recall that a sequence $x_1,\ldots,x_r$ of elements of R is a regular sequence for M if x_1 is not a zero divisor in M, and for all $i=2,\ldots,r$, x_i is not a zero in $M/(x_1,\ldots,x_{i-1})M$. Suppose R is a local ring with maximal ideal $\mathfrak{m}$, then the depth of M is the maximum length of a regular sequence, $x_1,\ldots,x_r$ for M with all $x_i\in\mathfrak{m}$. These definitions apply to the ring R itself, and a local Noetherian ring R is said to be Cohen-Macaulay if depth $R=\dim R$.

We now present some properties of the Cohen-Macaulay rings.

Theorem 2.1.20. *Let R be a local Noetherian ring with maximal ideal $\mathfrak{m}$.*

(i) If R is regular, then it is Cohen-Macaulay,

(ii) If R is Cohen-Macaulay, then any localization of R at a prime ideal is also Cohen-Macaulay.

(iii) If R is Cohen-Macaulay, then a set of elements $x_1,\ldots,x_r\in\mathfrak{m}$ forms a regular sequence for R if and only if $\dim R/(x_1,\ldots,x_r)=\dim R-r$.

(iv) If R is Cohen-Macaulay, and $x_1,\ldots,x_r\in\mathfrak{m}_r\in\mathfrak{m}$ is a regular sequence for R, then $R/(x_1,\ldots,x_n)$ is also Cohen-Macaulay.

(v) If R is Cohen-Macaulay, and $x_1,\ldots,x_r\in\mathfrak{m}$ is a regular sequence, Let I be the ideal $(x_1,\ldots,x_r)$. Then the natural map

$$(R)/[t_1,\ldots,t_r]\to gr_I=\oplus_{n\geq}I^n/I^{n+1},$$

defined by sending $t_i\mapsto x_i$, is an isomorphism. That is, I/I^2 is free R/I-module of rank r, and for every $n\geq 1$, the natural map $S^n(I/I^2)\to I^n/I^{n+1}$ is an isomorphism, where the n^{th} symmetric power is denoted by S^n.

For the proof check Matsumura [Mat89]

Remark 2.1.21. *We say that a Noetherian ring R is **normal** if for every prime ideal $\mathfrak{p}$, the localization $R_{\mathfrak{p}}$ is an integrally closed domain. A normal ring is a finite direct product of integrally closed domains.*

Theorem 2.1.22 (Serre)**.** *Let R be a Noetherian ring. We say R is normal if and only if it satisfies the following two conditions:*

(a) for each prime ideal $\mathfrak{p} \in R$ of height ≤ 1, $R_{\mathfrak{p}}$ is regular and

(b) for each prime ideal $\mathfrak{p} \in R$ of height ≥ 2, the depth $A_{\mathfrak{p}} \geq 2$.

For the proof of Theorem 2.1.8, check Matsumura [Mat89].

2.2 The language of categories.

A category $\mathbf{C}$ consists of the following data:

- A collection Obj $\mathbf{C}$ of objects;
- For every two objects C, D a set $\mathscr{H}(C,\, D)$ called morphisms from C to D. In Figure 3, a and b are elements of $\mathbf{C}$ while $\mathscr{H}(a)$ *and* $\mathscr{H}(b)$ are elements in $\mathbf{D}$.

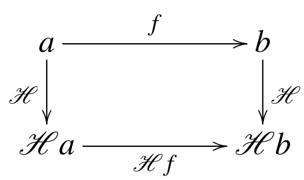

Figure 3. Data constituting a category.

- For every three objects $A, B, C,$ a composition map $\mathscr{H}(C,B) \times \mathscr{H}(B,A)$, $(g, f) \mapsto f \circ g,$ so that the following axioms are satisfied:

(1) For every object A, there is a distinguished morphism $\mathrm{Id}_A \in \mathscr{H}(A,A)$, called the identity;

(2) We have $f \circ \mathrm{Id}_B = f$ for every $f \in \mathscr{H}(B,\, A)$;

(3) We have $g \circ \mathrm{Id}_C = g$ for every $g \in \mathscr{H}(C,\, B)$;

(4) For every four objects $A,\, B,\, C,\, D$ and every three morphisms $h \in \mathscr{H}(D,\, C)$, $g \in \mathscr{H}(C,\, B)$,$f \in \mathscr{H}(B,\, A)$, the two morphisms $(f \circ g) \circ h)$ and $f \circ (g \circ h)$ in $\mathscr{H}(M,\, Q)$ are equal (the associativity of composition).

Remark 2.2.1 (Composition of morphisms). *If $g\colon C \to B$ and $f\colon B \to A$ are two morphisms, then one defines a new function $f \circ g$, the composition of g and f (g followed by f) by*

$$f \circ g\colon C \to A,\ x \mapsto f(g(x)).$$

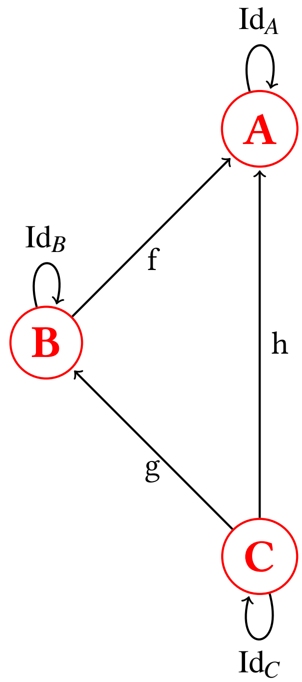

Figure 4. Identity morphisms.

A common notation for $\mathscr{H}(B,A)$ is also $\mathrm{Hom}_{\mathscr{H}}(B,A)$. Finally, we can often write $f\colon B \to A$ instead of $f \in \mathscr{H}(B,A)$.

Definition 2.2.2. *If $f\colon B \to A$ is a morphism in a category $\mathbf{C}$, then we say that f is left-invertible, resp. right-invertible, resp. invertible if there exists a morphism $g\colon A \to B$ such that $g \circ f = \mathrm{Id}_B$, resp. $f \circ g = \mathrm{Id}_A$, resp. $g \circ f = \mathrm{Id}_B$ and $f \circ g = \mathrm{Id}_A$.*

Remark 2.2.3. *If f is both left- and right-invertible, then we say it is invertible. An invertible morphism is also called an isomorphism.*

Below are examples of categories.

Example 2.2.4. *The category* **Set** *of sets consists of sets as the objects, and the usual maps between sets for the morphisms.*

Example 2.2.5. *The category* **Gr** *of groups contains the groups as the objects and the morphisms of groups for the morphisms. The category* **Ab** *of abelian groups consists of the abelian groups as the objects and the morphisms of groups for its morphisms. Take cognizance of the fact that objects of* **Ab** *are objects of* **Gr**; *therefore we could say that* **Ab** *is a full subcategory of* **Gr**.

Example 2.2.6. *The category* **Ring** *consists of rings as the objects and morphisms of rings as the morphisms.*

Example 2.2.7. *Similarly, there is the category* **Field** *of fields and if k is a field, the category $\mathbf{Vec_k}$ of k- vector spaces. More generally, for every ring A, there is a category $\mathbf{Mod_A}$ of right A-modules, and a category $_{\mathbf{A}}\mathbf{Mod}$ for the left A-modules.*

Example 2.2.8. *Let* $\mathbf{Q}$ *be a category; its opposite category* $\mathbf{Q}^0$ *contains the same objects as* $\mathbf{Q}$*, but the morphisms of* $\mathbf{Q}^0$ *are defined by* $\mathscr{H}^0(A,\ B) = \mathscr{H}(B,\ A)$ *and composed in the opposite direction. It is synonymous with the definition of an opposite group. However a category is always different from its opposite category.*

Example 2.2.9. *Let* $\mathbf{I}$ *be a partially ordered set. We attach to* I *a category* $\mathbb{I}$ *whose set of objects is* I *itself. Its morphisms are as follows: let* $i,\ j \in I$*; if* $i \leq j$*, then* $\mathscr{I}(i,\ j)$ *has a single element say the pair* $(i,\ j)$*; otherwise,* $\mathscr{I}(i,\ j)$ *is empty. The composition of morphisms is the obvious one:* $(j,\ k) \circ (i,\ j) = (i,\ k)$ *if* $i,\ j,\ k$ *are elements of* I *such that* $i \leq j \leq k$.

Remark 2.2.10. *Let* $\mathbf{Q}$ *be a category. We say that* $\mathbf{Q}$ *is small if* $\mathrm{Ob}\,\mathbf{Q}$ *is a set and* $\mathscr{H}\ (A,\ B)$ *is a set for every pair* (A,B) *of objects of* $\mathbf{Q}$*. A category* $\mathbf{Q}$ *such that the collection* $\mathscr{H}\ (A,\ B)$ *is a set for every pair* (A,B) *of objects is said to be locally small. Usually most categories considered in general Mathematics, such as the categories of sets, of groups, of abelian groups, of modules over a fixed ring, of vector spaces, e.t.c, are locally small, but not small [Cham15].*

Definition 2.2.11. *Let* $\mathbf{Q}$ *be a category, let* A,B *be objects of* $\mathbf{Q}$ *and let* $f \in \mathscr{H}(A,B)$*. We say that* f *is an epimorphism if for every object* P *of* $\mathbf{Q}$ *and every morphisms* $g_1, g_2 \in \mathscr{H}(B,P)$ *such that* $g_1 \circ f = g_2 \circ f$*, we have* $g_1 = g_2$*. We say* f *is a monomorphism if for every object* L *of* $\mathbf{Q}$ *and every morphisms* $g_1, g_2 \in \mathscr{H}(P,A)$ *such that* $f \circ g_1 = f \circ g_2$*, we have* $g_1 = g_2$.

2.3 Functors

Remark 2.3.1. *Functors are to categories what maps are to sets [Cham15].*

Let $\mathbf{C}$ and $\mathbf{D}$ be two categories. A functor $\mathscr{H}$ from $\mathbf{C}$ to $\mathbf{D}$ consists of the following data:

- An object $\mathscr{H}(M)$ of $\mathbf{D}$ for every object M of $\mathbf{C}$;
- a morphism $\mathscr{H}(f) \in \mathscr{H}(\mathscr{L}(X), \mathscr{L}(Y))$ for every objects X,Y of $\mathbf{C}$ and every $f \in \mathscr{L}(X,Y)$, subject to the conditions which follow:

(i) For any object X of $\mathbf{C}$, $\mathscr{H}(\mathrm{Id}_X) = \mathrm{Id}_{\mathscr{H}(X)}$;

(ii) For any objects X,Y,Z of $\mathbf{C}$ and every morphisms $f \in \mathscr{H}(X,Y)$ and $g \in \mathscr{H}(Y,Z)$, we have

$$\mathscr{H}(g \circ f) = \mathscr{H}(g) \circ \mathscr{H}(f).$$

A contravariant functor $\mathscr{H}$ from $\mathbf{C}$ to $\mathbf{D}$ is a functor from $\mathbf{C}^0$ to $\mathbf{D}$, consisting of the data as above.

Definition 2.3.2. *Such a functor $\mathscr{H}$ is said to be faithful, resp. full, resp. fully faithful if for each objects X,Y of $\mathbf{C}$ the map $f \mapsto \mathscr{H}(f)$ from $\mathscr{L}(X,Y)$ to $\mathscr{H}(\mathscr{L}(X),\mathscr{L}(Y))$ is injective, resp. surjective, resp. bijective.*

A similar definition applies for contravariant functors [Cham15].

A functor $\mathscr{H}$ is essentially surjective if for each object N of $\mathbf{D}$, there exists an object M of $\mathbf{C}$ such that $\mathscr{H}(X)$ is isomorphic to N in the category $\mathbf{D}$.

Example 2.3.3 (Forgetful functors)**.** *Many algebraic structures are defined by enriching other structures. Usually, forgetting this enrichment gives rise to a functor, called a forgetful functor [Cham15].*

For instance, a group already constitutes a set, and a map of groups is a morphism. There is therefore a functor relating to every group its underlying set, hence forgetting the group structure. We thus get a forgetful functor from **Gr** *to* **Set***. It is faithful, since a group morphism is determined by the map between the underlying sets. It is however not full because they are maps between two (non-trivial) groups which are not morphisms of groups [Cham15].*

Example 2.3.4. *The construction of the spectrum of a ring defines a contravariant functor from the category* **Ring** *of rings to the category* **Top** *of topological spaces.*

Conversely, set $\mathscr{O}(X)$ to be the ring of continuous complex-valued functions on a topological space X. If $f\colon X \to Y$ is a continuous map of topological spaces, and $f^* : \mathscr{O}(Y) \to \mathscr{O}(X)$ be the morphism of rings given by $f^*(u) = u \circ f$. We thus get a contravariant functor from the category **Top** *of topological spaces to the category of algebras over the field of complex numbers.*

If $\mathscr{H}$ and $\mathscr{G}$ are two functors from a category $\mathbf{C}$ to a category $\mathbf{D}$, then a morphism of functors α from $\mathscr{H}$ to $\mathscr{G}$ consists in the datum, for every object X of $\mathbf{C}$, a morphism $\alpha_X\colon \mathscr{H}(X) \to \mathscr{G}(X)$ with the following conditions obeyed:

For every morphism $f\colon X \to Y$ in $\mathbf{C}$, we have $\alpha_Y \circ \mathscr{H}(f) = \mathscr{G}(f) \circ \alpha_X$.

Remark 2.3.5.

(i) It is possible to compose morphisms of functors. Consider a functor $\mathscr{H}$, we have an identity morphism from $\mathscr{H}$ to itself. Consequently, functors from $\mathbf{C}$ to $\mathbf{D}$ form themselves a category, denoted $\mathscr{H}(\mathbf{C},\mathbf{D})$.

(ii) Let $\mathbf{C}$ and $\mathbf{D}$ be categories, let $\mathscr{H}$ be a functor from $\mathbf{C}$ to $\mathbf{D}$ and let $\mathscr{G}$ be a functor from $\mathbf{D}$ to $\mathbf{C}$. $\mathscr{H}$ and $\mathscr{G}$ are said to be quasi-inverse functors if the functors $\mathscr{G} \circ \mathscr{H} \cong \mathrm{Id}_C$ and $\mathscr{H} \circ \mathscr{G} \cong \mathrm{Id}_D$ respectively.

(iii) *We say that a functor* $\mathscr{H} : \mathbf{C} \to \mathbf{D}$ *is an equivalence of categories, if we have another functor* $\mathscr{G} : \mathbf{D} \to \mathbf{C}$ *and* $\mathscr{H}, \mathscr{G}$ *are quasi-inverse functors.*

Proposition 2.3.6. *To qualify to be an equivalence of categories, a functor* $\mathscr{H} : \mathbf{C} \to \mathbf{D}$ *needs to be fully faithful and essentially surjective (a necessary and sufficient condition).*

For the proof check,[Cham15].

Example 2.3.7 (Linear Algebra)**.** *Let* K *be a field. Traditionally undergraduate linear algebra only considers as vector spaces the subspaces of varying vector* K^n *and linear maps between them. This gives rise to a small category, because for every integer* $\mathbf{n}$*, the subspaces of* $K^{\mathbf{n}}$ *form a set.*

From this category to the category of finite dimensional K*-vector spaces, the obvious functor is an equivalence of categories which again is fully faithful and also essentially surjective: since vector spaces have bases, every finite dimensional* K*-vector space* V *is isomorphic to* $K^{\mathbf{n}}$ *with* $\mathbf{n} = \dim(V)$*. Consequently the (small) "Category of undergraduate linear algebra" is equivalent to the (large) category of finite dimensional vector spaces.[Cham15]*

Example 2.3.8. *Let* X *be a topological space, and let* $x \in X$*. Let* $\mathbf{Cov_X}$ *be the category of coverings of* X*. For every covering* $\rho : E \to X$*, the fundamental group* $\pi_1(X,x)$ *acts on the fiber* $p^{-1}(x)$*. This defines a functor* $\mathscr{H} : E \mapsto \mathscr{H}(E) = p^{-1}(x)$ *from the category* $\mathbf{Cov_X}$ *to the category of* $\pi_1(X,x)$*- sets.*

Remark 2.3.9. *The functor in Example* 2.3.8 *is said to be fully faithful if* X *is connected and locally pathwise connected. In addition, it is an equivalence of categories if* X *has simply connected cover.*

Example 2.3.10 (Galois Theory)**.** *Let* K *be a perfect field and let* ω *be an algebraic closer of* K*; let* G_K *be the group of* K*-automorphisms of* ω*. For every finite extension* L *of* K*, let* $s(L) = Hom_K(L, \omega)$*, the set of* K*-morphisms from* L *to* ω*. This gives a finite set, with cardinality* $[L : K]$*, and the group* G_K *acts on it by the formular* $g.\varphi = g \circ \varphi$*, for every* $\varphi \in s(L)$ *and every* $g \in G_K$*; additionally, the action of* G_K *is transitive.*

A map $f^* : s(L') \to s(L)$ *which is compartible with the actions of* G_K *is induced by Every morphism of extensions* $f : L \to L'$*. The assignments* $L \mapsto s(L)$ *and* $f \mapsto f^*$ *define a contravariant functor from the category of finite extensions of* K *to the category of finite sets endowed with a transitive action of* G_K

2.4 Sheaves

Through the concept of a sheaf, we find a systematic way of handling local algebraic data on a topological space. Furthermore, Sheaves play an essential role in the study of schemes as we shall see in a short while.

Definition 2.4.1. *Let X be a topological space. A presheaf $\mathscr{F}$ on X consists of the data*

(1) *for every open subset $U \subseteq X$, an abelian group $\mathscr{F}(U)$, and*

(2) *for every inclusion $V \subseteq U$ of open subsets of X, a morphism of abelian groups ρ_{UV} : $\mathscr{F}(U) \to \mathscr{F}(V)$ such that the following axioms are satisfied:*

   (i) *$\mathscr{F}(\emptyset) = 0$, where $\emptyset$ is the empty set.*

   (ii) *ρ_{UU} is the identity map $\mathscr{F}(U) \to \mathscr{F}(U)$, and*

   (iii) *if $W \subseteq V \subseteq U$ are those open subsets, then $\rho_{UW} = \rho_{VW} \circ \rho_{UV}$*

Definition 2.4.2. *A presheaf $\mathscr{F}$ on a topological space X is said to be a sheaf if it satisfies the following supplementary conditions:*

(3) *if U is an open set, and $\{V_i\}$ is an open covering of U, with $s \in \mathscr{F}(U)$ being an element such that $s|_{V_i} = 0$ for all i, then $s = 0$;*

(4) *if U is an open set, and $\{V_i\}$ is an open covering of U, and if we have elements $s_i \in \mathscr{F}(V_i)$ for each i, with the property that for each $i,\ j$, $s_i|_{V_i \cap V_j} = s_j|_{V_i \cap V_j}$, then there is an element $s \in \mathscr{F}(U)$ such that $s|_{V_i} = s_i$ for each i.*

Example 2.4.3. *Let X and Y be two topological spaces. For all open sets $U \subset X$, let $\mathscr{F} : U \to Y$ be the set of continuous maps. Then $\mathscr{F}(U)$ is a presheaf with the restriction maps given by simply restricting maps to smaller sets; it is a sheaf because a function is continuous on $\cup U_i$ if and only if its restrictions to each U_i are continuous.*

Example 2.4.4. *Consider two differentiable manifolds M and N. Let $\mathscr{F}(U)$ be the differentiable maps $U \to N$. This is a sheaf as differentiability is a local condition.*

Example 2.4.5. *Let X be a topological space, $\mathscr{F}(U)$ the vector space of locally constant real-valued functions on U modulo the constant functions on U. This is a presheaf. But every $s \in \mathscr{F}(U)$ goes to zero in $\prod \mathscr{F}(U_i)$ for some open covering (U_i), while if U is not connected, $\mathscr{F}(U) \neq (0)$. Thus it is not a sheaf.*

Example 2.4.6. *Let X be a variety over a field k. For each open set $U \subseteq X$, let $\mathscr{O}(U)$ be the ring of regular functions from $U \to k$, and for each $V \subseteq U$, let $\rho_{UV} : \mathscr{O}(U) \to \mathscr{O}(V)$ be the restriction map . Then $\mathscr{O}$ is a sheaf of rings on X. It is apparent that it is a presheaf of rings. To verify the supplementary conditions for a sheaf, we note that a function which is zero locally is 0, and a function which is regular locally is regular applying the definition of regular functions. Thus $\mathscr{O}$ is the sheaf of regular functions on X.*

Example 2.4.7. *X a topological space, B an abelian group. The constant sheaf $\mathscr{B}$ on X determined by B is defined as follows. Give B the discrete topology , and for any open set $U \subseteq X$, let $\mathscr{B}(U)$ be the group of all continuous maps of $U \to B$. Then from the usual restriction maps, one obtains a sheaf $\mathscr{B}$. Furthermore, if U is an open set whose connected components are open, then $\mathscr{B}(U)$ is a direct product of copies of B, one for each connected component of U.*

Remark 2.4.8. *For every connected open set U, $\mathscr{B}(U)$ is isomorphic to B, hence the name "constant sheaf."*

There are some important ideas on sheaves worth noting before we add other examples.

(i) **Stalks.** Let $\mathscr{F}$ be a sheaf on X, $x \in X$. The collection of $\mathscr{F}(U)$, U open which contains x, is an inverse system ,thus we can generate

$$\mathscr{F}_x = \underset{x \in U}{\xrightarrow{\lim}} \mathscr{F}(U),$$

which we call the stalk of $\mathscr{F}$ at x.

Example 2.4.9. *If $\mathscr{F}(U)$ is the continuous functions $U \to \mathbb{R}$, then $\mathscr{F}_x$ is the set of germs of continuous functions at x. It is $\cup_{x \in U} \mathscr{F}(U)$ modulo an equivalence relation: $f_1 \sim f_2$ if f_1 and f_2 coincide in a neighborhood of x.*

(ii) **Sheafification of a presheaf.** If $\mathscr{F}_0$ is a presheaf on X, then there exists a sheaf $\mathscr{F}$ and a map $f\colon \mathscr{F}_0 \to \mathscr{F}$ such that if $g\colon \mathscr{F}_0 \to \mathscr{F}'$ is any map with $\mathscr{F}'$ a sheaf, we have a unique map $h\colon \mathscr{F} \to \mathscr{F}'$ such that the diagram commutes.

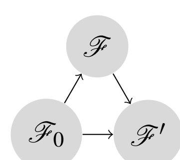

Figure 5. Sheafification of a presheaf.

Consider example 2.4.4 above, If M is locally compact, the sheafification of this presheaf is the sheaf of all continuous functions on M, and in example 2.4.5, the sheafification of this presheaf is (0).

It is worth mentioning something about the foregoing notations.

Notation One may write $\Gamma(U, \mathscr{F})$ for $\mathscr{F}(U)$, and term it the set of sections of $\mathscr{F}$ over U. $\Gamma(X, \mathscr{F})$ is the set of global sections of $\mathscr{F}$. We may denote $\mathscr{F}(X)$ by $H^0(X, \mathscr{F})$ in other contexts and call it the zeroth cohomology group.

Suppose that for all U, $\mathscr{F}(U)$ is a group [ring, e.t.c] and that all the restriction maps are group [ring, e.t.c] homomorphisms. Then we call $\mathscr{F}$ a sheaf of groups [rings, etc.]. In this scenario $\mathscr{F}_x$ is a group [rings, e.t.c] and so on.

Example 2.4.10. *If X is a topological space, $\mathscr{F}_{cont,X}(U)$ a set of continuous functions $U \to \mathbb{R}$, then $\mathscr{F}_{cont,X}(U)$ is a sheaf of rings.*

Remark 2.4.11. *Suppose $g\colon X \to Y$ is a continuous function and the operation $f \mapsto f \circ g$ gives us the following maps: for every open $U \subset Y$ a map $\mathscr{F}_{cont,Y}(U) \to \mathscr{F}_{cont,X}(g^{-1}U)$ such that*

$$\begin{array}{ccc} \mathscr{F}_Y(U) & \longrightarrow & \mathscr{F}_X(g^{-1}U) \\ {\scriptstyle \text{res}}\downarrow & & \downarrow{\scriptstyle \text{res}} \\ \mathscr{F}_Y(V) & \longrightarrow & \mathscr{F}_X(g^{-1}V) \end{array}$$

Figure 6. Commutative diagram with continuous functions.

commutes for all open sets $V \subset U$. Then we call this set up a morphism of the pair $(X, \mathscr{F}_X)$ to the pair $(Y, \mathscr{F}_Y)$ [Mum99].

Example 2.4.12. *Let M and N be differentiable manifolds. $\mathscr{F}_{diff,M}$ and $\mathscr{F}_{diff,N}$ be the sub-sheaves of $\mathscr{F}_{cont,M}$ and $\mathscr{F}_{cont,N}$ of differentiable functions. Suppose $g\colon M \to N$ is a continuous map. Then g is differentiable if and only if for all open sets $U \subset N$,*

$$f \in \mathscr{F}_{diff,N}(U) \to f \circ g \in \mathscr{F}_{diff,M}(g^{-1}U).$$

Example 2.4.13. *Let M, N be complex analytic manifolds. Let $\mathscr{F}_{an,M}$ and $\mathscr{F}_{an,N}$ be the sheaves of holomorphic functions. Then a continuous map $g\colon M \to N$ is holomorphic if and only if for all open sets, $U \subset N$,*

$$f \in \mathscr{F}_{an,N}(U) \to f \circ g \in \mathscr{F}_{an,M}(g^{-1}U).$$

Thus the idea of using a "structure sheaf" to describe an object is useful in many contexts, and it will solve our problems too [Mum99].

Definition 2.4.14. *Let $\mathscr{F}$ and $\mathscr{G}$ be any two presheaves on X, a morphism $\vartheta\colon \mathscr{F} \to \mathscr{G}$ consists of a morphism of abelian groups $\vartheta(U)\colon \mathscr{F}(U) \to \mathscr{G}(U)$ for each open set U, such that whenever $V \subseteq U$ is an inclusion, the diagram*

is commutative, where ρ and ρ' are the restriction maps in $\mathscr{F}$ and $\mathscr{G}$. If $\mathscr{F}$ and $\mathscr{G}$ are sheaves on X, for a morphism of sheaves, the same definition can be used. An isomorphism is a morphism which is both injective and surjective.

$$\begin{array}{ccc} \mathscr{F}(U) & \xrightarrow{\vartheta(U)} & \mathscr{G}(U) \\ {\scriptstyle \rho_{UV}}\downarrow & & \downarrow{\scriptstyle \rho'_{UV}} \\ \mathscr{F}(V) & \xrightarrow{\vartheta(V)} & \mathscr{G}(V) \end{array}$$

Figure 7. Commutative diagram of presheaves with an inclusion $V \subseteq U$.

A morphism $\vartheta\colon \mathscr{F} \to \mathscr{G}$ of presheaves on X induces a morphism $\vartheta_p\colon \mathscr{F}_p \to \mathscr{G}_p$ on the stalks, for any point $p \in X$. We illustrate the local nature of a sheaf by the corollary below, which happens to be false for presheaves [Hart77]

Corollary 2.4.15. *If $\vartheta\colon \mathscr{F} \to \mathscr{G}$ be a morphism of sheaves on a topological space X, then ϑ is said to be an isomorphism if and only if it induces an isomorphic map on the stalk $\vartheta_p\colon \mathscr{F}_p \to \mathscr{G}_p$ for each $p \in X$.*

For the proof of Corollary 2.4.15, check [Hart77].

We now proceed to give a definition of kernels, cokernels, and images of morphisms.

Definition 2.4.16. *If $\vartheta\colon \mathscr{F} \to \mathscr{G}$ is a morphism of presheaves, then we define the presheaf kernel on ϑ, presheaf cokernel of ϑ, and presheaf image of ϑ as the presheaves given by $U \mapsto \ker(\vartheta(U))$, $U \mapsto \operatorname{coker}(\vartheta(U))$ and $U \mapsto \operatorname{Im}(\vartheta(U))$ respectively.*

Remark 2.4.17. *Given a morphism of sheaves $\vartheta\colon \mathscr{F} \to \mathscr{G}$, the presheaf kernel of ϑ is a sheaf. On the contrary the presheaf of cokernel of ϑ and presheaf of the image of ϑ are generally not sheaves. This brings us to the notion of a sheaf associated to a presheaf [Hart77].*

Corollary 2.4.18. *For any presheaf $\mathscr{F}$, there is a sheaf $\mathscr{F}^+$ and a morphism $\theta\colon \mathscr{F} \to \mathscr{F}^+$, having the property that for every sheaf $\mathscr{G}$, and every morphism $\vartheta\colon \mathscr{F} \to \mathscr{G}$, we have a unique morphism $\Psi\colon \mathscr{F}^+ \to \mathscr{G}$ such that $\vartheta = \Psi \circ \theta$. Moreover the pair $(\mathscr{F}^+, \theta)$ is unique up to unique isomorphism. We call $\mathscr{F}^+$ the sheaf associated to the presheaf $\mathscr{F}$.*

`Proof.` $\mathscr{F}^+$ is constructed as follows. For every open set U, let $\mathscr{F}^+(U)$ be the set of functions s from U to the union $\bigcup_{p \in U} \mathscr{F}_p$ of the stalks of $\mathscr{F}$ over points of U, such that

(a) for every $p \in U$, $s(p) \in \mathscr{F}_p$, and

(b) for every $p \in U$, there is a neighborhood V of p , which is in U, and an element $t \in \mathscr{F}(V)$ such that for all $q \in V$, the germ t_q of t at q is equal to $s(q)$.

We can then verify that $\mathscr{F}^+$ with the natural restriction maps is a sheaf, that there is a natural morphism $\theta\colon \mathscr{F} \to \mathscr{F}^+$ and that it has the universal property described. The uniqueness of $\mathscr{F}^+$ is a formal consequence of the universal property. □

Proposition 2.4.19. *Suppose X is an irreducible algebraic set and let $R = \Gamma(\Sigma)$. Let $f \in R$, and $X_f = \{x \in X \mid f(x) \neq 0\}$. Then we have $\mathscr{O}_X(X_f) = R_f$.*

For the proof of Proposition 2.4.19 refer to Mumford [Mum99] . In particular, as a simple corollary we have;

Corollary 2.4.20. $\Gamma(X, \mathscr{O}_X) = R$.

Proposition 2.4.21. *For any two irreducible algebraic sets $X \subset k^n$ and $Y \subset k^m$, and a continuous map $f : X \to Y$, the following conditions are equivalent:*

(a) f is a morphism

(b) for all $g \in \Gamma(Y, \mathscr{O}_Y)$, $g \circ f \in \Gamma(X, \mathscr{O}_X)$

(c) for all open $U \subset Y$, and $\Gamma(U, \mathscr{O}_Y) \implies g \circ f \in \Gamma(f^{-1}U, \mathscr{O}_X)$

(d) for all $x \in X$ and $g \in \mathscr{O}_{f(x)} \implies g \circ f \in \mathscr{O}_X$.

For the proof of proposition 2.4.20 see Mumford [Mum99].

Definition 2.4.22. *A subsheaf of a sheaf $\mathscr{F}$ is a sheaf $\mathscr{F}'$ such that for any open set $U \subseteq X$, $\mathscr{F}'(U)$ is a subgroup of $\mathscr{F}(U)$, and the restriction maps of the sheaf $\mathscr{F}'$ are induced from the ones of $\mathscr{F}$. Clearly it follows that for every point p, the stalk $\mathscr{F}'_p$ is a subgroup of $\mathscr{F}_p$.*

Let $\vartheta : \mathscr{F} \to \mathscr{G}$ be a morphism of sheaves, we define the kernel of ϑ, denoted by $\ker\ \vartheta$ as the presheaf kernel of ϑ which is actually a sheaf. Therefore $\ker\ \vartheta$ is subsheaf of $\mathscr{F}$.

A morphism of sheaves $\vartheta : \mathscr{F} \to \mathscr{G}$ is said to be injective if $\ker\ \vartheta = 0$. Therefore ϑ is said to be injective if and only if the induced map $\vartheta(U) : \mathscr{F}(U) \to \mathscr{G}(U)$ is injective for each open set of X.

Let $\vartheta : \mathscr{F} \to \mathscr{G}$ be a morphism of sheaves, we define the image of ϑ denoted as $\mathrm{Im}\ \vartheta$, to be the sheaf associated to the presheaf image of ϑ. By the universal property of the sheaf associated to a presheaf, there exists a natural map $\mathrm{Im}\ \vartheta \to \mathscr{G}$ which is injective and therefore $\mathrm{Im}\ \vartheta$ can be identified with a subsheaf of $\mathscr{G}$.

A morphism $\vartheta : \mathscr{F} \to \mathscr{G}$ is said to be surjective if $\mathrm{Im}\ \vartheta = \mathscr{G}$.

Definition 2.4.23. *A sequence $\ldots \to \mathscr{F}^{i-1} \xrightarrow{\vartheta^{i-1}} \mathscr{F}^i \xrightarrow{\vartheta^i} \mathscr{F}^{i+1} \to \ldots$ of sheaves and morphisms is said to be exact if at each stage $\ker\ \vartheta^i = \mathrm{Im}\ \vartheta^{i-1}$. Therefore a sequence $0 \to \mathscr{F} \xrightarrow{\vartheta} \mathscr{G}$ is exact if and only if ϑ injective, and $\mathscr{F} \xrightarrow{\vartheta} \mathscr{G} \to 0$ is exact if and only if ϑ is surjective.*

If $\mathscr{F}'$ is a subsheaf of a sheaf $\mathscr{F}$. Then the quotient sheaf $\mathscr{F}/\mathscr{F}'$ is defined as the sheaf associated to the presheaf $U \to \mathscr{F}(U)/\mathscr{F}'(U)$. Thus for any point p, the stalk $(\mathscr{F}/\mathscr{F}')_p$ is given as the quotient $\mathscr{F}_p/\mathscr{F}'_p$.

If $\vartheta : \mathscr{F} \to \mathscr{G}$ is a morphism of sheaves, then, the cokenel of ϑ, denoted $\operatorname{coker} \vartheta$, *is said to be the sheaf associated to the presheaf cokernel of ϑ.*

Let us now define some operations on sheaves which are associated with a continuous map from one topological space to another.

Definition 2.4.24. *Let $f : X \to Y$ be a continuous map of topological spaces. For every sheaf $\mathscr{F}$ on X, the direct image sheaf $f_*\mathscr{F}$ on Y is defined by $(f_*\mathscr{F})(V) = \mathscr{F}(f^{-1}(V))$ for every open set $V \subseteq Y$. For every sheaf $\mathscr{G}$ on Y, the inverse image sheaf $f^{-1}\mathscr{G}$ on X is defined as the sheaf associated to the presheaf $U \mapsto \lim_{V \supseteq f(U)} \mathscr{G}(V)$, U being any open set in X and the limit is taken over all open sets V of Y containing $f(U)$.*

Definition 2.4.25. *Let T be a subset of X, regarded as a topological subspace with the induced topology, let $\omega : T \to X$ be the inclusion map, and let $\mathscr{F}$ be a sheaf on X. We call $\omega^{-1}\mathscr{F}$ the restriction of $\mathscr{F}$ to T, usually denoted $\mathscr{F}|_T$.*

Remark 2.4.26. *Note that f_* is a functor from the category $\mathfrak{Ub}(X)$ of sheaves on X to the category $\mathfrak{Ub}(Y)$ of sheaves on Y. In the same manner, f^{-1} is a functor from $\mathfrak{Ub}(Y)$ to $\mathfrak{Ub}(X)$ and the stalk of $\mathscr{F}|_T$ at any point $p \in T$ is just $\mathscr{F}_p$.*

3 Schemes

3.1 Spectrum of Rings and The Zariski topology

We start off by constructing the space **Spec** R associated to a ring R. As a set, the **Spec** R is said to be the set of all prime ideals of R. Let $\mathfrak{p}$ be any ideal of R, the subset $V\mathfrak{p} \subseteq \mathbf{Spec}\ R$ is defined as the set of all prime ideals which contain $\mathfrak{p}$. We give a lemma to illustrate the point.

Lemma 3.1.1.

(1) Let $\mathfrak{p}$ and $\mathfrak{m}$ be two ideals of R. We say $V(\mathfrak{p}\mathfrak{m}) = V(\mathfrak{p}) \cup V(\mathfrak{m})$

(2) Let $\{\mathfrak{p}_i\}$ be any set of ideals of R. We say $V(\sum \mathfrak{p}_i) = \bigcap V(\mathfrak{p}_i)$.

(3) Let $\mathfrak{p}$ and $\mathfrak{m}$ be two ideals, we say that $V(\mathfrak{p}) \subseteq V(\mathfrak{m})$ if and only if $\sqrt{\mathfrak{p}} \supseteq \sqrt{\mathfrak{m}}$.

Proof.

(1) Suppose $\mathfrak{a} \supseteq \mathfrak{p}$ or $\mathfrak{a} \supseteq \mathfrak{m}$. We have $\mathfrak{a} \supseteq \mathfrak{p}\mathfrak{m}$. In the other direction, suppose $\mathfrak{a} \supseteq \mathfrak{p}\mathfrak{m}$ and $\mathfrak{a} \not\supseteq \mathfrak{m}$ for example, then we have a $m \in \mathfrak{m}$ such that $m \notin \mathfrak{a}$. Now for any $p \in \mathfrak{p}$, $pm \in \mathfrak{a}$, there must be $p \in \mathfrak{a}$ because $\mathfrak{a}$ is a prime ideal. Therefore $\mathfrak{a} \supseteq \mathfrak{p}$.

(2) We say that $\sum \mathfrak{p}_i$ is contained in $\mathfrak{a}$ if and only if each $\mathfrak{p}_i$ is contained in $\mathfrak{a}$, since $\mathfrak{p}_i$ is the smallest ideal containing all of the ideals $\mathfrak{p}_i$

(3) The intersection of the set of all prime ideals which contains $\mathfrak{p}$ is the radical of $\mathfrak{p}$. Therefore $\sqrt{\mathfrak{p}} \supseteq \sqrt{\mathfrak{m}}$ if and only if $V(\mathfrak{p}) \subseteq V(\mathfrak{m})$.

□

Let us now define a topology on $\mathbf{Spec}(R)$ by taking the subsets of the form $V(\mathfrak{p})$ to be the closed subsets.

Definition 3.1.2. *The closed sets of $\mathbf{Spec}(R)$ are of the form $V(\mathfrak{p}) = \{\mathfrak{a} \in \mathbf{Spec}(R) \mid \mathfrak{p} \in \mathfrak{a}\}$, where $\mathfrak{p}$ is any ideal in R, called the Zariski topology on $\mathbf{Spec}(R)$.*

Remark 3.1.3. $V(R) = \emptyset$; $V((0)) = \mathbf{Spec}(R)$ *and from Lemma* 3.1.1 *we notice that the finite unions and arbitrary intersections of sets of the form* $V(\mathfrak{p})$ *are again of that form and thus form the set of closed sets for a topology on* $\mathbf{Spec}(R)$ *and so the collection of closed subsets* $\{V\mathfrak{p}\}$ *does define a topology.*

We introduce certain open sets of a spectrum that play a big role, thus the definition which follow.

Definition 3.1.4. *The distinguished open sets of a ring* R *are the open sets of the form* $\mathbf{Spec}(R)_f = \{\mathfrak{m} \in \mathbf{Spec}(R) \,|f \notin \mathfrak{m}\} = \mathbf{Spec}(R) \setminus V(f)$.

These sets form a basis for the Zariski topology on $\mathbf{Spec}(R)$

We then can summarise the topology of $\mathbf{Spec}(R)$ as follows.

1. Basic open set are defined as

$$\begin{aligned} D_f := &\{\mathfrak{m} \in \mathbf{Spec}(R) : f \notin \mathfrak{m}\} \\ = &\{\mathfrak{m} \in \mathbf{Spec}(R) : f(\mathfrak{m}) = R_{\mathfrak{m}}/\mathfrak{m},\ R \neq 0\} \end{aligned}$$

   It is the case that $D_f \cap D_g = D_{fg}$ and that

$$\begin{aligned} D_f \subseteq D_g \iff \mathbb{V}(g) \subseteq \mathbb{V}(f) &\iff \sqrt{f} \subseteq \sqrt{g} \\ &\iff f^N \in (g) \text{ some } N \\ &\iff g \in D_f \text{ invertile.} \end{aligned}$$

2. We can canonically identify $\mathbf{Spec}(R/I)$ with the **closed affine subscheme** $\mathbb{V}(I) \subseteq \mathbf{Spec}(R)$ through the inclusion map α on **Spec**s
3. Open sets $U_I := \mathbf{Spec}(R) \setminus \mathbb{V}(I) = \cup_{f \in I} D_f$,
   Maximal ideal $\mathfrak{m}_{\mathfrak{p}} \in U \subseteq \mathbf{Spec}(R_{\mathfrak{p}})$ open $\implies U = \mathbf{Spec}(R_{\mathfrak{p}})$.

Let us Now define a sheaf of rings $\mathscr{O}$ on $\mathbf{Spec}(R)$. For any prime ideal $\mathfrak{p} \subseteq R$, let $R_{\mathfrak{p}}$ be the localization of R at $\mathfrak{p}$. For an open set $U \subseteq \mathbf{Spec}(R)$, we define $\mathscr{O}(U)$ to be the set of functions $s : U \to \prod_{\mathfrak{p} \in U} R_{\mathfrak{p}}$, such that $s(\mathfrak{p}) \in R_{\mathfrak{p}}$ for every $\mathfrak{p}$, such that s is locally a quotient of elements of R : precisely, for every $\mathfrak{p} \in U$ there is neighborhood V of $\mathfrak{p}$, contained in U, and elements $a, f \in R$ such that for every $\mathfrak{m} \in V$, $f \notin \mathfrak{m}$ and $s(\mathfrak{m}) = a/f$ in $R_{\mathfrak{m}}$

From the foregoing, we note that sums and products of such functions are again of the same form, and that the element 1 giving 1 in any $R_{\mathfrak{p}}$ is an identity. Therefore $\mathscr{O}(U)$ is a commutative ring with identity. Let $V \subseteq U$ be two open sets, the natural restriction map $\mathscr{O}(U) \to \mathscr{O}(V)$ is a homomorphism of rings. Hence $\mathscr{O}$ is a presheaf. Finally, from the local nature of the definition, we can then draw a conclusion that $\mathscr{O}$ is a sheaf.

Definition 3.1.5. *The spectrum of a commutative ring R, is the set of prime ideals in R, and is denoted* $\mathbf{Spec}(R)$.

The definition 3.1.5 can be framed differently as below.

Definition 3.1.6. *Let R be a ring. The spectrum of R is the pair consisting of the topological space $\mathbf{Spec}(R)$ together with the sheaf of rings $\mathscr{O}$ defined on R.*

Example 3.1.7. *Consider* $\mathbf{Spec}(\mathbb{Z}) = \{0\} \cup \{(p) : p \in \mathbb{N} \text{ primes }\}$. $\mathbb{V}((0)) = \{\text{prime ideals containing } (0)\}$

(0) (2) (3) (5) ⋯

Figure 8. $\mathbf{Spec}(\mathbb{Z})$

$\mathbf{Spec}(\mathbb{Z}) \implies (0) \in \mathbf{Spec}(\mathbb{Z})$ *is a (dense)* ***generic point***.
$\mathbb{V}((p)) = \{(p)\}$ *are "**closed points**" ($= \{$maximal ideals of $\mathbb{Z}$, a PID $\}$).*

Example 3.1.8. *If k is a field, then $\mathbf{Spec}(k)$ is the one point space with $\mathscr{O}_{\mathbf{Spec}(k)}(\star) = k$.*

Example 3.1.9. *Let k be a field, and $R = \dfrac{k[x]}{(x^2)}$, then R has only one prime ideal, namely, (x), hence $\mathbf{Spec}\ R$ is one point, with $\dfrac{k[x]}{(x^2)}$ at that point.*

The main idea in Example 3.1.9 is that functions are no longer determined by their values. In particular, the function x is everywhere zero, though it is not the zero function.

We shall put forth some basic properties of the sheaf $\mathscr{O}$ on $\mathbf{Spec}(R)$. Let $f \in R$ be any element and let us denote by $D(f)$ the open complements of $V((f))$. The open sets of the form $D(f)$ form a basis for the topology of $\mathbf{Spec}(R)$. In fact, if $V(\mathfrak{p})$ is a closed set, and $\mathfrak{m} \notin V(\mathfrak{p})$, then $\mathfrak{m} \not\supseteq \mathfrak{p}$, thus there is an $f \in \mathfrak{p}$, $f \notin \mathfrak{m}$. It follows that $\mathfrak{m} \in D(f)$ and $D(f) \cap V(\mathfrak{p}) = \emptyset$

Proposition 3.1.10. *If R is a ring, and if $(\mathbf{Spec}(R), \mathscr{O})$ is its spectrum, then;*

(1) for any $\mathfrak{p} \in \mathbf{Spec}(R)$, the stalk $\mathscr{O}_\mathfrak{p}$ of the sheaf $\mathscr{O}$ and the local ring $R_\mathfrak{p}$ are isomorphic to each other, that is, $\mathscr{O}_p \cong R_p$.

(2) for any element $f \in R$, the ring $\mathscr{O}(D(f))$ and the localized ring R_f are isomorphic to each other, that is, $\mathscr{O}(D(f)) \cong R_f$

(3) in particular, $\Gamma(\mathbf{Spec}(R), \mathscr{O})$ *is isomorphic to* R.

For the proof of Proposition 3.1.10, check [Hart77]

Morphisms of ringed spaces

Definition 3.1.11. *We say that a ringed space is a pair* $(X, \mathscr{O}_X)$ *which consists of a topological space* X *and a sheaf of rings* $\mathscr{O}_X$ *on* X.

For examples on ringed spaces refer to the Examples 2.4.10, 2.4.12, 2.4.13 *covered earlier in chapter* 2.

We say a morphism of ringed spaces from $(X, \mathscr{O}_X)$ *to* $(Y, \mathscr{O}_Y)$ *is a pair* $(f, f^\sharp)$ *of continuous map* $f\colon X \to Y$ *and a map* $f^\sharp\colon \mathscr{O}_Y \to f_*\mathscr{O}_X$ *of sheaves of rings on* Y. *In other words*

$$
\begin{aligned}
(f, f^\sharp) : (X, \mathscr{O}_X) &\to (Y, \mathscr{O}_Y) \\
\varphi : X &\to Y \text{ is continuous} \\
f^\sharp : \mathscr{O}_Y &\to f_*\mathscr{O}_X \\
f^\sharp_y : \mathscr{O}_Y, y &\to (f_*\mathscr{O}_X)_y \to \mathscr{O}_{X,x}(\text{for all x, f(x)=y})
\end{aligned}
$$

We say the ringed space $(X, \mathscr{O}_X)$ *is a locally ringed space if for each point* $p \in X$, *the stalk* $\mathscr{O}_{X,p}$ *is a local ring.*

We say a morphism of locally ringed spaces is a morphism $(f, f^\sharp)$ *of ringed spaces, such that for any point point* $p \in X$, *the induced map of local rings* $f^\sharp_p\colon \mathscr{O}_{Y,f(p)} \to \mathscr{O}_{X,p}$ *is a local homomorphism of local rings.*

For an explanation to the last condition, check [Hart77].

The diagram below can be used to make sense from the foregoing discussion.

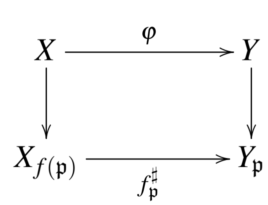

Figure 9. Morphism of rings

Definition 3.1.12. *we say an isomorphism of locally ringed spaces is a morphism with a two-sided inverse; that is a morphism which is both injective and surjective. Therefore we say* *a morphism $(f, f^\sharp)$ is an isomorphism if and only if f is a homeomorphism of the underlying topological spaces, and $f^\sharp$ is an isomorphism of sheaves.*

Proposition 3.1.13.

(1) *Let X be a ring. Then $(\mathbf{Spec}(X), \mathscr{O})$ is a locally ringed space.*

(2) *Let $\varphi\colon X \to Y$ be a homomorphism of rings. Then a natural morphism of locally ringed spaces*
$$(f, f^\sharp)\colon (\mathbf{Spec}(Y), \mathscr{O}_{\mathbf{Spec}(Y)} \to (\mathbf{Spec}(X), \mathscr{O}_{\mathbf{Spec}(X)})$$
is induced by φ.

(3) *Let X and Y be rings. Any homomorphism of rings $\varphi : X \to Y$ as in (2) above induces a morphism of locally ringed spaces from $\mathbf{Spec}(Y)$ to $\mathbf{Spec}(X)$.*

For the proof of Proposition 3.1.13, we can refer to the proof of Proposition 2.3 in [Hart77].

The upshot is captured by the following summary:

$$\begin{aligned}
\text{X being a ring} &\implies (\mathbf{Spec}\ X, \mathscr{O}) \text{ is locally a ringed space .}\\
\varphi : X \to Y \text{ ring Hom.} &\implies (f, f^\sharp) : (\mathbf{Spec}(Y), O_{\mathbf{Spec}(Y)}) \to (\mathbf{Spec}(X), \mathscr{O}_{\mathbf{Spec}(X)})\\
\varphi : X \to Y \text{ ring Hom.} &\implies f : \mathbf{Spec}(Y) \to \mathbf{Spec}(X) \text{ is an induced map.}
\end{aligned}$$

Let $\varphi : X \to Y$ be a ring homomorphism. Then a map $f : \mathbf{Spec}(Y) \to \mathbf{Spec}(X)$ is defined by $f(\mathfrak{p}) = \varphi^{-1}(\mathfrak{p})$ $(\mathfrak{p} \in \mathbf{Spec}\ Y.)$

3.2 First definition and examples of schemes.

We begin by giving the definition of a scheme.

Definition 3.2.1. *An affine scheme is a locally ringed space $(X, \mathscr{O}_X)$ which is isomorphic to the spectrum of some ring (as a locally ring space [Hart77].)*

Definition 3.2.2. *We say a scheme is a locally ringed space $(X, \mathscr{O}_X)$ in which every point has an open neighborhood U such that the topological space U, together with the restricted sheaf $\mathscr{O}_X|_U$, is an affine scheme.*

Remark 3.2.3. *X is the underlying topological space of the scheme $(X, \mathscr{O}_X)$, and $\mathscr{O}_X$ is its structure sheaf. We say a morphism of schemes is a morphism as locally ringed spaces.*

The following examples show what a scheme is.

Example 3.2.4. *Let k be a field.* **Spec** *k is an affine scheme whose topological space has one point, and whose structure sheaf contains the field k.*

Example 3.2.5. *Let A be a discrete valuation ring and let $W = \mathbf{Spec}\, A$ be an affine scheme whose topological space contains two points. One point w_0 is closed, with local ring equals A; the second point w_1 is open and dense, with local ring equal to K, the quotient field of ring A. The inclusion map $A \to K$ corresponds to the morphism $\mathbf{Spec}\, K \to W$ sending the unique point of $\mathbf{Spec}\, K$ to w_1. There is a second morphism of ringed spaces $\mathbf{Spec}\, K \to W$ which sends the unique point of $\mathbf{Spec}\, K$ to w_0 and applies the inclusion $A \to K$ to define the associated map $f^\sharp$ on structure sheaves. This morphism can not be induced by homomorphism $A \to K$, because it is not a morphism of locally ringed spaces.*

Example 3.2.6. *Consider $\mathbf{Spec}(\mathbb{Z})$. $\mathbb{Z}$ is principal ideal domain like $k[x]$, and $\mathbb{Z}$ is usually viewed as a line: It has one closed point for every prime number , plus a generic point (0). The stalk at $(\mathfrak{p})$ is $\mathbb{Z}_{(\mathfrak{p})}$ and at (0) is $\mathbb{Q}$, thus $\mathbb{Q}$ is the "function field" of $\mathbf{Spec}(\mathbb{Z})$.*

The nonempty open sets of $\mathbf{Spec}(\mathbb{Z})$ are obtained by omitting finitely many primes $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$. If $\mathfrak{m} = \prod \mathfrak{p}_i$, then this is the distinguished open $\mathbf{Spec}(\mathbb{Z}_\mathfrak{m})$, and

$$\Gamma(\mathbf{Spec}(\mathbb{Z}_\mathfrak{m}), \mathscr{O}_{\mathbf{Spec}(\mathbb{Z})}) = \{\frac{a}{k}\} | a \in \mathbb{Z}, k \geq 0.$$

The residue fields of the stalks $\mathscr{O}_X$ are $\mathbb{Z}/2\,\mathbb{Z}$, $\mathbb{Z}/3\,\mathbb{Z}, \ldots, \mathbb{Q}$. We get each prime field exactly once.

Example 3.2.7. *Almost a similar scenario apply to $\mathbf{Spec}(R)$ for any Dedekind domain R. All prime ideals are maximal or (0); hence again we have a "line" of closed points together with a generic point. A case in point is when R is a principal valuation ring. Such a ring has a unique maximal ideal $\mathfrak{m}$ hence $\mathbf{Spec}(R)$ has two points (0) and $(\mathfrak{m})$. Then (0) is an open point and $(\mathfrak{m})$ is closed.*

Example 3.2.8. *Let k be a field. We define the affine line over k, $\mathbb{A}^1_k$ as $\mathbf{Spec}\, k[x]$. It consists of a point η which corresponds to the (0) (the zero ideal), whose closure is the whole space. We call this the generic point. The other points, corresponding to the maximal ideals in $k[x]$, are all closed points and correspond one-to-one with the non-constant monic polynomials in x. Particularly, if k is algebraically closed, the closed points of $\mathbb{A}^1_k$ correspond one-to-one with elements of k.*

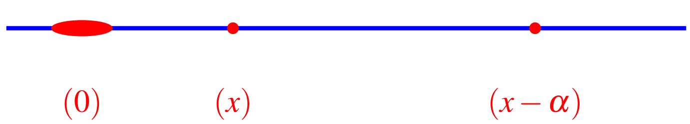

Figure 10. The affine line (**Scheme** $\mathbb{A}^1_K = \mathbf{Spec}(K[x])$)

Example 3.2.9. *If k is an algebraically closed field and if we consider the affine plane over k, defined as $\mathbb{A}^2_k = \mathbf{Spec}(k[x,y])$, then the closed points of $\mathbb{A}^2_k$ are in a 1-to-1 correspondence with the ordered pairs of elements of k. Moreover the set of all closed points of $\mathbb{A}^2_k$, with the* *induced topology, is homeomorphic to the variety called* $\mathbb{A}^2$*. Further, there is a generic point* η*, which corresponds to the* (0) *ideal of* $k[x,y]$*, whose closure is the whole space in addition to the closed points. We also have a point* ξ *whose closer consists of* ξ *together with all closed points* (a,b) *for which* $f(a,b) = 0$*, for each irreducible polynomial* $f(x,y)$*. We call* ξ *the generic point of the curve* $f(x,y) = 0$*.*

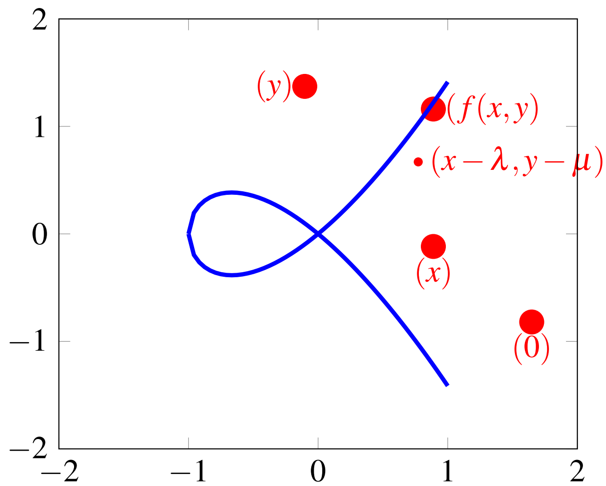

Figure 11. The affine plane(The scheme $\mathbb{A}^2_K = \mathbf{Spec}(K[x,y])$

Example 3.2.10. *Suppose* X_1 *and* X_2 *are schemes and if* $U_1 \subseteq X_1$ *and* $U_2 \subseteq X_2$ *are open subsets. Let* $\vartheta : (U_1, \mathscr{O}_{X_1}|_{U_1}) \to (U_2, \mathscr{O}_{X_2}|_{U_2})$ *be an isomorphism of locally ringed spaces. Define a scheme* X *obtained by gluing* X_1 *and* X_2 *along* U_1 *and* U_2 *through the isomorphism* ϑ*. The quotient of the disjoint union* $X_1 \cup X_2$ *by the equivalence relation* $x_1 \sim \vartheta(x_1)$ *for each* $x_1 \in U_1$*, with the quotient topology is the topological space of* X*.*

We then have maps $i_1 : X_1 \to X$ *and* $i_2 : X_2 \to X$*, and a subset* $V \subseteq X$ *is open if and only if* $i_1^{-1}(V)$ *is open in* X_1 *and* $i_2^{-1}(V)$ *is open in* X_2*. We define the structure sheaf* $\mathscr{O}_X$ *as follows: for every open set* $\mathscr{O}_X(V) = \{< s_1, s_2 > \mid s_1 \in \mathscr{O}_X, (i_1^{-1}(v))$ *and* $s_2 \in \mathscr{O}_X, (i_2^{-1}(v))$ *and* $\vartheta(s_1|_{i^{-1}(U) \cup U_1}) = s_2|_{i^{-1}(V) \cup U_2}\}$*.*

We notice that $\mathscr{O}_X$ *is a sheaf, and that* $(X, \mathscr{O}_X)$ *is a locally ringed space. Moreover, since* X_1 *and* X_2 *are schemes, every point of* X *has a an affine neighborhood, thus* X *is a scheme.*

Example 3.2.11 (A scheme which is non-separated and which is not an affine scheme)**.** *Let* k *be a field, let* $X_1 = X_2 = \mathbb{A}^1_k$*, and let* $U_1 = U_2 = \mathbb{A}^1_k \backslash \{p\}$*, where* p *is the point which corresponds to the maximal ideal* (x)*, and let* $\vartheta : U_1 \to U_2$ *be the identity map. Obtain a new scheme* X *by gluing* X_1 *and* X_2 *along* U_1 *and* U_2 *via* ϑ*. This is an "affine line with the point* p *doubled [Hart77]." (an example of a gluing construction.)*

Figure 12. Affine Line with point $\mathfrak{p}$ doubled

Example 3.2.12. *The* **Spec** $(\prod_{i=1}^{\infty} k)$*,* k *is a field. This topological space is the Stone-cech compactification of* $\mathbb{Z}_+$*.*

Example 3.2.13 (The "arithmetic surface"). *The* $\mathbf{Spec}(\mathbb{Z}[x])$. *This is an example which has a real mixing of arithmetic and geometric properties. The prime ideals in* $\mathbb{Z}[x]$ *are :*

(i) (0);

(ii) (p), *for* $p \in \mathbb{Z}$ *prime;*

(iii) principal prime ideals (f), *where* $f \in \mathbb{Z}[x]$ *is either a prime* p *or* $\mathbb{Q}$*-irreducible polynomial written such that its coefficients have g.c.d* 1; *and*

(iv) maximal ideals (p, f), $p \in \mathbb{Z}$ *is a prime and* $f \in \mathbb{Z}[x]$ *a monic integral polynomial irreducible modulo* p.

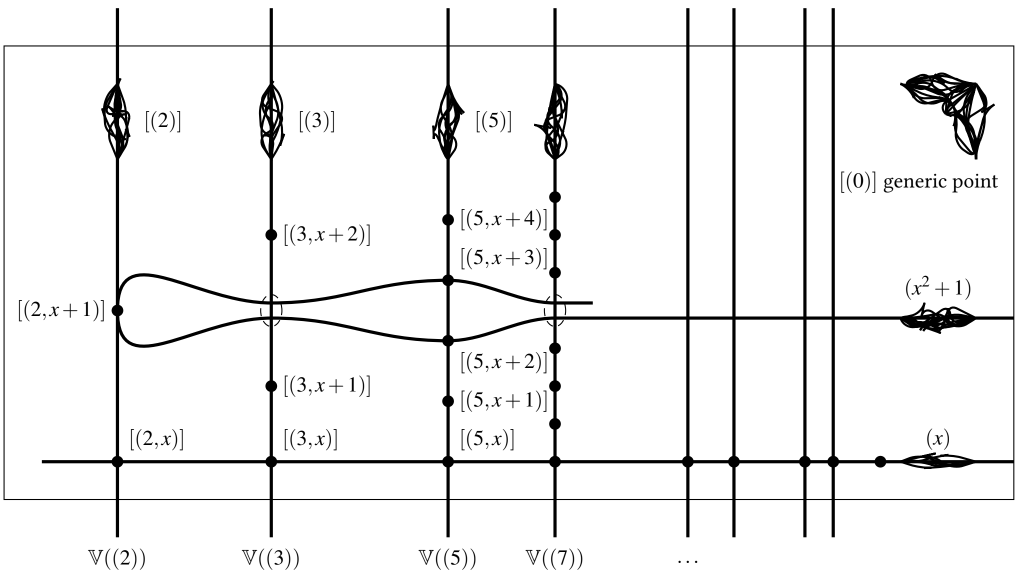

Figure 13. Mumford $\mathbf{Spec}(\mathbb{Z}[x])$ **diagram**

Morphisms of schemes

Definition 3.2.14. *A morphism between two schemes* X *and* Y *is a continuous map* $f: X \to Y$ *together with a map of sheaves on* Y, $f^{\sharp}: \mathscr{O}_Y \to f_\star \mathscr{O}_X$ *subject to the condition that if for any point* $p \in X$, *any neighbourhood* U *of* $q = f(p)$ *in* Y, *and any* $f_\star \in \mathscr{O}_Y$, $f_\star$ *varnishes at* q *if and only if* $f^{\sharp}$ *varnishes at* p.

To further explain the Definition 3.2.14, we give a corollary.

Theorem 3.2.15. *Let* X *be an arbitrary scheme and* R *a ring. Then there is a bijection*

$$\mathbf{Hom}(X, \mathbf{Spec}(R)) \cong \mathbf{Hom}(R, \mathscr{O}_X X)).$$

That is, the set of scheme morphisms from X *to* $\mathbf{Spec}(R)$ *can be identified with ring homomorphisms from* R *to the ring of global sections of* X.

In particular, if $X = \mathbf{Spec}(S)$ is also an affine scheme, then the maps $\mathbf{Spec}(S) \to \mathbf{Spec}(R)$ are basically the same thing as the maps $R \to S$ except that they go in the opposite direction.

Proj S Schemes

We now look at very fundamental class of schemes, constructed from graded rings, which are analogous to projective varieties.

Definition 3.2.16. *Suppose S is a graded ring. Denote by S_+ the ideal $\oplus_{d>0} S_d$. The set* **Proj** *S is defined as the set of all prime ideals $\mathfrak{p}$, which exclude all of S_+. Let a be a homogeneous ideal of S. We define the subset $V(a)$ as*

$$V(a) = \{\mathfrak{p} \in \mathbf{Proj}\ S \mid \mathfrak{p} \supseteq a\}$$

Lemma 3.2.17.

(1) Let $\mathfrak{p}$ and $\mathfrak{m}$ be homogeneous ideals in S. We have $V(\mathfrak{p}\,\mathfrak{m}) = V(\mathfrak{p}) \cup V(\mathfrak{m})$

(2) Let $\{\mathfrak{p}_i\}$ be any family of homogeneous ideals in S. We have $V(\sum \mathfrak{p}_i) = \bigcap V(p_i)$.

Proof. The proofs are the same as for Lemma $(3.1.1;1,2)$, when we consider the fact that a homogeneous ideal $\mathfrak{p}$ is prime if and only if for any two homogeneous elements $a, b \in S$, $ab \in \mathfrak{p}$ implies $a \in \mathfrak{p}$ or $b \in \mathfrak{p}$. □

Definition 3.2.18. *Let S be any graded ring. We define $(\mathbf{Proj}\ S, \mathscr{O})$ as the topological space together with the sheaf of rings defined on* **Proj** *S.*

Proposition 3.2.19. *Let S be a graded ring.*

(1) For every $\mathfrak{p} \in \mathbf{Proj}\ S$, the stalk $\mathscr{O}_\mathfrak{p} \cong S_{(\mathfrak{p})}$.

(2) For any homogeneous $f \in S_+$, let

$$D_+(f) = \mathfrak{p} \in \mathbf{Proj}\ S \mid f \notin \mathfrak{p}\}.$$

Then $D_+(f)$ is open in **Proj** *S.*

Additionally, these open sets cover **Proj** *S, and for any such open set, there is an isomorphism of locally ringed spaces*

$$(D_+(f), \mathscr{O}|_{D_+(f)}) \cong \mathbf{Spec}\ S_f,$$

such that $S_{(f)}$ is the sub-ring of elements of degree 0 in the localized ring S_f.

(3) **Proj** *S is a scheme.*

For the proof of Proposition 3.2.17 refer to [Hart77].

Example 3.2.20. *Let A be a ring , we define projective* $\mathbf{n}$*-space over A to be the scheme* $\mathbb{P}_A^{\mathbf{n}} = \mathbf{Proj}\ A\,[x_1,\ldots,x_{\mathbf{n}}]$. *In particular if* $A = k$ *is an algebraically closed field , then* $\mathbb{P}_k^{\mathbf{n}}$ *is a scheme whose subspace of closed points is naturally homeomorphic to the variety called projective* $\mathbf{n}$*-space denoted* $\mathbb{P}^{\mathbf{n}}$.

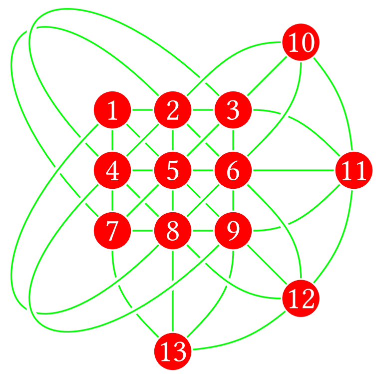

Figure 14. An example of a projective n-space

A variety can be converted into a scheme by naturally adding generic points for every irreducible subset of the variety.

The definition which follows helps to illustrate this result.

Definition 3.2.21. *If S is a fixed scheme, then a scheme over S is a scheme X, together with a morphism* $X \to S$.

If X and Y are schemes over S, a morphism of X to Y as scheme over S is a morphism $f : X \to Y$ *which is compatible with the given morphisms to S.*

The category of schemes over S is denoted by $\mathfrak{Sch}(S)$. *If A is a ring, we may write* $\mathfrak{Sch}(A)$ *for the category of schemes over* $\mathbf{Spec}(A)$. *from the category of varieties over k to the schemes over k. For any variety V, its topological space is homeomorphic to the set of closed points of* $\mathbf{sp}(t(V))$ *and its sheaf of regular functions is gotten by restriction of the structure sheaf of* $t(V)$ *through this homeomorphism.*

Proposition 3.2.22. *Let k be an algebraically closed field. There is a natural fully faithful functor:* $\mathfrak{Var}(k) \to \mathfrak{Sch}(k)$ *from the category of varieties over k to schemes over k. For any variety V, its topological space is homeomorphic to the set of closed points of* $\mathbf{sp}(t(V))$ *and* *its sheaf of regular functions is obtained by restricting the structure sheaf of $t(V)$ via this homeomorphism.*

For the proof of Proposition refer to [Hart77].

3.3 First properties of schemes.

In this section, we give some of the first properties of schemes.

Definition 3.3.1. *We say a scheme is connected if its topological space is connected. A scheme is said to be irreducible if its topological space is irreducible.*

Definition 3.3.2. *A scheme X is reduced if for every open set U, the ring $\mathscr{O}_X(U)$ has no nilpotent elements. Equivalently, X is reduced if and only if the local ring $\mathscr{O}_p$, for all $p \in X$, contain no nilpotent elements.*

Definition 3.3.3. *A scheme X is said to be integral if for every open set $U \subseteq X$, the ring $\mathscr{O}_X(U)$ is an integral domain.*

Example 3.3.4. *Let $X = \mathbf{Spec}(R)$ be an affine scheme. We say X is irreducible if and only if the nilradical* nil $R\,of\,R$ *is prime. Further, X is reduced if and only if* nil $R = 0$ *and X is integral if and only if R is an integral domain.*

Proposition 3.3.5. *A scheme is integral if and only if it is both reduced and irreducible.*

`Proof.` An integral scheme is clearly a reduced scheme. Suppose X is not irreducible, then we can find two non-empty disjoint open subsets U_1 and U_2 such that $\mathscr{O}(U_1 \cup U_2) = \mathscr{O}(U_1) \times \mathscr{O}(U_2)$ is not an integral domain. So irreducible is implied by integral.

In the other direction, suppose that X is reduced and irreducible. Let $U \subseteq X$ be an open subset, and suppose that there are elements $f, g \in \mathscr{O}(U)$ such that $fg = 0$. Let $Y = \{x \in U |\, f_x \in \mathfrak{m}_x\}$, and $Z = \{x \in U |\, g_x \in \mathfrak{m}_x\}$. Then Y and Z are closed subsets, and $Y \cup Z = U$. But X is irreducible so U is irreducible, so one of Y or Z is equal to U, say $Y = U$. But then the restriction of f to any open affine subset of U will be nilpotent, thus 0, therefore f is 0, an indication that X is integral. □

Definition 3.3.6. *If a scheme X can be covered by open subsets $\mathbf{Spec}(R_i)$ such that R_i is a Noetherian ring, then the scheme X is locally Noetherian. X is Noetherian if it is locally Noetherian and quasi-compact. Equivalently, X is Noetherian if it can be covered by a finite number of open affine subsets $\mathbf{Spec}(R_i)$, where R_i a Noetherian ring.*

Proposition 3.3.7. *A scheme X is locally Noetherian if and only if for every open affine Scheme $X = \mathbf{Spec}(R)$, R is a Noetherian ring. In particular, an affine scheme $X = \mathbf{Spec}(R)$ is Noetherrian scheme if and only if the ring R is a Noetherian ring.*

For the proof we can refer to [Hart77]

Example 3.3.8. *Quasi-projective varieties are Noetherian schemes: In particular we have the following:*

1. *algebraic curves.*
2. *elliptic curves.*
3. *Shimura varities*
4. *K3 surfaces*
5. *cubic surfaces.*

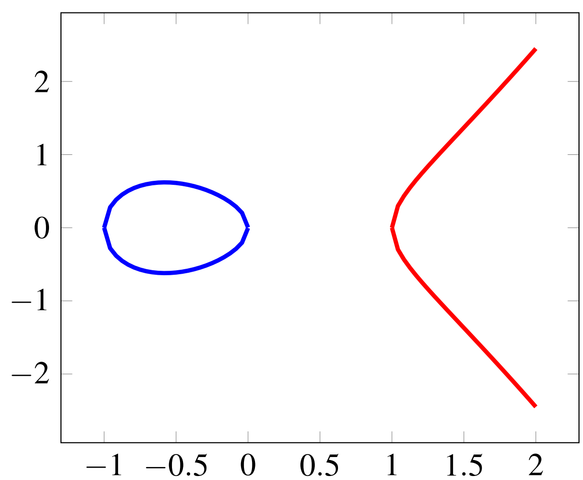

Figure 15. cubic plane curve

Lemma 3.3.9. *Let $X = \mathbf{Spec}\, R$ be an affine scheme. Then*

$$X \text{ irreducible} \iff \text{Nil}\, R \text{ is prime}$$
$$X \text{ reduced} \iff \text{Nil}\, R = 0$$
$$X \text{ integral} \iff R \text{ integral.}$$

Definition 3.3.10. *Let $f\colon X \to Y$ be a morphism of schemes. We say f is locally of finite type if there is a covering of Y by open affine subsets $V_i = \mathbf{Spec}(B_i)$, such that for each i, we can cover $f^{-1}(V_i)$ by open affine subsets $U_{ij} = \mathbf{Spec}(R_{ij})$, where each R_{ij} is finitely generated B_i- algebra. We say the morphism f is of finite type if in addition we can cover each $f^{-1}(V_i)$ by a finite number of the U_{ij}.*

Definition 3.3.11. *Let $f\colon X \to Y$ a morphism of schemes. We say f is a finite morphism if there is a covering of Y by open affine subsets $V_i = \mathbf{Spec}(B_i)$, such that for each i, $f^{-1}(V_i)$ is affine, equal to $\mathbf{Spec}(R_i)$, where R_i is a B_i- algebra which is a finitely generated B_i-module.*

Example 3.3.12. *Suppose V is a variety over algebraically closed field k, then the associated scheme $t(V)$ is an integral Noetherian scheme of finite type over k. In fact, we can cover V by a finite number of open affines of the form $\mathbf{Spec}(R_i)$, where each R_i is an integral domain and a finitely generated k-algebra, thus Noetherian.*

Example 3.3.13. *Suppose $p \in V$ is a point of a variety V, with local ring $\mathscr{O}_p$, then $\mathbf{Spec}(\mathscr{O}_p)$ is an integral Noetherian scheme, which is not of finite type over k in general.*

Open and closed sub-schemes.

Definition 3.3.14. *We say that an open sub-scheme of a scheme X is a scheme T, whose topological space is an open subset of X and whose structure sheaf $\mathscr{O}_T$ is isomorphic to the restriction $\mathscr{O}_X|_T$ of the structure sheaf of X.*

Definition 3.3.15. *An open immersion is a morphism $f\colon X \to Y$ which induces an isomorphism of X with an open subscheme of Y.*

A closed immersion is said to be a morphism $f\colon Y \to X$ of schemes such that a homeomorphism of $\mathbf{sp}(Y)$ onto a closed subset of $\mathbf{sp}(X)$ is induced by f, and moreover the induced map $f^\sharp : \mathscr{O}_X \to f_\star \mathscr{O}_Y$ of sheaves on X is surjective.

Definition 3.3.16. *We define a closed subscheme of a scheme X as an equivalence class of closed immersions, where $f\colon Y \to X$ and $i : Y' \to X$ such that $f' = f \circ i$.*

Example 3.3.17. *If R is a ring, and $\mathfrak{p}$ is an ideal of R. Let $X = \mathbf{Spec}(R)$ and $Y = \mathbf{Spec}(R/\mathfrak{p})$. Then a morphism of schemes $f\colon Y \to X$ which is a closed immersion is induced by the ring homomorphism $R \to R/\mathfrak{p}$. We say f is a homeomorphism of Y onto the closed subset $V(\mathfrak{p})$ of X, and the map of structure sheaves $\mathscr{O}_X \to f_* \mathscr{O}_Y$ is surjective since it is surjective on the stalks, which are localizations of R and $R/\mathfrak{p}$, respectively. So then, for any ideal $\mathfrak{p} \subseteq R$ there is a structure of closed subscheme on the closed set $V(\mathfrak{p}) \subseteq X$. In particular, every closed subset Y of X has many closed subscheme structures, which correspond to all the ideals $\mathfrak{p}$ where $V(\mathfrak{p}) = Y$. The upshot is, every closed subscheme structure Y of an affine scheme X originates from an ideal in the same manner.*

Example 3.3.18. *If $R = k[x,y]$, where k is a field, then $\mathbf{Spec}(R) = \mathbb{A}^2_k$ is the affine plane over k. The ideal $\mathfrak{p} = (xy)$ produces a reducible subscheme, which consists of the union of the x and y-axes. The ideal $\mathfrak{p} = (x^2)$ gives a subscheme structure which has the nilpotents on the y-axis.*

The ideal $\mathfrak{p} = (x^2, xy)$ produces another subscheme structure on the y-axis, which has nilpotents only in the local ring at the origin. This subscheme is said to have the origin as an embedded point.

Example 3.3.19. *If V is an affine variety over the field k, and Q a closed sub-variety, then Q corresponds to a prime ideal $\mathfrak{p}$ in the affine coordinate ring R of V. If $X = t(V)$ and $Y = t(Q)$*

are the associated schemes, then $X = \mathbf{Spec}(R)$ *and* $Y = t(Q)$ *is the closed sub-scheme defined by* $\mathfrak{p}$. *For every* $n \geq 1$ *let* Y_n *be the closed sub-scheme of* X *corresponding to the ideal* $\mathfrak{p}^n$, *then* $Y_1 = Y$, *but for every* $n > 1$, Y_n *is a non-reduced scheme structure on the closed set* Y, *which has no correspondence to any sub-variety of* V.

Y_n *is called the* n^{th} *infinitesimal neighborhood of* Y *in* X *and the scheme* Y_n *exhibit properties of the embedding of* Y *in* X.

Example 3.3.20. *If* X *is a scheme, and* Y *is a closed subset. Then in general* Y *contains many possible closed sub-scheme structures. However, there is one which is "smaller" than any other, which we call the reduced induced closed sub-scheme structure.*

We give a universal property of reduced induced sub-scheme structure.

Definition 3.3.21. *Let* S *be a scheme, and let* X, Y *be schemes over* S, *i.e., schemes with morphisms to* S. *We define the fibred product of* X *and* Y *over* S, *denoted* $X \times Y$, *to be a scheme, together with morphisms* $\rho_1 : X \times_S Y \to X$ *and* $\rho_2 : X \times_S Y \to Y$, *which makes a commutative diagram with the given morphisms* $X \to S$ *and* $Y \to S$, *such that given any scheme* Z *over* S, *and given morphisms commutative diagram with morphisms* $X \to S$ *and* $Y \to S$, *then there exists a unique morphism* $f \colon Z \to X$ *and* $g \colon Z \to Y$ *which makes a commutative diagram with the given morphisms* $X \to S$ *and* $Y \to S$, *then* $\theta \colon Z \to X \times_S Y$ *such that* $f = \rho_1 \circ \theta$, *and* $g = \rho_2 \circ \theta$. *The morphism* ρ_1 *and* ρ_2 *are called the projection morphisms of the fibred product onto its factors.*

Let X *and* Y *be schemes given without reference to any base scheme* S. *We take* $S = \mathbf{Spec}(\mathbb{Z})$ *and define the product of* X *and* Y, *denoted* $X \times Y$, *to be* $X \times_{\mathbf{Spec}(\mathbb{Z}} Y$.

Theorem 3.3.22. *Let* X *and* Y *be any two schemes over a scheme* S. *There exists a fibred product* $X \times_S Y$, *which is unique up to unique isomorphism.*

The proof of Theorem 3.3.20 can be found in [Hart77]

Definition 3.3.23. *If* $f \colon X \to Y$ *is a morphism of scheme, and* $y \in Y$ *is a point. If* $k(y)$ *is the residue field of* y, *and let* $\mathbf{Spec}\, k(y) \to Y$ *be the natural morphism. Then we define the fibre of the morphism* f *over the point* y *to be the scheme*

$$X_y = X \times_Y \mathbf{Spec}\, k(y).$$

3.4 Dimension of a scheme

3.4.1 Definition,first properties.

Definition 3.4.1. *Let X be a scheme. The dimension of X (denoted as* $\dim X$ *)is its dimension as a topological space.*

Let Y be an irreducible closed subset of X. The codimension of Y in X which we denote by codim (Y,X) is the supremum of all integers n such that there exists a chain

$$Y = Y_0 \subset Y_1 \ldots \subset Y_n$$

of distinct irreducible closed subsets of X, which begins from with Y.

Let Z be any closed subset of X, one defines

$$\mathrm{codim}(Z,X) = \inf_{Y \subseteq Z} (\mathrm{Y},\mathrm{X})$$

, the infimum taken over all closed irreducible subsets of Z.

Remark 3.4.2. *The* $\dim X$ *depends only on the topological space structure of X*

Remark 3.4.3. *In general the equality* $\dim Y + codim(Y,X) = \dim X$ *does not hold, even if X is an integral affine scheme. Take $X = \mathbf{Spec}(A)$ where $A = R[x]$ and $R = K[[t]]$. Then the prime ideal $\mu = (tx-1)$ of A satisfies ht $\mu = 1$, but $A/\mu \simeq R\left[\frac{1}{t}\right]$ is a field,hence is of dimension* 0. *Nevertheless* $\dim A = \dim R + 1 = 2$

Similarly the dimension of a dense open subset of $X = \mathbf{Spec}(A)$ might be strictly smaller than the dim X : for example,take $\mathbf{Spec}(A) = k[[t]]$, then $\dim\left(A\left[\frac{1}{t}\right]\right) = 0$, thus the dimension of $D(t) \subset \mathbf{Spec}(A)$ is 0.

Proposition 3.4.4. *Let $X = \mathbf{Spec}\ A$. The dimension of X,* $\dim X$ *is the Krull dimension* $\dim A$ *of A.*

Recall that the Krull dimension of a ring A is the supremum of ht μ for $\mu \in \mathbf{Spec}\,A$, where the ht μ of a prime ideal μ is the supremum of all n such that there exists a chain

$$\mu_0 \subset \ldots \subset \mu_n = \mu$$

of distinct prime ideals of A. Also *ht* $\mu = \dim A_\mu$

`Proof.` An irreducible closed subset of $\mathbf{Spec}\,A$ is of the form $V(\mu)$ with μ prime. Now for ideals $\mathfrak{p},\mathfrak{m}$ equal to their radicals (e.g. prime ideals),the equality $V(\mathfrak{p}) \subset V(\mathfrak{m})$ is equivalent to $\mathfrak{p} \supset \mathfrak{m}$,hence the result. □

Example 3.4.5. *The dimension of* $\mathbb{A}_k^n$ *is* $\mathbf{n}$ *(this follows from the fact that for any Noetherian ring* A, $\dim A[x_1,\ldots,x_n] = n + \dim A$*). The same is true for* $\mathbb{P}_k^n$.

Example 3.4.6. *The dimension of Spec* $\mathbb{Z}$ *is* 1 *(likewise for any principal ideal domain).*

Example 3.4.7. *The dimension of* $\mathbf{Spec}(k)$ *or* $\mathbf{Spec}(k[\xi])$ *is zero for any field k*

Example 3.4.8. *Some rings of dimension* 1 *are not Noetherian (take a ring of integers of* $\overline{QQ_p}$*, some Noetherian rings are not of finite dimension.*

3.4.2 Dimension and Schemes of finite type over a field

We give the main result in the following theorem; a consequence of a difficult but important result in commutative algebra.

Theorem 3.4.9. *Let* X *be an integral scheme of finite type over a field k, with function field k. Then*

(1) $\dim X$ *is finite, equal to the transcendence degree trdeg*(K/k) *of* K *over* k.

(2) For any non empty open subset U *of* X, $\dim X = \dim U$

(3) For any closed point $p \in X$, $\dim X = \dim \mathscr{O}_{X,p}$

`Proof.` We have that (2) follows from (1) since X and U have the same function field. To prove (1), we remark that if (U_i) is an open cover of X, then U and U_i have the same function and $\dim X = \sup_i(\dim U_i)$; therefore it is enough to prove (1), When $X = \mathbf{Spec}(A)$ is affine, where A is a finitely generated $k-$ algebra with quotient field k.

Now the formula $\dim A = \mathrm{trdeg}(K/k)$ is a classical result in commutative algebra. It is a consequence of Noether's normalization lemma: there exists $y_1,\ldots,y_r \in A$, algebraically independent over k, such that A is a finite module over $k[y_1,\ldots,y_r]$.

To prove (3) we may assume using (2) that X is affine. Then the result follows from the formula

$$\dim(A/\mu) + ht\mu = \dim A$$

which holds for any finitely generated $k-$ algebra A and any prime ideal μ of A (another consequence of Noether's normalization lemma). □

Remark 3.4.10. *If* X *is any scheme of finite type over a field k, write* $X = \cup_{i=1}^r Y_i$ *the decomposition of* X *into irreducible closed subsets, and give* Y_i *its structure of reduced scheme,* *then the dimension of each Y_i can be computed using the formula with the transcendence degree, because each Y_i can now be considered as an integral scheme. Then we have*

$$\dim X = \sup_{1\leq i\leq r} \dim Y_i$$

indeed any closed irreducible subset Y of X satisfies $Y = \cup_{1\leq i\leq r}(Y\cap Y_i)$, hence $Y\cap Y_i = Y$ for some i,that is $Y\subset Y_i$, therefore any descending chain of irreducible closed subsets of X is contained in some Y_i

We give another consequence of the above principle that extends theorem 3.4.10 to non integral schemes. Let us say that a Noetherian scheme is pure if each irreducible component of Y has the same dimension.

Proposition 3.4.11. *Let X be a scheme of finite type over a field k. Then*

(1) For a non empty open subset U, we have $\dim U = \dim X$ if U is dense or if X is pure.

(2) If X is pure, any closed irreducible subset Y of X satisfies

$$\dim Y + codim\,(Y,X) = \dim X$$

Proof.

(1) If X is irreducible, we may assume it is reduced, hence integral and we apply Theorem 3.4.10. In general, let $X = \cup_{1\leq i\leq r} Y_i$ be the decomposition of X into irreducible subsets. Then any non empty open subset U of X meets Y_i for some i. If X is pure, then $\dim X = \dim Y$ and $\dim U = \dim\,(U\cap Y_i)$ because $U\cap Y_i$ is a non empty open subset of the irreducible scheme Y_i. Now assume that U is dense (but not necessarily pure.) Then $U\cap Y_i = \emptyset$ for any $i = 1,\dots,r$ because each Y_i, contains a non empty open subset of X. Thus $\dim\,(U\cap Y_i) = \dim Y_i$ by the previous argument. Since $\dim U = \sup_{1\leq i\leq r}\dim(U\cap Y_i)$ and $\dim X = \sup_{1\leq i\leq r}\dim Y_i$ and we are done.

(2) Since Y is contained in some irreducible component of X is pure, we may assume X irreducible. Let U be an affine open subset of X containing some point of Y, then $\dim X = \dim U$ and $\dim Y = \dim(Y\cap U)$ by 1. Moreover codim $(Y,X) = \text{codim}(Y\cap U,\ X\cap U)$. Indeed $Z\longmapsto Z\cap U$ is a strictly increasing bijection between irreducible closed subsets of U. containing $Y\cap U$. Therefore we may assume $X = \mathbf{Spec}\ A$ affine and $Y = V(\mu)$ with $\mu = \mathbf{Spec}\ A$. Now the formula follows from

$$\dim(A/\mu) + ht\mu = \dim A$$

which holds for any k - algebra of finite type.

□

Proposition 3.4.12. *Let X be a scheme of finite type over a field k. Then the closed points of X are dense.*

Again, this is false in general, e.g., $X = \mathbf{Spec}((k[[t]])$.

Proof. We assume that X is integral. Let $U = \mathbf{Spec}(A)$ be affine open subset of X. Then A has a maximal ideal, that is there exists a point $x \in U$ which is closed in U. By a theorem, we have $\dim \mathscr{O}_{X,P} = \dim \mathscr{O}_{U,P} = \dim U = \dim X$. This shows that x is closed in any open affine subset $\mathbf{Spec}(B)$ of X. Therefore x is closed in X, and any non empty open subset of X has a closed point. Another approach consists of the fact that a point is closed in X if and only if its residue field is a finite extension of k. □

3.4.3 Morphisms and Dimension

Here again the 'intuitive' results are false in general. For a counter example we consider a morphism $f\colon Y \to X$ which might be surjective with $\dim Y \leq \dim X$ e.g. $Y = \mathbf{Spec}(k((t)) \oplus k)$, $X = \mathbf{Spec}(k[[t]])$. Then X is of dimension 1, Y is of dimension 0, but the morphism $Y \to X$ induced by the homomorphism $k[[t]] \to k((t)) \oplus k,\ f(t) \mapsto (f(t), f(0)$ is surjective. The situation is a bit better in 2 cases: finite morphisms and morphisms between schemes of finite type over a field. We capture the result in the Theorem 3.4.13 with the assumption that 'surjective' is of course necessary (for example a closed immersion is a finite morphism.)

Theorem 3.4.13. *Let $f\colon Y \to X$ be a finite, surjective morphism of Noetherian schemes. Then* $\dim Y = \dim X$.

sketch proof. We can reduce immediately to the case when X and Y are affine. Let $f\colon \mathbf{Spec}(Y) \to \mathbf{Spec}(X)$ be a finite and surjective morphism, we show that $\dim X = \dim Y$. Suppose that $\mathbf{Spec}(Y)$ and $\mathbf{Spec}(X)$ are reduced, replacing the homomorphism $i\colon X \to Y$ by $X_{\text{red}} \to Y_{\text{red}}$ (which is finite as well).In essence, it is related to the "going-up" theorem: If X is a subring of Y with Y/X finite, then for any pair of ideals $p_1 \subset p_2$ of X, and any ideal P_1 of Y lying over p_1, there is an ideal $\mathfrak{p}_2 \supset \mathfrak{p}_1$ of Y lying over p_2 □

We give another result in the Theorem 3.4.14 coming from commutative algebra.

Theorem 3.4.14. *Let $f\colon Y \to X$ be a morphism of Noetherian schemes. Let $y \in Y$ and $x \in f(y)$. Let Y_x be the fibre of Y at x. Then*

$$\dim\ \mathscr{O}_{Y,y} \leq \dim\ \mathscr{O}_{X,x} + \dim_x Y_x$$

($\dim_x Y_x$ is the dimension of the local ring of the fibre Y_x at x). A special case is when x, y are closed points of integral schemes of finite type over a field. Then the inequality means $\dim Y - \dim X \leq \dim Y_x$.

Proof. One reduces immediately to the affine case $Y = \mathbf{Spec}(B)$, $X = \mathbf{Spec}(A)$; then it is the formula $\dim B_\mu \leq \dim A_p + \dim (B_\mu \otimes_A k(p))$ which holds for each prime ideal μ of B and its inverse image $p \in \mathbf{Spec}(A)$ □

We give a more precise result for schemes of finite type over a field.

Theorem 3.4.15. *Let $f: Y \to X$ be a dominant morphism of integral schemes of finite type over a field k. Set $e = \dim Y - \dim X$. Then there is a non empty open subset U of Y such that for any $x \in f(U)$, the dimension of the fiber U_x is e.*

Sketch proof. Shrinking Y and X if necessary, we may assume that $Y = \mathbf{Spec}(B)$ and $X = \mathbf{Spec}(A)$ are affine. The generic fiber Y_n is an integral scheme with the same function field L as Y. Since

$$\mathrm{trdeg}(L/k) = \mathrm{trdeg}(L/K) + \mathrm{trdeg}(K/k)$$

we obtain that $\dim Y_n = e$ by Theorem 3.4.15, that is L/K is of transcendence degree e. Let $t_1, \ldots, t_e$ be a transcendence base of L/K. Localizing A and B if necessary, we can assume that $t_1, \ldots, t_e \in B$ and that B is a finite module over $A[t_1, \ldots, t_e]$ because L is a finite field extension of $K(t_1, \ldots, t_e)$. Set $X_1 = \mathbf{Spec}(A[t_1, \ldots, t_e])$, then the morphism f factors through a finite morphism $Y \to X_1$. Now for $x \in X$, the fiber Y_x has a finite and surjective morphism to the fiber of $X_1 \to X$ at x; the latter is isomorphic to $\mathbb{A}^e_{k(x)}$, hence is of dimension e. We conclude the proof by theorem 3.4.14 □

3.5 Constructions

In this chapter, we define a few useful constructions on schemes.

One of the basic ways to construct new topological spaces from the initial ones is to glue them together. Similarly, gluing can be done with schemes.

Consider a collection of schemes $\{X_\alpha\}$ and an open set $X_{\alpha\beta}$ in X_α for each $\beta \neq \alpha$. If we also have homomorphisms of schemes

$$\rho_{\alpha\beta}: X_{\alpha\beta} \to X_{\beta\alpha}$$

with the condition that $\rho_{\alpha\beta} = \rho_{\beta\alpha}^{-1}$,

$$\rho_{\alpha\beta}(X_{\alpha\beta} \cap X_{\alpha\gamma}) = X_{\beta\alpha} \cap X_{\beta\gamma}$$

and

$$\rho_{\beta\gamma} \circ \rho_{\alpha\beta}|_{(X_{\alpha\beta} \cap X_{\alpha\gamma}) = \rho_{\alpha\gamma}}|_{X_{\alpha\beta} \cap X_{\alpha\gamma}}$$

then we can define a new scheme X by identifying the X_α along the maps $\rho_{\alpha\beta}$.

Definition 3.5.1. *Given morphisms of schemes $f : X \to S$ and $g : Y \to S$, the fibre product of X and Y over S is a scheme $X \times_S Y$ together with maps $X \times_S Y \to X$ and $X \times_S Y \to Y$ that makes the following diagram a pullback:*

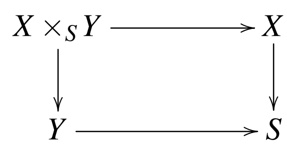

Figure 16. A pullback diagram.

By virtue of the fact that fiber products are pullbacks, they are unique if they exist. Also we need to note that the fiber product really does depend on the maps f and g, despite the terminology and notation. We begin our actual constructions by considering affine schemes.

Since affine schemes are dual to commutative rings, a pushout of commutative rings, dualized, would make a perfectly good fiber product of affine schemes. The following diagram is actually a pushout in the category of commutative rings:

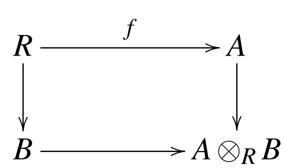

Figure 17. A pushout diagram in commutative rings

where the maps f and g give A and B R- algebra structures, and the tensor product is taken with this structure in mind. This diagram is a push-out by the universal property of the tensor product.

Dualizing this we get:

Definition 3.5.2. *Given maps ϑ : $\mathbf{Spec}(A) \to \mathbf{Spec}(R)$ and φ : $\mathbf{Spec}(B) \to \mathbf{Spec}(R)$, we define the fiber product to be*

$$\mathbf{Spec}(A) \times_R \mathbf{Spec}(B) := Spec\ (A \otimes_R B)$$

For arbitrary schemes, we simply decompose them into affine schemes, apply this definition, and glue them back together using the gluing construction.

4 Conclusion

4.1 Summary of the dissertation

In this dissertation we have :

1. built the Foundation to the theory of schemes beginning with some basic results on Algebraic Geometry and some local algebra, the language of categories with examples followed by an exhibition of Functors with examples then followed by the sheaf theory.

2. defined the spectrum of a ring, $\mathbf{Spec}(R) = \{\text{prime ideals } \mathfrak{p} \in R\}$ and also given the topology of the spectrum of a ring R.

3. Constructed structure sheaf on rings and subsequently defined an affine scheme thus a ringed space $(X, \mathscr{O}_X)$ which is isomorphic to the spectrum of some ring. We have also given the definition of a scheme thus: a scheme is a locally ringed space $(X, \mathscr{O}_X)$ such that $(U, \mathscr{O}_X|_U)$ is an affine scheme.

4. given numerous examples on the spectrum of rings and of affine schemes and provided some properties of schemes. The dissertation has further discussed the dimension of a scheme and concluded by exhibiting the gluing construction of schemes.

4.2 Future Research Direction

My future research interest is to study:

1. the application of finite rings based algebraic schemes of evolving S-boxes for images encryption.

2. the application of linear algebraic techniques to construct monochrome visual cryptographic schemes for general access structures and its application to colour images.

3. the numerical tests and theoretical estimations for a Lie-algebraic scheme of discrete approximations.

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