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NUMERICAL SOLUTIONS FOR PENNES' BIOHEAT EQUATIONS

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Declaration and Approval

I the undersigned declare that this dissertation is my original work and to the best of my knowledge, it has not been submitted in support of an award of a degree in any other university or institution of learning.

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Dedication

This project is dedicated to my father Geoffry Mutai and mother Selly Chepchumba Mutai for emotional,financial and spiritual support in the time of dire need. To my lovely husband Patrick Mwai for standing with me, suppporting me and motivating me during my graduate studies. To my lovely daughter Nina Maya whose birth has filled me with joy and motivation to work harder. To my brother Dr. Henry Ngenoh, Hunnington Ngeno and my sisters Mercy Mutai, vicky Mutai, Stella Mutai and Caro mutai for prayers and emotional support. And finally to my surpevisors Dr. Akwoyo and Dr. Victor Juma for the guidance throughout the whole project. May God bless you.

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Contents

1 INTRODUCTION

The world is changing drastically and diseases are evolving over time. One kind of such is cancer. Modern clinical treatments have come up to treat cancer and thus understanding of the temperature behaviour in human tissues is of essence. Thermal ablation is one kind of treatment to cancerous tissues by exposing them to very high temperature(hyperthermia) while protecting the temperatures of the surrounding tissue [1]. The main goal of thermal ablation is to raise the temperature of the cancerous tissue to a point where cancer cells are destroyed while maintaining normal temperature for tissues that surround. There are other modern clinical treatments that need the understanding of heat transfer in living tissues and these are cryosurgery, cryopreservation and thermal diagnostics [2]. Thus, study of bio heat transfer in living tissues has been an important topic to scientists over the last many years.

1.1 The Bio Heat Transfer

Bio heat transfer is the study of thermal energy transfer in living tissue. Biological processes are heat dependant hence heat transfer plays a critical role. In the living tissues Blood and skin plays a very important role in heat transfer.

1.1.1 Role of blood in heat transfer

In biological tissues blood plays an important role of transport of thermal energy. Some of the roles of blood in biological tissues are;

- Blood plays a role of body metabolism, this is by transporting oxygen to all the body parts and transporting of carbon-dioxide and waste from cells.
- Blood regulates blood pressure
- Blood transports heat which enhances thermo-regulation.

1.1.2 Role of skin in bio heat transfer

skin plays a very important role in thermal transfer. The skin has 3 layers: the dermis, epidermis and subcutaneous tissues shown in the figure below.

In these layers are blood vessels which are essential in blood circulation. The human body has a circulatory system that is composed of blood vessels which are majorly arteries and

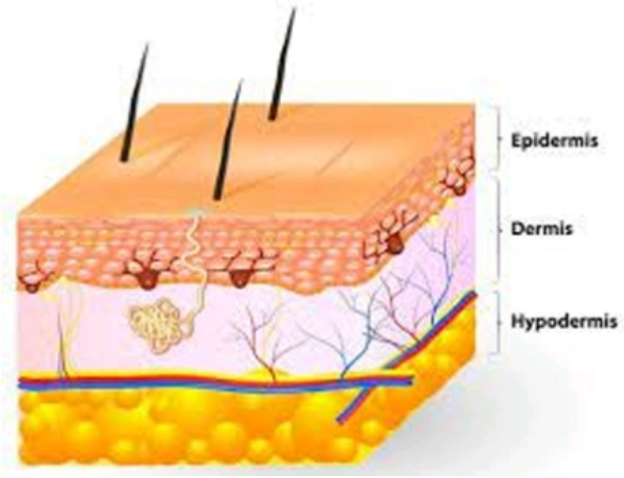

Figure 1. skin layers

veins, which carry blood from heart to the tissues and back to the heart. Blood leaves the heart through the largest artery called aorta. It then enters to the main supply arteries and veins, then to the primary arteries, to the secondary arteries, to the arterioles and finally to the capillaries which are in contact with the tissues and back to the heart through veins.This process of blood supply through the blood vessels is a major form of thermal transfer and thermal equilibrium.

Bio heat transfer includes evaporation, heat generation, heat absorption, heat transmission, evaporation and conduction. These biological processes takes place in the tissues which are solid and in the blood, which is the fluid. The table below shows significance of the thermal processes in a living tissue.

Thermal Processes			
Body Parts	Conduction	Convection	Radiation
Tissues	Significant	Not significant	Not significant
Bones	significant	Not significant	Not significant
Blood vessels	Not significant	significant	Not significant
Skins	Not significant	significant	significant

These processes are coupled up with physiological processes which includes blood circulation, metabolic heat generation and heat dissipation.[3] There are also various factors that include temperature distribution, tissues strain, tissue stress and damage of thermal tissues.

1.2 Importance of heat transfer

In living tissues, heat transfer plays an important role since biochemical processes are heat dependent and highly utilized in the medical field and physiological studies. For example, heat transfer in tissues is utilized in cryosurgery, frostbite, thermal ablation, hyperthermia/cancer treatment, skin burns and body thermal regulation. The temperature interact with tissues at different temperature ranges and are given different terminologies as shown in the table below;

Temperature ranges	
Terminology	Temperature range in °C
Cyrobiology	< -196°C
Hypothermia	< 35°
Hyperthermia	42-46
Irreversible cellular damage	46-48
Coagulation necrosis	50-52
Near Instantaneous necrosis	60-100
Tissue vaporization	>110

1.3 Biological background of cancer and treatment

Bio heat transfer has been used in diagnostic and therapeutic applications [4] which rely on advanced computerized techniques. This has enhanced the development of mathematical models to study and analyze various bio heat transfer processes. Cancer is one of the diagnostic application of the bio heat transfer.

The study of cancer in medicine field is called oncology. Cancer has its existence since ancient times. Cancer is the abnormal growth of cells in human body, whether in the breast, in the throat, in the stomach, in the bones and any other tissue in the body. Cancer was first discovered by Greek physician called Hippocrates(460-370BC). Hippocrates believed that the body has four fluids, the blood, phlegm, yellow bile, and black bile. He suggested that the imbalance of these fluids, with an excess of black bile caused cancer. Since then many theories and research on cancer have been studied. The latest theory in 1920's argued that trauma was the cause of cancer [5].

Tumors are detected in the body via temperature by evaluating temperature distribution around that particular tissue. Tissues with tumors are known to have a significant high temperatures than the body tissue [6].

1.3.1 Cancer treatment

In oncology, the term "hyperthermia" refers to the treatment of malignant tissues by use of very high temperatures [7]. The temperatures are usually in range of 40-42°C. This is also what we call thermal ablation, a process called coagulation necrosis. The use of hyperthermia to treat cancerous cells dates back to 1898, by Swedish gynecologist Westermark, who treated cervical cancer using hot water. [8].

In treatment of cancer we ensure that the high temperature only applies to the affected cells and not the normal cells. In cancer treatment, heating facilitates radio and chemotherapy therapy. Cancer treatment through hyperthermia includes killing of the cancerous cells, while taking care of the normal cells surrounding the cancerous tissue. It also involves decrease in oxyhemoglobin saturation which decreases the tumor tissue PH hence killing it [7]. In killing the cancerous cells, laser is used in order to protect the surrounding tissues. To study the heat transfer in killing the tumors, many scientist have come up with models to describe heat transfer and fluid flow in biological processes when a tissue is heated and vascularized.

1.4 The Pennes Bio Heat Equation

1.4.1 The Background of The Pennes' Bio Heat Equation

Several authors have come up to study the heat transfer within the tissues. Harry H. Pennes from the Department of Neurology, college of physicians and surgeons, Colombia University, and the Neurological Institute, New York is one of them who greatly contributed into this issue. In 1994, Harry H. Pennes published a study on temperature distribution in human body. He published "Analysis of tissue and arterial blood temperatures in the resting forearm" which appeared in volume 1, No. 2, published in August 1948. The purpose of Pennes' study was to "to evaluate the applicability of heat flow theory to the forearm in basic terms of the local rate of tissue heat production and volume flow of blood" [9]. Many authors have adopted pennes' work in developing mathematical models of heat transfer in human body for example analyzing digital cooling and developing a whole body human thermal model done by Eugene H. Wissler in 1958 and 1961 respectively.

1.4.2 Pennes' Bio Heat Equation

Pennes(1948) came up with his famous heat transfer model called Pennes' Bio heat equation. Pennes model considered the effect of blood flow in the region as a heat source or heat sink term added to the heat conduction equation.

Pennes formulated a model based on the temperatures at the forearm. He proposed a model to describe the effects of metabolism and blood perfusion on the energy balance within tissue. The Pennes Bio heat equation is given by;

$$\rho c \frac{\delta T}{\delta t} = k\Delta^2 T + \rho_b c_b \omega_b (T_a - T) + Q_{met} \tag{1}$$

where ;

1. T(x,t) is the sought temperature variable
2. T_b Temperature of arterial blood.
3. t is the time where t is greater than 0,
4. k is the thermal conductivity
5. is the mass density, c is the specific heat capacity
6. Q_{met} is the total internal heat generation per unit volume expressed as addition of heat generation and heat gain
$$Q_{met} = Q_m + Q_r,$$
and
$$Q_r,$$
is the metabolic heat generation and
$$Q_r,$$
is the heat deposition.
7. ρ_b, c_b, ω_b are the blood density,specific heat capacity of blood and blood perfusion rate respectively.

1.4.3 Definition of terms and derivation of Pennes'Bioheat Equation

The law of energy balance states that; the heat gained over a controlled volume is balanced by the heat the heat stored, the heat lost via conduction or convection through liquids and the rate of worked performed by the tissue [10]. The equation is written as follows;

$$\frac{d}{dt}Q_{gain} = \frac{d}{dt}Q_{stored} + \frac{d}{dt}Q_{lost} + \frac{d}{dt}W \tag{2}$$

1. The heat gain term

   Tissues generates heat per unit volume with respect to time. $Q_t(X,t)$ stands for the heat generated spatially across a tissue. The heat generated by tissues at a particular volume are summed to obtain the general heat gain as follows;

$$U_{gain} = \int_v Q_t(X,t)dV \tag{3}$$

2. Heat stored

   During heat consumption or production, heat maybe be unsteady and is therefore stored. This is controlled by density and specific heat capacity. The total rate of stored energy over a volume is given by;

$$U_{storage} = \int_v \rho c \frac{\partial T(X,t)}{\partial t} dv \tag{4}$$

3. Heat conduction term

   French mathematical physicist Joseph Fourier [11] formulated Fourier law which states that; the amount of thermal energy is directly proportional to the cross-sectional area A, temperature change in a medium ΔT, time difference over a length Δt, and inversely proportional to the length across a medium which conduction occurs [12]. There is also a thermal proportionality K. Fourier's law is written as follows;

$$U_c = -k\frac{A\Delta T\Delta t}{\Delta l} \tag{5}$$

   The negative sign is due to the temperature flow from higher regions of temperature to lower regions of temperature. Heat lost in conduction is gotten by dividing the the thermal energy with A and Δt which reduces equation (5) to;

$$\frac{U_c}{A\Delta t} = -k\frac{A\Delta T}{\Delta l} \tag{6}$$

   By letting $\Delta L \longrightarrow 0$ The fourier laws reduces to a standard Fourier law

$$U_c = -k\Delta T \tag{7}$$

   Integrating equation(7) over an area and perpendicular to the direction of the heat conduction, we obtain;

$$U_c = -\int_A k\Delta T.ndA \tag{8}$$

   where n is the normal unit vector.

4. Blood perfusion Term

   perfusion is the volumetric flow of fluids per unit rate across capillary tissues and extracellular spaces of living tissues. When talking about perfusion, we are particularly talking about convection, since its the most important for thermal energy transfer. Blood perfusion is so important since it transfers heat energy between blood and tissues. we express the total thermal energy over a controlled volume of a tissue as ;

$$U_b = \int_v \rho_b c_b \omega_b (T_a - T) dV \tag{9}$$

combining all the terms above we get the pennes' bio heat equation;

$$\rho c \frac{\delta T}{\delta t} = k\Delta^2 T + \rho_b c_b \omega_b (T_a - T) + Q_{met} \tag{10}$$

This is the Pennes Bio-heat Equation.

1.4.4 Assumptions in pennes' bio heat equation

In order to mathematically solve Pennes' equation, some assumptions are put forward.

They include;

1. The metabolic heat Q_{met} is assumed to be homogeneously/uniformly distributed throughout the tissue of interest as the rate of energy deposition per unit volume.
2. It is also assumed that the blood perfusion effect is homogeneous and isotropic and that thermal equalization occurs in the micro circulatory capillary bed. That said blood enters the capillaries at the temperature of the arterial blood, T_a,where heat exchange occurs to bring the temperature to that of the surrounding tissue, T.
3. It is also assumed that there is no energy transfer either before or after the blood passes through the capillaries, so that the temperature at which it enters the venous circulation is that of the local tissue.
4. The total energy exchange between blood and tissues is directly proportional to the density of blood ρ_b, specific heat capacity of blood c_b, and perfusion rate of blood ω_b throughout the tissue and is described in terms of sensible energy of blood.

1.4.5 Advantages of pennes' bio heat equation

The perfusion heat transfer is linear which allow us the solve the equation (1)

1.4.6 Inadequacies in pennes equation

Although Pennes' work has gained popularity there are a few questions about its validity that has remained unanswered. These includes;

1. Pennes' experimental data seems to be at variance with his theoretical results
2. Secondly Pennes focused his attention on heat transfer between capillary blood and tissue, but it is easily demonstrated that the temperature of blood in precapillary arterioles and post capillary venules is close to the temperature of then surrounding tissue.
3. Lastly, that the perfusion effect is not isotropic.

2 Literature Review

Bio heat transfer in tissues is a very important subject of discussion. Many scientists, mathematicians have come up with several models to study and do analysis on the topic.

2.1 The Pennes' model

One of the earliest model on thermal transfer was by Pennes [9]. This Pennes model marked the beginning of the study of heat transfer. Due to the simplicity of the Pennes' model, it has been a great backbone to the study of thermal heat. Most researchers have hence referred to pennes' model in their research work. The Equation that forms the Pennes' model is as follows;

$$\rho c \frac{\delta T}{\delta T} = k\Delta^2 T - \omega_b c_b (T_b - T) + Q_{met} \tag{11}$$

where; t is the time,

- ρ c is the specific heat capacity of the tissue the density of the tissue,
- ρ_b is the specific heat capacity of blood,
- ω_b is the blood perfusion rate, k is the thermal conductivity of the tissue,
- Q_{met} is the total heat sources and
- Δ is the gradient operator

[13, 9, 14] Pennes did the experiment "to evaluate the applicability of heat flow to the forearm in basic terms of the local rate of heat production and volume flow of blood". Pennes used a forearm to measure the radial temperature. Pennes proposed that heat transfer was by conduction and convection (by blood circulating through the region). He focused his attention capillary blood and tissue. He assumed that the arterial blood temperature was uniform everywhere and that the temperature of the vein was equal to that of the blood. According to Pennes heat was transferred through conduction by the tissue and convection by blood circulating in the region with temperature of the artery. He considered conduction and storage properties to be for tissue and blood properties to be for the perfusion term. The perfusion term was proportional to the difference between

the temperature of the blood in the artery and temperature of the tissue. He assumed that arterial blood temperature was uniform throughout and the temperature of the vein was equal to the temperature of the tissue. The added term blood perfusion rate ω_b and specific heat capacity of blood c_b represented represented the heat sink when positive and the heat source when negative. K is a constant.

2.2 Gautherie's model

This temperature dependent conductivity model by Gautherie in 1969 [10]. The model takes the equation below;

$$\rho c \frac{\delta T}{\delta t} = \Delta T(\text{(}T)\Delta T) + Q \tag{12}$$

In this model the blood perfusion term $\omega_b c_b$ is not there and the tissue conductivity K is a function of the local tissue temperature$K(T)$.

2.3 The Chen-Holmes bio heat transfer model

In 1980,Chen and Holmes [15] developed a model with the following equation;

$$\rho c \frac{\delta T}{\delta t} = \nabla(K\nabla T) + \rho_b \omega_b c_b (T_b - T) - \rho_b \omega_b u (T_b - T) + \nabla(k_p \nabla T) + Q \tag{13}$$

In this model, the heat transfer between blood and the tissue is separated into 3 parts; The first represent the equilibrium of the blood temperature, $\rho_b \omega_b c_b (T_b - T)$. The second term represent the convectional heat transfer and is denoted by $\rho_b \omega_b u (T_b - T$. The third term is the thermal contributions of vessels which are nearly in equilibrium with the surrounding tissue and is represented by $\nabla(k_p \nabla T)$ Compared to Pennes' model, this model has two added terms which are the convective heat transfer term and the enhanced tissue conductive term. This CH model has a more solid physical basis compared to Pennes' model. The conductivity depends on microvasculature and local blood perfusion rate. As much as Chen and Holmes made a huge contribution to the heat transfer they did not evaluate the actual thermal contribution of each term.

2.4 The Weinbaum and Jiji Model

In 1985, [15] Weinbaum and Jiji (WJ)proposed another model to solve Bio Heat Transfer Equation.Their study was from a vascular network of rabbit thighs that heat transfer happen in blood vessels in a counter current direction. Hence heat transfer between blood

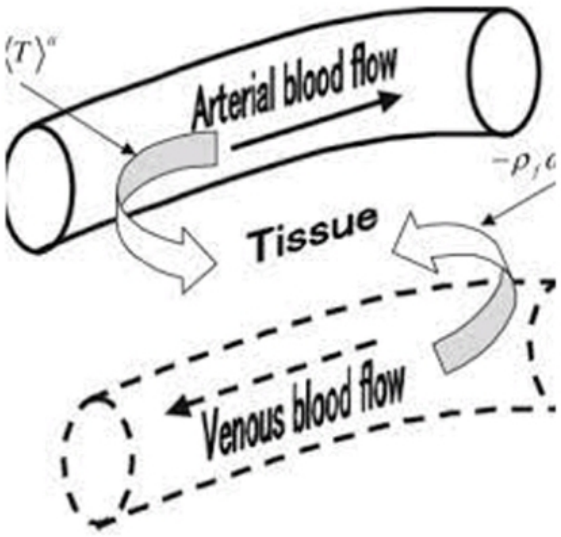

Figure 2. Blood in counter current direction

and tissues was in "incomplete countercurrent heat exchanger" between arteries and veinswith diameters about (50-500 micro metre). The above counter current flow is shown in the figure below [16]

The Equation was written as follows;

$$\rho c\frac{\delta\theta}{\delta t}-\frac{\delta}{\delta x_i}\left[(k_{ij})_{eff}\frac{\delta\theta}{\delta x_j}\right]=-\frac{\phi^2 na^2k_b^2}{4\sigma k}\rho e^2 l_j\frac{\delta l_i}{\delta k_j}\frac{\delta\theta}{\delta x_j}+Q_m \tag{14}$$

These are the description of the symbols in the above equation; θ is the local temperature, ρ is the tissue density, c is the tissues specific heat capacity, a is the radius of the blood vessel in the region, σ is the shape factor, n is the number of blood vessel with radius a, k_b is the blood thermal conductivity, P_e is the local blood paclet, u is the average velocity of blood in vessels, l_i is the directional cosine, $(k_{ij})_{eff}$ is the effective tensor conductivity element, which is given by;

$$(k_{ij})_{eff}=k\left(\delta_{ij}+\frac{\phi^2 na^2k_b^2}{4\sigma k}\rho e^2 l_j\frac{\delta l_i}{\delta k_j}\frac{\delta\theta}{\delta x_j}+Q_m\right) \tag{15}$$

where, δ_{ij} is the kronecker delta function, k is the thermal conductivity of the tissue. In this equation the focus is on the subcutaneous region. Heat transfer is represented in a single equation independent of any blood temperature. They argued that the small veins and arteries are parallel and flow is counter current which results in counter balanced heating and cooling. This makes the blood perfusion term in Pennes'equation negligible

and tissue therefore behaves as an anisotropic heat transfer medium. Weinbaum and jiji also modified the thermal conductivity in Pennes' bioheat equation by means of an "effective conductivity". Song et al. proposed the importance of K in Weinbaum-Jiji model. They stated that the conductivity K is a function of depth peclet number, which varies whether a person is at rest or doing some work.

Xu at al made comparison of the Pennes'model, the Chen-Holmes model and the Weinbaum model in pig kidney cortex. According to their analysis Pennes has only the blood perfusion related term, Weinbaum jiji has only the enhanced perfusion conduction term, while Chen-Helms equation has both and the added conduction term.

When we analyse this equation its a more detail form of the bioheat transfer. In this model WJ, put across two assumptions;

1. That the heat transfer between arteriole-venule pairs and tissue is negligible.
2. That the arithmetic mean of the arteriole venule blood can be approximated to the temperature of the tissue.

2.5 Baish et al

Baish et al tried to correct the assumption made by weinbaum and Jiji,[?]hat the heat transfer between the arterioles-venule and tissue was negligible. According to them heat transfer between the counter current vessels depended on;

1. $T_a - T_v$ that is the difference between the temperature of the artery and vein.
2. The difference of the tissue temperature, T and average blood temperature $\frac{T_a+T_v}{2}$

2.6 Preliminaries

In this section we will discuss numerical methods that have over the years been used in finding solutions of Pennes Bio heat Equation.

There are numerous numerical methods that have been developed to analyse bio heat equation. They include;

1. Finite Element Method (FEM) used by Marqa et al to investigate bioheat and thermal damage behaviour under laser irradiation[7].

2. Boundary Element Method(BEM) which was used by Ansari et al to study short-pulse laser propagation in biological tissue[8-9]
3. The Monte Carlo and Dual Reciprocity Boundary Elemnt Method(DRBEM) used to evaluate steady state thermal behaviours in biological tissues[10-11].
4. Laplace transform method
5. The Analog Equation Method(AEM)
6. An Axisymmetry Boundary formulation method derived by Majchrzak for analysis of freezing and thawing in biological tissues[12]
7. Finite volume method
8. Finite Difference Method
9. Cellular Neural Network(CNN) method applied by Niu to solve bio heat equation[13]

The above numerical methods can be used to solve bio heat transfer equation, however Finite Element Method and Finite Difference Method has achieved a great popularity.

2.6.1 Finite Element Method(FEM)

FEM has its origin from the works of Euler in the early 16th century.The earliest mathematical papers were from Schellback [1851] and Courant [1943].Engineers developed FEM to address mechanical problems in aerospace and civil engineering.In mid 1950s papers by Turner, Clough, Martin, and Topp [1956], Argyris [1957], and Babuska and Aziz [1972] were already being written.

The Finite Element Method(FEM) is a numerical technique used to perform Finite Element Analysis(FEA) of any physical phenomena. FEM has over time been used to model mechanical problems related to civil engineering and aerospace. In FEM the principle of energy minimization is highly regarded.This law states that when a boundary is applied,of the numerous possible configurations that a body can take only the one that the total energy is minimum is chosen. In the FEM,an object of 2 or 3 dimension can be subdivided into elements,which gives approximations of the original figure.The PDE is then expressed into an equivalent intergral and solved on each element and then collected into an approximate solution for the entire domain.

3 Solution of Pennes' Bio heat Equation

3.1 Finite Difference Method

Finite Difference method is widely known thanks to the Courant,Friedrichs and Lewy who first published an example five point difference approximations to derivatives for elliptic Laplace equation in 1928 [17]. Over the past decades several numerical methods have emerged to solve partial differential equations that are otherwise hard to solve analytically.Among them is the finite difference method(FDM).FDM is highly stable and accurate and has been used in the field of engineering for quite a long time.

Finite difference method is used to solve Ordinary differential boundary value problems. It is used to approximate solutions of boundary value problems. It consists of approximating the differential operator by replacing the derivatives in the equation by differential quotients. The derivatives are approximated using finite difference formulas that are derived from the Taylor's series expansion.

3.1.1 Discretization

Discretization is the process of approximation of solutions at each discrete point(s). By discretization we reduce a boundary value problem into a discrete problem which we can be able to solve.

The time space is uniformly partitioned as follows;

$$t_n = n\Delta t, n = 0, 1....., N_t \tag{16}$$

Equivalently the space domain is also uniformly partitioned as follows;

$$x_m = m\Delta x, m = 0, 1, ..., M_x \tag{17}$$

If we consider the diagram below labelled figure 3 above, There are discrete points on the grid represented by space and time step in the x and y direction with Δ t and Δ x as mesh discretization sizes in the y and x directions respectively.Δ t is time discrete time step size and Δx is the space time step.

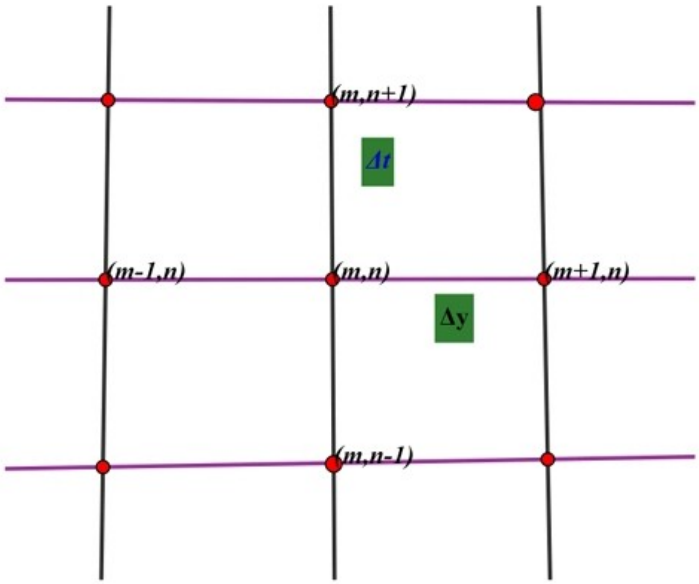

Figure 3. A figure to represent the nodal points

We refer to each point on the grid as a node. We have 5 nodal points and solutions are sought to be found at each nodal point. In the line above we subdivide into 5 nodes, 3 nodes that are intermediate and nodes at the ends. The node at the centre (m,n) is f(x), the one named (m+1,n) is f(x+h) and the one named (m-1,n) is f(x-h). The nodes have an interval of Δ x.

3.1.2 Deriving Finite Difference derivatives

We use the Taylor's series expansion to derive the finite difference derivatives.Consider the Taylor's series expansion as shown below;

$$f(x+\Delta x) = f(x) + \Delta x f'(x) + \Delta x^2 \frac{f''(x)}{2!} + ... \tag{18}$$

also;

$$f(x-\Delta x) = f(x) - \Delta x f'(x) + \Delta x^2 \frac{f''(x)}{2!} + ... \tag{19}$$

using equation (18) we can find the f'(x) as follows;

$$f'(x) = \frac{f(x+\Delta x) - f(x)}{\Delta x} + \Delta x \frac{f''(x)}{2!} \tag{20}$$

Equation (20) has and error of order 2.

$$error = O\Delta x^2 \tag{21}$$

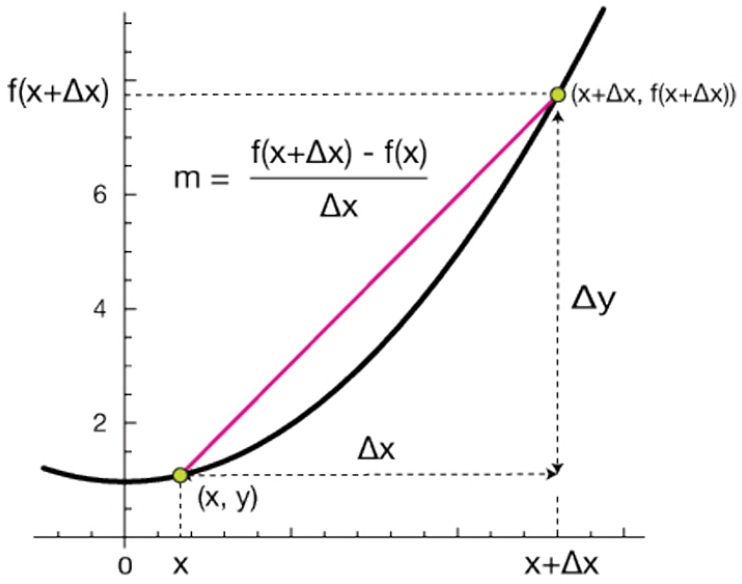

Figure 4. Illustration of Approximations

If the error term is eliminated, we have what is called the forward difference approximation.

$$f'(x) = \frac{f(x+\Delta x) - f(x)}{\Delta x} \tag{22}$$

Using equation (18), we can obtain the following equation;

$$f'(x) = \frac{f(x) - f(x+\Delta x))}{\Delta x} + h\frac{f''(x)}{2!} \tag{23}$$

Eliminating the error term we have the Backward difference approximation.

$$f'(x) = \frac{f(x) - f(x+\Delta x)}{\Delta x} \tag{24}$$

This is shown in the figure below;

If we add equation (18) and (19) we get the central difference approximation.

$$f'(x) = \frac{f(x+\Delta x) - f(x-\Delta x)}{2\Delta x} + \frac{\Delta x^2 f''}{3!} + ... \tag{25}$$

This is an approximation with an error of order Δx^2 When we eliminate the error we have a central difference approximation;

$$f'(x) = \frac{f(x+\Delta x) - f(x-\Delta x)}{2\Delta x} \tag{26}$$

For the second order derivative, we use the Taylor's series and truncate to the third order.

$$f(x+\Delta x) = f(x) + hf'(x) + \Delta x^2 \frac{f''(x)}{2!} + \Delta x^3 \frac{f'''(x)}{3!} + ... \tag{27}$$

$$f(x-\Delta x) = f(x) - \Delta x f'(x) + \Delta x^2 \frac{f''(x)}{2!} - \Delta x^3 \frac{f'''(x)}{3!} + ... \tag{28}$$

Adding equation (23) and (24), we get;

$$f(x+\Delta x) + f(x-\Delta x) = 2f(x) + 2\frac{\Delta x^2 f''(x)}{2!} + 2\frac{\Delta x^4 f''''(x)}{4!} + ... \tag{29}$$

Rearranging the terms we have the equation for the second derivatives as;

$$f''(x) = \frac{f(x+\Delta x) - 2f(x) + f(x-\Delta x)}{\Delta x^2} - \frac{\Delta x^2 f''''(x)}{12} + ... \tag{30}$$

Neglecting the other terms we have an equation of truncation error of order 2.

$$f''(x) = \frac{f(x+\Delta x) - 2f(x) + f(x-\Delta x)}{\Delta x^2} \tag{31}$$

The above equation is called the central difference formulation for the second order derivatives.

3.1.3 solving a BVP using Finite difference method

Example 1 solve The boundary value problem

$$U''(x) - 2xU'(x) - 2U = 0, U(0) = 1, U(1) = e \tag{32}$$

Using 8 intervals. This gives

$$h = \frac{1-0}{8} \tag{33}$$

Replace the function U in equation (32) with the finite difference derivatives.

$$\frac{U(x+h) - 2U(x) + U(x-h)}{h^2} - 2x\frac{U(x+h) - U(x-h)}{2h} - 2U = 0 \tag{34}$$

Rearranging and simplifying;

$$U(x+h) - 2U(x) + U(x-h) - xh\Big(U(x-h) - U(x-h)\Big) - 2h^2 U(x) = 0 \tag{35}$$

Grouping and rearranging we have;

$$U(x+h)\Big(1-xh\Big) + U(x)\Big(-2-2h^2\Big) + U(x-h)\Big(1-xh\Big) = 0 \tag{36}$$

Also we shall replace

$$x = x_n \tag{37}$$

$$U(x_n) = U_n \tag{38}$$

$$U(x_{n-h}) = U(x_{n-1}) = U_{n-1} \tag{39}$$

$$x_n = x_0 + \frac{1}{8}h \tag{40}$$

$$x_n = X_0 + hx - n \tag{41}$$

$$U(x_{n+h} = U(x_{n+1} = U_{n+1} \tag{42}$$

We have;

$$U_{n+1}\left(1 - \frac{1}{64}n\right) + U_n\left(-2 - \frac{1}{32}\right) + U_{n-1}\left(1 + \frac{1}{64}n\right) = 0 \tag{43}$$

Replacing n with 1;

$$U_2\frac{63}{64} + U_1\frac{-65}{32} + U_0\frac{65}{64} = 0 \tag{44}$$

Replacing n with 2;

$$U_3\frac{31}{32} + U_2\frac{-65}{32} + U_1\frac{33}{32} = 0 \tag{45}$$

$$U_4\frac{61}{64} + U_3\frac{-65}{32} + U_2\frac{67}{64} = 0 \tag{46}$$

$$U_5\frac{59}{64} + U_4\frac{-65}{32} + U_3\frac{17}{16} = 0 \tag{47}$$

$$U_6\frac{59}{64} + U_5\frac{-65}{32} + U_4\frac{69}{64} = 0 \tag{48}$$

$$U_7\frac{29}{32} + U_6\frac{-65}{32} + U_5\frac{35}{32} = 0 \tag{49}$$

$$U_8\frac{57}{64} + U_7\frac{-65}{32} + U_6\frac{71}{64} = 0 \tag{50}$$

$$U_9\frac{55}{64} + U_8\frac{-65}{32} + U_7\frac{9}{8} = 0 \tag{51}$$

We then generate a matrix of the form;

3.1.4 How do we apply the finite difference formulas into solving an BVP?

We re write the Boundary value problem in Finite difference form. We the apply the finite difference equation to every node, This will result to Linear algebraic equations. These linear algebraic equations are then solved using matrix inversion techniques. For example;

1. Gauss- Jordan Elimination
2. Gauss-seidel iteration
3. using excel, circular reference and iteration.

3.1.5 1D Pennes' Equation

In order to find the solution of the Pennes' Equation we first re-write the Pennes' Equation using the approximated derivatives of the first and second order using the time and space domain respectively. We then re-arrange the derivatives and come up with a matrix which will be solved completely using Matlab.

We shall try and adjust the Pennes'Bioheat Equation for it to allow us apply derivatives easier.Let's us consider the 1D Pennes' Equation as follows. Now let us consider Pennes'Bioheat Equation in one dimension;We shall use function $U(x,t)$ to replace the function $T(x,t)$ in our equation (1) of pennes' Equation.

$$\rho c\frac{\delta U}{\delta t} = k\frac{\delta^2 T}{\delta x^2} - \rho_b c_b \omega_b (U_a - U) + q_{met}$$

If we assume $k = 1$ and We divide the above equation by ρc,which simplifies the above equation into:

$$\frac{\delta U}{\delta t} + \frac{\rho_b c_b \omega_b}{\rho c}\left(U_a - \frac{q_{met}}{\rho_c c_b \omega_b}\right) = \frac{1}{\rho c}\frac{\delta^2 U}{\delta x^2} + \frac{\omega_b c_b \rho_c}{pc}U \tag{52}$$

Now, we will let $\frac{\rho_b c_b \omega_b}{\rho c}$ be a, the term $U_a - \frac{q_{met}}{\rho c c_b \omega_b}$ be U* also the term $\frac{1}{\rho c}$ be b. The the above equation simplifies to its simplest form;

$$\frac{\delta U}{\delta t} + aU* = b\frac{\delta^2 U}{\delta x^2} + aU \tag{53}$$

Now we write the Pennes'Equation and its initial and boundary conditions as follows;

$$\frac{\delta U}{\delta t} = b\frac{\delta^2 U}{\delta x^2} + aU - aU* \tag{54}$$

The initial condition is as follows;

$$U(x,0) = f(x), 0 \leq x \geq 1 \tag{55}$$

$$U(0,t) = g(t), 0 \leq t \geq T \tag{56}$$

and the boundary conditions;

$$U(1,t) = h(t) \tag{57}$$

3.1.6 Finite Difference Scheme for 1D Pennes' Bio Heat Equation

Now we want to apply the Finite difference method to approximate solutions of the equation (44) with boundary and initial conditions as shown above.We shall name our discrete points in the domains as (x_i, y_j), (x_{i+1}, y_j) moving a step forward in the space direction (x_{i-1}, y_j) moving 1 step backwards from the initial discrete points in the space direction;Also (x_i, y_{j+1}) moving a step forward in the direction of time and $(x_i, yj-1)$ and a step backwards in the direction of time from the initial discrete node.

Now we use the following;

$$(x_{i+1}, j) - (x_i, y_j)) = \Delta x \tag{58}$$

and

$$(x_i, y_{j+1}) - (x_i, y_j) = \Delta t \tag{59}$$

The function U(x,t) is defined as a mesh function $U(x_i, t_j)$ and the difference operator as;

$$\frac{\delta U}{\delta t} = U_t = \frac{U_{i,j+1} - U_{i,j}}{\Delta t} \tag{60}$$

This is by forward difference method. We use the central differences approximation for the second approximated derivative as ;

$$\frac{\delta^2 U}{\delta x^2} = U_{xx} = \frac{U_{i+1,j} - 2UI, J + Ui - 1, j + U_{i-1,j}}{\Delta x^2} \tag{61}$$

Applying equation (50) and (51) to the equation (44), we get the following;

$$\frac{U_{i,j+1} - U_{i,j}}{\Delta t} = b\left(\frac{U_{i+1,j} - 2U_{i,j} + U_{i-1,j}}{\Delta x^2}\right) + aU_{i,j} - aU* \tag{62}$$

and after rearranging we find;

$$U_{i,j+1} = \frac{b\Delta t}{\Delta x^2}\left(U_{i+1,j} + U_{i-1,j}\right) + \left(1 - \frac{2b\Delta t}{\Delta x^2} + a\Delta t\right)U_{i,j} - a\Delta t U* \tag{63}$$

3.2 Formulation of the problem

A body tissue that is initially at a temperature $T_0 = 37C$ is to be heated by electromagnetic radiators using 432 MHz antenna. While conducting thermal therapy, either we use a hot metallic disc or a temperature controlled probe and approximate it using 1D Cartesian coordinates and also put it at boundary conditions that are the constant temperature. It is

necessary to derive the characteristics of the heat transfer that are common to treatment methods. In this study we solve time space by heat transfer equation which governs the process of transfer of heat in the tissue

$$\rho C \frac{\partial^{\beta} T(r,t)}{\partial t^{\beta}} = K_t \frac{\partial^{\alpha} T(r,t)}{\partial t^{\alpha}} + Q_m + Q_b + Q_s \tag{64}$$

with initial condition

$$T(r,0) = T_0 \tag{65}$$

boundary condition

$$T(r,t)|_{r=R} = T_w \tag{66}$$

and symmetric condition

$$\frac{\partial T(r,t)}{\partial r}|_{r=0} = 0 \tag{67}$$

where T(r, t) is the temperature of the tissue locally. r is the coordinates of the space, t the time, density is ρ, C is specific heat and K_t is the thermal conductivity of tissues. The heat generation is made up of tissue temperature given by

$$Q_m = 0.17(2)^{(}T - 37)/10 \tag{68}$$

Where

$$Q_{m0} = Q_{00}[1 + 0.1(T_0 - 37], d = 0.1 \times T_0$$

and θ is dimensionless temperature. Q_b is a source of heat from blood circulation and could be given as

$$Q_b = W_b C_b (T_a - T) \tag{69}$$

Here T_a is the blood temperature of the alterial which remains constant.

The term Q_s stands for heat per unit volume of tissue caused by absorption of electromagnetic radiation.

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$$Q_s = \rho SPe^{a(\bar{r}-0.01)} \tag{70}$$

Here S and a are the constants of the antenna. P is the power transmitted, which depends on the requirements,

$$\bar{r} = R - r$$

is the distance between the tissue and the outer space.

Let us suppose that at some given radius R (arbitrarily chosen) the tissue is at a constant temperature T_0, by some processes phycologically happened in the entire body.

3.3 Solution to the problem

We introduce the dimensionless variable and the creteria of similarity

$$x = \frac{r}{R}; F_0 = (\frac{K_t}{\rho C R^\alpha})^\beta t; \theta_\alpha = \frac{T_\alpha - T_0}{T_0}; \theta = \frac{T - T_0}{T_0}; \theta_w = \frac{T_w - T_0}{T_0}; P_m = \frac{Q_{m0}}{T_0 K_t} R^2;$$

(71)

The system of Eqs **(64) - (67)** reduces to

$$\frac{\partial^\beta \theta(x, F_0)}{\partial F_0^\beta} = \frac{\partial^\alpha \theta(x, F_0)}{\partial x^\alpha} + C\theta(x, F_0) + P(x) \quad (0 < \beta \leq 1 < \alpha \leq 2$$

(72)

with initial condition

$$\theta(x, 0) = 0 \tag{73}$$

boundary conditions

$$\theta(x, F_0)_{x=0} = 0 \tag{74}$$

and the symmetric condition

$$\left.\frac{\partial \theta(x, F_0)}{\partial x}\right|_{x=0} = 0$$

where $C = (P_m d - P_f^2)$ and $P(x) = (P_m + P_f^2\theta_a + P_r \exp(a_0 - b_0 x))$.

(75)

applying equation **(75)** we have

$$\frac{\partial^\beta \theta(x_i, F_0)}{\partial F_0^\beta} = \frac{1}{h^\alpha}\sum_{j=0}^{i} g_j \theta(x_i - jh, F_0) + C\theta(x_i, F_0) + P(x_i), \quad i = 1, 2, \ldots, n-1,$$
$$\theta(x_i, 0) = 0$$
$$\theta(x_n, F_0) = \theta_w.$$

(76)

Denoting $\theta(x_i, F_0) = \theta_i(F_0)$, $P(x_i) = P_i(x)$

Taking $n = 10, h = 0.1$ and the symmetric condition equation we are able to write the scheme

$$\theta_0(F) = \frac{13}{21}\theta_1 + \frac{17}{21}\theta_2 - \frac{3}{7}\theta_3.$$

we get the differential matrix equations of order 9 by 9

$$D_{F_0}^\beta \theta(F_0) = A\theta(F_0) + P(x)$$

(77)

where

$$\theta(F_0) = [\theta_1 ... \theta_9]^T$$
$$P(x) = [P_1 ... P_9]^T$$
$$\theta(0) = 0$$

$$A = \begin{bmatrix}
\left(C + \frac{1}{h^\alpha} - \frac{13\alpha}{21h^\alpha}\right) & -\left(\frac{17\alpha}{21h^\alpha}\right) & \frac{3\alpha}{7h^\alpha} & 0 & 0 & 0 & 0 & 0 & 0 \\
\left(\frac{13F[\alpha,2]}{21h^\alpha} - \frac{\alpha}{h^\alpha}\right) & \left(C + \frac{17F[\alpha,2]}{21h^\alpha} + \frac{1}{h^\alpha}\right) & -\left(\frac{3F[\alpha,2]}{7h^\alpha}\right) & 0 & 0 & 0 & 0 & 0 & 0 \\
\left(\frac{F[\alpha,2]}{h^\alpha} - \frac{13F[\alpha,3]}{21h^\alpha}\right) & -\left(\frac{\alpha}{h^\alpha} + \frac{17F[\alpha,3]}{21h^\alpha}\right) & \left(C + \frac{1}{h^\alpha} + \frac{3F[\alpha,3]}{7h^\alpha}\right) & 0 & 0 & 0 & 0 & 0 & 0 \\
\left(\frac{13F[\alpha,4]}{21h^\alpha} - \frac{F[\alpha,3]}{h^\alpha}\right) & \left(\frac{17F[\alpha,4]}{21h^\alpha} + \frac{F[\alpha,2]}{h^\alpha}\right) & -\left(\frac{\alpha}{h^\alpha} + \frac{3F[\alpha,4]}{7h^\alpha}\right) & \left(C + \frac{1}{h^\alpha}\right) & 0 & 0 & 0 & 0 & 0 \\
\left(\frac{F[\alpha,4]}{h^\alpha} - \frac{13F[\alpha,5]}{21h^\alpha}\right) & -\left(\frac{17F[\alpha,5]}{21h^\alpha} + \frac{F[\alpha,3]}{h^\alpha}\right) & \left(\frac{F[\alpha,2]}{h^\alpha} + \frac{3F[\alpha,5]}{7h^\alpha}\right) & -\frac{\alpha}{h^\alpha} & \left(C + \frac{1}{h^\alpha}\right) & 0 & 0 & 0 & 0 \\
\left(\frac{13F[\alpha,6]}{21h^\alpha} - \frac{F[\alpha,5]}{h^\alpha}\right) & \left(\frac{17F[\alpha,6]}{21h^\alpha} + \frac{F[\alpha,4]}{h^\alpha}\right) & -\left(\frac{F[\alpha,3]}{h^\alpha} + \frac{3F[\alpha,6]}{7h^\alpha}\right) & \frac{F[\alpha,2]}{h^\alpha} & -\frac{\alpha}{h^\alpha} & \left(C + \frac{1}{h^\alpha}\right) & 0 & 0 & 0 \\
\left(\frac{F[\alpha,6]}{h^\alpha} - \frac{13F[\alpha,7]}{21h^\alpha}\right) & -\left(\frac{17F[\alpha,7]}{21h^\alpha} + \frac{F[\alpha,5]}{h^\alpha}\right) & \left(\frac{F[\alpha,4]}{h^\alpha} + \frac{3F[\alpha,7]}{7h^\alpha}\right) & -\frac{F[\alpha,3]}{h^\alpha} & \frac{F[\alpha,2]}{h^\alpha} & -\frac{\alpha}{h^\alpha} & \left(C + \frac{1}{h^\alpha}\right) & 0 & 0 \\
\left(\frac{13F[\alpha,8]}{21h^\alpha} - \frac{F[\alpha,7]}{h^\alpha}\right) & \left(\frac{17F[\alpha,8]}{21h^\alpha} + \frac{F[\alpha,6]}{h^\alpha}\right) & -\left(\frac{F[\alpha,5]}{h^\alpha} + \frac{3F[\alpha,8]}{7h^\alpha}\right) & \frac{F[\alpha,4]}{h^\alpha} & -\frac{F[\alpha,3]}{h^\alpha} & \frac{F[\alpha,2]}{h^\alpha} & -\frac{\alpha}{h^\alpha} & \left(C + \frac{1}{h^\alpha}\right) & 0 \\
\left(\frac{F[\alpha,8]}{h^\alpha} - \frac{13F[\alpha,9]}{21h^\alpha}\right) & -\left(\frac{17F[\alpha,9]}{21h^\alpha} + \frac{F[\alpha,7]}{h^\alpha}\right) & \left(\frac{F[\alpha,6]}{h^\alpha} + \frac{3F[\alpha,9]}{7h^\alpha}\right) & -\frac{F[\alpha,5]}{h^\alpha} & \frac{F[\alpha,4]}{h^\alpha} & -\frac{F[\alpha,3]}{h^\alpha} & \frac{F[\alpha,2]}{h^\alpha} & -\frac{\alpha}{h^\alpha} & \left(C + \frac{1}{h^\alpha}\right)
\end{bmatrix}$$

and

$$F[\alpha, n] = \frac{\Gamma(n+1)}{\Gamma(\alpha+1)\Gamma(n-\alpha+1)}.$$

(78)

To solve the Matrix of differential equations **(77)** we use the Perturbation method and come up with the following Homotopy

$$D^{\beta}_{F_0}\theta(F_0) = p\,[A\theta(F_0) + P(x)] \quad p \in [0, 1].$$

(79)

with Initial condition

$$\theta(0) = 0$$

Assuming that

$$\theta(F_0) = \theta^{(0)}(F_0) + p\theta^{(1)}(F_0) + p^2\theta^{(2)}(F_0) + p^3\theta^{(3)}(F_0) + \cdots$$

(80)

is the solution of **(79)**

Substituting **(80)** into **(79)**

we get the following set of equations after equating similar powers of p.

$$P^0: \quad D^{\beta}_{F_0}\theta^0(F_0) = 0$$

$$P^1: \quad D^{\beta}_{F_0}\theta^1(F_0) = A\theta^0(F_0) + P(x)$$

$$P^2: \quad D^{\beta}_{F_0}\theta^2(F_0) = A\theta^1(F_0)$$

$$P^3: \quad D^{\beta}_{F_0}\theta^3(F_0) = A\theta^2(F_0)$$

(81)

3.4 Mathematical simulation

Using matlab we will simulate the above results for

$$\theta^{(0)}(F_0) = 0$$

$$\theta^{(1)}(F_0)$$

$$\theta^{(2)}(F_0)$$

$$\theta^{(3)}(F_0)$$

From $\theta(x,F_0)$ at $\alpha=\beta=1$ as the starting point we proceed as follows.

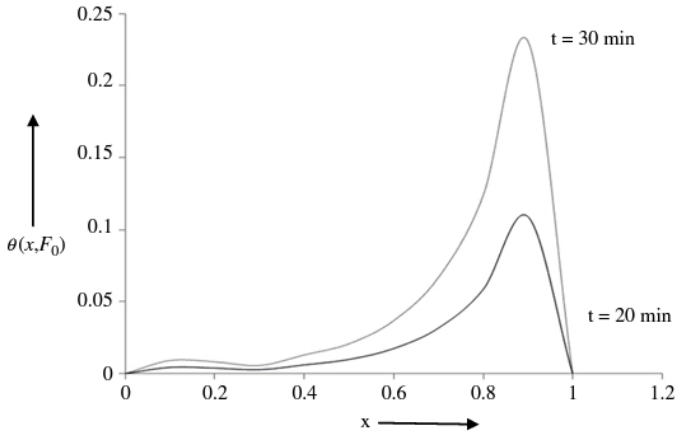

Figure 5. $\theta(x,F_0)$ **at** $\alpha=\beta=1$

The solution reads

$$\theta^{(0)}(F_0)=0$$

$$\theta^{(1)}(F_0)=P(x)\frac{F_0^{\beta}}{\Gamma(1+\beta)}$$

$$\theta^{(2)}(F_0)=AP(x)\frac{F_0^{2\beta}}{\Gamma(2\beta+1)}$$

$$\theta^{(3)}(F_0)=A^2P(x)\frac{F_0^{3\beta}}{\Gamma(1+3\beta)}$$

$$\ldots.$$

In the same way the rest of the components can be found. Consequently, we obtained the following approximated solutions taking $p=1$.

$$\theta(F_0)=P(x)\frac{F_0^{\beta}}{\Gamma(1+\beta)}+AP(x)\frac{F_0^{2\beta}}{\Gamma(2\beta+1)}+A^2P(x)\frac{F_0^{3\beta}}{\Gamma(1+3\beta)}+\cdots.$$

Now use the following parameters $S=12.5m^{-1}$, $P=5W$, $\alpha=127m^{-l}$, $\rho=1000kgm^{-3}$, $C_b=3344JKg^{-1}K^{-1}$, $K_t=0.5Wm^{-1}$, $T_0=T_\alpha=T_w=T_f=37^oC$

Case 1. standard equation $\alpha=2,\beta=1$
Case 2. Time fraction $\alpha=2,\beta=(0,1)$

Case 3. space fraction means $\alpha = (1,2), \beta = 1$

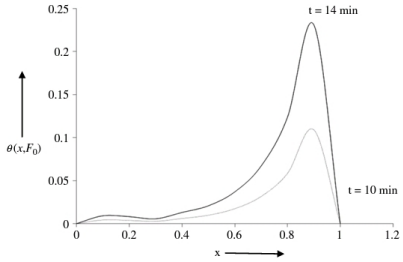

Figure 6. $\theta(x, F_0)$ **at** $\alpha = 2\beta = 0.9$

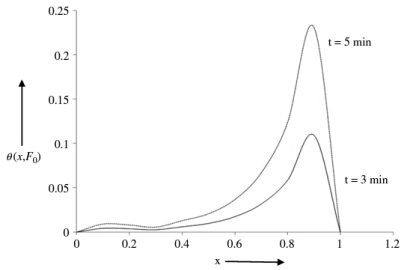

Figure 7. $\theta(x, F_0)$ **at** $\alpha = 2\beta = 0.8$

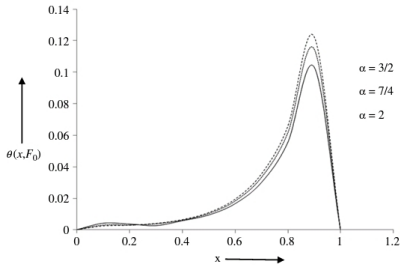

Figure 8. $\theta(x, F_0)$ **at** $(\alpha = 2, 7/4, 3/2)t = 20mins$

In this case we see that as α decreases the temperature inside the tissues slightly increases and there is at most same duration to reach the hyperthermia state. Temperature are shown graphically by the figures for different values of α

3.5 CONCLUSION

The fractional bioheat equation is shown and their effect presented in thermal therapy for cancer cell. In superconductivity case the time fractional derivative reduced and the subconductivity case the space fractional derivative is reduced with respect to space x. It has been noted that the time to achieve hyperthermia state decreases as the order two fractional derivative reduces. Further the time fractional derivative is more pronounced in comparison to space fractional derivative.

3.5.1 Future work

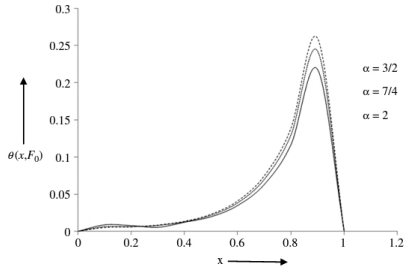

Figure 9. $\theta(x,F_0)$ **at** $(\alpha = 2,7/4,3/2)t = 30mins$

Going forward we will explore the alternative results that could be arrived at using the tridiagonal matrices and attempt to find the exact values.

Alternative RESULTS

From Equation (45) we can get a tridiagonal matrices when we substitute the exact values of i and j. Taking $N = 6$ we can find our tridiagonal matrix for the discretised linear equation(45). We first generate the linear equations when i and j have definite value.

For $i = 1$, we have

$$U_{1,j+1} = \frac{b\Delta t}{\Delta x^2}\left(U_{2,j} + U_{0,j}\right) + \left(1 - \frac{2b\Delta t}{\Delta x^2} + a\Delta t\right)U_{1,j} - a\Delta t U* \tag{71}$$

For $i = 2$, we have

$$U_{2,j+1} = \frac{b\Delta t}{\Delta x^2}\left(U_{3,j} + U_{1,j}\right) + \left(1 - \frac{2b\Delta t}{\Delta x^2} + a\Delta t\right)U_{2,j} - a\Delta t U* \tag{72}$$

For $i = 3$, we have

$$U_{3,j+1} = \frac{b\Delta t}{\Delta x^2}\left(U_{4,j} + U_{2,j}\right) + \left(1 - \frac{2b\Delta t}{\Delta x^2} + a\Delta t\right)U_{3,j} - a\Delta t U* \tag{73}$$

For $i = 4$, we have

$$U_{4,j+1} = \frac{b\Delta t}{\Delta x^2}\left(U_{5,j} + U_{3,j}\right) + \left(1 - \frac{2b\Delta t}{\Delta x^2} + a\Delta t\right)U_{4,j} - a\Delta t U* \tag{74}$$

For $i = 5$, we have

$$U_{5,j+1} = \frac{b\Delta t}{\Delta x^2}\left(U_{6,j} + U_{4,j}\right) + \left(1 - \frac{2b\Delta t}{\Delta x^2} + a\Delta t\right)U_{5,j} - a\Delta t U* \tag{75}$$

For $i = 6$, we have

$$U_{6,j+1} = \frac{b\Delta t}{\Delta x^2}\left(U_{7,j} + U_{5,j}\right) + \left(1 - \frac{2b\Delta t}{\Delta x^2} + a\Delta t\right)U_{6,j} - a\Delta t U* \tag{76}$$

We then have a matrix of the form;

$$\begin{bmatrix} 1 - \frac{2b\Delta t}{\Delta x^2} + a\Delta t & \frac{b\Delta t}{\Delta x^2} & U_{i,j} - a\Delta t U* & 0 & 0 & 0 & 0 \\ \frac{b\Delta t}{\Delta x^2} & 1 - \frac{2b\Delta t}{\Delta x^2} + a\Delta t & \frac{b\Delta t}{\Delta x^2} & U_{i,j} - a\Delta t U* & 0 & 0 & 0 \\ 0 & \frac{b\Delta t}{\Delta x^2} & \frac{b\Delta t}{\Delta x^2} & 1 - \frac{2b\Delta t}{\Delta x^2} + a\Delta t & \frac{b\Delta t}{\Delta x^2} & U_{i,j} - a\Delta t U* & 0 \end{bmatrix}$$

From equation (45) we can get the unknown terms U_1, j , $U_1, j, U_1, j,$

Appendices

A MATLAB codes

clear; clc; close all; N =300; % number of steps in space
M = 100; % number of steps in time
$maxt = 25$;
t=linspace(0,maxt,M);
zeta=(maxt/M); % time step size
% draw profile g = 9.8; % gravitational constant,
delta = 16/N;
u(:,:) = zeros(N + 1,M); % create space for velocity
h(:,:) = zeros(N + 1,M); % creat space for height
x = -8 : delta : 8; % space step and range
$u(1,:) = 0; u(N+1,:) = 0$; % velocity boundary conditions
u(:,1) = 0; % initial velocity
h(1,:) = 1; h(N + 1,:) = 1; % height boundary conditions
% initial displacement conditions for k = 2 : N
$h(k,1) = 1 + 1/5 * exp(-2 * x(k).^2)$; %Initial conditions end

% matrices for solving A = zeros(2 * (N - 1), 2 * (N - 1)); b = zeros(2 * (N - 1) 1);
for j = 2 : M A(1, N) = 2*delta ;
A(1,N + 1) = zeta*u(2,j-1);
A(1, 2) = zeta * h(2, j - 1);
%the first equation for i = 1 b(1, 1) = 2 * delta * h(2, j - 1) + zeta* u(2, j - 1) * h(1, j) + zeta * h(2, j - 1) * u(1, j)
A(2, 1) = 2 * delta * h(2, j - 1);
A(2, 2) = zeta * u(2, j -1) * h(2, j - 1);
%the second equation for i=1 A(2,N + 1) = zeta * g * h(2, j - 1);
b(2, 1) = 2 * delta * h(2, j - 1) * u(2, j -1) + zeta * u(2, j - 1) * h(2, j - 1) *u(1, j) + zeta * g * h(2, j - 1) * h(1, j);
A(2 * N - 3, 2 * N - 2) = 2 * delta;
A(2 * N - 3,N - 2 + N - 1) = -zeta * u(N,j - 1);
A(2 *N - 3,N - 1 - 1) = -zeta * h(N,j - 1);
%the first equation for i=3 b(2 *N -3, 1) = 2 * delta *h(N, j-1)-zeta *u(N, j-1) *h(N +1, j)-zeta*h(N,j-1) * u(N + 1, j);
A(2 * N - 2,N - 1) = 2 * delta * h(N, j - 1);
A(2 * N - 2,N - 1 - 1) = -zeta *u(N, j - 1) * h(N, j - 1);
A(2 * N - 2,N - 2 + N - 1) = -zeta * g * h(N, j - 1);

%the second equation for i=3 b(2 *N -2, 1) = 2 * delta * h(N, j -1) * u(N, j -1)-zeta * u(N, j -1) * h(N, j -1) *u(N + 1, j) - zeta * g * h(N, j- 1) * h(N + 1, j);
for i = 3 : N - 1

$A(2*i-3, N-2+i) = 2*delta;$
$A(2*i-3, N-2+i+1) = zeta*u(i, j-1);$
$A(2*i-3, N-2+i-1) = -zeta*u(i, j-1);$
$A(2*i-3, i+1-1) = zeta*h(i, j-1);$
$A(2*i-3, i-1-1) = -zeta*h(i, j-1);$
%the first equation for i = 2 $b(2*i-3, 1) = 2*delta*h(i, j-1);$
$A(2*i-2, i-1) = 2*delta*h(i, j-1);$
$A(2*i-2, i+1-1) = zeta*u(i, j-1)*h(i, j-1);$
$A(2*i-2, i-1-1) = -zeta*u(i, j-1)*h(i, j-1);$
$A(2*i-2, N-2+i+1) = zeta*g*h(i, j-1);$
$A(2*i-2, N-2+i-1) = -zeta*g*h(i, j-1);$
%the second equation for i = 2 $b(2*i-2, 1) = 2*delta*h(i, j-1)*u(i, j-1);$
end

% solving y=Ab;
y=Ab;
% applying the solution to the velocity and height spaces
$u(2:N, j) = y(1:N-1); h(2:N, j) = y(N:2*N-2);$
end

figure
plot(x, h(:, 1),' b-');
title('h at t=0');
axis([-8 8 0.8 1.8]);

figure
plot(x,h(:,25),'b-');
title('h at t=1.2060');
axis([-8 8 0.8 1.8]);

figure
plot(x,h(:,50),'b-');
title('h at t=2.4623');
axis([-8 8 0.8 1.8]);

```
figure
plot(x,h(:,75),'b-');
title('h at t=3.7186');
axis([-8 8 0.8 1.8]);


figure
plot(x,h(:,100),'b-');
title('h at t=4.9749');
axis([-8 8 0.8 1.8]);


figure
plot(x,h(:,1).*u(:,1),'r-');
title('h*u at t=0');
axis([-8 8 0.8 1.8]);


figure
plot(x,h(:,25).*u(:,25),'r-');
title('h*u at t=1.2060');
axis([-8 8 0.8 1.8]);


figure
plot(x,h(:,50).*u(:,50),'r-');
title('h*u at t=2.4623');
axis([-8 8 0.8 1.8]);


figure
plot(x,h(:,75).*u(:,75),'r-');
title('h*u at t=3.7186');
axis([-8 8 0.8 1.8]);


figure
plot(x,h(:,100).*u(:,100),'r-');
title('h*u at t=4.9749');
axis([-8 8 0.8 1.8]);
```

```
[T,X] = meshgrid(t,x);


figure
title('the meshgrid in 3 dimensions');
surf(T,X,h)
colorbar
xlabel('Time')
ylabel('Length')
zlabel('Height')


figure
surf(T,X,u)
colorbar
xlabel('Time')
ylabel('Length')
zlabel('Height')


figure
surf(T,X,h .*u)
colorbar
xlabel('Time')
ylabel('Length')
zlabel('Height')


figure
pcolor(T,X,h)
colormap(hsv(64))
colorbar
xlabel('Time')
ylabel('Length')
```

Bibliography

[1] A. Bartłomiejczyk, H. Leszczyński, and A. Poliński, "Thermal ablation modeling via bioheat equation," 2013.

[2] J. J. Zhao, J. Zhang, N. Kang, and F. Yang, "A two level finite difference scheme for one dimensional pennes' bioheat equation," *Applied Mathematics and Computation*, vol. 171, no. 1, pp. 320–331, 2005.

[3] J. Hristov, "Bio-heat models revisited: concepts, derivations, nondimensalization and fractionalization approaches," *Frontiers in Physics*, vol. 7, p. 189, 2019.

[4] J. W. Valvano, "Bioheat transfer," *Encyclopedia of Medical Devices and Instrumentation*, vol. 1, pp. 188–97, 2006.

[5] T. Tabassum, "Cancer risk factors: a critical review," 2016.

[6] G. R. Stroher and C. D. Santiago, "Numerical two-dimensional thermal analysis of the human skin using the multigrid method," *Acta Scientiarum. Technology*, vol. 42, pp. e40 992–e40 992, 2020.

[7] G. Hegyi, G. P. Szigeti, and A. Szász, "Hyperthermia versus oncothermia: cellular effects in complementary cancer therapy," *Evidence-Based Complementary and Alternative Medicine*, vol. 2013, 2013.

[8] F. Westermark, "Uber die behandlung des ulcerirenden cervix carcinoma mittels knonstanter warme," *Zentralblatt für Gynäkologie*, vol. 22, pp. 1335–7, 1898.

[9] H. H. Pennes, "Analysis of tissue and arterial blood temperatures in the resting human forearm," *Journal of applied physiology*, vol. 1, no. 2, pp. 93–122, 1948.

[10] M. Gautherie, "Etude par thermométrie infrarouge des properties thermiques de tissues humains 'in vivo'. influence de la température et de la vascularization," *Revue Francaise d'Etudes Cliniques et Biologiques*, vol. 14, pp. 885–901, 1969.

[11] L. Wang, "Generalized fourier law," *International journal of heat and mass transfer*, vol. 37, no. 17, pp. 2627–2634, 1994.

[12] M. N. Ã-zisik, M. N. Özışık, and M. N. Özısık, *Heat conduction.* John Wiley & Sons, 1993.

[13] J. Chato, J. Eckburg, and E. Hurlburt, "Comparison of three bioheat transfer models using finite difference techniques," *ASME Heat Transfer Div.*, vol. 126, pp. 17–21, 1989.

[14] A. Nakayama, F. Kuwahara, and W. Liu, "A macroscopic model for countercurrent bioheat transfer in a circulatory system," *Journal of Porous Media*, vol. 12, no. 4, 2009.

[15] H.-W. Huang and T.-L. Horng, "Bioheat transfer and thermal heating for tumor treatment," in *Heat transfer and fluid flow in biological processes*. Elsevier, 2015, pp. 1–42.

[16] C. Charny, S. Weinbaum, and R. Levin, "An evaluation of the weinbaum-jiji bioheat equation for normal and hyperthermic conditions," 1990.

[17] G. D. Smith, G. D. Smith, and G. D. S. Smith, *Numerical solution of partial differential equations: finite difference methods*. Oxford university press, 1985.