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FRACTIONAL FACTORIAL DESIGNS WHICH ALLOW ESTIMATION UP TO THREE FACTOR INTERACTIONS

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Dominic Mutiso Muia

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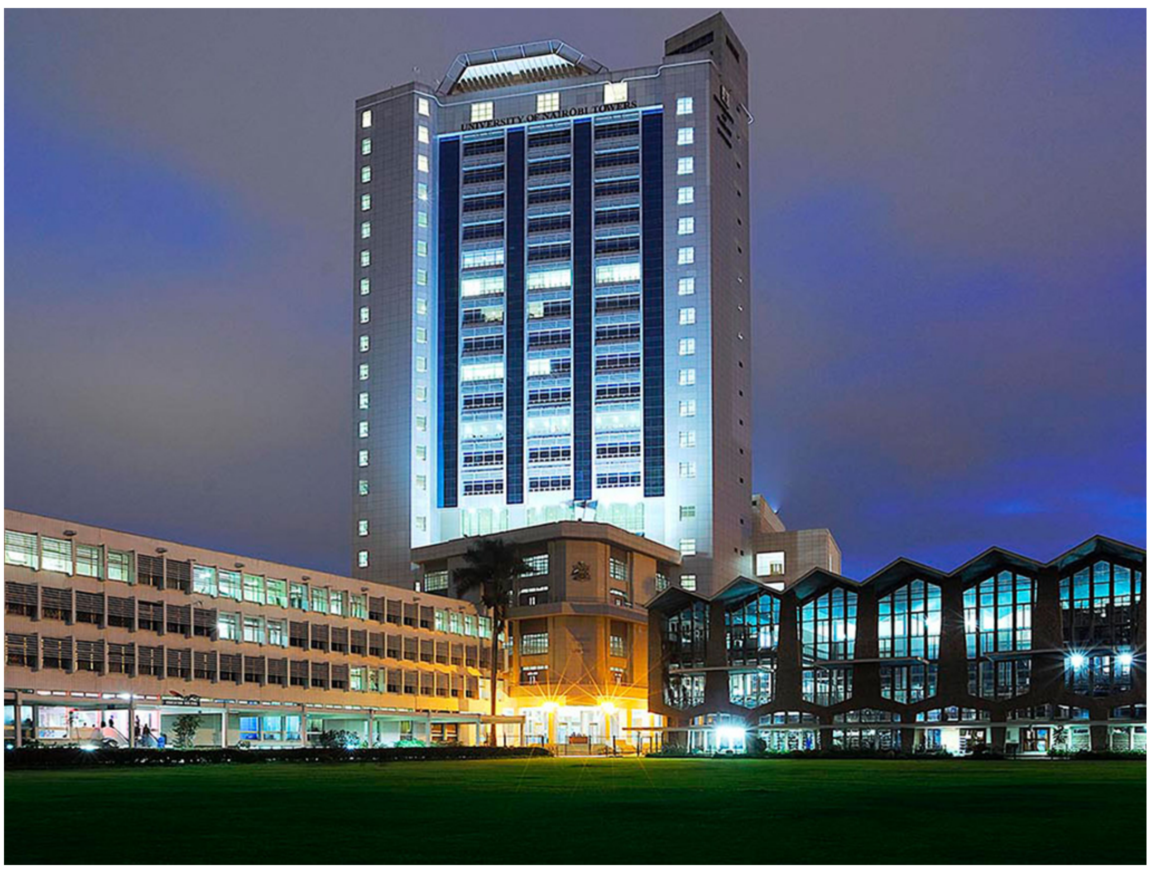

Submitted to the School of Mathematics in partial fulfilment for a degree in Master of Science in Pure Mathematics

FRACTIONAL FACTORIAL DESIGNS WHICH ALLOW ESTIMATION UP TO THREE FACTOR INTERACTIONS

Research Report in Mathematics, Number 15, 2021

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Master Thesis

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Abstract

In this project we focus on the construction of partially duplicated fractional factorial designs. Factors involved in these designs are at two levels. In the first part of this work we discuss partially duplicated fractional factorial designs that permit estimation of main effects and two factor interactions only. In part two of this work, we construct partially duplicated fractional factorial designs that permit estimation of main effects, two-factor and three-factor interactions only assuming high order interactions to be absent. Designs presented in this project have as many as ten factors. The method of construction, analysis, test procedure and block designs is illustrated and can be used in any of the designs presented in this work.

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Declaration and Approval

I the undersigned declare that this dissertation is my original work and to the best of my knowledge, it has not been submitted in support of an award of a degree in any other university or institution of learning.

Signature	Date
	19/08/2021

Dominic Mutiso Muia
Reg No. I56/34060/2019

In my capacity as a supervisor of the candidate's dissertation, I certify that this dissertation has my approval for submission.

Signature	Date
	19/08/2021

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Dedication

This project is dedicated to my loving parents; Benson Muia and Juliana Kanini and all those in the field of Design of experiments.

Contents

Acknowledgments

First of all is to thank the Almighty God for the gift of life and good health that has enabled me to successfully complete this project.

I owe many thanks to my supervisor Professor Manene Moses M. for his unflagging commitment and sacrifice throughout the preparation of this project. Thank you very much for the constructive comments and suggestions.

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Dominic Mutiso Muia

Nairobi, 2021.

1 Introduction

1.1 Concept of Factorial Designs

Factorial designs, sometimes referred to as Industrial designs were and are still widely used in experiments to study the effect of factors, that is, the significant factors. They also allow one to study the effect of their interactions. For Instance, in clinical trials, one would like to determine if the combination(Interaction) of different type of drugs is efficient in curing a disease.Factorial designs are also used in product development, designing processes and quality improvement of products.

Factorial designs are economical and saves on time as compared to experiments done factor by factor popularly known as one-factor-at-a-time (OFAT) experiments. Such experiments involve alternating the level of one factor at a time while adjusting for the other factor levels. Consequently, one will require additional resources and time. It is important to note that this type of experiment does not allow one to study for interactions. Complete factorials can be considered when the number of factors is sufficiently small. Full factorials contain all possible combination of levels for the factors involved.

In the case of an experiment involving large number of factors and/or limited resources to perform a full factorial, under reasonable assumption the experimenter can adopt to run a fraction of the complete factorial design. Consider a a^p Factorial design where a implies the number of levels and p the number of factors. These factors can be at two or more levels. Throughout we shall only discuss designs with factors at two levels. To illustrate the usefulness of a fractional factorial design, consider a 2^9 factorial design. A complete factorial will involve 512 runs which in real life may not be “practical” considering resources like money and time. However, one can adopt a fraction of the above design, say $\frac{1}{2^5}$ of the 2^9 factorial to get 16 runs which is much practical. Such designs are referred to as Fractional Factorial Designs and are much more economical and time-saving. Fractional factorial experiments are widely used in screening designs-designs meant to help identify significant or rather active factors from a number of factors.

Running an unreplicated complete or fractional factorial design with the focus of identifying significant effects only as a way of miminising on cost could lead to biased results while making inference during analysis. This is as a result of not obtaining an estimate for the pure error. Again, assumption made on high-order interactions to be considered as negligible so as to obtain an estimate for the experimental error could be misleading as some of the assumed interactions may not be negligible or absent. Disregarding some of these interactions could lead to an experimental error variance that is biased. One proposed method of obtaining a better estimate for the error variance is duplicating some of the treatment combinations in the design.

Certainly, complete duplication provides a better estimate for the error variance. However, it is not cost effective due to the additional runs and thus partial duplication is a good alternative. Partial duplication provides for an unbiased estimate of the error variance and estimates of effects which are more specific and reliable.

1.2 Literature review

The concept of factorial design was first presented by Fisher (1935) in his book "The Designs of Experiments".

Yates (1935) contributed to the concept of factorial design introduced by Fisher (1935) by suggesting an algorithm which could be used to estimate the effects involved in a particular factorial design. This algorithm is now widely used and is known as the Yate's algorithm.

An experiment can either be symmetrical factorial or asymmetrical factorial. Bose and Kishen (1940) studied in detail symmetrical factorial designs and addressed the problem of confounding in such designs highlighting the usefulness of partial confounding. Only symmetrical factorials will be addressed in this project, that is, experiments whose factors occur at the same number of levels.

Bose (1947) extended the work by Bose and Kishen (1940) on symmetrical factorial experiments giving the mathematical theory behind symmetric factorial designs.

Running a complete factorial experiment with many factors requires many observations(runs). To obtain the error d.f. we shall need to replicate the experiment. Fractional factorial designs have become extremely popular and have been explored widely by researchers such as Plackett and Burman, 1946; Dykstra, 1959; Patel, 1961 and Montgomery, 2001 among many others.

Plackett and Burman (1946) obtained orthogonal fractional factorial designs that could be used to estimate main effects only assuming that all other interactions were absent. These designs were used extensively by researchers like Ottieno (1984), Odhiambo (1985) and Manene (1987) among others for computing optimum expected number of runs in Group Screening Designs (GSD) when we have error in observations.

Daniel (1957) during the convocation of the American Society for Quality Control, proposed an error estimation method of partially duplicating a subset of the treatment combinations.

Later on Dykstra (1959) extended the conversation by giving an experimental plan and method of analysis of designs with factors at two levels with partial duplication involved.

Dykstra (1960) extended his work done in 1959 to partial duplication of Response Surface Designs to clear the uncertainty of whether variability remains constant or increases away from a center point resulting to a biased estimate of the error. Dykstra showed that obtaining duplicates over the experimental area could solve that uncertainty or the fear of variance increasing away from the center point.

Patel (1963) extended Dykstra's work (1959) giving the experimental plan, test procedures and block design for 2^p designs that had been duplicated partially. Patel's designs provided for fewer runs than the corresponding Dykstra's designs. Both Patel and Dykstra duplicated partially in their designs so as to allow for estimation of the error variance.

Pigeon and McAllister (1989) discussed how it was possible to have partial duplication without interfering with the orthogonality of main effects.

Liau (2008) extended the work by Pigeon and McAllister (1989) by presenting construction techniques on how to get the orthogonal main effect plan with some set of duplicated points.

Liau and Chai (2009) re-examined the $\frac{1}{8}$ fraction of 2^7 design by Snee (1985) where 2 refers to the number of levels and 7 the number of factors. Snee's design had four points duplicated. After Liau and Chai re-analysed the design they found out that had the points not been duplicated, the significant effects would have been different. They also concluded that partially duplicated designs are more robust and efficient in screening experiments.

Tsai and Liao (2011) extended the proposed 2^p symmetrical factorial by Liau and Chai (2009) to obtain optimality in partially replicated mixed factorial experiments.

Patel (1963) and Dykstra (1959) studied designs that allowed for estimation up to two-factor interactions. Plackett-Burman designs assumed no interaction in effects. In this project, we are going to include partial duplication in the proposed fractional factorial designs that allow for estimation of effects up to three-factor interaction as some of these interactions assumed to be absent could be actually active.

1.3 Statement of The Problem

It is a commonly accepted practice to obtain an estimate of the error by regarding high order interactions absent and pooling their d.f. and Sum of Squares to be for error. However, some of these interactions may actually be present leading to a biased estimate of the error. Moreover, we may not understand the extent to which the error term is biased (Dykstra,1959).

Here we are going to show how estimation of effects was done in Patel's work- estimation of effects up-to two-factor interactions using matrix approach- and extend to designs that estimate up to three factor interactions.

1.4 Objectives of the Study

The main aim of this project is to construct partially duplicated fractional factorial designs with as fewer runs as possible.

The specific objectives of the study are are:

i) To construct partially duplicated fractional factorial designs which allow for estimation up-to two factor interactions.

ii) To construct partially duplicated fractional factorial designs which allow for estimation up-to three factor interactions.

1.5 Methodology

1.5.1 Definition of Effects

Consider a factorial experiment with p factors $F_1,\ldots,F_p$ each at two levels and factors that appear at $x_1,\ldots,x_p$ levels. Let x_i the level of the i^{th} factor be coded as 0, 1 for $1 \leq i \leq p$. That is, $x_i = 0,1$ for $i = 1,\ldots,p$

Let the combination of levels of the p factors, that is, the treatment combinations be denoted by

$$f_1^{x_1},\ldots,f_P^{x_p} \qquad or \qquad (x_1,\ldots,x_P) \tag{1.1}$$

The design described above is a 2^p factorial design. Consider a 2^2 design. The treatment combinations are (1), f_1 , f_2 , f_1f_2. The same treatments can be represented as $(0,0)$, $(1,0)$, $(0,1)$ and $(1,1)$. In a 2^3 design the eight treatments are (1), f_1 , f_2 , f_1f_2 , f_3 , f_1f_3 , f_2f_3 and $f_1f_2f_3$ also presented as $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, $(1,1,0)$, $(0,0,1)$, $(1,0,1)$, $(0,1,1)$ and $(1,1,1)$.

The parameters in a 2^2 design are μ, F_1 , F_2 , F_1F_2 and their estimates are denoted by $\hat{\mu}$, $\hat{F_1}$, $\hat{F_2}$ and $F_1\hat{F}_2$ respectively. They are given by the following equations:-

$$\begin{aligned}
\hat{\mu} &= \frac{[\mu]}{4} = \frac{1}{4}[(1)+f_1+f_2+f_1f_2] \\
\hat{F_1} &= \frac{[F_1]}{4} = \frac{1}{4}[-(1)+f_1-f_2+f_1f_2] \\
\hat{F_2} &= \frac{[F_2]}{4} = \frac{1}{4}[-(1)-f_1+f_2+f_1f_2] \\
F_1\hat{F}_2 &= \frac{[F_1F_2]}{4} = \frac{1}{4}[(1)-f_1-f_2+f_1f_2]
\end{aligned} \tag{1.2}$$

where $[\mu]$, $[F_1]$, $[F_2]$ and $[F_1F_2]$ are contrasts in the actual observation values of the above treatments.

The parameters in a 3 factor experiment are μ, F_1 ,F_2 , F_3 , F_1F_2 , F_1F_3 , F_2F_3 and $F_1F_2F_3$. Their estimates are defined by the following eight equations:-

$$
\begin{aligned}
\hat{\mu} &= \frac{1}{8}[(1)+f_1+f_2+f_1f_2+f_3+f_1f_3+_2f_3+f_1f_2f_3] \\
\hat{F_1} &= \frac{1}{8}[(-1)+f_1-f_2+f_1f_2-f_3+f_1f_3-f_2f_3+f_1f_2f_3] \\
\hat{F_2} &= \frac{1}{8}[(-1)-f_1+f_2+_1f_2-f_3-f_1f_3+f_2f_3+f_1f_2f_3] \\
\hat{F_3} &= \frac{1}{8}[(-1)-f_1-f_2-f_1f_2+f_3+f_1f_3+f_2f_3+f_1f_2f_3] \\
F_1\hat{}F_2 &= \frac{1}{8}[(1)-f_1-f_2+f_1f_2+f_3-f_1f_3-f_2f_3+f_1f_2f_3] \\
F_1\hat{}F_3 &= \frac{1}{8}[(1)-f_1+f_2-f_1f_2-f_3+f_1f_3-f_2f_3+f_1f_2f_3] \\
F_2\hat{}F_3 &= \frac{1}{8}[(1)+f_1-f_2-f_1f_2-f_3-f_1f_3+f_2f_3+f_1f_2f_3] \\
F_1\hat{F_2}F_3 &= \frac{1}{8}[(-1)+f_1+f_2-f_1f_2+f_3-f_1f_3-f_2f_3+f_1f_2f_3]
\end{aligned}
\tag{1.3}
$$

From equations (1.2) we can see that $\hat{\mu}+\hat{F_1}$ is the average of the observations at high level of factor F_1 and $\hat{\mu}-\hat{F_1}$ is the average of observations corresponding to the low level of F_1. The same is true for F_2. Using the same logic we can reason for equations (1.3).

$\hat{\mu}+F_1\hat{}F_2$ is the average of the observations at the low level of both factors and at the high level of both factors whereas $\hat{\mu}-F_1\hat{}F_2$ is the average of observations at high level of F_1 and high level of F_2. The same argument can be used in equations (1.3).

Let B= $\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ and $B^{(2)} = B \oplus B$ be the direct product of matrix B.
Equation (1.2) can be derived as

$$
E\begin{bmatrix}(1)\\ f_1\end{bmatrix} \oplus \begin{bmatrix}(1)\\ f_2\end{bmatrix} = B^{(2)}\begin{bmatrix} I \\ F_1\end{bmatrix} \oplus \begin{bmatrix} I \\ F_2\end{bmatrix} \tag{1.4}
$$

where $(1).(1) = (1)$, $(1).f_1 = f_1 = f_1.(1)$, $I.I = I$, $I.F_1 = F_1 = F_1.I$ and $I = \mu$

Solving for equation (1.4) we get

$$E\begin{bmatrix}(1)\\ f_1\\ f_2\\ f_1f_2\end{bmatrix}=\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}\oplus\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}\begin{bmatrix}\mu\\ F_1\\ F_2\\ F_1F_2\end{bmatrix} \tag{1.5}$$

Now,

$$\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}\oplus\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}=\begin{bmatrix}1\times1 & 1\times-1 & -1\times1 & -1\times-1\\ 1\times1 & 1\times1 & -1\times1 & -1\times1\\ 1\times1 & 1\times-1 & 1\times1 & 1\times-1\\ 1\times1 & 1\times1 & 1\times1 & 1\times1\end{bmatrix}=\begin{bmatrix}1 & -1 & -1 & 1\\ 1 & 1 & -1 & -1\\ 1 & -1 & 1 & -1\\ 1 & 1 & 1 & 1\end{bmatrix} \tag{1.6}$$

Replace equation (1.6) in equation (1.5) to get

$$E\begin{bmatrix}(1)\\ f_1\\ f_2\\ f_1f_2\end{bmatrix}=\begin{bmatrix}1 & -1 & -1 & 1\\ 1 & 1 & -1 & -1\\ 1 & -1 & 1 & -1\\ 1 & 1 & 1 & 1\end{bmatrix}\begin{bmatrix}\mu\\ F_1\\ F_2\\ F_1F_2\end{bmatrix} \tag{1.7}$$

Getting the inverse of the above matrix in (1.7) gives us the equation of the estimates as

$$\begin{bmatrix}\hat{\mu}\\ \hat{F_1}\\ \hat{F_2}\\ F_1\hat{F}_2\end{bmatrix}=\frac{1}{4}\begin{bmatrix}1 & 1 & 1 & 1\\ -1 & 1 & -1 & 1\\ -1 & -1 & 1 & 1\\ 1 & -1 & -1 & 1\end{bmatrix}\begin{bmatrix}(1)\\ f_1\\ f_2\\ f_1f_2\end{bmatrix} \tag{1.8}$$

Note these equations are similar to those given in equation (1.2).

Similarly, for the 3-factor experiment we can proceed as follows.

Let B= $\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}$ and $B^{(3)}=B\oplus B\oplus B$ be the direct product of matrix B.

$$E\begin{bmatrix}(1)\\ f_1\end{bmatrix}\oplus\begin{bmatrix}(1)\\ f_2\end{bmatrix}\oplus\begin{bmatrix}(1)\\ f_3\end{bmatrix}=B^{(3)}\begin{bmatrix}I\\ F_1\end{bmatrix}\oplus\begin{bmatrix}I\\ F_2\end{bmatrix}\oplus\begin{bmatrix}I\\ F_3\end{bmatrix} \tag{1.9}$$

Solving the left-hand side of equation (1.9) to get

$$E\begin{bmatrix}(1)\\ f_1\\ f_2\\ f_1f_2\\ f_3\\ f_1f_3\\ f_2f_3\\ f_1f_2f_3\end{bmatrix}=\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}\oplus\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}\oplus\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}\begin{bmatrix}\mu\\ F_1\\ F_2\\ F_1F_2\\ F_3\\ F_1F_3\\ F_2F_3\\ F_1F_2F_3\end{bmatrix} \tag{1.10}$$

Equation (1.10) reduces to

$$E\begin{bmatrix}(1)\\ f_1\\ f_2\\ f_1f_2\\ f_3\\ f_1f_3\\ f_2f_3\\ f_1f_2f_3\end{bmatrix}=\begin{bmatrix}1 & -1 & -1 & 1 & -1 & 1 & 1 & -1\\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1\\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1\\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1\\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1\\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1\\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1\\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1\end{bmatrix}\begin{bmatrix}\mu\\ F_1\\ F_2\\ F_1F_2\\ F_3\\ F_1F_3\\ F_2F_3\\ F_1F_2F_3\end{bmatrix} \tag{1.11}$$

Getting the inverse of the matrix in (1.11), the equations below follow

$$\begin{bmatrix} \hat{\mu} \\ \hat{F_1} \\ \hat{F_2} \\ F_1\hat{F}_2 \\ \hat{F_3} \\ F_1\hat{F}_3 \\ F_2\hat{F}_3 \\ F_1\hat{F}_2F_3 \end{bmatrix} = \frac{1}{8} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\ -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} (1) \\ f_1 \\ f_2 \\ f_1f_2 \\ f_3 \\ f_1f_3 \\ f_2f_3 \\ f_1f_2f_3 \end{bmatrix} \tag{1.12}$$

The corresponding equations in (1.12) above are the same as those in equations (1.3).

Therefore, we can write the expectation equations in (1.4) and (1.9) in a more generalized form for more than two factors as

$$E\begin{bmatrix} (1) \\ f_1 \end{bmatrix} \oplus \begin{bmatrix} (1) \\ f_2 \end{bmatrix} \oplus \begin{bmatrix} (1) \\ f_3 \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} (1) \\ f_P \end{bmatrix} = B^{(p)} \begin{bmatrix} I \\ F_1 \end{bmatrix} \oplus \begin{bmatrix} I \\ F_2 \end{bmatrix} \oplus \begin{bmatrix} I \\ F_3 \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} I \\ F_p \end{bmatrix} \tag{1.13}$$

1.5.2 Construction of The Design

We are going to consider linear equations of the form $L = f_1x_1 + f_2x_2 + \cdots + f_px_p$ where $f_i = 0, 1$ for $i = 1, 2, \ldots, p$.
The number of non-zero co-efficients of the levels $x_1, \ldots, x_p$ is known as the weight of the linear form L.

Consider a set of k linearly independent equations

$$\begin{aligned} L_1 &= f_{11}x_1 + f_{12}x_2 + \cdots + f_{1p}x_p = b_1 \\ L_2 &= f_{21}x_1 + f_{22}x_2 + \cdots + f_{2p}x_p = b_2 \\ &\vdots \\ L_k &= f_{k1}x_1 + f_{k2}x_2 + \cdots + f_{kp}x_p = b_k \end{aligned}$$

where $f_{\tau j}$; $\tau = 1, 2, \ldots, k$ and $j = 1, 2, \ldots, p$.
The L_τ's in this project are chosen such that each linear function is of weight ≥ 3. All linear combinations of L_τ's gives $2^k - 1$ linear forms

$$\lambda_1 L_1 + \lambda_2 L_2 + \ldots + \lambda_k L_k$$

where $\lambda_\tau = 0,1$; $\tau = 1,2,\ldots,k$ and $(\lambda_1,\ldots,\lambda_k) \neq (0,\ldots,0)$.
The number of treatments which when written as column vectors constitute an orthogonal array of strength 2 is 2^{p-k} given by the $2^k - 1$ linear forms.

In treatment combinations each level of every factor appears the same number of times. Also, any combination of levels corresponding to a pair of factors occurs equally (Rao, 1950). $L_\tau = b_\tau$ (mod 2) generates the combination of levels denoted by $(x_1,\ldots,x_P)$ given in (1.1) where $b_\tau = 0,1$ for $\tau = 1,2,\ldots,k$. It therefore follows that L_τ for $\tau = 1,2,\ldots,k$ are the generators of the design giving the fractional design.

Now let $B = (b_{\tau m})$ be a non-singular matrix with entries 0, 1 (mod 2) for $\tau = 1,\ldots,k$ and $m = 1,\ldots,k$. Then the treatment combinations given by the k linearly independent equations

$$\begin{aligned} L_1 &= 0, b_{11}, b_{12}, \ldots, b_{1k} \\ L_1 &= 0, b_{21}, b_{22}, \ldots, b_{2k} \\ &\vdots \qquad\qquad\qquad\qquad \text{(mod 2)} \\ L_K &= 0, b_{k1}, b_{k2}, \ldots, b_{kk} \end{aligned} \tag{1.14}$$

gives the fraction of the design. The total number of treatment combination given by equation (1.14) is $(k+1)2^{p-k}$.

We are going to repeat any of the $k+1$ set of k equations in (1.14) so as to construct a partially duplicated fractional factorial design.

2 Partially Duplicated Fractional Factorial Designs which allow for Estimation up to Two-Factor Interactions

2.1 Introduction

In this chapter, fractional factorial designs which allow for estimation of effects up to two-fator interactions are presented. We consider experiments involving five factors up to ten factors. The construction plan, test method of significance and the block design is illustrated. The matrix method is used to obtain estimates for effects under consideration.

2.2 Five Factors Experiment involving 24+8 = 32 Runs

2.2.1 Constuction of the Design

Consider a design with treatment combinations $(x_1, x_2, x_3, x_4, x_5)$ which satisfy the simultaneous equations

$$\begin{aligned} x_1 + x_2 + x_3 &= 0, 1, 0 \\ x_1 + x_4 + x_5 &= 0, 0, 1 \end{aligned}$$

mod 2. The first set of treatment combination to the equations

$$\begin{aligned} x_1 + x_2 + x_3 &= 0 \\ x_1 + x_4 + x_5 &= 0 \end{aligned}$$

mod 2 are $(0,0,0,0,0)$, $(0,0,0,1,1)$, $(1,1,0,1,0)$, $(1,1,0,0,1)$, $(1,0,1,1,0)$, $(1,0,1,0,1)$, $(0,1,1,0,0)$ and $(0,1,1,1,1)$.

The second set of treatment combinations satisfying the equations

$$\begin{aligned} x_1 + x_2 + x_3 &= 1 \\ x_1 + x_4 + x_5 &= 0 \end{aligned}$$

mod 2 are $(0,1,0,0,0)$, $(0,0,1,0,0)$, $(0,0,1,1,1)$, $(0,1,0,1,1)$, $(1,0,0,0,1)$, $(1,1,1,0,1)$, $(1,1,1,1,0)$ and $(1,0,0,1,0)$. Similarly, for the third set the treatment combinations are $(0,0,0,1,0)$, $(0,0,0,0,1)$, $(1,1,0,1,1)$, $(1,1,0,0,0)$, $(1,0,1,1,1)$, $(1,0,1,0,0)$, $(0,1,1,1,0)$ and $(0,1,1,0,1)$ which satisfy the equations

$$\begin{aligned} x_1+x_2+x_3 &= 0 \\ x_1+x_4+x_5 &= 1 \end{aligned} \qquad \text{(mod 2)}$$

The defining relation is

$$I = F_1F_2F_3 = F_1F_4F_5 = F_2F_3F_4F_5$$

The aliased sets are

(F_1, F_2F_3, F_4F_5), (F_2, F_1F_3), (F_3, F_1F_2), (F_4, F_1F_5), (F_5, F_1F_4), (F_2F_5, F_3F_4), (F_2F_4, F_3F_5)
Listing only up to two factor interactions.

2.2.2 Method of Analysis

Expected responses $\varphi(x_1,x_2,x_3,x_4,x_5)$ expressed as a linear function of the grand average, main and two-factor interaction effects

$$
\varphi\begin{bmatrix}
0&0&0&0&0\\
0&0&0&1&1\\
1&1&0&1&0\\
1&1&0&0&1\\
1&0&1&1&0\\
1&0&1&0&1\\
0&1&1&0&0\\
0&1&1&1&1\\
0&0&0&0&0\\
0&0&0&1&1\\
1&1&0&1&0\\
1&1&0&0&1\\
1&0&1&1&0\\
1&0&1&0&1\\
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1&1&0&0&0\\
1&0&1&1&1\\
1&0&1&0&0\\
0&1&1&1&0\\
0&1&1&0&1
\end{bmatrix}
=
\begin{array}{c}
\begin{array}{cccccccccccccccc}
\mu & F_1 & F_2F_3 & F_4F_5 & F_3 & F_1F_2 & F_2 & F_1F_3 & F_4 & F_1F_5 & F_5 & F_1F_4 & F_2F_5 & F_3F_4 & F_2F_4 & F_3F_5
\end{array}\\
\begin{bmatrix}
1&-1&1&1&-1&1&-1&1&-1&1&-1&1&1&1&1&1\\
1&-1&1&1&-1&1&-1&1&1&-1&1&-1&-1&-1&-1&-1\\
1&1&-1&-1&-1&1&1&-1&1&-1&-1&1&-1&-1&1&1\\
1&1&-1&-1&-1&1&1&-1&-1&1&1&-1&1&1&-1&-1\\
1&1&-1&-1&1&-1&-1&1&1&-1&-1&1&1&1&-1&-1\\
1&1&-1&-1&1&-1&-1&1&-1&1&1&-1&-1&-1&1&1\\
1&-1&1&1&1&-1&1&-1&-1&1&-1&1&-1&-1&-1&-1\\
1&-1&1&1&1&-1&1&-1&1&-1&1&-1&1&1&1&1\\
1&-1&1&1&-1&1&-1&1&-1&1&-1&1&1&1&1&1\\
1&-1&1&1&-1&1&-1&1&1&-1&1&-1&-1&-1&-1&-1\\
1&1&-1&-1&-1&1&1&-1&1&-1&-1&1&-1&-1&1&1\\
1&1&-1&-1&-1&1&1&-1&-1&1&1&-1&1&1&-1&-1\\
1&1&-1&-1&1&-1&-1&1&1&-1&-1&1&1&1&-1&-1\\
1&1&-1&-1&1&-1&-1&1&-1&1&1&-1&-1&-1&1&1\\
1&-1&1&1&1&-1&1&-1&-1&1&-1&1&-1&-1&-1&-1\\
1&-1&1&1&1&-1&1&-1&1&-1&1&-1&1&1&1&1\\
1&-1&-1&1&-1&-1&1&1&-1&1&-1&1&-1&1&-1&1\\
1&-1&-1&1&1&1&-1&-1&-1&1&-1&1&1&-1&1&-1\\
1&-1&-1&1&1&1&-1&-1&1&-1&1&-1&-1&1&-1&1\\
1&-1&-1&1&-1&-1&1&1&1&-1&1&-1&1&-1&1&-1\\
1&1&1&-1&-1&-1&-1&-1&-1&1&1&-1&-1&1&1&-1\\
1&1&1&-1&1&1&1&1&-1&1&1&-1&1&-1&-1&1\\
1&1&1&-1&1&1&1&1&1&-1&-1&1&-1&1&1&-1\\
1&1&1&-1&-1&-1&-1&-1&1&-1&-1&1&1&-1&-1&1\\
1&-1&1&-1&-1&1&-1&1&1&1&-1&-1&1&-1&-1&1\\
1&-1&1&-1&-1&1&-1&1&-1&-1&1&1&-1&1&1&-1\\
1&1&-1&1&-1&1&1&-1&1&1&1&1&1&-1&1&-1\\
1&1&-1&1&-1&1&1&-1&-1&-1&-1&-1&-1&1&-1&1\\
1&1&-1&1&1&-1&-1&1&1&1&1&1&-1&1&-1&1\\
1&1&-1&1&1&-1&-1&1&-1&-1&-1&-1&1&-1&1&-1\\
1&-1&1&-1&1&-1&1&-1&1&1&-1&-1&-1&1&1&-1\\
1&-1&1&-1&1&-1&1&-1&-1&-1&1&1&1&-1&-1&1
\end{bmatrix}
\end{array}
\gamma
$$

where

$$\gamma^T = [\mu, F_1, F_2F_3, F_4F_5, F_3, F_1F_2, F_2, F_1F_3, F_4, F_1F_5, F_5, F_1F_4, F_2F_5, F_3F_4, F_2F_4, F_3F_5]$$

The first linear equation follows

$$\begin{aligned}\varphi[0,0,0,0,0] = \mu - F_1 - F_2 - F_3 - F_4 - F_5 + F_1F_2 + F_1F_3 + F_1F_4 \\ +F_1F_5 + F_2F_3 + F_2F_4 + F_2F_5 + F_3F_4 + F_3F_5 + F_4F_5\end{aligned} \tag{2.1}$$

The other equations can be written in a similar manner.

Consider the model given in (1.13). Performing the Kronecker Product on the Left-hand and Right-hand side of the model we obtain

$$\varphi[x_1,\ldots,x_p] = \mu + \sum_{i=1}^{p} \rho(x_i)F_i + \sum_{\substack{i,i\prime=1,2,\ldots,p \\ i<i\prime}} \rho(x_i)\rho(x_{i\prime})F_iF_{i\prime} \tag{2.2}$$

(for up to two-factor interaction)

$$\varphi[x_1,\ldots,x_p] = \mu + \sum_{i=1}^{p} \rho(x_i)F_i + \sum_{\substack{i,i\prime=1,2,\ldots,p \\ i<i\prime}} \rho(x_i)\rho(x_{i\prime})F_iF_{i\prime} + \sum_{\substack{i,i\prime,i\prime\prime=1,2,\ldots,p \\ i<i\prime<i\prime\prime}} \rho(x_i)\rho(x_{i\prime})\rho(x_{i\prime\prime})F_iF_{i\prime}F_{i\prime\prime} \tag{2.3}$$

(for up to three-factor effects)
where

$$E[y(x_1,\ldots,x_p)] = \varphi[x_1,\ldots,x_P]$$

for $\rho(x_i) = -1; x_i = 0$ and $\rho(x_i) = 1; x_i = 1$.

Using equation (2.2) one is able to obtain equation (2.1). The column vector of expected response is given by φ. Then from either equation (2.2) or (2.3) it follows

$$\varphi = X\gamma$$

where X is the matrix of constants and γ the column vector of factors. Let the estimate of γ be $\hat{\gamma}$. Then $\hat{\gamma}$ is obtained as follows

$$\hat{\gamma} = (X^TX)^{-1}(X^Ty) \tag{2.4}$$

The X matrix of co-efficients in our 2^5 design is

$$X = \begin{array}{c}
\begin{array}{cccccccccccccccc} \mu & F_1 & F_2F_3 & F_4F_5 & F_3 & F_1F_2 & F_2 & F_1F_3 & F_4 & F_1F_5 & F_5 & F_1F_4 & F_2F_5 & F_3F_4 & F_2F_4 & F_3F_5 \end{array} \\
\begin{bmatrix}
1 & -1 & 1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & 1 & 1 & 1 & 1 \\
1 & -1 & 1 & 1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & -1 & -1 & -1 \\
1 & 1 & -1 & -1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & -1 & 1 & 1 \\
1 & 1 & -1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & 1 & 1 & -1 & -1 \\
1 & 1 & -1 & -1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & 1 & -1 & -1 \\
1 & 1 & -1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & -1 & -1 & 1 & 1 \\
1 & -1 & 1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & -1 & -1 & -1 & -1 \\
1 & -1 & 1 & 1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & 1 & 1 & 1 \\
1 & -1 & 1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & 1 & 1 & 1 & 1 \\
1 & -1 & 1 & 1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & -1 & -1 & -1 \\
1 & 1 & -1 & -1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & -1 & 1 & 1 \\
1 & 1 & -1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & 1 & 1 & -1 & -1 \\
1 & 1 & -1 & -1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & 1 & -1 & -1 \\
1 & 1 & -1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & -1 & -1 & 1 & 1 \\
1 & -1 & 1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & -1 & -1 & -1 & -1 \\
1 & -1 & 1 & 1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & 1 & 1 & 1 \\
1 & -1 & -1 & 1 & -1 & -1 & 1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\
1 & -1 & -1 & 1 & 1 & 1 & -1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 \\
1 & -1 & -1 & 1 & 1 & 1 & -1 & -1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\
1 & -1 & -1 & 1 & -1 & -1 & 1 & 1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\
1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 \\
1 & 1 & 1 & -1 & 1 & 1 & 1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 \\
1 & 1 & 1 & -1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \\
1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\
1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & 1 & -1 & -1 & 1 & -1 & -1 & 1 \\
1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & -1 & -1 & 1 & 1 & -1 & 1 & 1 & -1 \\
1 & 1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & 1 & 1 & 1 & 1 & -1 & 1 & -1 \\
1 & 1 & -1 & 1 & -1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & 1 & -1 & 1 \\
1 & 1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & 1 & 1 & 1 & -1 & 1 & -1 & 1 \\
1 & 1 & -1 & 1 & 1 & -1 & -1 & 1 & -1 & -1 & -1 & -1 & 1 & -1 & 1 & -1 \\
1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & 1 & -1 & -1 & -1 & 1 & 1 & -1 \\
1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & -1 & -1 & 1 & 1 & 1 & -1 & -1 & 1
\end{bmatrix}
\end{array}$$

Thus

$$
X^T X = \begin{array}{c}
\begin{matrix} \mu & F_1 & F_2F_3 & F_4F_5 & F_3 & F_1F_2 & F_2 & F_1F_3 & F_4 & F_1F_5 & F_5 & F_1F_4 & F_2F_5 & F_3F_4 & F_2F_4 & F_3F_5 \end{matrix} \\
\begin{bmatrix}
32 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 32 & -16 & -16 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -16 & 32 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -16 & 0 & 32 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 32 & -16 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -16 & 32 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 32 & -16 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -16 & 32 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 32 & -16 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -16 & 32 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 32 & -16 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -16 & 32 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 32 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 32 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 32 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 32
\end{bmatrix}
\end{array}
$$

$$
X^T y = \begin{bmatrix} [\mu] \\ [F_1] \\ [F_2F_3] \\ [F_4F_5] \\ \vdots \\ [F_3F_5] \end{bmatrix}
$$

where

$$
\gamma^T = [\mu, F_1, F_2F_3, F_4F_5, F_3, F_1F_2, F_2, F_1F_3, F_4, F_1F_5, F_5, F_1F_4, F_2F_5, F_3F_4, F_2F_4, F_3F_5]
$$

We now have all the parts defined in (2.4), that is, $X^T X$ and $X^T y$ matrices.

Finding the inverse of the corresponding matrices in $X^T X$ matrix, we get estimates for the effects as shown below

$$\begin{bmatrix} \hat{F_1} \\ F_2\hat{F}_3 \\ F_4\hat{F}_5 \end{bmatrix} = \begin{bmatrix} 32 & -16 & -16 \\ -16 & 32 & 0 \\ -16 & 0 & 32 \end{bmatrix}^{-1} \begin{bmatrix} [F_1] \\ [F_2F_3] \\ [F_4F_5] \end{bmatrix} = \frac{1}{64} \begin{bmatrix} 4 & 2 & 2 \\ 2 & 3 & 1 \\ 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} [F_1] \\ [F_2F_3] \\ [F_4F_5] \end{bmatrix}$$

Effects F_2F_3 and F_4F_5 are estimated with the same efficiency which is higher than the efficiency used to estimate the main effect F_1.

$$\begin{bmatrix} \hat{F_3} \\ F_1\hat{F}_2 \end{bmatrix} = \begin{bmatrix} 32 & -16 \\ -16 & 32 \end{bmatrix}^{-1} \begin{bmatrix} [F_3] \\ [F_1F_2] \end{bmatrix} = \frac{1}{48} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} [F_3] \\ [F_1F_2] \end{bmatrix}$$

$\begin{bmatrix} \hat{F_2} \\ F_1\hat{F}_3 \end{bmatrix}, \begin{bmatrix} \hat{F_4} \\ F_1\hat{F}_5 \end{bmatrix}, \begin{bmatrix} \hat{F_5} \\ F_1\hat{F}_4 \end{bmatrix}$ are estimated by the matrix $\frac{1}{48} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$.

The correlated effects in sets (F_2, F_1F_3), (F_3, F_1F_2), (F_4, F_1F_5), (F_5, F_1F_4) are estimated with the same efficiency.

It is clear that the effects above are estimable and thus we can term them as correlated effects as opposed to aliased effects.

Estimate of effects

$$\hat{\mu} = \frac{1}{32}[\mu]$$

$$\hat{F_1} = \frac{1}{64}[[4[F_1]+2[F_2F_3]+2[F_4F_5]]$$

$$\hat{F_2} = \frac{1}{48}[[2[F_2]+[F_1F_3]]$$

$$\hat{F_3} = \frac{1}{48}[[2[F_3]+[F_1F_2]]$$

$$\hat{F_4} = \frac{1}{48}[[2[F_4]+[F_1F_5]]$$

$$\hat{F_5} = \frac{1}{48}[[2[F_5]+[F_1F_4]]$$

$$F_3\hat{F}_4 = \frac{1}{32}[F_3F_4]$$

$$F_4\hat{F}_5 = \frac{1}{32}[F_4F_5]$$

$$F_1\hat{F}_2 = \frac{1}{48}[[F_3]+2[F_1F_2]]$$

$$F_1\hat{F}_3 = \frac{1}{48}[[F_2]+2[F_1F_3]]$$

$$F_1\hat{F}_4 = \frac{1}{48}[[F_5]+2[F_1F_4]]$$

$$F_1\hat{F}_5 = \frac{1}{48}[[F_4]+2[F_1F_5]]$$

$$F_2\hat{F}_5 = \frac{1}{32}[[F_2F_5]]$$

$$F_2\hat{F}_4 = \frac{1}{32}[[F_2F_4]]$$

$$F_3\hat{F}_4 = \frac{1}{32}[F_3F_5]$$

Significance Test

The error sum of squares is given by

$$SSe = \frac{1}{2}\{\sum_{i=1}^{8}(y_i^{(.)} - y_i^{(..)})^2\}$$

and

$$E(SSe) = 8\sigma^2$$

Here $y_1, \ldots, y_8$ are the duplicated treatments whereas $(.)$ and $(..)$ denote the two observations. The test statistic say for F_2 is given as

$$t = \left|\frac{\hat{F_2}}{\sqrt{var(\hat{F_2})}}\right| = \left|\frac{\frac{[2[F_2]+[F_1F_3]}{48}}{\sqrt{\frac{2\times SSe}{48\times 8}}}\right| > t_{\frac{\alpha}{2}} \tag{2.5}$$

which is said to be significant at α level of significance and non-significant otherwise. $t_{\frac{\alpha}{2}}$ is the value of t distribution with eight degrees of freedom. The t-distribution is used when the sample size is small say $p < 30$.

Block Design

There are two ways in which to obtain the block design. We can decide to have three blocks with unequal number of treatments. One of the blocks can be allocated 16 treatments that arise as a result of the 8 duplicated treatments while the other two blocks each get 8 treatment combinations.

Four blocks could also be considered in this design. Each block will have an equal number of treatments. One of the four blocks could result wholly from the repeated treatments, that is, the duplicates are put in one separate block from the others.

2.3 Six Factor Experiment involving 32+8 = 40 Runs

Consider a design with treatment combinations $(x_1,x_2,x_3,x_4,x_5,x_6)$ which satisfy the simultaneous equations

$$\begin{aligned} x_1+x_2+x_3 &= 0,1,0,0 \\ x_1+x_4+x_5 &= 0,0,1,0 \\ x_1+x_4+x_6 &= 0,0,0,1 \end{aligned} \qquad \text{(mod 2)}$$

2.3.1 Constuction of the Design

We first get the treatment combinations satisfying the equations

$$\begin{aligned} x_1+x_2+x_3 &= 0 \\ x_1+x_4+x_5 &= 0 \\ x_1+x_4+x_6 &= 0 \end{aligned} \qquad \text{(mod 2)}$$

which give the first set of treatment combinations that are duplicated. From this first set we can easily obtain the second, the third and the fourth set of treatment combinations satisfying the corresponding set of simultaneous equations.

Adding 1 (mod 2) in the x_1, x_4 and x_6 position of the first set we obtain the second set. Adding 1 (mod 2) in the x_1, x_2 and x_6 position of the first set we obtain the third set. Adding 1 (mod 2) in the x_1, x_2 and x_5 position of the first set we obtain the fourth set.

Below follows the treatment combinations in each set.

$\underline{st_1}$	$\underline{st_2}$	$\underline{st_3}$	$\underline{st_4}$
$(0,0,0,0,0,0)$	$(1,0,0,1,0,1)$	$(1,1,0,0,0,1)$	$(1,1,0,0,1,0)$
$(1,0,1,1,0,1)$	$(0,0,1,0,0,0)$	$(0,1,1,1,0,0)$	$(0,1,1,1,1,1)$
$(1,1,0,0,1,1)$	$(0,1,0,1,1,0)$	$(0,0,0,0,1,0)$	$(0,0,0,0,0,1)$
$(0,1,1,1,1,0)$	$(1,1,1,0,1,1)$	$(1,0,1,1,1,1)$	$(1,0,1,1,0,0)$
$(1,1,0,1,0,0)$	$(0,1,0,0,0,1)$	$(0,0,0,1,0,1)$	$(0,0,0,1,1,0)$
$(1,0,1,0,1,0)$	$(0,0,1,1,1,1)$	$(0,1,1,0,1,1)$	$(0,1,1,0,0,0)$
$(0,1,1,0,0,1)$	$(1,1,1,1,0,0)$	$(1,0,1,0,0,0)$	$(1,0,1,0,1,1)$
$(0,0,0,1,1,1)$	$(1,0,0,0,1,0)$	$(1,1,0,1,1,0)$	$(1,1,0,1,0,1)$

The defining relation is

$$I = F_1F_2F_3 = F_1F_4F_5 = F_2F_4F_6 = F_3F_5F_6 = F_2F_3F_4F_5 = F_1F_3F_4F_6 = F_1F_2F_5F_6$$

The correlated sets of factors are

$(F_1,F_2F_3,F_4F_5),(F_2,F_1F_3,F_4F_6),(F_3,F_1F_2,F_5F_6),(F_4,F_1F_5,F_2F_6),(F_5,F_1F_4,F_3F_6),(F_6,F_2F_4,F_3F_5),$
(F_1F_6,F_3F_4,F_2F_5).

2.3.2 Method of Analysis

Using equation (2.4),we have the X matrix of co-efficients, X^TX and X^Ty matrices as

$$
X = \begin{array}{c}
\begin{array}{cccccccccccccccccccccc}
\mu & F_1 & F_2F_3 & F_4F_5 & F_2 & F_1F_3 & F_4F_6 & F_4 & F_1F_5 & F_2F_6 & F_3 & F_1F_2 & F_5F_6 & F_5 & F_1F_4 & F_3F_6 & F_6 & F_2F_4 & F_3F_5 & F_1F_6 & F_3F_4 & F_2F_5
\end{array} \\
\begin{bmatrix}
1 & -1 & 1 & 1 & -1 & 1 & 1 & -1 & 1 & 1 & -1 & 1 & 1 & -1 & 1 & 1 & -1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & -1 & -1 & -1 & 1 & 1 & 1 & -1 & -1 & 1 & -1 & -1 & -1 & 1 & 1 & 1 & -1 & -1 & 1 & 1 & 1 \\
1 & 1 & -1 & -1 & 1 & -1 & -1 & -1 & 1 & 1 & -1 & 1 & 1 & 1 & -1 & -1 & 1 & -1 & -1 & 1 & 1 & 1 \\
1 & -1 & 1 & 1 & 1 & -1 & -1 & 1 & -1 & -1 & 1 & -1 & -1 & 1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & -1 & -1 & 1 & -1 & -1 & 1 & -1 & -1 & -1 & 1 & 1 & -1 & 1 & 1 & -1 & 1 & 1 & -1 & -1 & -1 \\
1 & 1 & -1 & -1 & -1 & 1 & 1 & -1 & 1 & 1 & 1 & -1 & -1 & 1 & -1 & -1 & -1 & 1 & 1 & -1 & -1 & -1 \\
1 & -1 & 1 & 1 & 1 & -1 & -1 & -1 & 1 & 1 & 1 & -1 & -1 & -1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 \\
1 & -1 & 1 & 1 & -1 & 1 & 1 & 1 & -1 & -1 & -1 & 1 & 1 & 1 & -1 & -1 & 1 & -1 & -1 & -1 & -1 & -1 \\
1 & -1 & 1 & 1 & -1 & 1 & 1 & -1 & 1 & 1 & -1 & 1 & 1 & -1 & 1 & 1 & -1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & -1 & -1 & -1 & 1 & 1 & 1 & -1 & -1 & 1 & -1 & -1 & -1 & 1 & 1 & 1 & -1 & -1 & 1 & 1 & 1 \\
1 & 1 & -1 & -1 & 1 & -1 & -1 & -1 & 1 & 1 & -1 & 1 & 1 & 1 & -1 & -1 & 1 & -1 & -1 & 1 & 1 & 1 \\
1 & -1 & 1 & 1 & 1 & -1 & -1 & 1 & -1 & -1 & 1 & -1 & -1 & 1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & -1 & -1 & 1 & -1 & -1 & 1 & -1 & -1 & -1 & 1 & 1 & -1 & 1 & 1 & -1 & 1 & 1 & -1 & -1 & -1 \\
1 & 1 & -1 & -1 & -1 & 1 & 1 & -1 & 1 & 1 & 1 & -1 & -1 & 1 & -1 & -1 & -1 & 1 & 1 & -1 & -1 & -1 \\
1 & -1 & 1 & 1 & 1 & -1 & -1 & -1 & 1 & 1 & 1 & -1 & -1 & -1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 \\
1 & -1 & 1 & 1 & -1 & 1 & 1 & 1 & -1 & -1 & -1 & 1 & 1 & 1 & -1 & -1 & 1 & -1 & -1 & -1 & -1 & -1 \\
1 & 1 & 1 & -1 & -1 & -1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & 1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 \\
1 & -1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & 1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & -1 & 1 \\
1 & -1 & -1 & 1 & 1 & 1 & -1 & 1 & -1 & -1 & -1 & -1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\
1 & 1 & 1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & 1 & 1 & 1 & 1 & -1 & 1 & 1 & -1 & 1 & 1 & -1 & 1 \\
1 & -1 & -1 & 1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\
1 & -1 & -1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & 1 & -1 \\
1 & 1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & -1 & 1 & -1 \\
1 & 1 & 1 & -1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & -1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 \\
1 & 1 & -1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & -1 & 1 & -1 & -1 & -1 & -1 & 1 & -1 & 1 & 1 & 1 & -1 \\
1 & -1 & 1 & -1 & 1 & -1 & -1 & 1 & 1 & -1 & 1 & -1 & 1 & -1 & -1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \\
1 & -1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & -1 & 1 & -1 & 1 & 1 & 1 & -1 & 1 & -1 & 1 & 1 & -1 \\
1 & 1 & -1 & 1 & -1 & 1 & 1 & 1 & 1 & -1 & 1 & -1 & 1 & 1 & 1 & 1 & 1 & -1 & 1 & 1 & 1 & -1 \\
1 & -1 & 1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 & 1 & -1 & -1 & -1 & -1 & 1 & -1 & 1 & -1 & -1 & 1 \\
1 & -1 & 1 & -1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & -1 & 1 & 1 & 1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 \\
1 & 1 & -1 & 1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & -1 & -1 & 1 & -1 & -1 & -1 & 1 \\
1 & 1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & 1 & -1 & 1 & -1 & -1 & -1 & 1 \\
1 & 1 & -1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\
1 & -1 & 1 & 1 & 1 & -1 & 1 & 1 & -1 & 1 & 1 & -1 & 1 & 1 & -1 & 1 & 1 & 1 & 1 & -1 & 1 & 1 \\
1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & 1 & -1 & 1 & 1 \\
1 & 1 & -1 & -1 & -1 & 1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & 1 & -1 & -1 & -1 & -1 & -1 & 1 & 1 \\
1 & -1 & 1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & -1 & -1 & 1 & -1 & -1 \\
1 & -1 & 1 & 1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & -1 & -1 & -1 & 1 & -1 & -1 \\
1 & 1 & -1 & -1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 \\
1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & 1 & 1 & -1 & -1
\end{bmatrix}
\end{array}
$$

$$X^TX = \begin{array}{c} \begin{matrix} \mu & F_1 & F_2F_3 & F_4F_5 & F_2 & F_1F_3 & F_4F_6 & F_4 & F_1F_5 & F_2F_6 & F_3 & F_1F_2 & F_5F_6 & F_5 & F_1F_4 & F_3F_6 & F_6 & F_2F_4 & F_3F_5 & F_1F_6 & F_3F_4 & F_2F_5 \end{matrix} \\ \begin{bmatrix}
40 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 40 & -24 & -24 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -24 & 40 & 8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -24 & 8 & 40 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 40 & -24 & -24 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -24 & 40 & 8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -24 & 8 & 40 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 40 & -24 & -24 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -24 & 40 & 8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -24 & 8 & 40 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 40 & -24 & 8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -24 & 40 & 8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 8 & 8 & 40 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 40 & -24 & 8 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -24 & 40 & 8 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 8 & 8 & 40 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 40 & -24 & 8 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -24 & 40 & 8 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 8 & 8 & 40 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 40 & 8 & 8 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 8 & 40 & 8 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 8 & 8 & 40
\end{bmatrix} \end{array}$$

$$X^Ty = \begin{bmatrix} [\mu] \\ [F_1] \\ [F_2F_3] \\ [F_4F_5] \\ \vdots \\ [F_3F_5] \end{bmatrix}$$

$$\gamma^T = [\mu, F_1, F_2F_3, F_4F_5, F_2, F_1F_3, F_4F_6, F_3, F_1F_2, F_5F_6, F_4, F_1F_5, F_2F_6, F_5, F_1F_4, F_3F_6, F_6, F_2F_4, F_3F_5, F_1F_6, F_3F_4, F_2F_5]$$

We have now defined all the matrices as per equation (2.4). Finding the inverse of the subsequent matrices in X^TX we can estimate the effects as shown below

$$\begin{bmatrix} 40 & -24 & -24 \\ -24 & 40 & 8 \\ -24 & 8 & 40 \end{bmatrix}^{-1} = \frac{1}{96}\begin{bmatrix} 6 & 3 & 3 \\ 3 & 4 & 1 \\ 3 & 1 & 4 \end{bmatrix}, \qquad \begin{bmatrix} 40 & -24 & 8 \\ -24 & 40 & 8 \\ 8 & 8 & 40 \end{bmatrix}^{-1} = \frac{1}{64}\begin{bmatrix} 3 & 2 & -1 \\ 2 & 3 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

and

$$\begin{bmatrix} 40 & 8 & 8 \\ 8 & 40 & 8 \\ 8 & 8 & 40 \end{bmatrix}^{-1} = \frac{1}{244}\begin{bmatrix} 6 & -1 & -1 \\ -1 & 6 & -1 \\ -1 & -1 & 6 \end{bmatrix}$$

The matrix:

$$\frac{1}{244}\begin{bmatrix} 6 & -1 & -1 \\ -1 & 6 & -1 \\ -1 & -1 & 6 \end{bmatrix} \text{ gives estimates for } \begin{bmatrix} F_1F_6 \\ F_3F_4 \\ F_2F_5 \end{bmatrix}$$

The correlated effects in set (F_1F_6, F_3F_4, F_2F_5) are estimated with the same efficiency.

$$\frac{1}{96}\begin{bmatrix} 6 & 3 & 3 \\ 3 & 4 & 1 \\ 3 & 1 & 4 \end{bmatrix} \text{ gives estimates for } \begin{bmatrix} F_1 \\ F_2F_3 \\ F_4F_5 \end{bmatrix}, \begin{bmatrix} F_2 \\ F_1F_3 \\ F_4F_6 \end{bmatrix}, \begin{bmatrix} F_4 \\ F_1F_5 \\ F_2F_6 \end{bmatrix}$$

The efficiency used to estimate the interactions F_2F_3, F_4F_5, F_1F_3, F_4F_6, F_1F_5 and F_2F_6 is higher than the efficiency used to estimate the main effects F_1, F_2 and F_4. The main effects F_1, F_2 and F_4 are estimated with the same efficiency.

$$\frac{1}{64}\begin{bmatrix} 3 & 2 & -1 \\ 2 & 3 & -1 \\ -1 & -1 & 2 \end{bmatrix} \text{ gives estimates for } \begin{bmatrix} F_3 \\ F_1F_2 \\ F_5F_6 \end{bmatrix}, \begin{bmatrix} F_5 \\ F_1F_4 \\ F_3F_6 \end{bmatrix}, \begin{bmatrix} F_6 \\ F_2F_4 \\ F_3F_5 \end{bmatrix}$$

The main effects F_3, F_5, F_6 and the interactions F_1F_2, F_1F_4 and F_2F_4 are all estimated with the same efficiency which is lower than the efficiency attained for effects F_5F_6, F_3F_6 and F_3F_5.

Using equation (2.4) we obtain the estimates of the effects as

$$\begin{bmatrix} \hat{F}_1 \\ \hat{F_2F_3} \\ \hat{F_4F_5} \end{bmatrix} = \frac{1}{96}\begin{bmatrix} 6 & 3 & 3 \\ 3 & 4 & 1 \\ 3 & 1 & 4 \end{bmatrix} \begin{bmatrix} [F_1] \\ [F_2F_3] \\ [F_4F_5] \end{bmatrix}$$

$$\begin{bmatrix} \hat{F}_2 \\ F_1\hat{}F_3 \\ F_4\hat{}F_5 \end{bmatrix} = \frac{1}{96} \begin{bmatrix} 6 & 3 & 3 \\ 3 & 4 & 1 \\ 3 & 1 & 4 \end{bmatrix} \begin{bmatrix} [F_2] \\ [F_2F_3] \\ [F_4F_6] \end{bmatrix}$$

$$\begin{bmatrix} \hat{F}_3 \\ F_1\hat{}F_2 \\ F_5\hat{}F_6 \end{bmatrix} = \frac{1}{64} \begin{bmatrix} 3 & 2 & -1 \\ 2 & 3 & -1 \\ -1 & -1 & 2 \end{bmatrix} \begin{bmatrix} [F_3] \\ [F_1F_2] \\ [F_5F_6] \end{bmatrix}$$

$$\begin{bmatrix} \hat{F}_4 \\ F_1\hat{}F_5 \\ F_2\hat{}F_6 \end{bmatrix} = \frac{1}{96} \begin{bmatrix} 6 & 3 & 3 \\ 3 & 4 & 1 \\ 3 & 1 & 4 \end{bmatrix} \begin{bmatrix} [F_4] \\ [F_1F_5] \\ [F_2F_6] \end{bmatrix}$$

$$\begin{bmatrix} \hat{F}_5 \\ F_1\hat{}F_4 \\ F_3\hat{}F_6 \end{bmatrix} = \frac{1}{64} \begin{bmatrix} 3 & 2 & -1 \\ 2 & 3 & -1 \\ -1 & -1 & 2 \end{bmatrix} \begin{bmatrix} [F_5] \\ [F_1F_4] \\ [F_3F_6] \end{bmatrix}$$

$$\begin{bmatrix} \hat{F}_6 \\ F_2\hat{}F_4 \\ F_3\hat{}F_5 \end{bmatrix} = \frac{1}{64} \begin{bmatrix} 3 & 2 & -1 \\ 2 & 3 & -1 \\ -1 & -1 & 2 \end{bmatrix} \begin{bmatrix} [F_6] \\ [F_2F_4] \\ [F_3F_5] \end{bmatrix}$$

$$\begin{bmatrix} F_1\hat{}F_6 \\ F_3\hat{}F_4 \\ F_2\hat{}F_5 \end{bmatrix} = \frac{1}{244} \begin{bmatrix} 6 & -1 & -1 \\ -1 & 6 & -1 \\ -1 & -1 & 6 \end{bmatrix} \begin{bmatrix} [F_1F_6] \\ [F_3F_4] \\ [F_2F_5] \end{bmatrix}$$

Estimate of effects

$$\hat{\mu} = \frac{1}{40}[\mu]$$
$$\hat{F_1} = \frac{1}{96}[6[F_1]+3[F_2F_3]+3[F_4F_5]]$$
$$\hat{F_2} = \frac{1}{96}[6[F_2]+3[F_1F_3]+3[F_4F_6]]$$
$$\hat{F_3} = \frac{1}{64}[3[F_3]+2[F_1F_2]-[F_5F_6]]$$
$$\hat{F_4} = \frac{1}{96}[6[F_4]+3[F_1F_5]+3[F_2F_6]]$$
$$\hat{F_5} = \frac{1}{64}[3[F_5]+2[F_1F_4]-[F_3F_6]]$$
$$\hat{F_6} = \frac{1}{64}[3[F_6]+2[F_2F_4]-[F_3F_5]]$$
$$\widehat{F_1F_2} = \frac{1}{64}[2[F_3]+3[F_1F_2]-[F_5F_6]]$$
$$\widehat{F_1F_3} = \frac{1}{96}[3[F_2]+4[F_1F_3]+[F_4F_6]]$$
$$\widehat{F_1F_4} = \frac{1}{64}[2[F_5]+3[F_1F_4]-[F_3F_6]]$$
$$\widehat{F_1F_5} = \frac{1}{96}[3[F_4]+4[F_1F_5]+[F_2F_6]]$$
$$\widehat{F_1F_6} = \frac{1}{244}[6[F_1F_6]-[F_3F_4]-[F_2F_5]]$$
$$\widehat{F_2F_3} = \frac{1}{96}[3[F_1]+4[F_2F_3]+[F_4F_5]]$$
$$\widehat{F_2F_4} = \frac{1}{64}[2[F_6]+3[F_2F_4]-[F_3F_5]]$$
$$\widehat{F_2F_5} = \frac{1}{244}[[F_1F_6]-[F_3F_4]+6[F_2F_5]]$$
$$\widehat{F_2F_6} = \frac{1}{96}[3[F_4]+[F_1F_5]+4[F_2F_6]]$$
$$\widehat{F_3F_4} = \frac{1}{224}[-[F_1F_6]+6[F_3F_4]-[F_2F_5]]$$
$$\widehat{F_3F_5} = \frac{1}{64}[-[F_6]-[F_2F_4]+2[F_3F_5]]$$
$$\widehat{F_3F_6} = \frac{1}{64}[-[F_5]-[F_1F_4]+2[F_3F_6]]$$
$$\widehat{F_4F_5} = \frac{1}{96}[3[F_1]+[F_2F_3]+4[F_4F_5]]$$
$$\widehat{F_4F_6} = \frac{1}{96}[3[F_2]+[F_1F_3]+4[F_4F_6]]$$
$$\widehat{F_5F_6} = \frac{1}{64}[-[F_3]-[F_1F_2]+2[F_5F_6]]$$

2.4 Seven Factor Experiment involving 40+8 = 48 Runs

Consider a design with treatment combinations $(x_1, x_2, x_3, x_4, x_5, x_6, x_7)$ which satisfy the simultaneous equations

$$\begin{aligned} x_1+x_4+x_5 &= 0,0,1,1,1 \\ x_1+x_3+x_7 &= 0,1,0,1,1 \\ x_1+x_2+x_6 &= 0,1,1,0,1 \\ x_2+x_3+x_4 &= 0,1,1,1,0 \end{aligned}$$

(mod 2)

2.4.1 Constuction of the Design

We first get the treatment combinations satisfying the equations

$$x_1+x_4+x_5 = 0$$

$$\begin{aligned} x_1+x_3+x_7&=0\\ x_1+x_2+x_6&=0\\ x_2+x_3+x_4&=0 \end{aligned} \qquad \text{(mod 2)}$$

which give the first set of treatment combinations that are then duplicated. From this first set we can easily obtain the second, third, fourth and fifth set of treatment combinations satisfying the corresponding set of simultaneous equations.

Adding 1 (mod 2) in x_1 and x_4 position of the first set we obtain the second set. Adding 1 (mod 2) in the x_1 and x_3 position of the first set we obtain the third set. Adding 1 (mod 2) in the x_1 and x_2 position of the first set we obtain the fourth set. Adding 1 (mod 2) in x_1 position of the first set we obtain the fifth set. Below follows the treatment combinations for each set.

$\underline{st_1}$	$\underline{st_2}$	$\underline{st_3}$	$\underline{st_4}$	$\underline{st_5}$
$(0,0,0,0,0,0,0)$	$(1,0,0,1,0,0,0)$	$(1,0,1,0,0,0,0)$	$(1,1,0,0,0,0,0)$	$(1,0,0,0,0,0,0)$
$(0,0,1,1,1,0,1)$	$(1,0,1,0,1,0,1)$	$(1,0,0,1,1,0,1)$	$(1,1,1,1,1,0,1)$	$(1,0,1,1,1,0,1)$
$(0,1,0,1,1,1,0)$	$(1,1,0,0,1,1,0)$	$(1,1,1,1,1,1,0)$	$(1,0,0,1,1,1,0)$	$(1,1,0,1,1,1,0)$
$(0,1,1,0,0,1,1)$	$(1,1,1,1,0,1,1)$	$(1,1,0,0,0,1,1)$	$(1,0,1,0,0,1,1)$	$(1,1,1,0,0,1,1)$
$(1,0,0,0,1,1,1)$	$(0,0,0,1,1,1,1)$	$(0,0,1,0,1,1,1)$	$(0,1,0,0,1,1,1)$	$(0,0,0,0,1,1,1)$
$(1,0,1,1,0,1,0)$	$(0,0,1,0,0,1,0)$	$(0,0,0,1,0,1,0)$	$(0,1,1,1,0,1,0)$	$(0,0,1,1,0,1,0)$
$(1,1,0,1,0,0,1)$	$(0,1,0,0,0,0,1)$	$(0,1,1,1,0,0,1)$	$(0,0,0,1,0,0,1)$	$(0,1,0,1,0,0,1)$
$(1,1,1,0,1,0,0)$	$(0,1,1,1,1,0,0)$	$(0,1,0,0,1,0,0)$	$(0,0,1,0,1,0,0)$	$(0,1,1,0,1,0,0)$

The defining relation is

$$\begin{aligned} I &= F_1F_4F_5 = F_1F_3F_7 = F_1F_2F_6 = F_2F_3F_4 = F_2F_5F_7 = F_3F_5F_6 = F_4F_6F_7 = F_3F_4F_5F_7 \\ &= F_2F_4F_5F_6 = F_1F_2F_3F_5 = F_2F_3F_6F_7 = F_1F_2F_4F_7 = F_1F_3F_4F_6 = F_1F_5F_6F_7 \end{aligned}$$

The correlated sets of factors are

$(F_1,F_4F_5,F_2F_6,F_3F_7)$, $(F_2,F_3F_4,F_1F_6,F_5F_7)$, $(F_3,F_2F_4,F_1F_7,F_5F_6)$, $(F_4,F_2F_3,F_1F_5,F_6F_7)$, $(F_5,F_1F_4,F_3F_6,F_2F_7)$, $(F_6,F_1F_2,F_3F_5,F_4F_7)$, $(F_7,F_1F_3,F_4F_6,F_2F_5)$.

2.4.2 Method of Analysis

Using equation (2.4),we have the X matrix of constants, X^TX and X^Ty matrices as

$$X=\begin{array}{c}
\begin{matrix} \mu & 1 & 45 & 26 & 37 & 2 & 34 & 16 & 57 & 3 & 24 & 17 & 56 & 4 & 23 & 15 & 67 & 5 & 14 & 36 & 27 & 6 & 12 & 35 & 47 & 7 & 13 & 46 & 25 \end{matrix} \\
\begin{bmatrix}
1 & -1 & 1 & 1 & 1 & -1 & 1 & 1 & 1 & -1 & 1 & 1 & 1 & -1 & 1 & 1 & 1 & -1 & 1 & 1 & 1 & -1 & 1 & 1 & 1 & -1 & 1 & 1 & 1 \\
1 & -1 & 1 & 1 & 1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & 1 & -1 & -1 & -1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 \\
1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & 1 & -1 & -1 & -1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 \\
1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & 1 & -1 & -1 & -1 \\
1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & -1 & 1 & 1 & 1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & 1 & -1 & -1 & -1 & 1 & -1 & -1 & -1 \\
1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 \\
1 & 1 & -1 & -1 & -1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 \\
1 & 1 & -1 & -1 & -1 & 1 & -1 & -1 & -1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & -1 & 1 & 1 & 1 \\
1 & -1 & 1 & 1 & 1 & -1 & 1 & 1 & 1 & -1 & 1 & 1 & 1 & -1 & 1 & 1 & 1 & -1 & 1 & 1 & 1 & -1 & 1 & 1 & 1 & -1 & 1 & 1 & 1 \\
1 & -1 & 1 & 1 & 1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & 1 & -1 & -1 & -1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 \\
1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & 1 & -1 & -1 & -1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 \\
1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & 1 & -1 & -1 & -1 \\
1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & -1 & 1 & 1 & 1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & 1 & -1 & -1 & -1 & 1 & -1 & -1 & -1 \\
1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 \\
1 & 1 & -1 & -1 & -1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 \\
1 & 1 & -1 & -1 & -1 & 1 & -1 & -1 & -1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & -1 & 1 & 1 & 1 \\
1 & 1 & -1 & 1 & 1 & -1 & -1 & -1 & 1 & -1 & -1 & -1 & 1 & 1 & 1 & -1 & 1 & -1 & 1 & 1 & 1 & -1 & -1 & 1 & -1 & -1 & -1 & -1 & 1 \\
1 & 1 & -1 & 1 & 1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & 1 & -1 & 1 & -1 & -1 & -1 & -1 & -1 & 1 & -1 & 1 & 1 & 1 & -1 \\
1 & 1 & -1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & -1 & -1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & -1 & 1 \\
1 & 1 & -1 & 1 & 1 & 1 & 1 & 1 & -1 & 1 & 1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & 1 & 1 & 1 & 1 & 1 & -1 & 1 & 1 & 1 & 1 & -1 \\
1 & -1 & 1 & -1 & -1 & -1 & -1 & -1 & 1 & -1 & -1 & -1 & 1 & 1 & 1 & -1 & 1 & 1 & -1 & -1 & -1 & 1 & 1 & -1 & 1 & 1 & 1 & 1 & -1 \\
1 & -1 & 1 & -1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & 1 & -1 & -1 & 1 & 1 & 1 & 1 & 1 & -1 & 1 & -1 & -1 & -1 & 1 \\
1 & -1 & 1 & -1 & -1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & -1 & -1 & 1 & -1 & -1 & 1 & 1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & 1 & -1 \\
1 & -1 & 1 & -1 & -1 & 1 & 1 & 1 & -1 & 1 & 1 & 1 & -1 & 1 & 1 & -1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & 1 & -1 & -1 & -1 & -1 & 1 \\
1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & -1 & 1 & -1 & -1 & -1 & 1 & -1 & -1 & -1 & 1 & -1 & -1 & -1 & 1 & -1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & 1 & -1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & -1 & -1 & -1 \\
1 & 1 & 1 & 1 & -1 & 1 & 1 & 1 & -1 & 1 & 1 & -1 & 1 & 1 & 1 & 1 & -1 & 1 & 1 & 1 & -1 & 1 & 1 & 1 & -1 & -1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & -1 & 1 & 1 & 1 & -1 & -1 & -1 & 1 & -1 & -1 & -1 & -1 & 1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & 1 & -1 & -1 & -1 \\
1 & -1 & -1 & -1 & 1 & -1 & -1 & -1 & 1 & 1 & 1 & -1 & 1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & 1 & 1 & 1 & -1 & 1 & -1 & -1 & -1 \\
1 & -1 & -1 & -1 & 1 & -1 & -1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 & 1 & 1 & 1 \\
1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & 1 & 1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & -1 & -1 & -1 & 1 & 1 & -1 & -1 & -1 \\
1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & -1 & 1 & 1 & 1 \\
1 & 1 & 1 & -1 & 1 & 1 & 1 & -1 & 1 & -1 & -1 & -1 & 1 & -1 & -1 & -1 & 1 & -1 & -1 & 1 & -1 & -1 & 1 & 1 & 1 & -1 & -1 & 1 & -1 \\
1 & 1 & 1 & -1 & 1 & 1 & 1 & -1 & 1 & 1 & 1 & 1 & -1 & 1 & 1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & 1 & 1 & 1 & 1 & 1 & -1 & 1 \\
1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & 1 & 1 & -1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & 1 & -1 \\
1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & -1 & -1 & -1 & 1 & 1 & -1 & 1 \\
1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & -1 & 1 & -1 & -1 & -1 & 1 & 1 & 1 & -1 & 1 & 1 & -1 & -1 & -1 & 1 & 1 & -1 & 1 \\
1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & 1 & 1 & 1 & -1 & 1 & 1 & 1 & -1 & -1 & -1 & 1 & -1 & 1 & -1 & -1 & -1 & -1 & -1 & 1 & -1 \\
1 & -1 & -1 & 1 & -1 & -1 & -1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & 1 & -1 & -1 & 1 & 1 & 1 & 1 & 1 & -1 & 1 \\
1 & -1 & -1 & 1 & -1 & -1 & -1 & 1 & -1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & -1 & 1 & -1 & 1 & 1 & 1 & -1 & -1 & 1 & -1 \\
1 & 1 & 1 & 1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 \\
1 & 1 & 1 & 1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\
1 & -1 & -1 & -1 & -1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\
1 & -1 & -1 & -1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\
1 & -1 & -1 & -1 & -1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 \\
1 & -1 & -1 & -1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1
\end{bmatrix}
\end{array}$$

$$X^TX = \begin{array}{c}
\begin{array}{ccccccccccccccccccccccccccccc}
\mu & 1 & 45 & 26 & 37 & 2 & 34 & 16 & 57 & 3 & 24 & 17 & 56 & 4 & 23 & 15 & 67 & 5 & 14 & 36 & 27 & 6 & 12 & 35 & 47 & 7 & 13 & 46 & 25
\end{array} \\
\begin{bmatrix}
48 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 48 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 48 & 16 & 16 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 16 & 48 & 16 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 16 & 16 & 48 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 48 & 0 & 0 & -32 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 48 & 16 & 16 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 16 & 48 & -16 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -32 & 16 & -16 & 48 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 48 & 0 & 0 & -32 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 48 & 16 & 16 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 16 & 48 & -16 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -32 & 16 & -16 & 48 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 48 & 0 & 0 & -32 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 48 & 16 & 16 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 16 & 48 & -16 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -32 & 16 & -16 & 48 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 48 & 0 & -32 & -32 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 48 & 16 & 16 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -32 & 16 & 48 & 16 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -32 & 16 & 16 & 48 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 48 & 0 & -32 & -32 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 48 & 16 & 16 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -32 & 16 & 48 & 16 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -32 & 16 & 16 & 48 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 48 & 0 & -32 & -32 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 48 & 16 & 16 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -32 & 16 & 48 & 16 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -32 & 16 & 16 & 48
\end{bmatrix}
\end{array}$$

$$X^T y = \begin{bmatrix} [\mu] \\ [F_1] \\ [F_4F_5] \\ [F_2F_6] \\ \vdots \\ [F_2F_5] \end{bmatrix}$$

$$\gamma^T = [\mu, F_1, F_4F_5, F_2F_6, F_3F_7, F_2, F_3F_4, F_1F_6, F_5F_7, F_3, F_2F_4, F_1F_7, F_5F_6, F_4, F_2F_3, F_1F_5, F_6F_7, F_5, F_1F_4, F_3F_6,$$
$$F_2F_7F_6, F_1F_2, F_3F_5, F_4F_7, F_7, F_1F_3, F_4F_7, F_7, F_1F_3, F_4F_6, F_2F_5]$$

We have now defined the X^TX and X^Ty matrices as per equation (2.4). Finding the inverse of the corresponding matrices in X^TX we estimate for effects as

$$\begin{bmatrix} 48 & 16 & 16 \\ 16 & 48 & 16 \\ 16 & 16 & 48 \end{bmatrix}^{-1} = \frac{1}{160}\begin{bmatrix} 4 & -1 & -1 \\ -1 & 4 & -1 \\ -1 & -1 & 4 \end{bmatrix}, \quad \begin{bmatrix} 48 & 0 & 0 & -32 \\ 0 & 48 & 16 & 16 \\ 0 & 16 & 48 & -16 \\ -32 & 16 & -16 & 48 \end{bmatrix}^{-1} = \frac{1}{64}\begin{bmatrix} 4 & -2 & 2 & 4 \\ -2 & 3 & -2 & -3 \\ 2 & -2 & 3 & 3 \\ 4 & -3 & 3 & 6 \end{bmatrix}$$

and

$$\begin{bmatrix} 40 & 0 & -32 & -32 \\ 0 & 48 & 16 & 16 \\ -32 & 16 & 48 & 16 \\ -32 & 16 & 16 & 48 \end{bmatrix}^{-1} = \frac{1}{96}\begin{bmatrix} 10 & -4 & 6 & 6 \\ -4 & 4 & -3 & -3 \\ 6 & -3 & 6 & 3 \\ 6 & -3 & 3 & 6 \end{bmatrix}$$

The matrix:

$$\frac{1}{160}\begin{bmatrix} 4 & -1 & -1 \\ -1 & 4 & -1 \\ -1 & -1 & 4 \end{bmatrix} \text{ gives estimates for } \begin{bmatrix} F_4F_5 \\ F_2F_6 \\ F_3F_7 \end{bmatrix}$$

The correlated effects in set (F_4F_5, F_2F_6, F_3F_7) are estimated with the same efficiency.

$$\frac{1}{64}\begin{bmatrix} 4 & -2 & 2 & 4 \\ -2 & 3 & -2 & -3 \\ 2 & -2 & 3 & 3 \\ 4 & -3 & 3 & 6 \end{bmatrix} \text{gives estimates for} \begin{bmatrix} F_2 \\ F_3F_4 \\ F_1F_6 \\ F_5F_7 \end{bmatrix}, \begin{bmatrix} F_3 \\ F_2F_4 \\ F_1F_7 \\ F_5F_6 \end{bmatrix}, \begin{bmatrix} F_4 \\ F_2F_3 \\ F_1F_5 \\ F_6F_7 \end{bmatrix}$$

The effects F_3F_4, F_1F_6, F_2F_4, F_1F_7, F_2F_3 and F_1F_5 are estimated with a higher efficiency than the corresponding effects in the same set. Factors F_5F_7, F_5F_6 and F_6F_7 are estimated with a lower efficiency. Main effects F_2, F_3 and F_4 are estimated with a higher efficiency than F_5F_7, F_5F_6 and F_6F_7 but a lower efficiency than the efficiency attained for F_3F_4, F_1F_6, F_2F_4, F_1F_7, F_2F_3 and F_1F_5.

$$\frac{1}{96}\begin{bmatrix} 10 & -4 & 6 & 6 \\ -4 & 4 & -3 & -3 \\ 6 & -3 & 6 & 3 \\ 6 & -3 & 3 & 6 \end{bmatrix} \text{gives estimates for} \begin{bmatrix} F_5 \\ F_1F_4 \\ F_3F_6 \\ F_2F_7 \end{bmatrix}, \begin{bmatrix} F_6 \\ F_1F_2 \\ F_3F_5 \\ F_4F_7 \end{bmatrix}, \begin{bmatrix} F_7 \\ F_1F_3 \\ F_4F_6 \\ F_2F_5 \end{bmatrix},$$

Effects F_1F_4, F_1F_2 and F_1F_3 are estimated with a higher efficiency than the corresponding effects in the same set. Effects F_3F_6, F_2F_7, F_3F_5, F_4F_7, F_4F_6 and $F_2F_5$are estimated with a higher efficiency than the main effects F_5, F_6 and F_7 but a lower efficiency than the efficiency attained for effects F_1F_4, F_1F_2 and F_1F_3.

$$\hat{\mu} = \frac{1}{48}[\mu] \qquad \hat{F}_1 = \frac{1}{48}[F_1]$$

2.5 Eight Factor Experiment involving 48+16 = 64 Runs

Consider a design with treatments $(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8)$ which satisfy the simultaneous equations

$$\begin{aligned} x_1 + x_3 + x_6 &= 0, 0, 1 \\ x_2 + x_5 + x_7 &= 0, 0, 1 \\ x_4 + x_6 + x_7 &= 0, 1, 0 \\ x_3 + x_5 + x_8 &= 0, 1, 1 \end{aligned}$$

(mod 2)

2.5.1 Construction of The Design

The B matrix obtained here is dissimilar to that described in equation (1.14) that gives the construction plan. We are going to get a B matrix that is non-singular for a subset of the linear equations given. For example, linear equations $x_2+x_5+x_7$ and $x_4+x_6+x_7$ infer aliasing of F_7, F_2F_5 and F_4F_6 effects. Clearly, the B matrix in equations

$$
\begin{aligned}
x_2+x_5+x_7 &= 0,0,1\\
x_4+x_6+x_7 &= 0,1,0
\end{aligned}
$$

is non-singular, thus the effects under consideration are said to be estimable.

The treatment combinations satisfying

$$
\begin{aligned}
x_1+x_3+x_6 &= 0\\
x_2+x_5+x_7 &= 0\\
x_4+x_6+x_7 &= 0\\
x_3+x_5+x_8 &= 0
\end{aligned}
\pmod 2
$$

are given in set one denoted as st_1. The treatments in st_1 are repeated and thus 16 d.f are used to estimate the error variance.

If we add 1 (mod2) in x_1, x_6 and x_8 position of the first set, we get the second set. If we add 1 (mod2) in x_6, x_7 and x_8 position of the first set, we get the third set.

Treatment combinations

$\underline{st_1}$	$\underline{st_2}$	$\underline{st_3}$
$(0,0,0,0,0,0,0,0)$	$(1,0,0,0,0,1,0,1)$	$(0,0,0,0,0,1,1,1)$
$(0,0,0,1,1,0,1,1)$	$(1,0,0,1,1,1,1,0)$	$(0,0,0,1,1,1,0,0)$
$(0,0,1,0,1,1,1,0)$	$(1,0,1,0,1,0,1,1)$	$(0,0,1,0,1,0,0,1)$
$(0,0,1,1,0,1,0,1)$	$(1,0,1,1,0,0,0,0)$	$(0,0,1,1,0,0,1,0)$
$(0,1,0,0,1,0,0,1)$	$(1,1,0,0,1,1,0,0)$	$(0,1,0,0,1,1,1,0)$
$(0,1,0,1,0,0,1,0)$	$(1,1,0,1,0,1,1,1)$	$(0,1,0,1,0,1,0,1)$
$(0,1,1,0,0,1,1,1)$	$(1,1,1,0,0,0,1,0)$	$(0,1,1,0,0,0,0,0)$
$(0,1,1,1,1,1,0,0)$	$(1,1,1,1,1,0,0,1)$	$(0,1,1,1,1,0,1,1)$
$(1,0,0,0,1,1,1,1)$	$(0,0,0,0,1,0,1,0)$	$(1,0,0,0,1,0,0,0)$
$(1,0,0,1,0,1,0,0)$	$(0,0,0,1,0,0,0,1)$	$(1,0,0,1,0,0,1,1)$
$(1,0,1,0,0,0,0,1)$	$(0,0,1,0,0,1,0,0)$	$(1,0,1,0,0,1,1,0)$
$(1,0,1,1,1,0,1,0)$	$(0,0,1,1,1,1,1,1)$	$(1,0,1,1,1,1,0,1)$
$(1,1,0,0,0,1,1,0)$	$(0,1,0,0,0,0,1,1)$	$(1,1,0,0,0,0,0,1)$
$(1,1,0,1,1,1,0,1)$	$(0,1,0,1,1,0,0,0)$	$(1,1,0,1,1,0,1,0)$
$(1,1,1,0,1,0,0,0)$	$(0,1,1,0,1,1,0,1)$	$(1,1,1,0,1,1,1,1)$
$(1,1,1,1,0,0,1,1)$	$(0,1,1,1,0,1,1,0)$	$(1,1,1,1,0,1,0,0)$

The defining relation is

$$I = F_1F_3F_6 = F_2F_5F_7 = F_4F_6F_7 = F_3F_5F_8 = F_1F_3F_4F_7 = F_1F_5F_6F_8 = F_2F_4F_5F_6$$
$$=\mathsf{F}_2F_3F_7F_8 = F_1F_2F_4F_8F$$

The correlated sets of factors are

(F_1, F_3F_6), (F_2, F_5F_7), (F_3, F_1F_6, F_5F_8), (F_4, F_6F_7) (F_5, F_2F_7, F_3F_8), (F_6, F_1F_3, F_4F_7), (F_7, F_2F_5, F_4F_6), (F_2F_8, F_1F_4, F_3F_7), (F_1F_8, F_5F_6, F_2F_4), (F_1F_5, F_6F_8), (F_1F_6, F_5F_8), (F_2F_3, F_7F_8), (F_1F_2, F_4F_8).

The factorial effects F_8, F_3F_5, F_1F_7 and F_3F_4 are estimated orthogonally.

2.5.2 Method of Analysis

We define our matrices as per equation (2.4).

$$X^TX = \begin{array}{c}
\begin{array}{*{35}{c}}
\mu & 3 & 16 & 58 & 5 & 27 & 38 & 6 & 13 & 47 & 7 & 25 & 46 & 28 & 14 & 37 & 18 & 56 & 24 & 2 & 57 & 4 & 67 & 1 & 36 & 15 & 68 & 23 & 78 & 12 & 48 & 8 & 35 & 17 & 34
\end{array} \\
\left[\begin{array}{*{35}{c}}
64 & 0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0 \\
0&64&-32 & 0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0&0 \\
0&-32&64&32 & 0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0 \\
0&0&32&64 & 0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0 \\
0&0&0&0&64&-32 & 0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0 \\
0&0&0&0&-32&64&32 & 0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0&0&0 \\
0&0&0&0&0&32&64 & 0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0&0&0 \\
0&0&0&0&0&0&0&64&-32&-32 & 0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 \\
0&0&0&0&0&0&0&-32&64 & 0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0 \\
0&0&0&0&0&0&0&-32&0&64 & 0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&64&-32&-32 & 0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0&0 \\
0&0&0&0&0&0&0&0&0&0&-32&64 & 0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0&0&0 \\
0&0&0&0&0&0&0&0&0&0&-32&0&64 & 0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&64&32&32 & 0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&32&64 & 0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&32&0&64 & 0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&64&32&32 & 0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&32&64 & 0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&32&0&64 & 0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&64&-32 & 0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&-32&64 & 0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&64&-32 & 0&0&0&0&0 & 0&0&0&0&0 & 0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&-32&64 & 0&0&0&0&0 & 0&0&0&0&0 & 0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&64&-32 & 0&0&0&0&0 & 0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&-32&64 & 0&0&0&0&0 & 0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&64&32 & 0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&32&64 & 0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&64&32 & 0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&32&64 & 0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&64&32 & 0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&32&64 & 0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&64&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&64&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&64&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&64
\end{array}\right]
\end{array}$$

From our X^TX matrix we get the inverse of the matrices as follows

$$\begin{bmatrix} 64 & -32 & 0 \\ -32 & 648 & 32 \\ 0 & 32 & 64 \end{bmatrix}^{-1} = \frac{1}{128}\begin{bmatrix} 3 & 2 & -1 \\ 2 & 4 & -2 \\ -1 & -2 & 3 \end{bmatrix}, \qquad \begin{bmatrix} 64 & -32 & -32 \\ -32 & 64 & 0 \\ -32 & 0 & 64 \end{bmatrix}^{-1} = \frac{1}{128}\begin{bmatrix} 4 & 2 & 2 \\ 2 & 3 & 1 \\ 2 & 1 & 3 \end{bmatrix},$$

$$\begin{bmatrix} 64 & 32 & 32 \\ 32 & 64 & 0 \\ 32 & 0 & 64 \end{bmatrix}^{-1} = \frac{1}{128}\begin{bmatrix} 4 & -2 & -2 \\ -2 & 3 & 1 \\ -2 & 1 & 3 \end{bmatrix}, \qquad \begin{bmatrix} 64 & -32 \\ -32 & 64 \end{bmatrix}^{-1} = \frac{1}{96}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix},$$

and

$$\begin{bmatrix} 64 & 32 \\ 32 & 64 \end{bmatrix}^{-1} = \frac{1}{96}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

The matrix:

$$\frac{1}{128}\begin{bmatrix} 3 & 2 & -1 \\ 2 & 4 & -2 \\ -1 & -2 & 3 \end{bmatrix} \text{ gives estimates for } \begin{bmatrix} F_3 \\ F_1F_6 \\ F_5F_8 \end{bmatrix}, \begin{bmatrix} F_5 \\ F_2F_7 \\ F_3F_8 \end{bmatrix}$$

Effects F_3, F_5, F_5F_8 and F_3F_8 are estimated with the same efficiency which is higher than the efficiency attained for F_1F_6 and F_2F_7. F_1F_6 and F_2F_7 are estimated with the same efficiency.

$$\frac{1}{128}\begin{bmatrix} 4 & 2 & 2 \\ 2 & 3 & 1 \\ 2 & 1 & 3 \end{bmatrix} \text{ gives estimates for } \begin{bmatrix} F_6 \\ F_1F_3 \\ F_4F_7 \end{bmatrix}, \begin{bmatrix} F_7 \\ F_2F_5 \\ F_4F_6 \end{bmatrix}$$

The effects F_1F_3, F_4F_7, F_2F_5 and F_4F_6 are estimated with the same efficiency which is higher than the efficiency attained for F_6 and F_7. F_6 and F_7 are estimated with the same lower efficiency.

$$\frac{1}{128}\begin{bmatrix} 4 & -2 & -2 \\ -2 & 3 & 1 \\ -2 & 1 & 3 \end{bmatrix} \text{ gives estimates for } \begin{bmatrix} F_2F_8 \\ F_1F_4 \\ F_3F_7 \end{bmatrix}, \begin{bmatrix} F_1F_8 \\ F_5F_6 \\ F_2F_4 \end{bmatrix}$$

The effects F_1F_4, F_3F_7, F_5F_6 and F_2F_4 are estimated with the same efficiency which is higher than the efficiency attained by F_2F_8 and F_1F_8. Effects F_2F_8 and F_1F_8 are estimated with the same efficiency.

$$\frac{1}{96}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \text{ gives estimates for } \begin{bmatrix} F_2 \\ F_5F_7 \end{bmatrix}, \begin{bmatrix} F_4 \\ F_6F_7 \end{bmatrix}, \begin{bmatrix} F_1 \\ F_3F_6 \end{bmatrix}$$

$$\frac{1}{96}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \text{ gives estimates for } \begin{bmatrix} F_1F_5 \\ F_6F_8 \end{bmatrix}, \begin{bmatrix} F_1F_6 \\ F_5F_8 \end{bmatrix}, \begin{bmatrix} F_2F_3 \\ F_7F_8 \end{bmatrix}, \begin{bmatrix} F_1F_2 \\ F_4F_8 \end{bmatrix}$$

The sets of correlated effects, (F_1,F_3F_6), (F_2,F_5F_7), (F_4,F_6F_7), (F_1F_2,F_4F_8), (F_1F_5,F_6F_8), (F_1F_6,F_5F_8) and (F_2F_3,F_7F_8) are estimated with the same efficiency.

$$\hat{\mu} = \frac{1}{64}[\mu] \qquad \hat{F_3F_5} = \frac{1}{64}[F_3F_5]$$

$$\hat{F_1F_7} = \frac{1}{64}[F_1F_7] \qquad \hat{F_3F_5} = \frac{1}{64}[F_3F_5]$$

$$\hat{F_3F_4} = \frac{1}{64}[F_3F_4]$$

2.6 Nine Factor Experiment involving 64+16 = 80 Runs

Consider a design with treatments $(x_1,x_2,x_3,x_4,x_5,x_6,x_7,x_8,x_9)$ which satisfy the simultaneous equations

$$\begin{aligned} x_1+x_2+x_3 &= 0,0,1,1 \\ x_1+x_4+x_5 &= 0,1,0,1 \\ x_1+x_6+x_7 &= 0,1,1,0 \\ x_2+x_4+x_8 &= 0,0,0,1 \\ x_4+x_6+x_9 &= 0,1,1,1 \end{aligned}$$

(mod 2)

2.6.1 Constuction of The Design

The B matrix is obtained using the same procedure described in the eight factor experiment.

We first get the treatment combinations belonging to the first set. If we add 1 (mod2) in x_4, x_7 and x_8 position of the first set, we get the second set. If we add 1 (mod2) in x_1, x_4 and x_8 position of the first set, we get the third set. If we add 1 (mod2) in x_1, x_6 and x_8 position of the first set, we obtain the fourth set.

Treatment combinations

$\underline{st_1}$	$\underline{st_2}$	$\underline{st_3}$	$\underline{st_4}$
$(0,0,0,0,0,0,0,0,0)$	$(0,0,0,1,0,0,1,1,0)$	$(1,0,0,1,0,0,0,1,0)$	$(1,0,0,0,0,1,0,1,0)$
$(0,1,1,0,0,0,0,1,0)$	$(0,1,1,1,0,0,1,0,0)$	$(1,1,1,1,0,0,0,0,0)$	$(1,1,1,0,0,1,0,0,0)$
$(0,1,1,0,0,1,1,1,1)$	$(0,1,1,1,0,1,0,0,1)$	$(1,1,1,1,0,1,1,0,1)$	$(1,1,1,0,0,0,1,0,1)$
$(0,1,1,1,1,0,0,0,1)$	$(0,1,1,0,1,0,1,1,1)$	$(1,1,1,0,1,0,0,1,1)$	$(1,1,1,1,1,1,0,1,1)$
$(0,1,1,1,1,1,1,0,0)$	$(0,1,1,0,1,1,0,1,0)$	$(1,1,1,0,1,1,1,1,0)$	$(1,1,1,1,1,0,1,1,0)$
$(0,0,0,0,0,1,1,0,1)$	$(0,0,0,1,0,1,0,1,1)$	$(1,0,0,1,0,1,1,1,1)$	$(1,0,0,0,0,0,1,1,1)$
$(0,0,0,1,1,0,0,1,1)$	$(0,0,0,0,1,0,1,0,1)$	$(1,0,0,0,1,0,0,0,1)$	$(1,0,0,1,1,1,0,0,1)$
$(0,0,0,1,1,1,1,1,0)$	$(0,0,0,0,1,1,0,0,0)$	$(1,0,0,0,1,1,1,0,0)$	$(1,0,0,1,1,0,1,0,0)$
$(1,1,0,1,0,1,0,0,0)$	$(1,1,0,0,0,1,1,1,0)$	$(0,1,0,0,0,1,0,1,0)$	$(0,1,0,1,0,0,0,1,0)$
$(1,1,0,0,1,0,1,1,0)$	$(1,1,0,1,1,0,0,0,0)$	$(0,1,0,1,1,0,1,0,0)$	$(0,1,0,0,1,1,1,0,0)$
$(1,1,0,1,0,0,1,0,1)$	$(1,1,0,0,0,0,0,1,1)$	$(0,1,0,0,0,0,1,1,1)$	$(0,1,0,1,0,1,1,1,1)$
$(1,1,0,0,1,1,0,1,1)$	$(1,1,0,1,1,1,1,0,1)$	$(0,1,0,1,1,1,0,0,1)$	$(0,1,0,0,1,0,0,0,1)$
$(1,0,1,1,0,1,0,1,0)$	$(1,0,1,0,0,1,1,0,0)$	$(0,0,1,0,0,1,0,0,0)$	$(0,0,1,1,0,0,0,0,0)$
$(1,0,1,1,0,0,1,1,1)$	$(1,0,1,0,0,0,0,0,1)$	$(0,0,1,0,0,0,1,0,1)$	$(0,0,1,1,0,1,1,0,1)$
$(1,0,1,0,1,0,1,0,0)$	$(1,0,1,1,1,0,0,1,0)$	$(0,0,1,1,1,0,1,1,0)$	$(0,0,1,0,1,1,1,1,0)$
$(1,0,1,0,1,1,0,0,1)$	$(1,0,1,1,1,1,1,1,1)$	$(0,0,1,1,1,1,0,1,1)$	$(0,0,1,0,1,0,0,1,1)$

The defining relation for the fraction is

$$I = F_1F_2F_3 = F_1F_4F_5 = F_1F_6F_7 = F_2F_4F_8 = F_4F_6F_9 = F_3F_5F_8$$
$$= F_5F_7F_9 = F_2F_3F_4F_5 = F_2F_3F_6F_7 = F_4F_5F_6F_7 = F_1F_3F_4F_8$$
$$= F_1F_2F_5F_8 = F_1F_5F_6F_9 = F_1F_4F_7F_9 = F_2F_6F_8F_9 = F_3F_7F_8F_9$$

The correlated sets of factors are

$(F_1, F_2F_3, F_4F_5, F_6F_7)$, (F_2, F_1F_3, F_4F_8), (F_3, F_1F_2, F_5F_8), $(F_4, F_1F_5, F_2F_8, F_6F_9)$ $(F_5, F_1F_4, F_7F_9, F_3F_8)$, (F_6, F_1F_7, F_4F_9), (F_7, F_1F_6, F_5F_9), (F_8, F_2F_4, F_3F_5), (F_9, F_5F_7, F_4F_6), (F_1F_8, F_3F_4, F_2F_5), (F_1F_9, F_5F_6, F_4F_7), (F_2F_6, F_3F_7, F_8F_9), (F_2F_7, F_3F_6), (F_2F_9, F_6F_8), (F_3F_9, F_7F_8).

2.6.2 Method of Analysis

We define our matrices as per equation (2.4). The inverse of the sub-matrices in X^TX matrix are used to obtain estimates for the corresponding effects as shown below;

The matrix:

$$\begin{bmatrix} 80 & -16 & -16 & -16 \\ -16 & 80 & 16 & 16 \\ -16 & 16 & 80 & 16 \\ -16 & 16 & 16 & 80 \end{bmatrix}^{-1} = \frac{1}{512}\begin{bmatrix} 7 & 1 & 1 & 1 \\ 1 & 7 & -1 & -1 \\ 1 & -1 & 7 & -1 \\ 1 & -1 & -1 & 7 \end{bmatrix} \text{is used to estimate} \begin{bmatrix} F_1 \\ F_2F_3 \\ F_4F_5 \\ F_6F_7 \end{bmatrix} \tag{2.6}$$

The effects in equation (2.6) are estimated using the same efficiency.

In a similar way, the matrix

$$\begin{bmatrix} 80 & -16 & -48 & 16 \\ -16 & 80 & 48 & 48 \\ -48 & 48 & 80 & 16 \\ -16 & 48 & 16 & 80 \end{bmatrix}^{-1} = \frac{1}{128}\begin{bmatrix} 3 & 0 & 2 & -1 \\ 0 & 4 & -2 & -2 \\ 2 & -2 & 4 & 0 \\ -1 & -2 & 0 & 3 \end{bmatrix} \text{is used to estimate} \begin{bmatrix} F_4 \\ F_1F_5 \\ F_2F_8 \\ F_6F_9 \end{bmatrix}, \begin{bmatrix} F_5 \\ F_1F_4 \\ F_7F_9 \\ F_3F_8 \end{bmatrix} \tag{2.7}$$

In equation (2.7), the effects F_4, F_5, F_6F_9 and F_3F_8 are estimated with a higher efficiency than the efficiency attained for F_1F_5, F_2F_8, F_1F_4 and F_7F_9 . Effects F_1F_5, F_2F_8, F_1F_4 and F_7F_9 are estimated with the same efficiency.

$$
X^T X = \begin{array}{c}
\begin{array}{cccccccccccccccccccccccccccccccccccccccccccccc}
\mu & 1 & 23 & 45 & 67 & 4 & 15 & 28 & 69 & 5 & 14 & 79 & 38 & 2 & 13 & 48 & 7 & 16 & 59 & 6 & 17 & 49 & 3 & 12 & 58 & 8 & 24 & 35 & 9 & 57 & 46 & 18 & 34 & 25 & 19 & 56 & 47 & 26 & 37 & 89 & 27 & 36 & 29 & 68 & 39 & 78
\end{array} \\
\begin{bmatrix}
80 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 80 & -16 & -16 & -16 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -16 & 80 & 16 & 16 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -16 & 16 & 80 & 16 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -16 & 16 & 16 & 80 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 80 & -16 & -48 & 16 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -16 & 80 & 48 & 48 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -48 & 48 & 80 & 16 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 16 & 48 & 16 & 80 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 80 & -16 & -48 & 16 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -16 & 80 & 48 & 48 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -48 & 48 & 80 & 16 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 16 & 48 & 16 & 80 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 80 & -16 & -48 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -16 & 80 & 48 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -48 & 48 & 80 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 80 & -16 & -48 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -16 & 80 & 48 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -48 & 48 & 80 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 80 & -16 & 16 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -16 & 80 & 48 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 16 & 48 & 80 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 80 & -16 & 16 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -16 & 80 & 48 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 16 & 48 & 80 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 80 & -48 & 16 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -48 & 80 & 16 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 16 & 16 & 80 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 80 & -48 & 16 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -48 & 80 & 16 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 16 & 16 & 80 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 80 & 48 & 48 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 48 & 80 & 16 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 48 & 16 & 80 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 80 & 48 & 48 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 48 & 80 & 16 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 48 & 16 & 80 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 80 & 16 & 16 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 16 & 80 & 16 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 16 & 16 & 80 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 80 & 16 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 16 & 80 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 80 & 16 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 16 & 80 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 80 & 16 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 16 & 80
\end{bmatrix}
\end{array}
$$

Similarly the matrix

$$\begin{bmatrix} 80 & -16 & -48 \\ -16 & 80 & 48 \\ -48 & 48 & 80 \end{bmatrix}^{-1} = \frac{1}{192}\begin{bmatrix} 4 & -1 & 3 \\ -1 & 4 & -3 \\ 3 & -3 & 6 \end{bmatrix} \text{ is used to estimate } \begin{bmatrix} F_2 \\ F_1F_3 \\ F_4F_8 \end{bmatrix}, \begin{bmatrix} F_7 \\ F_1F_6 \\ F_5F_9 \end{bmatrix} \tag{2.8}$$

In equation (2.8), the effects F_2, F_7, F_1F_3 and F_1F_6 are estimated with a higher efficiency than the efficiency attained for effects F_4F_8 and F_5F_9 . Effects F_4F_8 and F_5F_9 are estimated with the same efficiency.

The other matrices used for estimating the effects follow below;

$$\begin{bmatrix} 80 & -16 & 16 \\ -16 & 80 & 48 \\ 16 & 48 & 80 \end{bmatrix}^{-1} = \frac{1}{128}\begin{bmatrix} 2 & 1 & -1 \\ 1 & 3 & -2 \\ -1 & -2 & 3 \end{bmatrix} \text{ is used to estimate } \begin{bmatrix} F_6 \\ F_1F_7 \\ F_4F_9 \end{bmatrix}, \begin{bmatrix} F_3 \\ F_1F_2 \\ F_5F_8 \end{bmatrix} \tag{2.9}$$

In equation (2.9), effects F_1F_7, F_4F_9, F_1F_2 and F_5F_8 are estimated with the same efficiency which is lower than the efficiency attained for F_6 and F_3.

$$\begin{bmatrix} 80 & -48 & 16 \\ -48 & 80 & 16 \\ 16 & 16 & 80 \end{bmatrix}^{-1} = \frac{1}{128}\begin{bmatrix} 3 & 2 & -1 \\ 2 & 3 & -1 \\ -1 & -1 & 2 \end{bmatrix} \text{ is used to estimate } \begin{bmatrix} F_8 \\ F_2F_4 \\ F_3F_5 \end{bmatrix}, \begin{bmatrix} F_9 \\ F_5F_7 \\ F_4F_6 \end{bmatrix} \tag{2.10}$$

In equation (2.10), effects F_3F_5 and F_4F_6 are estimated with the same efficiency. The effects F_8, F_9, F_2F_4 and F_5F_7 are estimated with a lower efficiency than the one attained for F_3F_5 and F_4F_6.

$$\begin{bmatrix} 80 & 48 & 48 \\ 48 & 80 & 16 \\ 48 & 16 & 80 \end{bmatrix}^{-1} = \frac{1}{192}\begin{bmatrix} 6 & -3 & -3 \\ -3 & 4 & 1 \\ -3 & 1 & 4 \end{bmatrix} \text{ is used to estimate } \begin{bmatrix} F_1F_8 \\ F_3F_4 \\ F_2F_5 \end{bmatrix}, \begin{bmatrix} F_1F_9 \\ F_5F_6 \\ F_4F_7 \end{bmatrix} \tag{2.11}$$

In equation (2.11), the effects F_3F_4, F_2F_5, F_5F_6 and F_4F_7 are estimated with a higher efficiency than the efficiency attained for F_1F_8 and F_1F_9. Effects F_1F_8 and F_1F_9 are estimated with same efficiency which is lower compared to the efficiency attained for other effects in the same set.

$$\begin{bmatrix} 80 & 16 & 16 \\ 16 & 80 & 16 \\ 16 & 16 & 80 \end{bmatrix}^{-1} = \frac{1}{448}\begin{bmatrix} 6 & -1 & -1 \\ -1 & 6 & -1 \\ -1 & -1 & 6 \end{bmatrix} \text{ is used to estimate } \begin{bmatrix} F_2F_6 \\ F_3F_7 \\ F_8F_9 \end{bmatrix} \quad \textbf{(2.12)}$$

Effects in equation (2.12) are estimated using the same efficiency.

$$\begin{bmatrix} 80 & 16 \\ 16 & 80 \end{bmatrix}^{-1} = \frac{1}{384}\begin{bmatrix} 5 & -1 \\ -1 & 5 \end{bmatrix} \text{ is used to estimate } \begin{bmatrix} F_2F_7 \\ F_3F_6 \end{bmatrix}, \begin{bmatrix} F_2F_9 \\ F_6F_8 \end{bmatrix}, \begin{bmatrix} F_3F_9 \\ F_7F_8 \end{bmatrix} \quad \textbf{(2.13)}$$

The effects in equation (2.13) are estimated using the same efficiency.

$$\hat{\mu} = \frac{1}{80}[\mu]$$

2.7 Ten Factor Experiment involving 64+16 = 80 Runs

Consider a design with treatments $(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10})$ which satisfy the simultaneous equations

$$\begin{aligned} x_1 + x_2 + x_3 &= 0,0,1,1 \\ x_1 + x_4 + x_5 &= 0,1,0,1 \\ x_1 + x_6 + x_7 &= 0,1,1,0 \\ x_2 + x_4 + x_8 &= 0,0,0,1 \\ x_4 + x_6 + x_9 &= 0,1,1,1 \\ x_2 + x_7 + x_{10} &= 0,1,1,1 \end{aligned}$$

(mod 2)

2.7.1 Construction of The Design

The B matrix is obtained using the same procedure described in the eight factor experiment.

We get the treatment combinations belonging to the first set that we then duplicate. If we add 1 (mod2) in x_5, x_6 and x_{10} position of the first set, we get the second set. If we add 1 (mod2) in x_3, x_7 and x_9 position of the first set, we get the third set. If we add 1 (mod2) in x_3, x_4 and x_{10} position of the first set, we obtain the fourth set.

Treatment combinations

$\underline{st_1}$	$\underline{st_2}$	$\underline{st_3}$	$\underline{st_4}$
$(0,0,0,0,0,0,0,0,0,0)$	$(0,0,0,0,1,1,0,0,0,1)$	$(0,0,1,0,0,0,1,0,1,0)$	$(0,0,1,1,0,0,0,0,0,1)$
$(0,0,0,0,0,1,1,0,1,1)$	$(0,0,0,0,1,0,1,0,1,0)$	$(0,0,1,0,0,1,0,0,0,1)$	$(0,0,1,1,0,1,1,0,1,0)$
$(0,0,0,1,1,0,0,1,1,0)$	$(0,0,0,1,0,1,0,1,1,1)$	$(0,0,1,1,1,0,1,1,0,0)$	$(0,0,1,0,1,0,0,1,1,1)$
$(0,0,0,1,1,1,1,1,0,1)$	$(0,0,0,1,0,0,1,1,0,0)$	$(0,0,1,1,1,1,0,1,1,1)$	$(0,0,1,0,1,1,1,1,0,0)$
$(0,1,1,0,0,0,0,1,0,1)$	$(0,1,1,0,1,1,0,1,0,0)$	$(0,1,0,0,0,0,1,1,1,1)$	$(0,1,0,1,0,0,0,1,0,0)$
$(0,1,1,0,0,1,1,1,1,0)$	$(0,1,1,0,1,0,1,1,1,1)$	$(0,1,0,0,0,1,0,1,0,0)$	$(0,1,0,1,0,1,1,1,1,1)$
$(0,1,1,1,1,0,0,0,1,1)$	$(0,1,1,1,0,1,0,0,1,0)$	$(0,1,0,1,1,0,1,0,0,1)$	$(0,1,0,0,1,0,0,0,1,0)$
$(0,1,1,1,1,1,1,0,0,0)$	$(0,1,1,1,0,0,1,0,0,1)$	$(0,1,0,1,1,1,0,0,1,0)$	$(0,1,0,0,1,1,1,0,0,1)$
$(1,1,0,0,1,0,1,1,0,0)$	$(1,1,0,0,0,1,1,1,0,1)$	$(1,1,1,0,1,0,0,1,1,0)$	$(1,1,1,1,1,0,1,1,0,1)$
$(1,1,0,0,1,1,0,1,1,1)$	$(1,1,0,0,0,0,0,1,1,0)$	$(1,1,1,0,1,1,1,1,0,1)$	$(1,1,1,1,1,1,0,1,1,0)$
$(1,1,0,1,0,0,1,0,1,0)$	$(1,1,0,1,1,1,1,0,1,1)$	$(1,1,1,1,0,0,0,0,0,0)$	$(1,1,1,0,0,0,1,0,1,1)$
$(1,1,0,1,0,1,0,0,0,1)$	$(1,1,0,1,1,0,0,0,0,0)$	$(1,1,1,1,0,1,1,0,1,1)$	$(1,1,1,0,0,1,0,0,0,0)$
$(1,0,1,0,1,0,1,0,0,1)$	$(1,0,1,0,0,1,1,0,0,0)$	$(1,0,0,0,1,0,0,0,1,1)$	$(1,0,0,1,1,0,1,0,0,0)$
$(1,0,1,0,1,1,0,0,1,0)$	$(1,0,1,0,0,0,0,0,1,1)$	$(1,0,0,0,1,1,1,0,0,0)$	$(1,0,0,1,1,1,0,0,1,1)$
$(1,0,1,1,0,0,1,1,1,1)$	$(1,0,1,1,1,1,1,1,1,0)$	$(1,0,0,1,0,0,0,1,0,1)$	$(1,0,0,0,0,0,1,1,1,0)$
$(1,0,1,1,0,1,0,1,0,0)$	$(1,0,1,1,1,0,0,1,0,1)$	$(1,0,0,1,0,1,1,1,1,0)$	$(1,0,0,0,0,1,0,1,0,1)$

The defining relation for the fraction is

$$
\begin{gathered}
I = F_1F_2F_3 = F_1F_4F_5 = F_1F_6F_7 = F_2F_4F_8 = F_4F_6F_9 = F_3F_5F_8 = F_5F_7F_9 = F_2F_3F_4F_5 \\
= F_2F_3F_6F_7 = F_4F_5F_6F_7 = F_1F_3F_4F_8 = F_1F_2F_5F_8 = F_1F_5F_6F_9 = F_1F_4F_7F_9 = F_2F_6F_8F_9 \\
= F_3F_7F_8F_9 = F_2F_7F_{10} = F_3F_6F_{10} = F_1F_3F_7F_{10} = F_1F_2F_6F_{10} = F_4F_7F_8F_{10} = F_5F_6F_8F_{10} \\
= F_3F_4F_9F_{10} = F_1F_8F_9F_{10} = F_2F_5F_9F_{10}
\end{gathered}
$$

The correlated sets of factors are

$(F_1, F_2F_3, F_4F_5, F_6F_7)$, $(F_2, F_1F_3, F_4F_8, F_7F_{10})$, $(F_3, F_1F_2, F_6F_{10}, F_5F_8)$, $(F_4, F_1F_5, F_2F_8, F_6F_9)$ $(F_5, F_1F_4, F_7F_9, F_3F_8)$, $(F_6, F_1F_7, F_3F_{10}, F_4F_9)$, $(F_7, F_1F_6, F_5F_9, F_2F_{10})$, (F_8, F_2F_4, F_3F_5), (F_9, F_5F_7, F_4F_6), (F_{10}, F_3F_6, F_2F_7), $(F_1F_8, F_3F_4, F_2F_5, F_9F_{10})$, $(F_1F_9, F_5F_6, F_4F_7, F_8F_{10})$, $(F_1F_{10}, F_3F_7, F_2F_6, F_8F_9)$, $(F_2F_9, F_6F_8, F_5F_{10})$, $(F_3F_9, F_7F_8, F_4F_{10})$.

2.7.2 Method of Analysis

The effects in this design are estimated using the following equations.

The set of effects in $(F_1, F_2F_3, F_4F_5, F_6F_7)$ are estimated using equation (2.6). The set of effects in $(F_2, F_1F_3, F_4F_8, F_7F_{10})$, $(F_3, F_1F_2, F_6F_{10}, F_5F_8)$, $(F_4, F_1F_5, F_2F_8, F_6F_9)$ $(F_5, F_1F_4, F_7F_9, F_3F_8)$, $(F_6, F_1F_7, F_3F_{10}, F_4F_9)$ and $(F_7, F_1F_6, F_5F_9, F_2F_{10})$ are estimated using equation (2.7). Effects in (F_8, F_2F_4, F_3F_5), (F_9, F_5F_7, F_4F_6) and (F_{10}, F_3F_6, F_2F_7) are estimated using equation (2.10). The effects $(F_2F_9, F_6F_8, F_5F_{10})$ and $(F_3F_9, F_7F_8, F_4F_{10})$ are estimated using (2.12).

The matrix

$$\begin{bmatrix} 80 & 48 & 48 & -16 \\ 48 & 80 & 16 & 16 \\ 48 & 16 & 80 & 16 \\ -16 & 16 & 16 & 80 \end{bmatrix}^{-1} = \frac{1}{128}\begin{bmatrix} 7 & -4 & -4 & 3 \\ -4 & 4 & 2 & -2 \\ -4 & 2 & 4 & -2 \\ 3 & -2 & -2 & 3 \end{bmatrix} \text{ gives estimates for } \begin{bmatrix} F_1F_8 \\ F_3F_4 \\ F_2F_5 \\ F_9F_{10} \end{bmatrix}, \begin{bmatrix} F_1F_9 \\ F_5F_6 \\ F_4F_7 \\ F_8F_{10} \end{bmatrix} \text{ and } \begin{bmatrix} F_1F_{10} \\ F_3F_7 \\ F_2F_6 \\ F_8F_9 \end{bmatrix}$$

The effects F_9F_{10}, F_8F_{10} and F_8F_9 are estimated with the same efficiency which is higher than the efficiency attained for other effects in similar sets. Effects F_3F_4, F_2F_5, F_5F_6, F_4F_7, F_3F_7 and F_2F_6 are estimated with the same efficiency which higher than the efficiency attained for effects F_1F_8, F_1F_9 and F_1F_{10} but lower than the efficiency attained for the effects F_9F_{10}, F_8F_{10} and F_8F_9. Effects F_1F_8, F_1F_9 and F_1F_{10} are estimated with the lowest efficiency in comparison to the efficiency attained for other effects in similar sets.

$$\hat{\mu} = \frac{1}{80}[\mu]$$

3 Partially Duplicated Fractional Factorial Designs which allow for Estimation up to Three-Factor Interactions

3.1 Six Factor Experiment involving 48+16 = 64 Runs

Consider a design with treatment combinations $(x_1,x_2,x_3,x_4,x_5,x_6)$ which satisfy the simultaneous equations

$$\begin{aligned} x_1+x_2+x_3+x_4 &= 0,1,0 \\ x_1+x_2+x_5+x_6 &= 0,0,1 \end{aligned} \pmod 2$$

3.1.1 Constuction of the Design

We first get the treatment combinations satisfying the equations

$$\begin{aligned} x_1+x_2+x_3+x_4 &= 0 \\ x_1+x_2+x_5+x_6 &= 0 \end{aligned} \pmod 2$$

which give the first set of treatment combinations that are then repeated. From this first set we can easily obtain the second and third set of treatment combinations satisfying the corresponding set of simultaneous equations.

If we add 1 (mod2) in x_3 or x_4 position of the first set, we get the second set. If we add 1 (mod2) in x_5 or x_6 position of the first set, we get the third set. Below follows the treatment combinations that are obtained.

$\underline{st_1}$	$\underline{st_2}$	$\underline{st_3}$
$(0,0,0,0,0,0)$	$(0,0,0,1,0,0)$	$(0,0,0,0,0,1)$
$(0,0,0,0,1,1)$	$(0,0,0,1,1,1)$	$(0,0,0,0,1,0)$
$(0,0,1,1,1,1)$	$(0,0,1,0,1,1)$	$(0,0,1,1,1,0)$
$(0,0,1,1,0,0)$	$(0,0,1,0,0,0)$	$(0,0,1,1,0,1)$
$(0,1,1,0,1,0)$	$(0,1,1,1,1,0)$	$(0,1,1,0,1,1)$
$(0,1,1,0,0,1)$	$(0,1,1,1,0,1)$	$(0,1,1,0,0,0)$
$(0,1,0,1,1,0)$	$(0,1,0,0,1,0)$	$(0,1,0,1,1,1)$
$(0,1,0,1,0,1)$	$(0,1,0,0,0,1)$	$(0,1,0,1,0,0)$
$(1,1,0,0,0,0)$	$(1,1,0,1,0,0)$	$(1,1,0,0,0,1)$
$(1,1,0,0,1,1)$	$(1,1,0,1,1,1)$	$(1,1,0,0,1,0)$
$(1,1,1,1,0,0)$	$(1,1,1,0,0,0)$	$(1,1,1,1,0,1)$
$(1,0,1,0,0,1)$	$(1,0,1,1,0,1)$	$(1,0,1,0,0,0)$
$(1,0,1,0,1,0)$	$(1,0,1,1,1,0)$	$(1,0,1,0,1,1)$
$(1,0,0,1,1,0)$	$(1,0,0,0,1,0)$	$(1,0,0,1,1,1)$
$(1,0,0,1,0,1)$	$(1,0,0,0,0,1)$	$(1,0,0,1,0,0)$
$(1,1,1,1,1,1)$	$(1,1,1,0,1,1)$	$(1,1,1,1,1,0)$

The defining relation is

$$I = F_1F_2F_3F_4 = F_1F_2F_5F_6 = F_3F_4F_5F_6$$

The correlated sets of factors are

$(F_1, F_2F_3F_4, F_2F_5F_6), (F_2, F_1F_3F_4, F_1F_5F_6), (F_3, F_1F_2F_4, F_4F_5F_6), (F_4, F_1F_2F_3, F_3F_5F_6), (F_5, F_1F_2F_6, F_3F_4F_6),$ $(F_6, F_1F_2F_5, F_3F_4F_5), (F_1F_2, F_3F_4, F_5F_6), (F_1F_3, F_2F_4), (F_1F_4, F_2F_3), (F_1F_5, F_2F_6), (F_1F_6, F_2F_5),$ $(F_3F_5, F_4F_6), (F_3F_6, F_4F_5), (F_2F_3F_5, F_1F_4F_5, F_1F_3F_6, F_2F_4F_6), (F_1F_3F_5, F_1F_4F_6, F_2F_3F_6, F_2F_4F_5).$

3.1.2 Method of Analysis

Using equation (2.4), we partition our X^TX matrix and find the inverse of each sub-matrix so as to obtain the estimates of the effects.

$$
X^T X = \begin{array}{c}
\begin{array}{cccccccccccccccccccccccccccccccccccccccccc}
\mu & 1 & 234 & 256 & 2 & 134 & 156 & 3 & 124 & 456 & 4 & 123 & 356 & 5 & 126 & 346 & 6 & 125 & 345 & 12 & 34 & 56 & 13 & 24 & 14 & 23 & 15 & 26 & 16 & 25 & 35 & 46 & 36 & 45 & 235 & 246 & 136 & 145 & 135 & 146 & 236 & 245
\end{array} \\
\begin{bmatrix}
64 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 64 & 32 & 32 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 32 & 64 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 32 & 0 & 64 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 64 & 32 & 32 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 32 & 64 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 32 & 0 & 64 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 64 & 32 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 32 & 64 & 32 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 32 & 64 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 64 & 32 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 32 & 64 & 32 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 32 & 64 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 64 & 32 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 32 & 64 & 32 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 32 & 64 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 64 & 32 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 32 & 64 & 32 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 32 & 64 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 64 & 32 & 32 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 32 & 64 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 32 & 0 & 64 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 64 & 32 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 32 & 64 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 64 & 32 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 32 & 64 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 64 & 32 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 32 & 64 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 64 & 32 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 32 & 64 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 64 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 64 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 64 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 64 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 64 & 0 & 32 & 32 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 64 & 32 & 32 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 32 & 32 & 64 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 32 & 32 & 0 & 64 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 64 & 0 & 32 & 32 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 64 & 32 & 32 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 32 & 32 & 64 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 32 & 32 & 0 & 64
\end{bmatrix}
\end{array}
$$

The matrix:

$$\begin{bmatrix} 64 & 32 & 32 \\ 32 & 64 & 0 \\ 32 & 0 & 64 \end{bmatrix}^{-1} = \frac{1}{128}\begin{bmatrix} 4 & -2 & -2 \\ -2 & 3 & 1 \\ -2 & 1 & 3 \end{bmatrix} \text{ gives estimates for } \begin{bmatrix} F_1 \\ F_2F_3F_4 \\ F_2F_5F_6 \end{bmatrix}, \begin{bmatrix} F_2 \\ F_1F_3F_4 \\ F_1F_5F_6 \end{bmatrix}, \begin{bmatrix} F_1F_2 \\ F_3F_4 \\ F_5F_6 \end{bmatrix}$$

Factors $F_2F_3F_4$, $F_2F_5F_6$, $F_1F_3F_4$, $F_1F_5F_6$, F_3F_4 and F_5F_6 are estimated with the same efficiency. Factors F_1, F_2 and F_1F_2 are also estimated with the same efficiency.

However, factors (F_1, F_2, F_1F_2) are estimated with a lower efficiency than the efficiency attained for $(F_2F_3F_4, F_2F_5F_6, F_1F_3F_4, F_1F_5F_6, F_3F_4, F_5F_6)$.

$$\begin{bmatrix} 64 & 32 & 0 \\ 32 & 64 & 32 \\ 0 & 32 & 64 \end{bmatrix}^{-1} = \frac{1}{128}\begin{bmatrix} 3 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 3 \end{bmatrix} \text{ estimates } \begin{bmatrix} F_3 \\ F_1F_2F_4 \\ F_4F_5F_6 \end{bmatrix}, \begin{bmatrix} F_4 \\ F_1F_2F_3 \\ F_3F_5F_6 \end{bmatrix}, \begin{bmatrix} F_5 \\ F_1F_2F_6 \\ F_3F_4F_6 \end{bmatrix}, \begin{bmatrix} F_6 \\ F_1F_2F_5 \\ F_3F_4F_5 \end{bmatrix}$$

Factors F_3, $F_4F_5F_6$, F_4, $F_3F_5F_6$, F_5, $F_3F_4F_6$, F_6 and $F_3F_4F_5$ are estimated with the same efficiency. Factors $F_1F_2F_4$, $F_1F_2F_3$, $F_1F_2F_6$ and $F_1F_2F_5$ are also estimated with the same efficiency.

Factors $(F_3, F_4F_5F_6, F_4, F_3F_5F_6, F_5, F_3F_4F_6, F_6, F_3F_4F_5)$ are estimated with a higher efficiency than the efficiency attained by factors $(F_1F_2F_4, F_1F_2F_3, F_1F_2F_6, F_1F_2F_5)$.

$$\begin{bmatrix} 64 & 32 \\ 32 & 64 \end{bmatrix}^{-1} = \frac{1}{96}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \text{ is used to estimate } \begin{bmatrix} F_1F_3 \\ F_2F_4 \end{bmatrix}, \begin{bmatrix} F_1F_4 \\ F_2F_3 \end{bmatrix}, \begin{bmatrix} F_1F_5 \\ F_2F_6 \end{bmatrix}, \begin{bmatrix} F_1F_6 \\ F_2F_5 \end{bmatrix}$$

The factors $(F_1F_3, F_2F_4, F_1F_4, F_2F_3, F_1F_5, F_2F_6, F_1F_6, F_2F_5)$ above are estimated with the same efficiency.

The matrix:

$$\begin{bmatrix} 64 & 0 & 32 & 32 \\ 0 & 64 & 32 & 32 \\ 32 & 32 & 64 & 0 \\ 32 & 32 & 0 & 64 \end{bmatrix}^{-1} \text{ is used to estimate } \begin{bmatrix} F_2F_3F_5 \\ F_2F_4F_6 \\ F_1F_3F_6 \\ F_1F_4F_5 \end{bmatrix}, \begin{bmatrix} F_1F_3F_5 \\ F_1F_4F_6 \\ F_2F_3F_6 \\ F_2F_4F_5 \end{bmatrix}$$

However, factors $(F_2F_3F_5, F_2F_4F_6)$ are only estimable with the assumption of factors $(F_1F_3F_6, F_1F_4F_5)$ being absent. Similarly, factors $(F_1F_3F_6, F_1F_4F_5)$ are estimable with the assumption of factors $(F_2F_3F_5, F_2F_4F_6)$ being absent. That is,

The matrix

$\begin{bmatrix} 64 & 0 \\ 0 & 64 \end{bmatrix}^{-1} = \frac{1}{64}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ estimates $\begin{bmatrix} F_2F_3F_5 \\ F_2F_4F_6 \end{bmatrix}$ assuming $\begin{bmatrix} F_1F_3F_6 \\ F_1F_4F_5 \end{bmatrix}$ to be absent. The vice versa is true.

The factors involved in this matrix are estimated with the same efficiency.

$$\hat{\mu} = \frac{1}{64}[\mu]$$

3.2 Seven Factor Experiment involving 64+16 = 80 Runs

Consider a design with treatment combinations $(x_1, x_2, x_3, x_4, x_5, x_6)$ which satisfy the simultaneous equations

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 &= 0,1,0,0 \\ x_1 + x_2 + x_5 + x_7 &= 0,0,1,0 \\ x_1 + x_3 + x_5 + x_6 &= 0,0,0,1 \end{aligned} \quad \text{(mod 2)}$$

3.2.1 Constuction of the Design

We first get the treatment combinations satisfying the equations

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 &= 0 \\ x_1 + x_2 + x_5 + x_7 &= 0 \\ x_1 + x_3 + x_5 + x_6 &= 0 \end{aligned} \quad \text{(mod 2)}$$

which give the first set of treatment combinations that are then repeated. From this first set we can easily obtain the second, third and fourth set of treatment combinations satisfying the corresponding set of simultaneous equations.

If we add 1 (mod2) in x_4 position of the first set, we get the second set. If we add 1 (mod2) in x_7 position of the first set, we get the third set. If we add 1 (mod2) in x_6 position of the first set, we get the fourth set. Below follows the sets of treatment combinations.

$\underline{st_1}$	$\underline{st_2}$	$\underline{st_3}$	$\underline{st_4}$
$(0,0,0,0,0,0,0)$	$(0,0,0,1,0,0,0)$	$(0,0,0,0,0,0,1)$	$(0,0,0,0,0,1,0)$
$(0,0,0,0,1,1,1)$	$(0,0,0,1,1,1,1)$	$(0,0,0,0,1,1,0)$	$(0,0,0,0,1,0,1)$
$(0,0,1,1,0,1,0)$	$(0,0,1,0,0,1,0)$	$(0,0,1,1,0,1,1)$	$(0,0,1,1,0,0,0)$
$(0,0,1,1,1,0,1)$	$(0,0,1,0,1,0,1)$	$(0,0,1,1,1,0,0)$	$(0,0,1,1,1,1,1)$
$(0,1,0,1,0,0,1)$	$(0,1,0,0,0,0,1)$	$(0,1,0,1,0,0,0)$	$(0,1,0,1,0,1,1)$
$(0,1,0,1,1,1,0)$	$(0,1,0,0,1,1,0)$	$(0,1,0,1,1,1,1)$	$(0,1,0,1,1,0,0)$
$(0,1,1,0,0,1,1)$	$(0,1,1,1,0,1,1)$	$(0,1,1,0,0,1,0)$	$(0,1,1,0,0,0,1)$
$(0,1,1,0,1,0,0)$	$(0,1,1,1,1,0,0)$	$(0,1,1,0,1,0,1)$	$(0,1,1,0,1,1,0)$
$(1,1,0,0,0,1,0)$	$(1,1,0,1,0,1,0)$	$(1,1,0,0,0,1,1)$	$(1,1,0,0,0,0,0)$
$(1,1,0,0,1,0,1)$	$(1,1,0,1,1,0,1)$	$(1,1,0,0,1,0,0)$	$(1,1,0,0,1,1,1)$
$(1,0,1,0,0,0,1)$	$(1,0,1,1,0,0,1)$	$(1,0,1,0,0,0,0)$	$(1,0,1,0,0,1,1)$
$(1,0,1,0,1,1,0)$	$(1,0,1,1,1,1,0)$	$(1,0,1,0,1,1,1)$	$(1,0,1,0,1,0,0)$
$(1,0,0,1,0,1,1)$	$(1,0,0,0,0,1,1)$	$(1,0,0,1,0,1,0)$	$(1,0,0,1,0,0,1)$
$(1,0,0,1,1,0,0)$	$(1,0,0,0,1,0,0)$	$(1,0,0,1,1,0,1)$	$(1,0,0,1,1,1,0)$
$(1,1,1,1,0,0,0)$	$(1,1,1,0,0,0,0)$	$(1,1,1,1,0,0,1)$	$(1,1,1,1,0,1,0)$
$(1,1,1,1,1,1,1)$	$(1,1,1,0,1,1,1)$	$(1,1,1,1,1,1,0)$	$(1,1,1,1,1,0,1)$

The defining relation is

$$I = F_1F_2F_3F_4 = F_1F_2F_5F_7 = F_1F_3F_5F_6 = F_3F_4F_5F_7 = F_2F_4F_5F_6 = F_2F_3F_6F_7 = F_1F_4F_6F_7$$

The correlated sets of factors are

$(F_1, F_2F_3F_4, F_2F_5F_7, F_3F_5F_6, F_4F_6F_7)$, $(F_2, F_1F_3F_4, F_1F_5F_7, F_4F_5F_6, F_3F_6F_7)$,
$(F_3, F_1F_2F_4, F_1F_5F_6, F_4F_5F_7, F_2F_6F_7)$, $(F_4, F_1F_2F_3, F_3F_5F_7, F_2F_5F_6, F_1F_6F_7,)$,
$(F_5, F_1F_2F_7, F_1F_3F_6, F_3F_4F_7, F_2F_4F_6)$, $(F_6, F_1F_3F_5, F_2F_4F_5, F_2F_3F_7, F_1F_4F_7)$,
$(F_7, F_1F_2F_5, F_3F_4F_5, F_2F_3F_6, F_1F_4F_6)$, (F_1F_2, F_3F_4, F_5F_7), (F_1F_3, F_2F_4, F_5F_6), (F_1F_4, F_2F_3, F_6F_7),
(F_1F_5, F_2F_7, F_3F_6), (F_1F_6, F_3F_5, F_4F_7), (F_1F_7, F_2F_5, F_4F_6), (F_2F_6, F_4F_5, F_3F_7),
$(F_1F_2F_6, F_3F_4F_6, F_5F_6F_7, F_2F_3F_5, F_1F_4F_5, F_1F_3F_7, F_2F_4F_7)$

3.2.2 Method of Analysis

This design involves both singular and non-singular matrices used to obtain estimates of the effects. In the case of singular matrices, that is, non-invertible matrices, we are going to use Moore-Penrose inverse.

Let $x \in \mathbb{R}^{m \times n}$ and $y \in \mathbb{R}^{m}$ be given. We find $x \in \mathbb{R}^{n}$ that solves the linear equation

$$Ax = y$$

Suppose $m = n$ and A is an invertible matrix, then the unique solution is

$$x = A^{-1}y$$

Now we consider a case where the solution does not exists. A possible alternative is getting the set of all vectors x' that minimise $\|Ax^{'} - y\|$, that is,

$$\min_{x \in \mathbb{R}^{n}} \|Ax^{'} - y\|$$

Moore-Penrose inverse gives the set $x^{'} \in \mathbb{R}^{n}$ that minimize $\|Ax' - y\|$. It can be shown $\min_{x \in \mathbb{R}^{n}} \|Ax^{'} - y\|$ always has a solution.

Therefore, if the linear equation $Ax = y$ has solutions, then $x^{'} = A^{+}y$ is an exact solution and has the least possible value where $A^{+} \in \mathbb{R}^{n \times m}$ is the Moore-Penrose pseudoinverse of A.

The matrix A^{+} is called the pseudoinverse of matrix A if it satisfies the following conditions:

1. $AA^{+}A = A$
2. $A^{+}AA^{+} = A^{+}$
3. $(AA^{+})^{T} = AA^{+}$
4. $(A^{+}A)^{T} = A^{+}A$

Moore-Penrose inverse is used in equations that lack solutions like those involving singular matrices.

For $\begin{bmatrix} 80 & 48 & 48 & 48 & -16 \\ 48 & 80 & 16 & 16 & 16 \\ 48 & 16 & 80 & 16 & 16 \\ 48 & 16 & 16 & 80 & 16 \\ -16 & 16 & 16 & 16 & 80 \end{bmatrix}$, the pseudoinverse is $\frac{1}{8192}\begin{bmatrix} 28 & 10 & 10 & 10 & -26 \\ 10 & 95 & -33 & -33 & 9 \\ 10 & -33 & 95 & -33 & 9 \\ 10 & -33 & -33 & 95 & 9 \\ -26 & 9 & 9 & 9 & 79 \end{bmatrix}$ **(3.1)**

For $\begin{bmatrix} 80 & 48 & 48 & 16 & 16 \\ 48 & 80 & 16 & 48 & -16 \\ 48 & 16 & 80 & -16 & 48 \\ 16 & 48 & -16 & 80 & 16 \\ 16 & -16 & 48 & 16 & 80 \end{bmatrix}$, the pseudoinverse is $\frac{1}{256}\begin{bmatrix} 8 & -4 & -4 & 0 & 0 \\ -4 & 4 & 3 & 0 & -1 \\ -4 & 3 & 1 & -1 & 0 \\ 0 & 0 & -1 & 2 & 1 \\ 0 & -1 & 0 & 1 & 2 \end{bmatrix}$ **(3.2)**

For $\begin{bmatrix} 80 & 48 & 16 & 16 & -16 \\ 48 & 80 & 48 & 48 & 16 \\ 16 & 48 & 80 & 16 & 48 \\ 16 & 48 & 16 & 80 & 48 \\ -16 & 16 & 48 & 48 & 80 \end{bmatrix}$, the pseudoinverse is $\frac{1}{2048}\begin{bmatrix} 48 & -28 & -4 & -4 & 20 \\ -28 & 35 & 5 & 5 & -25 \\ -4 & 5 & 19 & -13 & 0 \\ -4 & 5 & -13 & 19 & 0 \\ 20 & -25 & 0 & 0 & 27 \end{bmatrix}$ **(3.3)**

For $\begin{bmatrix} 80 & 48 & 48 & 48 & 16 & 16 & -16 \\ 48 & 80 & 16 & 16 & 48 & -16 & 16 \\ 48 & 16 & 80 & 16 & -16 & 48 & 16 \\ 48 & 16 & 16 & 80 & 48 & 48 & 16 \\ 16 & 48 & -16 & 48 & 80 & 16 & 48 \\ 16 & -16 & 48 & 48 & 16 & 80 & 48 \\ -16 & 16 & 16 & 16 & 48 & 48 & 80 \end{bmatrix}$

, the pseudoinverse is $\frac{1}{8192}\begin{bmatrix} 28 & 10 & 10 & 14 & 0 & 0 & -22 \\ 10 & 47 & 15 & -27 & 10 & -22 & 15 \\ 10 & 15 & 47 & -27 & -22 & 10 & 15 \\ 14 & -27 & -27 & 55 & 14 & 14 & -27 \\ 0 & 10 & -22 & 14 & 28 & 0 & 10 \\ 0 & -22 & 10 & 14 & 0 & 28 & 10 \\ -22 & 15 & 15 & -27 & 10 & 10 & 47 \end{bmatrix}$ **(3.4)**

The effects in $(F_1, F_2F_3F_4, F_2F_5F_7, F_3F_5F_6, F_4F_6F_7)$ are estimated using equation (3.1). F_1 is estimated using a higher efficiency compared to the other effects. Effects $F_2F_3F_4$, $F_2F_5F_7$ and $F_3F_5F_6$ are estimated with the same efficiency which is lower compared to the efficiency attained by effects in the same set.

Effects in the following sets $(F_2, F_1F_3F_4, F_1F_5F_7, F_4F_5F_6, F_3F_6F_7)$, $(F_3, F_1F_2F_4, F_1F_5F_6, F_4F_5F_7, F_2F_6F_7)$ and $(F_5, F_1F_2F_7, F_1F_3F_6, F_3F_4F_7, F_2F_4F_6)$ are estimated using equation (3.2). The main effects F_2, F_3 and F_5 are estimated with the lowest efficiency. The interations $F_4F_5F_6$, $F_3F_6F_7$, $F_4F_5F_7$, $F_2F_6F_7$, $F_3F_4F_7$ and $F_2F_4F_6$ are estimated with the same efficiency. Effects $F_1F_5F_7$, $F_1F_5F_6$ and $F_1F_3F_6$ are estimated with the highest efficieny in comparison to the other effects in the same set.

The correlated effects in sets $(F_4, F_1F_2F_3, F_3F_5F_7, F_2F_5F_6, F_1F_6F_7,)$, $(F_6, F_1F_3F_5, F_2F_4F_5, F_2F_3F_7, F_1F_4F_7)$ and $(F_7, F_1F_2F_5, F_3F_4F_5, F_2F_3F_6, F_1F_4F_6)$, are estimated using equation (3.3). The main effects F_4, F_6 and F_7 are estimated with the lowest efficiency. The three-factor interactions $F_3F_5F_7$, $F_2F_5F_6$, $F_2F_4F_5$, $F_2F_3F_7$, $F_3F_4F_5$, and $F_2F_3F_6$ are estimated with the same efficiency which is higher compared to the efficiency attained by effects in the same set.

The effects in the set $(F_1F_2F_6, F_3F_4F_6, F_5F_6F_7, F_2F_3F_5, F_1F_4F_5, F_1F_3F_7, F_2F_4F_7)$ are estimated using equation (3.4). The factors $F_1F_2F_6$, $F_1F_4F_5$, $F_1F_3F_7$ are estimated with a higher efficiency compared to the other effects in the same set. Effects $F_3F_4F_6$, $F_5F_6F_7$ and $F_2F_4F_7$ are estimated with the same efficiency. $F_2F_3F_5$ is estimated with the lowest efficiency among the effects in the same set.

The matrix:

$$\begin{bmatrix} 80 & 48 & 48 \\ 48 & 80 & 16 \\ 48 & 16 & 80 \end{bmatrix}^{-1} = \frac{1}{192}\begin{bmatrix} 6 & -3 & -3 \\ -3 & 4 & 1 \\ -3 & 1 & 4 \end{bmatrix} \text{gives estimates for} \begin{bmatrix} F_1F_2 \\ F_3F_4 \\ F_5F_7 \end{bmatrix}, \begin{bmatrix} F_1F_3 \\ F_2F_4 \\ F_5F_6 \end{bmatrix}, \begin{bmatrix} F_1F_5 \\ F_2F_7 \\ F_3F_6 \end{bmatrix}$$

Factors F_1F_2, F_1F_3 and F_1F_5 are estimated with the same efficiency. Factors F_3F_4, F_5F_7, F_2F_4, F_5F_6, F_2F_7 and F_3F_6 are estimated with same efficiency which is higher than the efficiency attained for F_1F_2, F_1F_3 and F_1F_5.

The matrix:

$$\begin{bmatrix} 80 & 48 & -16 \\ 48 & 80 & 16 \\ -16 & 16 & 80 \end{bmatrix}^{-1} = \frac{1}{128}\begin{bmatrix} 3 & -2 & 1 \\ -2 & 3 & -1 \\ 1 & -1 & 2 \end{bmatrix} \text{gives estimates for} \begin{bmatrix} F_1F_4 \\ F_2F_3 \\ F_6F_7 \end{bmatrix}, \begin{bmatrix} F_1F_6 \\ F_3F_5 \\ F_4F_7 \end{bmatrix}, \begin{bmatrix} F_1F_7 \\ F_2F_5 \\ F_4F_6 \end{bmatrix}$$

Factors F_6F_7, F_4F_7 and F_4F_6 are estimated with the same efficiency. The efficiency used is higher than the efficiency attained for the corresponding effects in the same set.

Factors F_1F_4, F_2F_3, F_1F_6, F_3F_5, F_1F_7 and F_2F_5 are estimated with the lowest efficiency.

The matrix:

$$\begin{bmatrix} 80 & 16 & 16 \\ 16 & 80 & 16 \\ 16 & 16 & 80 \end{bmatrix}^{-1} = \frac{1}{448} \begin{bmatrix} 6 & -1 & -1 \\ -1 & 6 & -1 \\ -1 & -1 & 6 \end{bmatrix} \text{gives estimates for} \begin{bmatrix} F_2F_6 \\ F_4F_5 \\ F_3F_7 \end{bmatrix}$$

Effects F_2F_6, F_4F_5 and F_3F_7 are estimated with the same efficiency.

$$\hat{\mu} = \frac{1}{80}[\mu]$$

3.3 Eight Factor Experiment involving 96+32 = 128 Runs

Consider a design with treatments $(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8)$ which satisfy the simultaneous equations

$$\begin{aligned} x_1 + x_2 + x_3 + x_6 &= 0, 0, 1 \\ x_2 + x_4 + x_6 + x_8 &= 0, 1, 0 \\ x_1 + x_2 + x_5 + x_7 &= 0, 1, 1 \end{aligned}$$

(mod 2)

3.3.1 Construction of The Design

The B matrix is obtained using the same procedure described in the eight factor experiment in the Two-Factor Interactions Designs.

For example, the linear equations $x_1 + x_2 + x_3 + x_6$ and $x_1 + x_2 + x_5 + x_7$ infer aliasing in the effects F_1F_2, F_3F_6 and F_5F_7.

Clearly, the B matrix in equations

$$x_1 + x_2 + x_3 + x_6 = 0, 0, 1$$
$$x_1 + x_2 + x_5 + x_7 = 0, 1, 1$$

is non-singular.

We first get the treatment combinations belonging to the first set that we then duplicate. If we add 1 (mod2) in x_5 or x_7 and x_4 or x_8 position of the first set, we get the second set. If we add 1 (mod2) in x_1 position of the first set or 1 (mod2) in x_3 and x_5 or x_7 position of the first set, we obtain the third set.

Treatment combinations

st_1	st_2	st_3
$(0,0,0,0,0,0,0,0)$	$(0,0,0,0,0,0,1,1)$	$(0,0,0,0,0,0,0,0)$
$(0,0,1,0,0,1,0,1)$	$(0,0,1,0,0,1,1,0)$	$(1,0,1,0,0,1,0,1)$
$(0,0,1,0,1,1,1,1)$	$(0,0,1,0,1,1,0,0)$	$(1,0,1,0,1,1,1,1)$
$(0,0,1,1,0,1,0,0)$	$(0,0,1,1,0,1,1,1)$	$(1,0,1,1,0,1,0,0)$
$(0,0,1,1,1,1,1,0)$	$(0,0,1,1,1,1,0,1)$	$(1,0,1,1,1,1,1,0)$
$(0,0,0,1,0,0,0,1)$	$(0,0,0,1,0,0,1,0)$	$(1,0,0,1,0,0,0,1)$
$(0,0,0,1,1,0,1,1)$	$(0,0,0,1,1,0,0,0)$	$(1,0,0,1,1,0,1,1)$
$(0,0,0,0,1,0,1,0)$	$(0,0,0,0,1,0,0,1)$	$(1,0,0,0,1,0,1,0)$
$(0,1,0,0,0,1,1,0)$	$(0,1,0,0,0,1,0,1)$	$(1,1,0,0,0,1,1,0)$
$(0,1,0,0,1,1,0,0)$	$(0,1,0,0,1,1,1,1)$	$(1,1,0,0,1,1,0,0)$
$(0,1,0,1,0,1,1,1)$	$(0,1,0,1,0,1,0,0)$	$(1,1,0,1,0,1,1,1)$
$(0,1,0,1,1,1,0,1)$	$(0,1,0,1,1,1,1,0)$	$(1,1,0,1,1,1,0,1)$
$(0,1,1,1,1,0,0,0)$	$(0,1,1,1,1,0,1,1)$	$(1,1,1,1,1,0,0,0)$
$(0,1,1,0,0,0,1,1)$	$(0,1,1,0,0,0,0,0)$	$(1,1,1,0,0,0,1,1)$
$(0,1,1,0,1,0,0,1)$	$(0,1,1,0,1,0,1,0)$	$(1,1,1,0,1,0,0,1)$
$(0,1,1,1,0,0,1,0)$	$(0,1,1,1,0,0,0,1)$	$(1,1,1,1,0,0,1,0)$
$(1,1,0,1,0,0,0,0)$	$(1,1,0,1,0,0,1,1)$	$(0,1,0,1,0,0,0,0)$
$(1,1,0,1,1,0,1,0)$	$(1,1,0,1,1,0,0,1)$	$(0,1,0,1,1,0,1,0)$
$(1,1,0,0,0,0,0,1)$	$(1,1,0,0,0,0,1,0)$	$(0,1,0,0,0,0,0,1)$
$(1,1,0,0,1,0,1,1)$	$(1,1,0,0,1,0,0,0)$	$(0,1,0,0,1,0,1,1)$
$(1,1,1,1,0,1,0,1)$	$(1,1,1,1,0,1,1,0)$	$(0,1,1,1,0,1,0,1)$
$(1,1,1,1,1,1,1,1)$	$(1,1,1,1,1,1,0,0)$	$(0,1,1,1,1,1,1,1)$
$(1,1,1,0,0,1,0,0)$	$(1,1,1,0,0,1,1,1)$	$(0,1,1,0,0,1,0,0)$
$(1,1,1,0,1,1,1,0)$	$(1,1,1,0,1,1,0,1)$	$(0,1,1,0,1,1,1,0)$
$(1,0,1,0,0,0,1,0)$	$(1,0,1,0,0,0,0,1)$	$(0,0,1,0,0,0,1,0)$
$(1,0,1,0,1,0,0,0)$	$(1,0,1,0,1,0,1,1)$	$(0,0,1,0,1,0,0,0)$
$(1,0,1,1,1,0,0,1)$	$(1,0,1,1,1,0,1,0)$	$(0,0,1,1,1,0,0,1)$
$(1,0,1,1,0,0,1,1)$	$(1,0,1,1,0,0,0,0)$	$(0,0,1,1,0,0,1,1)$
$(1,0,0,0,0,1,1,1)$	$(1,0,0,0,0,1,0,0)$	$(0,0,0,0,0,1,1,1)$
$(1,0,0,0,1,1,0,1)$	$(1,0,0,0,1,1,1,0)$	$(0,0,0,0,1,1,0,1)$
$(1,0,0,1,0,1,1,0)$	$(1,0,0,1,0,1,0,1)$	$(0,0,0,1,0,1,1,0)$
$(1,0,0,1,1,1,0,0)$	$(1,0,0,1,1,1,1,1)$	$(0,0,0,1,1,1,0,0)$

The defining relation for the fraction is

$$I = F_1F_2F_3F_6 = F_2F_4F_6F_8 = F_1F_2F_5F_7 = F_3F_5F_6F_7 = F_1F_3F_4F_8 = F_1F_4F_5F_8F_7F_8$$
$$= F_2F_3F_4F_5F_7F_8$$

The correlated sets of factors are

$(F_1, F_2F_3F_6, F_2F_5F_7, F_3F_4F_8)$, $(F_2, F_1F_3F_6, F_4F_6F_8, F_1F_5F_7)$, $(F_3, F_1F_2F_6, F_5F_6F_7, F_1F_4F_8)$, $(F_4, F_2F_6F_8, F_1F_3F_8)$ $(F_5, F_1F_2F_7, F_3F_6F_7)$, $(F_6, F_1F_2F_3, F_2F_4F_8, F_3F_5F_7)$, $(F_7, F_1F_2F_5, F_3F_5F_6)$, $(F_8, F_2F_4F_6, F_1F_3F_4)$, (F_1F_2, F_3F_6, F_5F_7), (F_1F_3, F_2F_6, F_4F_8), (F_1F_6, F_2F_3), (F_2F_4, F_6F_8), (F_2F_8, F_4F_6), (F_3F_5, F_6F_7), (F_3F_7, F_5F_6), $(F_1F_2F_4, F_3F_4F_6, F_1F_6F_8, F_4F_5F_7, F_2F_3F_8)$, $(F_1F_2F_8, F_3F_6F_8, F_1F_4F_6, F_5F_7F_8, F_2F_3F_4)$, $(F_1F_3F_5, F_2F_5F_6, F_2F_3F_7, F_1F_6F_7, F_4F_5F_8)$, $(F_1F_3F_7, F_2F_6F_7, F_2F_3F_5, F_1F_5F_6, F_4F_7F_8)$, $(F_1F_4F_5, F_2F_4F_7, F_3F_5F_8, F_6F_7F_8)$, $(F_1F_4F_7, F_2F_4F_5, F_3F_7F_8, F_5F_6F_8)$, $(F_1F_5F_8, F_2F_7F_8, F_3F_4F_5, F_4F_6F_7)$, $(F_1F_7F_8, F_2F_5F_8, F_3F_4F_7, F_4F_5F_6)$.

3.3.2 Method of Analysis

Effects F_1F_4, F_3F_8, F_1F_5, F_2F_7, F_1F_7, F_2F_5, F_1F_8, F_3F_4, F_4F_5, F_4F_7, F_5F_8 and F_7F_8 are orthogonally estimated.

For $\begin{bmatrix} 128 & 64 & 0 & 0 \\ 64 & 128 & 64 & 64 \\ 0 & 64 & 128 & 128 \\ 0 & 64 & 128 & 128 \end{bmatrix}$, the pseudoinverse is $\frac{1}{1024}\begin{bmatrix} 12 & -8 & 2 & 2 \\ -8 & 16 & -4 & -4 \\ 2 & -4 & 3 & 3 \\ 2 & -4 & 3 & 3 \end{bmatrix}$ **(3.5)**

For $\begin{bmatrix} 128 & 64 & 64 & 0 \\ 64 & 128 & 0 & 64 \\ 64 & 0 & 128 & 64 \\ 0 & 64 & 64 & 128 \end{bmatrix}$, the pseudoinverse is $\frac{1}{1024}\begin{bmatrix} 5 & 1 & 1 & -3 \\ 1 & 5 & -3 & 1 \\ 1 & -3 & 5 & 1 \\ -3 & 1 & 1 & 5 \end{bmatrix}$ **(3.6)**

For $\begin{bmatrix} 128 & 64 & 64 & 64 \\ 64 & 128 & 0 & 0 \\ 64 & 0 & 128 & 128 \\ 64 & 0 & 128 & 128 \end{bmatrix}$, the pseudoinverse is $\frac{1}{1024}\begin{bmatrix} 16 & -8 & -4 & -4 \\ -8 & 12 & 2 & 2 \\ -4 & 2 & 3 & 3 \\ -4 & 2 & 3 & 3 \end{bmatrix}$ **(3.7)**

$$\text{For } \begin{bmatrix} 128 & 64 & 64 & 0 & 0 \\ 64 & 128 & 0 & 64 & 64 \\ 64 & 0 & 128 & 64 & 64 \\ 0 & 64 & 64 & 128 & 128 \\ 0 & 64 & 64 & 128 & 128 \end{bmatrix}, \text{ the pseudoinverse is } \frac{1}{12544}\begin{bmatrix} 52 & 18 & 18 & -16 & -16 \\ 18 & 59 & -39 & 0 & 0 \\ 18 & -39 & 59 & 0 & 0 \\ -16 & 0 & 0 & 20 & 20 \\ -16 & 0 & 0 & 20 & 20 \end{bmatrix} \quad \textbf{(3.8)}$$

$$\text{For } \begin{bmatrix} 128 & 64 & 0 & 64 & 0 \\ 64 & 128 & 64 & 0 & 64 \\ 0 & 64 & 128 & 64 & 128 \\ 64 & 0 & 64 & 128 & 64 \\ 0 & 64 & 128 & 64 & 128 \end{bmatrix}, \text{ the pseudoinverse is } \frac{1}{12544}\begin{bmatrix} 52 & 18 & -16 & 18 & -16 \\ 18 & 59 & 0 & -39 & 0 \\ -16 & 0 & 20 & 0 & 20 \\ 18 & -39 & 0 & 59 & 0 \\ -16 & 0 & 20 & 0 & 20 \end{bmatrix} \quad \textbf{(3.9)}$$

$$\text{For } \begin{bmatrix} 128 & 0 & 0 & 64 \\ 0 & 128 & 128 & 64 \\ 0 & 128 & 128 & 64 \\ 64 & 64 & 64 & 128 \end{bmatrix}, \text{ the pseudoinverse is } \frac{1}{1024}\begin{bmatrix} 12 & 2 & 2 & -8 \\ 2 & 3 & 3 & -4 \\ 2 & 3 & 3 & -4 \\ -8 & -4 & -4 & 16 \end{bmatrix} \quad \textbf{(3.10)}$$

The effects in sets $(F_1, F_2F_3F_6, F_2F_5F_7, F_3F_4F_8)$ are estimated using equation (3.5). Effects $F_2F_5F_7$ and $F_3F_4F_8$ are estimated with the same efficiency which is the highest in that set. $F_2F_3F_6$ is estimated with the lowest efficiency among the effects in the same set.

The effects in set $(F_2, F_1F_3F_6, F_4F_6F_8, F_1F_5F_7)$ and $(F_3, F_1F_2F_6, F_5F_6F_7, F_1F_4F_8)$ are estimated using equation (3.6). Effects in these sets are estimated with the same efficiency.

The effects in set $(F_6, F_1F_2F_3, F_2F_4F_8, F_3F_5F_7)$ are estimated using equation (3.7). The main effect F_6 is estimated with the lowest efficiency. Effects $F_2F_4F_8$ and $F_3F_5F_7$ are estimated with the same efficiency which is the highest in that set.

Effects in set $(F_1F_2F_4, F_3F_4F_6, F_1F_6F_8, F_4F_5F_7, F_2F_3F_8)$ and $(F_1F_2F_8, F_3F_6F_8, F_1F_4F_6, F_5F_7F_8, F_2F_3F_4)$ are estimated using equation (3.8). The effects $F_3F_4F_6$, $F_1F_6F_8$, $F_3F_6F_8$ and $F_1F_4F_6$ are estimated with the lowest efficiency. Effects $F_4F_5F_7$, $F_2F_3F_8$, $F_5F_7F_8$, $F_2F_3F_4$ are estimated with the same efficiency which is higher compared to the efficiency attained for other effects in the same set.

Effects in set $(F_1F_3F_5, F_2F_5F_6, F_2F_3F_7, F_1F_6F_7, F_4F_5F_8)$ and $(F_1F_3F_7, F_2F_6F_7, F_2F_3F_5, F_1F_5F_6, F_4F_7F_8)$ are estimated using equation (3.9). The effects $F_2F_5F_6$, $F_1F_6F_7$, $F_2F_6F_7$ and $F_1F_5F_6$ are estimated with the lowest efficiency. Effects $F_2F_3F_7$, $F_4F_5F_8$, $F_2F_3F_5$ and $F_4F_7F_8$ are estimated with the same efficiency which is higher compared to the efficiency attained for other effects in the same set.

Factors in the set $(F_1F_4F_5, F_2F_4F_7, F_3F_5F_8, F_6F_7F_8)$, $(F_1F_4F_7, F_2F_4F_5, F_3F_7F_8, F_5F_6F_8)$, $(F_1F_5F_8, F_2F_7F_8, F_3F_4F_5, F_4F_6F_7)$ and $(F_1F_7F_8, F_2F_5F_8, F_3F_4F_7, F_4F_5F_6)$ are estimated using equation (3.10). Factors $F_2F_4F_7$, $F_3F_5F_8$, $F_2F_4F_5$, $F_3F_7F_8$, $F_2F_7F_8$, $F_3F_4F_5$, $F_2F_5F_8$ and $F_3F_4F_7$ are estimated with the same efficiency which is much higher compared to the efficiency attained for the remaining factors in similar sets. $F_6F_7F_8$, $F_5F_6F_8$, $F_4F_6F_7$ and $F_4F_5F_6$ are estimated with the same efficiency. The efficiency used is lower in comparison to the efficiency attained for other factors in the same set.

The matrix:

$$\begin{bmatrix} 128 & 0 & 64 \\ 0 & 128 & 64 \\ 64 & 64 & 128 \end{bmatrix}^{-1} = \frac{1}{256}\begin{bmatrix} 3 & 1 & -2 \\ 1 & 3 & -2 \\ -2 & -2 & 4 \end{bmatrix} \text{gives estimates for} \begin{bmatrix} F_5 \\ F_1F_2F_7 \\ F_3F_6F_7 \end{bmatrix}, \begin{bmatrix} F_7 \\ F_1F_2F_5 \\ F_3F_5F_6 \end{bmatrix}$$

Effects F_5, F_7, $F_1F_2F_7$ and $F_1F_2F_5$ are estimated using the same efficiency that is higher than the efficiency attained for $F_3F_6F_7$ and $F_3F_5F_6$.

The matrix:

$$\begin{bmatrix} 128 & 64 & 0 \\ 64 & 128 & 64 \\ 0 & 64 & 128 \end{bmatrix}^{-1} = \frac{1}{256}\begin{bmatrix} 3 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 3 \end{bmatrix} \text{gives estimates for} \begin{bmatrix} F_8 \\ F_2F_4F_6 \\ F_1F_3F_4 \end{bmatrix}, \begin{bmatrix} F_1F_2 \\ F_3F_6 \\ F_5F_7 \end{bmatrix}, \begin{bmatrix} F_1F_3 \\ F_2F_6 \\ F_4F_8 \end{bmatrix}$$

Effects F_8,$F_1F_3F_4$, F_1F_2, F_5F_7, F_1F_3 and F_4F_8 are estimated using the same efficiency that is higher than the efficiency attained for $F_2F_4F_6$, F_3F_6 and F_2F_6.

The matrix:

$$\begin{bmatrix} 128 & 64 \\ 64 & 128 \end{bmatrix}^{-1} = \frac{1}{192}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \text{gives estimates} \begin{bmatrix} F_1F_6 \\ F_2F_3 \end{bmatrix}, \begin{bmatrix} F_2F_4 \\ F_6F_8 \end{bmatrix}, \begin{bmatrix} F_2F_8 \\ F_4F_6 \end{bmatrix}, \begin{bmatrix} F_3F_5 \\ F_6F_7 \end{bmatrix}, \begin{bmatrix} F_3F_7 \\ F_5F_6 \end{bmatrix}$$

The effects here are estimated using the same efficiency.

$$\hat{\mu} = \frac{1}{128}[\mu], \quad F_1\hat{F}_4 = \frac{1}{128}[F_1F_4], \quad F_3\hat{F}_8 = \frac{1}{128}[F_3F_8], \quad F_1\hat{F}_5 = \frac{1}{128}[F_1F_5], \quad F_2\hat{F}_7 = \frac{1}{128}[F_2F_7],$$

$$\hat{F_1F_7} = \frac{1}{128}[F_1F_7], \quad \hat{F_2F_5} = \frac{1}{128}[F_2F_5], \quad \hat{F_1F_8} = \frac{1}{128}[F_1F_8], \quad \hat{F_3F_4} = \frac{1}{128}[F_3F_4], \quad \hat{F_4F_5} = \frac{1}{128}[F_4F_5],$$

$$\hat{F_4F_7} = \frac{1}{128}[F_4F_7], \quad \hat{F_5F_8} = \frac{1}{128}[F_5F_8], \quad \hat{F_7F_8} = \frac{1}{128}[F_7F_8]$$

3.4 Nine Factor Experiment involving 96+32 = 128 Runs

Consider a design with treatments $(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9)$ which satisfy the simultaneous equations

$$\begin{aligned} x_1 + x_3 + x_6 + x_9 &= 0,0,1 \\ x_2 + x_5 + x_7 + x_9 &= 0,0,1 \\ x_4 + x_6 + x_7 + x_9 &= 0,1,0 \\ x_3 + x_5 + x_8 + x_9 &= 0,1,1 \end{aligned}$$

(mod 2)

3.4.1 Construction of The Design

The B matrix is obtained using the same procedure described in the eight factor experiment in the Two-Factor Interactions Designs.

We first get the treatment combinations belonging to the first set which we then repeat. If we add 1 (mod2) in x_4 and x_8 position of the first set, we obtain the second set. If we add 1 (mod2) in x_1 and x_5 position of the first set, we obtain the third set.

Treatment combinations

$\underline{st_1}$	$\underline{st_2}$	$\underline{st_3}$
$(0,0,0,0,0,0,0,0,0)$	$(0,0,0,1,0,0,0,1,0)$	$(1,0,0,0,1,0,0,0,0)$
$(0,0,0,0,1,1,0,0,1)$	$(0,0,0,1,1,1,0,1,1)$	$(1,0,0,0,0,1,0,0,1)$
$(0,0,0,1,0,1,1,1,1)$	$(0,0,0,0,0,1,1,0,1)$	$(1,0,0,1,1,1,1,1,1)$
$(0,0,0,1,1,0,1,1,0)$	$(0,0,0,0,1,0,1,0,0)$	$(1,0,0,1,0,0,1,1,0)$
$(0,0,1,1,0,1,0,1,0)$	$(0,0,1,0,0,1,0,0,0)$	$(1,0,1,1,1,1,0,1,0)$
$(0,0,1,1,1,0,0,1,1)$	$(0,0,1,0,1,0,0,0,1)$	$(1,0,1,1,0,0,0,1,1)$
$(0,0,1,0,0,0,1,0,1)$	$(0,0,1,1,0,0,1,1,1)$	$(1,0,1,0,1,0,1,0,1)$
$(0,0,1,0,1,1,1,0,0)$	$(0,0,1,1,1,1,1,1,0)$	$(1,0,1,0,0,1,1,0,0)$
$(0,1,0,0,0,1,0,1,1)$	$(0,1,0,1,0,1,0,0,1)$	$(1,1,0,0,1,1,0,1,1)$
$(0,1,0,0,1,0,0,1,0)$	$(0,1,0,1,1,0,0,0,0)$	$(1,1,0,0,0,0,0,1,0)$
$(0,1,0,1,0,0,1,0,0)$	$(0,1,0,0,0,0,1,1,0)$	$(1,1,0,1,1,0,1,0,0)$
$(0,1,0,1,1,1,1,0,1)$	$(0,1,0,0,1,1,1,1,1)$	$(1,1,0,1,0,1,1,0,1)$
$(0,1,1,1,0,0,0,0,1)$	$(0,1,1,0,0,0,0,1,1)$	$(1,1,1,1,1,0,0,0,1)$
$(0,1,1,1,1,0,0,0,0)$	$(0,1,1,0,1,0,0,1,0)$	$(1,1,1,1,0,0,0,0,0)$
$(0,1,1,0,0,1,1,1,0)$	$(0,1,1,1,0,1,1,0,0)$	$(1,1,1,0,1,1,1,1,0)$
$(0,1,1,0,1,0,1,1,1)$	$(0,1,1,1,1,0,1,0,1)$	$(1,1,1,0,0,0,1,1,1)$
$(1,1,0,1,0,0,0,1,1)$	$(1,1,0,0,0,0,0,0,1)$	$(0,1,0,1,1,0,0,1,1)$
$(1,1,0,1,1,1,0,1,0)$	$(1,1,0,0,1,1,0,0,0)$	$(0,1,0,1,0,1,0,1,0)$
$(1,1,0,0,0,1,1,0,0)$	$(1,1,0,1,0,1,1,1,0)$	$(0,1,0,0,1,1,1,0,0)$
$(1,1,0,0,1,0,1,0,1)$	$(1,1,0,1,1,0,1,1,1)$	$(0,1,0,0,0,0,1,0,1)$
$(1,1,1,1,0,0,1,1,0)$	$(1,1,1,0,0,0,1,0,0)$	$(0,1,1,1,1,0,1,1,0)$
$(1,1,1,1,1,1,1,1,1)$	$(1,1,1,0,1,1,1,0,1)$	$(0,1,1,1,0,1,1,1,1)$
$(1,1,1,0,0,1,0,0,1)$	$(1,1,1,1,0,1,0,1,1)$	$(0,1,1,0,1,1,0,0,1)$
$(1,1,1,0,1,0,0,0,0)$	$(1,1,1,1,1,0,0,1,0)$	$(0,1,1,0,0,0,0,0,0)$
$(1,0,1,0,0,0,0,1,0)$	$(1,0,1,1,0,0,0,0,0)$	$(0,0,1,0,1,0,0,1,0)$
$(1,0,1,0,1,1,0,1,1)$	$(1,0,1,1,1,1,0,0,1)$	$(0,0,1,0,0,1,0,1,1)$
$(1,0,1,1,0,1,1,0,1)$	$(1,0,1,0,0,1,1,1,1)$	$(0,0,1,1,1,1,1,0,1)$
$(1,0,1,1,1,0,1,0,0)$	$(1,0,1,0,1,0,1,1,0)$	$(0,0,1,1,0,0,1,0,0)$
$(1,0,0,0,0,0,1,1,1)$	$(1,0,0,1,0,0,1,0,1)$	$(0,0,0,0,1,0,1,1,1)$
$(1,0,0,0,1,1,1,1,0)$	$(1,0,0,1,1,1,1,0,0)$	$(0,0,0,0,0,1,1,1,0)$
$(1,0,0,1,0,1,0,0,0)$	$(1,0,0,0,0,1,0,1,0)$	$(0,0,0,1,1,1,0,0,0)$
$(1,0,0,1,1,0,0,0,1)$	$(1,0,0,0,1,0,0,1,1)$	$(0,0,0,1,0,0,0,0,1)$

The defining relation for the fraction is

$$
\begin{gathered}
I = F_1F_3F_6F_9 = F_2F_5F_7F_9 = F_4F_6F_7F_9 = F_3F_5F_8F_9 = F_1F_2F_3F_5F_6F_7 = F_1F_3F_4F_7 \\
= F_1F_5F_6F_8 = F_2F_4F_5F_6 = F_2F_3F_7F_8 = F_3F_4F_5F_6F_7F_8 = F_1F_2F_3F_4F_5F_9 = F_1F_2F_6F_7F_8F_9 \\
= F_1F_4F_5F_7F_8F_9 = F_2F_3F_4F_6F_8F_9 = F_1F_2F_4F_8
\end{gathered}
$$

The correlated sets of factors are

1. $(F_1, F_5F_6F_8, F_3F_6F_9, F_2F_4F_8, F_3F_4F_7)$, $(F_2, F_3F_7F_8, F_5F_7F_9, F_1F_4F_8, F_4F_5F_6)$, $(F_3, F_1F_6F_9, F_5F_8F_9, F_1F_4F_7, F_2F_7F_8)$, $(F_4, F_6F_7F_9, F_1F_3F_7, F_2F_5F_6, F_1F_2F_8)$ $(F_5, F_2F_7F_9, F_3F_8F_9, F_2F_4F_6, F_1F_6F_8)$, $(F_6, F_1F_3F_9, F_4F_7F_9, F_1F_5F_8, F_2F_4F_5)$, $(F_7, F_2F_5F_9, F_4F_6F_9, F_2F_3F_8, F_1F_3F_4)$, $(F_8, F_1F_2F_4, F_1F_5F_6, F_2F_3F_7, F_3F_5F_9)$, $(F_9, F_4F_6F_7, F_1F_3F_6, F_2F_5F_7, F_3F_5F_8)$, (F_1F_2, F_4F_8), (F_1F_3, F_6F_9, F_4F_7), (F_1F_4, F_2F_8, F_3F_7), (F_1F_5, F_6F_8), (F_3F_9, F_1F_6, F_5F_8), (F_1F_7, F_3F_4), (F_5F_6, F_1F_8, F_2F_4), (F_1F_9, F_3F_6), (F_2F_3, F_7F_8), (F_2F_6, F_4F_5), (F_5F_9, F_2F_7, F_3F_8), (F_2F_9, F_5F_7), (F_3F_5, F_8F_9), (F_4F_6, F_7F_9, F_2F_5), (F_4F_9, F_6F_7)
2. $(F_1F_2F_3, F_2F_6F_9, F_5F_6F_7, F_2F_4F_7, F_1F_7F_8, F_4F_5F_9, F_3F_4F_8)$, $(F_1F_2F_5, F_1F_7F_9, F_3F_6F_7, F_2F_6F_8, F_1F_4F_6, F_3F_4F_9, F_4F_5F_8)$, $(F_1F_2F_7, F_1F_5F_9, F_3F_5F_6, F_2F_3F_4, F_1F_3F_8, F_6F_8F_9, F_4F_7F_8)$, $(F_1F_3F_5, F_5F_6F_9, F_1F_8F_9, F_2F_6F_7, F_4F_5F_7, F_3F_6F_8, F_2F_4F_9)$, $(F_1F_4F_9, F_3F_4F_6, F_1F_6F_7, F_3F_7F_9, F_2F_3F_5, F_5F_7F_8, F_2F_8F_9)$.
3. $(F_1F_2F_9, F_2F_3F_6, F_1F_5F_7, F_3F_4F_5, F_6F_7F_8, F_4F_8F_9)$.
4. $(F_1F_2F_6, F_2F_3F_9, F_3F_5F_7, F_2F_5F_8, F_1F_4F_5, F_7F_8F_9, F_4F_6F_8)$.

3.4.2 Method of Analysis

Effects F_1F_7, F_3F_4, F_2F_6, F_4F_5, F_3F_5 and F_8F_9 are orthogonally estimated.

$$
\text{For } \begin{bmatrix} 128 & 64 & 64 & 64 & 0 \\ 64 & 128 & 0 & 0 & 64 \\ 64 & 0 & 128 & 128 & 64 \\ 64 & 0 & 128 & 128 & 64 \\ 0 & 64 & 64 & 64 & 128 \end{bmatrix} \text{, the pseudoinverse is } \frac{1}{12544} \begin{bmatrix} 59 & 18 & 0 & 0 & -39 \\ 18 & 52 & -16 & -16 & 18 \\ 0 & -16 & 20 & 20 & 0 \\ 0 & -16 & 20 & 20 & 0 \\ -39 & 18 & 0 & 0 & 59 \end{bmatrix} \quad \textbf{(3.11)}
$$

$$\text{For} \begin{bmatrix} 128 & 64 & 0 & 0 & 64 \\ 64 & 128 & 64 & 64 & 0 \\ 0 & 64 & 128 & 128 & 64 \\ 0 & 64 & 128 & 128 & 64 \\ 64 & 0 & 64 & 64 & 128 \end{bmatrix}, \text{ the pseudoinverse is } \frac{1}{12544} \begin{bmatrix} 52 & 18 & -16 & -16 & 18 \\ 18 & 59 & 0 & 0 & -39 \\ -16 & 0 & 20 & 20 & 0 \\ -16 & 0 & 20 & 20 & 0 \\ 18 & -39 & 0 & 0 & 59 \end{bmatrix} \quad \textbf{(3.12)}$$

Equation (3.11) estimates effects $\begin{bmatrix} F_1 \\ F_5F_6F_8 \\ F_3F_6F_9 \\ F_2F_4F_8 \\ F_3F_4F_7 \end{bmatrix}, \begin{bmatrix} F_2 \\ F_3F_6F_8 \\ F_5F_7F_9 \\ F_1F_4F_8 \\ F_4F_5F_6 \end{bmatrix}, \begin{bmatrix} F_6 \\ F_1F_3F_9 \\ F_4F_7F_9 \\ F_1F_5F_8 \\ F_2F_4F_5 \end{bmatrix}, \begin{bmatrix} F_7 \\ F_2F_5F_9 \\ F_4F_6F_9 \\ F_2F_3F_8 \\ F_1F_3F_4 \end{bmatrix}, \begin{bmatrix} F_8 \\ F_1F_2F_4 \\ F_1F_5F_6 \\ F_2F_3F_7 \\ F_3F_5F_9 \end{bmatrix}, \begin{bmatrix} F_9 \\ F_4F_6F_7 \\ F_1F_3F_6 \\ F_2F_5F_7 \\ F_3F_5F_8 \end{bmatrix}$

The main effects estimated using equation (3.11) are estimated with the lowest efficiency efficiency. The effects $F_3F_6F_9$, $F_2F_4F_8$, $F_5F_7F_9$, $F_1F_4F_8$, $F_4F_7F_9$, $F_1F_5F_8$, $F_4F_6F_9$, $F_2F_3F_8$, $F_1F_5F_6$, $F_2F_3F_7$, $F_1F_3F_6$ and $F_2F_5F_7$ are estimated with the highest efficiency.

Equation (3.12) is used to estimate the effects $\begin{bmatrix} F_3 \\ F_1F_6F_9 \\ F_5F_8F_9 \\ F_1F_4F_7 \\ F_2F_7F_8 \end{bmatrix}, \begin{bmatrix} F_4 \\ F_6F_7F_9 \\ F_1F_3F_7 \\ F_2F_5F_6 \\ F_1F_2F_8 \end{bmatrix}, \begin{bmatrix} F_5 \\ F_2F_7F_9 \\ F_3F_8F_9 \\ F_2F_4F_6 \\ F_1F_6F_8 \end{bmatrix}$

The main effects estimated using equation (3.12) are estimated with the same efficiency. The effects $F_5F_8F_9$, $F_1F_4F_7$, $F_1F_3F_7$, $F_2F_5F_6$, $F_3F_8F_9$ and $F_2F_4F_6$ are estimated with the highest efficiency. Effects $F_1F_6F_9$, $F_2F_7F_8$, $F_6F_7F_9$, $F_1F_2F_8$, $F_2F_7F_9$ and $F_1F_6F_8$ are estimated with the lowest efficiency.

The matrix:

$$\begin{bmatrix} 128 & 64 & 0 \\ 64 & 128 & 64 \\ 0 & 64 & 128 \end{bmatrix}^{-1} = \frac{1}{256} \begin{bmatrix} 3 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 3 \end{bmatrix} \text{ estimates } \begin{bmatrix} F_1F_3 \\ F_6F_9 \\ F_4F_7 \end{bmatrix}, \begin{bmatrix} F_4F_6 \\ F_7F_9 \\ F_2F_5 \end{bmatrix}, \begin{bmatrix} F_1F_4 \\ F_2F_8 \\ F_3F_7 \end{bmatrix}, \begin{bmatrix} F_3F_9 \\ F_1F_6 \\ F_5F_8 \end{bmatrix}, \begin{bmatrix} F_5F_6 \\ F_1F_8 \\ F_2F_4 \end{bmatrix}$$

$$\text{and} \begin{bmatrix} F_5F_9 \\ F_2F_7 \\ F_3F_8 \end{bmatrix}$$

Effects F_1F_3,F_4F_7, F_4F_6, F_2F_5, F_1F_4, F_3F_7, F_3F_9, F_5F_8, F_5F_6, F_2F_4, F_5F_9 and F_3F_8 are estimated with a higher efficiency than the efficiency attained for effects F_6F_9, F_7F_9, F_2F_8, F_1F_6, F_1F_8 and F_2F_7.

The matrix:

$$\begin{bmatrix} 128 & 64 \\ 64 & 128 \end{bmatrix}^{-1} = \frac{1}{192}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \text{ estimates } \begin{bmatrix} F_1F_2 \\ F_4F_8 \end{bmatrix}, \begin{bmatrix} F_1F_5 \\ F_6F_8 \end{bmatrix}, \begin{bmatrix} F_1F_9 \\ F_3F_6 \end{bmatrix}, \begin{bmatrix} F_2F_3 \\ F_7F_8 \end{bmatrix}, \begin{bmatrix} F_2F_9 \\ F_5F_7 \end{bmatrix}, \begin{bmatrix} F_4F_9 \\ F_6F_7 \end{bmatrix}$$

The effects here are estimated using the same efficiency.

For $\begin{bmatrix} 128 & 64 & 128 & 0 & 64 & 64 & 64 \\ 64 & 128 & 64 & 64 & 0 & 0 & 128 \\ 128 & 64 & 128 & 0 & 64 & 64 & 64 \\ 0 & 64 & 0 & 128 & 64 & 64 & 64 \\ 64 & 0 & 64 & 64 & 128 & 128 & 0 \\ 64 & 0 & 64 & 64 & 128 & 128 & 0 \\ 64 & 128 & 64 & 64 & 0 & 0 & 128 \end{bmatrix}$

, the pseudoinverse is $\frac{1}{5120}\begin{bmatrix} 8 & 0 & 8 & -8 & 0 & 0 & 0 \\ 0 & 7 & 0 & 4 & -3 & -3 & 7 \\ 8 & 0 & 8 & -8 & 0 & 0 & 0 \\ -8 & 4 & -8 & 16 & 4 & 4 & 4 \\ 0 & -3 & 0 & 4 & 7 & 7 & -3 \\ 0 & -3 & 0 & 4 & 7 & 7 & -3 \\ 0 & 7 & 0 & 4 & -3 & -3 & 7 \end{bmatrix}$ **(3.13)**

The matrix in equation (3.13) estimates effects in set 2. Effects $F_2F_6F_9$, $F_1F_7F_8$, $F_4F_5F_9$ $F_3F_4F_8$, $F_1F_7F_9$, $F_2F_6F_8$ $F_3F_4F_9$, $F_4F_5F_8$, $F_1F_2F_7$, $F_2F_3F_4$, $F_6F_8F_9$, $F_3F_5F_6$, $F_1F_3F_5$, $F_1F_8F_9$, $F_4F_5F_7$ $F_2F_6F_7$, $F_3F_4F_6$, $F_1F_6F_7$, $F_2F_3F_5$ and $F_2F_8F_9$ are estimated with the highest efficiency compared to any other effect in the corresponding sets. Effects $F_2F_4F_7$, $F_5F_6F_9$, $F_3F_7F_9$,$F_1F_4F_6$ and $F_1F_3F_8$ are estimated with the lowest efficiency. Effects $F_1F_2F_3$, $F_5F_6F_7$, $F_1F_2F_5$ $F_3F_6F_7$, $F_1F_5F_9$, $F_4F_7F_8$ $F_3F_6F_8$, $F_2F_4F_9$, $F_1F_4F_9$ and $F_5F_7F_8$ are estimated with the same efficiency.

For $\begin{bmatrix} 128 & 64 & 0 & 64 & 0 & 0 & 64 \\ 64 & 128 & 0 & 0 & 64 & 64 & 128 \\ 0 & 0 & 128 & 0 & 0 & 0 & 0 \\ 64 & 0 & 0 & 128 & 64 & 64 & 0 \\ 0 & 64 & 0 & 64 & 128 & 128 & 64 \\ 0 & 64 & 0 & 64 & 128 & 128 & 64 \\ 64 & 128 & 0 & 0 & 64 & 64 & 128 \end{bmatrix}$

, the pseudoinverse is $\frac{1}{36864}\begin{bmatrix} 140 & 0 & 0 & 76 & -50 & -50 & 0 \\ 0 & 59 & 0 & -50 & 0 & 0 & 59 \\ 0 & 0 & 288 & 0 & 0 & 0 & 0 \\ 76 & -50 & 0 & 140 & 0 & 0 & -50 \\ -50 & 0 & 0 & 0 & 59 & 59 & 0 \\ -50 & 0 & 0 & 0 & 59 & 59 & 0 \\ 0 & 59 & 0 & -50 & 0 & 0 & 59 \end{bmatrix}$ **(3.14)**

The matrix in equation (3.14) estimates effects in set 4. Effect $F_3F_5F_7$ is estimated with the lowest efficiency. Effects $F_2F_3F_9$, $F_1F_4F_5$, $F_7F_8F_9$ and $F_4F_6F_8$ are estimated with the highest efficiency. $F_1F_2F_6$ and $F_2F_5F_8$ are estimated with the same efficiency.

For $\begin{bmatrix} 128 & 64 & 64 & 64 & 0 & 64 \\ 64 & 128 & 128 & 0 & 64 & 128 \\ 64 & 128 & 128 & 0 & 64 & 128 \\ 64 & 0 & 0 & 128 & 64 & 0 \\ 0 & 64 & 64 & 64 & 128 & 64 \\ 64 & 128 & 128 & 0 & 64 & 128 \end{bmatrix}$

, the pseudoinverse is $\frac{1}{1280}\begin{bmatrix} 6 & 0 & 0 & 2 & -4 & 0 \\ 0 & 1 & 1 & -1 & 0 & 1 \\ 0 & 1 & 1 & -1 & 0 & 1 \\ 2 & -1 & -1 & 5 & 2 & -1 \\ -4 & 0 & 0 & 2 & 6 & 0 \\ 0 & 1 & 1 & -1 & 0 & 1 \end{bmatrix}$ **(3.15)**

The matrix in equation (3.15) estimates effects in set 3. Effects $F_1F_2F_9$ and $F_6F_7F_8$ are estimated with the lowest efficiency. Effects $F_2F_3F_6$, $F_1F_5F_7$ and $F_4F_8F_9$ are estimated with the highest efficiency.

$$\hat{\mu} = \frac{1}{128}[\mu], \quad F_1\hat{F}_7 = \frac{1}{128}[F_1F_7], \quad F_3\hat{F}_4 = \frac{1}{128}[F_3F_4], \quad F_2\hat{F}_6 = \frac{1}{128}[F_2F_6], \quad F_4\hat{F}_5 = \frac{1}{128}[F_4F_5],$$

$$F_3\hat{F}_5 = \frac{1}{128}[F_3F_5], \quad F_8\hat{F}_9 = \frac{1}{128}[F_8F_9].$$

3.5 Ten Factor Experiment involving 96+32 = 128 Runs

Consider a design with treatments $(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10})$ which satisfy the simultaneous equations

$$\begin{aligned} x_1 + x_3 + x_6 + x_{10} &= 0,1,0 \\ x_2 + x_5 + x_7 + x_9 &= 0,0,1 \\ x_4 + x_6 + x_7 + x_9 &= 0,1,0 \\ x_3 + x_5 + x_8 + x_{10} &= 0,0,1 \\ x_1 + x_5 + x_9 + x_{10} &= 0,1,1 \end{aligned}$$

(mod 2)

3.5.1 Construction of The Design

The B matrix is obtained using the same procedure described in the eight factor experiment in the Two-Factor Interactions Designs.

We first get the treatment combinations belonging to the first set which we then repeat. If we add 1 (mod2) in x_1 and x_4 position of the first set, we obtain the second set. If we add 1 (mod2) in the x_5 position of the first set, we obtain the third set.

Treatment combinations

$\underline{st_1}$	$\underline{st_2}$	$\underline{st_3}$
$(0,0,0,0,0,0,0,0,0,0)$	$(1,0,0,1,0,0,0,0,0,0)$	$(0,0,0,0,1,0,0,0,0,0)$
$(0,0,0,0,1,1,1,0,0,1)$	$(1,0,0,1,1,1,1,0,0,1)$	$(0,0,0,0,0,1,1,0,0,1)$
$(0,0,0,1,0,1,1,1,1,1)$	$(1,0,0,0,0,1,1,1,1,1)$	$(0,0,0,1,1,1,1,1,1,1)$
$(0,0,0,1,1,0,0,1,1,0)$	$(1,0,0,0,1,0,0,1,1,0)$	$(0,0,0,1,0,0,0,1,1,0)$
$(0,0,1,1,0,1,0,1,0,0)$	$(1,0,1,0,0,1,0,1,0,0)$	$(0,0,1,1,1,1,0,1,0,0)$
$(0,0,1,1,1,0,1,1,0,1)$	$(1,0,1,0,1,0,1,1,0,1)$	$(0,0,1,1,0,0,1,1,0,1)$
$(0,0,1,0,0,0,1,0,1,1)$	$(1,0,1,1,0,0,1,0,1,1)$	$(0,0,1,0,1,0,1,0,1,1)$
$(0,0,1,0,1,1,0,0,1,0)$	$(1,0,1,1,1,1,0,0,1,0)$	$(0,0,1,0,0,1,0,0,1,0)$
$(0,1,0,0,0,1,0,1,1,1)$	$(1,1,0,1,0,1,0,1,1,1)$	$(0,1,0,0,1,1,0,1,1,1)$
$(0,1,0,0,1,0,1,1,1,0)$	$(1,1,0,1,1,0,1,1,1,0)$	$(0,1,0,0,0,0,1,1,1,0)$
$(0,1,0,1,0,0,1,0,0,0)$	$(1,1,0,0,0,0,1,0,0,0)$	$(0,1,0,1,1,0,1,0,0,0)$
$(0,1,0,1,1,1,0,0,0,1)$	$(1,1,0,0,1,1,0,0,0,1)$	$(0,1,0,1,0,1,0,0,0,1)$
$(0,1,1,1,0,0,0,0,1,1)$	$(1,1,1,0,0,0,0,0,1,1)$	$(0,1,1,1,1,0,0,0,1,1)$
$(0,1,1,1,1,1,1,0,1,0)$	$(1,1,1,0,1,1,1,0,1,0)$	$(0,1,1,1,0,1,1,0,1,0)$
$(0,1,1,0,0,1,1,1,0,0)$	$(1,1,1,1,0,1,1,1,0,0)$	$(0,1,1,0,1,1,1,1,0,0)$
$(0,1,1,0,1,0,0,1,0,1)$	$(1,1,1,1,1,0,0,1,0,1)$	$(0,1,1,0,0,0,0,1,0,1)$
$(1,1,0,1,0,0,1,1,0,1)$	$(0,1,0,0,0,0,1,1,0,1)$	$(1,1,0,1,1,0,1,1,0,1)$
$(1,1,0,1,1,1,0,1,0,0)$	$(0,1,0,0,1,1,0,1,0,0)$	$(1,1,0,1,0,1,0,1,0,0)$
$(1,1,0,0,0,1,0,0,1,0)$	$(0,1,0,1,0,1,0,0,1,0)$	$(1,1,0,0,1,1,0,0,1,0)$
$(1,1,0,0,1,0,1,0,1,1)$	$(0,1,0,1,1,0,1,0,1,1)$	$(1,1,0,0,0,0,1,0,1,1)$
$(1,1,1,1,0,0,0,1,1,0)$	$(0,1,1,0,0,0,0,1,1,0)$	$(1,1,1,1,1,0,0,1,1,0)$
$(1,1,1,1,1,1,1,1,1,1)$	$(0,1,1,0,1,1,1,1,1,1)$	$(1,1,1,1,0,1,1,1,1,1)$
$(1,1,1,0,0,1,1,0,0,1)$	$(0,1,1,1,0,1,1,0,0,1)$	$(1,1,1,0,1,1,1,0,0,1)$
$(1,1,1,0,1,0,0,0,0,0)$	$(0,1,1,1,1,0,0,0,0,0)$	$(1,1,1,0,0,0,0,0,0,0)$
$(1,0,1,0,0,0,1,1,1,0)$	$(0,0,1,1,0,0,1,1,1,0)$	$(1,0,1,0,1,0,1,1,1,0)$
$(1,0,1,0,1,1,0,1,1,1)$	$(0,0,1,1,1,1,0,1,1,1)$	$(1,0,1,0,0,1,0,1,1,1)$
$(1,0,1,1,0,1,0,0,0,1)$	$(0,0,1,0,0,1,0,0,0,1)$	$(1,0,1,1,1,1,0,0,0,1)$
$(1,0,1,1,1,0,1,0,0,0)$	$(0,0,1,0,1,0,1,0,0,0)$	$(1,0,1,1,0,0,1,0,0,0)$
$(1,0,0,0,0,0,0,1,0,1)$	$(0,0,0,1,0,0,0,1,0,1)$	$(1,0,0,0,1,0,0,1,0,1)$
$(1,0,0,0,1,1,1,1,0,0)$	$(0,0,0,1,1,1,1,1,0,0)$	$(1,0,0,0,0,1,1,1,0,0)$
$(1,0,0,1,0,1,1,0,1,0)$	$(0,0,0,0,0,1,1,0,1,0)$	$(1,0,0,1,1,1,1,0,1,0)$
$(1,0,0,1,1,0,0,0,1,1)$	$(0,0,0,0,1,0,0,0,1,1)$	$(1,0,0,1,0,0,0,0,1,1)$

The defining relation for the fraction is

$$I = F_1F_3F_6F_{10} = F_2F_5F_7F_9 = F_4F_6F_7F_9 = F_3F_5F_8F_{10} = F_1F_5F_9F_{10} = F_1F_3F_4F_7F_9F_{10}$$
$$= F_1F_5F_6F_8 = F_3F_5F_6F_9 = F_2F_4F_5F_6 = F_2F_3F_7F_8F_9F_{10} = F_1F_2F_7F_{10} = F_1F_4F_5F_6F_7F_{10}$$
$$= F_1F_3F_8F_9 = F_1F_2F_3F_4F_5F_{10} = F_1F_2F_6F_7F_8F_9 = F_2F_3F_6F_7 = F_1F_4F_5F_7F_8F_9 = F_3F_4F_5F_7$$
$$= F_6F_8F_9F_{10} = F_2F_3F_4F_6F_8F_{10} = F_1F_2F_4F_6F_9F_{10} = F_1F_2F_3F_5F_7F_8 = F_1F_3F_4F_6F_7F_8$$
$$= F_1F_2F_4F_8 = F_2F_3F_4F_9 = F_2F_5F_6F_7F_8F_{10} = F_4F_7F_8F_{10} = F_2F_4F_5F_8F_9F_{10}$$

The correlated set of factors are

1. $(F_1, F_2F_4F_8, F_2F_7F_{10}, F_3F_6F_{10}, F_3F_8F_9, F_5F_6F_8, F_5F_9F_{10})$,
   $(F_4, F_1F_2F_8, F_2F_3F_9, F_6F_7F_9, F_7F_8F_{10}, F_2F_5F_6, F_3F_5F_7)$
2. $(F_2, F_1F_4F_8, F_1F_7F_{10}, F_3F_4F_9, F_3F_6F_7, F_4F_5F_6, F_5F_7F_9)$,
   $(F_7, F_2F_5F_9, F_4F_6F_9, F_1F_2F_{10}, F_2F_3F_6, F_3F_4F_5, F_4F_8F_{10})$,
   $(F_8, F_3F_5F_{10}, F_1F_5F_6, F_1F_3F_9, F_6F_9F_{10}, F_1F_2F_4, F_4F_7F_{10})$,
   $(F_{10}, F_1F_2F_7, F_1F_3F_6, F_1F_5F_9, F_3F_5F_8, F_4F_7F_8, F_6F_8F_9)$.
3. $(F_3, F_1F_6F_{10}, F_1F_8F_9, F_2F_4F_9, F_2F_6F_7, F_4F_5F_7, F_5F_6F_9, F_5F_8F_{10})$,
   $(F_6, F_1F_3F_{10}, F_4F_7F_9, F_1F_5F_8, F_3F_5F_9, F_2F_4F_5, F_2F_3F_7, F_8F_9F_{10})$.
4. $(F_5, F_1F_6F_8, F_1F_9F_{10}, F_2F_4F_6, F_2F_7F_9, F_3F_4F_7, F_3F_6F_9, F_3F_8F_{10})$.
5. $(F_9, F_1F_3F_8, F_1F_5F_{10}, F_2F_3F_4, F_2F_5F_7, F_3F_5F_6, F_4F_6F_7, F_6F_8F_{10})$.
6. $(F_1F_2F_3, F_1F_4F_9, F_1F_6F_7, F_2F_6F_{10}, F_2F_8F_9, F_3F_4F_8, F_3F_7F_{10}, F_4F_5F_{10}, F_5F_7F_8)$.
7. $(F_1F_2F_5, F_1F_4F_6, F_1F_7F_9, F_2F_6F_8, F_2F_9F_{10}, F_3F_4F_{10}, F_3F_7F_8, F_4F_5F_8, F_5F_7F_{10})$.
8. $(F_1F_3F_5, F_1F_4F_7, F_1F_6F_9, F_1F_8F_{10}, F_2F_4F_{10}, F_2F_7F_8, F_3F_6F_8, F_3F_9F_{10}, F_5F_6F_{10}, F_5F_8F_9)$.
9. $(F_1F_2, F_4F_8, F_7F_{10})$, $(F_6F_{10}, F_8F_9, F_1F_3)$, (F_2F_3, F_6F_7, F_4F_9).
10. $(F_1F_6, F_3F_{10}, F_5F_8)$, $(F_1F_9, F_3F_8, F_5F_{10})$, (F_2F_5, F_7F_9, F_4F_6), (F_3F_4, F_2F_9, F_5F_7).
11. $(F_1F_8, F_2F_4, F_3F_9, F_5F_6)$.
12. $(F_1F_{10}, F_2F_7, F_3F_6, F_5F_9)$, $(F_3F_5, F_6F_9, F_8F_{10}, F_4F_7)$.
13. (F_1F_4, F_2F_8), (F_6F_8, F_9F_{10}).
14. (F_1F_7, F_2F_{10}), (F_4F_{10}, F_7F_8).
15. (F_2F_6, F_3F_7, F_4F_5).

3.5.2 Method of Analysis

Effects F_1F_7, F_3F_4, F_2F_6, F_4F_5, F_3F_5 and F_8F_9 are orthogonally estimated.

For $\begin{bmatrix} 128 & 128 & 64 & 64 & 64 & 0 & 0 \\ 128 & 128 & 64 & 64 & 64 & 0 & 0 \\ 64 & 64 & 128 & 128 & 128 & 64 & 64 \\ 64 & 64 & 128 & 128 & 128 & 64 & 64 \\ 64 & 64 & 128 & 128 & 128 & 64 & 64 \\ 0 & 0 & 64 & 64 & 64 & 128 & 128 \\ 0 & 0 & 64 & 64 & 64 & 128 & 128 \end{bmatrix}$, the pseudoinverse is

$$\frac{1}{1024}\begin{bmatrix} 3 & 3 & -1 & -1 & -1 & 1 & 1 \\ 3 & 3 & -1 & -1 & -1 & 1 & 1 \\ -1 & -1 & 2 & 2 & 2 & -1 & -1 \\ -1 & -1 & 2 & 2 & 2 & -1 & -1 \\ -1 & -1 & 2 & 2 & 2 & -1 & -1 \\ 1 & 1 & -1 & -1 & -1 & 3 & 3 \\ 1 & 1 & -1 & -1 & -1 & 3 & 3 \end{bmatrix} \quad \textbf{(3.16)}$$

The matrix in equation (3.16) estimates effects in set 1. Effects $F_2F_7f_{10}, F_3F_6F_{10}$, $F_3F_8F_9$, $F_2F_3F_9$, $F_6F_7F_9$ and $F_7F_8F_{10}$ are estimated with a higher efficiency than the efficiency attained for corresponding effects in the same sets. Factors F_1, $F_2F_4F_8$, $F_5F_6F_8$, $F_5F_9F_{10}$, F_4, $F_1F_2F_8$, $F_2F_5F_6$ and $F_3F_5F_7$ are estimated with the same efficiency which is lower than the efficiency attained for $F_2F_7f_{10}, F_3F_6F_{10}$, $F_3F_8F_9$, $F_2F_3F_9$, $F_6F_7F_9$ and $F_7F_8F_{10}$

For $\begin{bmatrix} 128 & 128 & 64 & 64 & 128 & 0 & 64 \\ 128 & 128 & 64 & 64 & 128 & 0 & 64 \\ 64 & 64 & 128 & 128 & 64 & 64 & 0 \\ 64 & 64 & 128 & 128 & 64 & 64 & 0 \\ 128 & 128 & 64 & 64 & 128 & 0 & 64 \\ 0 & 0 & 64 & 64 & 0 & 128 & 64 \\ 64 & 64 & 0 & 0 & 64 & 64 & 128 \end{bmatrix}$, the pseudoinverse is

$$\frac{1}{73984}\begin{bmatrix} 59 & 59 & 0 & 0 & 59 & -61 & 0 \\ 59 & 59 & 0 & 0 & 59 & -61 & 0 \\ 0 & 0 & 120 & 120 & 0 & 0 & -100 \\ 0 & 0 & 120 & 120 & 0 & 0 & -100 \\ 59 & 59 & 0 & 0 & 59 & -61 & 0 \\ -61 & -61 & 0 & 0 & -61 & 259 & 166 \\ 0 & 0 & -100 & -100 & 0 & 166 & 276 \end{bmatrix} \quad \textbf{(3.17)}$$

The matrix in equation (3.17) is used to estimate effects in set 2. The effects F_2, $F_1F_4F_8$, $F_3F_6F_7$, $F_1F_2F_{10}$, $F_4F_6F_9$, $F_4F_8F_{10}$, F_8, $F_1F_2F_4$, $F_6F_9F_{10}$, $F_1F_2F_7$, $F_1F_3F_6$ and $F_4F_7F_8$ are estimated with the highest efficiency. Effects $F_5F_7F_9$, $F_3F_4F_5$,$F_3F_5F_{10}$ and $F_1F_5F_9$ are estimated with the lowest efficiency. Effects $F_4F_5F_6$, $F_2F_5F_9$,$F_1F_5F_6$ and $F_3F_5F_8$ are estimated with the same efficiency.

For $\begin{bmatrix} 128 & 64 & 64 & 64 & 128 & 0 & 64 & 64 \\ 64 & 128 & 128 & 128 & 64 & 64 & 0 & 0 \\ 64 & 128 & 128 & 128 & 64 & 64 & 0 & 0 \\ 64 & 128 & 128 & 128 & 64 & 64 & 0 & 0 \\ 128 & 64 & 64 & 64 & 128 & 0 & 64 & 64 \\ 0 & 64 & 64 & 64 & 0 & 128 & 64 & 64 \\ 64 & 0 & 0 & 0 & 64 & 64 & 128 & 128 \\ 64 & 0 & 0 & 0 & 64 & 64 & 128 & 128 \end{bmatrix}$, the pseudoinverse is

$$\frac{1}{28672}\begin{bmatrix} 45 & 0 & 0 & 0 & 45 & -46 & 0 & 0 \\ 0 & 20 & 20 & 20 & 0 & 0 & 0 & 0 \\ 0 & 20 & 20 & 20 & 0 & 0 & 0 & 0 \\ 0 & 20 & 20 & 20 & 0 & 0 & 0 & 0 \\ 45 & 0 & 0 & 0 & 45 & -46 & 0 & 0 \\ -46 & 0 & 0 & 0 & -46 & 84 & 26 & 26 \\ 0 & 0 & 0 & 0 & 0 & 26 & 37 & 37 \\ 0 & 0 & 0 & 0 & 0 & 26 & 37 & 37 \end{bmatrix} \quad \textbf{(3.18)}$$

The matrix in equation (3.18) is used to estimate effects in set 3. The effects $F_1F_6F_{10}$, $F_1F_8F_9$, $F_2F_4F_9$, F_6, $F_8F_9F_{10}$ and $F_2F_3F_7$ are estimated with the highest efficiency compared to the efficiency attained for other effects in corresponding sets. Effects $F_5F_6F_9$,

$F_5F_8F_{10}$, $F_2F_4F_5$ and $F_1F_5F_8$ are estimated with the same efficiency. The effects F_3, $F_2F_6F_7$, $F_1F_3F_{10}$ and $F_4F_7F_9$ are estimated with the same efficiency. Effects $F_3F_5F_9$ and $F_4F_5F_7$ are estimated with the lowest efficiency.

For $\begin{bmatrix} 128 & 0 & 0 & 0 & 64 & 0 & 64 & 64 \\ 0 & 128 & 128 & 128 & 64 & 128 & 64 & 64 \\ 0 & 128 & 128 & 128 & 64 & 128 & 64 & 64 \\ 0 & 128 & 128 & 128 & 64 & 128 & 64 & 64 \\ 64 & 64 & 64 & 64 & 128 & 64 & 128 & 128 \\ 0 & 128 & 128 & 128 & 64 & 128 & 64 & 64 \\ 64 & 64 & 64 & 64 & 128 & 64 & 128 & 128 \\ 64 & 64 & 64 & 64 & 128 & 64 & 128 & 128 \end{bmatrix}$, the pseudoinverse is

$$\frac{1}{4096}\begin{bmatrix} 48 & 4 & 4 & 4 & -11 & 4 & -11 & -11 \\ 4 & 3 & 3 & 3 & -3 & 3 & -3 & -3 \\ 4 & 3 & 3 & 3 & -3 & 3 & -3 & -3 \\ 4 & 3 & 3 & 3 & -3 & 3 & -3 & -3 \\ -11 & -3 & -3 & -3 & 7 & -3 & 7 & 7 \\ 4 & 3 & 3 & 3 & -3 & 3 & -3 & -3 \\ -11 & -3 & -3 & -3 & 7 & -3 & 7 & 7 \\ -11 & -3 & -3 & -3 & 7 & -3 & 7 & 7 \end{bmatrix} \quad \textbf{(3.19)}$$

The matrix in equation (3.19) is used to estimate effects in set 4. The effects $F_1F_6F_8$, $F_1F_9F_{10}$, $F_2F_4F_6$ and $F_3F_4F_7$ are estimated with the highest efficiency compared to the efficiency attained for other effects in the same set. Effects $F_2F_7F_9$, $F_3F_6F_9$ and $F_3F_8F_{10}$ are estimated with the same efficiency. The main effect F_5 is estimated with the lowest efficiency compared to the efficiency attained for other effects in the same set.

For $\begin{bmatrix} 128 & 64 & 0 & 64 & 64 & 64 & 64 & 0 \\ 64 & 128 & 64 & 128 & 0 & 0 & 128 & 0 \\ 0 & 64 & 128 & 64 & 64 & 64 & 64 & 0 \\ 64 & 128 & 64 & 128 & 0 & 0 & 128 & 0 \\ 64 & 0 & 64 & 0 & 128 & 128 & 0 & 0 \\ 64 & 0 & 64 & 0 & 128 & 128 & 0 & 0 \\ 64 & 128 & 64 & 128 & 0 & 0 & 128 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 128 \end{bmatrix}$, the pseudoinverse is

$$\frac{1}{73984}\begin{bmatrix} 315 & 0 & -263 & 0 & 38 & 38 & 0 & 0 \\ 0 & 52 & 0 & 52 & 0 & 0 & 52 & 0 \\ -263 & 0 & 315 & 0 & 38 & 38 & 0 & 0 \\ 0 & 52 & 0 & 52 & 0 & 0 & 52 & 0 \\ 38 & 0 & 38 & 0 & 100 & 100 & 0 & 0 \\ 38 & 0 & 38 & 0 & 100 & 100 & 0 & 0 \\ 0 & 52 & 0 & 52 & 0 & 0 & 52 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 578 \end{bmatrix} \quad \textbf{(3.20)}$$

The matrix in equation (3.20) is used to estimate effects in set 5. The effects $F_1F_3F_8$, $F_2F_3F_4$ and $F_4F_6F_7$ are estimated with the highest efficiency compared to the efficiency attained for other effects in the same set. Effects $F_2F_5F_7$ and $F_3F_5F_6$ are estimated with the same efficiency. Effects F_9 and $F_1F_5F_{10}$ are estimated with the same efficiency. Effect $F_6F_8F_{10}$ is estimated with the lowest efficiency in comparison to the efficiency attained for other effects in the same set.

For $\begin{bmatrix} 128 & 64 & 128 & 64 & 0 & 128 & 64 & 64 & 0 \\ 64 & 128 & 64 & 128 & 0 & 64 & 128 & 0 & 64 \\ 128 & 64 & 128 & 64 & 0 & 128 & 64 & 64 & 0 \\ 64 & 128 & 64 & 128 & 0 & 64 & 128 & 0 & 64 \\ 0 & 0 & 0 & 0 & 128 & 0 & 0 & 0 & 0 \\ 128 & 64 & 128 & 64 & 0 & 128 & 64 & 64 & 0 \\ 64 & 128 & 64 & 128 & 0 & 64 & 128 & 0 & 64 \\ 64 & 0 & 64 & 0 & 0 & 64 & 0 & 128 & 64 \\ 0 & 64 & 0 & 64 & 0 & 0 & 64 & 64 & 128 \end{bmatrix}$, the pseudoinverse is

$$\frac{1}{6144}\begin{bmatrix} 5 & 0 & 5 & 0 & 0 & 5 & 0 & 0 & -5 \\ 0 & 5 & 0 & 5 & 0 & 0 & 5 & -5 & 0 \\ 5 & 0 & 5 & 0 & 0 & 5 & 0 & 0 & -5 \\ 0 & 5 & 0 & 5 & 0 & 0 & 5 & -5 & 0 \\ 0 & 0 & 0 & 0 & 48 & 0 & 0 & 0 & 0 \\ 5 & 0 & 5 & 0 & 0 & 5 & 0 & 0 & -5 \\ 0 & 5 & 0 & 5 & 0 & 0 & 5 & -5 & 0 \\ 0 & -5 & 0 & -5 & 0 & 0 & -5 & 21 & 15 \\ -5 & 0 & -5 & 0 & 0 & -5 & 0 & 15 & 21 \end{bmatrix} \quad \textbf{(3.21)}$$

The matrix in equation (3.21) is used to estimate effects in set 6. The effects $F_1F_2F_3$, $F_1F_4F_9$, $F_1F_6F_7$, $F_2F_6F_{10}$, $F_3F_4F_8$ and $F_3F_7F_{10}$ are estimated with the highest efficiency compared to the efficiency attained for other effects in the same set. Effects $F_4F_5F_{10}$ and $F_5F_7F_8$ are estimated with the same efficiency. $F_2F_8F_9$ is estimated with the lowest efficiency in comparison to the efficiency attained for other effects in the same set.

For $\begin{bmatrix} 128 & 0 & 64 & 0 & 0 & 64 & 0 & 128 & 64 \\ 0 & 128 & 64 & 128 & 128 & 64 & 128 & 0 & 64 \\ 64 & 64 & 128 & 64 & 64 & 128 & 64 & 64 & 0 \\ 0 & 128 & 64 & 128 & 128 & 64 & 128 & 0 & 64 \\ 0 & 128 & 64 & 128 & 128 & 64 & 128 & 0 & 64 \\ 64 & 64 & 128 & 64 & 64 & 128 & 64 & 64 & 0 \\ 0 & 128 & 64 & 128 & 128 & 64 & 128 & 0 & 64 \\ 128 & 0 & 64 & 0 & 0 & 64 & 0 & 128 & 64 \\ 64 & 64 & 0 & 64 & 64 & 0 & 64 & 64 & 128 \end{bmatrix}$, the pseudoinverse is

$$\frac{1}{82944}\begin{bmatrix} 104 & 0 & 0 & 0 & 0 & 0 & 0 & 104 & 80 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 131 & 0 & 0 & 131 & 0 & 0 & -134 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 131 & 0 & 0 & 131 & 0 & 0 & -134 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 104 & 0 & 0 & 0 & 0 & 0 & 0 & 104 & 80 \\ 80 & 0 & -134 & 0 & 0 & -134 & 0 & 80 & 236 \end{bmatrix} \quad \textbf{(3.22)}$$

The matrix in equation (3.22) is used to estimate effects in set 7. The effects $F_1F_4F_6$, $F_2F_6F_8$, $F_2F_9F_{10}$ and $F_3F_7F_8$ are estimated with the highest efficiency compared to the efficiency attained for other effects in the same set. Effects $F_1F_2F_5$ and $F_4F_5F_8$ are estimated with the same efficiency. $F_2F_8F_9$ is estimated with the lowest efficiency. Effects $F_1F_7F_9$ and $F_3F_4F_{10}$ are estimated with the same efficiency. $F_5F_7F_{10}$ is estimated with the lowest efficiency in comparison to the efficiency attained for other effects in the same set.

For $\begin{bmatrix} 128 & 128 & 0 & 64 & 0 & 128 & 128 & 64 & 64 \\ 128 & 128 & 0 & 64 & 0 & 128 & 128 & 64 & 64 \\ 0 & 0 & 128 & 64 & 128 & 0 & 0 & 64 & 64 \\ 64 & 64 & 64 & 128 & 64 & 64 & 64 & 128 & 128 \\ 0 & 0 & 128 & 64 & 128 & 0 & 0 & 64 & 64 \\ 128 & 128 & 0 & 64 & 0 & 128 & 128 & 64 & 64 \\ 128 & 128 & 0 & 64 & 0 & 128 & 128 & 64 & 64 \\ 64 & 64 & 64 & 128 & 64 & 64 & 64 & 128 & 128 \\ 64 & 64 & 64 & 128 & 64 & 64 & 64 & 128 & 128 \end{bmatrix}$, the pseudoinverse is

$$\frac{1}{4096}\begin{bmatrix} 3 & 3 & 0 & -3 & 0 & 3 & 3 & -3 & -3 \\ 3 & 3 & 0 & -3 & 0 & 3 & 3 & -3 & -3 \\ 0 & 0 & 12 & -5 & 12 & 0 & 0 & -5 & -5 \\ -3 & -3 & -5 & 7 & -5 & -3 & -3 & 7 & 7 \\ 0 & 0 & 12 & -5 & 12 & 0 & 0 & -5 & -5 \\ 3 & 3 & 0 & -3 & 0 & 3 & 3 & -3 & -3 \\ 3 & 3 & 0 & -3 & 0 & 3 & 3 & -3 & -3 \\ -3 & -3 & -5 & 7 & -5 & -3 & -3 & 7 & 7 \\ -3 & -3 & -5 & 7 & -5 & -3 & -3 & 7 & 7 \end{bmatrix} \quad \textbf{(3.23)}$$

The set $(F_1F_2F_6, F_1F_3F_7, F_1F_4F_5, F_2F_3F_{10}, F_2F_5F_8, F_4F_6F_8, F_4F_9F_{10}, F_6F_7F_{10}, F_7F_8F_9)$ is estimated using equation (3.23). The effects $F_1F_2F_6$, $F_1F_3F_7$, $F_1F_2F_6$ and $F_1F_3F_7$ are estimated with a higher efficiency compared to the efficiency attained for other effects in the same set. Effects $F_2F_3F_{10}$, $F_6F_7F_{10}$ and $F_7F_8F_9$ are estimated with the same efficiency. $F_1F_4F_5$ and $F_2F_5F_8$ are estimated with the same efficiency which is lower than the efficiency attained for other effects in the same set.

For $\begin{bmatrix} 128 & 0 & 64 & 64 & 64 & 0 & 0 & 0 & 64 & 64 \\ 0 & 128 & 64 & 64 & 64 & 128 & 128 & 128 & 64 & 64 \\ 64 & 64 & 128 & 128 & 128 & 64 & 64 & 64 & 0 & 0 \\ 64 & 64 & 128 & 128 & 128 & 64 & 64 & 64 & 0 & 0 \\ 64 & 64 & 128 & 128 & 128 & 64 & 64 & 64 & 0 & 0 \\ 0 & 128 & 64 & 64 & 64 & 128 & 128 & 128 & 64 & 64 \\ 0 & 128 & 64 & 64 & 64 & 128 & 128 & 128 & 64 & 64 \\ 0 & 128 & 64 & 64 & 64 & 128 & 128 & 128 & 64 & 64 \\ 64 & 64 & 0 & 0 & 0 & 64 & 64 & 64 & 128 & 128 \\ 64 & 64 & 0 & 0 & 0 & 64 & 64 & 64 & 128 & 128 \end{bmatrix}$, the pseudoinverse is

$$\frac{1}{128000}\begin{bmatrix} 288 & -88 & 64 & 64 & 64 & -88 & -88 & -88 & 136 & 136 \\ -88 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 64 & 0 & 92 & 92 & 92 & 0 & 0 & 0 & 0 & 0 \\ 64 & 0 & 92 & 92 & 92 & 0 & 0 & 0 & 0 & 0 \\ 64 & 0 & 92 & 92 & 92 & 0 & 0 & 0 & 0 & 0 \\ -88 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -88 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -88 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 136 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 167 & 167 \\ 136 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 167 & 167 \end{bmatrix} \quad \textbf{(3.24)}$$

The matrix in equation (3.24) is used to estimate effects in set 8. The effects $F_1F_4F_7$, $F_2F_7F_8$, $F_3F_6F_8$ and $F_3F_9F_{10}$ are estimated with a higher efficiency in comparison to the efficiency attained for other effects in the same set. Effects $F_1F_6F_9$, $F_1F_8F_{10}$ and $F_2F_4F_{10}$ are estimated with th same efficiency. $F_5F_6F_{10}$ and $F_5F_8F_9$ are estimated with the same efficiency. The effect $F_1F_3F_5$ is estimated with a lower efficiency compared to the efficiency atttained for other effects in the same set.

For $\begin{bmatrix} 128 & 128 & 64 \\ 128 & 128 & 64 \\ 64 & 64 & 128 \end{bmatrix}$, the pseudoinverse is $\frac{1}{384}\begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ -1 & -1 & 4 \end{bmatrix}$ **(3.25)**

The matrix in equation (3.25) is used to estimate effects in set 9. The effects F_1F_2, F_4F_8, F_6F_{10}, F_8F_9, F_2F_3 and F_6F_7 are estimated with a higher efficiency than the efficiency attained for F_7F_{10}, F_1F_3 and F_4F_9. Effects F_7F_{10}, F_1F_3 and F_4F_9 are estimated with the same efficiency.

For $\begin{bmatrix} 128 & 128 & 64 & 0 \\ 128 & 128 & 64 & 0 \\ 64 & 64 & 128 & 64 \\ 0 & 0 & 64 & 128 \end{bmatrix}$, the pseudoinverse is $\frac{1}{1024}\begin{bmatrix} 3 & 3 & -4 & 2 \\ 3 & 3 & -4 & 2 \\ -4 & -4 & 16 & -8 \\ 2 & 2 & -8 & 12 \end{bmatrix}$ **(3.26)**

Equation (3.26) is used to estimate effects in set 11. Effects F_1F_8 and F_2F_4 are estimated with a higher efficiency than the efficiency attained for other effects in the same set. F_3F_9 is estimated with a lower efficiency compared to the efficiency attained for other effects in the same set.

For $\begin{bmatrix} 128 & 64 & 64 & 0 \\ 64 & 128 & 128 & 64 \\ 64 & 128 & 128 & 64 \\ 0 & 64 & 64 & 128 \end{bmatrix}$, the pseudoinverse is $\frac{1}{256}\begin{bmatrix} 3 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 3 \end{bmatrix}$ **(3.27)**

Equation (3.27) is used to estimate effects in set 12. Effects F_2F_7, F_3F_6, F_6F_9 and F_8F_{10} are estimated with a higher efficiency than the efficiency attained for F_1F_{10}, F_5F_9, F_3F_5 and F_4F_7. The effects F_1F_{10}, F_5F_9, F_3F_5 and F_4F_7 are estimated with the same efficiency.

For $\begin{bmatrix} 128 & 128 & 0 \\ 128 & 128 & 0 \\ 0 & 0 & 128 \end{bmatrix}$, the pseudoinverse is $\frac{1}{512}\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix}$ **(3.28)**

The matrix in equation (3.28) is used to estimate effects in set 15. The effects F_2F_6 and F_3F_7 are estimated with a higher efficiency than the efficiency attained for F_4F_5.

For $\begin{bmatrix} 128 & 128 \\ 128 & 128 \end{bmatrix}$, the pseudoinverse is $\frac{1}{512}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ **(3.29)**

The matrix in equation (3.29) is used to estimate effects in set 13. The effects here are estimated using the same efficiency.

The matrix:

$$\begin{bmatrix} 128 & 64 & 0 \\ 64 & 128 & 64 \\ 0 & 64 & 128 \end{bmatrix}^{-1} = \frac{1}{256}\begin{bmatrix} 3 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 3 \end{bmatrix} \text{ gives estimates for } \begin{bmatrix} F_1F_6 \\ F_3F_{10} \\ F_5F_8 \end{bmatrix}, \begin{bmatrix} F_1F_9 \\ F_3F_8 \\ F_5F_{10} \end{bmatrix}, \begin{bmatrix} F_2F_5 \\ F_7F_9 \\ F_4F_6 \end{bmatrix}, \begin{bmatrix} F_3F_4 \\ F_2F_9 \\ F_5F_7 \end{bmatrix}$$

Effects F_1F_3,F_4F_7, F_4F_6, F_2F_5, F_1F_4, F_3F_7, F_3F_9, F_5F_8, F_5F_6, F_2F_4, F_5F_9 and F_3F_8 are estimated with a higher efficiency. Effects F_6F_9, F_7F_9, F_2F_8, F_1F_6, F_1F_8 and F_2F_7 are estimated with a lower efficiency.

The matrix:

$$\begin{bmatrix} 128 & 64 \\ 64 & 128 \end{bmatrix}^{-1} = \frac{1}{192}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \text{ gives estimates for } \begin{bmatrix} F_1F_7 \\ F_2F_{10} \end{bmatrix} \text{ and } \begin{bmatrix} F_4F_{10} \\ F_7F_8 \end{bmatrix}$$

The effects here are estimated using the same efficiency.

4 Conclusions and Recommendations

4.1 Conclusions

This current study extends Patel's designs that permit estimation of factors up to two-factor interactions only to designs that permit estimation of factors up to three-factor interactions assuming higher order interactions to be absent.

The method of construction and analysis of these designs is given. The linear forms are chosen such that each is of weight ≥ 4. By the weight of a linear form we mean the number of non-zero co-efficients. The linear equations provided are arbitrary selected. Consequently, correlated effects will depend on the fraction (defining contrast) used for the duplicated designs. The motivation to choose equations of weight ≥ 4 is to allow estimation of the grand mean, main effects, two-factor and three-factor interactions.

In the construction of designs involving estimation of factors up to three factor interactions, some matrices involved tend to be singular. In such a case, we make use of Moore-Penrose inverse to estimate effects in that particular matrix. We also check on a universal construction method of fractional designs that enable estimation up to m factor interactions $(m < p)$.

The method of obtaining blocks and the test procedure discussed in subsection 2.2.2 on "Method of Analysis" can be applied to all designs in this study. Similar method of construction can be used to obtain designs involving more than ten factors.

Two ways of obtaining the block designs are provided. The test procedure on how to estimate σ^2 and how to test for the significance of an effect is shown. The efficiency used to estimate for each factor is discussed in the designs presented.

These type of designs can be used in screening experiments where there errors in observations to identify active factors. They can also be used in mixture experiments-experiments that involve mixing the proportion of two or more components to make an end product.

4.2 Recommendations

In this study, we consider designs whose factors occur only at two levels. One can extend these designs to fractional factorial designs whose factors occur at three levels and that allow estimation of factors up to three-factor interactions with partial duplication considered.

This work involves symmetrical factorial designs- designs whose factors occur at the same number of levels. It would be interesting to study asymmetrical factorial designs also known as mixed factorials- designs whose factors are not at the same number of levels- that are partially duplicated and that permit estimation of factors up to three-factor interactions.

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