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Master Project

ON THE FUGLEDE-PUTNAM THEOREM AND NORMALITY OF NON-NORMAL OPERATORS IN HILBERT SPACES

By

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Declaration and Approval

I, the undersigned declare that this dissertation is my original work and to the best of my knowledge, it has not been submitted in support of an award of a degree in any other university or institution of learning.

08.12.2021

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In my capacity as a supervisor of the candidate's dissertation, I certify that this dissertation has my approval for submission.

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Dedication

This project is dedicated to myself, my family, my parents,and my friends.

Contents

Acknowledgments

Firstly, I wish to thank the almighty God for enabling me to reach this far.
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To my supervisor Dr.Imagiri, I wish to thank you for the guidance at every juncture of this project without forgetting other lecturers who taught me in the course period.
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1 Introduction

In this chapter, We give terminologies, notations and definitions that will be used throughout the project and also some brief historical backgrounds of some normal and non-normal operators.

1.1 Notations, Terminologies and Definitions

Notations

H: Hilbert space over $\mathbb{C}$
$B(H)$: Banach algebra of bounded operators

T^*: the adjoint of an operator T

$\|Tx\|$: the operator norm of T

$\|x\|$: the norm of a vector x

$\sigma(T)$: the spectrum of an operator T

$\pi_0(T)$: point spectrum of an operator T

$Ran(T)$: the range of an operator T

$Ker(T)$:the kernel of an operator T

$M \oplus M^{\perp}$: the direct sum of the Subspaces M and $M^{\perp}$ of H.

Terminologies and Notations

Throughout this paper, H or K will denote a complex Hilbert space and $B(H)$ will denote the Banach algebra of bounded linear operators on H

We denote the kernel and range of an operator T by $Ker(T)$ and Ran (T) respectively.

M and $M^{\perp}$ stands for the closure and orthogonal complement of of a closed subspace M of H.

We denote by $\sigma(T), \pi_0(T), \|T\|$ and $W(T)$ the spectrum, point spectrum, norm and numerical range of $T \in B(H)$ respectively.

We write $w(T)$ for the Weyle spectrum of T and $\overline{w(T)}$ for the closure of $w(T)$.

We denote the essential numerical range of T by $W_e(T)$ and the set of all isolated points of the spectral of T that are eigenvalues of finite multiplicity by $\sigma_{00}(T)$.

Let M be a closed subspace of H and $T \in B(H)$ be an operator. We denote the restriction of T to M by $T|M$.

Definition 1.1.1 An inner product space is a vector space E together with a map$< . >$: $E \times E \to \mathbb{F}$ such that

(i) $< \lambda x + \mu y, z >= \lambda < x, z > + \mu < y, z >$

(ii)$< x, y >= \overline{< y, x >}$

(iii) $< x, x > \geq 0, < x, x >= 0$ if and only if $x = 0 \forall x, y, z \in E$ and $\lambda \in \mathbb{F}$.

Definition 1.1.2 let X be a vector space and $\|.\| : X \to \mathbb{R} \cdots *$ be a real valued function. Then the function $*$ is called a norm if it satisfies the following.

(i) $\|x\| \geq 0$ and $\|x\| = 0$ if and only if $x = 0$

(ii)$\|\lambda x\| = |\lambda| \|X\| \forall x \in X, \lambda \in \mathbb{R}$

(iii) $\|x + y\| \leq \|x\| \, \|y\| \forall x, y \in X$.

The pair $(X, \|x\|)$ is called a normed space.

Proposition 1.1.3(Cauchy-Schwarz's inequality)

For any two elements x, y in an inner product space X,

$|< x, y >| \leq \|x\| \, \|y\|$.

Definition 1.1.4 An operator $T \in B(H)$ is said to be normal if it commutes with its adjoint(i.e, $T^*T = TT^*$, equivalently, $T^*T - TT^* = 0$)

Preposition 1.1.5 Let T be an operator on a Hilbert Space H. The following assertions are equivalent

(i) T is normal

(ii) $\|T^*x\| = \|Tx\|$ for any $x \in H$

(iii) T^n is normal for any integer ,$n \geq 1$

(iv) $\|T^{*n}x\|^2 = \|T^n x\|^2$.

Remark 1.1.6 By relaxing some conditions of normality of operators, we obtain non normal operators. The results below we check some of these nonnormal operators.

definition 1.1.7 An operator $T \in B(H)$ is said to be quasinormal if $T(T^*T) = (T^*T)T$ $i.e(T^*T - TT^*)T = 0$.

Proposition 1.1.8 A unilateral and bilateral shift operators are quasinormal operators.

Definition 1.1.9 An operator $T \in B(H)$ is said to be subnormal if it has a normal extension.
Proposition 1.1.10 Every Subnormal operator $T \in B(H)$ on a finite dimensional Hilbert Space is normal.
Definition 1.1.11 An operator $T \in B(H)$ is said to be hyponormal if $T^*T \geq TT^*$.
Proposition 1.1.12 An operator $T \in B(H)$ is hyponormal if and only if $\|T^*x\| \leq \|Tx\|$.
Proof
$\Longrightarrow$ If T is hyponormal, then $TT^* \leq T^*T$ if and only if

$$< TT^*x, x > \leq < T^*Tx, x > i.e < T^*x, T^*x > \leq < Tx, Tx >$$

that is $\|T^*x\|^2 \leq \|Tx\|^2$ which implies
$\|T^*x\| \leq \|Tx\|$.
Definition 1.1.13 An operator $T \in B(H)$ is said to be hyponormal if its adjoint T^* is hyponormal.
That is, $TT^* \geq T^*T$.
Proposition 1.1.14. An Operator $T \in B(H)$ is normal if and only if it is both hyponormal and cohyponormal.
Definition 1.1.15. An operator $T \in B(H)$ is semi-normal if it is either hyponormal or cohyponormal.
Definition 1.1.16 An operator $T \in B(H)$ is said to be paranormal if $\left\|T^2x\right\| \geq \|Tx\|^2$ for every $x \in H$.
Definition 1.1.17 An operator T is called normaloid if

$$\|T\| = \sup_{\|x\|} | < Tx, x > |$$

.
Definition 1.1.18 An operator T is called convexoid if the closure of the numerical range equals the convex hull of the spectrum of T.
That is

$$\overline{W(T)} = \overline{\{< Tx, x >: \|x\| = 1\}} = \sigma(T)$$

.
Definition 1.1.19. An operator $T \in B(H)$ is n-normal if $T^nT^* = T^*T^n$.
Remark 1.1.20. The class of all n-normal operators is denoted by $[n\mathbb{N}]$.
Definition 1.1.21. An operator $T \in B(H)$ is said to be binormal if T^*T commutes with TT^*.
Definition 1.1.22. An operator T is said to be w-hyponormal if $|\Delta(T)| \geq |T| \geq |\Delta^*(T)|$.
Definition 1.1.23. An operator $T \in B(H)$ is said to be spectraloid if $W(T) = r(T)$.
Definition 1.1.24. An operator $T \in B(H)$ is said to be a scalar if it is a scalar multiple of the identity operator (i.e $T = \alpha I, \alpha \in \mathbb{C}$).
Definition 1.1.25 An operator T is said to be an isloid if any isolated point of $\delta(T)$ is an eigenvalue of T.

Definition 1.1.26 An operator $T \in B(H)$ is said to be p-hyponormal if $(T^*T)^p \geq (TT^*)^p$.
Definition 1.1.27. An operator $T \in L(H)$ is posinormal if there exists a positive operator $P \in B(H)$ such that $TT^* = T^*PT$.
Definition 1.1.28 An operator $T \in L(H)$ is coposinormal if T^* is posinormal.
Remark 1.1.29. Every hyponormal operator is posinormal but the converse is not generally true.
Proposition1.1.30 Let T be a posinormal operator. Then T is hyponormal if and only if $KerT = KerT^*$.
Definition 1.1.31. An operator $T \in B(H)$ is said to be log-hyponormal if T is invertible and $\log T^*T \geq logTT^*$.
Definition 1.2.32 An operator $T \in L(H)$ is said to be p-posinormal if $(TT^*)^P \leq \alpha(T^*T)^p$ for some $\alpha > 1$ and $P > 0$.
Definition 1.1.33. An operator $T \in B(H)$ is said to be semi-hyponormal if $(T^*T)^{\frac{1}{2}} \geq (TT^*)^{\frac{1}{2}}$.
Definition1.1.34. An operator $T \in L(H)$ is said to be polaroid if every isolated point of the spectrum of T is a pole of the resolvent of T.
Definition 1.1.35. An operator is said to be an adjoint of an operator T if there exists a unique operator $T^* \in B(K,H)$ such that $< Tx, y >=< x, T^*y > \forall x \in H, y \in (K)$. In this case,$T^*$ is called the adjoint of T.
Theorem 1.1.36. For $S,T \in B(H,K)$, the following holds
(i) $\alpha S + T)^* = \bar{\alpha}S^* + T^*$
(ii)$(s^*)^* = S$
(iii$(ST)^* = T^*S^*$
(iv) $I^* = I$,where I is the identity operator in H
(v) $\|T^*x\| = \|T\|^2$ hence $\|T^*\| = \|T\|$.
Definition 1.1.37. An operator $T \in B(H)$ is said to be Hermitian (or self-adjoint) if $T = T^*$.
Remark 1.1.38 Every Hermitian Operator is normal.
Definition 1.1.39. An operator $P \in B(H)$ is said to be idempotent if $P^2 = P$.
Definition 1.1.40. An operator $P \in B(H)$ is said to be a projection if P is idempotent and $Ker(P) = Ran(P)^{\perp}$.
Theorem 1.1.41. Let $P \in B(H)$ be an idempotent operator. Then the following properties are equivalent
(i) P is a projection
(ii) P is the orthogonal projection of H onto Ran P
(iii) $\|P\| = 1$
(iv) P is Hermitian
(v) P is normal
(vi) P is positive
Definition 1.1.42 An operator $U \in B(H)$ is said to be an isometry if $\|Ux\| = \|x\| \forall x \in H$.

Theorem 1.1.43 For an operator $U \in B(H)$, the following are equivalent.
(i) U is an isometry
(ii) $U^*U = I$,the identity in H
(iii) $< Ux, Uy >=< x, y > \forall x, y \in H$
Proof
$(i) \implies (ii)$
$\forall x \in H, < (U^*U - I_H)x, x >= \|Ux\|^2 - \|x\| = 0$
Thus $U^*U - I_H = 0$
which implies $U^*U = I_H$
$(ii) \implies (iii)$
$< Ux, Uy >=< U^*Ux, y >=< x, y >$
$(iii) \implies (i)$
$\|Ux\|^2 =< Ux, Ux >=< x, x >= \|x\|^2$
Definition 1.1.44 An operator $U \in B(H, K)$ is said to be a partial isometry if it satisfies the following conditions.
(i) $U = UU^*U$
(ii) $P = U^*U$ is a projection
(iii) $U|_{Ker^{\perp}U}$ is an isometry.
Definition 1.1.45. An operator $T \in B(H)$ is said to be co-isometry if $TT^* = I$ (that is ,T is a co-Isometry if T^* is an Isometry).
Definition 1.1.46. An operator $U \in B(H)$ is called unitary if $T^*T = TT^* = I$.
Remark 1.1.47. An operator $U \in B(H, K)$ is unitary if and only if U is both an isometry and co-isometry.
Remark 1.1.48. A unitary operator is normal.
definition 1.1.49. An operator $T \in B(H)$ is said to be an essential isometry if $T^*T - I$ is compact.
Definition 1.1.50. An operator $T \in B(H)$ is said to be an essential co-isometry if $TT^* - I$ is compact.
definition 1.1.51. An operator $t \in B(H)$ is said to be positive if $\forall x, y \in H, < Tx, y >\geq 0$ and T is Hermitian.
Remark 1.1.52. Every projection operator is positive
Definition 1.1.53 An operator $T \in B(H, K)$ is invertible if there exists an operator $S \in B(K, H)$ such that $ST = I$ and $TS = I$.
Remark 1.1.54 An invertible operator is denoted by T^{-1}.
Proposition 1.1.55. For S and T invertible operators, then the following equality holds true.
$(TS)^{-1} = S^{-1}T^{-1}$
Proof

$$(TS)(S^{-1}T^{-1}) = T(SS^{-1})T^{-1} = TIT^{-1} = TT^{-1} = I$$

and

$$(S^{-1}T^{-1})(TS) = S^{-1}(T^{-1}T)S = S^{-1}IS = S^{-1}S = I$$

Remark 1.1.56 A unitary operator is invertible.
Definition 1.1.57 An operator $T \in B(H)$ is said to compact if for every bounded sequence,$\{x_n\}$ in H, the sequence $\{Tx_n\}$ has a subsequence which converges in H.
Definition 1.1.58 An operator $T \in B(H,K)$ is said to be a Hilbert-Schmidt operator if it satisfies the following conditions.
(i) $\sum_n \|T_{en}\|^2 < \infty$ for some orthonormal basis $\{e_n\}$ of H
(ii)$\sum_m \|T^* f_m\|^2 < \infty$ for some orthonormal basis $\{f_m\}$ in K
(iii) $\sum_n \|Te_n\|^2 < \infty$ for all orthonormal basis $\{e_n\}$ of H.
Definition 1.1.59. An operator $T \in B(H)$ is called Fredholm if there exists an operator S such that the operators $ST - I$ and TS_I are compact
Remark 1.1.60 Let T be a Fredholm operator. The index of T denoted by $IndT$ is defined by
$IndT = dimKerT - dimKerT^*$.
Remark 1.1.61 If T and S are Fredholm operators, then TS is a Fredholm operator and $IndTS = IndT + IndS$.
Definition 1.1.62 An operator S on H is called (Unilateral)shift operator if $S_{e_n} = e_n + 1, n = 1, 2\cdots$ for some orthonormal basis $\{e_n\}$ of H.
definition 1.1.63 An operator $T \in B(H)$ is called anti-adjoint (equivalently skew Hermitian) if $T^* = -T$.
Definition 1.1.64 An operator $T \in B(H)$ is called a semi-shift operator if
(i) T is an Isometry
(ii) $\cap_{i=1}^{\infty} RanT = 0$.
definition 1.1.65. An operator T is said to be Backward shift if it satisfies the following conditions
(i) dim $(kerT) = I$.
(ii) The Induced operator $\hat{T} : x/kerT \to X$ defined by $\hat{T}(x + KerT) = Tx$ is an isometry.
(iii) $\cup_{n=1\infty} KerT^n$ is dense in X
Definition 1.1.66 An operator $T \in B(H)$ is called a left shift operator if $Tx = y$ where $x = (x_1, x_2\cdots)$ and $y = (x_2, x_3\cdots) \in l^2$.
Definition 1.1.67. An operator $T \in B(H)$ is called a right shift operator if $T(x_1, x_2, \cdots) = (0, x_1, x_2, \cdots) \in l^2$.
Definition 1.1.68 An operator V for a function $f \in l^2[0,1]$ and a value $t \in [0,1]$ defined by $V_(f)(t) = \int_0^t f(s)d_s$ is called a volterra operator.
Remark 1.1.69 V is bounded. We note that V is a Hilbert-Schmidt operator and hence in particular compact.
Definition 1.1.70. The rank of an operator $T \in L(H,K)$ is the dimension of range of T.
1.1.71 An operator $T \in L(H)$ is a finite rank operator if Ran(T) is finite dimension.
Remark 1.1.72 A finite rank operator need not be bounded.

Proposition 1.1.73 If an Operator T is bounded, linear and has a finite rank, then T is compact.
Remark 1.1.74. Every $T \in B(H)$ finite rank operator is compact.
Definition 1.1.75. Suppose $1 \leq P \leq \infty$, the Hardy space H^p is defined by $H^p = \{f \in L^p(T) | \hat{f}(n) = 0, n < 0\}$.
Remark 1.1.76 H^2 endowed with the l^2-scalar product is a Hilbert space with an orthonormal basis.
Definition 1.1.77 An operator T_α is said to be Toeplitz if $T_\alpha : H^2 \to H^2, f \to P(\alpha f)$ where p is the projection of l^2 ontoH^2 and $\alpha \in l^\infty(T)$.
Remark 1.1.78. A Toeplitz operator is self adjoint
Definition 1.1.79 An operator $T \in B(H)$ is said to be involutive if if $T^2 = I$.
Definition 1.1.80 An operator $T \in B(H)$ is said to be a contraction if $\|T\| \leq 1$.
Definition 1.1.81. An operatot $T \in B(H)$ is called diagonalisable if there exists an orthonormal basis $\{e_n\}$ for H consisting of eigenvectors of T.
Remark 1.1.82 Any unitary operator on a finite dimensional complex Hilbert Space is diagonalisable.
Definition 1.1.83 An operator $T \in B(H)$ is called a numeroid if $W(T)$ is a spectral set for T.
Definition 1.1.84 An operator $T \in B(H)$ is said to be co-subnormal if its adjoint is subnormal.
Definition 1.1.85 An operator $T \in B(H)$ is called co-paranormal if its adjoint is paranormal.
Definition 1.1.86 Operator radius of an operator $T \in B(H)$ is defined by $w_p(T) = \inf\{U : U > 0, U^{-1}T \in C_p\}, 0 < p < \infty$.
Definition 1.1.87 An operator $T \in B(H)$ is said to be of class $C_p(p \geq 0)$ if there exists a unitary operator U on $B \in B(K)$ such that $T^n x = {}_p pU^n x$ for $n = 1, 2\cdots, x \in H$.
Definition 1.1.88 An operator $T \in B(H)$ is called $p - oid$ if $W(T^K) = (W(T))^k, k = 1, 2, \cdots$.
Remark 1.1.89$1 - oid$ and $2 - oid$ operators are normaloids and spectraloid operators respectively.
Definition 1.1.90 An operator $T \in B(H)$ is called a p-convexoid if $W(T) = conv.\sigma(T)$.
Definition 1.1.91 An operator $T \in B(H)$ is said to be reduction-p if the restriction of T to every invariant subspace of T has property p.
Definition 1.1.92 The Aluthge transformation of an operator $T \in B(H)$ (denoted by $\tilde{T}$) is defined as
$\tilde{T} = |T|^{\frac{1}{2}} U |T|^{\frac{1}{2}}$ for a unitary operator U.
Definition 1.1.93 An operator $T \in B(H)$ is called n-power quasinormal if $T^n T^* T = T^* T T^n = T^* T^{n+1}$.
Remark 1.1.94 The class of n-power quasinormal operators is denoted by $[nQN]$.
Remark 1.1.95 When $n = 1$, an n-power quasinormal operator is quasinormal

Definition 1.1.96 An operator $T \in B(H)$ is called m-partial isometry if it satisfies
$TB_m(T) = T\sum_{k=0}^{m} \begin{bmatrix} m \\ k \end{bmatrix} (-1)^k T^{*m-k}T^{m-k} = 0$
where $B_m(T)$ is obtained from the binomial expansion of $B_m(T) = (T^*T - I)^m$.
Remark 1.1.97 When $m = 1$, T is called partial isometry
Definition 1.1.98 An operator $T \in B(H)$ is called dominant if $Ran(A - \lambda I) \subseteq Ran(A - \lambda I)^* \forall \lambda \mathbb{C}$.
Definition 1.1.99 An operator $T \in B(H)$ is called Browder if T is Fredholm and $T - \lambda I$ is invertible for sufficiently small $\lambda \neq 0 \in \mathbb{C}$.
Definition 1.1.100 The essential spectrum of T (denoted by $\sigma_e(T)$) is defined by
$\sigma_e(T) = \{\lambda \in \mathbb{C} : T - \lambda I$ is not Fredholm$\}$.
1.1.101 The Browder spectrum (denoted by $\sigma_b(T)$) of T is defined by
$\sigma_b(T) = \{\lambda \in \mathbb{C} : T - \lambda I$ is not Browder$\}$.
Definition 1.1.102 An operator $T \in B(H)$ is said to be resuloid if $T - \lambda I$ is regular for each $\lambda \in Iso\sigma(T)$.
Definition 1.1.103 An operator $T \in B(H)$ is said to be closoid if $Ran(T - \lambda I)$ is closed for each $\lambda \in Iso\sigma(T)$.
Definition 1.1.104 An operator $T \in B(H)$ is called $(\alpha, \beta) - normal(0 \leq \alpha \leq 1 \leq \beta)$ if $\alpha^2 T^*T \leq TT^* \leq \beta^2 T^*T$.
Definition 1.1.105 An operator $T \in B(H)$ is said to be m-hyponormal if there exists a positive number m such that
$m^2(T - \lambda I)^*(T - \lambda I) - (T - \lambda I)(T - \lambda I)^* \leq 0 \forall \lambda \in \mathbb{C}$.
Definition 1.1.106 An operator $T \in B(H)$ is said to be quasi-invertible if T has zero kernel and dense range.
Definition 1.1.107 An operator $T \in B(H)$ is called transloid if $aT + bI$ is normaloid $\forall a, b \in \mathbb{C}$.
Remark 1.1.108 Every Transloid is convexoid
Definition 1.1.109 An operator $T \in B(H)$ is said to be a class $\mathbf{y}_\alpha$ operator(for $\alpha \geq 0$) if there exists a positive number $k_\alpha$such that
$|TT^* - T^*T|^\alpha \leq K^2{}_\alpha(T - \alpha)^*(T - \alpha) \forall \lambda \in \mathbb{C}$.
Remark 1.1.110 A class $\mathbf{y}$ operator id m-hyponormal.
Definition 1.1.111 An operator N is called Julia operator (denoted by J(N)) if

$$J(N) = \begin{bmatrix} (1 - NN^*)^{\frac{1}{2}} & N \\ -N^* & (1 - N^*N)^{\frac{1}{2}} \end{bmatrix}$$

. **Definition 1.1.112** An operator $T \in B(H)$ is called p-quasihyponormal if $T^*\left((T^*T)^p - (TT^*)^p\right)T \geq 0$ for $p \geq 0$.
Remark 1.1.113 If $p = 1$, Then T is quasihyponormal
If $p = \frac{1}{2}$,then T is semi-quasihyponormal.
Definition 1.1.114 An operator A is said to be $p - w-$hyponormal operator if $|\tilde{A}|^p \geq |A|^p \geq |\tilde{A^*}|^p$.

Definition 1.1.115 Let $A,B \in B(H)$ be operators. We define the generalized derivation(denoted by $\delta_{A,B}(X)$) induced by A and B as
$\delta_{A,B}(X) = AX - XB \forall X \in B(H)$.
Remark 1.1.116 The class of Hilbert-Schmidt operators is denoted by $C_2(H)$.
Remark 1.1.117 $C_2(H)$ is a Hilbert space .The Hilbert-Schmidt norm of $X \in C_2(H)$ is given by $\|x\|_2 = <X,X>^{\frac{1}{2}}$
Definition 1.1.118 An operator A is said to be $(p,k)-$quasiposinormal if $A^{*k}\left(C^2(A^*A)^p - (AA^*)^p\right)A^k \geq 0$ for some positive integer $0 < p \leq 1$, some $C > 0$ and a positive integer k.
Definition 1.1.119 Two operators $A,B \in B(H,K)$ are said to be similar if there exists an invertible operator $N \in B(H,K)$ such that $NA = BN$.
Definition 1.1.120 Two operators $A,B \in B(H,K)$ are said to be unitarily equivalent if there exists a positive unitary operator $U \in B(H,K)$ such that $UA = BU$.
Definition 1.1.121 A subspace $M \subseteq H$ is said to be invariant under $T \in B(H)$ if $TM \subseteq M$.
Definition 1.1.122 A subspace $M \subseteq H$ is said to be a reducing subspace of $T \in B(H)$ if it is invariant under both T and T^*.

1.2 Normal and Non-normal operators

Normal operators

The study of normal operators has been very successful in the sense that a lot of interesting results have been obtained concerning these operators.
One of the main results of these operators is the classical Fuglede-Putnam theorem that we will discuss in detail in this research paper and the spectral theorem that only holds for normal operators.
Many authors have defined new classes of operators by making them satisfy certain known properties of normal operators in the hope that some of the results which holds for normal operators will also hold for these new classes of operators. This has led to a new area of research on non-normal operators which are as a result of relaxing normality of normal operators.

It is well known that given two normal operators $A,B \in B(H)$,$A+B$, and AB are not normal in general. For example, consider the operators

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

and

$$B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

A simple computation shows that A and B are normal but AB is not.

The question on characterizing those pairs of normal operators for which their products are normal was studied for finite dimensional spaces by Gantmaher and Krein[20] in 1930 and Weigmann[63] for compact operators. However, it is important to point that normality of AB does not always imply normality of BA.
In 1953, Kaplansky[27] showed that under the compactness assumptions, the normality of AB and BA are equivalent. Later on, Kittaneh considered this question in [31] and showed that it is sufficient to assume that A and B^* be hyponormal and that AB be compact in order to conclude that BA is normal.
Gheondea proved the Gantmaher-Krein-Weigmann theorem in [21].
On the normality of any pair of normal operators, it is well known that if each of two normal operators commute with the adjoint of each other, then their sum is normal and so is their product. That is, if A and B are normal operators such that A commute with B^*, then $A+B$ is normal and so is AB.
Yadav and Ramanujan[66] proved that if the real part of each of two normal operators commutes with the imaginary part of the other, then their sum is normal. Mortad also showed in [40] that for two bounded operators $A, B \in B(H)$, $A+B$ is normal if AB and AB^* are normal such that A is positive.
In 1970, Embry[15] introduced the concept of similarities of normal operators by stating that:
if S and T are two commuting normal operators and $AS = TA$, where $0 \notin W(A)$ for $A \in B(H)$, then $S = T$.
Mortad [41] generalized Embry's theorem by imposing a self-adjointeness condition on A and dropping the commutativity of S and T and came up with the following result.
Let A be bounded self-adjoint operator such that $0 \notin W(A)$. If S and T are bounded normal operators such that $AS = TA$, then $S = T$.
Apart from the operations of normal operators, other authors have given more properties of normal operator. For instance, Putnam[48]has given sufficient condition that a square root of a norma operator is normal. Stampfli[58] also showed a result that an nth root of of an invertible normal operator is similar to a normal operator. Radjavi and Rosenthal[51] gave a clear representation of all square roots of normal operators.
We also note that if T is normal, then any polynomial of T is normal but the converse is not true in general. Kittaneh[32] has shown that if T^n is normal foe some $n > 1$, then T is quasi-similar to a direct sum of a normal operator and a compact operator. The author also showed that if $P(T)$ is normal for some nonzero polynomial P and T is essentially normal, then T can be written as a sum of a normal operator and a compact operator.

Thus, the class of normal operators has given birth to various areas of research and many authors today are still trying to relax their normalities to obtain new classes of operators that satisfies some results are only satisfied by normal operators.

Non-normal operators

One of the main results of normal operators is the class of non-normal operators which is achieved by relaxing some normality conditions of normal operators. This class has led to a wide area of research notably the Fuglede-Putnam theorem where many authors have come up with various classes of non-normal which under some conditions, satisfy the Fuglede-Putnam theorem.
As we have seen from section 1.1, there are many classes of non-normal operators and many more are still introduced even today and each of these operators has their own properties which makes them unique from the others. We shall see some of these properties in section 2.
One of the main classes of these operators is the class of subnormal operators. This class was introduced by Hamos in [23] who later defined the concept of hyponormal operators in 1950 by bringing the definition $T^*T \leq TT^*$. By considering the case where $TT^* \leq T^*T$, the class of cohyponormal was thus introduced. This enabled Brown[?] to introduce and study the class of quasinormal operators. As we shall show later on, it was proved that every quasinormal operator is subnormal.
The class of posinormal operators (or positive normal operators) was introduced by Rhaly in [52] and further studied in [24] where the authors studied more properties of this class and showed that Weyl's theorem holds for some totally posinormal operators.
Jibril[26] introduced the class of n-power normal operators and proved that an operator $T \in L(H)$ is n-power normal if and only if $\|T^n x\| = \|(T^n)^* x\|$ for all $x \in H$ and also gave some properties of n-power operators. Alzuraiqi and Patel studied further properties in of this class in [5]. As seen in subsection 1.1, this class is denoted by $[nN]$ for all positive intergers n.
Ahmed [1] continued the work of Jibril by introducing the class of n-power quasinormal operators and showed some relations between n-normal and quasi-normal operators. These relations will be investigated in section 2. This class is denoted by $[nQN]$.

1.3 Historical development of Hilbert Spaces

Functional analysis is a very important branch of mathematics that has found numerous applications both in mathematics world and other fields such as engineering and computer science among many others .

One of the cornerstones of Functional analysis is the notion of a Hilbert Space.
Hilbert Space emerged from the German mathematician David Hilbert's(1862-1943) efforts to generalize the concept of Euclidean Space to an infinite dimensional space.
He formulated the theory of square summable spacel^2.

However, it would be interesting to note that although Hilbert is considered to be the father of Hilbert Spaces, it was not until years later that von Neumann (1903-1957) gave the definition of a Hilbert space in 1927.
In his work, Neumann formulated an axiomatic theory of Hilbert Space and developed the modern theory of operators in Hilbert spaces .
Although these two great mathematicians contributed highly to the growth and development of Hilbert spaces, we note that other mathematicians contributed a lot to this development
The notion of a 'Space' was introduced by Riemann in his work in 1856 and was also the one who conceived the idea of a closed subspace of a Hilbert space(manifold).
Between 1844 and 1862, Herman Grassman(1809-1877) introduced the concept of a finite dimensional vector space.
Karl Heinstrass (1815-1897)considered the distance between two functions in the context of the calculus of variations but it was not until 1897 that Jacques Hadamand(1865-1963) gave a boost to Hilbert space theory by connecting the set theoretic ideas of Cantor with the notion of a space of functions.
However it was not until 1906 that Hadamand's PhD student, Maurice frechet, (1878-1973) astounded the mathematics world by introducing the concept of a metric space.
In 1916, the notion of a topological space was introduced by Felix Hausdorff (1868-1942) which was a crucial boost to the Hilbert theory. Topological spaces have become widely applicable in functional analysis and their contributions cannot be underrated.
Many other modern mathematicians, the likes of Schmidt, have greatly contributed to what we now love and know as the Hilbert Space Theory.The theory has not only enriched the world of mathematics but has proven extremely useful scientific theories. Hilbert spaces play a central role in analysis, mostly functional, geometry group theory and and number theory among others.

1.4 Historical development of Fuglede-putnam's theorem

The original paper of Fuglede first appeared in 1950[16] where the author proved the Fuglede's theorem. This was in answering a problem posed by John Von Neumann [44] in 1942.
In his theorem, Fuglede was able to prove the following result;
Let A and B be bounded operators on a complex Hilbert space with B being normal. If $AB = BA$,then $AB^* = B^*A$.

However, in 1958, Putnam generalized[49]Fuglede's theorem by proving the following;
If A,B,X are linear operators on a complex Hilbert Space and suppose X and B are normal, B is bounded and $BA = AX$,then $B^*A = AX^*$.
Berberian[10] proved that the Fuglede theorem was actually equivalent to that of Putnam by a nice operator matrix derivation trick. Thus Fuglede -Putnam theorem was born and it states as follows;
Let A and B be normal operators and X be an operator such such such that $AX = XB$, then $A^*X = XB^*$. Berberian was able to relax the hypothesis on A and B by requiring X to be a Hilbert -Schmidt operator(i.e $X \in C_2(H)$).
After the work of these great mathematicians, several authors have relaxed the normality of A and B in the Fuglede-Putnam's theorem over the years in various ways.
In 1958, Rosenblum[56] gave a simple and clear proof of Fuglede Putnams' theorem by using lioville's theorem.
Later on, M. Radjabalipour [50] (1987)showed that Fuglede-Putnam's theorem holds for hyponormal operators.
In 1994, Cha[13] showed that the hyponormality hypothesis can be replaced by the quasi-hyponormality of A and B^* under some conditions in the Fuglede-Putnam's theorem.
B.P. Duggal [14] showed that if A, B^* are p-hyponormal operators , then A and B satisfy Fuglede-Putnam's theorem.
In 1997, Lee[33] proved that if if A is p-quasihyponormal operator and B^* is an invertible p-quasihyponormal operator such that AX=XB for $X \in C_2(H)$ and

$$\left\| |A|^{1-p} \right\| . \left\| |B^{-1}|^{1-p} \right\| \leq 1$$

, then Fuglede-Putnam theorem holds(that is $A^*X = XB^*$).
In 1996, Patel[46] proved that for a Hilbert Schmidt operator X and A and B^* being p-hyponormal operators such that $AX = BX$, then $A^*X = XB^*$.
Uchiyama and Tanahashi[62](2002) showed that the Fuglede-Putnam theorem holds for p-hyponormal and log-hyponormal operators.
In 2005, Mecheri[38] showed that Lee's results remain the same without the condition

$$\left\| |A|^{1-p} \right\| . \left\| |B^{-1}|^{1-p} \right\| \leq 1$$

.
The author further showed that Lee's results remain true for $(p,k)-$quasihyponormal operators without the additional condition

$$\left\| |A|^{1-p} \right\| . \left\| |B^{-1}|^{1-p} \right\| \leq 1$$

.
Kim [28] showed that the result $A^*X = XB^*$ remains remains valid for an injective $(p,k)-$quasihyponorm
and log-hyponormal operator.

In 2009, Bakiri [9] showed that that if A is an injective (p,k)-quasihyponormal in H and B is a dominant operator in H such that $AX = XB$ for some $X \in B(H)$, then $A^*X = XB^*$. The author also showed that the above result remains valid for injective $(p,k)-$quasihyponormal and log hyponormal operators.
Mecheri and Uchiyama [39] showed that normality in the Fuglede-Putnam theorem can be replaced by A and B^* class operators.
Rashid and Noorani[54] showed that the result by Mecheri and Uchiyama for A and B^* quasi-class A operators with the additional condition

$$\||A^*|\| . \left\||B|^{-1}\right\| \leq 1$$

satisfies the Fuglede-putnam theorem.
As recent as 2012, Bashir et al. [6]proved that the Fuglede-Putnam theorem hold for w-hyponormal operators.
Clearly, Fuglede-Putnam theorem has fascinated many mathematicians in the mathematics world and many mathematicians are working day and night to try and relax the normality of A and B in the theorem.

1.5 Series of inclusion of classes of operators

In this section, we set to investigate some classes of operators and show some inclusion relationship of these operators.
In 1962, Stampfli[59] introduced hyponormal operators and was able to show that any normal operator is hyponormal.
In 1978, Campbell and Gupta[12] introduced k-quasinormal operators for some $k \in \mathbb{C}$. The authors were able to show that by letting $k = 1$, then a hyponormal operator would become a k-quasihyponormal.
It was in 1990 that Aluthge [3] astounded the mathematics world by introducing p-hyponormal operators and illustrated that if we let $p = 1$ in the definition of a p-hyponormal operator, then we get a hyponormal operator.
Another class of operators is the p-quasihyponormal and by letting $p = 1$,we get quasi-hyponormal operators.
Tanahashi[60],in 1999 introduced another class of operators called log-hyponormal operators which contains all invertible hyponormal operators. He was able to demonstrate that invertible p-hyponormal operators are log-hyponormal operators.
Then, in 2000, Aluthge et al.[4] was able to generalise both log-hyponormal and p-hyponormal operators to w-hyponormal operators which contains all p-hyponormal operators.
In 2003, Hyoun[28] introduced $(p,k)-$quasihyponormal operators and showed that if we let $p = 1$and $k = 1$, in the definition of $(p,k)-$quasihyponormal operator, then we get $k-$quasihyponormal and $p-$quasihyponormal operators respectively.

We note that a q-quasihyponormal operator is a $(p,k)-$quasihyponormal since $0<q<p$ and thus (p,k)-quasihyponormal operators contain all p-hyponormal operators.
In 2007, Jibril[26] introduced he class of 2-power normal operators and later generalized the class of 2-power normal operators in the class of n-power normal operators[25].
Later on, in 2011, Ahmed[1] generalised the work of Jibrii on n-power normal operators into the class of n-power quasinormal operators and through this he was able to show that every n-power normal operator is n-power quasinormal.
In 2012, Panayan[45] introduced an extension of all normal operators which he later call the n-power class operator.
We therefore have;
Projection⊆Self-Adjoint⊆Normal$subseteq$Hyponormal.
Normal⊆Quasinormal⊆Subnormal⊆Hyponormal⊆m-hyponormal.
Unitary⊆Isometry⊆Partial Isometry⊆Contraction.
Unitary⊆Isometry⊆2-normal⊆Binormal.
Normal⊆Quasinormal⊆Subnormal⊆Hyponormal⊆P-hyponormal⊆Log-hyponormal.
Normal⊆Hyponormal⊆P-hyponormal⊆w-hyponormal.
Normal⊆Quasihyponormal⊆Subnormal⊆Hyponormal⊆m-hyponormal.

2 Normality of non normal operators

Normal operators are a major class of both bounded and unbounded operators in Hilbert Spaces. There are many classes of non normal operators such as hyponormal,subnormal operators and many more. In this section, we set to investigate some of the conditions under which some non normal operators are normalized

. To set the stage, we first invsetigate some properties of these normal and non-normal operators.

2.1 Some general properties of normal and non normal operators

In this section, we investigate some properties of normal operators and non normal operators as discussed in chapter 1.

Properties of normal operators

Theorem 2.1.1 (Spectral theorem) If $A \in B(H)$ is normal, then there exists a finite measure space (X,μ) and a $\phi \in L^{\infty}(X,\mu)$ such that A is unitarily equivalent to the operator M_{ϕ} on $L^2(X,\mu)$.
Corollary 2.1.2 Let A be normal. Then A is
(i). Hermitian
(ii). Unitary
(iii). Positive
(iv). A projection
if and only if its spectrum is
(i). real
(ii). On the unit circle
(iii). on the non-negative real axis (iv). in the set $\{0,1\}$

We now represent square roots of normal operators.
Definition 2.1.3 (Square root of a normal operator) An operator is the square root of

an operator $T \in B(H)$ if and only if it is of the form

$$A \oplus \begin{bmatrix} B & C \\ 0 & -B \end{bmatrix}$$

where A and B are normal and C is a positive one-to-one operator commuting with B. Furthermore B can be chosen such that $\sigma(B)$ lies in the closed upper half plane and the Hermitian part of B is non-negative.

Theorem 2.1.4, (Radjavi and Rosenthal, [51]) If S and T are two operators with respective representations

$$A \oplus \begin{bmatrix} B & C \\ 0 & -B \end{bmatrix}$$

and

$$D \oplus \begin{bmatrix} E & F \\ 0 & -E \end{bmatrix}$$

where A,B,D,E are normal and the C and F are as in definition 2.1.3, then (i). A is the normal part of S

(ii). S and T are unitarily equivalent if and only if A is unitarily equivalent to D and the pair (B,C) is simultaneously unitarily equivalent to the pair (E,F)

Proof

(i). To prove this, it suffices to show that S_0 is completely non-normal. This will follow if we show that ${S_0}^*S_0 - S_0{S_0}^*$ has trivial null space. But ${S_0}^*S_0 - S_0{S_0}^* = \begin{bmatrix} C & 0 \\ 0 & C \end{bmatrix} \begin{bmatrix} -C & 2B^* \\ 2B & C \end{bmatrix}$ where both of the factors on the right are one-to-one, positive operators and hence our result.

(ii). Suppose S and T are unitarily equivalent. Then their normal parts are also unitarily equivalent. Thus A is unitarily equivalent to D and S_0 is unitarily equivalent to T_0.

Since C commutes with B, the null space of B reduces C. Hence S_0 can be put in the form

$$\begin{bmatrix} B_1 & C_1 \\ 0 & -B_1 \end{bmatrix} \oplus \begin{bmatrix} 0 & C_0 \\ 0 & 0 \end{bmatrix}$$

, whereB_1 has trivial null space. Similarly, T_0 can be put in the form

$$\begin{bmatrix} E_1 & F_1 \\ 0 & -E_1 \end{bmatrix} \oplus \begin{bmatrix} 0 & F_0 \\ 0 & 0 \end{bmatrix}$$

.

The unitarily equivalence of S_o and T_0 implies that of ${S_0}^2$ and ${T_0}^2$. Since the second direct

summand in the two sums are the restrictions of S_0 and T_0 to the null spaces ${S_0}^2$ and ${T_0}^2$, we conclude they are unitarily equivalent and so are the two first summands.
The proof of the other parts follows similarly.

We now investigate operations of normal operators. Recall that in general, the sum and product of two normal operators are not normal in general. Furthermore, in the case where it happens that AB is normal for two normal operators $A, B \in B(H)$, it does not always imply that BA is also normal. That is , normality of AB does not imply normality of BA.
The following result by Kaplansky gives the condition under which normality of AB implies normality of BA.
Proposition 2.1.5 [27] Let $A, B \in B(H)$ be such that A and AB are normal. Then BA is normal if and only if B commutes with $|A|$.
Proof
Suppose by hypothesis that A and AB are normal. Let $A = U|A|$ be the polar decomposition of A where $U \in B(H)$ is unitary and commutes with $A = (A^*A)^{\frac{1}{2}}$. In addition, suppose B commutes with $|A|$. Then
$U^*ABU = U^*U|A|BU = B|A|U = BU|A| = BA$ and hence BA is normal.
Conversely, suppose BA is normal. Let $M = AB$ and $N = BA$.
Since $MA = ABA = AN$,it follows that $M^*A = AN^*$(by Putnam theorem). That is, $B^*A^*A = AA^*B^*$ and taking into account that $A^*A = AA^*$, this means that B^* commutes with A^*A and so does B.

In the result below, we now give the form of those normal operators $A, B \in B(H)$ such that AB and BA are normal. Before this, we consider the following definition.
Definition 2.1.6 An operator $T \in B(H)$ is homogeneously normal if there exists $P \geq 0$ such that $T^*T = TT^* = P^2I$, equivalently, for some unitary operator $U \in B(H)$, we have $A = PU$.
Theorem 2.1.7 Let $S, T \in B(H)$ be normal compact operators. The following assertions are equivalent
(i). ST is normal.
(ii). TS is normal.
(iii). There exists at most countable family of mutually orthogonal subspaces $(Hi)_{i \in J}$ such that $H = \oplus_{i \in J} Hi$, the subspace (Hi) reduces both S and T and in addition, $S|Hi$ and $T|Hi$ are homogeneously normal for all $i \in J$.
Proof
See[64]

We now show the conditions under which $A+B$ is normal for $A, B \in B(H)$ normal.

Proposition 2.1.8 Let $A, B \in B(H)$ be self-adjoint operators and let $S = A + iB$. If A or B is strictly positive and if AB is normal, then S is also normal.

Proof

Suppose A is strictly positive. Then $0 \notin W(A)$. Since AB is normal and A and B are self-adjoint by hypothesis, it follows that S is normal.

Theorem 2.1.9 Let A and B be two bounded normal operators. If AB and AB^* are normal such that A is positive, then $A+B$ is normal.

Proof

Follows from the fact that $BAB = (AB^*)^*B$ and the Fuglede-Putnam theorem.

Properties of non-normal operators

One of the main classes of non-normal operators is the class of hyponormal operators introduced by Hamos[23] as shown in subsection 1.3. In this study, the author showed that the product and sum of two hyponormal operators need not be hyponormal. It was also shown that if S and T are hyponormal operators on a Hilbert space H such that S commutes with T^*, the $S+T$ is hyponormal.

However, if S and T are commuting hyponormal operators, then their sum, $S+T$, need not be hyponormal. Hamos also showed that this scenario also holds true for the product of two hyponormal opereators, that is, the product of two hyponormal operators need not be hyponormal even if they commute.

The following result gives some conditions under which this is true.

Remark 2.2.1 We first note that the positive part of an operator $A \in B(H)$ is given by $(A^*A)^{\frac{1}{2}}$.

Theorem 2.2.2 Let T_1 and T_2 be hyponormal operators. Suppose that T_1 commutes with the positive part of T_2 and T_2 commutes with the positive part of ${T_1}^*$. Then T_1T_2 and T_2T_1 are hyponormal.

Proof

In this proof, we will show the hyponormality of T_1T_2 only because the proof of hyponormality of T_2T_1 is similar.

Now by hypothesis, $T_1({T_2}^*T_2) = ({T_2}^*T_2){T_1}^*$ and ${T_2}^*(T_1{T_1}^*) = (T_1{T_1}^*){T_2}^*$.

Since for a positive operator P, R^*PR is a positive operator for every operator R, we have

$(T_1T_2)^*(T_1T_2) - (T_1T_2)(T_1T_2)^*$

$= {T_2}^*{T_1}^*T_1T_2 - T_1T_2{T_2}^*{T_1}^*$

$\geq {T_2}^*T_1{T_1}^*T_2 - T_1T_2{T_2}^*{T_1}^*$

$\geq T_1{T_1}^*{T_2}^*T_2 - T_1{T_2}^*T_2{T_1}^*$

$= T_1{T_1}^*{T_2}^*T_2 - T_1{T_1}^*{T_2}^*T_2$

$= 0$

Hence T_1T_2 is hyponormal.

Corollary 2.2.3 Let T_1 and T_2 be normal operators. Then each of T_1 and T_2 commutes with the positive part of each other if and only if T_1T_2 and T_2T_1 are normal.
Remark 2.2.4 Corollary 2.2.3 is an immediate result of Theorem 2.2.2

We now give sonme properties of quasihyponormal and subnormal operators and show how they relate.
Proposition 2.2.5 If $S = UA$ is the polar decomposition of S, then S is quasihyponormal if and only if $AU = UA$ where U is unitary.
Proof
If A and U commute, then $SA^2 = A^2S$ and so S is quasihyponormal.
Conversely, suppose S is quasihyponormal, then by definition, S commutes with A^2.
Since A can be approximated by the polynomials in A^2, $SA = AS$
Hence $(UA - AU)A = SA - AS = 0$
Thus $(UA - AU) = 0$ on $Ran(A)$.
But if $f \in (RanA)^{\perp} = KerA$, then by definition, $Uf = 0$
Thus $UA - AU = 0$.
proposition 2.2.6 Every quasinormal operator is subnormal.
Proof
Let $S \in B(H)$ be quasinormal. We need to show that S is subnormal. In this proof, we will consider two cases;where $KerS = \{0\}$ and $KerS \neq \{0\}$.
Suppose $KerS = \{0\}$. If $S = UA$ is the polar decomposition of S, then U must be an isometry.
If $E = UU^*$, then E is the projection onto the final space U_j thus, $E^{\perp}U = U^*E^{\perp} = 0$(where $E^{\perp} = I - E$)
Define operators V and B on $H = H_1 \oplus H_2$ by

$$V = \begin{bmatrix} U & E^{\perp} \\ 0 & U^* \end{bmatrix}, B = \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix}$$

and let $N = VB$. Since $UA = AU$ and $U^*A = AU^*$, it is easily seen that N is normal.
Since

$$N = \begin{bmatrix} S & E^{\perp}A \\ 0 & U^*A \end{bmatrix} = \begin{bmatrix} S & E^{\perp}A \\ 0 & S^* \end{bmatrix}$$

It follows that N leaves $H = H_1 \oplus \{0\}$ invariant and $N|H = S$.
Now suppose that $KerS \neq \{0\}$. Here $KerS = l \subseteq KerS^*$, since $S^* = AU^* = UA^*$.
Let $S_1 = S|l^{\perp}$; so $S = S_1 \oplus \{0\}$ on $l^{\perp} \oplus l = H$.
Now $S^*S = {S_1}^*S_1 \oplus \{0\}$ and it is easy to see that S_1 is quasinormal.
By the first part, S_1 is subnormal.

Clearly, S is subnormal.

In the next result, we show the relation of k-quasi-M-hyponormal operators with M-hyponormal.
We recall that an operator $T \in B(H)$ is k-quasi-M-hyponormal for a positive integer k, if there exists $M > 0$ such that
$T^{*k}M(T-\lambda)^*(T-\lambda)T^k \geq T^{*k}(T-\lambda)(T-\lambda)^*T^k$ for all $\lambda \in \mathbb{C}$.
It is clear that the following inclusion holds
Remark 2.2.7 M-hyponomality implies k-quasi-M-hyponormality but the converse is not generally true. For instance, the matrix

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

on $H = \mathbb{C}^2$ is $2-$quasi-M-hyponormal but not M-hyponormal.
The following result shows the conditions under which a k-quasi-M-hyponormal operator is M-hyponormal.
Theorem 2.2.8 Let $T \in B(H)$ be a k-quasi-M-hyponormal operator. If T has a dense range, then T is M-hyponormal.
Proof
Suppose $T \in B(H)$ is k-quasi-M-hyponormal and suppose $\overline{Ran(T)} = H$. Then there exists a sequence $(x_n)_n$ in H such that
$x = \lim_{n\to\infty} Tx_n$ for $x \in H$
By continuity of T, we get $\lim_{n\to\infty} T^k x_n = \lim_{n\to\infty} T^{k-1}Tx_n = T^{k-1}x$
Since T is k-quasi-M-hyponormal,

$$\left\|\sqrt{M}(T-\lambda)T^k x_n\right\| \geq \left\|(T-\lambda)^*T^k x_n\right\| \forall\lambda \in \mathbb{C}$$

Thus

$$\left\|\sqrt{M}(T-\lambda)T^{k-1}x\right\| = \left\|\sqrt{M}\lim_{n\to\infty}(T-\lambda)T^k x_n\right\| = \left\|\lim_{n\to\infty}\sqrt{M}(T-\lambda)T^k x_n\right\| \geq \left\|\lim_{n\to\infty}(T-\lambda)^*T^k x_n\right\| = \left\|\lim_{n\to\infty}\right.$$

Hence, T is $(k-1)-$quasi-M-hyponormal.
Since T has a dense range, T is $(k-2)-$quasi-M-nyponormal operator.
By iteration, T is M-hyponormal.

We finish this section with the class of n-normal operators.

Proposition 2.2.9 Let $T \in L(H)$. $T \in [nN]$ if and only if T^n is normal for any positive integer $n \geq 1$.

Proof

Suppose $T \in [nN]$. Then by definition, $T^nT^* = T^*T^n$

Then, $T^n(T^*)^n = T^*T^n(T^*)^{n-1}$

$i.eT^n(T^n)^* = T^*T^nT^*(T^*)^{n-2} = (T^n)^*T^n$

Conversely, suppose T^n is normal, then $T^nT = TT^n$ implies that $(T^n)^*T = T(T^n)*$

$i.eT^*T^n = T^nT^*$ (by Fuglede theorem).

Corollary 2.2.10 Let $T \in L(H)$. Then T is n-power normal if and only if $\|T^nx\| = \|(T^n)^*x\| \, \forall x \in H$.

Corollary 2.2.11 The class $[nN]$ of n-power normal operators on H is closed under scalar multiplication, unitary equivalence, and taking adjoints.

Moreover, the inverse, it it exists, and the restriction to a closed subspace of H of an n-power normal operator is n-power normal.

Proof

Follows immediately from proposition 2.2.9.

Remark 2.2.12 Unitary equivalence in corollary 2.2.9 cannot be replaced by similarity.

The operator

$$T = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

and

$$S = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$$

acting on $H = \mathbb{C}^2$ are similar since

$S = X^{-1}TX$ where $X = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$, but T is $3-$normal and S is not.

Proposition 2.2.13 Let $T \in B(H)$ be n-normal. Then

(i). T^* is n-normal

(ii). If T^{-1} exists, then $(T^{-1})^n$ is n-normal.

(iii). If $S \in B(H)$ is unitarily equivalent to T, then S is n-normal.

(iv). If M is closed subspace of H such that M reduces T, then $S = T|M$ is an n-normal operator.

Proof

(i). Since T is n-normal, T^n is normal.

So, $(T^n)^* = (T^*)^n$ is normal, T^* is an n-normal operator.

(ii). Since T is n-normal, then T^n is normal

Since $(T^n)^{-1}) = (T^{-1})^n$ is normal, T^{-1} is an n-normal operator.

(iii). Let $T \in B(H)$ be an n-normal operator and S be unitary equivalent to T. Then there exists a unitary operator U such that $S = UTU^*$ so

$S^n = UT^nU^*$.

Since T^n is normal, S^n is normal.
So $T|M$ is n-normal.
(iv). Since $T \in B(H)$ is n-normal, T^n is normal.
So$T^n|M$ is normal and since M is invariant under T, $T^n|M = (T|M)^n$
Thus $(T|M)^n$ is normal.
So $T|M$ is n-normal.

2.2 Normality of hyponormal,p-hyponormal,log-hyponormal,subnormal and semi-normal operators

In this section, we start by showing the conditions under which the above named non-normal operators are normal

Theorem 2.3.1 [11] If T is a semi-normal operator such that $T^p = ST^{*p}S^{-1} + C$ for some positive integer p, C compact and $0 \notin W(S)$, then T is normal.

Remark 2.3.2 We note that for a semi-normal$T \in B(H)$ operator such that $w(T)$ lies on finitely many lines through the origin, then T is normal.

Theorem 2.3.3 [51] If A, B and K are operators on H such that A and B^* are subnormal and K is positive(not necessarily invertible) and one-to-one, and if $AK = KB$, then A and B are normal and $A = B$.

Proof

See [51] **Corollary 2.3.4 [51]** If A, B and K are operators on H such that A and B^* are subnormal and K is one-to-one and $\overline{Ran(K)} = K$, and if $AK = KB$, then A and B are normal and unitarily equivalent.

Proof

Let$K = UH$ be the polar decomposition of K where U is unitary and H is positive. Then $U^*AUH = HB$. Since U^*AU is subnormal.

It follows from Theorem 2.3.3 that U^*AU and B are normal and $U^*AU = B$.

Remark 2.3.5 Any polynomial of a normal operator is normal but the converse is not generally true.

By recalling the definition of an Aluthge transformation, the following result shows the condition under which $\tilde{T}$ is normal.

Theorem 2.3.6 Let $T \in B(H)$ be an invertible operator and $T = U|T|$ be its polar decomposition.

Let Sp(U) be contained in some open semicircle. Then $\tilde{T}$ is normal if and only if T is normal.

Theorem 2.3.7 Let $T \in B(H)$ be a p-hyponormal operator and let $S^* \in B(H)$ be a p-hyponormal operator. If $TX = XS$ for some $X \in B(H)$ injective and has a dense range, then T is normal and unitary equivalent to S.

Theorem 2.3.8 Let $T \in B(H)$ be a $(p,k)-$ quasihyponormal operator and let $S^* \in B(H)$ be a p-hyponormal operator. If $TX = XT$ for some $X \in B(H)$ injective with a dense range, then T is a normal operator and unitarily equivalent to S.
Proposition 2.3.9 Let $T \in B(H)$ be a hyponormal operator. If T^n is normal for some integer n, then T is normal.
Remark 2.3.10 If T^* is hyponormal, then T is normal
Propositio 2.3.11 Let $T \in B(H)$ be a hyponormal operator. If the spectrum of T ($\sigma(T)$) only contains a finite number of limited points or has zero area, then T is normal
Theorem 2.3.12 [43] Let $T \in B(H)$ be log-hyponormal and $T = U|T|$ be its polar decomposition such that $U^m = U^*$for some positive integer m.
Then T is normal.
Proof
Suppose T$\in B(H)$ is log-hyponormal. Then we have
$log|T| \geq log|T^*| = U(log|T|)U^* \cdots (i)$
By multiplying both sides of (i) by U and U^*, we have

$$U(log|T|)U^* \geq U^2(log|T|)U^{2*}$$

whence
$log|T| \geq U(log|T|)U^* \geq U^2(log|T|)U^{2*}$
By proceeding in this way,we get
$log|T| \geq log|T^*| = U(log|T|)U^* \geq U^2(log|T|)U^{2*} \geq \cdots \geq U^{m+1}(log|T|)U^{(m+1)*}$
Since by hypothesis $U^m = U^*$, we have
$U^{m+1} = U^*U = U^{(m+1)*}$ is the projection onto $\overline{Ran(|T|)}$
Thus $U^{m+1}(log|T|)U^{(m+1)*} = log|T|$ and hence we get
$log|T| = log|T^*|$.
Thus $|T|^2 = |T^*|^2$ which shows that T is normal.

In the results below, we extend the results of proposition 2.3.9 and show that it still holds for T^{n+1}.
Lemma 2.3.13 Let $T \in B(H)$ be p-hyponormal with the polar decomposition $T = U|T|$ for U a partial isometry.
Then for $n \geq 0, n \geq p$, the following inequalities hold

$$\left(T^{n+1*}T^{n+1}\right)^{\frac{p}{n+1}} \geq \cdots\cdots (T^*T)^p \geq (TT^*)^p \geq \cdots \geq \left(T^{n+1}T^{n+1*}\right)^{\frac{p}{n+1}}$$

.
Theorem 2.3.14 [57] Let $T \in B(H)$ be p-hyponormal. If T^{n+1} is normal, then T is normal.
Proof
Suppose T is p-hyponormal. Then from lemma 2.3.12, T^{n+1} is $(\frac{p}{n+1})$-hyponormal.
Since T^{n+1} is normal by our hypothesis, then we have by definition,

$$T^{n+1*}T^{n+1} = T^{n+1}T^{n+1*} \cdots (i)$$

.

Thus we have that

$$\left(T^{n+1*}T^{n+1}\right)^{\frac{p}{n+1}} = (T^*T)^p = (TT^*)^p = \left(T^{n+1}T^{n+1*}\right)^{\frac{p}{n+1}}$$

.

Thus we get $T^*T = TT^*$ which shows the normality of T.

Remark 2.3.15 Theorem 2.3.14 holds true for $T \in B(H)$ a log-hyponormal operator. **Theorem 2.3.16 [53]** Let $A, B, X \in B(H)$ be operators such that A^* is p-hyponormal operator, B is a dominant operator and X is an invertible operator. If $XA = BX$, then there is a unitary operator U such that $AU = BU$ and hence A and B are normal.

Proof

Let $XA = BX$, then by Uchiyama's and Tanahashi's results on the Fuglede-Putnam's theorem, we have that $B^*X = XA^*$ and so $X^*B = AX^*$.

Since $AX^*X = X^*BX = X^*XA$, we let $X = UP$ be the polar decomposition of X.

Since by the hypothesis X is invertible, it follows that P is invertible and unitary. Now, $AP^2 = P^2$ and by the positivity of P, we get $AP = PA$.

Clearly, $BUP = UPA \implies BUP = UAP$. But P is invertible. Thus $BU \underset{\sim}{} UA$.

Hence, we have that A, B are unitary equivalent which implies that A is dominant and B is p-hyponormal

Thus A, B are normal.

Remark 2.3.17 Theorem 2.3.16 holds for A^* a log-hyponormal operator.

Theorem 2.3.18 [53] Let $T = A + iB$ be the cartesian decomposition of T. If T^* is hyponormal and AB is p-hyponormal, then T is a normal operator.

Proof

Let $S = AB$, then $SA = AS^* = ABA$

By the Fuglede-Putnam's theorem for p-hyponormal operator we have $S^* = AS$ which implies $BA^2 = A^2B$.

$$Now(S + S^*)A = A(S + S^*)$$

and

$$(S - S^*)A = A(S^* - S)$$

.

Since T^* is hyponormal, we have $TT^* - T^*T = 2i(BA - AB) = 2i(S^* - S) \geq 0$.

Letting $W = 2i(BA - AB)$, then $W \geq 0$ and $WA = -AWi$

Now,$W^2A = W(WA) = W(-AW) = -WAW = (-AW)W = AW^2$.

But W is positive, then $WA = AW = 0$.

Thus $A(AB-BA)=(AB-BA)A$
$\implies \delta(AB-BA)=\{0\}$
Thus $AB-BA$is quasinilpotent skew-Hermitian.
Thus $AB-BA=0$.So T is normal.
Theorem 2.3.19 [38] Let $T \in (WN)$. If T^P are T^q are normal operators for some co-prime integers p,q, then T is normal.
Remark 2.3.20 The normality of T^2 for a certain operator $T \in (WN)$ is not sufficient to ensure the normality of T.
The following example is a clear illustration of this.
Example 2.3.21 Let $T = \begin{bmatrix} i & 1 \\ 0 & -1 \end{bmatrix}$ on $H = dim2$.
A simple computation shows that T is normal but T^2 is not.

Theorem 2.3.22 [38] Let $T \in (WN)$. If T is a partial isometry and $0 \notin W(T)$, then T is Normal.
Remark 2.3.23 Theorem 2.3.22 is the converse of Theorem 2.3.19.

2.2.1 Normality of n-power normal operators and n-power quasinormal operators

In this section, conditions under which n-power normal and n-power quasinormal operators are investigated.
We recall that the class of all n-power normal operators is denoted by $[nN]$ and that of n-power quasinormal operators by $[nQN]$.
Proposition 2.4.1 Some properties of n-power quasinormal operators.
(i) The class$[nQN]$ is closed under unitary equivalence and scalar multiplication.
(ii) If T is of the class $[nQN]$ and M a subspace of H that reduces T, then T_M is of class $[nQN]$.
(iii) Every quasinormal operator is n-power quasinormal for each n.
(iv) a n-power normal operator is also n-power quasinormal.
Proof
We will prove (i) and (ii) since the proof of the other two is trivial.
Suppose $S \in B(H)$ is unitary equivalent to $T \in B(H)$ then, there is a unitary operator $U \in B(H)$ such that $T = U^*SU$ which implies
$T^* = US^*U^*$
Since $T^n = U^*S^nU$, and noting $I = UU^*$, we have

$$U^*S^nS^*SU = T^nT^*T = T^*T^{n+1} = U^*S^*S^{n+1}U$$

and hence the proof of part (i) follows.
Now since $(T|M)^{\Delta} = T^{\Delta}|M$ for Δ as the n-th power or the adjoint, it follows that the left side of $T^nT^*T = T^*TT^n = T^*T^{n+1}\cdots(i)$ for $(T|M)$ reads $(T^nT^*T|M)$ which is

$$T^*Tn+1|M = (T|M)^*(T|M)^{n+1}$$

which is the right hand side of (i). Thus shows that $T|M$ is of class $[nQN]$.
Remark 2.4.2 We note that the converse of property (iv) need not hold in general. The following example gives an illustration of this.

Example 2.4.3
Consider the operator

$$T = \begin{bmatrix} 0 & 0 & 0 & \ldots\ldots\ldots \\ 1 & 0 & 0 & \ldots\ldots\ldots \\ 0 & 1 & 0 & \ldots\ldots\ldots \end{bmatrix}$$

which is a unilateral Shift in $H = l^2$.

A simple computation shows that $T^2T^* - T^*T^2 \neq 0$ and
$(T^2T^* - T^*T^2)T = 0$ which shows that T is not 2-power normal but is a 2-power quasi-normal operator.
Remark 2.4.4 The classes $[2QN]$ and $[3QN]$ are not the same. The following examples show a clear illustration.
Example 2.4.5
Let $H = \mathbb{C}^3$ and $T \in B(H)$ be given by

$$T = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

A simple computation shows that T is of class $[2QN]$ but not of $[3QN]$.
Example 2.4.6
Let $H = \mathbb{C}^3$ and $S \in B(H)$ be given by

$$S = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & -1 \end{bmatrix}$$

A simple computation shows that S is of class $[3QN]$ but not of class $[2QN]$.

Proposition 2.4.7 Let $T \in B(H) \in [2QN] \bigcap [3QN]$.
Then T is of class $[nQN]$for all positive integer $n \geq 4$.
Proof
We use mathematical induction to prove proposition 2.2.6. We note that the case $n = 4$ is trivial and thus we start with the case $n = 5$.
Now, for $n = 5$, we have that since $T \in [2QN]$, then
$T^2T^*T = T^*T^3 \cdots *$
Multiplying $(*)$ to the left by T^3 we get
$T^5T^*T = T^3T^*T^3$
$= T^*T^4T^2$
T^*T^6.
Now suppose the result is true for $n \geq 5$
i.e $T^nT^*T = T^*TT^n$ then

$$T^{n+1}T^*T = TT^*T^{n+1} = TT^*T^3T^{n-2} = T^3T^*TT^{n-2} = T^*T^{n+2}$$

Thus T is of class $[(n+1)QN]$ and hence our result.
Theorem 2.4.8 [35] Let$T \in [nQN]$. If T and $T - I$ are of class $[2QN]$, then T is normal.
Proof
The condition on $T - I$ implies

$$T^2(T^*T) - T^2T^* - 2T(T^*T) + 2TT^* = (T^*T)T^2 - T^*T^2 - 2(T^*T)T + 2T^*T$$

Since T is of class $[2QN]$, we have

$$-TT^{*2} - 2T(T^*T) + 2TT^* = -T^*T^2 - 2(T^*T)T + 2T^*T$$

or

$$-TT^{*2} - 2(T^*T)T^* + 2TT^* = -T^{*2}T - 2T^*(T^*T) + 2T^*T \cdots *$$

Which shows that $*$ implies $N(T^*) \subseteq N(T) \cdots (i)$
Suppose $T^*x = 0$, from $*$, we get
$-3T^{*2}Tx + 2T^*Tx = 0 \cdots **$
Then $-3T^{*3}Tx + 2T^{*2}Tx = 0$
Thus since $T \in [2QN]$,
$-3T^*TT^{*2}x + 2T^{*2}Tx = 0$ and hence
$2T^*Tx = 0$.
Consequently, $**$ gives $2T^{*2}Tx = 0$ or $Tx = 0$
Thus $-T(T^*T) + TT^* = -(T^*T)T + T^*T$ or
$T^*(T^*T - TT^*) = T^*T - TT^* \cdots ***$

Thus, if $N(T^*-I)=0$, then $***$ implies T is normal.
If $N(T^*-I)\neq 0$,let $T^*x=x$ then $**$ gives
$T^{*2}Tx-T^*Tx=T^*Tx-Tx$ since $T^{*2}T=TT^{*2}$,we have $T^*Tx=Tx$
Therefore$\|Tx\|^2=<T^*Tx,x>=<T^*x,x>=\|x\|^2$
which implies $\|x\|^2=\|x\|^2$
Thus $\|Tx-x\|^2=\|Tx\|^2+\|x\|^2-2Re<Tx,x>$by (Cauchy-Schwarz's inequality)
$=\|x\|^2-\|x\|^2$
$=0$ or $Tx=0$
Thus $N(T^*-I)\subset N(T-I)\cdots(ii)$
Hence (ii) together with $**$ gives
$T(T^*T-TT^*)=T^*T-TT^*$ and so
$T(T^*T-TT^*)T=(T^*T-TT^*)T$ or
$TT^*T^2-T^2T^*T=T^*T^2-TT^*T$
since $T^2T^*=T^*T$ and $T^3T^*=T^*T^3$ we deduce that
$T^*T2=TT^*T$.
Thus T is quasinormal.
from (i), it follows that T is normal.
Theorem 2.4.9 If T is of class $[2QN]\bigcap[3QN]$ such that $T-I$ is of class $[nQN]$, then T is normal.
Proof
Follows easily from the proof of Theorem 2.4.8.

2.2.2 Normality of other non-normal operators

Proposition 2.5.1 Let T be an invertible operator such that the following holds
(i)$ST^*=T^{-1p}S+K$, where K is compact, $0\not\subset W_e(S)$ and $p\neq -1$ is an integer
(ii) T is p-oid,T^{-1} is δ-oid$(p,\delta\geq 1)$
(iii)$\sigma_{00}(T)=\emptyset$
Then T is unitary.
Proposition 2.5.2 If T is convexoid and satisfies
$ST^*=TS+K$, where K is compact, $0\not\subset W_e(S)$ and $\sigma_{00}=\emptyset$, then T is Self-adjoint.
Theorem 2.5.3 (Thakare,[61]) Let $T\in B(H)$ be an operator and suppose T satisfies the following.
(i)T is restriction-convexoid.
(ii) T is reduced by each of its eigenspaces.
(iii) $T=S^{-1}A^pS+K$,where $\sigma(A)$ is real, K is compact and p is a positive integer.
Then T is normal.
Proof
See [61].
Theorem 2.5.4 (Kim,[30]) If T is restriction convexoid and is reduced by each of its eigenspaces corresponding to its isolated eigenspaces and $\sigma(T)$ is countable, then T is diagonal and normal.

Proof
See [30].
Corollary 2.5.5 If T is restriction -convexoid and is reduced by each of its eigenspaces corresponding to isolated eigenspaces and $\sigma_e(T) = \{0\}$, then T is compact and normal.

Proof
Now since by hypothesis we have $\sigma_e(T) = \{0\}$, then we have$\sigma(T) \subset \{0\} \bigcup p_{00}(T)$, $\sigma(T)$ is countable
Thus by Theorem 2.5.4, we note that T is normal. We skip the proof for T as compact.
Corollary 2.5.6 If T is restriction-convexoid and is reduced by each of its eigenspaces corresponding to isolated eigenspaces and all but a finite number of elements of $\sigma(T)$ are real, then T is normal.

3 ON THE FUGLEDE-PUTNAM THEOREM

In this chapter, we generalize the Fuglede-Putnam theorem by relaxing the normality hypotheses of operators A and B. Many authors have relaxed this normality hypotheses of A and B in order to generalize the theorem.
The following result by Berberian was the first generalization of the Fuglede-Putnam theorem where we require that A and B^* are hyponormal operators and that X is a Hilbert-Schmidt operator.
Theorem 3.1 (Berberian [10]) If A and B^* are hyponormal, then $AX = XB$ implies $A^*X = XB^*$ for some $X \in C_2(H)$.
Proof
See [10]
Corollary 3.2 Suppose $A, B \in B(H)$ are operators such that $AX = XB$. Then for some $X \in B(H), A^*X = XB^*$ under either of the following hypotheses.
(i) A and B^* are hyponormal
(ii) B is invertible and $\|A\|.\left\|B^{-1}\right\| < 1$.
Proof
Follows easily from the proof of Theorem 3.1
Theorem 3.3 (Furuta,[17]) If A and B^* are subnormal and if $X \in C_2(H)$ such that $AX = XB$, then $A^*X = XB^*$
Proof
Let

$$N_A = \begin{bmatrix} A & A_{12} \\ 0 & A_{22} \end{bmatrix}$$

be the normal extension of $A \in B(H)$ on a lager Hilbert space K_A than H and

$$N_{B^*} = \begin{bmatrix} B^* & B_{12} \\ 0 & B_{22} \end{bmatrix}$$

be the normal extension on a larger Hilbert space K_{B^*} that contains H .
We consider the operator $T = \begin{bmatrix} B_{22}^* & B_{12} \\ 0 & B \end{bmatrix}$ acting on $(K_{B^*} \ominus H) \oplus H$.
Clearly, we note that T is normal since N_{B^*} is also normal. Now consider $\hat{A}$ and $\hat{X}$ acting on a lager Hilbert space $H \oplus (K_A \ominus H) \oplus (K_{B^*} \ominus H) \oplus H$ as shown below.

$$\hat{A} = \begin{bmatrix} A & A_{12} & 0 & 0 \\ 0 & A_{22} & 0 & 0 \\ 0 & 0 & B^*{}_{22} & B_{12} \\ 0 & 0 & 0 & B \end{bmatrix}, \hat{X} = \begin{bmatrix} 0 & 0 & 0 & X \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

.

Clearly, $\hat{A}$ is normal and we have $\hat{A}\hat{X} = \hat{X}\hat{A}$

Since $AX = XB$, we get $\hat{A^*}X = \hat{X}\hat{A^*}$.

Thus $A^*X = XB^*$ and thus the Fuglede-Putnam theorem is satisfied.

Theorem 3.4 If $A, B \in B(H)$ are operators such that $AX = XB$ for some $X \in C_2(H)$, then $A^*X = XB^*$ under any of the following hypotheses.

(i) A is k-quasihyponormal and B^* is invertible hyponormal

(ii) A is quasihyponormal and B^* is invertible hyponormal

(iii) A is nilpotent and B^* is invertible hyponormal

Remark 3.5 We arrive at Theorem 3.4 by relaxing the hypothesis on A in corollary 3.2 and keeping the hypotheses on B and X constant.

In the next result, we extend theorem 3.1 by using a Hilbert-Schmidt norm inequality.

Theorem 3.6 (Furuta,[18]) If A and B^* are hyponormal, then the following inequality holds.

$$\|AX - XB\|_2 > \|A^*X - XB^*\|_2$$

for $X \in C_2(H)$

$$\|AX - XB\|_2 = \|A^*X - XB^*\|_2$$

if and only if A and B are normal.

Proof

Let $J \in C_2(H)$ be defined as

$JX = AX - XB$

Then J^* exists(since $C_2(H)$ is a Hilbert space) and is given by

$J^*X = A^*X - XB^*$.

Also,

$$(J^*J - JJ^*)X = A^*((AX - XB) - (AX - XB))B^* - \{A(A^*X - XB^*) - (A^*X - XB^*)B\} = (A^*A - AA^*)X + X(BB^* - B$$

.

Clearly, J is hyponormal and hence

$$\|JX\|_2 \geq \|J^*X\|_2$$

That is

$$\|AX - XB\|_2 > \|A^*X - XB^*\|_2 \cdots **$$

Equality follows from inclusions $*$ and $**$.

Lemma 3.7 If A and B^* are p-quasihyponormal, the operator $J \in C_2(H)$ defined by.
$JX = AXB$ for some $X \in C_2(H)$,then JX is also p-quasihyponormal

Theorem 3.8 (Lee,[34]) Let A and B be P-quasihyponormal and B^* be invertible quasihyponormal such that
$AX = XB$ for $X \in C_2(H)$ and $\left\||A^*|^{1-p}\right\| . \left\||B^{-1}|^{1-p}\right\| \leq 1$.
Then $A^*X = XB^*$

Proof Define $J \in C_2(H)$ by
$JY = AYB^{-1} \forall Y \in C_2(H)$. Since B^* is invertible by hypothesis and we note that $(B^*)^{-1} = (B^{-1})$, thus (B^{-1}) is p-quasihyponormal and thus by lemma 3.7, J is p-quasihyponormal. Now, since $JX = X$ and since we have that J is a P-quasihyponormal, we have the following,

$$< (J^*J)^p X, X > \geq < (JJ^*)^p X, X >$$

and thus we get

$$\left\||J^*|^P X\right\|^2 \leq < (J^*J)^p X, X > \leq \|X\|^{2(1-p)} < J^*JX, X >^p = \|X\|^2$$

and therefore

$$\|J^*X\| \leq \left\||J^*|^{1-p} X\right\| . \left\||J^*|^P X\right\| \leq \left\||A^*|^{1-p}\right\| . \left\||B^{-1}|^{1-p}\right\| \leq \|X\|$$

.
Thus $\|J^*X - X\|^2 \leq 0$
So,$A^*X(B^{-1})^* = X$
which implies that $A^*X = XB^*$

Corollary 3.9 Let A be quasihyponormal and let B^* be invertible quasihyponormal such that $AX = XB$ for $X \in C_2(H)$
Then $A^*X = XB^*$.

Remark 3.10 Corollary 3.9 is an immediate result of Theorem 3.8.

Theorem 3.11 (Lee,[33]) If A is a (p,k)-quasihyponormal operator and B^* is an invertible (p,k)-quasihyponormal operator such that $AX = XB$ for $X \in C_2(H)$ and
$\left\||A|^{1-p}\right\| . \left\||B^{-1}|^{1-p}\right\| \leq 1$, then $A^*X = XB^*$.

Remark 3.12 Theorem 3.11 is an extension of Theorem 3.8 by requiring that A is a (p,k)-quasihyponormal and B^* is an invertible (p,k)-quasihyponormal operator.

We show that we can generalize the Fuglede-Putnam theorem by relaxing the normality hypotheses of A and B by requiring A^* be P-hyponormal or log-hyponormal and B be a dominant operator.

Lemma 3.13 Let $A^*, B \in B(H,K)$ be p-hyponormal operators. If $XA = BX$ for some $X \in B(H,K)$, then $XA^* = B^*X$

Proof

Let

$$W = \begin{bmatrix} A^* & 0 \\ 0 & B \end{bmatrix}$$

and

$$V = \begin{bmatrix} 0 & 0 \\ X & 0 \end{bmatrix}$$

be operators on $H \oplus K$

A simple computation shows that W is a p-hyponormal operator on $H \oplus K$ that satisfies $VW^* = WV$.

Hence we have $VW = W^*V$ and therefore $XA^* = B^*X$

Theorem 3.14 (Uchiyama and Tanahashi,[62]) Let $A \in B(H)$ be such that A^* is p-hyponormal or log-hyponormal.

Let $B \in B(K)$ be dominant. Then if $XA = BX$, then $XA^* = B^*X$ for some $X \in B(K,H)$.

Remark 3.15 In the result below, we show that the Fuglede-Putnam theorem still holds if A^* is m-hyponormal in theorem 3.13.

Theorem 3.16 Let $A^* \in B(H)$ be m-hyponormal and $B \in B(K)$ be dominant. Then if $XA = BX$ for some $X \in B(K,H)$, then $XA^* = B^*X$.

In the following result we show that theorem 3.8 still holds without the additional condition

$$\left\| |A|^{1-p} \right\| . \left\| |B^{-1}|^{1-p} \right\| \leq 1$$

Theorem 3.18 (Mecheri,[38]) Let A be p-quasihyponormal operator and B^* be an invertible p-quasihyponormal operator such that $AX = XB$ for $X \in C_2(H)$

Then $A^*X = XB^*$.

Proof

Let the operator $J \in C_2(H)$ be defined by $JY = AYB^{-1}$

Then $\forall Y \in C_2(H)$, we note from theorem 3.8 that $(B^*)^{-1}$ is p-quasihyponormal and also that J is p-quasihyponormal.

Also, $JX = AXB^{-1} = X$ and so X is an eigenvector of J

Thus we have $J^*X(B^{-1})^* = X$ which implies that $A^*X = XB^*$

Theorem 3.19 Let A be (p,k)-quasihyponormal operator and B^* be an invertible (p,k)-quasihyponormal such that $AX = XB$ for $X \in C_2(H)$.
Then $A^*X = XB^*$.
Remark 3.20 Theorem 3.19 is a result of dropping the additional condition $\left\||A|^{1-p}\right\| . \left\||B^{-1}|^{1-p}\right\| \leq 1$ in Theorem 3.11.
The following are the immediate results of Theorems 3.18 and 3.19.
Corollary 3.21 Let A be quasihyponormal operator and B^* be an invertible quasihyponormal operator such that $AX = XB$ for $X \in C_2(H)$
Then $A^*X = XB^*$.
Corollary 3.22 If $A \in B(H)$ is a p-quasihyponormal operator and $B^* \in B(H)$ is an invertible p-quasihyponormal operator such that $AX = XB$ for $X \in C_2(H)$ $(0 < p \leq 1)$ and $\left\||A|^{1-p}\right\| . \left\||B^{-1}|^{1-p}\right\| \leq 1$, then $A^*X = XB^*$.
Corollary 3.23 Let $A, B \in B(H)$ be operators such that A is p-hyponormal and B^* is an invertible p-hyponormal operator such that $Ax = XB$ for some $X \in C_2(H)$, then $A^*X = XB^*$.
Theorem 3.24 (Mecheri and Uchiyama,[39]) Let $A, B \in B(H)$ be operators and $S \in C_2(H)$ be a Hilbert-Schmidt operator.
Then

$$\left\|\delta_{A,B}(X) + S\right\|^2{}_2 = \left\|\delta_{A,B}(X)\right\|^2{}_2 + \|S\|^2{}_2$$

and

$$\left\|\delta^*_{A,B}(X) + S\right\|^2{}_2 = \left\|\delta^*_{A,B}(X)\right\|^2{}_2 + \|S\|^2{}_2$$

if and only if A and B^* are class A operators.
Remark 3.25 BY replacing the (p,k)-quasihyponormality of A and B^* in Theorem 3.18 with the class A operators A and B^*, we arrive at Theorem 3.24.

We now then try to generalize the Fuglede-Putnam theorem by introducing class $\mathbf{y}$ operators.
Lemma 3.26 Let $A, B \in B(H, K)$ be operators. Then the following are equivalent.
(i)A, B satisfy Fuglede-Putnam theorem
(ii) If $AX = XB$ for some operator $X \in B(H, K)$.then $\overline{Ran(X)}$ reduces A, $(KerX)^{\perp}$ reduces B and $A|_{\overline{Ran(X)}}, B|_{KerX^{\perp}}$ are normal.
Lemma 3.27 Let $A, B \in B(H, K)$ be such that A is an injective p-hyponormal and B^* be a class $\mathbf{y}$ operator.
If $AX = XB$ for some operator $X \in B(H, K)$, then $A^*X = XB^*$.
Moreover, $\overline{Ran(X)}$ reduces A, $(kerC)^{\perp}$ reduces B and $A|_{\overline{Ran(X)}}, B|_{(KerX)^{\perp}}$ are unitarily equivalent normal operators.
Remark 3.28 Lemmas 3.26 and 3.27 are generalization of the Fuglede-Putnam theorem by using class $\curvearrowright$ operators.

Theorem 3.29 (Mecheri,Tanahashi and Uchiyama,[38, ?]) Let $A \in B(H)$ and $B^* \in B(K)$. If either
(i) A is P-hyponormal and B^* is a class $\mathbf{y}$ operator (ii) A is a class $\mathbf{y}$ operator and B^* is p-hyponormal
Then if $AX = XB$ for some $X \in B(H,K)$, then $A^*X = XB^*$
Moreover, $\overline{Ran(X)}$ reduces A,$(Ker(X))^{\perp}$ reduces B, and $A|_{\overline{RanX}}, B|_{(KerX)^{\perp}}$ are unitarily equivalent normal operators.
Proof
Let's decompose A into two parts ,i.e, the normal part A_1 and the pure part A_2.
I.e $A = A_1 \oplus A_2$on $H = H_1 \oplus H_2$ and write $X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$.
We note that $KerA_2 \subset KerA_2^*$ and since A_2 is pure, then A_2 is injective.
Thus $AX = XB$ implies

$$\begin{bmatrix} A_1X_1 \\ A_2X_2 \end{bmatrix} = \begin{bmatrix} X_1B \\ X_2B \end{bmatrix}$$

Thus

$$A^*X = \begin{bmatrix} {A_1}^*X_1 \\ A_2^*X_2 \end{bmatrix} = \begin{bmatrix} X_1B^* \\ X_2B^* \end{bmatrix} = XB^*$$

Thus $A^*X = XB^*$.
We leave the proof of the second part of Theorem 3.29 because it's almost similar to that of the first part.
Theorem 3.30 (Kim,[29]) If $A^* \in B(H)$ is p-hyponormal ,$B \in B(H)$ is injective (p,k)-quasihyponormal and if $XA = BX$ for $X \in B(H)$, then $XA^* = B^*X$.
Proof
We note that since $AX = BX$,then $(KerX)^{\perp}$ and $\overline{RanX}$ are invariant Subspaces of A^* and B respectively
Clearly,$A^*|_{\overline{RanX}}$ is p-hyponormal and $B|_{\overline{RanX}}$ is a (p,k)-quasihyponormal.
We consider the decompositions $H = (KerX)^{\perp} \oplus KerX$ and $H = \overline{RanX} \oplus (RanX)^{\oplus}$.
Then we get the following matrix representations

$$A = \begin{bmatrix} A_1 & 0 \\ A_2 & A_3 \end{bmatrix}, B = \begin{bmatrix} B_1 & B_2 \\ 0 & B_3 \end{bmatrix}, X = \begin{bmatrix} X_1 & 0 \\ 0 & 0 \end{bmatrix}$$

A simple computation shows that ${A^*}_1$ is p-hyponormal and X_1 is injective with dense range.
Thus, $X_1A_1 = B_1X_1$ and hence A_1 and B_1 are normal and $X_1{A^*}_1 = {B^*}_1X_1$.
Thus we obtain $XA^* = B^*X$.

Corollary 3.31 Let $A \in B(H)$ be p-hyponormal operator and $B \in B(H)$ be a p-hyponormal operator. If $XA = BX$ for $X \in B(H)$, then
$XA^* = B^*X$.

In the following result, we generalize the Fuglede-Putnam theorem by considering the case when A is dominant and B^* is either p-hyponormal or log-hyponormal or w-hyponormal.
Theorem 3.32 Let $A \in B(H)$ be dominant and $B^* \in B(H)$ be p-hyponormal operator. Then if $AX = XB$ for some $X \in B(H)$, then
$A^*X = XB^*$.
Remark 3.33 Theorem 3.32 holds for B^* log-hyponormal operator.
Theorem 3.34 (Bachir and Lombarkia,[6]) Let $A \in B(H)$ be dominant operator and $B^* \in B(H)$ be w-hyponormal such that $KerB^* \subset KerB$, then if $AX = XB$ for some $X \in B(H)$,
Then $A^*X = XB^*$
Proof
We prove theorem 3.34 by considering two cases, i.e , when B^* is injective and when its not injective.
However, since the proof of the two cases are almost the same, we will prove the first case in which B^* is injective.
Suppose B^* is injective. Now by hypothesis, $AX = XB$ for some $X \in B(H)$.
We note that $\overline{RanX}$ is invariant for A and $(KerX)^{\perp}$ is invariant for B^*.
Consider the following decompositions
$H = \overline{RanX} \oplus \overline{(RanX)^{\perp}}, H = (KerX)^{\perp} \oplus KerX$
and

$$A = \begin{bmatrix} A_1 & A_2 \\ 0 & A_3 \end{bmatrix}, B = \begin{bmatrix} B_1 & 0 \\ B_2 & B_3 \end{bmatrix}, X = \begin{bmatrix} X_1 & 0 \\ 0 & 0 \end{bmatrix}$$

Then we get $A_1X_1 = X_1B_1 \cdots *$.
Now, we suppose $B^*{}_1 = U^*|B^*{}_1|$ is the polar decomposition of $B^*{}_1$
By multiplying the left hand side and the right hand side of $*$ by $|B^*{}_1|^{\frac{1}{2}}$, we obtain,

$$A_1X_1|B^*{}_1|^{\frac{1}{2}} = X_1B_1|B^*{}_1|^{\frac{1}{2}}$$

, hence

$$A_1X_1|B^*{}_1|^{\frac{1}{2}} = X_1|B^*{}_1|^{\frac{1}{2}}(B^*{}_1)^*$$

Now since A_1 is dominant, and $B^*{}_1$ is w-hyponormal, then $\widetilde{B^*}_1$ is semi-normal.
Clearly, $A_1X = \widetilde{B^*}_1X$ implies $A^*{}_1X = (\widetilde{B^*}_1)^*X$
Therefore, $A_1|_{\overline{RanX_1|_{|B^*{}_1|^{\frac{1}{2}}}}}$ and $\widetilde{B^*}_1|_{Ker(X_1{}_{|B^*{}_1|^{\frac{1}{2}}})^{\perp}}$ are normal operators.

Since X_1 is injective with dense range and $|{B^*}_1|^{\frac{1}{2}}$ is injective, thus
$\overline{Ran(X_1|{B^*}_1|^{\frac{1}{2}})} = \overline{Ran(X_1)} = \overline{RanX}$ and
$Ker(X_1|{B^*}_1|^{\frac{1}{2}}) = Ker(X_1) = KerX$
Clearly, ${B^*}_1|_{Ker(X)^{\perp}}$ is normal and $Ker(X)^{\perp}$ reduces B^*.
Therefore $\overline{Ran(X)}$ reduces A and $Ker(X)^{\perp}$ reduces B.
It thus follows that $A_2 = A_3 = 0$
Since we have shown that A_1 and B_1 are normal, it therefore follows that ${A^*}_1X_1 = X_1{B^*}_1$.
Hence $A^*X = XB^*$.
Theorem 3.35 Let $A^* \in B(H)$ be w-hyponormal and $B \in B(H)$ be w-hyponormal with $Ker(A^*) \subset Ker(A)$ and $Ker(B) \subset Ker(B^*)$.
If $AX = XB$ for some $X \in B(H)$, then $A^*X = XB^*$.
Theorem 3.36(Rashid,[55]) Let $A^* \in B(H)$ be an injective w-hyponormal operator and $B \in B(H)$ be dominant. If $XA = BX$ for some $X \in B(H)$, then $XA^* = B^*X$.
Proof
See [55]
Theorem 3.37 Let $A \in B(H)$ be w-hyponormal operator and $B^* \in B(H)$ be injective-w-hyponormal, then if $AX = XB$ for some $X \in B(H)$, then $A^*X = XB^*$.
Theorem 3.38 Let $A \in B(H)$ be w-hyponormal operator such that $KerA \subset KerA^*$ and $B^* \in B(H)$ be w-hyponormal such that $KerB^* \subset KerB$. If $AX = XB$, then $A^*X = XB^*$.
Remark 3.39 We note that the proofs of Theorems 3.37 and 3.38 follows easily from that of Theorem 3.34.
In the following result, by replacing the condition on A from being dominant to being a class $\mathbf{y}$ operator in Theorem 3.29, we show that that the theorem still holds.
Theorem 3.40 (Bachir,[8]) Let $A \in B(H)$ be a class $\mathbf{y}$ operator and $B^* \in B(K)$ be w-hyponormal such that $KerB^* \subset KerB$. If $AX = XB$ for some $X \in B(H,K)$, then $A^*X = XB^*$.
Proof
We note that the proof of theorem 3.40 is similar to that of theorem 3.34.
In this theorem, we will proof the second case that we skipped in Theorem 3.34 (i.e the case when B^* is not injective).
Suppose B^* is not injective. Then $kerB^* \subset KerB$ implies that $KerB^*$ reduces B^*. Now, since $KerA$ reduces A, we note that the operators A and B can be written as the following decomposition.
$H = (KerA)^{\perp} \oplus KerA, K = (kerB^*)^{\perp} \oplus KerB^*$ as follows

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} B_1 & 0 \\ 0 & 0 \end{bmatrix}$$

.
We thus have that A_1 is injective class $\mathbf{y}$ operator and B_1^* is injective w-hyponormal operator.
Now let $X : (KerB^*)^{\perp} \oplus KerB^* \to (kerA)^{\perp} \oplus KerA$ and let $X = [X_{ij}]^2 i, j = 1$ be the matrix representation, then $AX = XB$ implies that $A^*X = XB^*$.

Corollary 3.41 Let $A, B \in B(H,K)$ be operators such that A is an injective w-hyponormal operator and B^* is a class $\mathbf{y}$ operator.
If $AX = XB$ for some $X \in B(H,K)$, then $A^*X = XB^*$.
Remark 3.42 Corollary 4.41 is an immediate consequence of Theorem 3.40 by reversing the conditions on A and B^*.
Theorem 3.43 Let $A \in B(H)$ be w-hyponormal operator such that $KerA \subset KerA^*$ and $B^* \in B(K)$ be a class $\mathbf{y}$ operator. If $AX = XB$ for some $X \in B(K,H)$, then $A^*X = XB^*$
Proof
By decomposing A into the normal and pure parts, we get
$A = A_1 \oplus A_2$ on $H = H_1 \oplus H_2$.
Let $X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} : K \to H$
Since $KerA_2 \subset KerA^*$ is injective ,then $AX = XB$ implies

$$\begin{bmatrix} A_1X_1 \\ A_2X_2 \end{bmatrix} = \begin{bmatrix} X_1B \\ X_2B \end{bmatrix}$$

Thus $A^*X = \begin{bmatrix} A^*{}_1X_1 \\ A^*{}_2X_2 \end{bmatrix} = \begin{bmatrix} X_1B^* \\ X_2B^* \end{bmatrix} = XB^*.$

In the next result, we extend the Fuglede-Putnam theorem to posinormal operators.
Theorem 3.44 (Bachir,[7]) If $A \in B(H)$ is hyponormal and $B^* \in B(H)$ is an invertible posinormal operator such that $AX = XB$ for some $X \in C_2(H)$, then $A^*X = XB$.

We now generalize the Fuglede-Putnam theorem by relaxing the normality of A to p-w-hyponormal operator and that of B^* to class $\mathbf{y}$ operator.
Theorem 3.45(Prasad and Bachir,[47]) Let $A \in B(H)$ be an injective p-w-hyponormal and $B^* \in B(K)$ be class $\mathbf{y}$ operator. If $AX = XB$ for some $X \in B(K,H)$, then $A^*X = XB^*$.
Theorem 3.46 If $A \in B(H)$ is a p-w-hyponormal operator such that $KerA \subset KerA^*$ and $B^* \in B(K)$ a class $\mathbf{y}$ operator, such that $AX = XB$ for some $X \in B(K,H)$, then $A^*X = XB^*$.
Proof Follows easily from the proof of Theorem 3.43
That is, by decomposing A into normal part A_1 and pure part A_2 as follows,
$A = A_1 \oplus A_2$ on $H = H_1 \oplus H_2$ and by letting $X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} : K \to H_1 \oplus H_2$.
We find that since $KerA_2 \subset KerA^*{}_2$ and A_2 is pure, A_2 is injective.
Thus $AX = XB$ implies

$$\begin{bmatrix} A_1X_1 \\ A_2X_2 \end{bmatrix} = \begin{bmatrix} X_1B_1 \\ X_2B_2 \end{bmatrix}$$

.

Hence,

$$A^*X = \begin{bmatrix} A^*{}_1X \\ A^*{}_2X \end{bmatrix} = \begin{bmatrix} X_1B^*{}_1 \\ X_2B^*{}_2 \end{bmatrix} = XB^*$$

Which shows that $A^*X = XB^*$.

Generalization of the Fuglede-Putnam theorem by putting conditions to X.

So far, we have tried to generalize the Fuglede-Putnam theorem by relaxing the normality hypotheses of the operators A and B.

In this section, a generalization of the Fuglede-Putnam theorem is done by putting some conditions on the operator X.

Lemma 3.47 Let $X \in B(H)$ be unitary and $A, B \in B(H)$ be operators.If $XA = BX$, then $XA^* = B^*X$

Remark 3.48 By dropping the unitary hypothesis on X in Lemma 3.47, we use a matrix operator trick by use of the Julia operator to arrive to the result below.

Theorem 3.49(Mortad,[42]) Let $A, B \in B(H)$ be operators and suppose $X \in B(H)$ is a contraction such that

$$(1-X^*X)^{\frac{1}{2}}A = B(1-XX^*)^{\frac{1}{2}} = (1-X^*X)^{\frac{1}{2}}A^* = B^*(1-XX^*)^{\frac{1}{2}} = 0$$

If $XA = BX$, then $XA^* = B^*X$.

Proof

Consider the matrix operators defined on $H \oplus H$ as

$$\hat{A} = \begin{bmatrix} 0 & 0 \\ 0 & A \end{bmatrix}, \hat{B} = \begin{bmatrix} B & 0 \\ 0 & 0 \end{bmatrix}$$

and

$$\hat{N} = J(N) = \begin{bmatrix} (1-NN^*)^{\frac{1}{2}} & N \\ -N^* & (1-N^*N)^{\frac{1}{2}} \end{bmatrix}$$

Then $\hat{N}\hat{A} = \begin{bmatrix} 0 & NA \\ 0 & (1-N^*N)^{\frac{1}{2}}A \end{bmatrix} = \begin{bmatrix} 0 & NA \\ 0 & 0 \end{bmatrix}$ by hypothesis. We also have $\hat{B}\hat{N} = \begin{bmatrix} B(1-NN^*)^{\frac{1}{2}} & BN \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & BN \\ 0 & 0 \end{bmatrix}$

Then $\hat{B}\hat{N} = \hat{N}\hat{A}$.

But $\hat{N}$ is unitary so that we have

$$\hat{B}^*\hat{N} = \begin{bmatrix} B^*(1-NN^*)^{\frac{1}{2}} & BN \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & NA^* \\ 0 & (1-N^*N)^{\frac{1}{2}}A^* \end{bmatrix} = \hat{N}\hat{A}$$

.

Thus we get $B^*N = NA^*$.

Corollary 3.50 Let A and B be two bounded operators. If N is an isometry such that $B(1-NN^*)^{\frac{1}{2}} = B^*(1-NN^*)^{\frac{1}{2}} = 0(1)$then $BN = NA \implies B^*N = NA^*$.

Theorem 3.51 Let $A, B \in B(H)$ be operators and $X \in B(H)$ be a partial isometry. If

(i) $XA = XB$

(ii)$\|A\| \geq \|B\|$

(iii) $(X^*X)A = A(X^*X)$

(iv) $X(\|A\|^2 - AA^*)^{\frac{1}{2}} = 0$

then $XA^* = B^*X$.

4 Applications and conclusions

In this chapter, we give some of the numerous applications of the Fuglede-Putnam theorem and give a detailed summary of our work.

4.1 Applications of Fuglede-Putnam theorem

In this section, we look at some of the applications of the Fuglede-Putnam theorem.
In the following result, we use the Fuglede-Putnam theorem to prove that if we have self-adjoint operators(bounded or unbounded) and if their product is normal, then it is self-adjoint provided a certain condition is satisfied.
Albrecht and P. G. Spain [2] proved that if we have two bounded self-adjoint operators A and B and if B satisfies $\sigma(B) \cap \sigma(-B) \subseteq \{0\} \ldots.. **$, then AB normal implies AB is self-adjoint.
By using the Fuglede-Putnam theorem, we show that the above condition is satisfied even when one operator is unbounded. That is, if B is is a bounded self-adjoint operator satisfying condition $**$, and if A is any unbounded self adjoint operator, then the result holds.
We also show that when both A and B are unbounded and B satisfies condition $**$, then the result holds.
Theorem 4.1.1 (Albrecht and Spain) Let A and B be two bounded self-adjoint operators. Let B satisfy $\sigma(B) \cap \sigma(-B) \subseteq \{0\}$. If AB is normal, then it is self-adjoint.
Proof
Let $N = AB$. Then we have $BAB = BN = N^*B$. Using the Fuglede theorem we obtain $BN^* = NB$ or $B^2A = AB^2$ and because $f : \sigma(B^2) \to \sigma(B) : \lambda^2 \to \lambda$ is well defined and continuous, $f(B^2)A = Af(B^2)$ or $BA = AB$ which implies that AB is self-adjoint.
Remark 4.1.2 In the following result, we show that the result above holds for A densely defined self-adjoint operator and B a bounded self adjoint operator satisfying condition $**$.
Theorem 4.1.3 Let A be densely defined self-adoint operator and let B be a bounded self-adjoint operator such that $\sigma(B) \cap \sigma(-B) \subseteq \{0\}$. If AB is normal, then it is self-adjoint.
Proof
Let $N = AB$ be normal. Then since $N^* = (AB)^* \supset B^*A^* = BA$
we have $BAB = (BA)B = B(AB) \implies BN = (BA)B \subset N^*B$. But N and $N^*$are normal, so by the Fuglede-Putnam theorem we get $BN^* \subset NB = \overline{N}B = NB$ since N is closed.
It thus follows that $B^2A = B(BA) \subseteq BN^* \subset NB = (BA)B = AB^2$ that is, B^2 and A commute.

Since the function $f:\sigma(B^2)\to\sigma(B):\lambda^2\to\lambda$ is well defined and continuous, $f(B^2)$ and Acommute or B and A commute i.e $BA=AB$ i.e $BA\subset AB$.
Since AB is normal, then $D(AB)=D((AB)^*)$ and on $D((AB)^*)$ we have $(AB)^*=AB$ which shows that AB is self adjoint.
Remark 4.1.3 Theorem 4.1.2 still holds if instead of assuming that AB is normal, we assume that BA is normal and the other assumptions remains, then BA is normal.
Lemma 4.1.4 If N is an unbounded normal operator and if B is a self-adjoint operator such that $D(N)\subset D(B)$, then

$$BN\subset N^*B \implies BN^*\subset NB$$

.
Proposition 4.1.5 Let A,B be two unbounded self-adjoint operators. If $N=AB$ is normal, then

$$BN\subset N^*B \implies BN^*\subset NB$$

.
Proof
The proof is trivial since $D(AB)\subset D(B)$.
Theorem 4.1.5 [40] Let A,B be two unbounded self-adjoint operators such that $\sigma(B)\cap\sigma(-B)\subseteq\{0\}$. If AB is normal, then it is self-adjoint.
Proof
Let $N=AB$. We have $BAB=B(AB)=(BA)B\subset(AB)^*B$ which implies that $BN\subset N^*B$ but $D(N)\subset D(K)$.
From Proposition 4.1.5, we have that $BN\subset N^*B$ or $B^2A\subset B(AB)^*\subset AB^2$ and $B^2A\alpha=NB^2\alpha$ for $\alpha\in D(B^2A)$. Using the same argument as in the proof of lemma 4.1.4, we can say that for $\alpha\in RanP_{B_n}$we have $B^2N\alpha=NB^2\alpha$.
Now we take the function $f:\sigma(B^2)\to\sigma(B):\lambda^2\to\lambda$ to get $f(B^2)N\alpha=N^*B\alpha$ and hence $BN\alpha=NB\alpha$. But $BN\alpha=N^*B\alpha$ on H_n.
Hence $N^*B\alpha=NBalpha$.
Using the orthogonal decomposition $H_n=\overline{RanB}\oplus KerB$ for the B restricted to H_n, we have $N=N^*$ on H_n. This shows that N_n, where N_n is N restricted to H_n, is self-adjoint. Hence $\sigma(N_n)\subseteq\mathbf{R}\forall n$ and then $\sigma(N)\subseteq\mathbf{R}$ and a normal operator with a real spectrum is self adjoint. Thus AB is self-adjoint.
Remark 4.1.6 We have clearly seen that the result is true for any couple of self-adjoint operators regardless of their boundedness and provided condition $(**)$ is satisfied.

Another application of the Fuglede-Putnam theorem is in the rectangular matrix version of the Fuglede-Putnam theorem where it is used to prove that for rectangular complex matrices A and B, both AB and BA are normal if and only if $A^*AB=BAA^*$ and $B^*BA=ABB^*$.

In [63] , the author proved that if A and B are normal $n \times n$ complex matrices, AB and BA are normal if and only if $A^*AB = BAA^*$.
Later on, the author improved this in [65] by omitting the requirement that B be normal. In this work, by using the Fuglede-Putnam , we show that the assumption on the normality of A can also be removed.
By recalling that C^{mn} denote the set of all $m \times n$ complex matrices, we start by having Fuglede-Putnam theorem in matrix notation.
Theorem 4.1.7 (Fuglede-Putnam) Let $P \in C^{mm}, Q \in C^{nn}, T \in C^{mn}$. If P and Q are normal and $PT = TQ$, then $P^*T = TQ^*$.
Proof
Since the matrix $P \oplus Q$ is normal, there exists a scalar polynomial g such that $(P \oplus Q)^* = g(P \oplus Q)$.
This implies that $P^* = g(p)$ and $Q^* = g(Q)$. Hence

$$P^*T = g(P)T = TG(Q) = TQ^*$$

Theorem 4.1.8 Let $A \in C^{mn}$ and $B \in C^{nm}$. Then AB and BA are normal if and only $A^*AB = BAA^*$ and $ABB^* = B^*BA$.
Proof
Suppose AB and BA are normal. Then $(AB)^*$ and $(BA)^*$ are normal.
Hence, since

$$A^*(AB)^* = A^*B^*A^* = (BA)^*A^*$$

, then by the Fuglede-Putnam theorem, we have $A^*AB = BAA^*$.
Similarly, from $(AB)^*B^* = B^*(BA)^*$, we obtain $ABB^* = B^*BA$
Conversely, from $A^*AB = BAA^*$ and $ABB^* = B^*BA$, by multiplying the first equation by B^* and the second one by A^*, we have that AB and BA are normal.
Remark 4.1.9 We note that every $A \in C^{nn}$ has a polar decomposition as $A = UH$ where $H \in C^{nn}$ is positive semidefinite Hermitian and $U \in C^{nm}$ is unitary. If A is singular, U is not unique. We have the following theorem.
Theorem 4.1.10 ([22]) Let $A = UH$ where $H \in C^{nn}$ is positive semi definite Hermitian and $U \in C^{nn}$ is unitary and let $B \in C^{nn}$.
(a). If BU is normal and $HBU = BUH$, then AB and BA are normal.
(b). If AB and BA are normal, then $HBU = BUH$.
Proof
Suppose that BU is normal and $HBU = BUH$. Then

$$BAA^* = BUH(UH)^* = BUH^2U^* = H^2BUU^* = H^2B = (UH)^*UHB = A^*AB$$

Since BU is normal and $HBU = BUH$, from the Fuglede-Putnam theorem, we also have

$H(BU)^* = (BU)^*H$.
Hence,

$$ABB^* = UHBU(BU)^* = UBU(BU)^*H = U(BU)^*BUH = UU^*B^*BUH = B^*BA$$

.
Therefore by Theorem 4.1.8, AB and BA are normal and hence the proof of (a).
Now to prove (b), we let AB and BA be normal and note that there exists a positive semi definite Hermitian $K \in C^{nn}$ such that $A = KU$.
Using Theorem 4.1.8, we obtain $H^2B = A^*AB = BAA^* = BK^2$.
Hence, since H and K are positive semidefinite Hermitian, $HB = BK$.
Then $HBU = BKU = BUH$.

4.2 Conclusion

In this thesis, we have found that under some certain conditions, some non-normal operators are normal.
Some of the non-normal operators we have observed are hyponormal,p-hyponormal, log-hyponormal, semi-normal, n-power normal, n-power quasihyponormal and restriction-convexoid operators and we have clearly shown some conditions under which these operators are normal. In this section, we give a brief summary of our work where we have generalized the Fuglede-Putnam theorem to non-normal operators.
Remark 4.2.1 In Theorem 3.8, we have shown that if A is $P-$quasihyponormal operator and B^* is an invertible $P-$quasihyponormal operator such that $AX = XB$ for $X \in C_2(H)$ and $\left\| |A^*|^{1-p} \right\| . \left\| |B^{-1}|^{1-p} \right\| \leq 1$. Then $A^*X = XB^*$. However, we have proved in Theorem 3.18 that Theorem 3.8 still holds without the additional condition $\left\| |A^*|^{1-p} \right\| . \left\| |B^{-1}|^{1-p} \right\| \leq 1$..
We have also shown in Theorem 3.11 that if A is $(P,k)-$quasihyponormal operator and B^* is an invertible $(P,K)-$quasihyponormal such that $AX = XB$ for $X \in C_2(H)$ and $\left\| |A^*|^{1-p} \right\| . \left\| |B^{-1}|^{1-p} \right\| \leq 1$., then $A^*X = XB^*$. However, we have shown that the results of Theorem 3.18 still holds true if we consider a $(p,k)-$ quasihyponormal operator instead of a $p-$qusihyponormal. Since an invertible $(p,k)-$quasihyponormal is $(p,k)-$ quasihyponormal operator, thus in Theorem 3.19, we have shown that Theorem 3.11 remains true with $(p,k)-$ quasihyponormal operator without the additional condition $\left\| |A^*|^{1-p} \right\| . \left\| |B^{-1}|^{1-p} \right\| \leq 1$..
Therefore, as a consequence of Theorem 3.19, we obtain.
Corollary 4.2.2 Let $A, B, \in B(H)$ and $X \in C_2(H)$ such that $AX = XB$. Then $A^*X = XB^*$ under either of the following hypotheses
Remark 4.2.3 In Theorem 3.14, we have shown that for $A \in B(H)$ with A^* p-hyponormal or log-hyponormal and $B \in B(K)$ being dominant such that $XA = BX$, then $XA^* = BX^*$ for $X \in B(K,H)$.In [19], it is shown that every p-hyponormal and every log-hyponormal operator is class A. However, the following example shows that the assertion of Theorem

3.14 does not necessary hold true for class A operators.

Example 4.2.4 Let$(E_n)_{n=-\infty}^{\infty}$ be a complete orthonomal system for H. We denote the orthogonal projection onto $\mathbb{C}e_n$by Pn. Let W be weighted shift on H defined by

$$We_n = \left\{ \sqrt{2e_n} + 1(n \geq 0)en + 1(n < 0) \right\}$$

Then $W^*W - WW^* = P0$.

Define an operator T on a Hilbert space $K = H \oplus \mathbb{C}e_0$ by

$$T = \begin{bmatrix} W & 0 \\ 0 & 0 \end{bmatrix}$$

.

Then $T^{*2}T^2 = (T^*T)^2$ and therefore $|T|^2 = T^*T$.

This shows that T is class A. It is easy to see that

$KerT = \mathbb{C}\,(-e_, \oplus e_0)$ and $KerT^*\{0\} \oplus \mathbb{C}e_0$.

Hence T does not reduce T and therefore the assertions of Theorem 3.14 are not necessarily true for class A operators.

Remark 4.2.4 In Theorem 3.49, we have shown that we can also generalize the Fuglede-Putnam theorem by putting some conditions to X. Also, we note that the condition $B(1-NN^*)^{\frac{1}{2}} = B^*(1-NN^*)^{\frac{1}{2}} = 0$ in Corollary 3.50 cannot be completely eliminated. For instance, if we take the unilateral shift$U$ on l^2 and set $N = B = U$, then N is an isometry and one can check that does not verify equation$B(1-NN^*)^{\frac{1}{2}} = B^*(1-NN^*)^{\frac{1}{2}} = 0$. If we also set $A = U$, then $BN = U^2 = NA$ while $B^*N = U^*U \neq UU^* = NA^*$.

We now give an example satisfying the hypothesis of Corollary 3.50 and not satisfied by any other known version of the Fuglede-Putnam theorem.

Example 4.2.5 Consider the infinite matrices

$$X = \begin{bmatrix} 0 & 0 & & & & 0 \\ 0 & 0 & & & & \\ 1 & 0 & & & & \\ 0 & 1 & 0 & & & \\ & 0 & 1 & & & \\ & & & & & \\ 0 & & & & & \end{bmatrix}, B = \begin{bmatrix} 0 & 0 & 0 & 0 & & & & 0 \\ & 0 & 0 & 0 & 0 & & & \\ & & 0 & 0 & 2 & 0 & & \\ & & & 0 & 0 & 1 & 0 & \\ & & & & 0 & 0 & 1 & \\ & & & & & & & \\ 0 & & & & & & & \end{bmatrix}, A = \begin{bmatrix} 0 & 0 & 2 & 0 & & 0 \\ 0 & 0 & 0 & 1 & 0 & \\ & 0 & 0 & 0 & 1 & \\ & & 0 & 0 & 0 & \\ & & & 0 & & \\ & & & & & \\ 0 & & & & & \end{bmatrix}$$

.

A simple computation shows that A and B are non normal and N is an isometry.

Further we have

$$B(1-XX^*)^{\frac{1}{2}} = B^*(1-XX^*)^{\frac{1}{2}} = 0$$

.
Thus, we can verify that $BX = XA$ and hence $XA^* = B^*X$.

In Theorem 3.51 , we have shown that by requiring $X \in B(H)$ to be a partial isometry, then the Fuglede-Putnam theorem is satisfied given that the conditions in Theorem 3.51 are satisfied.
In examples 4.2.6 we show that without necessarily putting conditions on A and B and requiring that X is an isometry, the Fuglede-Putnam theorem is satisfied if the conditions of Theorem 3.50 are met.
Example 4.2.4 Let

$$A = \begin{bmatrix} 0 & 0 & a_1 & & & \\ & 0 & 0 & a & & \\ & & 0 & 0 & a_2 & \\ & & & 0 & 0 & a \\ & & & & & \\ & & & & & \\ 0 & & & & & \end{bmatrix}, B = \begin{bmatrix} 0 & 0 & a & & & 0 \\ & 0 & 0 & b_1 & & \\ & & 0 & 0 & a & \\ & & & 0 & 0 & b_2 \\ & & & & & \\ & & & & & \\ 0 & & & & & \end{bmatrix}, X = \begin{bmatrix} 0 & 1 & 0 & & & 0 \\ & 0 & 0 & 0 & & \\ & & 0 & 0 & 0 & \\ & & & & & \\ & & & & & \\ 0 & & & & & \end{bmatrix}$$

A simple computation shows that A, B and X satisfies properties $(i), (ii), (iii)$ and (iv) of Theorem 3.51 and thus $XA^* = B^*X$.

4.3 Summary

In summary, we have been able to prove that the Fuglede-Putnam theorem can be generalised to non-normal operators under the following condtions:
(1). A and B^* are hyponormal. (2). B is invertible and $\|A\| . \left\|B^{-1}\right\| \leq 1$.
Also, we have shown that the Fuglede-Putnam theorem holds under the following conditions for some non-normal operators:
(i) If A and B^* are hyponormal and $X \in B(H)$.
(ii) If A and B^* are subnormal and $X \in C_2(H)$.
(iii) If A is quasihyponormal and B^* is invertible quasi-hyponormal and $X \in C_2(H)$.
(iv)If A^* is p-hyponormal or log-hyponormal , B is dominant and $X \in B(H)$.
(v) If A^* is m-hyponormal, B is dominant and $X \in B(H)$.
(vi) If A is (p,k)-quasihyponormal, B^* is invertible (p,k)-quasihyponormal and $X \in C_2(H)$.
(vii) If A is p-quasihyponormal, B^* is invertible p-quasihyponormal and $X \in B(H)$.

(viii) If A is an injective p-hyponormal operator, B^* is a class$\mathbf{y}$ and $X \in B(H)$.
(ix) If A^* is p-hyponormal, B is injective (p,k)-quasihyponormal and $X \in B(H)$.
(x) If A is dominant, B^* is p-hyponormal or log-hyponormal and $X \in B(H)$.
(xi) If A is dominant, B^* is w-hyponormal such that $KerB^* \subset KerB$ and $X \in B(H)$.
(xii) If A is w-hyponormal, B^* is injective w-hyponormal and $X \in B(H)$.
(xiii) If A is w-hyponormal such that $KerA \subset KerA^*, B^*$ is w-hyponormal and $X \in B(H)$.
(xiv) If A is a class $\mathbf{y}$ operator, B^* is w-hyponormal such that $KerB^* \subset KerB$ and $X \in B(H)$.
(xv) If A is w-hyponormal such that $KerA \subset KerA^*$, B^* is a class $\mathbf{y}$ operator and $X \in B(H)$.
(xvi) If A is injective p-w-hyponormal for $0 < P \leq 1, B^*$ a class $\mathbf{y}$ operator and $X \in B(H)$.
(xvii)If A^* is w-hyponormal, B w-hyponormal with $KerA^* \subset KerA$ and $KerB \subset KerB^*$ and $X \in B(H)$.
(xviii) If A^* is an injective w-hyponormal operator, B is dominant and $C_2(H)$.
(xix) If A is hyponormal, B^* is invertible posinormal and $X \in C_2(H)$.

4.4 Open Problem

In our research, we found that an open problem is to find more classes of non-normal operators that satisfies the Fuglede-Putnam theorem.

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