UNIVERSITY OF NAIROBI

On Decomposition Of Operators in Hilbert Spaces

BY

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I56/38471/2020

A Thesis Submitted to the Department of Mathematics for Examination in Partial Fulfillment of the Requirements for the Award of Degree of Master of Science in Pure Mathematics of the University of Nairobi

August 2022

Abstract

In this project, we investigate the direct sum decomposition of some classes of operators in Hilbert spaces with the aim of defining properties of the direct summands of these
operators.
We show that an arbitrary operator T decomposes into a normal and a completely nonnormal parts. The properties for which an operator T has nontrivial normal and direct summands are given. In addition, we study this decomposition of operators in some equivalence classes (similar, unitarily equivalent, quasisimilar and almost-similar) of operators.
We also investigate the properties of the direct decomposition of a contraction into a unitary and a completely nonunitary parts. We show that an arbitrary operator T decomposes this way upon dividing the operator by its norm (re-normalization).

Master Thesis in Mathematics at the University of Nairobi, Kenya.
ISSN 2410-1397: Research Report in Mathematics

DISTRIBUTOR: Department of Mathematics, University of Nairobi, Kenya

Declaration and Approval

I the undersigned declare that this dissertation is my original work and to the best of my knowledge, it has not been submitted in support of an award of a degree in any other university or institution of learning.

01/09/2022

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Dedication

This project is dedicated to my spouse Fiona and our daughters Luna and Ella.

Contents

Acknowledgments

I am grateful to my supervisors Prof. Nzimbi and Dr. Wafula for their support and guidance in making this project a success. I must also thank my lecturers for their endless effort and insightful ideas during my graduate studies. Special thanks goes to Prof. Luketero, Dr. Ongaro, Dr. Imagiri, Dr. Katende, Dr. Muriuki and Dr. Rao.

I am thankful to the University of Nairobi - School of Mathematics, through which I participated in East African Universities Mathematics Programme (EAUMP) and International Centre for Theoretical Physics (ICTP) seminars. It is also fair to thank my classmates Kipng'ong'o, Omondi and Masinde who were a real help in my graduate level studies.

Finally, I wish to acknowledge my family for their moral support during the period I have been working on this project. Without their support, I would not have finished this work.

William Mwangi

Nairobi, 2022.

1 PRELIMINARIES

1.0.1 Introduction

In this project, we study the direct sum decomposition of some classes of operators in Hilbert spaces. Decomposition of operators is an important tool for operator theory in functional analysis. The idea of decomposing an operator is to isolate the parts into direct summands of the operator. As one of many decompositions, direct sum decomposition has been largely determined by the work of Nagy and Foias [30]. A major result from this work is that any operator can be decomposed into a direct sum of normal and completely non-normal (c.n.n) parts. Moreover, a contraction can be decomposed into direct sum of a unitary and completely non-unitary (c.n.u) parts (where any of the direct summands could be missing). Another important discovery in the decomposition of operators came up from Neumann Wold [40]. The von Neumann-Wold decomposition of an isometry is where the operator (isometry) decomposes into unitary and completely non-unitary parts. The c.n.u part in this case is a unilateral shift. Fuhrmann [12] proved a similar result that any contraction $T \in B(H)$ has a unique decomposition with respect to the decomposition of H into a direct sum $H = H_0 \oplus H_1$ of reducing subspaces of T such that $T\mid_{H_0}$ is unitary and $T\mid_{H_1}$ is completely non-unitary.
Williams [39] is another Mathematician that has shown important results on decomposition of operators. He has proved that every operator T is unitarily equivalent to the sum $T_1 \oplus T_2$ where T_1 is normal and T_2 is completely non-normal. Moreover, when M is a T_2 reducing subspace and $T_2\mid_M$ is normal, then $M = \{0\}$. On hyperinvariance results, Kubrusly [27] has proved that similarity preserves non-trivial invariant subspaces while quasisimilarity preserves hyperinvariant subspaces. He has also shown that if a contraction has no non-trivial invariant subspace, then it is either a C_{00}, a C_{01} or a C_{10} contraction. Duggal and Kubrusly [9] described the completely non-unitary part of a contraction using the Putnam-Fuglede (PF) Theorem. We investigate the connection between the direct sum decomposition of a contraction operator and an arbitrary operator to this decomposition.

1.0.2 Notations, Terminologies and Definitions

Notations

H, H_1, K, K_1: Hilbert spaces or subspaces of Hilbert spaces over the complex numbers $\mathbb{C}$.
T, T_1, T_2, A, B: Bounded linear operators.
$B(H)$: Banach algebra of bounded linear operators on H.
T^*: The adjoint of T.
$\|T\|$: The operator norm of T.

$B(H,K)$: The set of bounded linear operators from H to K and equipped with the norm.
$\|x\|$:The norm of a vector x.
$\langle x,y\rangle$: The inner product of x and y on a Hilbert Space H.
$Ran(T)$: The range of an operator T.
$Ker(T)$: The kernel of an operator T.
$\overline{M}$: The closure of subspace M of H.
$M^{\perp}$: The orthogonal complement of a closed subspace M of H.
$M \oplus N$: The direct sum of subspaces M and N of H.
0 and I: The zero and identity operator on H respectively.
c.n.n: Completely non-normal
c.n.u: Completely non-unitary
l^2: Hilbert space of all square summable infinite sequence of complex numbers ($\mathbb{C}$)

Terminologies and Definitions

Definition 1.0.1. *An* ***operator*** *means a bounded (i.e. continuous) linear transformation from H into K (equivalently, with domain H and range a subset of K).*

Definition 1.0.2. *The set* $\sigma(T) = \{\lambda \in \mathbb{C} : \lambda I - T$ *is not invertible* $\}$ *(equivalently,* $Ker(\lambda I - T) \neq \{0\}$ *or* $Ran(\lambda I - T) \neq H$*) is the* ***spectrum of*** T.

Definition 1.0.3. *The* ***spectral radius of*** T *is given by* $r(T) = sup\{|\lambda| : \lambda \in \sigma(T)\} = max\{|\lambda| : \lambda \in \sigma(T)\} = lim_n\{\| T \|^{\frac{1}{n}}\}$.

Definition 1.0.4. *The* ***numerical range*** *(field of values of* T*) is denoted by* $W(T) = \{\langle Tx, x\rangle : x \in H, \|x\| = 1\}$.

Definition 1.0.5. *The* ***point spectrum of*** T *(i.e. the set of all eigenvalues of* T*) is defined as* $\sigma_P(T) = \{\lambda \in \mathbb{C} : Ker(\lambda I - T) \neq \{0\}\}$.

Definition 1.0.6. *The* ***continuous spectrum of*** T *(i.e. the set of* $\lambda \in \mathbb{C}$ *where* $(\lambda I - T)$ *has a densely but unbounded inverse) is defined as* $\sigma_C(T) = \{\lambda \in \mathbb{C} : Ker(\lambda I - T) = \{0\}, \overline{Ran(\lambda I - T)} = H$ *and* $Ran(\lambda I - T) \neq H\}$.

Definition 1.0.7. *The* ***residual spectrum of*** T *(i.e. the set of* λ *where* $(\lambda I - T)$ *has an inverse that is not densely defined), is given by* $\sigma_R(T) = \{\lambda \in \mathbb{C} : Ker(\lambda I - T) = \{0\}$ *and* $\overline{Ran(\lambda I - T)} \neq H\}$.

Remark 1.0.8. *We have* $\sigma(T) = \sigma_P(T) \cup \sigma_C(T) \cup \sigma_R(T)$ *where the elements in the right are pairwise disjoint.*

Definition 1.0.9. *A subspace* $M \subset H$ *is said to be* ***invariant*** *under an operator* $T \in B(H)$ *if* $TM \subseteq M$ *(equivalently, if* $x \in M \Rightarrow Tx \in M$*).*

Definition 1.0.10. T *is said to have* ***nontrivial invariant subspace*** *if there is a subspace* $\{0\} \neq M \neq H$ *invariant for* T.

Definition 1.0.11. *A subspace* $M \subseteq H$ ***reduces*** *an operator* T *if* M *is invariant under both* T *and* T^* *(equivalently, if both* M *and* $M^{\perp}$ *are invariant under* T*).*

Definition 1.0.12. *An operator is* ***reducible*** *if it has a nontrivial reducing subspace.*
An operator T on H is ***reductive*** *if each invariant subspace of T reduces T.*

Definition 1.0.13. *The* ***commutant*** *of $T \in B(H)$ is the set of all operators that commute with T, denoted by $\{T\}' = \{S \in B(H) : ST = TS\}$.*

Definition 1.0.14. *M is a* ***hyperinvariant subspace*** *for T if it is invariant for every operator that commutes with T.*

Remark 1.0.15. *If M is an invariant subspace under $T \in B(H)$, then relative to the decomposition $H = M \oplus M^{\perp}$, T can be written as*

$$T = \begin{pmatrix} T\mid_M & X \\ 0 & Y \end{pmatrix}$$

where operators $X : M^{\perp} \to M$, $Y : M^{\perp} \to M^{\perp}$ and $T\mid_M : M \to M$. Conversely, if $T \in B(H)$ can be written as

$$T = \begin{pmatrix} Z & X \\ 0 & Y \end{pmatrix}$$

with respect to the decomposition $H = M \oplus M^{\perp}$, then $Z = T\mid_M$ is a part of T. The operator $X = 0$ if and only if M reduces T. In this case, T is reduced into the orthogonal direct sum of the operators $Z = T\mid_M$ and $Y = T\mid_{M^{\perp}}$ such that $T = Z \oplus Y$.
If $\{T_k \in B(H_k)\}$ is a bounded set of operators, then the direct sum of $\{T_k\}$ is the operator $T \in B(H)$: $T\mid_{H_k} = T_k$ for each k. We denote this by

$$T = \bigoplus_k T_k$$

Definition 1.0.16. *A* ***direct summand*** *is a restriction of an operator to a reducing subspace of it.*

Definition 1.0.17. *A bounded operator T on H is said to be* ***nilpotent*** *if $T^n = 0$ for some positive integer n.*

Definition 1.0.18. *An operator $T \in B(H)$ is said to be:*
normal *if $T^*T = TT^*$.*
involution *if $T^2 = I$.*
self-adjoint *if $T^* = T$.*
unitary *if $T^*T = TT^* = I$.*
isometry *if $T^*T = I$.*
co-isometry *if $TT^* = I$.*
projection *if $T^* = T$ and $T^2 = T$.*
partial isometry *if $T = TT^*T$ (equivalently, if T^*T is a projection).*
symmetry *if $T = T^* = T^{-1}$ (equivalently, T is self-adjoint unitary).*
quasinormal *if $T(T^*T) = (T^*T)T$ (equivalently, if T commutes with T^*T, i.e $[T, T^*T] = 0$).*
binormal *if $(T^*T)(TT^*) = (TT^*)(T^*T)$.*

2-normal *if* $T^*T^2 = T^2T^*$.
hyponormal *if* $T^*T \geq TT^*$ *(equivalently, if* $T^*T - TT^* \geq 0$ *(a positive operator).*
cohyponormal *if its adjoint is hyponormal, i.e,* T *is cohyponormal if* $TT^* \geq T^*T$. *Obviously, if* $T \in B(H)$ *is both hyponormal and cohyponormal, then* T *must be normal.*
p-hyponormal *if* $(T^*T)^p \geq (TT^*)^p$, *where* $0 < p \leq 1$.
M-hyponormal *if* $\|(zI - T)^*x\| \leq M\|(zI - T)\|$, $\forall$ *complex numbers* z *and* $\forall\, x \in M \subset H$ *and* M *a positive number.*
quasihyponormal *if* $T^{*2}T^2 - (T^*T)^2 \geq 0$ *(equivalently, if* $T^*(T^*T - TT^*)T \geq 0$).
paranormal *if* $\|Tx\|^2 \leq \|T^2x\|\ \forall\, x \in H$*(equivalently, if* $\|Tx\| \leq \|T\|\|x\|\ \forall\, x \in H$*).*
k-quasihyponormal *if* $T^{*k}(T^*T - TT^*T^k \geq 0$, *for some integer* $k \geq 1$ *and* $x \in H$.
p-quasihyponormal *if* $T^*((T^*T)^p - (TT^*)^p)T \geq 0$.
p,k-quasihyponormal *if* $T^{*k}((T^*T)^p - (TT^*)^p)T^k \geq 0$, *where* $0 < p \leq 1$ *and* k *is a positive number.*
positive *if* $\langle Tx, x\rangle > 0$ *for all* $0 \neq x \in H$ *and* T *is self-adjoint.*
dominant *if for any* $\lambda \in \mathbb{C}$ *corresponds a number* $M_\lambda \geq 1$ *such that* $\|(T - \lambda I)^*x\| \leq M_\lambda\|(T - \lambda I)x\|$ $\forall\, x \in H$.
seminormal *if it is either hyponormal or cohyponormal (equivalently, if either* T *or* T^* *is hyponormal). Clearly, every hyponormal operator is seminormal but the converse is not true.*
subnormal *if it has a normal extension, i.e. if there exists a normal operator* N *on a Hilbert space* K *where* $H \subset K$ *and* H *is* N*-invariant and* $T = N|_H$.

Remark 1.0.19. *From the above definitions, we get the following inclusions*
Unitary operators $\subseteq$ *Isometric operators* $\subseteq$ *Partial isometries.*
Normal $\subseteq$ *Quasinormal* $\subseteq$ *Subnormal* $\subseteq$ *Hyponormal* $\subseteq$ *Seminormal.*

Definition 1.0.20. *An operator* $T \in B(H)$ *is a*
left shift *if* $Tx = y$ *where* $x = (x_1, x_2, \cdots)$ *and* $y = (x_2, x_3, \cdots)$.
right shift *if* $Tx = y$ *where* $x = (x_1, x_2, \cdots)$ *and* $y = (0, x_1, x_2, \cdots)$.
unilateral shift *if there exists a sequence of (pairwise) orthogonal subspaces* $\{H_n : n \geq 0\}$ *such that* $H = \bigoplus_{n=0}^{\infty} H_n$ *and* S *maps each* H_n *isometrically onto* H_{n+1}.
bilateral shift *if there exist orthogonal subspaces* $\{H_n : n = 0, \pm 1, \pm 2, \cdots\}$ *such that* $H = \bigoplus_{n=-\infty}^{\infty} H_n$ *and* S *maps each* H_n *isometrically onto* H_{n+1}.
contraction *if* $\|Tx\| \leq \|x\|$ *for every* $x \in H$.
scalar *if it is a scalar multiple of the identity operator (i.e if* $T = \alpha I$ *where* $\alpha \in \mathbb{C}$*).*

Definition 1.0.21. *A* ***unitary subspace*** *(respectively, normal) in* H *of an operator* T *is the maximum (largest) subspace in* H *which reduces* T *to a unitary (normal) operator.*

Definition 1.0.22. $T \in B(H)$ *is* ***completely non-normal (c.n.n)*** *(pure) if there exist no nontrivial reducing subspace* $M \subset H$ *such that* $T|_M$ *is normal (equivalently, if* T *has no direct normal summand or if the normal subspace is* $\{0\}$*).*

Definition 1.0.23. *A contraction* $T \in B(H)$ *is called* ***completely non-unitary (c.n.u)*** *if there exist no nontrivial reducing subspace* $M \subset H$ *of* T *on which* T *acts unitarily (equivalently, if its unitary part acts on the zero space* $\{0\}$*).*

Definition 1.0.24. *Let H and K be Hilbert spaces, then $T \in B(H,K)$ is **invertible** if it is both injective (one-to-one) and surjective (onto) (equivalently, if $Ker(T) = \{0\}$ and $Ran(T) = K$).*

Remark 1.0.25. *Definition 1.0.24 is not true in infinite dimensional Hilbert spaces.*

Definition 1.0.26. *Two operators $T \in B(H)$ and $S \in B(K)$ are*
***similar** (denoted by $T \sim S$) if there exist an operator $X \in B(H,K)$ such that $XT = SX$ (equivalently, $T = X^{-1}SX$ or $S = XTX^{-1}$.*
***unitarily equivalent** (denoted by $T \cong S$) if there exist a unitary operator $U \in B(H,K)$ such that $UT = SU$ (equivalently, $T = U^*SU$ or $S = UTU^*$).*

Definition 1.0.27. *An operator $X \in B(H,K)$ is a **quasiinvertible** (quasiaffinity) if it is an injective operator with dense range (equivalently, $Ker(X) = \{0\}$ and $\overline{Ran(X)} = K$ or $Ker(X) = \{0\}$ and $Ker(X^*) = \{0\}$. This means that $X \in B(H,K)$ is quasiinvertible if and only if $X^* \in B(K,H)$ is quasiinvertible.*

Definition 1.0.28. *An operator $T \in B(H)$ is a **quasiaffine transform** of $S \in B(K)$ if there exists a quasiinvertible $X \in B(H,K)$ such that $XT = SX$.*

Definition 1.0.29. *Two operators $T \in B(H)$ and $S \in B(K)$ are*
***quasisimilar** (denoted by $T \approx S$) if they are quasiaffine transforms of each other (equivalently, if there exist quasiinvertible operators $X \in B(H,K)$ and $Y \in B(K,H)$ such that $XT = SX$ and $YS = TY$).*

Remark 1.0.30. *It is easily verified that similar operators are quasisimilar but the converse is not true. Additionally, quasisimilarity is an equivalence relation and that T^* is quasisimilar to S^* whenever $T \approx S$ ([1]).*

Definition 1.0.31. *Two operators T and S are said to be **almost-similar** (denoted by $T \approx^{a.s} S$) if there exist an invertible operator N such that the following properties hold*

$$T^*T = N^{-1}(S^*S)N$$

$$T^* + T = N^{-1}(S^* + S)N$$

Definition 1.0.32. *An operator $X \in B(H,K)$ **intertwines** $T \in B(H)$ to $S \in B(K)$ if $XT = SX$.*

Definition 1.0.33. *T is **densely intertwined** to S if there exist an operator with dense range intertwining T to S.*

Definition 1.0.34. *The **multiplicity** of $T \in B(H)$, denoted by $\mu(T)$, is the minimum cardinality of a set $K \subset H$ such that*

$$H = \vee_{n=0}^{\infty} T^n K$$

Definition 1.0.35. *A **lattice**, $\mathscr{L}$, is a partially ordered set where each pair of elements $a, b \in \mathscr{L}$ has a least upper bound and a greatest lower bound.*
The lattice of all invariant subspaces of T will be denoted by $Lat(T)$ while for all reducing subspaces will be denoted as $Red(T)$.

Definition 1.0.36. ***Polar decomposition*** *of an operator refers to the factorization of an operator into the product of a partial isometry and a nonnegative operator.*

Definition 1.0.37. ***Cartesian decomposition*** *of an operator refers to the direct sum decomposition where every operator T can be written as $T = Re(T) + iIm(T)$ such that $Re(T) = \frac{1}{2}(T + T^*)$ and $Im(T) = -\frac{i}{2}(T - T^*)$ are self-adjoint operators.*

1.0.3 Convergence and stability

Suppose $\{T_n \in B(H) : n \geq 1\}$ is a sequence of operators on a Hilbert space H. Then from the Banach-Steinhaus Theorem (which states that if $\mathscr{F}$ is an arbitrary set of bounded linear transformations between Banach spaces X and Y, then $sup_{F \in \mathscr{F}} \|F\| < \infty$ whenever $sup_{F \in \mathscr{F}} \|Fx\| < \infty$ $\forall x \in X$), the following conditions are pairwise equivalent.

1. $sup_n \|T_n\| < \infty$.
2. $sup_n \|T_n x\| < \infty \; \forall x \in H$.
3. $sup_n \mid \langle T_n x; y \rangle \mid < \infty \; \forall x, y \in H$.
4. $sup_n \mid \langle T_n x; x \rangle \mid < \infty \; \forall x \in H$.

Definition 1.0.38. *If one of the conditions holds true, then the sequence $\{T_n; n \geq 1\}$ is called a* ***bounded sequence****.*

Consider the following conditions which are also pairwise equivalent

1. There exists $T \in B(H) : \langle T_n x; y \rangle \longrightarrow \langle Tx; y \rangle$ as $n \longrightarrow \infty \; \forall x, y \in H$.
2. There exists $T \in B(H) : \langle T_n x; x \rangle \longrightarrow \langle Tx; x \rangle$ as $n \longrightarrow \infty \; \forall x \in H$.
3. The scalar sequence $\{\langle T_n x; x \rangle \in \mathbb{C}; n \geq 1\}$ converges $\forall x \in H$.
4. The scalar sequence $\{\langle T_n x; y \rangle \in \mathbb{C}; n \geq 1\}$ converges $\forall x, y \in H$.

Definition 1.0.39. *If any of the conditions above holds true, then the sequence $\{T_n; n \geq 1\}$ is* ***weakly convergent*** *(denoted as $T_n \xrightarrow{w} T$).*

Definition 1.0.40. *The sequence $\{T_n; n \geq 1\}$ is* ***strongly convergent****($T_n \xrightarrow{s} T$) if one of the following equivalent conditions holds true:*

1. *$\exists \; T \in B(H)$: $\|(T_n - T)x\| \longrightarrow 0$ as $n \longrightarrow \infty \; \forall x \in H$.*
2. *$\{T_n x \in H; n \geq 1\}$ converges in H $\forall x \in H$.*

Definition 1.0.41. *A sequence* $\{T_n; n \geq 1\}$ *is* ***uniformly convergent*** *(*$T_n \xrightarrow{u} T$*) if it converges in* $B(H)$ *(i.e, if* $\|T_n - T\| \longrightarrow 0$ *as* $n \longrightarrow \infty$ *for some* $T \in B(H)$*).*

Remark 1.0.42. *Uniform convergence means convergence in the operator norm.*

Definition 1.0.43. *An operator* $T \in B(H)$ *is* ***weakly stable*** *if the power sequence* $\{T^n; n \geq 1\}$ *converges to the null operator (i.e.* $T^n \xrightarrow{w} 0$*) or if* $\langle T^n x; y\rangle \longrightarrow 0$ *as* $n \longrightarrow \infty$ $\forall x, y \in H$ *or if* $\langle T^n x; x\rangle \longrightarrow 0$ *as* $n \longrightarrow \infty$ $\forall x \in H$.

Definition 1.0.44. T *is called* ***power bounded*** *when the power sequence is bounded (i.e.,* $sup_n \|T^n\| < \infty$*). An operator* T *is* ***uniformly stable*** *(*$T^n \xrightarrow{u} 0$*) if* $\|T^n\| \longrightarrow 0$ *as* $n \longrightarrow \infty$.

Theorem 1.0.45. ***(Fuglede's Theorem)*** *If* $T \in B(H)$ *is normal and if* $TS = ST$ *for some* $S \in B(H)$*, then* $T^*S = ST^*$.

Remark 1.0.46. *Weak stability, strong stability, uniform stability and spectral radius are related as follows, according to the* ***Gelfand-Beurling formula*** *(*$lim\|T^n\|^{\frac{1}{n}} = r(T) \leq \|T^n\|^{\frac{1}{n}}$ *for every* $n \geq 1$*), that is,*

$$r(T) < 1 \Longleftrightarrow T^n \xrightarrow{u} 0 \Longrightarrow T^n \xrightarrow{s} 0 \Longrightarrow T^n \xrightarrow{w} 0 \Longrightarrow sup_n\|T^n\| < \infty \Longrightarrow r(T) \leq 1$$

We note that strong stability is not preserved under the adjoint operation. Indeed, T *is uniformly or weakly stable if and only if* T^* *is.*

2 LITERATURE REVIEW

The study of the structures and properties of any operator on Hilbert spaces is basically equivalent to the study of its complementary parts, its invariant and hyperinvariant subspaces. One of the major steps in investigating linear non-normal operators, has been that of finding ways of decomposing such operators into different parts which are easier to handle. The analysis of invariant subspaces is an ordinary first step in the attempt to comprehend the structure of operators. The Jordan form (for finite dimensional operators) and the spectral theorem (for normal operators) are the fundamental structure theorems that provide decompositions into invariant subspaces of special kinds.

Several researchers have discovered that in order to study the structures of any operator in Hilbert spaces, we examine how the operator can be defined by decomposing it into simple forms; for example, direct sum decomposition, polar decomposition or Cartesian decomposition, with respect to the separable Hilbert spaces. A separable Hilbert space has its invariant subspaces, that is, $H = H_1 \oplus H_2$ where H_1 is a closed subspace and H_2 is the orthogonal complement subspace.

In Furuta [13], a subspace H_1 of H is invariant under $T \in B(H)$ if for every vector $\{x \in H_1 : Tx \in H_1\}$ and a subspace H_1 of H reduces T if H_1 and H_2 are both invariant under T. The operator T has a decomposition given as

$$T_1 = T \setminus H_1$$

and

$$T_2 = T \setminus H_2$$

hence T has a direct sum decomposition as

$$T = T_1 \oplus T_2.$$

As one of many known forms of decompositions, the direct sum decomposition (orthogonal decomposition) has been largely motivated by Nagy and Foias' [30] work from which it results that an arbitrary operator can be decomposed as a direct sum of normal and completely non-normal parts.

In the literature, many researchers have demonstrated some tremendous work on the decomposition of operators. Williams [39] has shown that every operator T is always unitarily equivalent to the direct sum $T_1 \oplus T_2$ such that T_1 is normal and T_2 is completely non-normal (pure). In addition, if M is a reducing subspace for T_2 and $T_2 |_M$ is normal, then $M = \{0\}$.

Nagy and Foias [30] on their theory of contraction operators, proved that a contraction $T \in B(H)$ is a direct sum of a unitary part and a completely non-unitary part. This decomposition concurs with the von Neumann Wold decomposition for isometries, where the pure part in this case is a unilateral shift (Wold [40]). By Duggal [10], the direct sum decomposition decomposes a bounded linear operator into its normal and pure parts.

Stampfli and Wadhwa [36] proved that a hyponormal operator must be normal if it is similar to a normal operator. Fuhrmann [12] attested a similar result where any contraction $T \in B(H)$ has a unique decomposition relating to the decomposition of H into a direct sum $H = H_1 \oplus H_2$ of reducing subspaces of T where $T\mid_{H_1}$ is unitary and $T\mid_{H_2}$ is completely non-unitary. Wu [41] showed that if T is a contraction with finite defect indices, then T is quasisimilar to an isometry if and only if the completely non-unitary part is quasisimilar to an isometry.

The open question of the existence of nontrivial invariant subspaces has been studied by some operator theorists. For instance, Kubrusly [27] has proved that if a contraction has no nontrivial invariant subspace, then it is either a C_{00}, a C_{01} or a C_{10}. Kubrusly and Levan [24] proved a similar result for the class of hyponormal contractions such that if a hyponormal contraction has no nontrivial invariant subspace, then it is either a C_{00} or a C_{10} contraction. Hoover [20] showed that quasisimilarity preserves the existence of nontrivial hyperinvariant subspaces and Herrero [19] has proved that quasisimilarity does not preserve full hyperlattice.

However, despite of all these research on the decomposition of operators, there exist few results in the literature on the core properties of the pure part of the decomposed operator. There are also few results which connect the decomposition of a contraction operator into unitary and completely non-unitary parts to an arbitrary operator. Hence, from the invariant subspace problem: Does every operator on a (separable) Hilbert space of dimension greater than one have a nontrivial invariant subspace? there are some questions to-date remain unanswered. For example, does every operator decompose into a direct sum? Which classes of operators decompose into nontrivial direct summands?

In this project, we establish a connection between direct sum decompositions, invariant and reducing subspaces of an operator. For example, we demonstrate that any direct sum decomposition of a contraction operator into unitary and completely non-unitary parts can be directly determined from the direct sum decomposition of an operator into normal and completely non-normal parts. We show that for any non-zero operator T, the invariant subspace problem is reduced to the class of contractions (i.e. Does every contraction have a nontrivial invariant subspace?).

3 ON NORMAL AND COMPLETELY NON-NORMAL SUMMANDS OF AN OPERATOR

In this chapter, we study the decomposition of an operator into a direct sum of its normal and completely non-normal parts. We investigate the properties of normal and c.n.n summands for $T \in B(H)$. An operator T is classified by properties of its direct summands.
The direct sum decomposition of operator has the property that it transfers invariant subspaces from the parts (ordinary summands) to the original (decomposed) operator. Other forms of decomposition such as polar or cartesian decompositions do not have this property.
Every bounded linear operator T on H has an orthogonal decomposition $T = T_1 \oplus T_2$ whereby T_1 is normal and T_2 is c.n.n. This decomposition is implemented through T restricted to a reducing subspace. This implies that no restriction or part of T_2 to a reducing subspace is normal. Clearly, either of the two summands may be missing or absent.
Duggal and Kubrusly [9] showed that quasinormality, subnormality and hyponormality all reduce to normality in a finite-dimensional setting. This shows that such operators will have no pure (c.n.n) direct summands. We begin with the discussion of the following known result.

Lemma 3.0.1. [30] *For any operator $T \in B(H)$, if $\|\lambda\| = \|T\|$ is an eigenvalue of T then $Ker(T-\lambda I)$ is reducing.*

Example 3.0.2. *For a finite-dimensional H, eigenvalues of self-adjoint and normal operators are $\mathbb{R}$. So suppose*

$$T = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

and let $\lambda = 1$. We have

$$\sigma(T) = \{0,1\} = \sigma_P(T)$$

$$\|T\| = 1 = \mid 1 \mid$$

$Ker(T-\lambda I)$ is a reducing subspace of T. $\Rightarrow \mid \lambda \mid = \|T\|$. If T is normal, then $T^*T = TT^*$, where $\lambda \in \sigma(T), \lambda^* \in \sigma(T)$ and $T^*T = \|T\|^2$. So if T is invertible, then $Ker(T)$ is T-invariant. By extension, if T is invertible, then $Ker(T)$ is T-reducing. Therefore, $H = Ker(T) \oplus Ker(T)^{\perp}$.

From Lemma 3.0.1, we get the following result.

Corollary 3.0.3. *If T is pure or c.n.n and if $\|T\| = r(T)$, then there are no eigenvalues λ for which $\|\lambda\| = \|T\|$.*

Remark 3.0.4. *We note that $\sigma(T) = \sigma_P(T)$ for operator T acting on a finite dimensional space but $\sigma_P(T)$ may be empty in an infinite-dimensional space.*

Example 3.0.5. *Let T be a unilateral shift given by $T : l^2 \longrightarrow l^2$, such that $T(v_1,v_2,v_3,\ldots) = (0,v_1,v_2,v_3,\ldots)$ for every $(v_1,v_2,v_3,\ldots) \in l^2$. Assume that $\lambda \in \mathbb{C}$ is an eigenvalue of T. So there exists a nonzero eigenvector $(v_1,v_2,v_3,\ldots) \in l^2$ such that $(0,v_1,v_2,v_3,\ldots) = \lambda(v_1,v_2,v_3,\ldots) = (\lambda v_1,\lambda v_2,\lambda v_3,\ldots)$, so that $\lambda v_1 = 0$ and $\lambda v_i = v_{i-1}$ for each $i > 1$. If $\mid \lambda \mid= 0$, then the second condition shows that $v_1 = v_2 = v_3 = \ldots = 0$, a contradiction again! It follows that T (a unilateral shift),does not have eigenvalues and hence $\sigma_P(T) = \emptyset$.*

Lemma 3.0.6. *If T is a normal operator, then $\sigma_R(T) = \emptyset$.*

Proof. Let $\sigma_R(T) \neq \emptyset$ and suppose $\lambda \in \sigma_R(T)$. But $\lambda \in \sigma_R(T)$ if $(\lambda I - T)^{-1})$ exist as a map bounded or unbounded (i.e. not densely defined). This implies that there exists $x \neq 0$ such that
$(\overline{\lambda} I - T^*)x = 0$ ....(1)
$(\lambda I - T)$ is normal since T is normal. Thus
$\|(\lambda I - T)x\| = (\overline{\lambda} I - T^*)x \; \forall x \in H$ ....(2)
From equations (1) and (2), we get
$\|(\lambda I - T)x\| = 0$ for $x \neq 0$ or $(\lambda I - T)x = 0$ for $x \neq 0$.
Hence $x \in \sigma_P(T)$.
A contradiction since $\sigma_R(T) \cap \sigma_P(T) = \emptyset$. Hence $\sigma_R(T) = \emptyset$. □

3.0.1 Direct summands of similar and unitarily equivalent operators

Proposition 3.0.7. [27] *If an operator $T \in B(H)$ is similar (unitarily equivalent) to a part of an operator $L \in B(K)$, then it is a part of an operator similar (unitarily equivalent) to L.*

Proof. Let M be a subspace of K, and L be an operator on K. Suppose M is an invariant for L. With respect to the decomposition $K = M \oplus M^{\perp}$, we can write L as

$$L = \begin{pmatrix} L\mid_M & X \\ 0 & Y \end{pmatrix}$$

for operators $X : M^{\perp} \to M$, $Y : M^{\perp} \to M^{\perp}$ and $L\mid_M : M \to M$ which is a part of L.If $T \in B(H)$ is similar to $L\mid_M \in B(M)$, then there exists an invertible operator $U \in B(H,M)$ such that

$$T = U^{-1}(L\mid_M)U$$

Now suppose invertible operator $W = U \oplus I : H \oplus M^{\perp} \to M \oplus M^{\perp}$ so that

$$W^{-1}LW = \begin{pmatrix} U^{-1}(L\mid_M)U & U^{-1}X \\ 0 & Y \end{pmatrix}$$

Thus, $W^{-1}LW : H \oplus M^{\perp} \to H \oplus M^{\perp}$ is an operator similar to L for which T is a part, hence

$$T = W^{-1}LW\mid_H$$

$\square$

Remark 3.0.8. *We note that W is unitary whenever U is unitary. The next result shows that direct sums and direct summands are preserved under unitary equivalence only.*

Proposition 3.0.9. [27] *If an operator $T \in B(H)$ is unitarily equivalent to a direct sum $L \in B(K)$, then it is a direct sum itself with direct summands unitarily equivalent to each direct summand of L (that is, if $T \cong \bigoplus_k L_k$, then $T = \bigoplus_k T_k$ with $T_k \cong L_k$ for each k).*

Corollary 3.0.10. [27] *Every operator unitarily equivalent to a reducible operator is reducible.*

Example 3.0.11. *Consider the matrix operator*

$$T = \begin{pmatrix} 0 & 1 \\ 0 & 2 \end{pmatrix}$$

on $H = \mathbb{R}^2 = M \oplus N$. Using the standard basis in $\mathbb{R}^2$, we have $\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\}$. Let $M = span\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}\}$ and $N = span\{\begin{pmatrix} 0 \\ 1 \end{pmatrix}\}$.

$$TM = \begin{pmatrix} 0 & 1 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \in span\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}\}$$

Therefore M is T-invariant.

$$TN = \begin{pmatrix} 0 & 1 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \notin span\{\begin{pmatrix} 0 \\ 1 \end{pmatrix}\}$$

$\Rightarrow N \notin Lat(T)$. Hence, $Lat(T) = \{\{0\}, M, \mathbb{R}^2\}$
Now consider $T^* = \begin{pmatrix} 0 & 0 \\ 1 & 2 \end{pmatrix}$. Then by computing

$$T^*M = \begin{pmatrix} 0 & 0 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \notin span\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}\}$$

Therefore $M \notin Lat(T^*) \Rightarrow M$ *is not* T*-reducing.*

$$T^*N = \begin{pmatrix} 0 & 0 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix} \in span\{\begin{pmatrix} 0 \\ 1 \end{pmatrix}\}$$

Therefore $N \in Lat(T^*)$. *Hence,* $Lat(T^*) = \{\{0\}, N, \mathbb{R}^2\}$

So $T = \begin{pmatrix} 0 & 1 \\ 0 & 2 \end{pmatrix}$ *is not reducible since it has no non-trivial subspace i.e.*

$$Red(T) = \{\{0\}, \mathbb{R}^2\} \neq Lat(T) = \{\{0\}, M, \mathbb{R}^2\}$$

Clearly, T *is not reductive and* $T^*T \neq TT^*$*(not normal).*

Remark 3.0.12. *Self-adjoint operators must be reducible and reductive. For example orthogonal projections where* $Red(T) = Lat(T)$.
Corollary 3.0.10 *and Proposition* 3.0.9 *do not hold under similarity.*

Example 3.0.13. *Consider the following* 3 *by* 3 *matrices representing operators on* $\mathbb{C}^3$.

$$T = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, L = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, W = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We have that $WT = LW$, *where* W *is invertible,* L *is a direct sum such that*

$$L = 1 \oplus \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

and T *is irreducible (since the only one-dimensional* T*-invariant subspace is not* T^**-invariant).*

3.0.2 Direct summands of normal and quasinormal operators

Lemma 3.0.14. [37] *Let $T \in B(H_1)$ be a p-quasihyponormal operator and $N \in B(H_2)$ be a normal operator. If $X \in B(H_2.H_1)$ has dense range and satisfies $TX = XN$, then T is also a normal operator.*

Proof. By Lemma 3.0.32, $T = \begin{pmatrix} T_1 & T_2 \\ 0 & 0 \end{pmatrix}$ and $N = \begin{pmatrix} N_1 & 0 \\ 0 & 0 \end{pmatrix}$ with respect to the decomposition $H_1 = \overline{Ran(T)} \oplus Ker(T^*)$ and $H_2 = \overline{Ran(N)} \oplus Ker(N^*)$ respectively. Since $TX = XN$ and X has dense range, we have $\overline{X(\overline{Ran(N)})} = \overline{Ran(T)}$. Let X_1 denotes the restriction of X to $\overline{Ran(N)}$, then $X_1 : \overline{Ran(N)} \to \overline{Ran(T)}$ has dense range and for every $x \in \overline{Ran(N)}$, we have

$$X_1N_1x = XNx = TXx = T_1X_1x$$

so that

$$X_1N_1 = T_1X_1$$

Since T_1 is p-hyponormal by Lemma 3.0.32, there exists a hyponormal operator $\widehat{T}$ corresponding to T_1 and a quasiaffinity Y such that $\widehat{T_1}Y = YT_1$, where

$$\widehat{T_1} = |\widehat{T_1}|^{\frac{1}{2}}V|\widehat{T_1}|^{\frac{1}{2}}$$

with $T_1 = U \mid T_1 \mid$ and $\widehat{T_1} = |T_1|^{\frac{1}{2}}U|T_1|^{\frac{1}{2}}$. Hence, we have

$$\widehat{T_1}YX_1 = YT_1X_1 = YX_1N_1$$

Since YX_1 has dense range, $\widehat{T_1}$ is normal, and so T_1 is normal.Thus the inequality

$$(T_1^*T_1)^p \geq (T_1T_1^* + T_2T_2^*)^p \geq (T_1T_1^*) = (T_1^*T_1)^p$$

implies that $T_2 = 0$.Hence T is normal. □

Remark 3.0.15. *For any arbitrary operator $T \in B(H)$, the self-commutator of T ($[T^*, T] = T^*T - TT^*$) is always self-adjoint and hence normal. We characterize normal and quasinormal operators using this notion.*

Theorem 3.0.16. *Let $T \in B(H)$ such that $T = T_1 \oplus T_2$ with T_1 normal and T_2 pure. T is normal if and only if $[T_2^*, T_2] = 0$.*

Proof. Suppose $T \in B(H)$ is normal. Then $T^*T - TT^* = [T_2^*, T_2] = 0$. Conversely, let $[T_2^*, T_2] = T_2^*T_2 - T_2T_2^* = 0$. Since T_2 is pure and this holds only if $T_2 = 0$. Hence $T^*T = TT^*$ and therefore T is normal. □

Remark 3.0.17. *Theorem* 3.0.16 *can be proved easily using the fact that T has no pure part. Recall that an operator $T \in B(H)$ is quasinormal if $(T^*T - TT^*)T = 0$ (equivalently, if it commutes with T^*T).*

Theorem 3.0.18. *$T \in B(H)$ is quasinormal if and only if $[T^*, T]T = 0$.*

Proof. The proof follows from imitating Theorem 3.0.16. □

Theorem 3.0.19. *Every direct summand of a quasinormal operator is again quasinormal.*

Proof. Let M be a reducing subspace for $T \in B(H)$. Assume that $T = T_1 \oplus T_2$ on $H = M \oplus M^{\perp}$ where $T_1 = T \mid_M$ and $T_2 = T \mid_{M^{\perp}}$. By quasinormality of T, $(T^*T - TT^*)T = 0$ and writing in terms of direct summands of T_1 and T_2, we have,

$$T^*TT = (T_1^* \oplus T_2^*)(T_1 \oplus T_2)(T_1 \oplus T_2) = T_1^*T_1T_1 \oplus T_2^*T_2T_2 = T_1T_1^*T_1 \oplus T_2T_2^*T_2 = TT^*T$$

$\Rightarrow T_1^*T_1T_1 = T_1T_1^*T_1$ and $T_2^*T_2T_2 = T_2T_2^*T_2$ (that is, $[T_1^*, T_1]T_1 = 0$ and $[T_2^*, T_2]T_2 = 0$). By Theorem 3.0.18, T_1 and T_2 are both quasinormal. □

Remark 3.0.20. *Theorem* 3.0.19 *implies that the restriction of a quasinormal operator to a reducing subspace is always quasinormal.*
The following corollary is a consequence of Theorem 3.0.19.

Corollary 3.0.21. *Let $T \in B(H)$ have direct sum decomposition $T = T_1 \oplus T_2$, with T_1 normal and T_2 c.n.n. Then T is quasinormal if and only if T_2 is quasinormal.*

Corollary 3.0.22. *Let $T \in B(H)$ be hyponormal where its c.n.n summand has finite multiplicity. Then T is quasisimilar to an isometry if and only if its normal summand is unitary and its c.n.n summand is quasisimilar to a unilateral shift.*

Proof. Let T be hyponormal with the decomposition $T = T_1 \oplus T_2$, where T_1 is normal and T_2 is c.n.n. Suppose that T is quasisimilar to an isometry $V = U \oplus S$, where U is unitary and S is unilateral shift. Thus T_1 is unitarily equivalent to U [18, Proposition 3.5] and hence unitary. Since by assumption T is quasisimilar to V, and by Clary [6] quasisimilar hyponormal operators have the same spectra, then by [17], $\| T \| = r(T) = r(V) = 1$. Hence, this shows that T_2 is quasisimilar to S. □

Corollary 3.0.23. *Assume A and B are hyponormal operators. Let the c.n.n summand of A have finite multiplicity. If A is quasisimilar to B then their normal summands are unitarily equivalent.*

Proof. From Corollary 3.0.22, the result follows easily. Also, the result follows from the fact that quasisimilar normal operators are unitarily equivalent (Hastings [18], Williams [39]). □

3.0.3 Direct sum of dominant and (p,k)-quasihyponormal operators

The class of (p,k)-quasihyponormal operators was introduced by Kim [23]. These operators are extension of p-hyponormal, k-quasihyponormal and p-quasihyponormal operators. These operators share many properties with hyponormal operators. We note that a hyponormal operator which is similar to a normal operator must be normal. We also note that every hyponormal operator is dominant.

Corollary 3.0.24. *If $T \in B(H)$ is dominant with $T = T_1 \oplus T_2$ where T_1 is normal and T_2 is pure, then T_2 is dominant.*

Remark 3.0.25. *Corollary* 3.0.24 *applies to all the subclasses of dominant operators (i.e. hyponormal, M-hyponormal).*

Theorem 3.0.26. [23] *If $T \in B(H)$ is (p,k)-quasihyponormal and $S^* \in B(H)$ is p-hyponormal, and if $TX = XS$ where $X : K \to H$ is an injective operator with dense range(a quasiaffinity), then T is normal unitarily equivalent to S.*

Remark 3.0.27. *Theorem* 3.0.26 *implies that a (p,k)-quasihyponormal operator that is a quasiaffine transform of co-p-hyponormal is always normal.*

Proposition 3.0.28. [35] *If $T \in B(H)$ is hyponormal and $S^{-1}TS = T^*$ for an operator S, such that $0 \notin \overline{W(S)}$, then T is self-adjoint.*

Remark 3.0.29. *From the Proposition* 3.0.28*, we deduce that T is normal since a self-adjoint operator is normal. We also deduce that if a hyponormal operator is similar to its adjoint, then it must be normal.*

Lemma 3.0.30. [38] *If $T \in B(H)$ is any operator where $S^{-1}TS = T^*$, such that $0 \notin \overline{W(S)}$, then $\sigma(T) \subseteq \mathbb{R}$.*

Theorem 3.0.31. [2] *If T or T^* is p-hyponormal, S is an operator where $0 \notin \overline{W(S)}$ and $ST = T^*S$, so T is self-adjoint and hence normal.*

Proof. Suppose that T or T^* is hyponormal. Since $\sigma(S) \subseteq \overline{W(S)}$, S is invertible and hence $ST = T^*S$ becomes $S^{-1}T^*S = T = (T^*)^*$. By Lemma 3.0.30, we get $\sigma(T^*) \subset \mathbb{R}$. So $\sigma(T) = \overline{\sigma(T^*)} = \sigma(T^*) \subset \mathbb{R}$. Therefore, the planar Lebesgue measure for p-hyponormal operators T or T^* is zero. It follows that T or T^* is normal. T must be self-adjoint since $\sigma(T) = \sigma(T^*) \subset \mathbb{R}$. $\square$

Lemma 3.0.32. [23] *If $T \in B(H)$ is a (p,k)-quasihyponormal operator, then T has the following matrix representation*

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$$

with respect to the decomposition $H = Ran(T^k) \oplus Ker(T^{*k})$, where T_1 is p-hyponormal on $\overline{Ran(T^k)}$ and $T_3^k = 0$. Furthermore, $\sigma(T) = \sigma(T_1) \cup \{0\}$.
Since for quasihyponormal, we have $T^{*2}T^2 - (T^*T)^2 \geq 0$.

Theorem 3.0.33. [23] *If T is (p,k)-quasihyponormal and S is any operator for which $0 \notin \overline{W(S)}$ and $ST = T^*S$, so T is an orthogonal sum of self-adjoint (hence normal) and a nilpotent operator.*

Corollary 3.0.34. *If T or T^* is p-quasihyponormal and S is any operator where $0 \notin \overline{W(S)}$ and $ST = T^*S$, so T is self-adjoint and hence normal.*

Proof. Let T be p-quasihyponormal, so by Lemma 3.0.32, for $k = 1$, T has the matrix representation

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & 0 \end{pmatrix}$$

where T_1 is p-hyponormal on $\overline{Ran(T^k)}$ and $\sigma(T) = \sigma(T_1) \cup \{0\}$. Since T_1 is self-adjoint and $T_2 = 0$ by Theorem 3.0.31,

$$T = \begin{pmatrix} T_1 & 0 \\ 0 & 0 \end{pmatrix}$$

is also self-adjoint. Conversely, when T^* is (p,k)-quasihyponormal, so by Theorem 3.0.31, T is self-adjoint and hence normal. $\square$

Theorem 3.0.35. *(**Lowner-Heinz Theorem** [21]) If A and B are operators where $A \geq B \geq 0$ then $A^\alpha \geq B^\alpha$ for every $\alpha \in [0,1]$.*

Theorem 3.0.36. *(**Hansen's Inequality** [21]) If $A \geq 0$ and $B \leq 1$, then $(B^*AB)^\delta \geq B^*A^\delta B$ for all $\delta \in (0,1]$.*

Lemma 3.0.37. *If $T \in B(H)$ is (p,k)-quasihyponormal and M is T-invariant where $T|_M$ is an injective normal operator, then M reduces T.*

Proof. Suppose that P is an orthogonal projection of H onto $\overline{Ran(T^k)}$. Since T is (p,k)-quasihyponormal, we have $T^{*k}((T^*T)^p - (TT^*)^p)T^k \geq 0$. Let $S = PT|_M$, then clearly,$P((T^*T)^p - (TT^*)^p)P \geq 0$. Put $T_1 = T|_M$ and

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$$

on $H = M \oplus M^\perp$. Clearly, $S = T_1$, if $M = \overline{Ran(T^k)}$. By assumption that T_1 is an injective normal operator, then we get $Q \leq P$ for the orthogonal projection Q of H onto M and $\overline{Ran(T_1^k)} = M$, since T_1 has dense range. Hence, $M \subseteq \overline{Ran(T^k)}$ and therefore $Q((T^*T)^p - (TT^*)^p)Q \geq 0$. By the Lowner-Heinz and Hansen's inequalities, we get

$$\begin{pmatrix} (T_1T_1^*)^p & 0 \\ 0 & 0 \end{pmatrix} = Q(TQT^*)^pQ \leq Q(TT^*)^pQ \leq (QT^*TQ)^p = \begin{pmatrix} (T_1^*T_1)^p & 0 \\ 0 & 0 \end{pmatrix}$$

Since T_1 is normal, then by Lowner's inequality

$$(TT^*)^{\frac{p}{2}} = \begin{pmatrix} (T_1T_1^*)^{\frac{p}{2}} & A \\ A^* & B \end{pmatrix}$$

Thus,

$$\begin{pmatrix} (T_1T_1^*)^p & 0 \\ 0 & 0 \end{pmatrix} = Q(TT^*)^pQ = \begin{pmatrix} (T_1T_1^*)+AA^* & 0 \\ 0 & 0 \end{pmatrix}$$

and therefore $A = 0$ and $TT^* = \begin{pmatrix} T_1T_1^* & 0 \\ 0 & B^{\frac{2}{p}} \end{pmatrix}$. Since $TT^* = \begin{pmatrix} T_1T_1^*+T_2T_2^* & T_2T_3^* \\ T_3T_2^* & T_3T_3^* \end{pmatrix}$ It follows that $T_2 = 0$ and hence T is reduced by M. □

Remark 3.0.38. *Lemma* 3.0.37 *implies that*

$$T = \begin{pmatrix} T_1 & 0 \\ 0 & T_3 \end{pmatrix}$$

where $T_1 = T\mid_M$. This implies that T decomposes into a direct sum of non-trivial complementary parts.

Theorem 3.0.39. [25] *Let $T \in B(H)$. The following statements are pairwise equivalent*
(a) M reduces T
(b) $T = T\mid_M \oplus T\mid_{M^\perp} = \begin{pmatrix} T\mid_M & 0 \\ 0 & T\mid_{M^\perp} \end{pmatrix} : H = M \oplus M^\perp \to H = M \oplus M^\perp$
(c) $PT = TP$, where $P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} : H = M \oplus M^\perp \to H = M \oplus M^\perp$ is the orthogonal projection onto M.

Proof. (a) If M is invariant for $T \in B(H)$ and P is the orthogonal projection of H onto M, generally we get $PTP = TP$ and $T = \begin{pmatrix} T_1 & X \\ 0 & T_2 \end{pmatrix}$.
(b) If M is reducing for $T \in B(H)$ and P is the orthogonal projection of H onto M, then generally $TP = PT$ and $T = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix} \Rightarrow T = T_1 + T_2$. □

Lemma 3.0.40. *If $T \in B(H)$ is paranormal, then $T\mid_M$ is also paranormal where M is an invariant subspace.*

Proof. If M is invariant under T, then T has a matrix representation $T = \begin{pmatrix} T_1 & X \\ 0 & T_2 \end{pmatrix}$.Thus $T_1 = T\mid_M$ where T is paranormal.
Let $x \in M$ be any vector. Then we have

$$\|T\mid_M x\|^2 = \|Tx\|^2 \leq \|T^2x\|\|x\| = \|(T\mid_M)^2 x\|\|x\|$$

$\Rightarrow T\mid_M$ is paranormal. □

Remark 3.0.41. *We note that any operator which is M-hyponormal acting on a finite-dimensional Hilbert space can be extended to the class of dominant operators. The next result is useful.*

Theorem 3.0.42. [29] *If $T \in B(H)$, then there exists a reducing subspace $M \subset H$ (possibly trivial) where $T\mid_M$ is normal and $T\mid_{M^\perp}$ is c.n.n. Furthermore, the decomposition is unique and*

$$M = \bigcap_{m,n=0}^{\infty} Ker(T^nT^{*m} - T^{*m}T^n) = \bigcap_{m=0}^{\infty}\bigcap_{n=0}^{\infty} Ker(T^nT^{*m} - T^{*m}T^n)$$

Remark 3.0.43. *Theorem* 3.0.42 *gives the uniqueness of the decomposition and it is used to prove the following theorem.*

Theorem 3.0.44. *Let $T \in B(H)$. If $K = Ran(T^*T - TT^*)$ is the smallest reducing subspace of T, then $T\mid_K$ is the c.n.n summand of T.*

Proof. Let $K = Ran(T^*T - TT^*)$. By Theorem 3.0.42, $T = T_1 \oplus T_2$ on $M \oplus M^\perp$, given T_1 is c.n.n and T_2 is normal. Since $[T^*, T] = [T_1^*, T_1] \oplus [T_2^*, T^2]$ and $[T_2^*, T_2] = 0$, then clearly $K \subseteq M$ because M is a reducing subspace of T containing the range of $[T^*, T]$. If $K \subset M$ (proper), then T_1 itself could further be reduced into $T_{11} \oplus T_{12}$ on $K \oplus K^\perp$. But from the definition of K we have $[T_{12}^*, T_{12}] = 0$. This is a contradiction since T_1 is c.n.n. Hence, $K = M$.
Special case: If T is normal, then $T^*T - TT^* = 0$. So, $K = Ran(0) = \{0\}$ (which is always the smallest reducing subspace of any operator). If T decomposes as $T = T_1 \oplus T_2$ where T_2 is c.n.n with respect to the decomposition $H = M \oplus N$, then $T\mid_N = 0$ □

Proposition 3.0.45. *(**Wold decomposition** [27]) Every isometry is a direct sum of a unitary operator and a unilateral shift.*
The following result is a consequence of the Proposition 3.0.45.

Proposition 3.0.46. *An isometry is pure (c.n.n) if and only if it is a unilateral shift.*

Proof. Let T be an isometry and that $T = T_1 \oplus T_2$, where T_1 is normal and T_2 is c.n.n. Since T is an isometry, then $T^*T = I$. Let T be pure, then T_1 is missing. Hence, $T = T_2$. Thus $TT^* \neq T^*T = I = T_2^*T_2 \Rightarrow T^*T = T_2^*T_2$. Hence T is an isometry which is not a co-isometry and therefore must be a unilateral shift. Conversely, let T be a unilateral shift, then $T^*Tx \neq TT^*x$ for every $0 \neq x \in M \subset H$. This implies that T is pure. □

Example 3.0.47. *Suppose $H = l^2$ (space of all square-summable sequences) and S the shift operator (unilateral shift) such that $S(x_1, x_2, \ldots) = (0, x_1, x_2, \ldots)$.*
Then $\|Sx\| = \|x\| \; \forall \, x \in l^2$. Since S^* is the left shift operator, thus we get $S^*(S(x)) = S^*(0, x_1, x_2, \ldots) = (x_1, x_2, \ldots) = x$.
Conversely, we have $S(S^*(x)) = S(x_2, x_3, \ldots) = (x_2, x_3, \ldots) \neq x$. Hence $S^*S \neq SS^* \implies S$ is not normal (non-normal isometry). In fact $SS^* = P$, where P is a projection and $P \neq I$ i.e $S^*S = I \neq P = SS^*$.
Thus, a unilateral shift operator is hyponormal but not normal (i.e. it has no normal direct summand).

Remark 3.0.48. *Proposition* 3.0.46 *implies that any pure isometry is a unilateral shift or a direct sum of unilateral shifts.*

Corollary 3.0.49. [36] *Let $T \in B(H)$ is hyponormal. If T is similar to a normal operator, then T is normal.*

Proof. Let T be hyponormal and $T = T_1 \oplus T_2$, where T_1 is normal and T_2 is c.n.n. T_2 is absent if T is similar to a normal operator. Suppose $T = X^{-1}NX$ such that N is normal. Then $N = N_1 \oplus N_2$ (where $N_2 = 0$ or absent). Thus,

$$XT = NX$$

$$X(T_1 \oplus T_2) = (N_1 \oplus N_2)X$$

$$XT_1 \oplus XT_2 = N_1X \oplus N_2X.$$

Equating the summands, we get $XT_1 = N_1X$ and $XT_2 = N_2X$
Let $T \sim N$. By normality $T^*T = TT^*$, then

$$(T_1 \oplus T_2)^*(T_1 \oplus T_2) = (T_1 \oplus T_2)(T_1 \oplus T_2)^*$$

$$T_1^*T_1 \oplus T_2^*T_2 = T_1T_1^* \oplus T_2T_2^*.$$

By hyponormality, $T^*T \geq TT^*$,

$$T_1^*T_1 \oplus T_2^*T_2 \geq T_1T_1^* \oplus T_2T_2^*.$$

□

Corollary 3.0.50. [36] *Let $T \in B(H)$ and $TW = WN$ where N is normal and W is any non-zero operator in $B(H)$. Then T has a non-trivial invariant subspace.*

Remark 3.0.51. *Corollary* 3.0.50 *applies to quasiaffine transforms of all reducible operators with a finite-dimensional direct summand.*

Corollary 3.0.52. [10] *Let $A \in B(H_1)$, $B \in B(H_2)$ and $X \in B(H_2, H_1)$ be such that $AX = XB$. If either A is a pure dominant operator or B^* is a pure M-hyponormal operator, then $X = 0$.*

Theorem 3.0.53. *An operator $T \in B(H)$ is k-quasihyponormal if and only if*

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$$

with respect to the decomposition $H = Ran(T^k) \oplus Ker(T^{*k})$, where $T_1^* T_1 - T_1 T_1^* \geq T_2 T_2^*$ and $T_3^k = 0$.

`Proof.` The result follows easily from Lemma 3.0.32. □

Corollary 3.0.54. *If T is k-quasihyponormal and the spectrum of T has zero Lebesgue measure, then T is a direct sum of normal and nilpotent operators.*

`Proof.` From the hypothesis,

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$$

and the spectrum of T_1 is of zero area measure. Hence, T_1 is normal and thus $T_2 = 0$. Therefore,

$$T = \begin{pmatrix} T_1 & 0 \\ 0 & T_3 \end{pmatrix}$$

where T_1 is normal and $T_3^k = 0$. This shows that T_3 is nilpotent.
Alternatively
Let discrete set of $M = \{a_1, a_2, ...\}$ and $\sigma(T) = \{\lambda_1, \lambda_2, ...\}$. Then Lebesgue measures, $\mu(M) = 0$ and $\mu(\sigma(T)) = 0$. Thus

$$T = \begin{pmatrix} T_1 & A \\ 0 & T_2 \end{pmatrix}$$

on $M \oplus M^\perp$, where T_1 and T_2 are diagonal components and square matrices. We have $T_1 : M \longrightarrow M$. Similarly, $T_2 : M^\perp \longrightarrow M^\perp$. By Normalization, we get $\sigma(T_1) = \{\lambda_1, \lambda_2, ...\}$ and

$$T_1 = \begin{pmatrix} \lambda_1 & 0 & \\ & \lambda_2 & \\ 0 & & \ddots \end{pmatrix}$$

$\Rightarrow T_1$ is normal. □

Remark 3.0.55. *It is obvious the direct summand T_3 is c.n.n. Also, every diagonal matrix must be normal.*

Corollary 3.0.56. *If T is k-quasihyponormal and the spectrum of T has zero Lebesgue area measure, and $Ker(T) \subset Ker(T^*)$ (equivalently, $Ker(T) \cap Ran(T) = \{0\}$), then T is normal.*

`Proof.` Let T satisfies all the conditions in Corollary 3.0.54. Then $T = T_1 \oplus T_3$ where T_1 is normal. If $T_3^k = 0$ and $T_3 \neq 0$, then $Ran(T) \cap Ker(T) = \{0\}$. Thus, $T_3 = 0$ and $T = T_1 \oplus 0$ which is normal. □

3.0.4 Decomposition for quasitriangular operators

Definition 3.0.57. *An operator $T \in B(H)$ is called* ***quasitriangular*** *if there exists an increasing sequence $\{P_n\}_{n=1}^{\infty}$ of finite rank (orthogonal) projections such that $P_n \longrightarrow I$ (strongly, $n \longrightarrow \infty$) and $\|TP_n - P_nTP_n\| \longrightarrow 0$ as $n \longrightarrow \infty$ (see [19], [32]).*

Theorem 3.0.58. [20] *If A is c.n.n and reductive, and if T commutes with A, then T is quasitriangular.*

`Proof.` Let A be reductive, then $AT = TA$. So T is reductive and thus each T-invariant subspace is reducing. Hence, each invariant subspace of T is also T^*-invariant subspace. Suppose λ is an eigenvalue for T^* such that $M_\lambda = Ker(\lambda I - T^*)$. This shows that M_λ is T^*-invariant and A^*-invariant. Hence, M_λ is hyperinvariant for T^* and therefore reduces T. Now let T be nonquasitriangular and let M be the span of all eigenvectors of T^*. We have M reduces T and $T\mid_M$ is diagonal. Thus $T\mid_{M^\perp}$ is nonquasitriangular. But $T^*\mid_{M^\perp}$ must have an eigenvector. This is a contradiction. Hence T is quasitriangular. □

Remark 3.0.59. *Theorem 3.0.58 is not generally true for all reductive operators. The following example illustrates this fact.*

Example 3.0.60. *Consider a 3×3 matrix $T = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ and two projections of rank 1, P onto $< 0,1,0 >$ and Q onto $< a,b,0 >$, where $|a|^2 + |b|^2 = 1$ and a is small (but $\neq 0$). Suppose P is invariant under T and Q is nearly invariant under T. If $R = P \vee Q$, then $\|TR - RTR\| = 1$. It can be shown that*

$$P = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, Q = \begin{pmatrix} |a|^2 & ab^* & 0 \\ a^*b & |b|^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}, R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\|TP - PTP\| = 0, \|TQ - QTQ = |a|, \|TR - RTR\| = 1.$$

Corollary 3.0.61. *Every reductive operator $T \in B(H)$ is quasitriangular.*

`Proof.` Let T be reductive. Then $T = T_1 \oplus T_2$ where T_1 is normal hence quasitriangular and T_2 is c.n.n which commutes with itself. By Theorem 3.0.58, then T is quasitriangular. □

3.0.5 Direct sum decomposition of 2-normal operators

Proposition 3.0.62. [33] *Let $T \in B(H)$ have the direct sum decomposition $T = T_1 \oplus T_2$ relative to the decomposition $H = H_1 \oplus H_2$. If T is a 2-normal operator (i.e, $T \in [2N]$), then each direct summand $T_i, i = 1,2$ is 2-normal.*

Proof. Let $T^*T^2 = T^2T^*$. From a simple operator multiplication, we get $T^*T^2 = T_1^*T_1^2 \oplus T_2^*T_2^2$ and $T^2T^* = T_1^2T_1^* \oplus T_2^2T_2^*$. Since $T \in [2N]$, we get $T_1^*T_1^2 \oplus T_2^*T_2^2 = T_1^2T_1^* \oplus T_2^2T_2^*$. By equating the respective direct summands, we have $T_1^*T_1^2 = T_1^2T_1^*$ and $T_2^*T_2^2 = T_2^2T_2^*$. Thus $T_i \in [2N]$, $i = 1,2$. □

Remark 3.0.63. *Nzimbi, Pokhariyal and Khalaghai* [33] *have shown that the converse of Proposition* 3.0.62 *is also true.*

Corollary 3.0.64. *Let $T \in B(H)$ be 2-normal and $T = T_1 \oplus T_2$, where T_1 is normal and T_2 is pure. Then T_2 is 2-normal.*

Proof. Follows from Proposition 3.0.62. □

Remark 3.0.65. *Corollary* 3.0.64 *shows that T normal implies 2-normal.*
Additionally, T normal implies that T^* is normal i.e. From $T^*T - TT^* = 0$ and taking adjoints, $T^*(T^*)^* - (T^*)^* = 0$. This implies that T^* is normal.

Proposition 3.0.66. [33] *Let T be a normal operator. Then T is 2-normal.*

Proof. T^* is normal since T is. Hence

$$T^*T^2 = (T^*T)T = (TT^*)T = T(T^*T) = T(TT^*) = T^2T^*.$$

□

Remark 3.0.67. *In general, the converse of Proposition* 3.0.66 *does not hold.*

Example 3.0.68. *Normal $\subset$ 2-normal.*
Suppose $T = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. From a simple matrix calculation, it shows that T is 2-normal but not normal (in fact, T is pure in this case). This implies that a 2-normal operator T decomposes as $T = T_1 \oplus T_2$ where T_1 is normal and T_2 is c.n.n and any of these summands could be missing.

Remark 3.0.69. *If T is normal and $AB = 0$, but neither A nor $B = 0$, we don't conclude $(A, B = 0)$.*

Example 3.0.70. *Let $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$. Then*

$$AB = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Question:
If T and T^* are quasinormal, is T normal?

Proposition 3.0.71. *If $T \in B(H)$ is a 2-normal and quasinormal operator and injective on $Ran([T^*,T])$, then T is normal.*

Proof. As T is 2-normal and quasinormal, it shows that $T^*T^2 = T^2T^*$ and $(T^*T - TT^*)T = 0$ $\Rightarrow T$ is normal. $\square$

Example 3.0.72. *By Proposition 3.0.71, an operator which is both 2-normal and quasinormal has no non-zero c.n.n direct summand. If T is the unilateral shift on l^2, then T has an infinite matrix representation* $T = \begin{pmatrix} 0 & 0 & 0 & \dots \\ 1 & 0 & 0 & \dots \\ 0 & 1 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$.
A simple calculation shows that T is quasinormal but not 2-normal. Also, $T^*T - TT^* = diag(1,0,0,\cdots) \neq diag(0,0,0,\cdots)$. Hence T is not normal.

Remark 3.0.73. *We note that the 2-normality or quasinormality conditions in Proposition 3.0.71 can not be dropped.*

4 ON UNITARY AND COMPLETELY NON-UNITARY SUMMANDS OF A CONTRACTION OPERATOR

In this chapter, we investigate the decomposition of a contraction operator into a direct sum of unitary and completely non-unitary parts (c.n.u). We also study the properties of the c.n.u summands of a contraction.
We relate a contraction T to a pair of operators A and V such that A denotes a positive contraction while V an isometry. Similarly, the operators A_* and V_* denote the respective positive contraction and isometry associated with the operator T^*.
For every contraction $T \in B(H)$, there exists operators A and A_* on H that are the strong limits of $\{T^{*n}T^n; n \geq 1\}$ and $\{T^nT^{*n}; n \geq 1\}$ respectively.

4.0.1 Some classes of contractions and quasisimilarity

Remark 4.0.1. *We recall that if $A = A_* = 0$, then contractions T and T^* are strongly stable. If $A = I$, then a contraction is an isometry. If $A = A_* = I$ it is unitary.*
Moreover, if A commutes with T, then it is a projection. If T is a normal contraction then $A = A_*$.
The reducing subspace for T is the subspace $Ker(I-A) \cap Ker(I-A_*)$.
If A is invertible, then T is similar to the isometry V. In addition, T is similar to V and again similar to a unitary operator if A and A_* are invertible.

We let $C_{0\cdot}$ be the class of all strongly stable contractions while $C_{\cdot 0}$ be the class of all adjoints which are strongly stable contractions. Let $C_{1\cdot}$ and $C_{\cdot 1}$ be the classes of all contractions for which T^nx and $T^{*n}x$ do not converge to zero respectively,$\forall x \in H$.
A contraction $T \in C_{0\cdot}$ if and only if $A = 0$ and $T \in C_{1\cdot}$ if and only if $Ker(A) = \{0\}$. Thus,

$$T \in C_{00} \Longleftrightarrow A = A_* = 0.$$

$$T \in C_{01} \Longleftrightarrow A = 0, Ker(A_*) = \{0\}.$$

$$T \in C_{10} \Longleftrightarrow Ker(A) = \{0\}, A_* = 0.$$

$$T \in C_{11} \Longleftrightarrow Ker(A) = Ker(A_*) = \{0\}.$$

Remark 4.0.2. *A unitary operator is a C_{11}-contraction.*
Every isometric operator is a contraction.
Every unitary is invertible.

Proposition 4.0.3. [27] *Suppose T is a contraction such that $T^{*n}T^n \xrightarrow{s} A$. The following properties characterize operator A.*

1. $0 \leq A \leq I$ *where* A *is positive contraction.*
2. $\| T^n x \| \longrightarrow \| A^{\frac{1}{2}} x \|$ *as* $n \longrightarrow \infty$, $\forall x \in H$.
3. $T^{*n} A T^n = A$ *for every* $n \geq 1$.
4. $\| A^{\frac{1}{2}} T^n x \| = \| A^{\frac{1}{2}} x \|$, $\forall x \in H$ *and every* $n \geq 1$.
5. $(I - A)T^n \xrightarrow{s} 0$, $(I - A^{\frac{1}{2}})T^n \xrightarrow{s} 0$.
6. $\| AT^n x \| \longrightarrow \| A^{\frac{1}{2}} x \|$ *as* $n \longrightarrow \infty$, $\forall x \in H$.
7. $\| A \| = 1$ *whenever* $A \neq 0$.
8. $AT \neq 0$ *and* $TA \neq$ *if* $A \neq 0$.
9. $Ker(A) = \{x \in H : T^n x \longrightarrow 0\}$.
10. $Ker(I - A) = \{x \in H : \| T^n x \| = \| x \|, \forall n \geq 1\}$.

Remark 4.0.4. *Properties (9) and (10) in Proposition* 4.0.3 *imply that the subspaces* $Ker(A)$ *and* $Ker(I - A)$ *are* T*-invariant. The following proposition is useful.*

Proposition 4.0.5. [27] $Ker(A - A^2) = Ker(A) \oplus Ker(I - A)$.

Proof. Since $Ker(A) \cup Ker(I - A) \subseteq Ker(A - A^2)$ and $Ker(A) \perp Ker(I - A)$ (since A is self-adjoint), then
$$Ker(A) \oplus Ker(I - A) \subseteq Ker(A - A^2).$$
Conversely, since $Ker(A - A^2)$ is A-invariant, then it reduces A. Hence
$$A = A_0 \oplus A_1$$
where $A_0 = A \mid_{Ker(A - A^2}$ and $A_1 = A \mid_{Ker(A - A^2)^{\perp}}$.
Since A is a projection on $Ker(A - A^2)$ (for $0 \leq A_0$ and $A_0 = A_0^2$), thus
$$Ker(A - A^2) = Ker(A_0) \oplus Ker(A_0)^{\perp} = Ker(A_0) \oplus Ker(I - A_0) \subseteq Ker(A) \oplus Ker(I - A)$$
where the inclusion is trivially showed once $Ker(A_0) \subseteq Ker(A)$, $Ker(I - A_0) \subseteq Ker(I - A)$ and $Ker(A) \perp Ker(I - A)$. □

Proposition 4.0.6. [27] *If a contraction is quasisimilar to a unitary operator, then it is of class* C_{11}.

Proof. Let $T \in B(H)$ be a contraction, $U \in B(K)$ a unitary operator such that $XT = UX$ and $YU = TY$ where $X \in B(H, K)$ and $Y \in B(K, H)$ are quasiinvertible operators, then $XT^n = U^n X$ and $Y^* T^{*n} = U^{*n} Y^*$ for every $n \geq 1$. Hence, if $x \in H$ is such that $lim_{n \longrightarrow \infty} T^n x = 0$, then $lim_{n \longrightarrow \infty} U^n X x = 0$. Thus $Xx = 0$ so that $x = 0$. This implies that $Ker(A) = 0$. Dually (replacing U by U^* and X by Y^*), $Ker(A_*) = \{0\}$. So, if a contraction T is quasisimilar to a unitary operator, then $Ker(A) = Ker(A_*) = \{0\} \Rightarrow T \in C_{11}$. □

Remark 4.0.7. *The converse of Proposition* 4.0.6 *holds true, that is, every C_{11}-contraction is quasisimilar to a unitary operator.*

Question: If a contraction T is similar to a unitary operator U, does it mean that T is unitary? The answer is NO. There exists non-unitary operators similar to a unitary operator. Recall that $T \in B(H)$ is unitary if T is invertible and $\|T\| \leq 1$ and $\|T^{-1}\| \leq 1$.

Corollary 4.0.8. *If T is a contraction for which $A \neq 0$ and $A_* \neq 0$, then either T has a non-trivial hyperinvariant subspace or T is a scalar unitary operator.*

Proof. If $Ker(A) = Ker(A_*) = \{0\}$, then T is a C_{11}-contraction and it either has a non-trivial hyperinvariant subspace or it is a scalar unitary. If $Ker(A) \neq \{0\}$, then $Ker(A)$ is a non-trivial hyperinvariant subspace for T. We know that $Ker(A) \neq H$ since $A \neq 0$. Dually, if $Ker(A_*) \neq \{0\}$, then $Ker(A_*)$ is a nontrivial hyperinvariant subspace for T^*. Thus, $Ker(A_*)^{\perp}$ is a non-trivial hyperinvariant subspace for T. □

4.0.2 On decomposition of contractions

Kubrusly [27] has proved the largest reducing subspace for a contraction on which it is unitary. This is known as the Nagy-Foias-Langer decomposition for contractions.

Theorem 4.0.9. *(**Nagy-Foias-Langer decomposition** [27, Theorem 5.1])*
Let T be a contraction on a Hilbert space H and

$$\mathcal{M} = Ker(I - A) \cap Ker(I - A_*)$$

where $\mathcal{M}$ is a reducing subspace for T. Moreover, the decomposition $T = U \oplus C$ on $H = \mathcal{M} \oplus \mathcal{M}^{\perp}$ is such that $U = T \mid_{\mathcal{M}}$ is unitary and $C = T \mid_{\mathcal{M}^{\perp}}$ is a c.n.u contraction.

Proof. We recall that $Ker(I - A) = \{x \in H : \| T^n x \| = \| x \|, \forall n \geq 1\}$, which is an invariant subspace for T. Hence

$$\mathcal{M} = \{x \in H : \| T^n x \| = \| T^{*n} x \| = \| x \|, \forall n \geq 1\}$$

is a subspace of H (intersection of subspaces is a subspace) that reduces T since $Ker(I - A)$ and $Ker(I - A_*)$ are invariant for T and T^* respectively, hence their intersection is both T and T^*-invariant. Therefore $(T \mid_{\mathcal{M}})^n x = T^n x$ and $(T \mid_{\mathcal{M}})^{*n} x = T^{*n} x$ so that

$$\| (T \mid_{\mathcal{M}})^n x \| = \| (T \mid_{\mathcal{M}})^{*n} x \| = \| x \|$$

, $\forall x \in \mathscr{M}$ and every $n \geq 1$. Thus $T \mid_{\mathscr{M}}$ is unitary on $\mathscr{M}$. Suppose $\mathscr{U}$ is unitary on and a reducing subspace for T, then

$$\| (T \mid_{\mathscr{U}})^n x \| = \| (T \mid_{\mathscr{U}})^{*n} x \| = \| x \|$$

$$(T \mid_{\mathscr{U}})^n x = T^n x$$

$$(T \mid_{\mathscr{U}})^{*n} x = T^{*n} x$$

$$\Rightarrow \| T^n x \| = \| T^{*n} x \| = \| x \|$$

$\forall x \in \mathscr{U}$ and $n \geq 1 \Rightarrow \mathscr{U} \subseteq \mathscr{M}$. Hence $\mathscr{M}$ is the largest reducing subspace for T on which it is unitary. Therefore, $T \mid_{\mathscr{M}^\perp}$ is completely non unitary (i.e, it has no unitary direct summand). □

Remark 4.0.10. *Nagy-Foias-Langer decomposition holds for isometries since isometries are contractions. We note that the restriction of an isometry to a reducing subspace is also an isometry. Thus from the decomposition $T = U \oplus C$, C stands for a c.n.u isometry. The following result is important and it is used to prove that a c.n.u isometry is a unilateral shift.*

Proposition 4.0.11. *Every c.n.u coisometry is strongly stable (i.e, if T is a c.n.u isometry then $A_* = 0$).*

Proof. Let T be an isometry on H. Then $A_* = TA_*T^* \Rightarrow T^*A_* = A_*T^*$ (for $T^*T = 1$ since T is an isometry). Since $A_* = A_*^2$, then $Ker(I - A_*) = Ran(A_*)$. In addition, $A = I$ since T is an isometry and so $Ker(I - A) = H$. Moreover, $Ker(I - A) \cap Ker(I - A_*) = \{0\}$ since isometry T is c.n.u by Nagy-Foias-Langer decomposition. Hence, $Ran(A_*) = \{0\}$. Similarly, $A_* = 0$. □

Remark 4.0.12. *Proposition 4.0.11 on coisometry can be extended to cohyponormal contractions since isometries are indeed hyponormal contractions.*

Theorem 4.0.13. *Every c.n.u cohyponormal contraction is strongly stable.*

Proof. (see [27], pg 83). □

Remark 4.0.14. *An isometry is pure (c.n.n) if it has no normal isometry as a direct summand. But a normal isometry is indeed a unitary operator. Hence, a pure isometry is a c.n.u isometry which in turn implies it is a unilateral shift.*

Corollary 4.0.15. *(**von Neumann-Wold decomposition** [27], Corollary 5.6) If T is an isometry on H, then $Ker(I - A_*)$ is a reducing subspace for T. Additionally, the decomposition*

$$T = U \oplus S_+$$

on $H = Ker(I - A_*) \oplus Ker(I - A_*)^\perp$ is such that $U = T \mid_{Ker(I-A_*)}$ is unitary and $S_+ = T \mid_{Ker(I-A_*)^\perp}$ is a unilateral shift.

`Proof.` Since T is an isometry (i.e, $A = I$), then T is a contraction where $Ker(I - A) = H$ $\Rightarrow Ker(I-A) \cap Ker(I-A_*) = Ker(I-A_*)$. Applying Nagy-Foias-Langer decomposition for contractions with $M = Ker(I-A_*)$, then $T|_M$ is unitary and $T|_{M^\perp}$ is a c.n.u isometry on $M^\perp$ which implies is a unilateral shift. □

Remark 4.0.16. *From Corollary* 4.0.15, *we can deduce that any isometry $T \in B(H)$ is either:*
(i) a unitary operator (c.n.u part is absent).
(ii) a unilateral shift (unitary part is absent).
(iii) a direct sum of a unitary operator and a unilateral shift operator.
The next result shows that unilateral shifts can be extended to arbitrary isometries if bilateral shifts are substituted by unitary operators.

Lemma 4.0.17. *Every unitary operator is a part of isometry.*

`Proof.` Let T be an isometry on H. By the decomposition $T = U \oplus S_+$ on $H = M \oplus M^\perp$ for subspace $M \subset H$ that reduces T; such that U is unitary on M and S_+ is a unilateral shift on $M^\perp$. But S_+ is part of a bilateral shift S acting on Hilbert space K containing $M^\perp$. Hence, with respect to the decomposition $K = M^\perp \oplus \mathscr{U}$ where $\mathscr{U} = K \ominus M^\perp$,

$$S = \begin{pmatrix} S_+ & X \\ 0 & Y \end{pmatrix}$$

where $X : \mathscr{U} \longrightarrow M^\perp$ and $Y : \mathscr{U} \longrightarrow \mathscr{U}$. Now consider the unitary operator $W = U \oplus S$ acting on $M \oplus K$. With respect to the decomposition $(M \oplus K) = M \oplus M^\perp \oplus \mathscr{U}$, then

$$W = \begin{pmatrix} U & 0 & 0 \\ 0 & S_+ & X \\ 0 & 0 & Y \end{pmatrix}$$

so that T is a part of W. In fact,

$$T = U \oplus S_+ = W|_{M \oplus M^\perp} = W|_H .$$

□

Proposition 4.0.18. *If a contraction $T = T_1 \oplus T_2$, where T_1 and T_2 are unitary, then T must be unitary.*

`Proof.` Let $T = T_1 \oplus T_2$. Since both T_1 and T_2 are unitary, we have $T_1^* T_1 = T_1 T_1^* = I$ and $T_2^* T_2 = T_2 T_2^* = I$. Thus,

$$TT^* = (T_1^* \oplus T_2^*)(T_1 \oplus T_2)$$

$$= T_1^*T_1 \oplus T_2^*T_2$$
$$= T_1T_1^* \oplus T_2T_2^*$$
$$= TT^* = I \oplus I = I.$$

□

Remark 4.0.19. *The following theorem is useful in decomposing a contraction with A(or A_*) being a projection.*

Theorem 4.0.20. *([27], Theorem 5.8) Assume T is a contraction on H. If $A = A^2$, then*

1. *$T = G \oplus S_+ \oplus U$, where G, S_+ and U represent a strongly stable contraction acting on $Ker(A)$, a unilateral shift acting on $Ker(I-A) \cap Ker(A_*)$, and a unitary acting on $Ker(I-A) \cap Ker(A_*)$ respectively. In addition, if $A = A^2$ and $A_* = A_*^2$, then*
2. *$T = B \oplus S_- \oplus S_+ \oplus U$, where B is a C_{00}-contraction on $Ker(A) \cap Ker(A_*$ and S_- is a backward unilateral shift on $Ker(A) \cap Ker(I-A_*)$.*
3. *Furthermore, if $A = A_*$, then $T = B \oplus U$.*

`Proof.` Assume T is a contraction on H. If $A = A^2$, then from Proposition 4.0.5 we get $H = Ker(A) \oplus Ker(I-A)$. From properties (9) and (10) in Proposition 4.0.3, the subspaces $Ker(A)$ and $Ker(I-A)$ are T-invariant and T-reducing. So we get the decomposition

$$T = G \oplus K$$

on $H = Ker(A) \oplus Ker(I-A)$where $G = T\mid_{Ker(A)}$ is strongly stable contraction on $Ker(A)$ while $K = T\mid_{Ker(I-A)}$ is an isometry on $Ker(I-A)$. But by von Neumann-Wold decomposition

$$K = U \oplus S_+$$

on $Ker(I-A) = M^\perp \oplus M$ where $U = K\mid_M$ is a unitary operator acting on M and $S_+ = K\mid_{M\perp}$ is a unilateral shift acting on $M^\perp$. Additionally, by Proposition 4.0.3 property (10), $M = \{x \in Ker(I-A) : \| K^{*n}x \| = \| x \|, \forall n \geq 1\}$. We note that $K^{*n}x = T^{*n}x \ \forall x \in Ker(I-A)$ and $n \geq 1$, for $Ker(I-A)$ reduces T and $K = T\mid_{Ker(I-A)}$. Hence $M = \{x \in Ker(I-A) : \| T^{*n}x \| = \| x \|, \forall n \geq 1\}$, so that

$$M = Ker(I-A) \cap Ker(I-A_*).$$

Thus $K\mid_M = T\mid_{Ker(I-A)\cap Ker(I-A_*)}$. From the decompositions above we get $T^* = G^* \oplus S_+^* \oplus U^*$ on $H = Ker(A) \oplus M^\perp \oplus M$. It follows that $M^\perp = Ker(I-A) \ominus M \subseteq Ker(A_*) \subseteq Ker(A) \oplus M^\perp$ since $S_+^{*n} \xrightarrow{s} 0$. Therefore $Ker(I-A) \cap Ker(A_*) \subseteq Ker(I-A) \cap (Ker(A) \oplus M^\perp) = M^\perp \subseteq Ker(I-A) \cap Ker(A_*)$. Hence

$$M^\perp = Ker(I-A) \cap Ker(A_*)$$

and thus $K|_{M^{\perp}} = T|_{Ker(I-A)\cap Ker(A_*)}$ which concludes proof of (1). Now suppose $Ker(A) \neq \{0\}$ and let operator A'_* on $Ker(A)$ be the strong limit of $\{G^nG^{*n}; n \geq 1\}$. We note that $A_* = A'_* \oplus 0 \oplus I$ and $(I - A_*) = (I - A'_*) \oplus I \oplus 0$ on $H = Ker(A) \oplus M^{\perp} \oplus M$. Hence $Ker(A_*) = Ker(A'_*) \oplus M^{\perp} \subseteq Ker(A) \oplus M^{\perp}$

$$\Longrightarrow Ker(A'_*) = Ker(A_*) \cap Ker(A)$$

and

$$Ker(I - A_*) = Ker(I - A'_*) \oplus M \subseteq Ker(A) \oplus M$$

$\Longrightarrow$

$$Ker(I - A'_*) = Ker(I - A_*) \cap Ker(A).$$

Next, assume that $A_* = A_*^2$ (in addition to $A = A^2$), then $A'_* = A'^2_*$. Thus from Proposition 4.0.5 we get $Ker(A) = Ker(A'_*) \oplus Ker(I - A'_*)$. By properties (9) and (10) in Proposition 4.0.3, both subspaces $Ker(A'_*)$ and $Ker(I - A'_*)$ are invariant for G^*, so they reduce G. Hence we get the decomposition

$$G = B \oplus S_-$$

on $Ker(A) = Ker(A'_*) \oplus Ker(I - A'_*)$ where $B = G|_{Ker(A'_*)} = T|_{Ker(A_*)\cap Ker(A)}$ and $S_- = G|_{Ker(I-A'_*)} = T|_{Ker(I-A_*\cap Ker(A)} \Longrightarrow B$ is a C_{00}-contraction acting on $Ker(A) \cap Ker(A_*$ and S_- is a strongly stable coisometry acting on $Ker(A) \cap Ker(I - A_*)$ (again from Proposition 4.0.3 (9) and (10) properties). Hence S_- is a c.n.u coisometry, and therefore its adjoint is a c.n.u isometry (a unilateral shift). This completes proof of (2). We know if $A = A_*$, then $A = A^2$ and hence the result in (3). □

Remark 4.0.21. *The decompositions in (1), (2) and (3) in Theorem* 4.0.20*, trivially imply that* $A = A^2$, $A = A^2$ *and* $A_* = A_*^2$ *; and* $A = A_*$ *respectively.*

4.0.3 Nature of direct summands of some c.n.u contractions

Remark 4.0.22. *The following result is useful in giving and proving conditions under which a p-quasihyponormal contraction is normal. We note, by [13], the class of p-quasihyponormal operators is contained in the class of paranormal operators.*

Theorem 4.0.23. *The c.n.u summand of a paranormal contraction is of class* $C_{\cdot 0}$.

Remark 4.0.24. *From Theorem* 4.0.23*, we have* $T^n = (T_1 \oplus T_2)^n = T_1^n \oplus T_2^n$. *Thus,* $T_2^{*n} \longrightarrow 0$ *as* $n \longrightarrow \infty$ *(i.e* T_2^* *is strongly stable). The following is a consequence of Theorem* 4.0.23.

Corollary 4.0.25. *If* T *is a paranormal contraction and* $T^{*n} = T_1^{*n} \oplus T_2^{*n}$. *Then* T_1^n *and* T_1^* *do not converge to* 0. *Hence* $T_1 \in C_{11}$.

Claim: If $T = T_1 \oplus T_2$ is a paranormal contraction where T_1 is unitary and T_2 is c.n.u, T_2 is also paranormal.

Proposition 4.0.26. *If* $T \in B(H)$ *is a normal contraction, then the c.n.u part of* T *is of class* C_{00}.

Proof. Let T be normal and $T = T_1 \oplus T_2$, where T_1 is unitary and T_2 is c.n.u. Then we have $T^{*n}T^n = T^nT^{*n}$ for every $n \geq 1$ (by induction). By Theorem 4.0.23, T_2 is of class $C_{\cdot 0}$. It remains to show that T_2 is of class $C_{0\cdot}$. Since T_2 is of class $C_{\cdot 0}$, then $\|T_2^{*n}\| \longrightarrow 0$. Moreover, since T is normal $A = lim_n T_2^{*n}T_2^n = lim_n T_2^nT_2^{*n} = A_* = 0$. Hence $T_2 \in C_{00}$. □

Remark 4.0.27. *We note that Proposition* 4.0.26 *follows from the fact that T has the Putnam-Fuglede (PF) property (i.e, T and T^* have PF-property iff $A = A_*$)* (*Nzimbi, et.al* [31]).

Corollary 4.0.28. *Assume T is a contraction on H. If T has no non-trivial invariant subspace, then it is either a C_{00}-contraction, a C_{01}-contraction such that $\| A_*x \| < \| x \| \; \forall x \in H$, or a C_{10}-contraction such that $\| Ax \| < \| x \| \; \forall x \in H$.*

Proof. Assume T is a contraction on H. When $\{0\} \neq Ker(A - A^2) \neq H$, then $Ker(A - A^2)$ is a non-trivial invariant subspace for T. Dually, if $\{0\} \neq Ker(A_* - A_*^2) \neq H$, then $Ker(A_* - A_*^2)$ is a non-trivial T^*-invariant subspace. Hence, $Ker(A_* - A_*^2)^{\perp}$ is a non-trivial invariant subspace for T. This implies that there are 4 cases where T has no non-trivial invariant subspace.

1. $Ker(A - A^2) = \{0\}$ and $Ker(A_* - A_*^2) = \{0\}$.
2. $Ker(A - A^2) = \{0\}$ and $Ker(A_* - A_*^2) = H$.
3. $Ker(A - A^2) = H$ and $Ker(A_* - A_*^2) = \{0\}$.
4. $Ker(A - A^2) = H$ and $Ker(A_* - A_*^2) = H$.

For case (1) it is impossible. In fact, from Proposition 4.0.5 we get $Ker(A) = Ker(A_*) = \{0\}$. Equivalently, it means that T is a C_{11}-contraction (But a C_{11}-contraction has a non-trivial invariant subspace whenever $dim(H) > 1$).
If $Ker(A_* - A_*^2) = H$, then $T^* = G$ (a strongly stable contraction) by Theorem 4.0.20 since T^* has no non-trivial invariant subspace and thus no shift and no unitary as direct summands.
If $Ker(A - A^2) = \{0\}$, then $Ker(A) = \{0\}$ and $Ker(I - A) = \{0\}$ by again Proposition 4.0.5. Hence, case (2) means T is a C_{10}-contraction such that $\| Ax \| < \| x \| \; \forall x \in H$ (recall, $Ker(I - A) = \{x \in H : \| Ax \| = \| x \|\}$.
Dually, case (3) implies that T is a C_{01}-contraction such that $\| A_*x \| < \| x \| \; \forall x \in H$. Finally, if $Ker(A - A^2) = Ker(A_* - A_*^2) = H$, then by Theorem 4.0.20 we have

$$T = B \oplus S_- \oplus S_+ \oplus U.$$

Hence $T = B$ which is of class C_{00}, since S_-, S_+ and U clearly have non-trivial invariant subspaces. Therefore, case (4) leads to a C_{00}-contraction. □

Proposition 4.0.29. *If $A \in B(H)$ is a normal contraction and $B \in B(H)$ is similar to A, then the c.n.u summand of B is of class C_{00}.*

Proof. Let $A = A_1 \oplus A_2$ where A_1 is unitary and A_2 is c.n.u be a normal operator. From previous results we have $T_2 \in C_{00}$. Let $B = B_1 \oplus B_2$ be similar to A, so $B_2 \in C_{00}$.

$$B = X^{-1}AX = X^{-1}(A_1 \oplus A_2)X$$

$$B_1 \oplus B_2 = X^{-1}A_1X \oplus X^{-1}A_2X$$
$$B_1 = X^{-1}A_1X$$

and

$$B_2 = X^{-1}A_2X \in C_{00}$$
$$B_2^n = (X^{-1}A_2X)^n = (X^{-1}A_2X)(X^{-1}A_2X)\cdots(X^{-1}A_2X) = X^{-1}A_2^nX \longrightarrow 0$$

Thus $A_2^n \longrightarrow 0 \Rightarrow B_2^n \longrightarrow 0$. It remains to show that $B_2 \in C_{0\cdot}$

$$B_2 = X^{-1}A_2X$$

$$B_2^{*n} = X^*A_2^{*n}X^{-1*}.$$

But $A_2^{*n} \longrightarrow 0$, thus $B_2^{*n} \longrightarrow 0$. Hence $B_2 \in C_{00}$. □

Remark 4.0.30. *By replacing similarity with either unitary equivalence or quasisimilarity, Proposition* 4.0.29 *holds.*

Lemma 4.0.31. [9, *Lemma* 1] *If A is a normal contraction such that A^n is normal for any integer $n \geq 2$, then there exists direct sum decompositions $H = H_n \oplus H_p$ and $A_n = A\mid_{H_n}$ is a normal $C_{11} \oplus C_{00}$ type contraction and $A_p = A\mid_{H_p}$ is a pure C_{00}-contraction.*

Remark 4.0.32. *We note that if a contraction is pure then it must be c.n.u but the converse is not generally true.*

Example 4.0.33. *Consider the matrix* $T = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}$. *Then* $T^2 = \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{4} \end{pmatrix}$, $T^n \longrightarrow \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ *as* $n \longrightarrow \infty$. *Therefore,* $T \in C_{0\cdot}$. *Clearly,* $T^*T \Rightarrow T \in C_{\cdot 0}$. *Thus* $T \in C_{0\cdot} \cap C_{\cdot 0} = C_{00} \Rightarrow T \in C_{00}$ *and T is normal. Hence, not all C_{00}-contractions are pure. Equivalently, there is no C_{00}-contraction with a unitary part. Thus a pure C_{00}-contraction is c.n.u.*

Example 4.0.34. *Consider the matrix* $T = \begin{pmatrix} 0 & \frac{1}{2} \\ 0 & \frac{1}{2} \end{pmatrix}$, $T^2 = \begin{pmatrix} 0 & \frac{1}{4} \\ 0 & \frac{1}{4} \end{pmatrix}$. *Then* $T^*T = \begin{pmatrix} 0 & 0 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} 0 & \frac{1}{2} \\ 0 & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$. *By mathematical induction,* $T^n = \begin{pmatrix} 0 & \frac{1}{2^n} \\ 0 & \frac{1}{2^n} \end{pmatrix} \longrightarrow \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0$. *Therefore,* $T \in C_{0\cdot}$, $T^{*2} = \begin{pmatrix} 0 & 0 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} 0 & 0 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ \frac{1}{4} & \frac{1}{4} \end{pmatrix}$. *Thus,* $T^{*n} \longrightarrow \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \Rightarrow T \in C_{\cdot 0}$. *Hence, T is a C_{00}-contraction and so T has no unitary part i.e T is c.n.u.*

Remark:
Not normal does not necessarily mean c.n.n. For instance, from Example 4.0.34 we have $T^*T = \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$, $TT^* = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} \end{pmatrix}$. So $T^*T \neq TT^*$ (not normal).

Lemma 4.0.35. [9] *Let T be a C_{11}-contraction on H and U be a unitary operator on K. If there exists an injective operator $X : H \longrightarrow K$ such that $XT = UX$, then T is quasisimilar to the unitary operator $U \mid_{\overline{XH} \subset K}$.*

Proof. As T is a C_{11}-contraction, so it is quasisimilar to a unitary operator U, hence the assertion follows. □

Remark 4.0.36. *Every unitary operator is of class C_{11}. However, there are C_{11}-contractions which are not normal (and hence, not unitary).*

Example 4.0.37. *We find a contraction T such that T^n and T^{*n} do not converge to 0 and $T^*T \neq TT^*$. Consider the matrix operator $T = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$. Then $T^2 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$. Therefore, $T^n \longrightarrow \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \neq \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0$. So T^n does not converge to $0 \Rightarrow T \in C_{1}.$. Similarly, $T^* = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$, $T^{*2} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$. Therefore, $T^{*n} \longrightarrow \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \neq \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0 \Rightarrow T \in C_{.1}$. Hence $T \in C_{11}$ but T is not normal.*

Corollary 4.0.38. *A non-unitary C_{11}-contraction is similar to a unitary operator if it is invertible.*

Proof. Let a c.n.u $T \in C_{11}$ be such that $T = X^{-1}UX$, for which U is unitary and let T be not invertible. Then this is contradiction since the right hand side is invertible while left hand side is not. □

4.0.4 Summands of partial isometry, normal and subnormal partial isometries

Theorem 4.0.39. *Let $T \in B(H)$. Then T is a quasinormal partial isometry if and only if T is the orthogonal (direct) sum of an isometry and zero.*

Proof. Let T be a partial isometry and quasinormal, thus $T = PT = TP$, such that $P = T^*T$ is the projection on $M = \overline{Ran(\mid T \mid)}$. Hence the space M reduces T and $T \mid_M$ is an isometry. This implies that $T = S \oplus 0$ where S is an isometry. Conversely, let $T = S \oplus 0$, where S is an isometry. Then

$$T^*TT = (S^*S \oplus 0)(S \oplus 0) = S \oplus 0 = T = (S \oplus 0)(S^*S \oplus 0) = TT^*T.$$

□

Theorem 4.0.40. [13] *Let $T \in B(H)$. Then*

1. *T is normal partial isometry if and only if T is the direct (orthogonal) sum of a unitary operator and zero.*

2. *T is subnormal partial isometry if and only if T is the direct (orthogonal) sum of an isometry and zero.*

Proof. (1) As $T^*T = TT^*$ and $Ker(T)^{\perp}$ coincides with $Ran(T)$ and hence $T\mid_{Ker(T)^{\perp}}$ is unitary, then $T = U \oplus 0$ on $Ker(T)^{\perp} \oplus Ker(T)$. The converse proof is trivial.
(2) When T is subnormal, so T is hyponormal (i.e, $T^*T \geq TT^*$). Therefore $Ker(T)^{\perp} \supset Ran(T)$. This implies that $Ker(T)^{\perp}$ is invariant under T, and thus reduces T. It is obvious that $T\mid_{Ker(T)^{\perp}}$ is an isometry, thus $T = S \oplus 0$ on $Ker(T)^{\perp} \oplus Ker(T)$, where S is an isometry. The converse follows from [13, §2.6.2]. □

4.0.5 Unitary and c.n.u summands of almost similar contractions

The following results are due to [32].

Proposition 4.0.41. *Let $A \in B(H)$ such that A is almost similar to an isometry T. Then the unitary and c.n.u summands of A are isometric.*

Proof. By the von Neumann-Wold decomposition, if T is an isometry, then $T = S_+ \oplus U$, such that U is unitary and S_+ is the forward shift (unilateral shift). Since $A \approx^{a.s} T$, there exists an operator N where

$$\begin{aligned} A^*A &= N^{-1}[(S_+ \oplus U)^*(S_+U)]N \\ &= N^{-1}(S_+^*S_+ \oplus U^*U)N \\ &= N^{-1}(I \oplus I)N. \end{aligned}$$

Now, let $A = A_1 \oplus A_2$, then $A^*A = (A_1^*A_1 \oplus A_2^*A_2) \Longrightarrow (A_1^*A_1 \oplus A_2^*A_2) \approx I \oplus I$. From this equation, we have that $A_i^*A_i \approx I$, $i = 1,2$. This implies that there exists N where $N^{-1}IN = I$. Hence $A_i^*A_i = I$. Therefore, the direct summands of A are isometric. □

Remark 4.0.42. *The following results from Proposition* 4.0.41

Corollary 4.0.43. *If an operator $A \in B(H)$ is such that A^* is almost similar to a c.n.u coisometry, then A has no unitary direct summand.*

Proof. Let $A = A_1 \oplus A_2$. Applying the proof of Proposition 4.0.41, we get that the direct summands of A are unitary. But the c.n.u part of an operator cannot be unitary. Thus $A_1 = 0$ or A_1 acts on the null space $\{0\}$. Hence A has no unitary direct summand. $\square$

Remark 4.0.44. *An operator which is unitarily equivalent to a unitary operator has no c.n.u direct summand* [32, *Corollary* 2.13].

Proposition 4.0.45. *If $A, B \in B(H)$ are contractions such that $A \approx^{a.s} B$ and B is c.n.u, then A is c.n.u.*

Proof. From Nagy-Foias-Langer decomposition, we have $B = U \oplus C$ on $H = H_1 \oplus H_2$, where $U = B\mid_{H_1}$ is the unitary part of B and $C = B\mid_{H_2}$ is the c.n.u summand of B. Since B is c.n.u, the unitary direct summand U is missing or $H_1 = \{0\}$. Without loss of generality, we let $B = C$. Then $A^*A = N^{-1}(B^*B)N = N^{-1}(C^*C)N$. This shows that A^*A is similar to C^*C (that is, $A^*A \approx C^*C$). Now let $A = A_1 \oplus A_2$, where A_1 is unitary and A_2 is c.n.u. Thus, $(A_1^*A_1 \oplus A_2^*A_2) \approx C^*C$. This is true if and only if direct summand A_1 is missing $\Rightarrow A = A_2$.Therefore A is c.n.u. $\square$

Corollary 4.0.46. *If $A \in B(H)$ is normal, then $A \approx^{a.s} A^*$.*

Proof. From the fact that $AA^* = A^*A = N^{-1}(AA^*)N = N^{-1}(A^*A)N$ and $A + A^* = A^* + A = N^{-1}(A + A^*)N = N^{-1}(A^* + A)N$, the result follows. $\square$

Remark 4.0.47. *In general, the converse of Corollary* 4.0.46 *does not hold. The following example illustrates this fact.*

Example 4.0.48. *Consider* $A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ *and* $N = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. *By matrix calculation, we get* $A^*A = N^{-1}(AA^*)N$ *and* $A^* + A = N^{-1}(A + A^*)N \Rightarrow A \approx^{a.s} A^*$, *but A is not normal.*

Theorem 4.0.49. *(**Foguel decomposition** [27, Theorem 7.3]) Suppose T is a contraction on H and set*

$$\mathscr{X} = \{x \in H : \langle T^n x; y\rangle \longrightarrow 0, n \longrightarrow \infty, \forall y \in H\}$$

is a reducing subspace for T. Moreover, the decomposition

$$T = Z \oplus U$$

on $H = \mathscr{X} \oplus \mathscr{X}^\perp$ given as $Z = T\mid_{\mathscr{X}}$ is a weakly stable contraction whereas $U = T\mid_{\mathscr{X}^\perp}$ is a unitary operator.

Proof. see [1,pg 104]. $\square$

Definition 4.0.50. *A unitary operator is **absolutely continuous** when its constant spectral measure is absolutely continuous relative to Lebesgue measure on the unit circle.*

Corollary 4.0.51. *Any c.n.u contraction is weakly stable.*

Proof. As $\mathscr{X}^{\perp} \subseteq \mathscr{M} = \{x \in H : \|T^n x\| = \|T^{*n}x = \|x\| \forall n \geq 1\}$, then $\mathscr{M}^{\perp} \subseteq \mathscr{X}^{\perp\perp} = \overline{\mathscr{X}} = \mathscr{X} \subseteq H$. Thus, if $\mathscr{M}^{\perp} = H$ (i.e, when T is a c.n.u contraction), then $\mathscr{X} = H$. This implies that T is weakly stable. $\square$

Remark 4.0.52. *The converse of the above result is not true (i.e there exists weakly stable unitary operator). Therefore, Foguel decomposition is not unique unlike Nagy-Foias-Langer decomposition. For example, a bilateral shift is weakly stable and unitary, including any direct summand of it.*
According to the above corollary, each direct part of a bilateral shift (itself included) that is similar to a c.n.u contraction is weakly stable.
Each unitary operator is uniquely the orthogonal sum of a singular unitary and an absolutely continuous unitary. In addition, if a unitary operator is a direct part of a bilateral shift then it is absolutely continuous.

5 CONCLUSION AND RECOMMENDATIONS

5.1 Conclusion

In this project, we have seen that every linear operator acting on a Hilbert space can be expressed as a direct sum decomposition of normal and completely non-normal operator (and that either direct summand may be missing). Likewise, every contraction operator can be expressed as a direct sum decomposition of a unitary part and a completely lion-unitary part.

We have shown conditions under which some higher classes of operators are normal. For instance, in Theorem 3.0.26 it has been shown that a p-quasihyponormal operator which is a quasi-affine transform of a normal operator is normal. Similarly, we have seen in Lemma 3.0.37 that an operator decomposes into a direct sum of nontrivial normal and c.n.n (complementary) parts, if the operator is (p,k)-quasihyponormal in which the restriction of the operator to an invariant subspace is injective and normal.Furthermore, in Proposition 3.0.46, using Example 3.0.47, we showed that an isometry is pure (c.n.n) if and only if it is a unilateral shift. A result showing that any linear operator T that is 2-normal, quasinormal and injective on $Ran([T^*,T])$ has no c.n.n part. The c.n.u part of a contraction operator has been investigated. For example, in Proposition 4.0.26, we have shown that the c.n.u. part of an operator which is similar to a normal contraction is of class C_{00}.

Chapter 4 is a special case of chapter 3. For instance, the results in Corollary 3.0.22 and Lemma 4.0.31 cut across chapters 3 and 4.

5.2 Recommendations

Decomposition of operators is applicable in the study of mathematical systems theory. It is easier to study the parts of a system than the entire 'complicated' system. For example, Let $T,S \in B(H)$ decompose as $T = T_1 \oplus T_2$ and $S = S_1 \oplus S_2$ respectively where T_1, T_2, S_1, S_2 are the direct summands of T and S.

Question 1

If S,T are similar, what can we say about the direct summands of both T and S? Is it true that T_1 is similar to S_1 and T_2 similar to S_2?

Question 2

How is $\sigma(T)$ related to $\sigma(T_1)$ and $\sigma(T_2)$? Is it true that

1. $\sigma(T) = \sigma(T_1) \cup \sigma(T_2)$?
2. $\sigma(T) \subseteq \sigma(T_1) \cup \sigma(T_2)$?

In our research , we were able to show that if two operators of finite multiplicity are hyponormal and are quasisimilar, then their normal parts are unitarily equivalent. However, we were unable to conclude that quasisimilar hyponormal operators have quasisimilar pure parts. Hence, more research on this problem is recommended. In addition, we were able to show that each direct part of a bilateral shift similar to a c.n.u contraction is weakly stable. However, we did not conclude whether the absolutely continuous unitary operator is the only weakly stable operator. Therefore, further research on this is also recommended.

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