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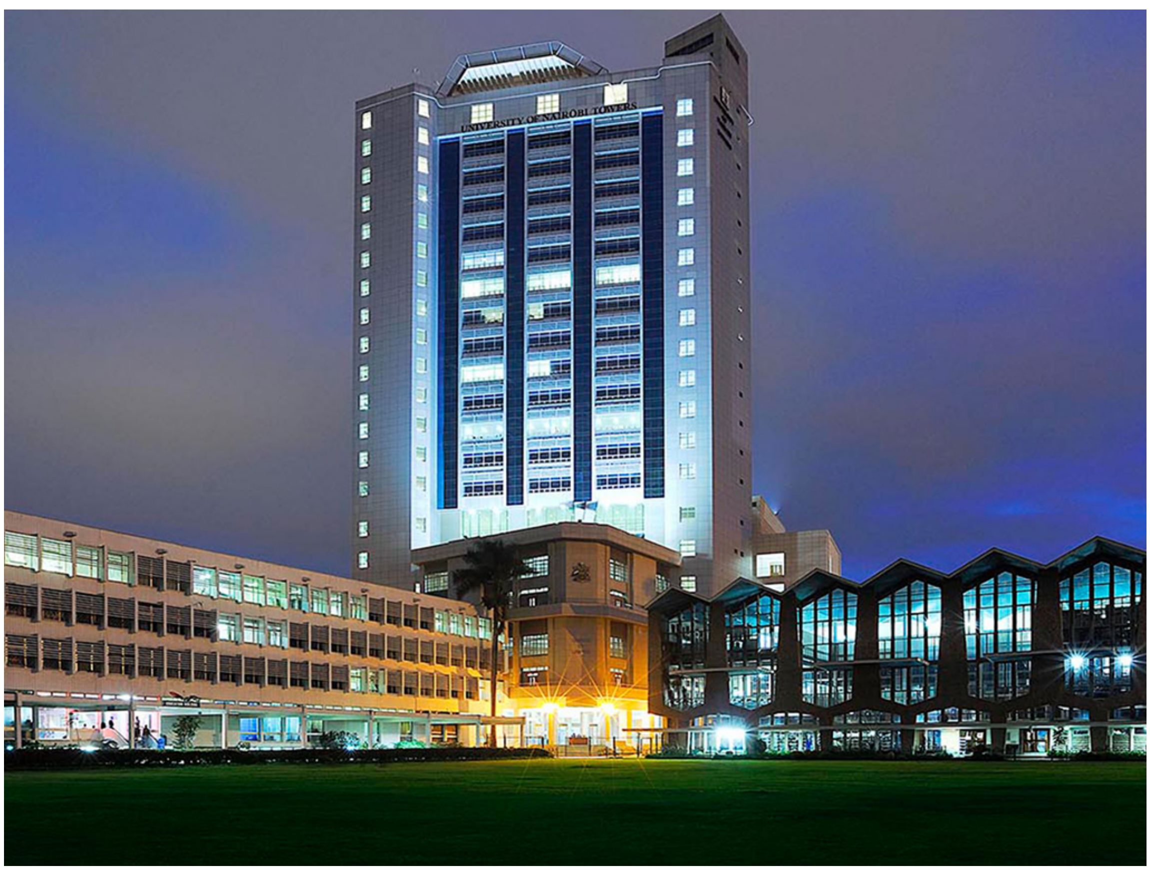

Submitted to the School of Mathematics in partial fulfilment for a degree in Masters of Science in Applied Mathematics

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STUDY OF NEW CURVATURE TENSORS ON PARA KENMOTSU MANIFOLD AND OTHER RELATED MANIFOLDS.

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Masters Thesis
Submitted to the School of Mathematics in partial fulfilment for a degree in Masters of Science in Applied Mathematics

Submitted to: The Graduate School, University of Nairobi, Kenya

Abstract

The objective of this paper is to study a Para Kenmotsu manifold admitting the W_7-curvature tensor. We will investigate the geometry of a Para Kenmotsu manifold when it is W_7-flat, W_7-semisymmetric, W_7-symmetric and W_7-recurrent. Also, we will establish the necessary condition for a P-Kenmotsu manifold to be W_7-irrotational.

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Declaration and Approval

I the undersigned declare that this dissertation is my original work and to the best of my knowledge, it has not been submitted in support of an award of a masters degree in any other university or institution of learning.

23.08.2021

Signature Date

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In my capacity as a supervisor of the candidate's dissertation, I certify that this dissertation has my approval for submission.

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Dedication

This project is dedicated to my family and my future kids.

Contents

Acknowledgments

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Esther Nafula

Nairobi, 2021.

1 Chapter 1

1.1 Introduction

Mathematics is a natural science with a peculiar mode of operation. It is safe to say that Geometry is an ancient creation of the Greeks. Geometry is a branch of mathematics that deals with individual shapes, properties of space, and spatial relationships among several objects. A geometric property independent of the geometric curve configuration or of a surface under review as a whole but relies only on the configuration form is referred to as the *local property* and the corresponding geometry called *local geometry*. The study of surfaces and curves is called *global property* and the related geometry is called *global geometry*. The main branches of Geometry are projective Geometry, Euclidean Geometry, analytic Geometry, topology, non-Euclidean Geometry, and Differential Geometry.

Differential geometry is a branch of geometrical mathematics that deals with surfaces and space curves by means of differential calculus. Differential Geometry is not a new concept in the field of Mathematics due to its historical background. Between the 19^{th} and 20^{th} centuries, differential geometry came about from the theory of curves, planes, and surfaces in the Euclidean space.

The major theme of both contemporary geometrical dynamics and geometry is based on the concept of a manifold; an abstract mathematical space locally resembling the Euclidean geometry spaces.

Differential geometry's precepts are relatively new in modern Mathematics. German philosopher, Carl Gauss used differential geometry to characterize the intrinsic properties of curves and surfaces. Carl showed that the intrinsic curvature of a cylinder is similar to that of a plane, but different from that of a sphere.

Riemann Geometry is the most advanced section of differential geometry of manifolds. A Riemann space is a differentiable manifold on a Euclidean metric, depending on a smooth point. Other concepts in differential Geometry include manifolds, fibre bundles, groups, and groupoids. Generally, curvature differential geometry deals with smooth curves in the plane and in the Euclidean Space. The rapid evolution in the 20th century has made

Riemann geometry to be one of the most vital mathematical concepts in modern times. The concept of differentiable manifold generalizes surfaces and curves in R3 that are described in differential geometry. The essentials of differential geometry and topology include multi-linear algebra manifolds, differentiation and integration of manifolds, Lie algebras and Lie groups, homotopy, Rham co-homology, vector bundles, homology, Riemann and pseudo-Riemann geometry, and degree theory. It is largely known that the major distinction between the geometry of sub manifolds in Riemann manifolds and in semi-Riemann manifolds is that in the case of semi-Riemann manifolds, the semi-Riemann metric induced metric tensor field on the ambient space is not non-degenerate.

Authors defined para-contact geometrical structure on pseudo-Riemann manifold, and some of its remarkable sub-classes such as the para-Sasaskian manifolds. Local symmetries with respect to local geodesic geometries have been used to classify some particular classes of Riemann manifolds; and the curvature geodesic-led local symmetries contributed to classification of the famous local symmetric manifolds and constant curvature spaces. The study of local symmetries leads to outstanding geometrical results founded on the geodesic manifolds.

1.2 Definitions and Notations

Definition 1.2.1. *Given a C^{k+1} manifold M, with $k \geq 1$. For any subset, U of M, a vector field on U is, any section, ξ of T(M) over U, that is any function.*

$\xi : U \longrightarrow T(M)$, such that $\pi o \xi = idU$ That is $\xi(P) \in T_p(M)$ for every $p \in U$. The set of all vector fields constitute a vector space.

Definition 1.2.2. *A 1-form vector $\vec{r}$ defined at ρ is a linear scalar operator acting as a vector space V_ρ to real numbers $\Re$.*

This means that ,

(1) $\vec{r} : V_\rho \longrightarrow \Re$

(2) For any $(u, v) \in V_\rho$ and if $a, b \in \Re \Rightarrow \vec{r}(au + bv) = a\vec{r}u + b\vec{r}v$

The set of all 1-forms defined at ρ is called *a co-vector or a dual space of* V_ρ, and it is denoted by $V\rho^*$. This is also *an n-dimensional vector space.*

Definition 1.2.3. *Tensors are a generalization of vectors and 1-forms (co-vectors) since any vector* $u \in V_\rho$ *can be associated with a linear scalar operator acting on 1-form* $u \in V_\rho$ *to* $\Re$ *i.e.* $u\vec{r} \neq \vec{r}u : V_\rho^* \longrightarrow \Re$.

Definition 1.2.4. *Let* $c : (-\in, \in) \subset \Re \rightarrow M$ *be a differentiable curve on a manifold M. Consider all the functions* $C^\infty(p), f : M \rightarrow \Re$ *that are differentiable at* $c(0) = p$. *We say that the tangent vector to the curve c at p is the operator* $\dot{c}(0) : C^\infty(p) \rightarrow \Re$ *defined by*

$$\dot{c}(0)(f) = \frac{\delta(foc)}{\delta t}(0).$$

A tangent vector to M at p is a tangent vector to some differentiable curve c in M such that $c(0) = p$. The tangent space at p is the space of all tangent vectors at p and is denoted by T_pM

Definition 1.2.5. *Let M be a smooth manifold. Then by a Riemannian metric on this manifold we mean a tensor field* $g \in T^2(M)$ *that is symmetric and positive definite symmetric in the sense that* $g(X,Y) = g(Y,X)$ *positive definite implies that* $g(X,X) > 0$ *If* $X \neq 0$

A Riemannian metric therefore determines an inner product on each tangent space T_m. A manifold together with a given Riemannian metric is called *a Riemannian manifold.*

The tangent space of a smooth manifold M at p is a vectorial space of dimension n, and the operators $\left(\frac{\delta}{\delta x^i}p\right)_{i\in 1,\ldots,n}$ is determined by coordinate chart at p and is called *associated basis* to that chart or *holonomic frame*. We take a general basis e_i that is not associated to the previous chart.

The disjoint union of all tangent space T_pM of M at all points is called *the tangent bundle* and is denoted by $TM = U_pT_p$. We can define functions on TM that give a tamgent vector for each $p \in M$

Definition 1.2.6. *Consider a smooth manifold M and its tangent bundle TM. A vector field X is defined as a map* $X : M \rightarrow TM$ *that assigns a tangent vector to a point p, i.e.* $X(p) := X_p \in T_pM$. *The vector field is differentiable if this map is differentiable. The set of all vector fields on M is denoted by* $\chi(M)$

Definition 1.2.7. *Let M be a smooth manifold, an Affine connection (Levi-Civita) ∇ on M is a differential operator, sending smooth vector fields ∇X and Y to a smooth vector field , which then satisfies the following conditions*

1. $\nabla_{X+Y} Z = \nabla_x Y + \nabla_y Z$
2. $\nabla_x (Y+Z) = \nabla_x Y + \nabla_x Z$
3. $\nabla_{fx} Y = f \nabla_x Y$
4. $\nabla_x (fY) = X(f)Y + f(\nabla_x Y)$

$\forall$ *vector fields X,Y and Z, and real valued function f on M.*

The vector field $\nabla_x Y$ is known as the covariant derivative of the vector field Y along X with respect to ∇

Definition 1.2.8. *By S and R, where S denote Ricci Tensor and R, Riemannian curvature tensor of an n-dimensional Riemannian manifold (M,g), then S can be defined as*

$$S(X,Y) =$$

$$\sum_{i=1}^{\infty} g(R(e_i, X)Y, e_i)$$

Where e1, e2,...en are orthonormal basis vector fields in $TM, and X, Y, Z \in TM$.

Definition 1.2.9. *A curve $\gamma(s)$ is a geodesic if its tangent vector $\gamma(s)$ at each point are parallel.*

Definition 1.2.10. *Let X and Y be topological spaces. A homeomorphism $f : X \to Y$ is a continuous bijection whose inverse $f^{-1} : Y \to X$ is also continuous.*

Definition 1.2.11. *Two smooth manifolds X and Y are called diffeomorphic if there exists a homeomorphism $f : X \to Y$ such that $X = f^{*}Y$.*

Definition 1.2.12. *Let M be an n-dimensional contact manifold with contact form η, that is, $\eta\Lambda(d\lambda)^n \neq 0$, then, a contact manifold admits a vector field ξ called characteristic vector such that $\eta(\xi) = 1$ for any field $X \in \chi(M)$.*

Furthermore, if M admits a Riemannian metric g, and a tensor field ϕ of type (1,1), such that,

$$\phi^2 X = X - \eta(X)\xi$$

$$g(X,\xi) = \eta(X)$$

$$g(X,\phi Y) = d\eta(X,Y)$$

Then we can say that $(\phi,\eta.\xi,g)$ is a contact metric structure.

Definition 1.2.13. *We say that a contact metric manifold is sasakian if*

$$(\nabla_x \phi)Y = g(X,Y)\xi - \eta(Y)X$$

where,

$$\nabla_x \xi = -\phi X$$

$$R(X,Y)\xi = \eta(Y)X - \eta(X)Y$$

For all vector fields $X,Y \in M$.

Definition 1.2.14. *An n-dimensional differentiable manifold M is said to admit an almost para-contact Riemannian structure (ϕ,η,ξ,g) such that*

$$\phi^2 X = -\eta(X) + X$$

$$\phi\xi = 0$$

$$\eta(\xi) = 1$$

$$\eta(\phi X) = 0$$

$$g(X,\xi) = \eta(X),$$

$$g(\phi X,\phi Y) = g(X,Y) - \eta(X)\eta(Y),$$

for all vector fields X,T on M.

If an almost para-contact Riemannian structure (ϕ,η,ξ,g) satisfy the following equations

$$d\eta = 0, \nabla_x\xi = \phi X$$

$$(\nabla_x\phi) = -g(X,Y)\xi - \eta(Y)X + 2\eta(X)\eta(Y)\xi$$

Then M is referred to as Para-sasakian manifold.

If M admits 1-form η, such that $(\nabla_x,\eta)Y = \eta(X)\eta(Y) - g(X,Y)$, for all $X,Y \in M$, then para-sasakian manifold is *a special manifold.*

Definition 1.2.15. *An n-dimensional differentiable manifold M^n is Lorentzian Para-Sasakian manifold of it admits a (1,1) -tensor field ϕ, vector field ξ,1-form η and a Lorentzian metric g which satisfies*

$$\phi^2 X = X + \eta(X)\xi$$

$$\phi\xi = 0$$

$$\eta(\xi) = -1$$

$$\eta(\phi X) = 0$$

$$g(X,\xi) = \eta(X)$$

$$g(\phi X,\phi Y) = g(X,Y) + \eta(X)\eta(Y),$$

$$(\nabla_\phi)Y = g(X,Y)\xi + \eta(Y)X + 2\eta(X)\eta(Y)\xi,$$

$$\nabla_x \xi = \phi X$$

Where X and Y are arbitrary vector fields, ∇_X *denote covariant differentiation in the direction of* X *with respect to g.*

2 Chapter 2

2.1 Preliminaries

Particularly in this chapter, we will venture into defining tensors, curvature tensors, kemontsu manifolds, para kemontsu manifolds and other related manifolds.

Terminologies and Definitions

2.1.1 Tensors

Given two non-negative intergers k and l we define $T^{k,l}(V)$ to be the vector space of all multilinear maps $f(V_1^*,...,V_k^*,v_1,...,),V^{*}*,...,V^{*}*V \to R$, with k arguments in V* and L arguments in V

The elements of $T^{K,L}(V)$ are called tensors of degree (or order) (K,L).

If we have a basis $e_1,e_2,...,e_n$ of V, we uniquely determine a tensor T of degree (K,L) by the n^{k+1} numbers referred to as the coefficients of the tensors

$$T^{i_1,...,i_k}_{j_1,...,j_k} = T(e^{i_1},...,e^{i_k},e_{j_1,...,e_{j_l}})$$

Conversly there exists a tensor of degree (K,L) for any choice of these numbers. Thus the vector space $T^{k,l}(V)$ has dimension n^{k+1}

A tensor of type (k,0) for some $k \geq 1$ is called contravariant and a tensor of type (0,1) for some $k \geq 1$ is called covariant.

For tensor $T^{i_1,...,i_k}_{j_1,...,j_k}$

the indices $i_1,...,i_k$ are contravariant and indices $j_1,...,j_k$ are covariant.

Superscripts are used to denote contravariant indices and subscripts are used to denote covariant indices.

Examples

1. For k=l=0, we have a tensor of degree (0,0). Tensors of degree (0,0) are referred to as scalars.
2. $T^{0,1}(V)$ This is the space of all linear maps $V \to R$. Therefore $T^{0,1}(V) = V^*$ and a tensor of degree (0,1) is a linear functional on V
3. $T^{1,0}(V)$

   Similarly, we have that $T^{1,0}(V)$ equals V^{**} which we identify with V.

   Thus, $T^{1,0}(V) = V$, and a tensor of degree (1,0) is an element of V.

4. A linear mapping $S : V \longrightarrow V$ defines a tensor T of degree (1,1) by $T(V^*,V) = [V^*,S(V)]; V* \in V^*, v \in V$

   In a basis $e_1,...,e_n$, this tensor has the coefficients $T^i_j = T(e^i,...,e_j) = [e^i,S(e_j)] = S_j$ where, $S_i j$ is the matrix representation of S in the basis $e_1,...,e_n$

Fundamental Properties of Tensors

1. Addition.

   The sum of to or more tensors of the same rank and type .i.e the same number of contravariant indices and the same number of covariant indices is also a tensor of the same rank and type.

   Thus if A^{mp}_q and B^{mp}_q are tensors then $C^{mp}_q = A^{mp}_q + B^{mp}_q$ is also a tensor (sum of tensors) or $\lambda A^{mp}_q + \mu B^{mp}_q$ can be formed, with the help of the above tensors, which will satisfy the law of transformation, where λ and μ are invariants.
2. Subtraction.

   The difference of two tensors of the same rank and type is also a tensors of the same rank and type.

   Thus if A^{mp}_q and B^{mp}_q are tensors, then $D^{mp}_q = A^{mp}_q - B^{mp}_q$ is also a tensor.

Example

The tensor A_{ij} can be written as $A_{ij} = \frac{1}{2}(A_{ij} + A_{ji}) + \frac{1}{2}(A_{ij} - A_{ji})$.
Now $(A_{ij} + A_{ji})$ is symmetric part and $(A_{ij} - A_{ji})$ is skew-symmetric part of the tensor A_{ij}.
Thus by covariant (or contavariant) tensor of the second order is the sum of a symmetric tensor and a skew symmetric tensor.

3. Outer Multiplication. The product of two tensors is a tensor whose rank is the sum of the ranks of the given tensors.
   This product which involves temporary multiplication of the component of the tensor is called *the outer product.*
   For example $A_q^{pr} B_s^m = C_q^{pm}$ is the outer product of A_q^{pr} and B_s^m

   However, it must be noted that not every tensor can be written as a product of two tensors of lower rank. Thus division in the usual sense of one tensor by another is not defined that is, division of tensors is not aways possible.

4. Contraction.

   If one contravariant and one covariant index of a tensor are set equal than the result indicated that a summation over the equal index is to be taken according to the summation convention, such that the resulting sum is a tensor of rank two less than that of the original tensor. This process is called *contraction.*

   For example in the tensor of rank 5 A_{qs}^{mpr}, by setting $r = s$ we obtain $A_{qr}^{mpr} = B_q^{mp}$, a tensor of rank 3. Further by setting $p = q$, we get $B_p^{mp} = C_n^m$, a tensor of rank 1.
   As another example from the mixed tensor A_j^i IF WE SET $i = j$, then we get an invariant A i.e. from a tensor (mixed) of rank 2, we get an invariant (rank zero).

5. Inner Multiplication. We can also combine outer multiplication and contraction to produce new tensors. By the process of outer multiplication of two tensors followed by a contraction, we obtain a tensor called *an inner product of the given tensor.* This process is called *Inner multiplication of two tensors.*

   For example, given the tensors A_q^{mp} and B_{st}^r, the outer product is $A_q^{mp} B_{st}^r$.
   Letting $q = r$, we obtain the inner product $A_q^{mp} B_{st}^r$. Again putting $q = r$ and $p = s$, another inner product $A_r^{mp} B_{pt}^r$ is obtained.

   Note: Inner and outer multiplication of tensors is commutative and associative.

   It may be noted that we never contract two indices of the same type, as the reslting sum is not necessarily a tensor.

6. Quotient Law.

   Given the set of functions, we want to show whether it forms the components of a tensor or not, the method that it satisfies the equation of transformation or not is troublesome. In practice a simple test is provided by the quotient law.

   The quotient law states that *If an inner product of any quantity say X with an arbitrary tensor is itself a tensor then X is also a tensor.*

   Example

   The set of N^3 functions A^{ijk} from the components of a tensor (rank 3 contravariant) if $A^{ijk}B^p_{ij} = C^{pk}$, provided that B^P_{ij} is an arbitrary tensor and C^{pk} a tensor.

   The equation $A^{-ijk}B^{-P}_{\bar{ij}} = C^{pk}$ (transforms x̄) is satisfied.

2.1.2 Vector and 1-Form

1 index tensors. In tensor notation, a 1-form (or a covector at a point) is a tensor with a single subscript index, e.g., α_i. If we have a 1-form α and a vector field v, then we can combine these to make a function:

$\alpha(v) := v^i\alpha_i.$

Denoting the space of vector fields as $\chi(M)$, this defines a linear map $\alpha : \chi(M) \to C^\infty(M)$ which is $C^\infty(M)$ -linear in the sense that for any $v \in \chi(M)$ and $f \in C^\infty(M), \alpha(fv) = f\alpha(v)$.

Any $C^\infty(M)$ -linear map $\alpha : \chi(M) \to C\infty(M)$ is given by a 1-form in this way, so we can alternately define 1-forms as maps of this kind. This is also denoted as a pairing,$\alpha, v := \alpha(v)$.The set of 1-forms over M is denoted $\Omega^1(M)$

Definition 2.1.1. *The curvature operator R of a connection ∇ is the association of two vector fields $X,Y \in \chi(M)$ to the map $R(X,Y) : \chi(M) \to \chi(M)$ defined by*

$$R(X,Y)Z := \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_(X,Y)Z$$

The curvature operator can be seen as a way of measuring the non-commutativity of the connection. This operator defines a $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$ tensor, the *Riemann curvature tensor R*

$$R = R^l_{ijk} \delta x^i \otimes \delta x^j \otimes \delta x^k \otimes \frac{\delta}{\delta x^l},$$

where coeffiecient R^l_{ijk} is the l-th coordinate of the vector $R\left(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}\right)\frac{\delta}{\delta x^k}$ and are found to be $R^l_{ijk} = \Gamma^l_{jk,i} - \Gamma^l_{ik,j} + \Gamma^m_{jk}\Gamma^l_{im} - \Gamma^m_{ik}\Gamma^l_{jk}$

We will assume that (M,g) is a Riemannian manifold and ∇ is the Levi-Civita connection. From the Riemannian curvature tensor, we can define a new 4-covariant tensor by *lowering* the last index of R with the metric $\boldsymbol{g}$:

$$R(X,Y,Z,W) = g(R(X,Y)Z,W)$$

In a coordinate system, the coordinates of this 4-covariant Riemann curvature tensor are $R_{ijkl} = R^m_{ijk} g_{ml}$.

The Riemann curvature tensor has the following symmetries:

label= Bianchi identity. $R_{ijkl} + R_{jkil} + R_{kijl} = 0$

lbbel= $R_{ijkl} = -R_{jikl}$

lcbel= $R_{ijkl} = -R_{ijlk}$

ldbel= $R_{ijkl} = R_{klij}$

Ricci tensor is an important geometric object in general relativity.

2.1.3 Manifolds

A (real) n-dimensional manifold is a topological space M for which every point $x \in M$ has a neighbourhood homeomorphic to Euclidean space R^n.

Definition 2.1.2. *Given a topological space M, a chart (local coordinate map) is a pair (U, φ), where U is an open subset of M and*

$\varphi : U \longrightarrow \Omega$ *is a homomorphism onto an open subset,* $\Omega = \varphi(U)$*, of* $\Re^{n_\varphi}$ *(for some* $n_\varphi \geq 1$*).*

For any $p \in M$ a chart, (U, φ), is a chart at p iff $p \in U$. If (U, φ) is a chart then the functions $x_1 = po\varphi$ are called local coordinate and for every $p \in U$ the tuple $(x_1(p), ..., x_n(p))$ is the set of coordinates of p with respect to the chart.

The inverse, (Ω, φ^{-1}), of a chart is called a local parametrization.

Given a topological space M and and any two intergers, $n \geq 1$ and $k \geq 1$ a C^k n-atlas A, is a family of charts, $(U : \varphi)$ such that

1. $\varphi(U_i)\Re$ for all i
2. The U_i cover M, i.e $M = Uu_i$
3. Whenever $U_i \cap U_j \neq \phi$, the transition map φ_{ij} (and φ) is a C^k-diffeomorphism.

Definition 2.1.3. *A manifold with boundary is smooth if the transition maps are smooth. For an arbitrary subset* $X \subseteq R^m$*, a function* $f : X \longrightarrow R^n$ *is called smooth if every point in X has some neighborhoodwhere f can be extended to a smooth function.*

Definition 2.1.4. *A Hausdorff topological space M is an n-dimensional topological manifold if it admits an atlas* $U_\alpha, \phi_\beta, \phi_\alpha \longrightarrow R^n, n \in N$

Definition 2.1.5. *Let M be the set of all e infinity vector field on A the brackets [] is defined by mapping*

$[] : M * M \rightarrow M$ *such that for* x, y *in M and* $[x, y]f = xyf - yxf$

where f is smooth function for x, y, z *in M we have*

1. $[X, Y] + = -[Y, X]$ skew commutative (symmetric)

2. $[X+Y,Z] = [X,Z] + [Y,Z]$
3. $[fX, gY] = fg[X,Y] + f(XgY) - g(Yf)X$
4. $[[X,Y],Z] + [[Y,Z],X] + [[Z,X],Y] = 0$

Note that the last equation 4 above is referred to as *Jacobs identity*

Definition 2.1.6. *Consider two vector fields $X, Y \in \chi$ The Lie bracket or commutator of X and Y is the vector field*

$[X,Y] = XoY - YoX.$

Considering a chart $x : U \subset M \to \Re^n$, the vector fields X and Y have the expressions $X = X^i \frac{\delta}{\delta x^i}$ and $Y = Y^i \frac{\delta}{\delta x^i}$. Computing the expression of the Lie bracket in coordinates yields to the result:

$[X,Y] = (X.Y^i - Y.X^i)\frac{\delta}{\delta x^i}.$

The commutator has the following properties: given $X,Y,Z \in \chi(M)$

1. Bilinearity: for $\alpha, \beta \in \Re$, $[\alpha X + \beta Y, Z] = \alpha[X,Z] + \beta[Y,Z]$ and $[Z, \alpha X + \beta Y] = \alpha[Z,X] + \beta[Z,Y]$
2. Antisymmetry: $[X,Y] = -[Y,X]$
3. *Jacobi identity*: $[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0$
4. *Leibnitz rule*: for any $f, g \in C^\infty$, $[fX, gY] = fg[X,Y] + f(X.g)Y - g(Y.f)X$

Differential Manifold

A manifold M is called *a differential manifold* of class C^k if there is an atlas of $M(U_\alpha, \phi_\alpha)|\alpha \in I$ such that, for any $\alpha, \beta \in I$, the composites

$\phi_\beta o \phi_\alpha^{-1} : \phi_\beta(U_\alpha \cap U_\beta) \to R^n$ is differentiable of class C^k on M. If instead, the atlas is of class C^k, then M is said to have a differentiable (smooth) structure and is called a smooth (diffential) manifold.

Riemannian Manifold

Let T_p be the tangent space at the point p of a differentiable manifold M. If we single out a real valued bilinear, symmetric and positive definite function g on the ordered pairs of tangent vectors at each point p in M, then M is considered to be a Riemannian manifold and g is called the metric tensor of M. Thus, for two vectors X, Y in Tp, we have

1. $g(X,Y) \in \Re$
2. $g(X,Y) = g(Y,X)$
3. $g(aX+bY,Z) = ag(X,Z)+bg(Y,Z)$
4. $g(X,X) > 0$
5. If X and Y are C^{∞} fields with domain A, then $g(X,y)$ is a C^{∞} function on A.

Complex Manifolds

Definition 2.1.7. *We say that a manifold is Complex if it is differentiable with a holomorphic atlas. Such manifolds are of even dimensions; $2n$ and having a collection of charts (U_j,Z_j) that are one to one maps of corresponding U_j to C^m such that every non-empty intersection $U_j \cap U_k$ the maps are $z_j z_k^{-1}$ are holomorphic.*

Given the subset of R^3, a *two sphere* S^2 defined by $x^2+y^2+z^2=1$ is a *complex manifold.* Here we can use a stereographic projection from the North pole to the real plane R^2 with coordinates X,Y given by

$$(X,Y) = \left(\tfrac{x}{1-z}, \tfrac{y}{1-z}\right)$$

This can be done for any point except the North Pole itself (corresponding to $z=1$). To include the North Pole, we introduce a second chart, in which we stereographically project from the South pole:

$$(U,V) = \left(\tfrac{x}{1+x}, \tfrac{y}{1+x}\right)$$

which holds for any point S^2 except for the South (pole at $z = 1$). In both patches, we can now define complex coordinates,

$$Z = X + iY, Z = X - iY, W = U - iV, W = U + iV,$$

and show that on the overlap of the patches, the transition is holomorphic indeed, on the overlap we compute that $W = \frac{1}{z}$.

Holomorphic functions on manifolds

Let $U \subset X$ be open, $f : U \to C$ be a function. Then f is holomorphic on U if, taken $(U\alpha, z\alpha)$ such that $U \cap U\alpha \neq \phi$, the function

$$f \circ z_\alpha^{-1} : z_\alpha(U_\alpha \cap U) \to C$$

is holomorphic. This definition does not depend on the choice of the coordinate $(U\alpha, z\alpha)$.

In addition, we define $O_X(U) := f : U \to C \mid f$ is holomorphic

Almost Complex Manifolds

Definition 2.1.8. *An almost complex structure on a manifold M is an operator* $1 : TM \longmapsto TM$ *such that* $I^2 = -Id$. *It is called integrable if I is induced by a complex structure.*

Let M be a Haudorf topological space. In order to analyze M locally, we use open charts, that is to say, pairs of the type (U, φ) where U is an open subset of M and $\varphi : U \longmapsto \varphi(U) \subset R^k$ is a homeomorphism of U onto an open subset of R^k. A collection of charts $(U_\infty, \varphi\alpha)\alpha \in A$ gives M the structure of a smooth manifold of dimension k if the open sets U^∞ cover M, and if for all pairs of indices α, β the transition function $\varphi_\beta o \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U^\beta) \longmapsto \varphi_\beta(U_\alpha \cap U_\beta)$ is a smooth map. When we say that $(U_\infty, \varphi\alpha)\alpha \in A$ is an atlas of M.

A complex structure on a topological space M consists of a family $(U^{\infty,\varphi\alpha})\alpha \in A$ where U_∞ is an open subset of M and $U_\infty : U_\infty \longmapsto C^m$ is a homeomorphism onto an open subset C^m, such that

1. $M = U_{\alpha t A} U_\infty$
2. For each pair of indices $\alpha, \beta \in A$ the function $\varphi_\beta o \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U^\beta) \longmapsto \varphi_\beta(U_\alpha \cap U_\beta)$ is holomorphic.

Each pair $(U_\infty, \varphi_\alpha)$ is called a *complex chart*, and the whole collection $(U_\infty, \varphi_\alpha)_\infty \in A$ is called a *comples atlas.* The integer n is the complex dimension of M.

A complex manifold of dimension n is, in a natural way, a real manifold of dimension 2n. For given a point $p \in M$, let us consider a complex chart (U, ∞) with $p \in U$ and $\varphi(q) = (z^1(q), ..., z^n(q))$. The complex valued function z^f can be decomposed in terms of their real and imaginary parts, $z^f(q) = x^j(q) + iy^j(q)$, decomposition that in turn induces a map.

$q \longmapsto (x^1(q), y^1(q), ..., x^n(q), y^n(q))$

from U onto an open subset of R^{2n}. This function defines a real local chart of M. It is easy to see that transition functions of these charts of M are smooth functions. Thus, the collection of all such charts on M as a real differentiable manifold of dimension 2n.

The set $\sigma_{xj}|p, \sigma_{yj}|p$ forms a basis of the tangent space T_pM. Using it, we define a linear isomorphism,

$$J = J_p : T_pM \longmapsto T_pM$$

by

$$J(\sigma_{xj}|p, \sigma_{yj}|p, J(\sigma_{yj}|p) = -\sigma_{xj}|p,$$

Kenmotsu manifolds

Definition 2.1.9. *Let* $M^{2n+1}(\phi, \xi, \eta, g)$ *be an almost contact Riemannian manifold, where* ϕ *is a (1,1)-tensor field* ϕ*, vector field* ξ*,1-form* η *and a Riemannian metric g.*

We know that;

$$\phi\xi = 0$$

$$\eta(\xi) = 1$$

$$\eta(\phi X) = 0$$

$$\phi^2 X = -X + \eta(X)\xi$$

$$g(X, \xi) = \eta(X)$$

$$g(\phi X, \phi Y) = g(X,Y) + \eta(X)\eta(Y)$$

for any vector fields X and Y on M.

Definition 2.1.10. *We call an almost contact metric manifold* $M^n(\phi, \eta, \Xi, g)$ ***a Kenmotsu manifold*** *if the following conditions hold:*

$(\nabla_x \phi)Y = -\eta(Y)\phi X + g(\phi X, Y)\xi$, where $X, Y \in \chi(M)$

$\nabla_x \xi = X - \eta(X)\xi$

Where ∇ is the Riemannain connection of g.

The following relations holds for Kenmotsu manifolds;

$(\nabla_x \phi)Y = g(\phi X, Y)$

$\eta(R(X,Y)Z) = \eta(Y)g(X,Z) - \eta(X)g(Y,Z)$

$R(X,Y)\xi = \eta(X)\lambda - \eta(Y)\lambda$

(a) $R(\xi,X)Y = \eta(Y)X - g(X,Y)\xi$

(b) $R(\xi,X)\xi = X - \eta(X)\xi$

where R is the Riemannian curvature tensor and S is the Rici-Tensor.

In a Riemannian Manifold we also have $g(R(W,X)Y,Z) + g(R(W,X)Z,Y) = O$, for every vector fields X,Y,Z

Almost paracontact manifold

Definition 2.1.11. *Let M_n be an n-dimensional differentiable manifold endowed with structure tensors (ϕ,ξ,η) where ϕ is a tensor of type (1,1), ξ is a vector field, η is a 1-form such that*

$\eta(\xi) = 0$ (1.1)

$\phi^2(X) = X - \eta(Y)\xi; = \phi X$ (1.2)

Then M is called an almost paracontact manifold.

Paracontact Riemannian Structure

Definition 2.2.3.2

Let g be the Riemannian metric satisfying such that, for all vector fields X and Y on M,

$g(X,\xi) = \eta(X)$ (1.3)

$\phi\xi = 0, \eta(\phi X) = 0, \phi = n-1$ (1.4)

$g(\phi X, \phi Y) = g(X,Y) - \xi(X)\xi(Y)$ (1.5)

Then the manifold M_n is said to admit an almost paraconract Riemannian structure (ϕ, ξ, η, g)

Para-Kenmotsu manifold

Definition 2.1.12. *A manifold of dimension 'n' with Riemannian metric 'g' admitting a tensor field ϕ of type (1,1), a vector field ξ and a 1-form η satisfying (1.1) (1.3) along with*

$$(\nabla_x, \eta)Y - (\nabla_y, \eta)x = 0 \text{ (1.6)}$$

$$(\nabla_x, \nabla_y Y)Z = [-g(X,Z) + \eta(X)\eta(Z)]\eta(Y + [-g(X,Y) + \eta(X)\eta(Y)]) \text{ (1.7)}$$

$$\nabla_x \xi = \phi^2 X = X - \eta(X)\xi \text{ (1.8)}$$

is called a Para kenmotsu manifold or brifely P-kenmotsu manifold.

Properties of Para-Kenmotsu Manifolds

Proposition

Let (M, ϕ, η, ξ, g) be a para-kenmotsu manifold, then we have

$$R(X,Y)\xi = \eta(X)Y - \eta(Y)X,$$

$$R(X,\xi)Y = g(X,Y)\xi - \eta(Y)X,$$

$$Ric(X,\xi) = -2n\eta(X)$$

$$k(X,\xi) = -1$$

$$(\nabla_z R)(X,Y,\xi) = R(X,Y)Z + g(Y,Z)X - g(X,Z)Y$$

where Ric is the Ricci tensor and $X,Y,Z \in T_pM$

2.1.4 Riemannian Connections

Definition 2.1.13. *We call a connection ∇ Riemannian if the following conditions holds;*

1. ∇ is symmetric or torsion free that is $\nabla_x Y - \nabla_y X = [X.Y]$
2. g is covariant constant with respect to ∇ $\nabla_x g = 0$

Definition 2.1.14. *The torsion tensor of a connection V is a vector valued bilinear function T which assigns to each pair of C^∞ fields X, Y , with domain A, a C^∞ vector field T(X,Y) with domain A defined by,*

$$T(X,Y) = \nabla_x Y - \nabla_y X - [X,Y]$$

If $T(X,Y) = 0$, then the connection ∇ is said to be torsion free or symmetric.

3 Chapter 3

3.1 Literature Review

3.1.1 Introduction

The purpose of this chapter is to review the literature, both the theoretical and the empirical literature. First, the chapter presents the theoretical literature of para-Kemonstu manifolds, in comparison to other manifold types. The chapter zooms in to discuss the empirical literature testing the relevance of the theories presented. Also, the chapter presents the conceptual framework and provides a summary to the reviewed literature.

3.1.2 Theoretical Review

Manifolds theory is basic for the theory of Riemannian, Einsteinian, and Lie groups geometries. The inception of manifold concept in the 1960 into general relativity by Martin Kruskal, shed more light on the topic of manifolds, and highlighted the topological characteristics, global and local, of space-time models. Manifolds exist everywhere, mostly in many physical phenomena, and they can be modelled mathematically.

S. Bochner studied Betti numbers of the Kahler manifolds, and introduced a tensor which dominated his theory of Weyl tensors in Riemann manifolds. Here, he considered a flat manifold as a real space form extension. The Weyl tensor is a conformal invariant of the Riemann manifolds, however, several attempts were made to find a geometrical interpretation. Tensors can be nicely introduced by decomposition of the curvature tensor spaces. Sasakian, co-Kahler, and Kenmotsu manifolds' classes are classified by the almost Hermitian manifolds $(M^{(2n+1)})$ for which there exists maximal dimension $(n+1)^2$ for the automorphism group. The properties of the Riemann connections of the co-Kahler, Kenmotsu, and Sasakian manifolds help in characterizing these manifolds.

3.1.3 The Conceptual framework

A conceptual framework is a written or visual presentation that explains either graphically, or in narrative form, the main things to be studied, the key factors, concepts or variables and the presumed relationship among them (Miles and Huberman, 1994). This research will look at curvature tensors in general, and explore Sasakian manifolds and curvature tensors in k-contact.

Mishra (1970), and Pokhariyal (1979) defined a new set of new curvature tensors relating to Weyl tensor in the study of curvature tensors' Relativistic significance. The definition of Weyl's projective curvature tensor was based on the geodesic correspondence since it contained a specific type of distribution. Correspondingly, the new tensors did not depend on the variation within the two spaces; instead, they indicated that the arrangement of the vectors in question prior to being acted on by the tensor of the vector field over the metric potentials and matter tensors have a significant role in the configuration of the different geometrical and physical tensor properties.

Tripath, Mukut Mani, and Gupta (2011) explored Sasakian manifolds and curvature tensor in k-contact and inspected the properties of the Sasakian manifolds where they posed the necessary conditions for the contact manifolds. $W5$ -curvature tensor is a field that has attracted several authors leading to rich outcome for application in geometric modelling. Deszcz (1992) initiated the aspect of pseudo-symmetric manifolds. Ojha (1986) studied the properties of the Sasakian and Kahler m-projective curvature manifolds. Additionally, he argued that such tensors linked con-harmonic curvature tensors, concircular curvature tensors, conformal curvature tensors from one side to H-projective curvature tensors on another.

Singh (1965) also explored the Relativistic importance of the projective curvature tensor initiated by Weyl, a concept that is very common in the Geometry subject. The equation for the tensor was given by;

$$W(X,Y,X,T) = R(X,Y,Z,T + \frac{1}{n-1}[g(X,Y)Ric(Y,T) - g(X,T)Ric(Y,Z)]$$

Some of its applications will be instrumental in the discussions in this paper. However, the applications are limited in the extent of their usage and the topics explored. Some of the applications including the minimal submanifolds, Morse Theory, and Kahler manifolds are covered implicitly. The curvature does not emerge explicitly in Mathematical studies, but accompanies the theory of curves and the Euclidean Space surfaces. Riemann's definition of curvature tensor is abstract and rigorous, taking Gauss' work as reference.

De and Bismas (2006) explored the flat contact metric manifold and ascertained that the k-contact metric manifold is conformally flat if it belongs to the Einstein manifold. However, Dwidedi and Kim (2010) could not find proof showing that if a Sasakian Manifold is in Einstein Manifold, then it is conharmanically flat. Pokhariyal and Mishra (1970) define several tensors that include;

$$W_1(X,Y,Z,T) = R(X,Y,Z,T + \frac{1}{n-1}[g(X,Z)Ric(Y,Z) - g(Y,T)Ric(X,Z)]$$

$$W_2(X,Y,Z,T) = R(X,Y,Z,T + \frac{1}{n-1}[g(X,Z)Ric(Y,T) - g(Y,T)Ric(X,T)]$$

$$W_3(X,Y,Z,T) = R(X,Y,Z,T + \frac{1}{n-1}[g(Y,Z)Ric(X,T) - g(Y,T)Ric(X,Z)]$$

$$W_4(X,Y,Z,T) = R(X,Y,Z,T + \frac{1}{n-1}[g(X,Z)Ric(Y,T) - g(Y,T)Ric(X,Z)]$$

$$W_5(X,Y,Z,T) = R(X,Y,Z,T + \frac{1}{n-1}[g(X,Z)Ric(Y,T) - g(Y,T)Ric(X,Z)]$$

$$W_6(X,Y)Z = R(X,Y) + \frac{1}{n-1}[g(X,Z)Y - S(Y,Z)X]$$

Pokhariyal G.P [20] later defined a m-projective curvature field tensor $W*$, as;

$$W*(X,Y,Z,U) =' R(X,Y,Z,U) - \frac{1}{2(n-1)}[S(Y,Z)g(X,U) - S(X,Z)g(Y,U)]$$

where,

$$W*(X,Y,Z,U) = g(W*(X,Y)Z,U)$$

and

$$'R(X,Y,Z,U) = g('R(X,Y)Z,U)$$

Also, Pokhariyal (1985) defines W_5 curvature tensor to be the equation;

$$W_5(X,Y)Z = R(X,Y)Z + \frac{1}{n-1}[g(X,Z)\phi Y - S(X,Z)y]$$

During the same period, Pokhariyal defined $W_7, W_8 and W_9$ as follows;

$$W_7(X,Y,Z,T) = R(X,Y,Z,T) + \frac{1}{n-1}[g(Y,Z)Ric(X,T) - g(X,T)Ric(Y,Z)]$$

and,

$$W_8(X,Y)Z = R(X,Y)Z + \frac{1}{n-1}[S(X,Y)Z - S(Y,Z)X]$$

where

$$S(X,Y) = g(QX,Y) = (n-1)g(X,Y) = R(X,Y),$$

Q is the Ricci operator, that is, the endomorphic linear tangent space at each of the points.

$$W_9(X,Y,Z,T) = R(X,Y,Z,T) + \frac{1}{n-1}[g(Z,Y)Ric(X,Y) - g(Y,Z)Ric(X,T)]$$

Prakasha D.G, Vasant, and Kakasab (2016) realized that a $\phi - W_5$ -generalized flat Sasakian Space form is conformally flat.

Matsumoto K introduced the Lorentzian para-Sasakian ideology in 1988, while Mihai I and Rosca R in 1992 defined the idea of LP Sasakian and got several outcomes. The other authors to have explored the LP-Sasakian manifolds' field include Shaikh A.A (2004) and U.C. De. (1999). On the other hand, Bagewadi C.S and Kumar K.T. (2011) studied the LP-Sasakian manifolds while Ahmet Yildiz and U.C. De. examined the tensors in Kenmotsu Sasakian manifolds. W_2 -curvature tensor and its related skew-symmetric and symmetric tensors in the Einstein Sasakian manifolds was also studied by Pokhariyal. U.C. De and Sarkar (2009) did an extensive analysis on the Para-Sasakian involving the W_2 – symmetric tensor and proved that the W_2 – symmetric para-Sasakian manifold have a constant curvature and thus it is an LP-Sasakian Manifold, similar to the recurrent W2 P-Sasakian. Other authors that contributed to the W_2- include Moindi S.K, Pokhariyal P.G and Njori P.W., , where they proved the W_2 recurrent P-Sasakian manifold theorem.

One of the most significant analytical methods for differential geometry is the famous Bochner technique, founded in the 1960s by S. Bochner, K.Yano, and others to investigate the connection topology and curvature of the Riemann manifold. Since the 1970s, the non-compact (complete) Riemann manifolds have been integrated in many researches using the Bochner technique. There are known results of the relations in Riemann geometry between topology and curvature, and between the local and global aspects of a Riemann manifold. At large, many results exist; on locally conformally flat Riemann manifolds. S. Goldberg supplemented his own theorem on conformally flat homogeneous Riemann manifold by arguing that there exists a constant curvature for a locally conformal flat Riemann manifold if its Ricci tensor is parallel and it is either positive or negative definite. However, completeness and compactness characteristics do not feature in Goldberg's theorems.

Ricci Flow

The idea of Ricci-flow was introduced by R. Hamilton (1982) and he explained the hurdles in manifold geometry. For example, Hamilton indicated that if there exist singular points in manifold geometry, they can be mitigated under Ricci-flow. Ricci-solitons are the stationary points under Ricci -flow. The Ricci-flow equation as given by Hamilton is as;

$$\frac{\delta g}{\delta t} = 2Ric(g)$$

Where Ric is S in the space of metric on M.

Ricci-Soliton

Hamilton then moved to define a Ricci-Soliton (g, V, λ) on a Riemann manifold M as;

$$L_v g + 2S + 2\lambda g = 0$$

Where S- is the Ricci tensor, L_v is Lie Derivative operator on M in direction V and λ is a scalar. This equation for the Ricci soliton is said to either shrink when λ is negative, steady when λ is zero, or expanding when λ is positive. Further, Hamilton realized that a compact Ricci-solitons belong to the points of the Ricci flow below;

$$\frac{\delta g}{\delta t} = -Ric(g)$$

Projected from the metric space to its modulo quotient, scallings, and diffeomorphism that rise often as blow-up limit for the Ricci-flow on compact manifold. In 1923, Einstein ascertained that a manifold is reducible if a positive definite Riemann manifold (M,g) allows another parallel symmetric covariant tensor apart from the constant multiple of the metric tensor. The sufficient conditions for the existence of such tensors were then obtained by Levi, and later; a generalization of Levi's results was done by Sharma and showed that a second order parallel tensor on an $n>0$ space of a constant curvature is a constant multiple of the metric tensor. Sharma also showed that there exists a non-zero parallel 2-form and while he was examining the k-contact manifolds Ricci solitons, he found out that there exists a condition where the initial derivative disappears, a condition that disrupts the Ricci solitons rule. He then moved on to prove that a comprehensive k-contact gradient result is compact Sasakian and Einstein.

3.1.4 Curvature tensor

Many scholars agree that the curvature tensor is the most significant isometry invariant of the Riemann metric. For instance, a study on the conditions on the curvature tensors in obtaining geometric and topological restrictions concluded that positive curvature can only exist on trivial topological manifolds. Previous studies have shown that the Riemann curvature tensor is clearly important in general relativity.

The fourteen invariants of the Riemann curvature tensor are divided into four, for the Weyl tensor, three, for the Einstein curvature tensor, and six, for the combined Einstein and Weyl tensors. According to Petrov and Sharma and Husain, the four invariants of the Riemann tensor in empty spacetimes have been calculated to classify the Riemann tensors. Petrov studied the space-matter tensor for which all the algebraic properties of the Riemann curvature holds more generally than the Weyl conformal curvature.
The space-time tensor is given as

$$P_{abcd} = R_{abcd} - A_{abcd} + \delta(g_{ac}g_{bd} - g_{ad}g_{bc}),$$

where $A_{abcd} = \frac{\lambda}{2}(g_{ac}T_{bd} + g_{bd}T_{ac} - g_{ad}T_{bc} - g_{bc}T_{ad})$ and T_{ab} is given by the *Einstein's* field equations $R_{ab} - \frac{1}{2}Rg_{ab} = \lambda T_{ab}$. Here λ is a constant and T_{ab} is the *energy-momentum tensor*. The tensor P_{abcd} is known as *space-matter tensor.*

4 Chapter 4

A Para Kenmotsu manifold is an n dimensional manifold with Reimannian metric g admitting a tensor field ϕ of type $(1,1)$, a vector field ξ and a 1-form η satisfying

$$\eta(\xi) = 1 \tag{1}$$
$$g(X,\xi) = \eta(X) \tag{2}$$
$$(\nabla_X \eta)Y - (\nabla_Y \eta)X = 0 \tag{3}$$
$$\begin{aligned}(\nabla_X \nabla_Y \eta)Z &= [-g(X,Z) + \eta(X)\eta(Y)]\eta(Z) \\ &+ [-g(X,Y) + \eta(X)\eta(Y)]\eta(Z)\end{aligned} \tag{4}$$
$$(\nabla_X \phi) = g(\phi X,Y)\xi - \eta(Y)\phi X \tag{5}$$

It is known that in a P-Kenmotsu manifold the following relations hold:

$$S(X,\xi) = -(n-1)\eta(X) \tag{6}$$
$$g(QX,Y) = S(X,Y) \tag{7}$$
$$\begin{aligned}g(R(X,Y)Z,\xi) &= \eta(R(X,Y,Z)) \\ &= g(X,Z)\eta(Y) - g(Y,Z)\eta(X)\end{aligned} \tag{8}$$
$$R(\xi,X)Y = \eta(Y)X - g(X,Y)\xi \tag{9}$$
$$R(X,Y,\xi) = \eta(X)Y - \eta(Y)X; \textit{when } X \textit{ is orthogonal to } \xi \tag{10}$$

where S is the Ricci tensor, r is the scalar curvature and Q is the symmetric endormorphism of the tangent space at each point corresponding to the Ricci tensor and R is the Reimannian curvature. Below we shall use A to denote the 1-form η.

4.1 W_7 Curvature Tensor in Para Kenmotsu Manifold

Mishra and Pokhariyal gave the definition of W_7 curvature tensor as

$$W_7 = R(X,Y)Z + \frac{1}{n-1}[g(Y,Z)QX - Ric(Y,Z)X] \tag{11}$$

or

$$W_7^1(X,Y,Z,T) = R^1(X,Y,Z,T) \qquad + \frac{1}{n-1}[g(Y,Z)Ric(X,T) - Ric(Y,Z)g(X,T)]$$

Definition 4.1.1. *A Para Kenmotsu manifold is said to be flat if the Riemannian curvature tensor vanishes identically i.e.* $R(X,Y)Z = 0$*.*

Definition 4.1.2. *A Para Kenmotsu manifold* M_n *is said to be* W_7 *flat if* W_7 *curvature tensor vanishes identically i.e.* $W_7(X,Y)Z = 0$*.*

Theorem 4.1.3. *A* W_7 *flat Para Kenmotsu manifold is a flat manifold*

Proof

Given a W_7 curvature tensor which is defined as

$$\begin{aligned} W_7^1(X,Y,Z,T) &= R^1(X,Y,Z,T) + \frac{1}{n-1}[g(Y,Z)QX - Ric(Y,Z)X] \\ &\quad or \\ W_7^1(X,Y,Z,T) &= R^1(X,Y,Z,T) \\ &\quad + \frac{1}{n-1}[g(Y,Z)Ric(X,T) - Ric(Y,Z)g(X,T)] \end{aligned}$$

If P-Kenmotsu space is W_7 flat then $W_7 = 0$,

$$0 = R^1(X,Y,Z,T) + \frac{1}{n-1}[g(Y,Z)Ric(X,T) - Ric(Y,Z)g(X,T)]$$

where $Ric(X,Y) = -(n-1)g(X,Y)$ we have:

$$\begin{aligned} R^1(X,Y,Z,T) &= \frac{1}{n-1}[Ric(Y,Z)g(X,T) - g(Y,Z)Ric(X,T)] \\ &= \frac{1}{n-1}[-(n-1)g(Y,Z)g(X,T) + g(Y,Z)(n-1)g(X,T)] \end{aligned}$$

That is;

$$R^1(X,Y,Z,T) = g(Y,Z)g(X,T) - g(Y,Z)g(X,T)$$

But in P-Kenmotsu manifold we have:

$$R^1(X,Y,Z,T) = g(Y,Z)g(X,T) - g(X,Z)g(Y,T)$$

$$\implies R^1(X,Y,Z,T) = 0$$

or

$$Ric(X,Y) = 0$$

Hence the proof.

Corollary 4.1.4. *A W_7-flat P-Kenmotsu manifold is neither Einstein nor η-Einstein manifold.*

Proof
A manifold is said to be Einstein manifold if $a \neq 0$ and $b = 0$ in the expression below

$$Ric(X,Y) = ag(X,Y) + b\eta(X)\eta(Y)$$

and η-Einstein if a and b are none zero. From the results of the above theorem we have shown that $Ric(X,Y) = 0$. This therefore, means both a and b are zero, hence, it is neither Einstein nor η-Einstein manifold.

4.2 A W_7-Semisymmetric P-Kenmotsu manifold

U.C.De and N. Guha gave the definition of semi symmetry as $R(X,Y)R(Z,T)V = 0$

Definition 4.2.1. *A P-Kenmotsu manifold is said to be W_7-semisymmetric if $R(X,Y)W_7(Z,T)V = 0$*

Theorem 4.2.2. *A W_7-semisymmetric P-Kenmotsu manifold is a W_7-flat manifold.*

Proof
If P-Kenmotsu space if W_7-semisymmetric then $R(X,Y)W_7(Z,T)V = 0$

$$\implies g(R(X,Y)W_7(Z,T)V,L) = R^1(X,Y,W_7(Z,T)V,L)$$

$$= g(X,L)g(W_7(Z,T)V,Y) - g(Y,L)g(W_7(Z,T)V,X)$$
$$= A(X)W_7^1(Y,Z,T)V - A(Y)W_7^1(X,Z,T)V = 0$$

but since $A(X) \neq 0$ and $A(Y) \neq 0$

$$W_7^1(Y,Z,T)V = 0$$

and

$$W_7^1(X,Y,T)V = 0$$

from

$$R(X,Y)W_7(Z,T)V = 0$$

Hence the theorem.

Corollary 4.2.3. *A W_7-semisymmetric P-Kenmotsu manifold is neither Einstein nor η-Einstein manifold.*

Proof
A manifold is said to be Einstein manifold if $a \neq 0$ and $b = 0$ in the expression below

$$Ric(X,Y) = ag(X,Y) + b\eta(X)\eta(Y)$$

and η-Einstein if a and b are none zero. From the results of the above two theorems, it is clear that the Riemann curvature tensor R is equal to zero and consequently the Ricci tensor Ric is also equal to zero. This therefore, means both a and b are zero, hence, it is neither Einstein nor η-Einstein manifold.

4.3 A W_7-symmetric P-Kenmotsu manifold

A P-Kenmotsu manifold is said to be W_7-symmetric if $\nabla_U W_7(X,Y)Z = W_7^1(U,X,Y)Z = 0$

Theorem 4.3.1. *A W_7-symmetric and W_7-flat P-Kenmotsu is a flat manifold.*

Proof
From the previous theorem, we found out that a W_7-semisymmetric manifold is a W_7-flat manifold and if P-Kenmotsu space is a W_7-symmetric this implies

$$R(X,Y,W_7(Z,T,V)) - W_7(R(X,Y,Z),T,V) - W_7(Z,R(X,Y,T),V) - W_7(Z,T,R(X,Y,V)) = 0$$

We expand the expressions as follows:

$$\begin{aligned} R^1(X,Y,W_7(Z,T,V),L) &= g(X,L)g(Y,W_7(Z,T,V)) \\ &- g(Y,L)g(X,W_7(Z,T,V)) \\ &= A(X)W_7^1(Y,Z,T,V) - A(Y)W_7^1(X,Z,T,V) \end{aligned} \tag{12}$$

$$\begin{aligned} W_7^1(R(X,Y,Z),T,V,L) &= R^1(R(X,Y,Z),T,V,L) \\ &+ \frac{1}{n-1}[Ric(R(X,Y,Z),L)g(T,V) \\ &- Ric(T,V)g(R(X,Y,Z),L)] \end{aligned} \tag{13}$$

Then using $Ric(X,Y) = S(X,Y) = -(n-1)g(X,Y)$ we get

$$\begin{aligned} W_7^1(R(X,Y,Z),T,V,L) &= R^1(R(X,Y,Z),T,V,L) + \frac{1}{n-1}[-(n-1)R^1(X,Y,Z,L)g(T,V) \\ &+ (n-1)g(T,V)R^1(X,Y,Z,L)] \\ &= R^1(R(X,Y,Z),T,V,L) - R^1(X,Y,Z,L)g(T,V) + g(T,V)R^1(X,Y,Z,L) \\ &= g(R(X,Y,Z),L)g(T,V) - g(T,L)g(R(X,Y,Z),V) \\ &= R^1(X,Y,Z,L)g(T,V) - A(T)R^1(X,Y,Z,V) \end{aligned}$$

$$\begin{aligned} W_7^1(Z,R(X,Y,T),V,L) &= R^1(Z,R(X,Y,T),V,L) \\ &+ \frac{1}{n-1}[g(V,R(X,Y,T)Ric(Z,L) \\ &- Ric(R(X,Y,T),V)g(Z,L)] \end{aligned} \tag{14}$$

Then using $Ric(V,Y) = -(n-1)g(V,Y)$ we have

$$\begin{aligned} &= R^1(Z,R(X,Y,T),V,L) + \frac{1}{n-1}[g(V,R(X,Y,T)(-(n-1))g(Z,L) \\ &\quad + (n-1)g(R(X,Y,T),V)g(Z,L)] \\ &= R^1(Z,R(X,Y,T),V,L) - R^1(X,Y,T,V)g(Z,L) + R^1(X,Y,T,V)g(Z,L) \\ &= R^1(Z,R(X,Y,T),V,L) \\ &= g(Z,L)g(R(X,Y,T),V) - g(R(X,Y,T),L)g(Z,V) \\ &= g(Z,L)R^1(X,Y,T,V) - R^1(X,Y,T,L)g(Z,V) \\ &= A(Z)R^1(X,Y,T,V) - R^1(X,Y,T,L)g(Z,V) \end{aligned}$$

$$\begin{aligned} W_7^1(Z,T,R(X,Y,V),L) &= R^1(Z,T,R(X,Y,V),L) \\ &+ \frac{1}{n-1}[g(T,R(X,Y,V))Ric(Z,L) \\ &- Ric(T,R(X,Y,V)g(Z,L)] \end{aligned} \tag{15}$$

Then using $Ric(X,Y) = -(n-1)g(X,Y)$ we have:

$$\begin{aligned} R^1(Z,T,R(X,Y,V),L) &+ \frac{1}{n-1}[g(T,R(X,Y,V)(-(n-1))g(Z,L) \\ &+ (n-1)g(T,R(X,Y,V)g(Z,L) \end{aligned}$$

Also using $g(T, R(X,Y,V) = R^1(X,Y,V,T)$ we have:

$$\begin{aligned} &R^1(Z,T,R(X,Y,V),L) - R^1(X,Y,V,T)g(Z,L) \\ &\qquad\qquad + R^1(X,Y,V,T)g(Z,L) \\ &= R^1(Z,T,R(X,Y,V),L) \\ &= g(Z,L)g(T,R(X,Y,V)) - g(T,L)g(Z,R(X,Y,V)) \\ &= A(Z)R^1(X,Y,V,T) - g(T,L)R^1(X,Y,V,Z) \\ &= A(Z)R^1(X,Y,V,T) - A(T)R^1(X,Y,V,Z) \end{aligned}$$

Putting together 12, 13, 14, and 15 we have:

$$A(X)W_7^1(Y,Z,T,V) - A(Y)W_7^1(X,Z,T,V)$$

$$-R^1(X,Y,Z,L)g(T,V) + A(T)R^1(X,Y,Z,V)$$

$$-A(Z)R^1(X,Y,T,V) + R^1(X,Y,T,L)g(Z,V)$$

$$-A(Z)R^1(X,Y,V,T) + A(T)R^1(X,Y,V,Z) = 0$$

From the requirement that $W_7^1(X,Y)Z$ be symmetric, the terms $W_7^1(Y,Z,T,V)$ and $W_7^1(X,Z,T,V)$ vanish. Due to the skew-symmetric property of $R^1(X,Y,Z,V)$ in its last two variables, the coefficients of $A(Z)$ cancel out. The same applies to the coefficients of $A(T)$. We then remain with the expression

$$R^1(X,Y,T,L)g(Z,V) - R^1(X,Y,Z,L)g(T,V) = 0$$

Since $g(Z,V)$ and $g(T,V) \neq 0$ for arbitrary vectors T,V,Z, we must have $R^1(X,Y,Z,L) = 0$. Hence the theorem.

5 Chapter 5

5.1 A W_7-Recurrent P-Kenmotsu manifold

Definition 5.1.1. *A manifold M^n is said to be a recurrent manifold if*

$$(\nabla_U R)(X,Y,Z) = B(U)R(X,Y,Z)$$

where B is the associated recurrence 1-form.

Definition 5.1.2. *Similarly, it is Ricci recurrent if*

$$(\nabla_U Ric)(X,Y) = B(U)Ric(X,Y)$$

Definition 5.1.3. *We shall refer to a P-Kenmotsu manifold as W_7-recurrent if*

$$(\nabla_U W_7^1)(X,Y,Z,T) = B(U)W_7^1(X,Y,Z,T) \tag{16}$$

Theorem 5.1.4. *If a P-Kenmotsu manifold is W_7-recurrent and Ricci recurrent, then for the same recurrence 1-form, it is recurrent.*

Proof
From 16, we have

$$\begin{aligned}
(\nabla_U W_7^1)(X,Y,Z,T) = B(U)W_7^1(X,Y,Z,T) &= (\nabla_U R^1)(X,Y,Z,T) \\
&+ \frac{1}{n-1}[g(Y,Z)(\nabla_U Ric)(X,T) \\
&- g(X,T)(\nabla_U Ric)(Y,Z)] \\
B(U)W_7^1(X,Y,Z,T) &= (\nabla_U R^1)(X,Y,Z,T) \\
&+ \frac{B(U)}{n-1}[g(Y,Z)Ric(X,T) - g(X,T)Ric(Y,Z)] \\
(\nabla_U R^1)(X,Y,Z,T) &= B(U)\{W_7^1(X,Y,Z,T) \\
&- \frac{1}{n-1}[g(Y,Z)Ric(X,T) - g(X,T)Ric(Y,Z)]\} \\
\implies (\nabla_U R^1)(X,Y,Z,T) &= B(U)R^1(X,Y,Z,T)
\end{aligned}$$

Hence the theorem.

5.2 A W_7-Irrotational P-Kenmotsu manifold

Definition 5.2.1. *The rotation (Curl) of a W_7 -curvature tensor on a P-Kenmotsu manifold is defined as*

$$RotW_7(X,Y)Z = (\nabla_Z UW_7)(X,Y)Z + (\nabla_X UW_7)(U,Y)Z + (\nabla_Y UW_7)(X,U)Z - (\nabla_Z UW_7)(X,Y)U$$

In consequence of Bianchi's second identity, equation 29 becomes

$$RotW_7(X,Y)Z = -(\nabla_Z W_7)(X,Y)U$$

If the W_7 -curvature tensor is irrotational, then curl $W_7(X,Y)Z = 0$ and hence $(\nabla_Z W_7)(X,Y)U = 0$

which gives

$$(\nabla_Z W_7)(X,Y)U = W_7(\nabla_Z X,Y)U + W_7(X,\nabla_Z Y)U + W_7(X,Y)\nabla_Z U \tag{17}$$

Theorem 5.2.2. *The W_7 -curvature tensor in P-Kenmotsu manifold is irrotational if and only if*

Proof
Let W_7 -curvature tensor be irrotational then putting $U = \xi$ in equation 11, we get

$$(\nabla_Z W_7)(X,Y)\xi = W_7(\nabla_Z X,Y)\xi + W_7(X,\nabla_Z Y)\xi + W_7(X,Y)\nabla_Z \xi \tag{18}$$

Putting $Z = \xi$ in equation (11) and using equations (5) and (10), we get

$$W_7(X,Y)\xi = \eta(X)Y - \eta(Y)X + \frac{1}{n-1}[\eta(Y)QX - S(Y,\xi)X] \tag{19}$$

Using equations (19) and (18), we obtain

$$\nabla_Z \eta(X)Y - \eta(Y)X + \frac{1}{n-1}[\eta(Y)QX - S(Y,\xi)X] = W_7(\nabla_Z X,Y)\xi + W_7(,\nabla_Z Y)\xi + W_7(X,Y)\nabla_Z \xi \tag{20}$$

$$\nabla_Z\eta(X)Y-\nabla\eta(Y)X+\tfrac{1}{n-1}[(\nabla_Z\eta)(Y)QX+(\nabla_ZQ)(X)\eta(Y)-(\nabla_ZS)(Y,\xi)X+(\nabla_ZX)S(Y,\xi)X]$$

$$=\eta(\nabla_ZX)Y-\eta(Y)\nabla_ZX+\tfrac{1}{n-1}[\eta(Y)Q\nabla_ZX-S(Y,\xi)\nabla_ZX]+\eta(X)\nabla_ZY-\eta(\nabla_ZY)X$$

$$+\tfrac{1}{n-1}[\eta(\nabla_ZY)QX-S(\nabla_ZY,\xi)X]+W_7(X,Y)Z-\eta(Z)[\eta(X)Y-\eta(Y)X+\tfrac{1}{n-1}[\eta(Y)QX-S(Y,\xi)X]$$

Using $(\nabla_X\eta)(Y)=g(X,Y)-\eta(X)\eta(Y)$ in the equation above, we btain

$$\begin{aligned} q(Z,X)Y-\eta(Z)\eta(X)Y-g(Z,Y)X+\eta(Z)\eta(Y)X & \\ &+\frac{1}{n-1}[g(Z,Y)QX-\eta(Z)\eta(Y)QX+(\nabla_ZQ)(X)\eta(Y) \\ &-(\nabla_ZS)(Y,\xi)X-\nabla_ZX)S(Y,\xi)] \\ =W_7(X,Y)Z-\eta(Z)\eta(X)Y+\eta(Z)\eta(Y)X & \\ &-\frac{1}{n-1}[\eta(Z)\eta(Y)QX+\eta(Z)S(Y,\xi)X] \end{aligned}$$

Simplifying the above equation and using equation (11), we obtain

$$g(Z,X)Y-g(Z,Y)X+\frac{1}{n-1}[g(Z,Y)QX+(\nabla_ZQ)(X)\eta(X)-(\nabla_ZS)(Y,\xi)X-(\nabla_ZX)S(Y,\xi)]=R(X,Y)Z+\frac{1}{n-1}[g \tag{21}$$

On rearranging we have

$$R(X,Y)Z=g(Z,X)Y-g(Z,Y)X+\frac{1}{n-1}[(\nabla_ZQ)(X)\eta(X)-(\nabla_ZS)(Y,\xi)X-(\nabla_ZX)S(Y,\xi)+S(Y,\xi)X] \tag{22}$$

Conversely, retreating the steps,we can show that W_7-curvature tensor is irrotational.

5.3 Conservative W_7-curvature tensor in a Para Kenmotsu manifold

W_7-curvature tensor is given by

$$W_7(X,Y)Z = R(X,Y)Z - \frac{1}{n-1}[g(Y,Z)QX - S(Y,Z)X] \tag{23}$$

Differentiating (35) covariantly with respect to U, we have

$$(\nabla_U W_7)(X,Y)Z = (\nabla_U R)(X,Y)Z - \frac{1}{n-1}[g(Y,Z)(\nabla_U Q)(X) - (\nabla_U S)(Y,Z)X] \tag{24}$$

Contracting equation (24), we obtain

$$(div\ W_7)(X,Y)Z = (\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z) + \frac{1}{n-1}[g(Y,Z)dr(X) - (\nabla_Y)S(X,Z)] \tag{25}$$

$$(div\ W_7)(X,Y)Z = (\nabla_X S)(Y,Z) - \frac{n}{n-1}(\nabla_Y S)(X,Z) + \frac{1}{n-1}g(Y,Z)dr(X) \tag{26}$$

Suppose that $W_7(X,Y)Z$ is conservative, i.e. $(div\ W_7)(X,Y)Z = 0$, then equation (26) reduces to

$$(\nabla_X S)(Y,Z) - \frac{n}{n-1}(\nabla_Y S)(X,Z) = -\frac{1}{n-1}g(Y,Z)dr(X) \tag{27}$$

Putting $X = \xi$ in (27), we get

$$(\nabla_\xi S)(Y,Z) - \frac{n}{n-1}(\nabla_Y S)(\xi,Z) = -\frac{1}{n-1}g(Y,Z)dr(\xi) \tag{28}$$

Expanding the first term of (28) we have

$$(\nabla_\xi S)(Y,Z) = \nabla_\xi S(Y,Z) - S(\nabla_\xi Y,Z) - S(Y,\nabla_\xi Z) \tag{29}$$

Since ∇ is torsion free i.e. $\nabla_X Y - \nabla_Y X = [X,Y]$ and using $\nabla_X \xi = \phi^2 X = X - \eta(X)\xi$ and $(L_\xi S)(X,Y) = 2S(X,Y) + 2(n-1)\eta(X)\eta(Y)$ in (29), we obtain

$$\begin{aligned}
(\nabla_\xi S)(Y,Z) &= \nabla_\xi S(Y,Z) - S([\xi,Y] + \nabla_Y \xi, Z) - S(Y, [\xi,Z] + \nabla_Z \xi) \\
&= \nabla_\xi S(Y,Z) - S([\xi,Y],Z) - S(\nabla_Y \xi, Z) - S(Y,[\xi,Z]) - S(Y, \nabla_Z \xi) \\
&= \nabla_\xi S(Y,Z) - S([\xi,Y],Z) - S(Y,[\xi,Z]) - S(Y - \eta(Y)\xi, Z) \\
&\quad - S(Y, Z - \eta(Z)\xi) \\
&= (L_\xi S)(Y,Z) - S(Y,Z) + S(\eta(Y)\xi, Z) - S(Y,Z) + S(\eta(Z)\xi, Y) \\
&= (L_\xi S)(Y,Z) - 2S(Y,Z) - 2(n-1)\eta(Y)\eta(Z) \\
&= 0 \qquad \textbf{(30)}
\end{aligned}$$

Also, expanding the second term of (28) we have

$$(\nabla_Y S)(\xi,Z) = \nabla_Y S(\xi,Z) - S(\nabla_Y \xi, Z) - S(\xi, \nabla_Y Z) \qquad \textbf{(31)}$$

Using $\nabla_X \xi = \phi^2 X = X - \eta(X)\xi$ and $S(X,\xi) = -(n-1)\eta(X)$ in (31), we get

$$\begin{aligned}
(\nabla_Y S)(\xi,Z) &= \nabla_Y[-(n-1)g(\xi,Z)] - S(Y - \eta(Y)\xi, Z) + (n-1)g(\xi, \nabla_Y Z) \\
&= -(n-1)[g(\nabla_Y \xi, Z) + g(\xi, \nabla_Y Z)] - S(Y,Z) + S(\eta(Y)\xi, Z) \\
&\quad + (n-1)g(\xi, \nabla_Y Z) \\
&= -(n-1)g(Y - \eta(Y)\xi, Z) - S(Y,Z) - (n-1)g(\eta(Y)\xi, Z) \\
&= -(n-1)g(Y,Z) + (n-1)g(\eta(Y)\xi, Z) - S(Y,Z) - (n-1)g(\eta(Y)\xi, Z) \\
&= -(n-1)g(Y,Z) - S(Y,Z) \qquad \textbf{(32)}
\end{aligned}$$

Now using equations (30) and (32) in (28), we obtain

$$\begin{aligned}
\frac{n}{n-1}[(n-1)g(Y,Z) + S(Y,Z)] &= -\frac{1}{n-1} g(Y,Z) dr(\xi) \\
S(Y,Z) &= -\frac{1}{n} g(Y,Z) dr(\xi) - (n-1)g(Y,Z) \\
S(Y,Z) &= [-\frac{dr(\xi)}{n} - (n-1)]g(Y,Z) \qquad \textbf{(33)}
\end{aligned}$$

Thus we have the following result.

Theorem 5.3.1. *On a Para Kenmotsu manifold M^n, if the $W_7(X,Y)Z$ curvature tensor of type (1,3) is conservative then M^n is Einstein.*

6 Chapter 6

6.1 Conclusion

We have studied some geometric properties of W_7-curvature tensor in Para Kenmotsu manifolds satisfying the conditions

$W_7(X,Y)Z = 0,$

$R(X,Y)W_7(Z,T)V = 0,$

$\nabla_U W_7(X,Y)Z = 0,$

$(\nabla_U W_7')(X,Y,Z,T) = B(U)W_7'(X,Y,Z,T)$

and

$$R(X,Y)Z = g(Z,X)Y - g(Z,Y)X + \frac{1}{n-1}[(\nabla_Z Q)(X)\eta(X) - (\nabla_Z S)(Y,\xi)X - (\nabla_Z X)S(Y,\xi) + S(Y,Z)X] \tag{34}$$

.

We have shown that a W_7-flat P-Kenmotsu manifold is a flat manifold. Its curvature vanishes identically everywhere. Also, we have shown that a W_7-semisymmetric P-Kenmotsu manifold is a W_7-flat manifold. Therefore a W_7-semisymmetric P-Kenmotsu manifold is neither η-Einstein nor Einsten. Similarly, we have established the necessary condition for a W_7-curvature tensor to be irrotational.

6.2 Future Research

The W_7 curvature tensor is given by

$$W_7(X,Y,Z,T) = R(X,Y,Z,T) + \frac{1}{n-1}[g(Y,Z)Ric(X,T) - g(X,T)Ric(Y,Z)] \quad (35)$$

Expressing equation (35) in index notation we get

$$W_{7ijkl} = R_{ijkl} + \frac{1}{n-1}[g_{jk}R_{il} - g_{il}R_{jk}] \quad (36)$$

Contracting equation (36) we obtain

$$g^{il}W_{ijkl} = g^{il}R_{ijkl} + \frac{1}{n-1}[g^{il}g_{jk}R_{il} - g^{il}g_{il}R_{jk}] \quad (37)$$

$$W_{jk} = R_{jk} + \frac{1}{n-1}[Rg_{jk} - nR_{jk}] \quad (38)$$

$$W_{jk} = \frac{1}{n-1}(R_{jk} + Rg_{jk}) \quad (39)$$

The contracted version of W_7-curvature tensor above does not vanish in an Einstein space. This shows that we can not extend the Pirani's formalism of gravitational wave to the Einstein space with the help of W_{7ijkl}.

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