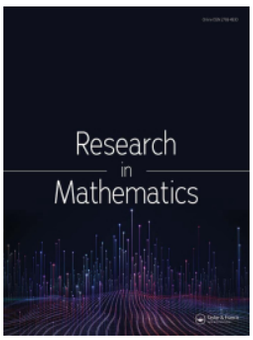

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Bühlmann credibility approach to systematic mortality risk modeling for sub-Saharan Africa populations (Kenya)

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ABSTRACT

The classical mortality models such as the Cairns-Blake-Dowd (CBD), Lee-Carter (LC), Linear Regression (LR) models are used to model Systematic Mortality Risk (SMR) for many developed countries populations for actuarial product valuations. This research study aims at incorporating the Bühlmann credibility approach (BCA) to improve the SMR models to fit sub-Saharan African populations like Kenya. Since the Kenyan population does not exhibit the Gaussian properties used in modeling the classical error terms, we proposed using Normal Inverse Gaussian distribution to model these error terms instead of a Gaussian distribution. We model the error terms of the classical models (LC, CBD, and LR) as a Normal Inverse Gaussian (NIG) distribution through the Bühlmann credibility approach. This novel approach demonstrates an improved precision of the predicted SMR as shown by the values of MAPE and RMSE measures compared to those under classical mortality risk models. Ultimately, we have done actuarial valuations of annuities and assurances using our determined SMR, thus concluding that this BCA approach improves the accuracy of actuarial products sold in the Kenyan market.

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SUBJECTS
Advanced Mathematics; Applied Mathematics; Mathematics Education; Statistics & Probability

1. Introduction

Mortality risk plays an essential role in determining the development strategies that countries worldwide may adopt when planning in several areas like health, industry establishment, and resource distribution (Li, 2010). Insurance companies and pension firms often charge premiums depending on the potential systematic mortality risk, thus modeling and predicting mortality risk becomes a vital concept for actuaries, demographers, and many other researchers (Dowd et al., 2006).

From the classical mortality risk model (Lee & Carter, 1992), many extensions exist today to reduce their shortcomings. From the LC model, crude death rates were determined by a constant age-specific change in age dynamic and mortality rate trends. (Cairns et al., 2006) The model proposed a logit function of specific year death probabilities, which considers both the general time trend and age-related trend. Both (Lee & Carter, 1992) and (Cairns et al., 2006) have two-time trends that generally assume a random walk with a drift during mortality prediction.

Many models in research papers such as (Alavi et al., 2021; Chen & Cox, 2009; Haberman & Renshaw, 2009; Hammal et al., 2020; Najafabadi, 2010; Pan et al., 2008; Plat, 2009; Tan et al., 2015) have been used in modeling mortality risks, with each of them making assumptions that the disturbances/error terms follow a Gaussian distribution. This assumption primarily considers the kind of population data being used, especially in European countries. (Tsai & Yang, 2015) proposed linear regression for a two-parameter with a time lag for a specific age when forecasting the mortality rates with Gaussian assumptions. Tsai and Lin (2017a) extended the (Lin et al., 2015) by including the copula and Generalized Autoregressive Conditional Heteroskedasticity model.

The purpose of this study is to incorporate the Bühlmann credibility approach introduced by (Buhlmann & Gisler, 2005) into three commonly used mortality models, namely, the (Lee & Carter, 1992) model, the (Dowd et al., 2006) model, and (Lin et al., 2015) model. This introduction will boost the prediction of SMR capabilities when considering the Kenyan population. The novelty in this study is the assumption that the error terms of the three models ((the (Lee & Carter, 1992) model, the (Dowd et al., 2006) model) follow a Normal Inverse Gaussian distribution as opposed to the Gaussian distribution commonly used in the original mortality models. The approach is applied because the Kenyan population does not exhibit the normality assumptions of the error terms in the classical models. (Tsai & Lin, 2017b) has had integrated the Bühlmann credibility into the three classical mortality models to

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test their accuracy while assuming the Normality of the randomness, which our study improves for novelty.

We applied a Jarque-Bera test to check whether the Kenyan population data have kurtosis and skewness. The results match a Gaussian distribution to confirm our choice of using NIG distribution data as it does not possess Gaussian distribution properties. Furthermore, we have offered a credibility estimates interpretation with a person aged exactly x time trend and a conglomeration of trends for all individual ages. Ultimately, we applied the projected mortality rates under with and without Bühlmann credibility approaches to value the net annual single premiums payable by policyholders to life assurance firms. We also compared their corresponding MAPEs and RMSEs errors to the Kenyan population data.

The novelty in the study is the assumption that the error terms of the three classical models follow a Normal Inverse Gaussian distribution as opposed to the Gaussian distribution commonly used in the original mortality models. Our proposal is anchored on the fact that the Kenyan population does not exhibit the normality assumptions applied in the classical models, thus necessitating this research study. In addition, the inclusion of Bühlmann's credibility approach improves the precision of forecasted systematic mortality risk used during life products valuation sold in the Kenyan markets.

We have structured this research study as follows: the introduction and literature review are in section 1. In section 2, Bühlman's credibility approach and Normal Inverse Gaussian distribution mathematical concepts are reviewed. In section 3, we have incorporated the Bühlmann Credibility Approach into stochastic mortality models by stating the original models, justification of non-normality of Kenyan population data, and application of NIG distribution when modeling the error terms for each of the mortality models. In section 4, we compare the forecasting performances of these models with and without the approach of Bühlmann's credibility theory. In section five, we have applied these predicted mortality rates both with and without the concept of Bühlmann's credibility. We use the idea to value the standard assurances and annuities of actuarial products sold in the Kenyan Insurance market. In section 6, we offered conclusions as well as recommended areas for further research.

2. Mathematical preliminaries of Bühlmann Credibility theory

Let $X_1, X_2, \ldots, X_{n-1}, X_n, X_{n+1}$ represent the observed data points that are independently and identically distributed. Let Θ denote the parameter of conditional risk. Also, let $\pi_\Theta(\theta)$ denote the risk characteristics distribution of the policyholders in (Buhlmann & Gisler, 2005; Kim & Jeon, 2013). The experience of policyholders, which can either be the size of claims or the claim's numbers for an individual policyholder with a risk parameter, is denoted as $(\Theta = \theta)$. In addition, it is modeled through a conditional distribution $g_{X/\Theta}(X/\theta)$ given $\Theta = \theta$ and θ is defined as the prior information on the distribution of the number of claim amounts of $X_1, X_2, \ldots, X_n$ as shown by (Pitselis, 2013).

Let observations $X_1, X_2, .., X_n$ denote the prior exposure periods. The Buhlmann Credibility Estimator, C, experienced at X_{n+1} is given by

$$C = Z\bar{X} + (1 - Z)\mu \tag{1}$$

where Z is the credibility factor estimate, which is assigned to the data of observed experience and μ is the unconditional mean $E[X]$ (the mean value that has been taken on all the risk parameters, Θ, members) and $\bar{X}$ is the sample mean. The Buhlmann credibility factor or estimator C is a linear function of all historical data written in the form:

$$C = Z\overline{X} + (1 - Z)\mu \approx w_0 + \sum_{i=1}^{n} w_i X_i \tag{2}$$

where $w_0 = (1 - Z)\mu$ and $w_i = \frac{Z}{n}$ for $i = 1, 2, 3, \ldots, n$ and due to limited data, the actual mean in equation (1) can be approximated from sample mean in equation (2) mean under the assumption of central limit theory (Blackburn et al., 2017).

Buhlmann credibility factor or estimator is an optimal linear estimator to the mean of Bayesian predictive, $\mathbb{E}[X_{n+1}|X_1, \cdots, X_n]$, and the hypothetical mean $\mathbb{E}[X_{n+1}|\Theta]$ ultimately minimizing the squared error loss (Najafabadi et al., 2012). Alternatively, the coefficients denoted as w_i are determined in such a way that the predictive expectations or the loss functions are minimized after taking all expectations over all the observations of Θ to get

$$L = \mathbb{E}\left(\left[\mathbb{E}[X_{n+1}|X_1, \cdots, X_n] - w_0 - \sum_{i=1}^{n} w_i X_i\right]^2\right) \tag{3}$$

where equation (3) represents quadratic loss function that minimizes the expected squared error of a linear estimator of the past observations of deaths in a given population.

The hypothetical mean and variance of conditional on the risk parameter, Θ, is denoted by $\mathbb{E}[X|\Theta]$ and $Var[X|\Theta]$, respectively. Then, $\mu = \mathbb{E}[X] = \mathbb{E}[\mathbb{E}[X|\Theta]]$ is the expected present value of the hypothetical means or the unconditional mean. Thus, the total variance of the random process is defined as

$$Var[X] = \mathbb{E}[Var[X|\Theta]] + Var[\mathbb{E}[X|\Theta]] \quad (4)$$

From equation (4), it has two parts namely, the total variance $E[Var[X|\Theta]]$ expectation known as the expected value of process variance (EPV), whereas the first part $Var[\mathbb{E}[X|\Theta]]$ is known as the variance of the hypothetical means (VHM). From the $\mathbb{E}[X|\Theta]$ and $Var[\mathbb{E}[X|\Theta]]$, the credibility factor Z is calculated as

$$Z = \left(\frac{n}{n+K}\right) \quad (5)$$

where $K = \frac{\mathbb{E}[X|\Theta]}{Var[\mathbb{E}[X|\Theta]]}$

This study assumes that the Error terms in the three classical stochastic mortality models follow Normal Inverse Gaussian Distribution (NIG; Lillestol, 2000) instead of a Normal distribution with parameters (μ, σ^2). Our models will now follow an NIG distribution. Estimation of the parameters of the NIG distribution using the Kenyan population data is then done. Thus, the distribution of the Error terms, say, X as $X \sim NIG(\hat{\alpha}, \hat{\beta}, \hat{\theta}, \hat{\sigma})$ with all the parameters to be estimated from the available Kenyan life table data.

3 Incorporation of the Bühlmann credibility approach into stochastic mortality models

Let **A, B**, and **C** denote LC (Lee-Carter), CBD and Linear Relational (LR) Models, respectively.

3.1. Mathematical modeling of mortality

Let $q(x,t)$ denote the instantaneous death rates associated with $\mu(x,t)$, which is the force of mortality. With the assumptions of UDD (Uniform distribution of deaths) and constant force of mortality (CFM) in between the integer age x as well as year t. The force of mortality is equivalent to the central death rates, $\mu(x,t) = m(x,t)$ with the rate $m(x,t)$ defined as the select central death rates.

We assumed an ordinary least squares method, during parameter estimations of the models since the Kenyan mortality data provided is in discrete form. We make an assumption of t years within the year of fitting span, say $[x_{low}, x_{high}]$ $(x_{low} - x_{high} + 1 = t)$ and k ages in the fitting age span $[t_{low}, x_{high}]$ $(t_{low} - t_{high} + 1] = m$. We note that models A and C use $ln(m(x,t))$ while B uses $logit(q(x,t)) = ln\frac{q(x,t)}{1-q(x,t)}$ during mortality risk modeling.

The empirical Kenyan mortality data show that $ln(m(x,t))$ in both models A and C and $ln\frac{q(x,t)}{1-q(x,t)}$ in B has displayed a downward trend during period x (see Figure 1). By denoting $Q(x,t) = ln(m(x,t)) - ln(m(x,t-1))$ for the A and C models and $Q(x,t) = logit(q(x,t)) - logit(q(x,t-1))$ for B, (as illustrated in Figure 1) for $x = x_{low} + 1, \ldots, t_{high}$, it eliminates this downward trend. This is an assumption that the $(t-1)$ observed values for all $Q_{x,k_{low}+1}$, …, Q_x, k_{high} are provided.

Bühlmann credibility estimate $\hat{Q}_{x,t_{high}} + 1$ for those ages x in the year $t_{high} + 1$ is defined as the weighted average or weighted proportion of the mean sample and $\hat{Q}_x = \frac{1}{m}\sum_{x=x_{low}+1}^{x_{high}} Q_{(x,t)}$, which is the exact mean, μ with weights of Z at the same time $(1-Z)$, in the following

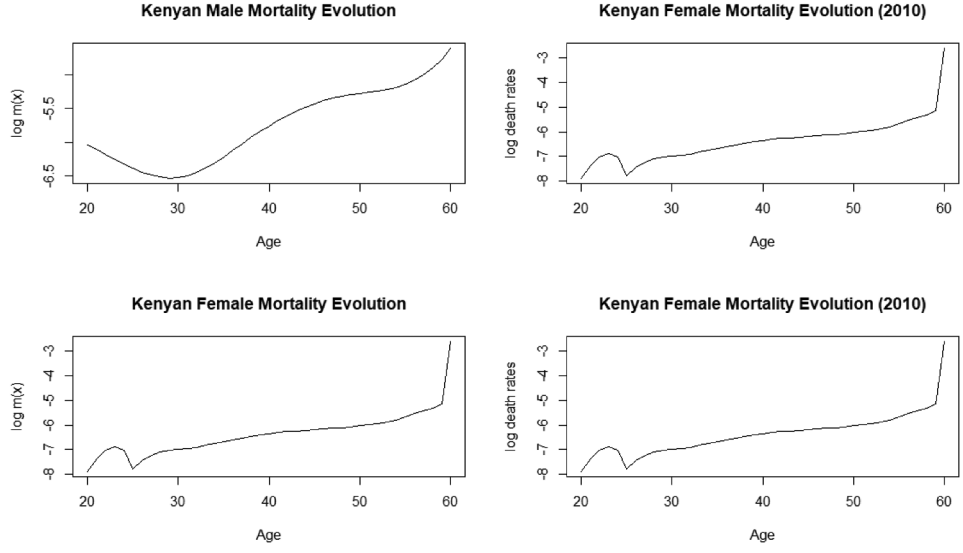

Figure 1. $ln(m(x,t))$ and $logit(q(x,t))$ against time for Kenyan Males(left) and Females(right), respectively, on top and $Q(x,t)$ against time for Kenya Males (left) and Females (right), respectively, at the bottom.

order. The distribution of the risk parameter, $Q_{x,t}$ determines the kinds of parameters used when determining the value of μ and Z.

3.2. Incorporation of the Bühlmann credibility approach into the LC mortality model

Let Model A (Lee & Carter, 1992) is defined as;

$$ln(m(x,t)) = \alpha_x + \beta_x\kappa_t + e(x,t) \tag{6}$$

where the values of $x = 1, 2, 3 \ldots . n$ and $t = 1, 2, 3 \ldots . n$ are subjected to the constraints $\sum_{x=1}^{n} \beta_x = 1$ and $\sum_{t=1}^{n} k_t = 0$. Moreover, α_x denotes the expected age-specific mortality rate, κ_t is the overall mortality trend that varies with time t, β_x is the sensitivity of age-specific mortality to time, and $e(x,t) \sim N(0, \sigma_e^2)$. This study models the error term associated with the model as a NIG distribution. See the appendix for the estimation of the parameters of equation (6).

3.2.1. Justification of non-normality feature of Kenyan population data

Jarque–Bera (JB) test for Lee-Carter Model is performed on Kenyan Data to test whether the kind of residuals of Kenya data used follow a normal statistical distribution. JB test needs pre-requisite conditions of a large number of data when doing the goodness-of-fit test of whether Kenyan population data possess kurtosis and skewness matching a Gaussian distribution. These conditions were all ascertained when the test was done on the Kenyan population data for normality assumptions.

This *JB* test statistic is non-negative meaning that it is much far from zero, which indicates that the specific data do not have the property of a Gaussian distribution (Thadewald & Büning, 2007). From the formulated null hypothesis against alternative hypothesis such as

JB test (P-value>0.05) = Accept H_0 (Normal Distribution) *Vs* JB test(P-value<0.05) = Reject H_0 (Non-Normal Distribution)

This tells us that the test statistic is 4.1831 and the $p - value$ of the test is 0.01235. In this case, we would reject the stated null hypothesis (H_0) that the residuals of data are normally distributed. This means that we have sufficient evidence that the Kenyan population dataset residuals are not normally distributed, thus justification of the choice of NIG distribution.

3.2.2. The LC mortality model under NIG assumptions

From the classical model in equation (6) and justification in Table 1, we assume that the error terms denoted as $e(x,t)$ are independent with mean of $(\theta + \frac{\beta}{\tau}\sigma)$ and variance of $\sigma\frac{\alpha^2}{\tau^3}$. The classical paper assumes that the overall mortality trend follows a simple random walk, with a drift ϑ for the prediction of mortality such that $\kappa_t = \kappa_{t-1} + \vartheta + e(t)$ where trend errors ($e(t)$) of the time follows a *NIG* and are *i.i.d.* for $e(t)$ such that $t = t_x + 1, \ldots . t_m$. From the work of (Hári et al., 2008), with assumption of *i.i.d.* on the error terms being white noises satisfying the martingale structure equation as;

$$\begin{bmatrix} e(x,t) \\ e(t) \end{bmatrix} / W_{t-1} \sim NIG \begin{Bmatrix} \theta + \frac{\beta}{\tau}\sigma,\ \sigma_x\frac{\alpha^2}{\tau^3} \\ \theta + \frac{\beta}{\tau}\sigma,\ \sigma_\varepsilon\frac{\alpha^2}{\tau^3} \end{Bmatrix} \tag{7}$$

where W_{t-1} provides the information about the process up to a time $t-1$ and co-variances of the two random errors is *zero*.

Let $Q_{x,t}$ denote a random variable of the differences between the central death rates t and $t-1$. This means that $Q_{x,t} = ln(m(x,t)) - ln(m(x,t-1))$

$$Q_{x,t} = \beta_x(k_t - k_{t-1}) + \Delta e(x,t)$$

$$Q_{x,t} = \beta_x\vartheta + \beta_x e_t + \Delta e(x,t)$$

where the value of $x = x_1 \ldots , x_m; t = t_{x1} + 1, \ldots , t_{xm}$ and $\Delta e(x,t) \sim NIG(\theta + \frac{\beta}{\tau}\sigma, 2\sigma\frac{\alpha^2}{\tau^3})$ are in equation (7). The result will lead to $Q_{x,t} \sim NIG(\vartheta_1\beta_x, \vartheta_2\beta_x, \beta_x^2\sigma_\varepsilon^2 + 2\sigma_x^2)$, which follows the sum of independent variables of *NIG* distribution and still remains a *NIG* distribution with the new parameters.

From both conditional expectation and variance of $Q_{x,t}$, we apply the Bühlmann credibility, such that $\theta(x) = \mathbb{E}[Q_{x,t}/X] = \beta_x\vartheta$ and $[\mathrm{Var}Q_{x,t}/X] = \beta_x^2\sigma_\varepsilon^2 + 2\sigma_x^2$ in the subsequent order. Since the expectation of value of the stated hypothetical mean, $\theta = \mathbb{E}[\theta(x)] = \mathbb{E}[\mathbb{E}[Q_{x,t}/X]] = \theta\mathbb{E}[\mathbb{E}[Q_{x,t}/X]]$, the estimated value of θ, denoted by $\hat{\theta}$ is given by;

Table 1. Jarque–Bera normality test for Lee-Carter Model

Distribution	Mean	Standard Deviation
Normal	0.56468	2.0568
The adjusted test statistic:	$JB =$	4.1831
Significance level:	$\alpha = 0.05$	
Critical value:	2.4538	
Critical region:	Reject H_0 if	$JB < 0.01235$

$$\hat{\theta} = \frac{\hat{\vartheta}}{n}\sum_{x_1}^{x_n}\hat{\theta}_x = \frac{\hat{\vartheta}}{n} \tag{8}$$

From the equation 8, the variance process of expected value is given as $a = \mathbb{E}[a(X)] = \mathbb{E}[\beta_x^2]\sigma_\varepsilon^2 + 2\mathbb{E}[\sigma_x^2]$. The value of $\hat{a}$ is used to calculate the credibility factor in equation (5). Thus,

$$\hat{a} = \frac{\hat{\sigma}^2}{n}\sum_{x_1}^{x_n}\hat{\theta}_x^2 + 2\sum_{x_1}^{x_n}\frac{\hat{\sigma}^2}{m} \tag{9}$$

while the hypothetical mean variance $c = Var[\theta(X)] = \vartheta^2 Var[\beta_X] = \vartheta^2\mathbb{E}[\beta_X^2] - \mathbb{E}[\beta_X]^2$. This can be estimated through

$$\hat{a} = \vartheta^2[\frac{\hat{\vartheta}}{n}\sum_{x_1}^{x_n}\theta_x^2 - (\frac{1}{n}\sum_{x_1}^{x_n}\theta_x)^2$$

By writing the equation in form of the $\bar{Z}X + (1-Z)\mu$, it is easy to estimate the value of $\hat{\mu}$ as $\frac{\hat{\vartheta}}{n}$.

3.3 Incorporation of the Bühlmann credibility approach into the CBD mortality model

Model B (Dowd et al., 2006) is defined as

$logit(q(x,t)) = ln(\frac{q(x,t)}{1-q(x,t)})$, which can be simplified to give

$$logit(q(x,t)) = k_t^{(1)} + k_t^{(2)}(x - \bar{x}) + e(x,t) \tag{10}$$

where $x = 1, 2, 3 \ldots .., n$, $t = 1, 2, 3, \ldots ..m$, and $\bar{x}$ is the expected age over the specific age span and $e(x,t) \sim N(0, \sigma_e^2)$. See the appendix for the estimation of the parameters of equation (10).

3.3.1 Validation of non-normality property of Kenyan life table data

We perform a Doornik-Hansen test for CBD Model to justify that the Kenyan population does not exhibit the Normality assumptions used in the classical models. Since the CBD model is a multivariate parameter model, we can model the model's errors using independent univariate normal inverse Gaussian Levy processes (Ainou, 2011). Also, we use the Doornik-Hansen test see, Doornik and Hansen (2008) when testing whether the residuals of Kenyan Mortality data follow a Gaussian distribution before applying the proposed NIG distribution.

Therefore, we rewrite equation (10) in the form

$$K_{t+1} = K_t + \theta + CZ_{t+1} \tag{11}$$

where K_t is a two-dimensional random walk with drift terms, where θ and C are constants (C is a 2×2 upper triangular matrix). Z_t is a two-dimensional standard Gaussian random variable. While (Dowd et al., 2006) applied OLS to estimate the parameters of the model, we apply the MLE method when estimating its two parameters.

From equation (11), we rewrite it as;

$$logitq(x,t) = K_{t+1}^1 + K_{t+1}^2(x) \tag{12}$$

where ages x rangefrom 60 to 90 and t covers from 2010 to 2020. Then, we apply linear regression in equation (12) to estimate the value of K_t. It is clear from (12) that

$$\begin{pmatrix} E[K_{t+1} - K_t] = \theta \\ Var[K_{t+1} - K_t] = CC' \end{pmatrix} \tag{13}$$

From equation (13), the mean and the variance of the first consecutive differences, $Z_{t+1} - A_t$, can be used to estimate θ and $W = CC'$, respectively. . Generally, those negative values for θ_1 show mortality improvement. From the same trend, the positive value for θ_2 indicates that mortality rates at significantly higher ages are now improving at a much slower rate. The Estimated Mean and Variance Matrices for the CBD Model have been determined as : $\tilde{\theta}$ is $\begin{bmatrix} -0.0868560 \\ 0.00082550 \end{bmatrix}$ and $\hat{W}$ is $\begin{bmatrix} 0.052567300 & -0.00066890 \\ -0.00018935 & 0.000023457 \end{bmatrix}$.

The multivariate section of Table 2 indicates that the test statistic is significantly based on the p-value of the test, which is a bi-variate normality assumption that has been rejected at a significance level of $\alpha = 0.05$. Subsequently, the multivariate/bi-variate normality assumption that has been made on the model does not hold. We confirm the Doornik-Hansen Normality test for CBD Model using a Multivariate Shapiro-Wilk Test for Normality, which is tabulated below as.

From table 2, test statistic of 0.8629 , which is less than the critical value at the level of significance of $\alpha = 0.05$, we therefore reject the null hypothesis, which is an assumption that the Kenyan data is normal. This prompts the use of NIG during errors modeling. Besides, it confirms the findings of the Doornik-Hansen test on the Normality for CBD Model, as in Table 2.

3.3.2. The CBD mortality model modification under NIG assumptions

From equation 10 and justifications on non-normality of Kenyan data from Table 2, the error term denoted as $e_{(x,t)}$ is assumed to be *NIG* distributed and independently and identically distributed with a mean of $\theta + \frac{\beta}{\tau}\sigma$ and variance of $\sigma_x \frac{\alpha^2}{\tau^3}$ for $_t$. Besides, $k_t^{(1)}$ and $k_t^{(2)}$ are time trends modeled by a bi-variate random walk having a drift of ϑ.

It is easy to model $k_t^{(1)}$ and $k_t^{(2)}$ time trends as Bi-variate random walk with the drift ϑ *i.e.* $k_t^{(i)} = k_t^{(i)} + \vartheta + e(x,t)$ where $k_t^{(i)}, i = 1,2$ equals $(k_t^{(1)}\ k_t^{(2)})'$, $\vartheta = (\vartheta_1, \vartheta_2)'$ and $e(t) = (e_t^{(1)}, e_t^{(2)})'$ thus satisfying the equation

$$\begin{bmatrix} e_t^{(1)} \\ e_t^{(2)} \end{bmatrix} / W_{t-1} \sim NIG \left\{ \begin{matrix} \theta_1 + \frac{\beta}{\tau}\sigma_1, & \sigma_{e,1}^2 & \sigma_{e,1}\sigma_{e,2} \\ \theta_2 + \frac{\beta}{\tau}\sigma_2, & \sigma_{e,1}\sigma_{e,2} & \sigma_{e,2}^2 \end{matrix} \right\} \tag{14}$$

with the two errors $e(x,t)$ as well as $e(t)$ are assumed to be identical and independent with the time trends for $_t$ are also *iid* when values of $i = 1,2$.

By considering a random variable

$$\begin{aligned} Q_{x,t} &= ln\left\{\frac{q(x,t) * p(x,t-1)}{p(x,t) * q(x,t-1)}\right\} \\ &= logit\{p(x,t) - logit(p(x,t-1)\} \end{aligned}$$

$$Q_{x,t} = (k_t^{(1)} - k_{t-1}^{(1)}) + (x-\bar{x})(k_t^{(2)} - k_{t-1}^{(2)}) + \Delta e(x,t)$$

$$\begin{aligned} Q_{x,t} &= \{(\vartheta_1 + \vartheta_2)(x-\bar{x})\} \\ &\quad + \{(e(t) + e(t-1)) \times (x-\bar{x})\} + \Delta e(x,t) \end{aligned}$$

for $x = 1,2,3\ldots..,n$, $t = 1,2,3,\ldots..m$, and $\Delta e(x,t) = e(x,t) - e(x) \sim NIG(0, 2\sigma_x^2)$ and from equation (14), $Q_{x,t} \tilde{N}IG(\vartheta_1 + \vartheta_2)(x-\bar{x})\}, \sigma_{e,1}^2(x-\bar{x}) + 2(x-\bar{x})\sigma_{e,1}\sigma_{e,2} + 2\sigma_x^2)$.

The conditional expectation variance of $[Q_{x,t}/X] = \vartheta_1 + \vartheta_2)(x-\bar{x})$ and the variance expectation $Var(v) = \hat{v} = \sigma_{e,1}^2(x-\bar{x}) + 2(x-\bar{x}) \times \sigma_{e,1}\sigma_{e,2} + 2\sigma_x^2$.

It is essential to estimate the hypothetical mean expected value, $\theta = \mathbb{E}[\theta(X)] = \vartheta_1 + \vartheta_2 1 + \mathbb{E}(x-\bar{x})$ such that;

$$\hat{\theta} = \hat{\vartheta}_1 + \hat{\vartheta}_2 + \frac{1}{n}\sum_{x_1}^{x_n}(x-\bar{x}) \tag{15}$$

where

$$\hat{\theta} = \frac{1}{n}\left(\frac{\sum_{x_1}^{x_n}[ln(m(x,t_1-1) - ln(m(x,t_1-1)]}{m-1}\right) = \frac{1}{n}\sum_{x_1}^{x_n}\hat{Q}_{x,t} = \bar{Q}_{x,t}.$$

Similarly, the process variance expected value, $Var(v) = \hat{v} = \sigma_{e,1}^2(x-\bar{x}) + 2(x-\bar{x})\sigma_{e,1}\sigma_{e,2} + 2\sigma_x^2$ is estimated as

$$Var(v) = \hat{\sigma}_{e,1}^2 + \left(\frac{2\sigma_e^2}{n}\right)\sum_{x_1}^{x_n}(x-\bar{x})^2 + \frac{2}{n}\sum_{x_1}^{x_n}\hat{\sigma}_x{}^2 \tag{16}$$

The expected value of $\hat{v}$ is estimated as

$$\hat{v} = \frac{\hat{\vartheta}_1^2}{n}\sum_{x_1}^{x_n}(x-\bar{x})^2$$

since $Var[\theta(\vartheta)] = \vartheta_2^2 Var[(x-\bar{x})]$.

3.4. Incorporation of the Bühlmann credibility approach into the Linear Relational (LR) mortality model

Model C (Tsai & Yang, 2015) is defined as

$$ln(m(x,t)) = \kappa_t^{(0)} + \kappa_t^{(1)} \times ln(m(x,t_x-1)) + e(x,t) \tag{17}$$

where $x = 1,2,3,\ldots.n$, $t = 1,2,3\ldots.m$, $t_x - 1$ denote the base year while the parameters $\kappa_t^{(0)}$ and $\kappa_t^{(1)}$ are obtained as the regression coefficients from $ln(m(x,t))$ on a line of $\ln(m(x,t-1))$ for values $t = 1,2,3,\ldots..m$ and it satisfies the precondition of the value.

$$\hat{k}_t^{(0)} + \hat{k}_t^{(1)} \times \frac{1}{n}\sum_{x_1}^{x_n} ln(m(x,t_x-1) = \frac{1}{n}\sum_{x_1}^{x_n} ln(m(x,t) \tag{18}$$

Moreover, the time trend in the model $\hat{k}_t^{(j)}$ $,j = 0,1$ is assumed to follow a simple random walk of drift ϑ_i, i.e. $\hat{k}_t^{(j)} = \hat{k}_{t-1}^{(j)} + \vartheta_i + e(j,t)$ since $j = 0,1$ and $e(x,t) \sim N(0,\sigma_e^2)$. See the appendix for the estimation of the parameters of equation (17).

3.4.1. Anderson-Darling test of non-normality property assumptions of Kenyan mortality data

We do the Anderson-Darling Test for the LR model to determine whether specific residuals of the data sample drawn from a particular type of statistical probability distribution (Evans et al., 2017). It means that we

Table 2. Doornik-Hansen Normality test for CBD Model (left) and Multivariate Shapiro-Wilk Test for Normality (right)

Normal	Distribution	Kenyan Data	Normal	Distribution	Kenyan Data
Adjusted test statistic:	$p-value =$	24.1831	Adjusted test statistic:	$p-value =$	0.8629
Significance level:	$a = 0.05$		Significance level:	$a = 0.05$	
Critical value:	8.1235		Critical value:	0.6235	
Critical region:	Reject H_0 if	$p-value < 0.0385$	Critical region:	Reject H_0 if	$p-value < 0.0455$

ascertain that our data from Kenyan mortality does not conform to the normal distribution, thus choosing Normal Inverse Gaussian distribution.

We define our hypothesis as follows:

H_0: Kenyan population data is following a Gaussian distribution Vs H_1: Kenyan population data do not follow a Gaussian distribution.

We conclude that since the test statistic of 0.8257 is greater than 0.752 from the Anderson-Darling test, we fail to reject the null hypothesis and conclude that there is no sifficient sufficient evidence at 5% to say that our residuals of data follow a Normal distribution. Hence, we choose to model the data follows a NIG distribution.

3.4.2. Mortality model modification under NIG assumptions

From equation 17 and justification from Table 3, the error terms denoted as $e_{(x,t)}$ is now assumed to be *iid* and *NIG* distributed with a mean of $(\theta+\frac{\beta}{\tau}\sigma)$ with a variance of $\sigma_x\frac{\alpha^2}{\tau^3}$ for $_t$. All the white Noises given in the (Tsai & Yang, 2015) must satisfy the following condition

$$\begin{bmatrix} e(x,t) \\ e(0,t) \\ e(1,t) \end{bmatrix} / W_{t-1} \sim NIG \begin{Bmatrix} \theta_1+\frac{\beta}{\tau}\sigma \\ \theta_2+\frac{\beta}{\tau}\sigma, \sigma^2, \sigma^2_{\varepsilon,0}, \sigma^2_{\varepsilon,1} \\ \theta_3+\frac{\beta}{\tau}\sigma \end{Bmatrix} \quad (19)$$

By considering the random variable given as $Q_{x,t}$ defined by $\ln(m(x,t)) - (m(x,t-1)$ and replacing it with the values of the (Tsai & Yang, 2015) model as written in equation (17), we obtain

$$\begin{aligned} Q_{x,t} &= k_t^{(0)} + k_{t-1}^{(0)} + k_t^{(1)} + k_{t-1}^{(1)} \\ &\quad + ln(m(x,t_1-1) + \Delta e(x,t) \end{aligned}$$

$$\begin{aligned} Q_{x,t} &= \vartheta_0 + \vartheta_1 \\ &\quad \times ln(m(x,t_1-1) \\ &\quad + [\varepsilon(0,t) + \varepsilon(0,1) \times ln(m(x,t_1-1)] + \Delta e(x,t) \end{aligned}$$

where $x = 1,2,3\ldots..,n,\quad t = 1,2,3,\ldots..m,$ and $\Delta e(x,t) = e(x,t) - e(x,t-1) \sim NIG(0,2\sigma^2)$ and from equation (19), we have

$$\begin{aligned} Q_{x,t} \sim NIG(&\vartheta_0 + \vartheta_1 \\ &\times ln(m(x,t_1-1), \sigma^2(\varepsilon,0) \\ &+ [ln(m(x,t_1-1)]^2 \times \sigma^2(\varepsilon,1) + 2\sigma^2] \end{aligned} \quad (20)$$

where $Q_{x,t}$ following similar assumptions under the Normal Inverse Gaussian distribution of a heavy-tailed distribution and convolution properties.

At the specific ages, we assume that $\Delta e(x,t)$ follows a white noise with $Q_{x,t}$ being *i.i.d.* for the values of $x = 1,2,3\ldots..,n$.

From the above model, we apply the approach of Bühlmann credibility theory by calculating the values of the hypothetical mean and variance where

$$\theta(X) = \mathbb{E}(Q_{X,t}/X) = \vartheta_0 + \vartheta_1 \times ln(m(x,t_1-1)$$

$$\begin{aligned} Var(v) &= Var(Q_{X,t}/X) \\ &= (\sigma^2(\varepsilon,0) \\ &\quad + [ln(m(x,t_1-1)]^2 \times \sigma^2(\varepsilon,1) + 2\sigma^2) \end{aligned}$$

We proceed ahead to estimate the values of the means and variance, $\theta(X)$ and Var (v), respectively, as follows

$$\hat{\theta} = \hat{\vartheta}_0 + \left\{ \frac{\hat{\vartheta}_1}{n} \sum_{x_1}^{x_n} ln(m(x,t_1-1) \right\} \quad (21)$$

From equation (20), it is important to rewrite equation (21) in the Bühlmann credibility formulae as:

$$\begin{aligned} \bar{Q}_{X,t} &= \left(\frac{ln(m(x,t_1-1) - ln(m(x,t_1-1)}{m-1} \right) \\ &= \frac{1}{m-1} \sum_{x_1}^{x_n} (Q_{X,t}) \end{aligned}$$

Consequently, we estimate the expected variance, Var (v) as applied in the Bühlmann credibility approach as;

$$\begin{aligned} Var(v) &= Var(Q_{X,t}/X) = (\sigma^2(\varepsilon,0) + \\ &[ln(m(x,t_1-1)]^2 \times \sigma^2(\varepsilon,1) + 2\sigma^2) \end{aligned} \quad \text{as}$$

$$\begin{aligned} Var(v) &= \hat{v} \\ &= \hat{\sigma}^2(\varepsilon,0) \\ &\quad + \frac{\hat{\sigma}^2(\varepsilon,1)}{n} \sum_{x_1}^{x_n} [ln(m(x,t_1-1)]^2 + 2\hat{\sigma}^2 \end{aligned}$$

while the variance of the stated hypothetical mean is defined as $v = Var[\theta(X)] = Var[ln(m(x,t_1-1)] = \vartheta_1^2 \times \mathbb{E}[ln(m(x,t_1-1)^2] - \mathbb{E}[ln(m(x,t_1-1)]^2$ which is the probability theory for the computation of the variance is estimated as

$$Var(v) = \hat{v} = \hat{\vartheta}_1{}^2 \left(\frac{1}{n} \sum_{x_1}^{x_n} ln(m(x,t_1-1))^2 - [\frac{1}{n} \sum_{x_1}^{x_n} ln(m(x,t_1-1))]^2 \right) \quad (22)$$

where the estimation of the parameters, $\hat{v}$, $\hat{\vartheta}$ and $\hat{\theta}$ as the Bühlmann credibility estimates of the $\bar{Q}_{X,t}$. Also, this has a value of

Table 3. Anderson-Darling Test for LR Model

Distribution	Mean	Standard Deviation
Normal	0.004360	1.001816
Adjusted test statistic:	$A^2 =$	0.8257
Significance level:	$\alpha = 0.05$	
Critical value:	0.752	
Critical region:	Reject H_0 if	$A^2 > 0.752$

$$\hat{\bar{Q}}_{(X,t+1)} = \left(ln(\hat{m}_{x,t_n+1}) - ln(m_{x,t_n+1}),\ \ forAC(\hat{q}_{x,t_n+1}) - logit(q_{x,t_n+1}), foB \right. \quad = Z\hat{Q}_x + (1-Z)\hat{\theta} \tag{23}$$

where Z of $logit(q_{x,t_n+1})$ and $ln(m_{x,t_n+1})$ for a given age x for a period of $t_n + 1$. Hence, the estimates are

$$\begin{pmatrix} ln(\hat{m}_{x,t_n+1}) = ln(m_{x,t_n}) + Z\hat{Q}_x + (1-Z)\hat{\theta} \\ logit(\hat{q}_{x,t_n+1}) = logit(q_{x,t_n}) + Z\hat{Q}_x + (1-Z)\hat{\theta} \end{pmatrix} \tag{24}$$

3.5 Bühlmann credibility estimate determination

When fitting and forecasting the respective A, B, and C models, we use the following strategies

Bühlmann credibility estimate determination

When dealing with the three A, B, and C models, the parameters under the Bühlmann credibility approach are determined under the MLE method as tabulated in the table.

Remark 1. It is essential to point out that the values are as follows $\mathbb{E}[Var[X|\Theta]] = \theta$ and $Var[\mathbb{E}[X|\Theta]] = b$ from Table 4 as the values of $\hat{b}$ could sometimes be negative because of subtraction. Whenever it happens, the value of $\hat{b}$ can be set, that implies that the value of $Z = 0$. Thus, the value of Bühlmann Credibility Estimate becomes $\hat{\theta} = \bar{Q}$. We use the following strategies to fit and forecast the respective A, B, and C models.

3.5.1 Strategy EW: Expansion of the window by a year

We apply the estimate $\{\hat{Q}_{X,t_n+1}$ to $\{Q_{x,t_0+1}, Q_{x,t_1+1}, \ldots Q_{x,t_n+N}\}$ to predict the estimates of Bühlmann's credibility for the year, $t_n + 1$, $t_n + 2$, ... and obtain the value of $\bar{Q}_{X,t_n+1} = \frac{1}{m}[\sum_{x_1}^{x_n} Q_{X,t} + \hat{Q}_{x,t_1+1}]$, $\hat{\theta}(t_n+2) = \frac{1}{n}\sum_{x_1}^{x_n} \bar{Q}_{X,t_n+2}$ and $Z(t_n+2) = \left\{\frac{n}{n+K}\right\}$. To determine the Bühlmann's credibility estimate for the different time lags i.e. $\tau = 2, 3 \ldots\ldots$, we apply the following equation

$$\bar{Q}_{X,t+\tau} = \frac{1}{m+\tau-2}\left(\sum_{x_1}^{x_n} Q_{X,t} + \sum_{x_1+1}^{x_n-\tau+1} \hat{Q}_{X,t}\right) \tag{25}$$

where the values of $\hat{\theta}(t_n+2) = \hat{Q}_{x,t_1+\tau}$ and $Z(t_n+\tau) = \frac{n+\tau-2}{n+\tau-2+K}$. For the values of $\bar{Q}_{X,t+\tau}$, $\tau = 2, 3, 4 \ldots$. the value of $Z(t_n+\tau)$ will be increasing for all τ for this EW strategy.

3.5.2 Strategy MW: Movement of the window by a year

To predict the estimates of Bühlmann credibility for the year, $t_n + 1$, we apply the estimate $\{\hat{Q}_{X,t_n+1}$ to $Q_{x,t_0+1}, Q_{x,t_1+1}, \ldots Q_{x,t_n+N}\}$, then obtain the value of estimate at time $t_n + 2$, from the value of $\bar{Q}_{X,t_n+1} = \frac{1}{m}\left(\sum_{x_1}^{x_n} Q_{X,t} + \hat{Q}_{x,t_1+1}\right)$, $\hat{\theta}(t_n+2) = \frac{1}{n}\sum_{x_1}^{x_n} \bar{Q}_{X,t_n+2}$ and $Z(t_n+2) = \frac{n}{n+K}$. To estimate the Bühlmann credibility estimate for the different time lags i.e. $\tau = 2, 3 \ldots\ldots$, we apply the following equation

$$\bar{Q}_{X,t+\tau} = \frac{1}{m-1}\sum_{x_1+1}^{x_n-\tau+1} \hat{Q}_{X,t} \tag{26}$$

and the value of

$$Z(t_n+\tau) = \left(\frac{n-1}{n-1+\hat{K}}\right) \tag{27}$$

where the values of $\hat{\theta}(t_n+2) = \hat{Q}_{x,t_1+\tau}$ and $\hat{K} = \frac{\mathbb{E}[Var[X|\Theta]]}{Var[\mathbb{E}[X|\Theta]]}$ are determined. The values of $\bar{Q}_{X,t+\tau}$, $\tau = 2, 3, 4 \ldots$. we obtain $Z(t_n+\tau)$ that is increasing value of τ for the MW strategy.

For all the two strategies, we get the values of $Z(t_n + \tau)$ where $\hat{Q}_{X,t} = ln(\hat{m}(x,t) - ln(\hat{m}(x,t-1)$ for the A and C while $\hat{Q}_{X,t} = ln(\hat{q}(x,t) - ln(\hat{q}(x,t-1)$ for B when dealing with values of $(t = t_1 + 1, t_1 + 2, \ldots, t_1 + \tau + 3)$. The forecasted mortality rates for an individual aged exactly x in year $t_{low} + \tau$ under the A, C, and B models without credibility incorporated approach will be all linear functions of τ having different slopes.

A is defined as:

$$ln(\hat{m}(x,t_n+\tau) = ln(\hat{m}(x,t_n) + \hat{\vartheta}\hat{\beta}_x \times \tau$$

$$ln(\hat{m}(x,t_n+\tau) = ln(\hat{m}(x,t_n) + \hat{Q}^{LC}_{X,t_n+1} \times \tau \tag{28}$$

B is defined as:

$$ln(\hat{q}(x,t_n+\tau) = ln(\hat{q}(x,t_n) + [\hat{\vartheta}_1 + \hat{\vartheta}_2(x-\bar{x})] \times \tau$$

$$ln(\hat{q}(x,t_n+\tau) = ln(\hat{q}(x,t_n) + \hat{Q}^{CBD}_{X,t_n+1} \times \tau \tag{29}$$

C is defined as:

$$ln(\hat{m}(x,t_n+\tau) = ln(\hat{m}(x,t_n) + (\hat{\vartheta}_1 + \hat{\vartheta}_1)ln(\hat{m}(x,t_n-1) \times \tau$$

Table 4. Estimations for the values of $\theta, v,$ and b

Estimation	for values of	θ,υ and b
$\bar{Q}_{xi} = (Q_{xi,t+1}, \ldots Q_{xi,t+n})$	$\bar{Q}_{xi} = \frac{1}{m-1}\sum_{x_i=1}^{x_n} Q_{xi+t}$	$\hat{v}_{xi} = \frac{1}{m-2}\sum_{x_i=1}^{x_n} (Q_{xi+t} - \bar{Q}_{xi+t})^2$
…	….	…
$\bar{Q}_{xni} = (Q_{xni,t+1}, \ldots Q_{xi,t+n})$	$\bar{Q}_{xni} = \frac{1}{m-1}\sum_{x_i=1}^{x_n} Q_{xi+t}$	$\hat{v}_{xni} = \frac{1}{m-2}\sum_{x_i=1}^{x_n} (Q_{xni+t} - \bar{Q}_{xni+t})^2$
$\hat{b} = \frac{1}{n-1}\sum_{x_i=1}^{x_n} (\bar{Q}_x - \bar{Q})^2 - \frac{\hat{v}}{m-1}$	$\hat{\theta} = \bar{Q} = \frac{1}{m}\sum_{x_i=1}^{x_n} \bar{Q}_x$	$\hat{v} = \frac{1}{n}\sum_{x_i=1}^{x_n} \hat{v}_x$

$$ln(\hat{m}(x, t_n + \tau) = ln(\hat{m}(x, t_n) + \hat{Q}^{LR}_{X,t_n+1} \times \tau \quad (30)$$

Remark 2. Using equations (28, 29, and 30), we get the values of predicted mortality rates under A, B, and C. Also, the EW strategy shows the downward trends of all estimated future mortality rates that are better for each of all ages x. From the two common invariant properties of the MW strategy, it is easier to do computations of Bühlmann's credibility estimates before comparing the EW strategy.

4 Fitting and forecasting of models

In this section, the fitting of A, B, and C is done both with and without credibility before making sample-based predictions for future consecutive years using Kenyan data. We use StMoMo R packages (Villegas et al., 2015). For the study period of $[T_1, T_2]$ where mortality rates are always available, we assume the end of year of t_n before making projections of the mortality rates. After the projections, we do an evaluation of the forecasting performances for the respective years $t_n + 1, \ldots, T_N$ through the application of the mortality data within the rectangle (window) defined as $[x_n, x_N] \times [t_n, t_N]$ where $T_1 \leq t_n$ and $t_N < T2$. An examination of the forecasting performances in two cases is done before and after incorporating the Bühlmann credibility approach.

MAPE (mean absolute percentage error) is used as a measure of forecasting error of the true rate of mortality (q) and predicted one $(\hat{q}$; Blake et al., 2019), and (Tsai & Lin, 2017a). To be precise, given a specified fitting year span $[t_n, t_N]$, we define *MAPE* and *RMSE* for a life aged x in year $t_N + t$ as

$$MAPE_{x,t_n+1} = |\frac{\hat{q}(x, t_n + 1) - q(x, t_n + 1)}{q(x, t_n + 1)}| \quad (31)$$

$$RMSE_{x,t_n+1} = \sqrt{(\frac{(\hat{q}(x, t_n + 1) - q(x, t_n + 1))^2}{T_N - t_n})} \quad (32)$$

From equation (31) and (32), we fit A, B, and C models using Kenyan population data for both males and females from the values of the tabulated central death rates.

The model is done before fitting yearly for the given sets of age span $[t_n, t_N]$, followed by forecast of mortality rates for the year under mortality. This helps in the calculation of the estimates of the Bühlmann credibility method for each of the years $t = 1, 2, 3 \ldots .N$ with the application of both EW as well as MW strategies. We measure the forecasting performances by calculation of the average of the *MAPE* over the ages 25, …, 100 and predicting years for the remainder of the years. The *AMAPE* is then defined as

$$AMAPE_{x,t_n+1} = \frac{1}{T_N - t_n}\sum_{x_1+1}^{x_n-\tau+1}\sum_{x=25}^{100} |\frac{\hat{q}(x, t_n + 1) - q(x, t_n + 1)}{q(x, t_n + 1)}| \quad (33)$$

Reduction ratio (RR) is done to measure how effectively the Bühlmann's credibility approach improves predicting performance, which measures the percentage reduction in AMAPE after incorporating the Bühlmann credibility into different sets of mortality models M for a given S strategy. Mathematically, it becomes

$$RR = 1 - \left(\frac{AMAPE_{x,t_n+1}}{AMAPE_{x,t_n+N}}\right) \quad (34)$$

Table 5 shows the measure values:

Remark 3. All models with and without Bühlmann Credibility for Kenyan Males and Females are shown in Table 4. In all the three models, the ratios have improved from the classical model for all the projected years. For example, in model A, values for males and females were 14.85 and 12.63, respectively, compared to 8.56 and 7.25 under the EW strategy. For Model B, in 2010, the values for males and females are 11.45 and 14.33, respectively, compared to the given values of 7.85 and 8.11 under the EW strategy. In Model C, for instance, in 2010, the values for males and females are 15.95 and 12.44, respectively, compared to 7.96 and 7.35 under the EW strategy.

We make conclusions from numerical calculations that incorporating the Bühlmann credibility method into the models A, B, and C have significantly improved

Table 5. Error Measures For Models A, B, and C Under with and without Bühlmann Credibility for Males and Females(In brackets), respectively

Kenya	Model A	AMAPE			RR		RMSE	
	Classical Model %	EW%	MW%	EW%	MW%	EW%	MW%	Z
2010	14.85(12.63)	8.56(7.25)	9.55(8.95)	36.88(33.45)	33.56(32.65)	123.50(118.26)	182.20(180.18)	0.857
2020	16.45(14.85	8.23(7.92)	10.16(9.14)	38.35(36.25)	36.89(34.75)	126.45(122.34)	193.47(188.22)	0.925
2030	18.86(15.24)	9.34(8.35)	10.89(9.85)	39.89(37.56)	37.25(36.25)	132.75(126.36)	195.89(190.26)	0.950
Average	**16.72(21.36)**	**8.71(7.84)**	**10.20(9.31)**	**38.37(35.75)**	**35.9(34.55)**	**127.56(122.32)**	**190.52(186.22)**	**0.910**
Kenya	Model B	AMAPE			RR		RMSE	
	Classical Model %	EW%	MW%	EW%	MW%	EW%	MW%	Z
2010	11.45(14.33)	7.85(8.11)	9.55(9.22)	36.88(35.55)	33.56(34.45)	122.33(116.75)	202.26(195.05)	0.857
2020	12.05(15.25)	8.25(8.44)	10.15(9.45)	37.85(36.45)	36.55(35.41)	128.36(117.82)	200.53(198.85)	0.925
2030	13.25(15.83)	9.45(9.45)	11.23(9.90)	38.15(37.10)	37.85(36.99)	134.37(118.27)	201.55(196.89)	0.950
Average	**12.25(15.14)**	**8.52(8.67)**	**10.31(9.52)**	**37.63(36.37)**	**35.99(35.62)**	**128.35(118.62)**	**201.45(196.93)**	**0.910**
Kenya	Model C	AMAPE			RR		RMSE	
	Classical Model %	EW%	MW%	EW%	MW%	EW%	MW%	Z
2010	15.95(12.44)	7.96(7.35)	8.65(9.15)	35.35(35.54)	32.40(32.85)	140.33(132.52)	210.15(201.22)	0.900
2020	16.20(12.90)	8.15(8.55)	8.96(9.35)	35.95(35.80)	33.15(33.00)	141.65(134.35)	209.45(199.35)	0.925
2030	17.28(13.35)	9.06(8.95)	9.45(9.88)	36.55(36.05)	34.25(33.55)	139.45(136.36)	211.35(199.95)	0.975
Average	**16.48(12.90)**	**8.39(8.28)**	**9.02(9.46)**	**35.95(35.80)**	**33.27(33.13)**	**140.48(134.42)**	**210.32(203.17)**	**0.933**

their forecasting performances using the two strategies, thus contributing to similar prediction performances.

5 Actuarial valuation of Kenyan life assurance and annuities products

5.1 Valuation of life assurance products

We apply the predicted cohort mortality to value life assurance/annuities products (whether whole-life or endowment (pure/term) life assurance/annuities during a specified period as issued to a given insured aged x in year $t_N + 1$ see, (Mitchell et al., 2013). For assurances, we define $A_{1\overline{x:n|}}$, as the expected present value of a temporary life assurance of amount 1 payable to a life aged x at the end of the year of death during n years, will be $A_{1\overline{x:n|}} = \sum_{k=1}^{N-1} {}_kp_x(q_{x+k:t+k+1})v^t$=. We calculate the value using the predicted probabilities with an assumption that interest rates are charged at 4 % per year and v as a discounting factor. The same procedure is followed when calculating the values of other types of Assurance.

We apply the use of predicted mortality rates, which are based on the stated models; A, B, and C without Bühlmann credibility and with the EW strategy when pricing these life insurance products for x = 25,...,100 with N = 10, 20 and 30, respectively. This is derived from the following equation:

$$AMAPE^{A_x}_{x,t_n+1} = \frac{\widehat{A}_{1\overline{x:n|}} - A_{1\overline{x:n|}}}{A_{1\overline{x:n|}}} \tag{35}$$

where $A_{1\overline{x:n|}}$ denotes the Expected Present Value (EPV) of temporary life assurance with the actual mortality rates and $\widehat{A}_{1\overline{x:n|}}$ is EPV of temporary life assurance with corresponding forecasted ones.

Similarly, we define an annuity denoted by $a_{1\overline{x:n|}}$, as a temporary annuity payable in arrears by a life aged x for a period of n years. Mathematically speaking, $a_{1\overline{x:n|}} = \sum_{k=1}^{N-1} {}_kp_x(v^k)$, with an assumption that interest rates being charged at 4 % per year, we can calculate the value using the predicted probabilities. Similarly, this can be done for the whole life annuity, a_x, whether payable in arrears, advanced or continuously. We also measure the relative error in between two expected net present values determined from the actual and predicted mortality rates as

$$AMAPE^{a_x}_{x,t_n+1} = \frac{\widehat{a}_{1\overline{x:n|}} - a_{1\overline{x:n|}}}{a_{1\overline{x:n|}}} \tag{36}$$

where $a_{1\overline{x:n|}}$ denotes the EPV of a temporary life annuity with the real mortality rates and $\widehat{A}_{1\overline{x:n|}}$ is EPV of temporary life assurance with corresponding predicted one.

Therefore, incorporation of the Bühlmann credibility has reduced the MAPEs and the differences between A, B, and C models; simultaneously, the MAPEs from the stated three models with credibility are almost similar for all cases. For both Kenyan Males and females where the MAPEs are under model A, it would differ slightly from those under models B and C.

From the numerical illustrations from Table 6, it is easy to note that incorporating the three Bühlman's credibility into the mortality models, namely LC, CBD, and LR, improves forecasting performances significantly. From all the models above for both males and females, the life assurances improved; for instance, in model A, we have 9.12 compared to 2.55 in EW strategy for security and 77.52 compared to 21.68 in EW strategy for an annuity. Besides, the proposed two strategies contribute to similar forecasting performances.

Incorporating the Bühlmann credibility drives the forecasting MAPEs from the three mortality models that ultimately converge to a consistent level.

Table 6 shows the numerical results that indicate strong evidence that incorporating the Bühlmann credibility approach improves the forecasting performances of the above-stated three underlying mortality models when calculating the values of Assurances sold in the Kenyan market.

6 Conclusion and recommendations

The study has shown that incorporating the Bühlmann credibility approach improves the accuracy of the SMR models (LC, CBD, and Linear relational (LR)) in Kenya from lower MAPE, RR, and RMSE. In addition, the EW strategy comes with two invariant properties, enhancing computation of the Bühlmann credibility estimates instead of using the MW strategy. In contrast, the MW computing strategy has a consistent Bühlmann credibility estimate or factor that generally produces lower AMAPE, RR, and RMSE ratios than the EW strategy, as shown in Table 4. Furthermore, the results have shown that life assurance and annuities are better estimated and priced under the Bühlmann credibility approach than the classical models, as shown in Table 5.

By modelling the error terms of the classical models (LC, CBD, and LR) as a Normal Inverse Gaussian (NIG) distribution through the Bühlmann credibility approach improves the precision of SMR that is important in actuarial valuation. This novel approach demonstrates an improved precision of the predicted SMR as shown by the values of MAPE and RMSE measures compared to those under classical mortality risk models. Ultimately, the calculated actuarial valuations of annuities and assurances using our determined SMR have shown that the BCA approach improves the accuracy of actuarial products sold in the Kenyan market, as shown by AMAPE, RR, and RMSE ratios.

On recommendations for further research, we proposethe application of Bühlmann Credibility-based approaches when Modeling mortality risks for the multi-dimensional populations, especially for those countries with similar demographic characteristics in Sub-Saharan African Populations such as Kenya. This area of the Hierarchical Credibility approach to modeling mortality is an area that future researchers can explore when looking at the interdependence of the different countries regarding their demographic features.

On policy change recommendations, the results of the study should inform policymakers like the Insurance regulatory authority (IRA) and Retirement Benefits Authority (RBA) to come up with measures on how to improve the precision of actuarial products (annuities and assurances) sold in the Kenyan market.

Author's statement

Systematic Mortality risk modeling in Sub-Saharan African countries is key in the revolution of the financial sectors for the safety of the policyholders. Systematic mortality risk modelling plays an important role in determining the values of actuarial products sold in the market especially in the insurance industry. However, the availability of data determines the accuracy of the models used in pricing. The use of Bühlmann credibility approach is vital in improving the precision of the forecasted systematic mortality risk (SMR) of Sub-Saharan African countries, such as Kenya. Through the incorporation of the Bühlmann credibility approach into three commonly used classical stochastic mortality modeling models can help increase their individual predicting capabilities that are needed in Sub-Saharan African countries. Furthermore, the aim of the research is to reduce losses associated with poor modelling of systematic mortality risk that has cost many Sub-Saharan Africans lose money through the financial products sold

Table 6. Actuarial Valuation of temporary life Assurance (top) and temporary annuities (bottom) for Males and Females, respectively

			Model			EW	Strategy		Reduction	Ratio
N	Sex	A	B	C	A	B	C	A	B	C
10	Male	9.12	16.66	13.55	2.55	3.45	3.88	62.44	80.55	74.56
	Female	9.45	14.54	13.95	2.45	2.11	2.22	78.15	63.45	86.45
20	Male	18.22	20.15	18.55	12.65	12.94	12.75	32.55	38.75	24.08
	Female	15.56	12.45	12.20	7.35	6.95	6.99	48.90	7.15	44.05
30	Male	30.54	26.80	25.50	26.87	25.50	25.45	19.86	2.95	−7.45
	Female	16.56	18.66	9.35	7.25	4.90	4.85	58.10	45.53	48.45
	Average	**16.58**	**16.06**	**15.52**	**9.85**	**9.31**	**9.35**	**50.00**	**39.73**	**45.02**
			Model			EW	Strategy		Reduction	Ratio
N	Sex	A	B	C	A	B	C	A	B	C
10	Male	77.52	16.66	115.18	21.68	29.33	32.98	530.44	683.55	632.76
	Female	80.25	123.60	118.58	20.84	17.94	18.87	664.25	636.85	734.85
20	Male	154.82	171.25	158.55	157.65	109.94	110.05	280.25	280.80	208.25
	Female	132.25	112.50	105.82	64.75	56.05	54.69	48.90	64.75	374.45
30	Male	253.84	226.90	215.85	26.87	225.35	215.85	169.60	21.55	8.95
	Female	136.68	158.90	79.55	61.75	41.95	42.00	508.20	345.50	418.50
	Average	**139.23**	**134.96**	**132.25**	**58.92**	**80.09**	**79.08**	**366.94**	**338.83**	**396.29**

in the respective countries.

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PUBLIC INTEREST STATEMENT

Like many other sub-Saharan African countries, Kenya lacks adequate data to help in mortality modeling and projections. Therefore, it calls for modifying the classical mortality models to improve the efficiency and accuracy during systematic mortality risk modeling and estimation. This paper enhances the precision of the mortality models used in developed countries rich in collected mortality data. We have improved some of the commonly used mortality models to fit the limited Kenyan mortality data, thus increasing the accuracy of the Systematic Mortality Risk (risk of death) by Integration of the Bühlmann Credibility Approach. In this study, we have determined the improved precision of the forecasted systematic mortality risk (SMR) of the Kenyan population by incorporating the Bühlmann credibility approach. The findings will help the insurance companies in Kenya during the pricing of many life assurance products (assurances and annuities) sold today for policyholders.

Data availability

The data used to support the findings of this study are available from the corresponding author upon request.

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Appendix: The Parameter estimations for the Three Mortality Models

1. *The LC Model*

On the LC model, it is important to note that the parameters are subjected to two constraints namely $\sum_{x=1}^{n}\beta_x = 1$ and $\sum_{t=1}^{n} k_t = 0$ as well as the estimations using the method of singular value decomposition (SVD). From the constraint that $\sum_{t=1}^{n} k_t = 0$, it is key to note that the parameter of a_x denoted as $\hat{a}_x$ can be estimated as;

$\hat{a}_x = \frac{1}{n}\sum_{t=1}^{n} ln(m(x,t))$ for values of $t = 1, 2, \ldots\ldots n$. In addition, the constraint of $\sum_{x=1}^{n}\beta_x = 1$ will lead to the estimates of k_t, which is $\hat{k}_t$ as follows;

$$\hat{k}_t = \sum_{t=1}^{n}[ln(m(x,t)) - \hat{a}_x]$$

for values of $t = 1, 2, \ldots\ldots n$. The value of $\hat{\beta}_x$ is obtained through the process of regression of the $[ln(m(x,t)) - \hat{a}_x]$ on the value of $\hat{k}_t$ without involving the constant term being included in all ages of x.

The drift parameter, ϑ, the variance of the time trend error, σ_e^2, and the variance of the model error, σ_x^2, are estimated by the process of;

$$\begin{cases} \hat{\vartheta} = & \frac{1}{n-1}\sum_{t=1}^{n}(k_t - k_{t-1}) = \frac{(k_t - k_1)}{n-1} \\ \sigma_e^2 = & \frac{1}{n}\sum_{t=1}^{n}(k_t - k_{t-1} - \hat{\vartheta})^2 \\ \sigma_x^2 = & \frac{1}{n}\sum_{t=0}^{n}[ln(m(x,t)) - \hat{\alpha}_x + \hat{\beta}_x\hat{k}_t]^2 \end{cases}$$

The logarithm of the predicted central death rates for all ages of x in year $t+\eta$ is given by $ln(\hat{m}(x, t+\eta)) = \hat{\alpha}_x + \hat{\beta}_x(\hat{k}_t + \eta(\hat{\vartheta}) = ln(\hat{m}(x,t)) + \hat{\beta}_x(\eta)\hat{\vartheta}$ for all values of $\eta = 1, 2, 3, \ldots$

2. The CBD Model

For the CBD model, the values of $k_t^{(1)}$ and $k_t^{(2)}$ are determined by the $logitq(x,t)$ on the value of $(x - \bar{x})$ at each value of t that satisfies the following condition;

$$\hat{k}_t^{(1)} = \frac{1}{n}\sum_{t=1}^{n}[logitq(x,t) - \frac{\hat{k}_t^{(2)}}{n}\sum_{t=1}^{n}(x - \bar{x})$$

$$\hat{k}_t^{(1)} = \frac{1}{n}\sum_{t=1}^{n} logitq(x,t)$$

The drift parameter, ϑ, the variance of the time trend error, $\sigma_{ei}^2, i = 1, 2$, and the variance of the model error, σ_x^2, are estimated by the process of;

$$\begin{cases} \hat{\vartheta} = & \frac{1}{n-1}\sum_{t=1}^{n}(k_t^{(i)} - k_{t-1}^{(i)}) = \frac{(k_t^{(i)} - k_1^{(i)})}{n-1} \\ \sigma_e^2 = & \frac{1}{n}\sum_{t=1}^{n}(k_t^{(i)} - k_{t-1}^{(i)} - \hat{\vartheta}_i)^2 \\ \sigma_x^2 = & \frac{1}{n}\sum_{t=0}^{n}[ln(m(x,t)) - k_t^{(1)} - k_{t-1}^{(2)}(x - \bar{x})]^2 \end{cases}$$

The *logit* function of the predicted mortality rate $logit\hat{q}(x,t)$ for all ages x in year $t + \eta$ is given by logit $\hat{q}(x,t) = k_t^{(1)} + \eta\hat{\vartheta}_1 + k_t^{(2)} + \eta\hat{\vartheta}_2(x - \bar{x}) = logit\hat{q}(x,t) + \hat{\vartheta}_1 + \hat{\vartheta}_2(x - \bar{x})\eta$ for all values of $\eta = 1, 2, 3, \ldots$

3. The LR Model

The drift parameter, $\vartheta_i, i = 0, 1,$ the variance of the time trend error, $\sigma_{ei}^2, i = 1, 2$, and the variance of the model error, σ_x^2, are estimated by the process of;

$$\begin{cases} \hat{\vartheta}_i = & \frac{1}{n-1}\sum_{t=1}^{n}(k_t^{(i)} - k_{t-1}^{(i)}) = \frac{(k_t^{(i)} - k_1^{(i)})}{n-1} \\ \sigma_{e,i}^2 = & \frac{1}{n}\sum_{t=1}^{n}(k_t^{(i)} - k_{t-1}^{(i)} - \hat{\vartheta}_i)^2 \\ \sigma^2 = & \frac{1}{n(m-1)}\sum_{t=0}^{n}\sum_{x=0}^{m}[ln(m(x,t)) - k_t^{(1)} - k_{t-1}^{(2)}ln(m(x,t-1))]^2 \end{cases}$$

The logarithm of the predicted central death rates for all ages of x in year $t + \eta$ is given by $ln(\hat{m}(x,t+\eta)) = k_t^{(1)} + \eta\hat{\vartheta}_1 + k_t^{(2)} + \eta\hat{\vartheta}_2 = ln(\hat{m}(x,t)) + [\hat{\vartheta}_1 + \hat{\vartheta}_2(\eta)ln(\hat{m}(x,t-1)]$ for all values of $\eta = 1, 2, 3, \ldots$