SHARED COMPOUND FRAILTY MODEL WITH APPLICATION IN JOINT LIFE ANNUITY INSURANCE

WALTER OMONYWA ONCHERE

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Department of Mathematics
University of Nairobi

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Declaration

I declare that this thesis is my original research and has not been submitted for a degree award in any other institution of learning. All the sources used herein are duly acknowledged.

Walter Omonywa Onchere
Reg. No. I80/56009/2019
Sign: **Date:** 2/12/2022

Approval

In our capacity as supervisors for the candidate, we declare that this thesis has been under our supervision and has our approval for submission.

Prof. Patrick G.O Weke
Department of Mathematics,
University of Nairobi,
Box 30196, 00100 Nairobi, Kenya.

Sign: **Date:** 02.12.2022

Dr. Carolyne A. Ogutu
Department of Mathematics,
University of Nairobi,
Box 30196, 00100 Nairobi, Kenya.

Sign: **Date:** 02.12.2022

Prof. Joseph A.M Ottieno
Department of Mathematics,
University of Nairobi,
Box 30196, 00100 Nairobi, Kenya.

Sign: **Date:** 2/12/2022

Dedication

To God's faithfulness upon my life.

To my wonderful family, my dad Dr. Samwel Onchere, my mum Mrs. Pauline Onchere and my brother Abraham. May our good Lord bless you all abundantly.

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Abbreviations and Definitions

MCMC: Markov Chain Monte Carlo.
OpenBUGS: Open Source Bayesian Inference using Gibbs Sampling.
BGR: Brooks-Gelman-Rubin.
DIC: Deviance Information Criteria.
AIC: Akaike Information Criteria.
BIC: Bayesian Information Criteria.
PVF: Power-Variance-Function.
GIG: Generalized Inverse-Gaussian.
NCG: Non-Central Gamma.
CPHM: Cox Proportional Hazards Model.
EPV: Expected Present Value.
CDF: Cumulative Density Function.
PDF: Probability Density Function.
U: Frailty random variable.
$\mathbb{L}_U(.)$: Laplace transform of the frailty random variable.
$\mathbb{P}(.)$: Probability function.
$\mathbb{F}(.)$: Probability generating function.
$\mathcal{L}(.)$: Likelihood function.
T: Random variable representing time-to-death.
$f(.)$: Probability density function.
$f_0(.)$: Base force of mortality probability density function.
$F(.)$: Cumulative density function.
$S(.)$: Survivor function.
$h(.)$: Hazard (intensity) function.
$h_0(.)$: Base force of mortality intensity function.
$H(.)$: Cumulative intensity function.
$H_0(.)$: Cumulative base force of mortality intensity function.

Continuous mixture:

$$f(t) = \int_0^\infty f(t|u)f(u)du.$$

Posterior distribution:

$$f(u|t) = \frac{f(t|u)f(u)}{\int_0^\infty f(t|u)f(u)du}.$$

Survivor function:

$$S(t) = \int_t^\infty f(u)du.$$

Intensity function:

$$h(t) = \lim_{\Delta t \to 0^+} \frac{\mathbb{P}(t \leq T < t + \Delta t | T \geq t)}{\Delta t}.$$

Cumulative intensity function:

$$H(t) = \int_0^t h(x)dx.$$

Mean actuarial value of a joint-life annuity payment of amount Ksh 1 per annum paid in arrears provided either (x) or (y) is alive.

$$a_{\overline{xy}} = \sum_{t=1}^{\infty} v^t S_{\overline{xy}}(t).$$

Abstract

Observable risk factors (e.g., health condition) can explain heterogeneity in mortality among assureds; but modeling the risk profile of a heterogeneous life from unobservable risk factors, such as genetics is complex. This limits the application of only reported rating classes adopted for underwritten annuities. Insurance firms routinely disregard unreported risk factors perhaps because of difficulties in modeling. Although a number of research has been done in univariate frailty modeling to account for unobserved risks, the widely applied frailty mixture is the gamma. One major drawback of the gamma is that it is time-invariant. The scientific interest of the study is to account for time-varying heterogeneity using compound processes. For single-life insurance contracts, the non-central gamma compound process is suggested with the generalized exponential and generalized Weibull baselines to account for time-varying frailties and carry out valuations. On the other hand, grouping insureds in clusters such as joint annuities imposes statistical dependence between lifetimes. The dependence is a result of an unreported risk factor called the frailty that represents a weighted sum of shared lifestyles on mortality risk of group members. In fact, standard insurance valuation considers independence when pricing joint-life products. Different approaches to dependence modeling have been proposed in literature. However, these models consider separately either only the negative effects of dependence alone or positive effect of dependence. The study further proposes to apply the shared compound frailty approach in valuation of joint annuities to address time-varying heterogeneity effects positively and negatively associated with dependence. The positive stable distribution used entails the frailty distribution with the weighted exponential, generalized exponential and weighted Weibull as the base force of mortality distributions. In this study, Bayes inference based on Gibbs sampling is used to calibrate the base force of mortality distributions using a large Kenyan insurer term insurance and joint-life last-survivor data. Subsequently, the performance of the candidate models is compared following the information criteria values. The findings shows that the gamma-generalized Weibull model overestimates the intensity rates at all ages compared to the non-central gamma generalized Weibull model. The non-central gamma generalized Weibull fits well to the insurers claims experience. Thus using the gamma as the frailty distribution may lead to inappropriate term assurance valuations resulting in high prices that negatively impacts marketability of term contracts. The non-central gamma is recommended for valuation of term assurance contracts. Further, application of the positive stable frailty mixture with generalized exponential baselines shows a declined policyholder's annuity payments at early stages when dependence is incorporated. Later on in the contract the annuity payments increase. A good explanation for this trend is that we expect the frail couples to have died early and the less frail ones to survive to extreme ages. It is therefore recommended to account for dependence in modeling joint-life products.

Chapter 1

General Introduction

This chapter introduces the topic and objectives of the thesis. It provides some background information to the main goal of the thesis. Both heterogeneity and dependence modeling is discussed. Section **1.2** describes the problem statement of the thesis. The main and specific objectives are outlined in Section **1.3** followed by the significance of the study i.e. applications in insurance valuation to account for both heterogeneity (in single life products) and dependence (in joint-life products).

1.1 Background Information

In actuarial valuations, any population or class of insureds is heterogeneous with respect to mortality. Sometimes, certain unobservable heterogeneity factors linked to an event of interest are disregarded because of limited modeling knowledge or economic reasons. Statistical dependence may further arise in clustered survival data when the individuals under study are in groups such as in joint-life insurance contracts for married couples. This thesis aims at outlining models that considers time-varying unobservable heterogeneity and dependence to avoid bias in pricing life insurance products. These two research themes that are of current interest in frailty modeling are described below.

Frailty models use mixture distributions to account for heterogeneity by considering the study population risk as a mixture of reported (e.g., health condition) and unreported (e.g., health-seeking behavior) mortality risk factors. The frailty model builds on the Cox Proportional Hazards Model (CPHM)

(Cox (1972)) that only considers reported risk factors affecting mortality. The frailty is a random effects model in which individual frailty information determines additional risks. Frailty theory was first proposed by Beard (1959) to account for age-pattern of mortality, where the term "longevity factor" was used instead of "frailty". It was Vaupel *et al.,* (1979) who coined the term "frailty" in his seminal work that modeled heterogeneity effect on individual mortality using Sweden mortality data-sets. Homogeneity in respect of observable risk factors is assumed in standard models. The implication is that study subjects are pooled in the same risk profile at a given age. However, statistical evidence suggest a different model as indicated by other researchers, such as Su & Sherris (2012); Gatzert *et al.,* (2012); Fong (2015) ; Olivieri & Pitacco (2016) and Pitacco (2018) with references therein. Further, observations of medical statistics e.g. Haberman (1996) shows that individuals differ greatly. Thus, due to unobservable risk factors, the study population must be regarded as heterogeneous. Hence, this one of the motivations of study.

Clayton (1978) proposed the multivariate frailty model to account for dependence when two or more lives are not independent and thus assumed to share frailty risk. Clusters based on survival times are considered conditionally independent sharing a common risk. This commonality introduces dependence to initially independent lives. Many authors have applied shared frailty models see e.g., Hanagal (2020) to model dependence but the application has been in medical and bio-statistical fields. This thesis scientific interest is in utilizing the shared frailty approach in insurance setting to explain both negative and positive effects of dependence. Dependence modeling in insurance (Frees *et al.,* (1996); Luciano *et al.,* (2016); D'Amato *et al.,* (2017); Yang (2017); Gildas *et al.,* (2018) and Arias & Cirillo (2021)) has mainly examined either the negative effects of association alone or positive effects of association. This thesis develops a model that can account for the effects of association both negatively and positively associated with dependence relations between life times.

1.2 Research Problem

Observable demographic risk factors can explain heterogeneity in mortality among assureds; but modeling the risk profile of a heterogeneous life from unobservable risk factors, such as survival-related health-seeking attitudes and/or genetics is complex. This limits the application of only reported risk factors for underwriting life annuities. Insurers routinely disregard unreported risk factors perhaps because of difficulties modeling heterogeneity due to these factors. The risk of adverse selection is inevitable if only annuitants in optimal health conditions bought standard life annuities whose pricing relies on a mortality table based on the assumption of above-average longevity. The outcome is joint-life annuities that are overpriced.

The standard assumption for the frailty distribution is the gamma, but this restriction implies constant frailty over time. For a more accurate valuation, better expression of time-varying heterogeneity due to unobserved risk factors is needed.

Grouping insureds in clusters such as joint or group insurance imposes statistical dependence between paired lifetimes. This dependence is a result of sharing aggregate effects of similar lifestyle or exposure to disaster. Standard insurance practice considers independence when valuing joint-life annuity insurances. This is because modeling the dependence structure based on survival data is complex. Researchers have developed various dependence models for actuarial evaluations. However, these approaches model separately the negative or positive effects of dependence.

1.3 Objectives

Main Objective:

The main objective of this study is to apply the shared compound frailty processes in modeling joint-life annuity insurance.

Specific Objectives:

The specific objectives of this study are

i. To construct compound shared frailty processes.

ii. To apply compound mixtures in explaining heterogeneity and dependence effects.

iii. To develop the non-central gamma compound frailty to account for univariate unobserved heterogeneity effects.

iv. To apply the positive stable frailty approach to account for positive and negative effects of dependence.

v. To construct the positive stable frailty life-table for collective valuation of joint-life annuity products.

1.4 Significance of Study

Risks between joint-life insurance contracts are heterogeneous but within the joint-life contracts the risks are dependent. Therefore, to adequately price and allocate reserves that represent the insurance contracts all relevant factors affecting mortality and dependence needs to be considered. Neglecting unobserved heterogeneity and dependence or usage of only reported risk factors could result in biased insurance products pricing and reserves allocation. This thesis outlines a heterogeneous and dependence model that will improve the underwriting process to ensure fair pricing and reserving of joint-life products consistent with the insured risk.

Annuity insurance pricing is determined by the expected present values (EPVs) that is applied in valuation. A suitable model for the intensity rates is needed when computing EPVs to minimize the risk of biased valuations. See for instance, Frees *et al.,* (1996), Olivieri & Pitacco (1999), Coppola *et al.,* (2000), Luciano *et al.,* (2016), Gildas *et al.,* (2018) and Arias & Cirillo (2021), to cite a few important contributions. Frailty models account for heterogeneity in insured lives due to unreported risk factors (Su & Sherris (2012); Gatzert *et al.,* (2012); Fong (2015)). Frailty methodology has been adopted by many researchers, see for instance, Gatzert *et al.,* (2012) applies a frailty model to represent mortality heterogeneity in risk classes optimization for sub-standard annuities that incorporates reported risks. Olivieri & Pitacco (2016) suggests frailty modeling to classify risk factors for life annuity portfolios. Specifically, the authors identify risk clusters in a population based on assigned frailty estimates for each cluster. Pitacco (2018) modeled the effect of frailty on life insurance variables such as cash flow and profitability for life annuity portfolios and risk groups. This thesis develops a model for adjusting insurance prices to account for heterogeneity and dependence in mortality risk.

The underwriting process in insurance policies considers heterogeneity in respect of reported risk factors to guarantee appropriate premiums for the insured risk. The purpose of underwriting is to assign each insured a frailty factor $\hat{U}$ as an estimate of U to determine the pricing intensity rates. This results in a split of the insureds into super preferred, preferred, standard and substandard risk classes for which additional mortality can be applied (Batty *et al.,* (2010)). While heterogeneity may be reduced due to underwriting, mortality heterogeneity still varies within risk classes (Meyricke & Sherris (2013)). Thus, to improve the underwriting process the frailty model is applied.

1.5 Outline

The research is organized as follows: In chapter two, detailed review of the literature is given together with similarities between frailty and copulas. Chapter three introduces the shared compound frailty distributions and baselines that will be applied in the analysis which forms the basis for Chapters four. Both the univariate and bivariate compound frailty models are discussed in Chapter four. Applications to real life insurance data is shown in Chapters five & six. Finally, Chapter seven contains the conclusions and recommendations of the thesis.

Chapter 2

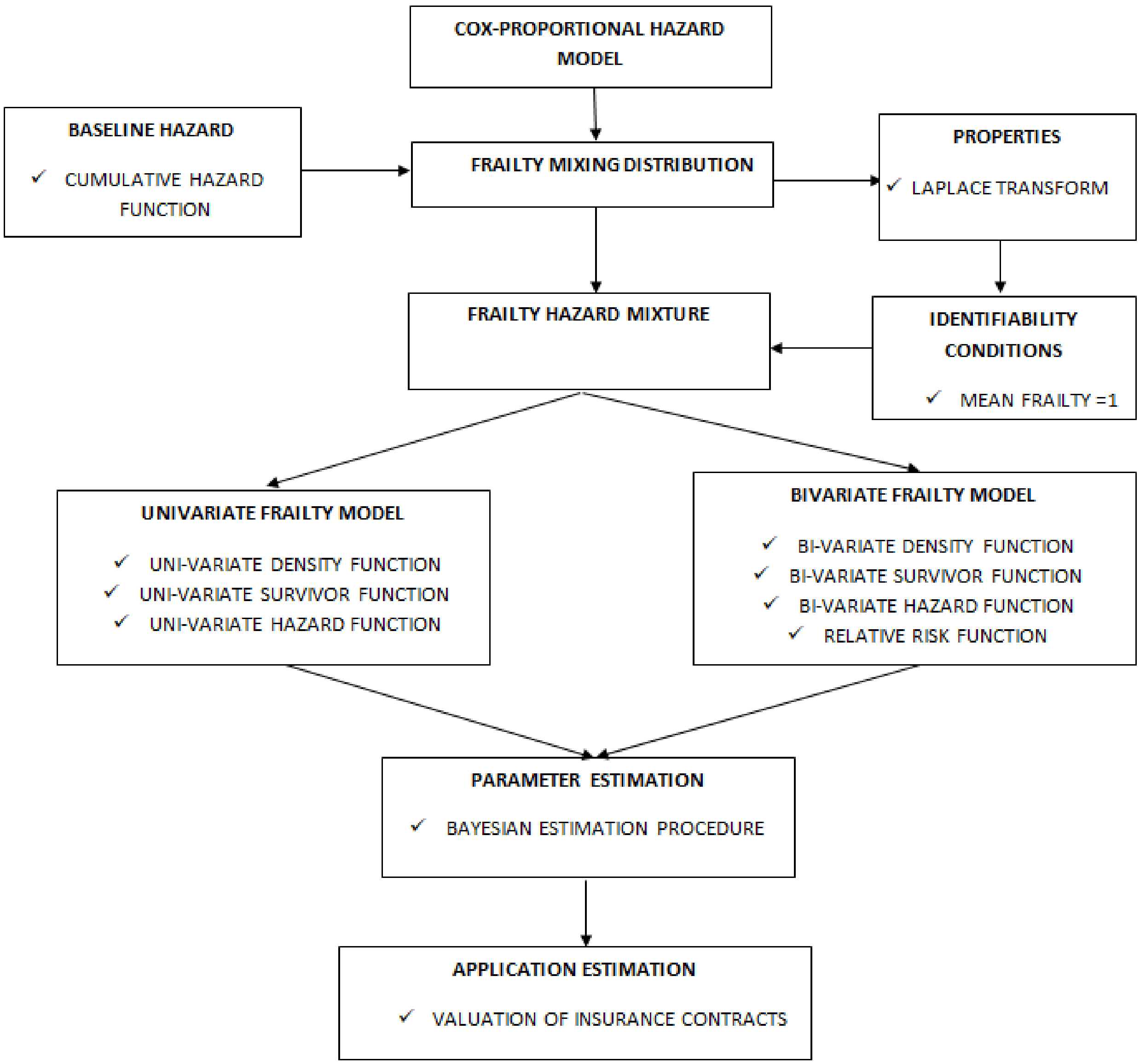
Literature Review

2.1 Introduction

The first part of this chapter describes relevant literature according to a review framework. This is followed by sections that explain the various choices of frailty models, frailty distributions and base force of mortality distributions. Similarity between frailty modeling and the Clayton Archimedean copula dependence modeling is also discussed.

The literature review is organized in the following review framework. Firstly, the Cox Proportional Hazards Model (CPHM) is described and its extension to frailty model framework outlined. An extensive review of the various types of frailty models and choices of frailty mixing distributions is discussed. Researchers reasons for selecting those distributions and the research gaps are identified. Secondly, the frailty distribution is mixed with the base force of mortality distribution to give the frailty mixture. This can either be a univariate frailty mixture or bivariate frailty mixture. Since the thesis is also aimed at dependence modeling, the univariate frailty is extended to bivariate frailty mixtures. Finally, the areas of applications and parameter estimation strategies for the reviewed models is discussed thus further research gaps are identified.

Figure 2.1: Model Framework.



2.2 Cox Proportional-Hazards

The CPHM (Cox (1972)) is a technique that explores the connection between several reported risk factors affecting mortality and the intensity rate of an individual. An individual's intensity rate positively correlates with the base force of mortality $h_0(t)$ and thus the intensity rate is fully determined by the covariate vector. The individual's intensity rate at time t, $h(t)$ is written as:

$$h(t|z) = h_0(t) \exp\left(\theta_1 z_1 + \theta_2 z_2 + \ldots + \theta_n z_n\right), \tag{2.1}$$

where $\theta's$ are the regression coefficients and $z's$ are covariate vectors.

Covariates are entered into this model to determine their effect on mortality. However, not all risk factors may be considered since some may be unobserved or due to cost implications. Frailty models are able to tackle such issues.

2.3 Frailty Model

Vaupel *et al.,* (1979) modeled the multiplicative frailty as:

$$h(t|u) = uh_0(t), \tag{2.2}$$

$h_0(t)$ denotes the standard intensity function assumed to have a frailty $U = 1$ that is equivalent to a "standard" individual. U includes all variables affecting mortality except age and can take either zero or a positive value. $U > 1$ denotes a higher proportional risk of mortality across a lifetime, whereas $U < 1$ denotes lower intensity rates.

In the literature various differential frailty models have been proposed. For instance, Aalen (1989) non-parametric additive frailty model describes the frailty effect as acting additive on the base force of mortality differing from the multiplicative representation. This model is useful in dealing with right censored times-to-event data, especially data that includes time-varying covariates. Eriksson & Scheike (2015) applies the gamma additive frailty model to analyse competing hazards in relatives.

The age shift approach presumes that mortality experience of a group of impaired lives accepted for life insurance should have an increased premium rating determined by assuming that the insured's age is higher than the real current age, hence adopting the "age shift". An alternative is the constant mortality model that represents mortality increments occurring at a fixed rate and is unrelated to initial age. This model is useful in dealing with occupational or sports accident-linked mortality increase.

Different types of frailty models have been applied to model unobserved risks depending on the nature of the problem and the study subjects. The univariate frailty models considers heterogeneity due to unreported risk factors for independent life times in a proportional-hazards model. The heterogeneity includes two components: a hypothetically predictable part that is dependent on reported risk factors and a component that cannot be predicted, even when there are no unknowns. This model has been used by many authors see (Wang & Brown (1998); Eriksson & Scheike (2015)) to show that unpredicted results can be explained by these two sources of heterogeneity. The frailty approach can also be applied in cases where observed covariates are not available or when only survival data is present. However, the model cannot be identified from survival data only, since different aggregations of the base force of mortality and frailty distributions results in similar marginal intensity rate. If two conditions are met, that is, the baseline parametric structure is fixed and the frailty variable is assumed to be parametrically distributed, then the model becomes identifiable.

Bivariate frailty models are used to analyse dependence effects in correlated lifespans. This model estimates the impact of dependence on the regression coefficients of the CPHM. Hanagal & Alok (2013), Hanagal & Pandey (2015) proposes the bivariate frailty model for analysing bivariate survival data due to kidney infection by McGilchrist & Aisbett (1991) using frailty model and proposes an improved model. An example of the bivariate approach is the correlated frailty model. In this model, the measured cross-ratio risk determines the frailty of each member of a pair whose frailty effect is determined using associated random variables. Two variables can be randomly assigned to the husband and the wife to remove the effect of common frailty. These linked variables would exhibit a joint distribution. For paired individuals, frailties may not be similar. Yashin & Iachine (1995) first used the correlated frailty approach with gamma mixtures on related lifetimes. Wienke *et al.,* (2003) applies correlated frailty models in an experimental research to investigate the effect of various estimation methods on the behavior of parametric estimates. The study detected significant dependence between variance and frailty correlation.

The nested frailty approach explains hierarchical grouping of data using two nested random variables with a multiplicative effect on the intensity function. The type of datasets that can be modeled this way include data grouped according to hierarchical levels, for example, geographical units (Rondeau *et al.,* (2006)). Joint frailty models provides a way to investigate how two study subjects evolve jointly by taking a major event occurring as informative censoring (Rondeau *et al.,* (2007)). In this case, the lifespan for a cohort in a study are limited by natural attrition, conclusion of study, or a terminal event like death. Here, the failure event could be related to recurrences of particular events.

The frailty factor normally exhibits a continuous distribution. Sometimes, it is reasonable to express heterogeneity as a discrete mixture using discrete frailty models. A zero frailty indicates immunity and population-level heterogeneity can be determined from discrete frailty models. Continuous frailty distributions must involve risks. Aalen (1992) used the discrete model to explain heterogeneity risks in a data-set concerning marriage rates and fertility. In this thesis however continuous frailty models will be considered since the application is in joint life annuity insurance where mortality risk must occur.

2.4 Shared Frailty Model

This model is proposed by Clayton (1978) and is applied to event times of related subjects or observations that fall into geographical clusters such as cities that are presumed to share the same frailty U. A fundamental assumption is that lifetimes are independent with regard to the shared risk. The shared frailty intensity rate is defined as:

$$h_{ij}(t|u_i) = u_i h_0(t) \exp(\underline{\theta}' \underline{z}_{ij}), \tag{2.3}$$

where $h_{ij}(t|u_i)$ is the conditional hazard function for the j^{th} individual in the i^{th} group and u_i is the frailty variable of the i^{th} group. Assuming: $i = 1, 2, ..., n; j = 1, 2, ..., k$; and $\underline{\theta}' = (\theta_1, \theta_2, \ldots, \theta_k)$.

Fulla & Laurent (2008) propose a shared frailty Gompertz model for the representation of lifetimes stochastic dependence. Non-linear pricing measures and reinsurance premiums are thus computed. Their results shows that even a small amount of dependence can dramatically increase risk measures. The shared model has been further adopted by Hanagal & Alok (2013) in the analysis of kidney infection data.

Shared Frailty and Archimedean Copula Approach
The copula approach is widely applied in actuarial literature to model joint-life's survivor rates (Carriere (2000); Arias & Cirillo (2021)). We discuss similarity between the copula and frailty methodologies by applying the Archimedean copula approach. The Archimedean copula families (Cossette *et al.,* (2017); Li & Lu (2018)) is outlined in relation to a generator function $\tau(.)$.

$$C[x_1,\ldots,x_n] = \tau\left(\sum_{j=1}^{n}\theta(x_j)\right).$$

In the bivariate case:

$$C_\phi[y,x] = \tau\left((\theta(y)+\theta(x))\right). \tag{2.4}$$

The generator $\tau(.)$ denotes a decreasing positive function and positive second derivative with $\tau(0) = 1$ and $\theta(.)$ its pseudo-inverse function. When $\tau(s) = \mathbb{L}_U(s)$ and $y = S(t_{i1}), x = S(t_{i2})$ the shared frailty model is obtained with U being the unreported shared risk. If $\tau(.)$ denotes a gamma (scale=1) distribution Laplace, the result is a Clayton copula model having a gamma-distributed frailty. If $\tau(.)$ denotes an identifiable positive stable distribution Laplace, then the Gumbel-Hougaard copula is obtained having a positive stable distributed frailty. However, the modeling of the marginals differs in both approaches leading to different joint survivor functions (Goethals *et al.,* (2010)).

To incorporate the copula methodology the researcher has to specify both the association structure and survivor marginals (Nelsen (2007); Czado *et al.,* (2012)). Whereas in the shared frailty approach association is introduced indirectly via continuous mixtures.

2.5 Frailty (Mixing) Distributions

A frailty distribution is selected considering mathematical suitability i.e. closed-form expression is required for Laplace transform. Choices can vary widely. Congdon (1995) shows that patterns of change in mortality and age slope against intensity rate is dependent on the intensities analytic form, the frailty distribution and the level of heterogeneity. Therefore the specific application of the model determines the choice of frailty model. Other criteria for choosing between different distributions include; checking for a low mean squared error and high goodness of fit indicator. All arguments and counter-arguments are based on mathematical scenarios no biological reason exists. For further explanation of the frailty methodology see Hougaard (2000) and Wienke (2011).

Vaupel *et al.,* (1979) discusses the effect of heterogeneity on Sweden times-to-death dataset using the gamma mixture. Their findings showed that the present standard life tables overestimate life expectancy and the effect of public health programs and safety precautions on life expectancy and underestimates aging rates. This model is applied by various researchers see for instance, Avanzi *et al.,* (2015) and Eriksson & Scheike (2015). Clayton (1978) further introduces a shared frailty approach to examine impact of the same type of failure in a related pair of individuals using the gamma mixture as frailty distribution. A unique property of the gamma frailty is that its coefficient of variation and cross-ratio function is constant with age. This implies constant frailty from birth to death.

Hougaard (2000) applies the work to other distributions to model heterogeneity that are consistent with the CPHM. Distributions that describe "U" as $\exp(\theta x)$. In particular, the inverse Gaussian is considered. Hanagal & Sharma (2015) compare the inverse Gaussian and gamma as shared frailty distributions for bivariate survivor data-sets. Their findings showed that the inverse Gaussian fits better to the bivariate data-set. The frailty mixture has also been applied by; Whitmore & Lee (1991), Su & Sherris (2012), Wienke (2011) and Hanagal & Pandey (2015). Unlike the gamma frailty, the inverse Gaussian coefficient of variation decreases with age implying that the population becomes homogeneous with time.

The stable Probability Density Function (PDF) is derived from the III parameter family mixtures see Hougaard (1986). The model is applied to life-table construction for heterogeneous lives. Qiou *et al.,* (1999) and Nalini & Dey (2000) discuss models for associated multivariate survival data-sets by applying positive stable mixtures to a data-set on survival times for patients with kidney infection. Their choice was based on the idea that the stable model permits the proportional-hazards to apply conditionally as well as unconditionally. Secondly, the first and second moments of the positive stable mixture is infinite. This enables greater level of heterogeneity to be modeled that could not have been possible to account for using a finite variance distribution. Further, when covariates are present the association and heterogeneity parameters are confounded (Clayton & Cuzick (1985)). Elbers & Ridder (1982) shows that this drawback is present when considering finite mean frailty model. Positive stable models have not been adopted in joint-life annuity valuation. However, most applications have been in medical and bio-statistical fields such as survival times for patients with kidney infection or in myocardial infarction studies.

McGilchrist & Aisbett (1991) considers regression in frailty modeling where times-to-event occurs more than once using the lognormal as a frailty distribution. One major drawback is that the unconditional likelihood has no explicit form. However, increased computational power has overcome such limitations. The distribution has also been adopted as a frailty mixture by Wienke *et al.,* (2003). The compound Poisson distribution (CPD) accounts for zero-susceptibility and susceptible risks on the event of interest. Aalen (1992) proposes the compound Poisson frailty and applies it to a zero-susceptible study subjects. The model is then fitted to a data-set concerning marriage rates and fertility. The CPD has also been applied in bivariate frailty models by Wienke *et al.,* (2010).Rocha (1994) considers modeling heterogeneity for a zero susceptible group using the non central chi-squared (NCC) frailty. The Non-Central Gamma (NCG) is regarded as a general form of the NCC mixture.

Aalen & Tretli (1999) considers the compound negative binomial distribution (CNBD) to analyse testis cancer where $X \sim$ denotes damages caused and $N \sim$ frequency of damages. The CNBD represents the number of damages until outset of cancer. Hanagal & Alok (2013) apply the CNBD to kidney infection-related joint survivor dataset adopted by McGilchrist & Aisbett (1991) to assess non-susceptible cases. The CNBD as applied is useful to cases where there is possibility of zero susceptibility on the event of interest. Hakon *et al.,* (2003) generalized the standard frailty models by modeling risk as a weighted Levy process. Thus, the individuals' risk is assumed to evolve with time and not constant value. Tron & Aalen (2009) have adopted the Levy distribution on hierarchical models in the study of infants mortality in a Norwegian data-set.

Table 2.1: Summary of Frailty Distributions in Literature.

Frailty Distributions	**Reasons**
1. Gamma. (Vaupel *et al.*, (1979); Hanagal & Pandey (2015))	Chosen due to mathematical tractability and convenience.
2. Inverse Gaussian. (Whitmore & Lee (1991); (Su & Sherris (2012))	To model frailty that is dependent on age e.g. when survivors become homogeneous with time. Also useful in modeling various dependence structures in the data.
3. Positive Stable (Hougaard (1986); Nalini & Dey (2000))	Preserves population's proportional-hazards assumption to apply conditionally as well as unconditionally.
4. Lognormal (McGilchrist & Aisbett (1991); (Gustafson (1997))	Easier estimation of partial likelihood procedures. Used to mimic random normal effects from a linear mixed model.
5. Compound Poisson (Aalen (1992); Hanagal (2020))	Has a positive probability of zero frailty, useful when individuals may be immune to the event of interest.
6. Non Central Chi-squared (Rocha (1994))	Has a positive probability of zero frailty, useful when individuals may be immune to the event of interest.
7. Compound Negative Binomial (Aalen & Tretli (1999); Hanagal & Alok (2013))	Useful when the frequency of events occurring is not constant.
8. Levy (Tron & Aalen (2009))	Used when risk is assumed to evolve with time and not a constant value.

Identifiability

The unreported and integrated frailty U results in a model with unspecified components. This raises the question of identifiability, that is, whether the frailty or distribution due to age is distinguishable. Following research by Elbers & Ridder (1982) we can explain identifiability if valid assumptions about frailty-related probability distributions are considered.

In the univariate case, the frailty distribution is presumed to have mean=1 and finite unknown variance to ensure the model is identifiable. However, for multivariate survival data, the intensity function can be calculated using mean frailty distributions that are finite or infinite.

Covariates for shared frailty distributions with finite mean have confounding effect on heterogeneity (Elbers & Ridder (1982)). To eliminate the effect of these confounds, a positive stable frailty model with infinite mean is used. The use of this distribution also allows for even greater heterogeneity among the groups, since the variance is theoretically infinite.

2.6 Base Force of Mortality Distributions

For annuity valuation purposes parametric base force of mortality estimation is desired (see Frees *et al.,* (1996)). The chosen distribution is largely selected as per the nature of the accessed data and the objectives of the study. For instance, the Gompertz distribution (Wienke (2011)) is widely applied to model human mortality as it fits well to middle and old age intensity rates. For cases where the intensity is presumed to remain constant e.g. mortality for individuals who remain in good health in a population then the exponential model (Qiou *et al.,* (1999)) is applied. For most applications in reliability studies where the intensity rate is monotonically rising, declining or unchanged the Weibull (Santos & Achcar (2010)) is utilized. The loglogistic distribution (Hanagal & Sharma (2015)) and lognormal provides a flexible functional form and can achieve non-monotonic shapes i.e. bathtub-shaped or hump-shaped. The humped hazard can model patients who are at a higher mortality risk at the initial stages of infection.

The generalized distributions on the other hand, such as the generalized Weibull (Hanagal & Pandey (2015)), generalized exponential (Hanagal & Alok (2013)),generalized loglogistic (Hanagal & Pandey (2015)) and generalized Pareto (Arvind *et al.,* (2018)) models are applied as an improvement of the aforementioned models. These distributions give higher flexibility in modeling and are used in instances where the intensity is expected to either rise, decline or remain unchanged. Further, representing non-monotonic and/or unimodal intensity rates.

Chapter 3

Construction of Generalized Inverse Gaussian Distribution and their Properties

3.1 Introduction

We wish to construct and obtain distributions that will be used in this study. The distributions are expressed in terms of probability density functions and Laplace transforms used to derive the means and variances. Generalized inverse-Gaussian distribution (GIG), its special cases and limiting cases are discussed. Other distributions considered are: positive stable, compound Poisson and compound negative binomial distributions.

In literature there are a number of parametrizations and transformations that have been used. In this chapter we are using Sichel (1974) approach since the other parametrizations are as a result of Sichel's parametrization and also due to mathematical convenience.

3.2 The GIG Frailty

The frailty can be constructed under various parametrizations. For instance; considering Sichel (1974) parametrization $\epsilon = \sqrt{\varpi\vartheta}$.
Let $K_\eta(\epsilon)$ denote a modified Bessel function of III kind, where ϵ and η are the order and index, respectively.

$$K_\eta(\epsilon) = \frac{1}{2}\int_0^\infty x^{\eta-1}\exp\left(-\frac{\epsilon}{2}(x+\frac{1}{x})\right)dx. \tag{3.1}$$

Under the parametrization

$$K_\eta(\sqrt{\varpi\vartheta}) = \frac{1}{2}\int_0^\infty x^{\eta-1}\exp\left(-\frac{\sqrt{\varpi\vartheta}}{2}(x+\frac{1}{x})\right)dx.$$

The transformation $x=\sqrt{\frac{\varpi}{\vartheta}}u$, $dx=\sqrt{\frac{\varpi}{\vartheta}}du$, yields

$$K_\eta(\sqrt{\varpi\vartheta}) = \frac{1}{2}\int_0^\infty \left(\sqrt{\frac{\varpi}{\vartheta}}\right)^\eta u^{\eta-1}\exp\left(-\frac{\sqrt{\varpi\vartheta}}{2}(\sqrt{\frac{\varpi}{\vartheta}}u+\frac{1}{\sqrt{\frac{\varpi}{\vartheta}}u})\right)du,$$

$$1 = \frac{1}{2}\int_0^\infty \frac{(\sqrt{\frac{\varpi}{\vartheta}})^\eta u^{\eta-1}\exp\left(-\frac{1}{2}(\varpi u+\frac{\vartheta}{u})\right)du}{K_\eta(\sqrt{\varpi\vartheta})}.$$

It can be seen that the above expression gives the density function say, $g(u)$.

$$g(u) = \frac{(\sqrt{\frac{\varpi}{\vartheta}})^\eta}{2K_\eta(\sqrt{\varpi\vartheta})}u^{\eta-1}\exp\left(-\frac{1}{2}(\varpi u+\frac{\vartheta}{u})\right); u>0, \vartheta\geq 0, \varpi\geq 0, -\infty<\eta<\infty. \tag{3.2}$$

The above Equation (3.2) is the GIG density function. Thus $U \sim GIG(\vartheta,\varpi,\eta)$

Proposition 3.1 *The Laplace transform of the GIG distribution is*

$$\mathbb{L}_U(s) = \left(\sqrt{\frac{\varpi}{2s+\varpi}}\right)^\eta \frac{K_\eta(\sqrt{(2s+\varpi)\vartheta})}{K_\eta(\sqrt{\varpi\vartheta})}.$$

Proof.

$$\begin{aligned}
\mathbb{L}_U(s) &= \int_0^\infty \exp(-su)\frac{(\sqrt{\varpi/\vartheta})^\eta}{2K_\eta(\sqrt{\varpi\vartheta})}u^{\eta-1}\cdot\exp\left(-\frac{1}{2}(\varpi u+\frac{\vartheta}{u})\right)du,\\
&= \frac{(\sqrt{\varpi/\vartheta})^\eta}{2K_\eta(\varpi\vartheta)}\int_0^\infty u^{\eta-1}\exp\left(-\frac{1}{2}[u(2s+\varpi)+\frac{\vartheta}{u}]\right)du,\\
&= \frac{(\sqrt{\varpi/\vartheta})^\eta}{2K_\eta(\varpi\vartheta)}\frac{2K_\eta\sqrt{(2s+\varpi)\vartheta}}{(\sqrt{\frac{2s+\varpi}{\vartheta}})^\eta},\\
&= \left(\sqrt{\frac{\varpi}{2s+\varpi}}\right)^\eta \frac{K_\eta(\sqrt{(2s+\varpi)\vartheta})}{K_\eta(\sqrt{\varpi\vartheta})}. \qquad (3.3)
\end{aligned}$$

□

3.3 Special Cases of GIG

The special cases of the GIG presented in this section includes: the inverse gaussian, reciprocal inverse gaussian, harmonic and positive hyperbolic distributions.

Inverse Gaussian (IG) Distribution:
Let $U \sim IG(\varpi, \vartheta)$ the PDF is obtained from the GIG when $\eta = -\frac{1}{2}$.

$$f(u) = \frac{(\sqrt{\frac{\varpi}{\vartheta}})^{-1/2}}{2K_{-1/2}(\sqrt{\varpi\vartheta})} u^{-3/2} \exp\left(-\frac{1}{2}(\varpi u + \frac{\vartheta}{u})\right); u > 0, \vartheta, \varpi \geq 0.$$

Applying the Bessel function's even and recursive relation (Sichel (1974))

$$K_{1/2}(\epsilon) = K_{-1/2}(\epsilon) = \sqrt{\frac{\pi}{2\epsilon}} \exp(-\epsilon),$$

substituting for $\epsilon = \sqrt{\varpi\vartheta}$ we get

$$\begin{aligned} f(u) &= \frac{(\sqrt{\frac{\varpi}{\vartheta}})^{-1/2}}{2[\frac{\pi}{2\sqrt{\varpi\vartheta}}]^{1/2} \exp(-\sqrt{\varpi\vartheta})} u^{-3/2} \exp\left(-\frac{1}{2}(\varpi u + \frac{\vartheta}{u})\right), \\ &= \sqrt{\frac{\vartheta}{2\pi u^3}} \exp(\sqrt{\varpi\vartheta}) \exp\left(-\frac{1}{2}(\varpi u + \frac{\vartheta}{u})\right). \end{aligned} \tag{3.4}$$

The subsequent, result gives the closed form of the corresponding IG Laplace transform derived from a GIG distribution when $\eta = -\frac{1}{2}$, see Sichel (1974).

Proposition 3.2 *To ensure the model is identifiable, the IG distribution Laplace transform is expressed as:*

$$\mathbb{L}_U(s) = \exp\left(\frac{1 - (1 + 2s\sigma^2)^{1/2}}{\sigma^2}\right).$$

Proof. When $\eta = -\frac{1}{2}$ in Equation (3.3)

$$
\begin{aligned}
\mathbb{L}_U(s) &= \left(\sqrt{\frac{\varpi}{\varpi+2s}}\right)^{-1/2} \frac{K_{-1/2}(\sqrt{(\varpi+2s)\vartheta})}{K_{-1/2}(\sqrt{\varpi\vartheta})}, \\
&= \left(\sqrt{\frac{\varpi}{2s+\varpi}}\right)^{-1/2} \frac{[\frac{\pi}{2\sqrt{(2s+\varpi)\vartheta}}]^{1/2}\exp\left(-\sqrt{(2s+\varpi)\vartheta}\right)}{[\frac{\pi}{2\sqrt{\varpi\vartheta}}]^{1/2}\exp\left(-\sqrt{\varpi\vartheta}\right)}, \\
&= \left(\sqrt{\frac{\varpi}{2s+\varpi}}\right)^{-1/2} \left(\sqrt{\frac{\varpi}{2s+\varpi}}\right)^{1/2} \exp\left(\sqrt{\varpi\vartheta} - \sqrt{(2s+\varpi)\vartheta}\right), \\
&= \exp\left(\sqrt{\varpi\vartheta} - \sqrt{(2s+\varpi)\vartheta}\right). \qquad (3.5)
\end{aligned}
$$

Using the parametrizations $\varpi = \frac{1}{\Theta^2}, \vartheta = \mu^2$ leads to

$$
\mathbb{L}_U(s) = \exp\left(-\frac{\mu}{\Theta}[(1+2\Theta^2 s)^{1/2} - 1]\right). \qquad (3.6)
$$

$$
E[U] = -\frac{d}{ds}\mathbb{L}_U(s)|_{s=0} = -\mathbb{L}'_U(s)|_{s=0} = \mu\Theta,
$$

$$
var[U] = \frac{d^2}{ds^2}\mathbb{L}_U(s)|_{s=0} - (E[U])^2 = \mathbb{L}''_U(s)|_{s=0} - (\mu\Theta)^2 = \mu\Theta^3.
$$

To ensure the model is identifiable set the mean $E[U] = 1$. That is; $\mu\Theta = 1$ implying $\mu = \frac{1}{\Theta}$ thus the variance is $\sigma^2 = \Theta^2$. The Laplace therefore becomes

$$
\mathbb{L}_U(s) = \exp\left(\frac{1 - (1+2s\sigma^2)^{1/2}}{\sigma^2}\right). \qquad (3.7)
$$

□

Reciprocal Inverse Gaussian (RIG) Distribution

Given that $U = \frac{1}{X}$ where $X \sim IG(\varpi\vartheta)$ then U is said to be the reciprocal of the IG. The density is obtained from the GIG when $\eta = \frac{1}{2}$.

$$
\begin{aligned}
f(u) &= \frac{\left(\sqrt{\frac{\varpi}{\vartheta}}\right)^{\frac{1}{2}}}{2K_{\frac{1}{2}}(\sqrt{\varpi\vartheta})} u^{-\frac{1}{2}} \exp\left(-\frac{1}{2}(\varpi u + \frac{\vartheta}{u})\right), \\
&= \frac{\left(\sqrt{\frac{\varpi}{\vartheta}}\right)^{\frac{1}{2}}}{2[\frac{\pi}{2\sqrt{\varpi\vartheta}}]^{1/2}\exp\left(-\sqrt{\varpi\vartheta}\right)} u^{-\frac{1}{2}} \exp\left(-\frac{1}{2}(\varpi u + \frac{\vartheta}{u})\right), \\
&= \sqrt{\frac{2\varpi}{u\pi}} \exp\left(\sqrt{\varpi\vartheta}\right) \exp\left(-\frac{1}{2}(\varpi u + \frac{\vartheta}{u})\right). \qquad (3.8)
\end{aligned}
$$

Proposition 3.3 *The RIG distribution Laplace transform allowing for identifiability is*

$$\mathbb{L}_U(s) = (1+2s\Theta^2)^{-1/2}\exp\left(\frac{1-\Theta^2}{\Theta^2}[1-(1+2\Theta^2 s)^{1/2}]\right).$$

Proof. When $\eta = \frac{1}{2}$ in Equation (3.3)

$$\begin{aligned}\mathbb{L}_U(s) &= \left(\sqrt{\frac{\varpi}{2s+\varpi}}\right)^{1/2}\frac{K_{1/2}(\sqrt{(2s+\varpi)\vartheta})}{K_{1/2}(\sqrt{\varpi\vartheta})},\\ &= \left(\sqrt{\frac{\varpi}{2s+\varpi}}\right)^{1/2}\frac{[\frac{\pi}{2\sqrt{(2s+\varpi)\vartheta}}]^{1/2}\exp(-\sqrt{(2s+\varpi)})}{[\frac{\pi}{2\sqrt{\varpi\vartheta}}]^{1/2}\exp(-\sqrt{\varpi\vartheta})},\\ &= \sqrt{\frac{\varpi}{2s+\varpi}}\exp(\sqrt{\varpi\vartheta}-\sqrt{(2s+\varpi)\vartheta}). \qquad (3.9)\end{aligned}$$

Substituting $\varpi = \frac{1}{\Theta^2}, \vartheta = \mu^2$ leads to

$$\mathbb{L}_U(s) = (1+2s\Theta^2)^{-1/2}\exp\left(\frac{\mu}{\Theta}[1-(1+2\Theta^2 s)^{1/2}]\right). \qquad (3.10)$$

$$E[U] = -\mathbb{L}'_U(s)|_{s=0} = \Theta^2 + \mu\Theta.$$

To ensure the model is identifiable set the mean $E[U] = 1$. That is; $\Theta^2+\mu\Theta = 1$ and $\therefore \quad \mu = \frac{1-\Theta^2}{\Theta}$.

$$\begin{aligned}var[U] &= \mathbb{L}''_U(s)|_{s=0} - (E[U])^2,\\ &= 3\Theta^4 + \mu\Theta^3 + 2\mu\Theta^3 + (\mu\Theta)^2 - (\Theta^2+\mu\Theta)^2 = \Theta^2(2\Theta^2+\mu\Theta).\end{aligned}$$

Substituting $\mu = \frac{1-\Theta^2}{\Theta}$ leads to $\sigma^2 = \Theta^2(\Theta^2+1)$.
Laplace transform becomes

$$\mathbb{L}_U(s) = (1+2s\Theta^2)^{-1/2}\exp\left(\frac{1-\Theta^2}{\Theta^2}[1-(1+2\Theta^2 s)^{1/2}]\right). \qquad (3.11)$$

□

Harmonic Distribution
The Harmonic is derived from the GIG distribution when $\eta = 0, \vartheta = an, \varpi = \frac{a}{n}$.

$$f(u) = \frac{u^{-1} \exp\{-\frac{a}{2}(\frac{u}{n} + \frac{n}{u})\}}{2K_0(a)}; u > 0, \varpi \geq 0, \vartheta \geq 0. \tag{3.12}$$

Proposition 3.4 *To ensure the model is identifiable, the harmonic distribution Laplace transform is described as*

$$\mathbb{L}_U(s) = \frac{K_1(\sqrt{2ans + a^2})}{nK_1(a)}.$$

Proof. Letting $\eta = 0, \vartheta = an, \varpi = \frac{a}{n}$ in the GIG Laplace we obtain

$$\mathbb{L}_U(s) = \frac{K_0(\sqrt{2ans + a^2})}{K_0(a)}. \tag{3.13}$$

To ensure the model is identifiable set the mean $E[U] = 1$.

$$f(u) = \frac{u^{-1} \exp\{-\frac{1}{2}(\frac{au}{n} + \frac{an}{u})\}}{2K_0(a)}.$$

$$\begin{aligned} E[U] &= \int_0^\infty \frac{\exp\{-\frac{1}{2}(\frac{au}{n} + \frac{an}{u})\}}{2K_0(a)} du \\ &= \frac{2K_1(\sqrt{\frac{a}{n} \times an})}{2K_0(a)\sqrt{\frac{a}{n} \times \frac{1}{an}}}, \\ &= \frac{nK_1(a)}{K_0(a)} = 1. \end{aligned}$$

The Laplace becomes

$$\mathbb{L}_U(s) = \frac{K_1(\sqrt{2ans + a^2})}{K_1(a)}. \tag{3.14}$$

□

Positive Hyperbolic Distribution
The Positive Hyperbolic is derived from the GIG distribution when $\eta = 1$.

$$f(u) = \frac{\sqrt{\frac{\varpi}{\vartheta}}}{2K_1(\sqrt{\varpi\vartheta})} \exp\left(-\frac{1}{2}[\varpi u + \frac{\vartheta}{u}]\right); u > 0, \vartheta, \varpi \geq 0. \tag{3.15}$$

Proposition 3.5 *For identifiability reasons, Equation (3.15) Laplace transform is given by*

$$\mathbb{L}_U(s) = \left(\frac{\varpi}{2s + \varpi}\right) \frac{K_2(\sqrt{(\varpi + 2s)\vartheta})}{K_2(\sqrt{\varpi\vartheta})}.$$

Proof. Letting $\eta = 1$ in the GIG Laplace we obtain

$$\mathbb{L}_U(s) = \left(\sqrt{\frac{\varpi}{2s + \varpi}}\right) \frac{K_1(\sqrt{(2s + \varpi)\vartheta})}{K_1(\sqrt{\varpi\vartheta})}. \tag{3.16}$$

To ensure the model is identifiable set the mean $E[U] = 1$.

$$\begin{aligned} E[U] &= \frac{\sqrt{\frac{\varpi}{\vartheta}}}{2K_1(\sqrt{\varpi\vartheta})} \int_0^\infty u \exp\left(-\frac{1}{2}[\varpi u + \frac{\vartheta}{u}]\right) du, \\ &= \frac{\sqrt{\frac{\varpi}{\vartheta}}}{2K_1(\sqrt{\varpi\vartheta})} \cdot \frac{2K_2(\sqrt{\varpi\vartheta})}{(\sqrt{\frac{\varpi}{\vartheta}})^2}, \\ &= \frac{\sqrt{\frac{\vartheta}{\varpi}} K_2(\sqrt{\varpi\vartheta})}{K_1(\sqrt{\varpi\vartheta})} = 1. \end{aligned}$$

The Laplace becomes

$$\mathbb{L}_U(s) = \left(\frac{\varpi}{2s + \varpi}\right) \frac{K_2(\sqrt{(\varpi + 2s)\vartheta})}{K_2(\sqrt{\varpi\vartheta})}. \tag{3.17}$$

□

3.4 Limiting Cases of GIG

The limiting cases of the GIG presented includes: the gamma, inverse gamma and levy distributions.

Gamma Distribution
The PDF is a limiting case of the GIG obtained when $\vartheta = 0; \eta, \varpi > 0$.

$$\begin{aligned} f(u) &= \frac{u^{\eta-1}\exp\{-\frac{1}{2}(\varpi u + \frac{\vartheta}{u})\}}{\int_0^\infty u^{\eta-1}\exp\{-\frac{1}{2}(\varpi u + \frac{\vartheta}{u})\}du}; u > 0, \vartheta, \varpi \geq 0, -\infty < \eta < \infty. \\ &= \frac{u^{\eta-1}\exp\{-\frac{1}{2}(\varpi u)\}}{\int_0^\infty u^{\eta-1}\exp\{-\frac{1}{2}(\varpi u)\}du}, \\ &= \frac{(\frac{1}{2}\varpi)^\eta}{\Gamma(\eta)} u^{\eta-1}\exp\{-\frac{1}{2}(\varpi u)\}. \end{aligned} \tag{3.18}$$

Proposition 3.6 *Equation (3.18) Laplace transform allowing for identifiability is*

$$\mathbb{L}(s) = (1 + s\sigma^2)^{-1/\sigma^2}.$$

Proof. When $\vartheta = 0; \eta, \varpi > 0$ in Equation (3.3)

$$\begin{aligned} \mathbb{L}_U(s) &= \frac{(\sqrt{\frac{\varpi}{\varpi+2s}})^\eta \cdot K_\eta(\sqrt{(\varpi+2s)\cdot 0})}{K_\eta(\sqrt{\varpi \cdot 0})}, \\ &= \left(\frac{\varpi}{\varpi + 2s}\right)^{\eta/2}. \end{aligned} \tag{3.19}$$

Using the parametrization $\varpi = 2\beta$; $\frac{\eta}{2} = k$. Equation (3.19) becomes

$$\mathbb{L}_U(s) = \left(1 + \frac{s}{\beta}\right)^{-k}. \tag{3.20}$$

The mean frailty at birth

$$E[U] = -\mathbb{L}'_U(s)|_{s=0} = \frac{k}{\beta}\left(1 + \frac{s}{\beta}\right)^{-k-1}|_{s=0} = \frac{k}{\beta}.$$

$$var[U] = \mathbb{L}_U''(s)|_{s=0} - (E[U])^2,$$
$$= \frac{-k}{\beta^2}(-k-1)\left(1+\frac{s}{\beta}\right)^{-k-2} - \left(\frac{k}{\beta}\right)^2|_{s=0},$$
$$= \frac{k}{\beta^2}.$$

The coefficient of variation (CV)

$$CV(U) = \frac{\sqrt{var[U]}}{E[U]} = \frac{1}{\sqrt{k}}.$$

If $k \to \infty$, then $CV(v) \to 0$, i.e. the population can be considered homogeneous.
To ensure the model is identifiable set the mean $E[U] = 1$ ($k = \beta$) and $\sigma^2 = \frac{1}{\beta}$.
Thus

$$\mathbb{L}_U(s) = (1+s\sigma^2)^{-1/\sigma^2}. \tag{3.21}$$

□

Inverse Gamma Distribution
The case when $\eta < 0, \vartheta > 0, \varpi = 0$

$$f(u) = \frac{u^{\eta-1}\exp(-\frac{\vartheta}{2u})}{\int_0^\infty u^{\eta-1}\exp(-\frac{\vartheta}{2u})du}. \tag{3.22}$$

Let

$$x = \int_0^\infty u^{\eta-1}\exp\left(-\frac{\vartheta}{2u}\right)du$$

Using the transformation

$$u = \frac{1}{y}, du = -\frac{dy}{y^2}.$$

$$x = \int_0^\infty \left(\frac{1}{y}\right)^{\eta-1}\exp\left(-\frac{\vartheta}{2}y\right)\left(\frac{dy}{y^2}\right),$$
$$= \int_0^\infty \left(\frac{1}{y}\right)^{\eta+1}\exp\left(-\frac{\vartheta}{2}y\right)dy = \frac{\Gamma(-\eta)}{(\vartheta/2)^{-\eta}},$$

$$\therefore \qquad f(u) = \frac{(\vartheta/2)^{-\eta}}{\Gamma(-\eta)} u^{\eta-1} \exp\left(-\frac{\vartheta}{2u}\right); u > 0. \tag{3.23}$$

Let $\eta = -\beta$ where $\beta > 0$

$$f(u) = \frac{(\vartheta/2)^{\beta}}{\Gamma(\beta)} u^{-\beta-1} \exp\left(-\frac{\vartheta}{2u}\right). \tag{3.24}$$

This is an inverse gamma density function.

Proposition 3.7 *The inverse gamma Laplace transform allowing for identifiability is*

$$\mathbb{L}_U(s) = 2\frac{(\beta-1)^{\beta}}{\Gamma(\beta)} \cdot \left(\sqrt{\frac{\beta-1}{s}}\right)^{-\beta} \cdot K_{-\beta}(\sqrt{4s(\beta-1)})$$

Proof. The Laplace transform is

$$\begin{aligned}
\mathbb{L}_U(s) &= \frac{(\vartheta/2)^{\beta}}{\Gamma(\beta)} \int_0^{\infty} u^{-\beta-1} \exp\left(-\frac{1}{2}[2su + \frac{\vartheta}{u}]\right) du, \\
&= 2\frac{(\vartheta/2)^{\beta}}{\Gamma(\beta)} \frac{K_{-\beta}(\sqrt{2s\vartheta})}{(\sqrt{\frac{2s}{\vartheta}})^{-\beta}}
\end{aligned}$$

$$\mathbb{L}_U(s) = 2\frac{(\sqrt{\vartheta s/2})^{\beta}}{\Gamma(\beta)} K_{-\beta}(\sqrt{2s\vartheta}). \tag{3.25}$$

To ensure the model is identifiable set the mean $E[U] = 1$.

$$E[U] = \frac{(\vartheta/2)^{\beta}}{\Gamma(\beta)} \int_0^{\infty} u^{-\beta} \exp\left(\frac{-\vartheta}{2u}\right) du.$$

Let

$$u = \frac{1}{y}, du = -\frac{1}{y^2} dy.$$

$$E[U] = \frac{(\vartheta/2)^{\beta}}{\Gamma(\beta)} \int_0^{\infty} y^{\beta-2} \exp\left(-\frac{y\vartheta}{2}\right) dy.$$

Let

$$x = \frac{y\vartheta}{2}, dx = \frac{\vartheta}{2} dy.$$

$$E[U] = \frac{(\vartheta/2)^{\beta}}{\Gamma(\beta)} \int_0^{\infty} \frac{x^{\beta-2}}{(\vartheta/2)^{\beta-1}} \exp(-x)dx = \frac{\vartheta\Gamma(\beta-1)}{2(\beta-1)\Gamma(\beta-1)}.$$

Since

$$E[U] = \frac{\vartheta}{2(\beta-1)} = 1 \therefore \quad \vartheta = 2(\beta-1)$$

The Laplace becomes

$$\mathbb{L}_U(s) = 2\frac{(\sqrt{(\beta-1)s})^{\beta}}{\Gamma(\beta)} K_{-\beta}(\sqrt{4s(\beta-1)}). \tag{3.26}$$

□

The Levy Distribution

The Levy is obtained from the inverse gamma whose PDF derived in Equation (3.23)

$$f(t) = \frac{(\vartheta/2)^{-\eta}}{\Gamma(-\eta)} t^{\eta-1} \exp\left(-\frac{\vartheta}{2t}\right),$$

when $\eta = -\frac{1}{2}; \vartheta = \mu^2$ the Levy density is obtained

$$\begin{aligned} f(t) &= \frac{(\mu^2/2)^{1/2}}{\Gamma(1/2)} \exp\left(\frac{-\mu^2}{2t}\right) t^{-1-1/2}, \\ &= \sqrt{\frac{\mu^2/2}{t^3 2\pi}} \exp\left(\frac{-\mu^2}{2t}\right). \end{aligned} \tag{3.27}$$

Proposition 3.8 *To ensure the model is identifiable, the Laplace transform for Equation (3.27) is*

$$\mathbb{L}_U(s) = \exp(-\sqrt{2s}).$$

Proof. Substituting $\varpi = 0$ in Equation (3.5) the Laplace of the Levy is obtained

$$\mathbb{L}_U(s) = \exp(-\sqrt{2s\mu^2}). \tag{3.28}$$

To ensure the model is identifiable set the mean $E[U] = 1$.

The Laplace becomes

$$\mathbb{L}_U(s) = \exp(-\sqrt{2s}). \tag{3.29}$$

□

3.5 Other Frailty Distributions

Other distributions presented are the compound Poisson, positive stable, compound negative binomial and non-central gamma distributions.

Compound Poisson Distribution
Let $N \sim$ Poisson $(\rho > 0)$ and let $X_j, j = 1, 2, \ldots$ be identically distributed random variables, independent of N. $U \sim$ CPD expressed as:

$$U = \begin{cases} X_1 + \ldots + X_N, & N > 0 \\ 0, & N = 0 \end{cases}$$

Proposition 3.9 *The CPD Laplace transform with gamma distributed random variables is*

$$\mathbb{L}_U(s) = \exp\left(\frac{-k}{\alpha}[(\beta + s)^\alpha - \beta^\alpha]\right). \tag{3.30}$$

Proof.

$$\begin{aligned} \mathbb{L}_U(s) &= E[\exp(-sU)] = E\{E[\exp(-s(X_1 + \ldots + X_N))|N = n]\}, \\ &= E\{E[(\exp(-sX) \times \exp(-sX) \times \ldots \times \exp(-sX))]\}, \\ &= E\{[E(\exp(-sX))]^n\}, \\ &= \mathbb{F}(\mathbb{L}_X(s)) = \exp\{\rho(\mathbb{L}_X(s) - 1)\}. \end{aligned} \tag{3.31}$$

Since $X \sim \Gamma(k, \beta)$ applying its Laplace transform we have

$$\mathbb{L}_U(s) = \exp\left(\rho[(1 + \frac{s}{\beta})^{-k} - 1]\right),$$

by parametrization substitute $\rho = \frac{-k\beta^\alpha}{\alpha}$ and $k = -\alpha$.

$$\begin{aligned} \mathbb{L}_U(s) &= \exp\left(\frac{-k\beta^\alpha}{\alpha}[(1 + \frac{s}{\beta})^\alpha - 1]\right), \\ &= \exp\left(\frac{-k}{\alpha}[(\beta + s)^\alpha - \beta^\alpha]\right). \end{aligned} \tag{3.32}$$

□

For $0 < \alpha \leq 1$ the Power-Variance-Function (PVF) distributions is attained. For $\alpha < 0$ the CPD is obtained, these two sub-classes are separated by the gamma distribution $\alpha = 0$.
The first raw moment and second central moment is given by

$$
\begin{aligned}
E[U] &= -\mathbb{L}'_U(s) = k(\beta+s)^{\alpha-1}\exp\left(\frac{-k}{\alpha}[(\beta+s)^{\alpha}-\beta^{\alpha}]\right),\\
&= -\mathbb{L}'_U(s)|_{s=0} = k\beta^{\alpha-1}.\\
var[U] &= \mathbb{L}''_U(s)|_{s=0} - (E[U])^2\\
&= -k(\alpha-1)(\beta+s)^{\alpha-2}\exp\left(\frac{-k}{\alpha}[(\beta+s)^{\alpha}-\beta^{\alpha}]\right) + (k(\beta+s)^{\alpha-1})^2\\
&\cdot \exp\left(\frac{-k}{\alpha}[(\beta+s)^{\alpha}-\beta^{\alpha}]\right)|_{s=0} - (k\beta^{\alpha-1})^2,\\
&= -k(\alpha-1)\beta^{\alpha-2}\\
&= \frac{1-\alpha}{\beta}
\end{aligned}
$$

Allowing for identifiability i.e. $E[U] = k\beta^{\alpha-1} = 1$ and $var[U] = \frac{1-\alpha}{\beta}$ the Laplace becomes

$$
\mathbb{L}_U(s) = \exp\left(\frac{\alpha-1}{\alpha\sigma^2}[(1+s\frac{\sigma^2}{1-\alpha})^{\alpha}-1]\right). \tag{3.33}
$$

Positive Stable Distribution
The stable mixture is defined using the parameters described below: skewness $\theta \in [-1,1]$, location μ, scale $k > 0$, and index $\alpha \in (0,2]$ which characterizes the peakedness and tail behavior.

When $\mu = 0$, $\theta = 1$ and $0 < \alpha \leq 1$ a series approximation of the PDF for the positive stable is obtained (Samorodnitsky & Taqqu (1994)):

$$
f(u) = -\frac{1}{\pi u}\sum_{j=1}^{\infty}\frac{\Gamma(j\alpha+1)}{j!}(-u^{-\alpha}k/\alpha)^j\sin(\alpha j\pi); k > 0, u > 0, 0 < \alpha \leq 1. \tag{3.34}
$$

Proposition 3.10 *The positive stable distribution Laplace is obtained as a special case of the three parameter PVF Laplace as*

$$\mathbb{L}(s) = \exp\left(-\frac{k}{\alpha}s^{\alpha}\right). \tag{3.35}$$

Proof. Let $N \sim$ Poisson $(\rho > 0)$ and $U_j, j = 1, 2, 3, \ldots$ denote independent and identically distributed risk variables. Y is CPD represented as:

$$Y = \begin{cases} U_1 + U_2 + \ldots + U_N, & N > 0 \\ 0, & N = 0 \end{cases}$$

The Laplace

$$\begin{aligned} \mathbb{L}_Y(s) &= E[\exp(-sY)] = E[E[\exp\{-s(U_1 + U_2 + \ldots + U_N)\}|N = n]], \\ &= E[E(\exp(-sU))]^n = \mathbb{F}(\mathbb{L}_U(s)), \\ &= \exp\{\rho(\mathbb{L}_U(s) - 1)\}. \end{aligned}$$

Since $U \sim \Gamma(k, \beta)$ applying its Laplace transform we have

$$\mathbb{L}_Y(s) = \exp\left(\rho[(1 + \frac{s}{\beta})^{-k} - 1]\right).$$

Using the parametrization substitute $\rho = \frac{-k\beta^{\alpha}}{\alpha}$ and $k = -\alpha$.

$$\begin{aligned} \mathbb{L}_Y(s) &= \exp\left(\frac{-k\beta^{\alpha}}{\alpha}[(1 + \frac{s}{\beta})^{\alpha} - 1]\right), \\ &= \exp\left(\frac{-k}{\alpha}[(\beta + s)^{\alpha} - \beta^{\alpha}]\right), \end{aligned}$$

when $\beta = 0$

$$\mathbb{L}_U(s) = \exp\left(-\frac{k}{\alpha}s^{\alpha}\right). \tag{3.36}$$

□

To ensure identifiability let $\alpha = k$.

$$\mathbb{L}_U(s) = \exp(-s^{\alpha}). \tag{3.37}$$

Compound Negative Binomial Distribution
The CNBD is represented as:

$$U = \begin{cases} X_1 + \ldots + X_N, & N > 0 \\ 0, & N = 0 \end{cases}$$

Where $X \sim$ Gamma and $N \sim$ Negative Binomial. If $N > 0$, U can be interpreted as collective heterogeneity due to failures before the first α^{th} success. The Laplace transform is

$$\begin{aligned} \mathbb{L}_U(s) &= E[\exp(-sU)] = E\{E[\exp(-s(X_1 + \ldots + X_N))|N = n]\}, \\ &= E\{[E(\exp(-sX))]^n\} = \mathbb{F}(\mathbb{L}_X(s)), \\ &= \left(\frac{p}{1 - q\mathbb{L}_X(s)}\right)^{\alpha} = \left(\frac{p}{1 - q(1 + \frac{s}{\beta})^{-k}}\right)^{\alpha}. \end{aligned} \tag{3.38}$$

Where p denotes success probability and q failure probability of the α success occurrences.
The first raw and second central moments is given by

$$\begin{aligned} E[U] &= -\mathbb{L}'_U(s)|_{s=0} = \frac{\alpha q k}{\beta p}, \\ var[U] &= \mathbb{L}''_U(s)|_{s=0} - (\mathbb{L}'_U(s)|_{s=0})^2, \\ &= \frac{\alpha q^2 k^2 + \alpha q p k + \alpha q p k^2}{(\beta p)^2}, \\ &= \frac{\alpha q^2 k^2 + \alpha q k^2(1-q) + \alpha q p k}{(\beta p)^2}, \\ &= \frac{\alpha q k (p + k)}{(\beta p)^2}. \end{aligned}$$

Allowing for identifiability i.e.

$$\begin{aligned} E[U] &= \frac{\alpha q k}{\beta p} = 1. \\ var[U] &= \frac{p + k}{\beta p}. \end{aligned}$$

The Laplace becomes

$$\mathbb{L}_U(s) = \left(\frac{p}{1 - q(1 + \frac{s\sigma^2 p}{p+k})^{-k}}\right)^{\alpha}. \tag{3.39}$$

Non-Central Gamma Distribution.
The NCG distribution is a special case of the CPD with gamma distributed random variables. The PDF of the variable Y, denoted $f(u)$, where $Y = U_1 + \ldots + U_N$ and $U \sim \Gamma(b, \beta)$ with shape parameter b and scale parameter β. $N \sim Poisson(a\beta)$ with the the non-centrality parameter a leads to the following proposition

Proposition 3.11 *Given that* $Y = U_1 + U_2 + \ldots + U_N$ *with respective weights* $\exp(-\beta)\frac{(\beta)^n}{n!}$ *such that* $U \sim \Gamma(b, \beta)$ *and* $N \sim Poisson(a\beta)$ leads to the convolution;

$$f(u) = \sum_{n=0}^{\infty} \frac{\exp(-a\beta)(a\beta)^n}{n!} \cdot \left[\frac{u^{b+n-1}\exp(-\frac{u}{\beta})}{\Gamma(b+n)\beta^{b+n}}\right]. \tag{3.40}$$

Proof.

$$\begin{aligned} f(u) &= \sum_{n=0}^{\infty} \mathbb{P}(U_1 + U_2 + \ldots + U_N \;\; and \;\; N = n), \\ &= \sum_{n=0}^{\infty} \mathbb{P}(U_1 + U_2 + \ldots + U_N | N = n) \cdot \mathbb{P}(N = n), \\ &= \sum_{n=0}^{\infty} \left[\frac{u^{b-1}\exp(-u/\beta)}{\Gamma(b)\beta^b}\right]^{n^*} \cdot \frac{\exp(-a\beta)(a\beta)^n}{n!}, \end{aligned}$$

where $\left[\frac{u^{b-1}\exp(-u/\beta)}{\Gamma(b)\beta^b}\right]^{n^*}$ is the n^{th} fold convolution power of $\left[\frac{u^{b-1}\exp(-u/\beta)}{\Gamma(b)\beta^b}\right]$.

$$f(u) = \sum_{n=0}^{\infty} \left[\frac{u^{b+n-1}\exp(-u/\beta)}{\Gamma(b+n)\beta^{b+n}}\right] \cdot \frac{\exp(-a\beta)(a\beta)^n}{n!}.$$

□

Proposition 3.12 *The Laplace transform for Equation (3.40) is*

$$\mathbb{L}_U(s) = \frac{1}{(1+\beta s)^b} \exp\left(-\frac{sa\beta^2}{1+\beta s}\right).$$

Proof.

$$\mathbb{L}_U(s) = E[\exp(-sU)] = \int_0^\infty \exp(-su) f(u) du,$$
$$= \exp(-a\beta) \sum_{n=0}^{\infty} \frac{(a\beta)^n}{n!\Gamma(b+n)\beta^{b+n}} \int_0^\infty u^{b+n-1} \exp\left(-u(\frac{1}{\beta}+s)\right) du,$$

Let $y = u\left(\frac{1}{\beta}+s\right)$ *and* $dy = du\left(\frac{1}{\beta}+s\right)$.

$$\mathbb{L}_U(s) = \exp(-a\beta) \sum_{n=0}^{\infty} \frac{(a\beta)^n}{n!\Gamma(b+n)\beta^{b+n}} \int_0^\infty \frac{y^{b+n-1}\exp(-y)}{(\frac{1}{\beta}+s)^{b+n}} dy,$$
$$= \frac{1}{(1+\beta s)^b} \exp\left(-\frac{sa\beta^2}{1+\beta s}\right). \tag{3.41}$$

□

When $b = 0$

$$\mathbb{L}_U(s) = \exp\left(-\frac{sa\beta^2}{1+\beta s}\right). \tag{3.42}$$

Allowing for identifiability i.e.

$$E[U] = -\mathbb{L}'_U(s)|_{s=0} = a\beta^2 = 1.$$

$$var[U] = \mathbb{L}''_U(s)|_{s=0} - (\mathbb{L}'_U(s)|_{s=0})^2 = 2a\beta^3.$$

Let mean=1 $\therefore \sigma^2 = 2\beta$.
The Laplace becomes

$$\mathbb{L}_U(s) = \exp\left(-\frac{s}{1+0.5\sigma^2 s}\right). \tag{3.43}$$

In this chapter we have shown the construction of the PDFs and identifiable Laplace transforms that will be used in the study. In particular, the GIG limiting distributions and the compound distributions will be applied in the subsequent chapters 4 & 5. The Laplace transform is the main tool that will be used to obtain both the univariate and bivariate frailty marginal survival functions.

Chapter 4

Frailty Model

4.1 Univariate Frailty

For completeness of our presentation, we first begin by describing the frailty approach in a univariate context. We start with the derivation of CPHM and then extend it to a univariate frailty approach which is a continuous mixture. The frailty mixture is then presented in terms of the density, intensity and survivor function. There is some similarity between frailty mixture and exponential mixture as shown in Section **4.3**.

4.2 Cox Proportional-Hazards Model

The exponential(β) PDF is

$$f(t|\beta) = \beta \exp\{-\beta t\}, t > 0, \beta > 0.$$

The survivor function

$$S(t|\beta) = \exp\{-\beta t\},$$

intensity function

$$h(t|\beta) = \beta, \tag{4.1}$$

which is a constant.

Suppose

$$\log \beta = \Theta_0 + \Theta_1 x_1 + \Theta_2 x_2 + \ldots + \Theta_n x_n,$$

Then

$$\beta = \beta_0 \exp\{\underline{\Theta}'\underline{x}\},$$

where $\underline{\Theta}' = (\Theta_1, \Theta_2, \ldots, \Theta_n)$ and $\underline{x}' = (x_1, x_2, \ldots, x_n)$.

Replace β by h

$$\therefore \quad h = h_0 \exp\{\underline{\Theta}'\underline{x}\},$$

which implies that

$$h(t) = h_0(t) \exp\{\underline{\Theta}'\underline{x}\}, \tag{4.2}$$

which is the CPHM where $h_0(t)$ is the base force of mortality function $\underline{\Theta}$ are the regressors and $\underline{x}$ the covariates.

4.3 Frailty

Observable demographic risk factors can explain heterogeneity in mortality among assureds; but modeling the risk profile of a heterogeneous life from unobservable risk factors, such as survival-related health-seeking attitudes and/or genetics is complex. The frailty methodology is aimed to account for unobservable covariates affecting mortality in the CPHM.

Construction of Frailty Mixture
Suppose the CPHM is extended to

$$h(t|x, z) = h_0(t) \exp\{\underline{\Theta'x} + \underline{c'z}\},$$

where $\underline{z}$ represents the unobserved covariates and $\underline{c}'$ vector of the corresponding regressors. Then

$$\exp\{\underline{\Theta'x} + \underline{c'z}\} = U, \quad say;$$

where $\underline{\Theta}' = (\Theta_1, \Theta_2, \ldots, \Theta_n)$ and $\underline{c}' = (c_1, c_2, \ldots, c_n)$ and U is a random variable called frailty.

$$\therefore \quad h(t|u) = h_0(t) \exp\{\underline{\Theta'x} + \underline{c'z}\} = h_0(t)u. \tag{4.3}$$

The frailty U is presumed to follow some distribution $g(u)$ with positive support and has a multiplicative effect on the base force of mortality.

$$\therefore \quad h(t) = \int_0^\infty h(t|u)g(u)du.$$

This is a continuous mixture where $h(t)$ is the mixed rate of mortality, $h(t|u)$ is the conditional rate of mortality and $g(u)$ is the mixing distribution which is the frailty distribution.
Thus the conditional survivor is

$$\begin{aligned} S(t|u) &= \exp\{-\int_0^t h(x|u)dx\}, \\ &= \exp\{-\int_0^t uh_0(x)dx\} = \exp\{-uH_0(t)\}, \end{aligned}$$

where

$$H_0(t) = \int_0^t h_0(x)dx.$$

$$\therefore \quad S(t) = \int_0^\infty S(t|u)g(u)du,$$

$$\begin{aligned} i.e. \quad S(t) &= E[S(t|u)] = E[\exp\{-uH_0(t)\}], \\ &= \mathbb{L}_U[H_0(t)]. \end{aligned} \tag{4.4}$$

Which is the Laplace of U at $H_0(t)$. Thus the survivor function frailty mixture is the Laplace of U assessed at the cumulative base force of mortality.

$$\begin{aligned}
f(t|u) &= h(t|u) \cdot S(t|u), \\
&= h_0(t)u \cdot \exp(-uH_0(t)), \\
\therefore \quad f(t) &= \int_0^\infty f(t|u)g(u)du, \\
&= \int_0^\infty h_0(t)u \cdot \exp(-uH_0(t))g(u)du, \\
&= h_0(t)\int_0^\infty u\exp(-uH_0(t))g(u)du, \\
&= h_0(t)E[u\exp(-uH_0(t))]. \\
But; \quad \mathbb{L}_U[H_0(t)] &= E[\exp(-uH_0(t))], \\
\therefore \quad \mathbb{L}'_U[H_0(t)] &= \frac{d}{dt}\mathbb{L}_U[H_0(t)], \\
&= \frac{d}{dt}E[\exp(-uH_0(t))], \\
&= -h_0(t)E[u\exp(-uH_0(t))]. \\
\therefore \quad f(t) &= -\mathbb{L}'_U[H_0(t)]. \qquad (4.5) \\
h(t) &= \frac{f(t)}{S(t)} = \frac{-\mathbb{L}'_U[H_0(t)]}{\mathbb{L}_U[H_0(t)]}. \qquad (4.6)
\end{aligned}$$

Similarity with Exponential Mixture
The exponential (β) PDF is

$$f(t|\beta) = \beta\exp(-\beta t), t > 0, \beta > 0,$$

where β is varying taking the distribution $h(\beta)$

$$\begin{aligned}
Since \quad f(t) &= \int_0^\infty f(t|\beta)h(\beta)d\beta, \\
&= \int_0^\infty \beta\exp(-\beta t)h(\beta)d\beta,
\end{aligned}$$

which is an exponential mixture expressed in-terms of a PDF. For survivor function we have

$$\begin{aligned} S(t) &= \int_0^\infty S(t|\beta)h(\beta)d\beta, \\ &= \int_0^\infty \exp{(-\beta t)}h(\beta)d\beta, \\ &= E[\exp{(-\beta t)}] = \mathbb{L}_\beta(t). \end{aligned} \tag{4.7}$$

Thus, the survivor function of an exponential mixture is the mixing distributions Laplace.

4.4 Positive Stable Distribution

Hougaard (2000) derived the positive-stable mixture from the PVF distribution. A distribution is strictly stable if the aggregation of independent random variables from the distribution normalized follows the same distribution. The choice was based on the fact that the stable mixture permits the proportional-hazards model to apply conditionally as well as unconditionally since the frailty mixture preserves the proportional-hazards assumption in the unconditional intensity rate after integrating out the frailty as shown below:
The conditional intensity rate is

$$h_\epsilon(t|u) = uh_0(t)\epsilon, \tag{4.8}$$

where u represents the unobserved covariates only and $\epsilon = \exp{(\underline{\Theta' x})}$ the observed covariates.

$$\therefore \quad S_\epsilon(t|u) = \exp\{\int_0^t -uh_0(x)\epsilon dx\} = \exp\{-uH_0(t)\epsilon\},$$

$$S_\epsilon(t) = E[S_\epsilon(t|u)] = E[\exp\{-uH_0(t)\epsilon\}].$$

Since $\mathbb{L}(s) = E[\exp{(-sU)}] = \exp{(-s^\alpha)}$ as derived in Equation (3.37) for the positive stable random variable U then

$$\begin{aligned} S_\epsilon(t) &= \exp\{-(H_0(t)\epsilon)^\alpha\}. \\ h_\epsilon(t) &= -\frac{d}{dt}\log S_\epsilon(t) = \alpha h_0(t)(H_0(t))^{\alpha-1}\epsilon^\alpha. \end{aligned}$$

The unconditional intensity rate is

$$
\begin{aligned}
h(t|u) &= uh_0(t). \qquad (4.9)\\
\therefore \quad S(t|u) &= \exp\{-uH_0(t)\},\\
S(t) &= E[S(t|u)],\\
&= E[\exp\{-uH_0(t)\}] = \exp\{-(H_0(t))^\alpha\}.\\
h(t) &= -\frac{d}{dt}\log S(t) = \alpha h_0(t)(H_0(t))^{\alpha-1}.
\end{aligned}
$$

Therefore

$$
\frac{h_\epsilon(t)}{h(t)} = \frac{\alpha h_0(t)(H_0(t))^{\alpha-1}\epsilon^\alpha}{\alpha h_0(t)(H_0(t))^{\alpha-1}} = \epsilon^\alpha. \qquad (4.10)
$$

Thus the positive stable permits the proportional-hazards approach to apply unconditionally as well as conditionally.

Construction of Positive Stable Univariate Frailty Model
The positive stable PDF as a series approximation is represented as:

$$
f(u) = -\frac{1}{\pi u}\sum_{j=1}^{\infty}\frac{\Gamma(j\alpha+1)}{j!}(-u^{-\alpha}k/\alpha)^j\sin(\alpha j\pi); k>0, u>0, 0<\alpha\leq 1.
$$

The Laplace derived in Equation (3.35) is

$$
\begin{aligned}
\mathbb{L}_U(s) &= \exp\left(-\frac{k}{\alpha}s^\alpha\right),\\
\textit{Since} \quad S(t) &= \mathbb{L}_U(H_0(t)) = \exp\left(-\frac{k}{\alpha}H_0(t)^\alpha\right).\\
f(t) &= -\mathbb{L}'_U(H_0(t)) = -\frac{d}{dt}\cdot\exp\left(-\frac{k}{\alpha}(H_0(t))^\alpha\right),\\
&= kH_0(t)^{\alpha-1}h_0(t)\cdot\exp\left(-\frac{k}{\alpha}(H_0(t))^\alpha\right).\\
h(t) &= \frac{kH_0(t)^{\alpha-1}h_0(t)\cdot\exp\left\{-\frac{k}{\alpha}(H_0(t))^\alpha\right\}}{\exp\left\{-\frac{k}{\alpha}H_0(t)^\alpha\right\}} = kH_0(t)^{\alpha-1}h_0(t).
\end{aligned}
$$

Considering various choices of $h_0(t)$ the base force of mortality function, we have the following positive stable frailty models.

Gompertz-Positive Stable Frailty Model
The Gompertz distribution is mostly adopted to describe adult lifespan distribution from a vast literature on human mortality (Frees *et al.,* (1996)). The intensity rate is represented as:

$$h_0(t) = A\exp(Bt), t > 0, A > 0, B > 0, \tag{4.11}$$

where A and B are constants.

$$\begin{aligned}
\therefore \quad H_0(t) &= \int_0^t A\exp(Bx)dx = \frac{A}{B}\exp(Bx)|_{x=0}^{x=t}, \\
&= \frac{A}{B}(\exp(Bt) - 1). \qquad (4.12) \\
S(t) &= \exp\left(-\frac{k}{\alpha}H_0(t)^\alpha\right) = \exp\left(-\frac{k}{\alpha}\left[\frac{A}{B}(\exp(Bt)-1)\right]^\alpha\right). \\
f(t) &= kH_0(t)^{\alpha-1}h_0(t)\cdot\exp\left(-\frac{k}{\alpha}(H_0(t))^\alpha\right), \\
&= k\left[\frac{A}{B}(\exp(Bt)-1)\right]^{\alpha-1} A\exp(Bt)\cdot\exp\left(-\frac{k}{\alpha}\left[\frac{A}{B}(\exp(Bt)-1)\right]^\alpha\right). \\
h(t) &= \frac{k(\frac{A}{B}(\exp(Bt)-1))^{\alpha-1}A\exp(Bt)\cdot\exp\{-\frac{k}{\alpha}(\frac{A}{B}(\exp(Bt)-1))^\alpha\}}{\exp\{-\frac{k}{\alpha}(\frac{A}{B}(\exp(Bt)-1))^\alpha\}}, \\
&= k\left[\frac{A}{B}(\exp(Bt)-1)\right]^{\alpha-1} A\exp(Bt).
\end{aligned}$$

For identifiability reasons we apply Laplace transform derived in Equation (3.37). The univariate marginal survivor, density and intensity rate for the Gompertz-positive stable mixture becomes:

$$\begin{aligned}
S(t) &= \exp(-H_0(t)^\alpha) = \exp\left(-\left[\frac{A}{B}(\exp(Bt)-1)\right]^\alpha\right). \\
f(t) &= \alpha h_0(t)H_0(t)^{\alpha-1}\cdot\exp(-H_0(t)^\alpha), \\
&= \alpha A\exp(Bt)\left[\frac{A}{B}(\exp(Bt)-1)\right]^{\alpha-1}\cdot\exp\left(-\left[\frac{A}{B}(\exp(Bt)-1)\right]^\alpha\right). \\
h(t) &= \alpha h_0(t)H_0(t)^{\alpha-1} = \alpha A\exp(Bt)\left[\frac{A}{B}(\exp(Bt)-1)\right]^{\alpha-1}.
\end{aligned}$$

Weibull-Positive Stable Frailty Model
The Weibull distribution function is represented as:

$$f_0(t) = \beta\rho t^{\rho-1}\exp(-\beta t^{\rho}); \beta, \rho > 0; t \geq 0.$$

We can express survivor, mortality and cumulative intensity rates as:

$$\begin{aligned}
S_0(t) &= \mathbb{P}(T > t), \\
&= \int_t^{\infty} \beta\rho x^{\rho-1}\exp(-\beta x^{\rho})dx. = \exp(-\beta t^{\rho}). \\
h_0(t) &= \frac{\beta\rho t^{\rho-1}\exp(-\beta t^{\rho})}{\exp(-\beta t^{\rho})} = \beta\rho t^{\rho-1}. && (4.13) \\
H_0(t) &= \int_0^t \beta\rho x^{\rho-1}dx = \beta t^{\rho}. && (4.14) \\
S(t) &= \exp\left(-\frac{k}{\alpha}H_0(t)^{\alpha}\right) = \exp\left(-\frac{k}{\alpha}(\beta t^{\rho})^{\alpha}\right). \\
f(t) &= kH_0(t)^{\alpha-1}h_0(t)\cdot\exp\left(-\frac{k}{\alpha}(H_0(t))^{\alpha}\right), \\
&= k(\beta t^{\rho})^{\alpha-1}\beta\rho t^{\rho-1}\cdot\exp\left(-\frac{k}{\alpha}(\beta t^{\rho})^{\alpha}\right). \\
h(t) &= \frac{k(\beta t^{\rho})^{\alpha-1}\beta\rho t^{\rho-1}\cdot\exp\{-\frac{k}{\alpha}(\beta t^{\rho})^{\alpha}\}}{\exp\{-\frac{k}{\alpha}(\beta t^{\rho})^{\alpha}\}} = k(\beta t^{\rho})^{\alpha-1}\beta\rho t^{\rho-1}.
\end{aligned}$$

For identifiability reasons we apply Laplace transform derived in Equation (3.37). The univariate marginal survivor, density and intensity rate for the Weibull-positive stable mixture becomes:

$$\begin{aligned}
S(t) &= \exp(-H_0(t)^{\alpha}) = \exp(-(\beta t^{\rho})^{\alpha}). \\
f(t) &= \alpha h_0(t)H_0(t)^{\alpha-1}\cdot\exp(-H_0(t)^{\alpha}), \\
&= \alpha\beta\rho t^{\rho-1}(\beta t^{\rho})^{\alpha-1}\cdot\exp(-(\beta t^{\rho})^{\alpha}). \\
h(t) &= \alpha h_0(t)H_0(t)^{\alpha-1} = \alpha\beta\rho t^{\rho-1}\cdot(\beta t^{\rho})^{\alpha-1}.
\end{aligned}$$

Generalized Weibull-Positive Stable Frailty Model
The generalized Weibull (GW) density function is represented as:

$$f_0(t) = b(1 - \exp(-\lambda t^\rho))^{b-1} \cdot \lambda\rho t^{\rho-1} \cdot \exp(-\lambda t^\rho); t > 0, \rho, b, \lambda > 0.$$

We can express the survivor, mortality and cumulative intensity rates as:

$$S_0(t) = \int_t^\infty b(1 - \exp(-\lambda x^\rho))^{b-1} \cdot \lambda\rho x^{\rho-1} \cdot \exp(-\lambda x^\rho) dx.$$

$$Let \quad y = \exp(-\lambda x^\rho); \quad and \quad \frac{dy}{dx} = -\lambda\rho x^{\rho-1}\exp(-\lambda x^\rho)$$

$$\therefore \quad S_0(t) = \int_{\exp(-\lambda t^\rho)}^0 -b(1-y)^{b-1}dy = (1-y)^b|_{y=\exp(-\lambda t^\rho)}^{y=0}$$

$$= 1 - [1 - \exp(-\lambda t^\rho)]^b,$$

$$h_0(t) = \frac{b(1 - \exp(-\lambda t^\rho))^{b-1}\lambda\rho t^{\rho-1}\exp(-\lambda t^\rho)}{1 - [1 - \exp(-\lambda t^\rho)]^b}, \tag{4.15}$$

$$H_0(t) = -\log S_0(t) = -\log(1 - [1 - \exp(-\lambda t^\rho)]^b). \tag{4.16}$$

$$S(t) = \exp\left(-\frac{k}{\alpha}H_0(t)^\alpha\right) = \exp\left(-\frac{k}{\alpha}(-\log(1 - [1 - \exp(-\lambda t^\rho)]^b))^\alpha\right).$$

$$f(t) = kH_0(t)^{\alpha-1}h_0(t) \cdot \exp\left(-\frac{k}{\alpha}(H_0(t))^\alpha\right),$$

$$= k(-\log(1 - [1 - \exp(-\lambda t^\rho)]^b))^{\alpha-1}\frac{b(1 - \exp(-\lambda t^\rho))^{b-1}\lambda\rho t^{\rho-1}\exp(-\lambda t^\rho)}{1 - [1 - \exp(-\lambda t^\rho)]^b}$$

$$\cdot \exp\left(-\frac{k}{\alpha}(-\log(1 - [1 - \exp(-\lambda t^\rho)]^b))^\alpha\right)$$

$$h(t) = \frac{k(-\log(1 - [1 - \exp(-\lambda t^\rho)]^b))^{\alpha-1}\frac{b(1-\exp(-\lambda t^\rho))^{b-1}\lambda\rho t^{\rho-1}\exp(-\lambda t^\rho)}{1-[1-\exp(-\lambda t^\rho)]^b}}{\exp\{-\frac{k}{\alpha} - \log(1 - [1 - \exp(-\lambda t^\rho)]^b)^\alpha\}}$$

$$\cdot \exp\left(-\frac{k}{\alpha}(-\log(1 - [1 - \exp(-\lambda t^\rho)]^b))^\alpha\right)$$

$$= k(-\log(1 - [1 - \exp(-\lambda t^\rho)]^b))^{\alpha-1}\frac{b(1 - \exp(-\lambda t^\rho))^{b-1}\lambda\rho t^{\rho-1}\exp(-\lambda t^\rho)}{1 - [1 - \exp(-\lambda t^\rho)]^b}.$$

For identifiability reasons we apply Laplace transform derived in Equation (3.37). The univariate marginal survivor, density and intensity rate for the Generalized Weibull-positive stable mixture becomes:

$$S(t) = \exp\left(-H_0(t)^\alpha\right) = \exp\left(-(-\log(1-[1-\exp(-\lambda t^\rho)]^b))^\alpha\right).$$

$$\begin{aligned}
f(t) &= \alpha h_0(t) H_0(t)^{\alpha-1} \cdot \exp\left(-H_0(t)^\alpha\right), \\
&= \alpha \left(\frac{b(1-\exp(-\lambda t^\rho))^{b-1} \lambda \rho t^{\rho-1} \exp(-\lambda t^\rho)}{1-[1-\exp(-\lambda t^\rho)]^b} \right) \cdot (-\log(1-[1-\exp(-\lambda t^\rho)]^b))^{\alpha-1} \\
&\quad \cdot \exp\left(-(-\log(1-[1-\exp(-\lambda t^\rho)]^b))^\alpha\right)
\end{aligned}$$

$$\begin{aligned}
h(t) &= \alpha h_0(t) H_0(t)^{\alpha-1}, \\
&= \alpha \left(\frac{b(1-\exp(-\lambda t^\rho))^{b-1} \lambda \rho t^{\rho-1} \exp(-\lambda t^\rho)}{1-[1-\exp(-\lambda t^\rho)]^b} \right) \cdot (-\log(1-[1-\exp(-\lambda t^\rho)]^b))^{\alpha-1}.
\end{aligned}$$

Exponential-Positive Stable Frailty Model

The Exponential function is derived from the Weibull distribution when $\rho = 1$. Putting $\rho = 1$ in Equations (4.13,4.14) leads to

$$\begin{aligned}
h_0(t) &= \beta. \\
H_0(t) &= \beta t. \\
S(t) &= \exp\left(-\frac{k}{\alpha} H_0(t)^\alpha\right) = \exp\left(-\frac{k}{\alpha}(\beta t)^\alpha\right). \\
f(t) &= k H_0(t)^{\alpha-1} h_0(t) \cdot \exp\left(-\frac{k}{\alpha}(H_0(t))^\alpha\right), \\
&= k(\beta t)^{\alpha-1}\beta \cdot \exp\{-\frac{k}{\alpha}(\beta t)^\alpha\}. \\
h(t) &= \frac{k(\beta t)^{\alpha-1}\beta \cdot \exp\{-\frac{k}{\alpha}(\beta t)^\alpha\}}{\exp\{-\frac{k}{\alpha}(\beta t)^\alpha\}} = k(\beta t)^{\alpha-1}\beta.
\end{aligned}$$

For identifiability reasons we apply Laplace transform derived in Equation (3.37). The univariate marginal survivor, density and intensity rate for the exponential-positive stable mixture becomes:

$$\begin{aligned}
S(t) &= \exp\left(-H_0(t)^\alpha\right) = \exp\left(-(\beta t)^\alpha\right). \\
f(t) &= \alpha h_0(t) H_0(t)^{\alpha-1} \cdot \exp\left(-H_0(t)^\alpha\right), \\
&= \alpha\beta(\beta t)^{\alpha-1} \cdot \exp\left(-(\beta t)^\alpha\right). \\
h(t) &= \alpha h_0(t) H_0(t)^{\alpha-1} = \alpha\beta \cdot (\beta t)^{\alpha-1}.
\end{aligned}$$

Generalized Exponential-Positive Stable Frailty Model

The generalized exponential (GE) function is derived from the GW distribution when $\rho = 1$. Putting $\rho = 1$ in Equations (4.15,4.16) leads to

$$h_0(t) = \frac{b\lambda(1-\exp(-\lambda t))^{b-1}\exp(-\lambda t)}{1-[1-\exp(-\lambda t)]^b}, \tag{4.17}$$

$$H_0(t) = -\log(1-[1-\exp(-\lambda t)]^b) \tag{4.18}$$

$$S(t) = \exp\left(-\frac{k}{\alpha}H_0(t)^\alpha\right) = \exp\left(-\frac{k}{\alpha}(-\log(1-[1-\exp(-\lambda t)]^b))^\alpha\right).$$

$$\begin{aligned} f(t) &= kH_0(t)^{\alpha-1}h_0(t)\cdot\exp\left(-\frac{k}{\alpha}(H_0(t))^\alpha\right), \\ &= k(-\log(1-[1-\exp(-\lambda t)]^b))^{\alpha-1}\cdot\left(\frac{b\lambda(1-\exp(-\lambda t))^{b-1}\exp(-\lambda t)}{1-[1-\exp(-\lambda t)]^b}\right) \\ &\cdot\exp\left(-\frac{k}{\alpha}(-\log(1-[1-\exp(-\lambda t)]^b))^\alpha\right) \end{aligned}$$

$$\begin{aligned} h(t) &= \frac{k(-\log(1-[1-\exp(-\lambda t)]^b))^{\alpha-1}(\frac{b\lambda(1-\exp(-\lambda t))^{b-1}\exp(-\lambda t)}{1-[1-\exp(-\lambda t)]^b})}{\exp\{-\frac{k}{\alpha}(-\log(1-[1-\exp(-\lambda t)]^b))^\alpha\}} \\ &\cdot\exp\left(-\frac{k}{\alpha}(-\log(1-[1-\exp(-\lambda t)]^b))^\alpha\right) \\ &= k(-\log(1-[1-\exp(-\lambda t)]^b))^{\alpha-1}\cdot\left(\frac{b\lambda(1-\exp(-\lambda t))^{b-1}\exp(-\lambda t)}{1-[1-\exp(-\lambda t)]^b}\right) \end{aligned}$$

For identifiability reasons we apply Laplace transform derived in Equation (3.37). The univariate marginal survivor, density and intensity rate for the generalized exponential-positive stable mixture becomes:

$$S(t) = \exp(-H_0(t)^\alpha) = \exp(-(-\log(1-[1-\exp(-\lambda t)]^b))^\alpha).$$

$$\begin{aligned} f(t) &= \alpha h_0(t)H_0(t)^{\alpha-1}\cdot\exp(-H_0(t)^\alpha), \\ &= \alpha\left(\frac{b\lambda(1-\exp(-\lambda t))^{b-1}\exp(-\lambda t)}{1-[1-\exp(-\lambda t)]^b}\right)\cdot\left(\exp\{-\frac{k}{\alpha}(-\log(1-[1-\exp(-\lambda t)]^b))^\alpha\}\right)^{\alpha-1} \\ &\cdot\exp\left(-\left(\exp\{-\frac{k}{\alpha}(-\log(1-[1-\exp(-\lambda t)]^b))^\alpha\}\right)^\alpha\right) \end{aligned}$$

$$h(t) = \alpha h_0(t) H_0(t)^{\alpha-1},$$
$$= \alpha \left(\frac{b\lambda(1-\exp(-\lambda t))^{b-1}\exp(-\lambda t)}{1-[1-\exp(-\lambda t)]^b} \right) \cdot \left(\exp\{-\frac{k}{\alpha}(-\log(1-[1-\exp(-\lambda t)]^b))^\alpha\} \right)^{\alpha-1}.$$

Loglogistic-Positive Stable Frailty Model
The loglogistic density is expressed as:

$$f_0(t) = \frac{(\Theta/\lambda)(t/\lambda)^{\Theta-1}}{(1+(t/\lambda)^\Theta)^2}; t, \Theta, \lambda > 0. \quad (4.19)$$

We can express survivor, mortality and cumulative intensity rates as:

$$S_0(t) = \int_t^\infty \frac{(\Theta/\lambda)(u/\lambda)^{\Theta-1}}{(1+(u/\lambda)^\Theta)^2} du.$$
$$Let \quad y = \left(\frac{u}{\lambda}\right)^\Theta \quad and \quad \frac{dy}{du} = \frac{\Theta}{\lambda}\left(\frac{u}{\lambda}\right)^{\Theta-1}.$$
$$S_0(t) = \int_{(\frac{t}{\lambda})^\Theta}^\infty \frac{dy}{(1+y)^2} = \frac{1}{1+(\frac{t}{\lambda})^\Theta}.$$
$$h_0(t) = \frac{(\Theta/\lambda)(t/\lambda)^{\Theta-1}}{1+(t/\lambda)^\Theta}. \quad (4.20)$$

The cumulative intensity rate is represented as:

$$H_0(t) = \int_0^t \frac{(\Theta/\lambda)(x/\lambda)^{\Theta-1}}{(1+(x/\lambda)^\Theta)} dx = \log(1+(\frac{t}{\lambda})^\Theta). \quad (4.21)$$
$$S(t) = \exp\left(-\frac{k}{\alpha}H_0(t)^\alpha\right) = \exp\left(-\frac{k}{\alpha}\left(\log\left[1+\left(\frac{t}{\lambda}\right)^\Theta\right]\right)^\alpha\right).$$
$$f(t) = kH_0(t)^{\alpha-1}h_0(t)\cdot\exp\left(-\frac{k}{\alpha}(H_0(t))^\alpha\right),$$
$$= k\left(\log\left[1+\left(\frac{t}{\lambda}\right)^\Theta\right]\right)^{\alpha-1}\frac{(\Theta/\lambda)(t/\lambda)^{\Theta-1}}{1+(t/\lambda)^\Theta}\cdot\exp\left(-\frac{k}{\alpha}\left(\log\left[1+\left(\frac{t}{\lambda}\right)^\Theta\right]\right)^\alpha\right).$$
$$h(t) = \frac{k(\log(1+(\frac{t}{\lambda})^\Theta))^{\alpha-1}\frac{(\Theta/\lambda)(t/\lambda)^{\Theta-1}}{1+(t/\lambda)^\Theta}\cdot\exp\{-\frac{k}{\alpha}(\log(1+(\frac{t}{\lambda})^\Theta))^\alpha\}}{\exp\{-\frac{k}{\alpha}(\log(1+(\frac{t}{\lambda})^\Theta))^\alpha\}},$$
$$= k\left(\log\left[1+\left(\frac{t}{\lambda}\right)^\Theta\right]\right)^{\alpha-1}\frac{(\Theta/\lambda)(t/\lambda)^{\Theta-1}}{1+(t/\lambda)^\Theta}.$$

For identifiability reasons we apply Laplace transform derived in Equation (3.37). The univariate marginal survivor, density and intensity rate for the loglogistic-positive stable frailty distribution becomes:

$$
\begin{aligned}
S(t) &= \exp\left(-H_0(t)^{\alpha}\right) = \exp\left(-\left(\log\left[1+\left(\frac{t}{\lambda}\right)^{\Theta}\right]\right)^{\alpha}\right),\\
f(t) &= \alpha h_0(t) H_0(t)^{\alpha-1}\cdot\exp\left(-H_0(t)^{\alpha}\right),\\
&= \alpha\frac{(\Theta/\lambda)(t/\lambda)^{\Theta-1}}{1+(t/\lambda)^{\Theta}}\left(\log\left[1+\left(\frac{t}{\lambda}\right)^{\Theta}\right]\right)^{\alpha-1}\cdot\exp\left(-\left(\log\left[1+\left(\frac{t}{\lambda}\right)^{\Theta}\right]\right)^{\alpha}\right).\\
h(t) &= \alpha h_0(t) H_0(t)^{\alpha-1},\\
&= \alpha\frac{(\Theta/\lambda)(t/\lambda)^{\Theta-1}}{1+(t/\lambda)^{\Theta}}\left(\log\left[1+\left(\frac{t}{\lambda}\right)^{\Theta}\right]\right)^{\alpha-1}.
\end{aligned}
$$

Lognormal-Positive Stable Frailty Model
The lognormal function is derived as;
let $y = \log x$ this implies $x = \exp(y)$, where $y \sim N(\mu, \sigma^2)$,

$$\therefore \quad \frac{dy}{dx} = \frac{1}{x},$$

$$
\begin{aligned}
f(x) = g(y)\frac{1}{x} &= \frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(-\frac{(y-\mu)^2}{2\sigma^2}\right)\frac{1}{x},\\
&= \frac{1}{x\sigma\sqrt{2\pi}}\exp\left(-\frac{(\log x-\mu)^2}{2\sigma^2}\right), 0<x<\infty, -\infty<\mu<\infty, \sigma>0.
\end{aligned}
\tag{4.22}
$$

Which is a lognormal distribution.

$$\therefore \quad f(t) = \frac{1}{\sigma t}\phi\left(\frac{\log t-\mu}{\sigma}\right),$$

where $\phi(.)$ is a standard normal distribution. The Cumulative Density Function (CDF) is given by

$$F(t) = \int_0^t \frac{1}{\sigma x}\phi\left(\frac{\log x - \mu}{\sigma}\right) dx.$$

$$Let, \quad y = \frac{\log x - \mu}{\sigma} \quad and \quad dy = \frac{1}{\sigma x}dx.$$

$$\therefore \quad F(t) = \int_{-\infty}^{\frac{\log t-\mu}{\sigma}} \phi(y)dy = \Phi\left(\frac{\log t - \mu}{\sigma}\right),$$

where $\Phi(.)$ is the CDF of a standard normal distribution.

$$\therefore \quad h(t) = \frac{f(t)}{1-F(t)} = \frac{\frac{1}{\sigma t}\phi(\frac{\log t-\mu}{\sigma})}{1-\Phi(\frac{\log t-\mu}{\sigma})},$$

$$= \frac{\phi(\frac{\log t-\mu}{\sigma})}{\sigma t[1-\Phi(\frac{\log t-\mu}{\sigma})]},$$

Since the lognormal represents the base force of mortality function we have

$$h_0(t) = \frac{\phi(\frac{\log t-\mu}{\sigma})}{\sigma t[1-\Phi(\frac{\log t-\mu}{\sigma})]}. \tag{4.23}$$

$$H_0(t) = -\log\left[1-\Phi\left(\frac{\log t-\mu}{\sigma}\right)\right]. \tag{4.24}$$

This implies that survivor and density functions become

$$S(t) = \exp\left(-\frac{k}{\alpha}H_0(t)^\alpha\right) = \exp\left(-\frac{k}{\alpha}\left(-\log\left[1-\Phi\left(\frac{\log t-\mu}{\sigma}\right)\right]\right)^\alpha\right).$$

$$f(t) = kH_0(t)^{\alpha-1}h_0(t)\cdot\exp\{-\frac{k}{\alpha}(H_0(t))^\alpha\},$$

$$= k\left(-\log\left[1-\Phi\left(\frac{\log t-\mu}{\sigma}\right)\right]\right)^{\alpha-1}\frac{\phi(\frac{\log t-\mu}{\sigma})}{\sigma t[1-\Phi(\frac{\log t-\mu}{\sigma})]}$$

$$\cdot\exp\left(-\frac{k}{\alpha}\left(-\log\left[1-\Phi\left(\frac{\log t-\mu}{\sigma}\right)\right]\right)^\alpha\right).$$

$$h(t) = \frac{k(-\log[1-\Phi(\frac{\log t-\mu}{\sigma})])^{\alpha-1}\frac{\phi(\frac{\log t-\mu}{\sigma})}{\sigma t[1-\Phi(\frac{\log t-\mu}{\sigma})]}\cdot\exp\{-\frac{k}{\alpha}(-\log[1-\Phi(\frac{\log t-\mu}{\sigma})])^\alpha\}}{\exp\{-\frac{k}{\alpha}(-\log[1-\Phi(\frac{\log t-\mu}{\sigma})])^\alpha\}},$$

$$= k\left(-\log\left[1-\Phi\left(\frac{\log t-\mu}{\sigma}\right)\right]\right)^{\alpha-1}\cdot\frac{\phi(\frac{\log t-\mu}{\sigma})}{\sigma t[1-\Phi(\frac{\log t-\mu}{\sigma})]}.$$

For identifiability reasons we apply Laplace transform derived in Equation (3.37). The univariate marginal survivor, density and intensity rate for the lognormal-positive stable frailty distribution becomes:

$$
\begin{aligned}
S(t) &= \exp\left(-H_0(t)^\alpha\right) = \exp\left(-\left(-\log\left[1-\Phi\left(\frac{\log t-\mu}{\sigma}\right)\right]\right)^\alpha\right).\\
f(t) &= \alpha h_0(t) H_0(t)^{\alpha-1}\cdot\exp\left(-H_0(t)^\alpha\right),\\
f(t) &= \alpha\frac{\phi(\frac{\log t-\mu}{\sigma})}{\sigma t[1-\Phi(\frac{\log t-\mu}{\sigma})]}\cdot\left(-\log\left[1-\Phi\left(\frac{\log t-\mu}{\sigma}\right)\right]\right)^{\alpha-1}\\
&\quad\cdot\exp\left(-\left(-\log\left[1-\Phi\left(\frac{\log t-\mu}{\sigma}\right)\right]\right)^\alpha\right).\\
h(t) &= \alpha h_0(t) H_0(t)^{\alpha-1} = \alpha\frac{\phi(\frac{\log t-\mu}{\sigma})}{\sigma t[1-\Phi(\frac{\log t-\mu}{\sigma})]}\cdot\left(-\log\left[1-\Phi\left(\frac{\log t-\mu}{\sigma}\right)\right]\right)^{\alpha-1}.
\end{aligned}
$$

Exponential-Power-Positive Stable Frailty Model
The exponential power mixture is a generalization of the normal distribution expressed as:

$$
f_0(t) = \exp(rt^\lambda) r\lambda t^{\lambda-1}\cdot(\exp\{1-\exp(rt^\lambda)\}); t,\lambda,r>0. \tag{4.25}
$$

$$
\begin{aligned}
S_0(t) &= \int_t^\infty \exp(rx^\lambda) r\lambda x^{\lambda-1}\cdot(\exp\{1-\exp(rx^\lambda)\})dx.\\
\textit{Let};\quad y &= \exp(rx^\lambda)\quad \textit{and}\quad \frac{dy}{dx} = \lambda r x^{\lambda-1}\exp(rx^\lambda).\\
\therefore\quad S_0(t) &= \int_{\exp(rt^\lambda)}^\infty \exp(1-y)dy = -\exp(1-y)\big|_{y=\exp(rt^\lambda)}^{y=\infty},\\
&= \exp\{1-\exp(rt^\lambda)\}.\\
h_0(t) &= \frac{\exp(rt^\lambda) r\lambda t^{\lambda-1}\cdot(\exp\{1-\exp(rt^\lambda)\})}{\exp\{1-\exp(rt^\lambda)\}} = \exp(rt^\lambda) r\lambda t^{\lambda-1},
\end{aligned}
\tag{4.26}
$$

The intensity rate can achieve a bathtub-shape.

$$
H_0(t) = -\log(S(t)) = \exp(rt^\lambda) - 1. \tag{4.27}
$$

This implies that survivor and density functions become

$$
\begin{aligned}
S(t) &= \exp\left(-\frac{k}{\alpha}H_0(t)^\alpha\right) = \exp\left(-\frac{k}{\alpha}(\exp(rt^\lambda)-1)^\alpha\right). \\
f(t) &= kH_0(t)^{\alpha-1}h_0(t)\cdot\exp\left(-\frac{k}{\alpha}(H_0(t))^\alpha\right), \\
&= k(\exp(rt^\lambda)-1)^{\alpha-1}\exp(rt^\lambda)r\lambda t^{\lambda-1}\cdot\exp\left(-\frac{k}{\alpha}(\exp(rt^\lambda)-1)^\alpha\right). \\
h(t) &= \frac{k(\exp(rt^\lambda)-1)^{\alpha-1}\exp(rt^\lambda)r\lambda t^{\lambda-1}\cdot\exp\{-\frac{k}{\alpha}(\exp(rt^\lambda)-1)^\alpha\}}{\exp\{-\frac{k}{\alpha}(\exp(rt^\lambda)-1)^\alpha\}}, \\
&= k(\exp(rt^\lambda)-1)^{\alpha-1}\exp(rt^\lambda)r\lambda t^{\lambda-1}.
\end{aligned}
$$

For identifiability reasons we apply Laplace transform derived in Equation (3.37). The univariate marginal survivor, density and intensity rate for the exponential-power-positive stable mixture becomes:

$$
\begin{aligned}
S(t) &= \exp(-H_0(t)^\alpha) = \exp(-(\exp(rt^\lambda)-1)^\alpha). \\
f(t) &= \alpha h_0(t)H_0(t)^{\alpha-1}\cdot\exp(-H_0(t)^\alpha), \\
&= \alpha\exp(rt^\lambda)r\lambda t^{\lambda-1}\cdot(\exp(rt^\lambda)-1)^{\alpha-1}\cdot\exp(-(\exp(rt^\lambda)-1)^\alpha). \\
h(t) &= \alpha h_0(t)H_0(t)^{\alpha-1} = \alpha\exp(rt^\lambda)r\lambda t^{\lambda-1}\cdot(\exp(rt^\lambda)-1)^{\alpha-1}.
\end{aligned}
$$

Pareto-Positive Stable Frailty Model
The Pareto I density is expressed as:

$$
f_0(t) = \frac{\vartheta r^\vartheta}{t^{\vartheta+1}}; t > r, r, \vartheta > 0. \tag{4.28}
$$

$$
S_0(t) = \int_t^\infty \frac{\vartheta r^\vartheta}{x^{\vartheta+1}}dx = -r^\vartheta\left[\frac{1}{x^\vartheta}\right]|_{x=t}^{x=\infty} = \frac{r^\vartheta}{t^\vartheta},
$$

$$
h_0(t) = \frac{\frac{\vartheta r^\vartheta}{t^{\vartheta+1}}}{\frac{r^\vartheta}{t^\vartheta}} = \frac{\vartheta}{t}. \tag{4.29}
$$

$$
\begin{aligned}
H_0(t) &= \int_r^t \frac{\vartheta}{x}dx = \vartheta\log x|_{x=r}^{x=t} \\
&= \vartheta\log\left(\frac{t}{r}\right) = \log\left(\frac{t}{r}\right)^\vartheta. \qquad (4.30)
\end{aligned}
$$

$$\therefore \quad S(t) = \exp\left(-\frac{k}{\alpha}H_0(t)^\alpha\right) = \exp\left(-\frac{k}{\alpha}\left[\log\left(\frac{t}{r}\right)^\vartheta\right]^\alpha\right).$$

$$f(t) = kH_0(t)^{\alpha-1}h_0(t)\cdot\exp\left(-\frac{k}{\alpha}(H_0(t))^\alpha\right),$$

$$= k\left[\log\left(\frac{t}{r}\right)^\vartheta\right]^{\alpha-1}\frac{\vartheta}{t}\cdot\exp\left(-\frac{k}{\alpha}\left[\log\left(\frac{t}{r}\right)^\vartheta\right]^\alpha\right).$$

$$h(t) = \frac{k(\log(\frac{t}{r})^\vartheta)^{\alpha-1}\frac{\vartheta}{t}\cdot\exp\{-\frac{k}{\alpha}(\log(\frac{t}{r})^\vartheta)^\alpha\}}{\exp\{-\frac{k}{\alpha}(\log(\frac{t}{r})^\vartheta)^\alpha\}}$$

$$= k\left[\log\left(\frac{t}{r}\right)^\vartheta\right]^{\alpha-1}\frac{\vartheta}{t}. \tag{4.31}$$

For identifiability reasons we apply Laplace transform derived in Equation (3.37). The univariate marginal survivor, density and intensity rate for the Pareto-positive stable mixture becomes:

$$S(t) = \exp\left(-H_0(t)^\alpha\right) = \exp\left(-\left[\log\left(\frac{t}{r}\right)^\vartheta\right]^\alpha\right).$$

$$f(t) = \alpha h_0(t)H_0(t)^{\alpha-1}\cdot\exp\left(-H_0(t)^\alpha\right),$$

$$= \alpha\frac{\vartheta}{t}\left[\log\left(\frac{t}{r}\right)^\vartheta\right]^{\alpha-1}\cdot\exp\left(-\left[\log\left(\frac{t}{r}\right)^\vartheta\right]^\alpha\right).$$

$$h(t) = \alpha\frac{\vartheta}{t}\left[\log\left(\frac{t}{r}\right)^\vartheta\right]^{\alpha-1} = \alpha\frac{\vartheta}{t}\left[\log\left(\frac{t}{r}\right)^\vartheta\right]^{\alpha-1}. \tag{4.32}$$

4.5 Compound Poisson Distribution

Aalen (1992) proposes the compound Poisson mixture and applies it to a sub-group of zero-susceptibility. The choice is based on the fact that the CPD has a positive probability of zero frailty and therefore useful in cases where individuals may be immune to the event of interest.

Construction of Compound Poisson Univariate Frailty Model
Let $N \sim Poisson(\rho)$ and $X_i, i = 1, 2, \ldots$ be identically distributed random variables independent of N. $U \sim$ CPD expressed as:

$$U = \begin{cases} X_1 + \ldots + X_N, & N > 0 \\ 0, & N = 0 \end{cases}$$

The Laplace transform derived in Equation (3.32) is

$$\mathbb{L}_U(s) = \exp\left(\frac{-k}{\alpha}[(\beta + s)^\alpha - \beta^\alpha]\right).$$

$$\textit{Since} \quad S(t) = \mathbb{L}_U(H_0(t)) = \exp\left(\frac{-k}{\alpha}[(\beta + H_0(t))^\alpha - \beta^\alpha]\right).$$

$$\begin{aligned} f(t) &= -\mathbb{L}'_U(H_0(t)) = -\frac{d}{dt}\exp\left(\frac{-k}{\alpha}[(\beta + H_0(t))^\alpha - \beta^\alpha]\right), \\ &= kh_0(t)(\beta + H_0(t))^{\alpha-1}\exp\left(\frac{-k}{\alpha}[(\beta + H_0(t))^\alpha - \beta^\alpha]\right). \\ h(t) &= \frac{kh_0(t)(\beta + H_0(t))^{\alpha-1}\exp\{\frac{-k}{\alpha}[(\beta + H_0(t))^\alpha - \beta^\alpha]\}}{\exp\{\frac{-k}{\alpha}[(\beta + H_0(t))^\alpha - \beta^\alpha]\}}, \\ &= kh_0(t)(\beta + H_0(t))^{\alpha-1}. \end{aligned} \tag{4.33}$$

Considering various choices of $h_0(t)$ the base force of mortality function, we have the following CPD frailty models.

Gompertz-CPD Frailty Model
The Gompertz function base force of mortality and cumulative intensity rate is given in Equations (4.11,4.12) respectively.

$$S(t) = \exp\left(\frac{-k}{\alpha}[(\beta + H_0(t))^\alpha - \beta^\alpha]\right) = \exp\left(\frac{-k}{\alpha}\left[\left(\beta + \frac{A}{B}(\exp(Bt) - 1)\right)^\alpha - \beta^\alpha\right]\right).$$

$$\begin{aligned} f(t) &= kh_0(t)(\beta + H_0(t))^{\alpha-1}\exp\left(\frac{-k}{\alpha}[(\beta + H_0(t))^\alpha - \beta^\alpha]\right), \\ &= kA\exp(Bt)\left(\beta + \frac{A}{B}(\exp(Bt) - 1)\right)^{\alpha-1} \cdot \exp\left(\frac{-k}{\alpha}\left[\left(\beta + \frac{A}{B}(\exp(Bt) - 1)\right)^\alpha - \beta^\alpha\right]\right). \\ h(t) &= \frac{kA\exp(Bt)(\beta + \frac{A}{B}(\exp(Bt) - 1))^{\alpha-1}\exp\{\frac{-k}{\alpha}[(\beta + \frac{A}{B}(\exp(Bt) - 1))^\alpha - \beta^\alpha]\}}{\exp\{\frac{-k}{\alpha}[(\beta + \frac{A}{B}(\exp(Bt) - 1))^\alpha - \beta^\alpha]\}}, \\ &= kA\exp(Bt)\left(\beta + \frac{A}{B}(\exp(Bt) - 1)\right)^{\alpha-1}. \end{aligned}$$

For identifiability reasons we apply Laplace transform derived in Equation (3.33). The univariate marginal survivor, density and intensity rate for the

Gompertz-CPD mixture becomes:

$$
\begin{aligned}
S(t) &= \exp\left(\frac{\alpha-1}{\alpha\sigma^2}\left[\left(1+H_0(t)\frac{\sigma^2}{1-\alpha}\right)^{\alpha}-1\right]\right), \\
&= \exp\left(\frac{\alpha-1}{\alpha\sigma^2}\left[\left(1+\frac{A}{B}(\exp(Bt)-1)\frac{\sigma^2}{1-\alpha}\right)^{\alpha}-1\right]\right). \\
f(t) &= h_0(t)\cdot\left(1+H_0(t)\frac{\sigma^2}{1-\alpha}\right)^{\alpha-1}\cdot\exp\left(\frac{\alpha-1}{\alpha\sigma^2}\left[\left(1+H_0(t)\frac{\sigma^2}{1-\alpha}\right)^{\alpha}-1\right]\right), \\
&= A\exp(Bt)\cdot\left(1+\frac{A}{B}(\exp(Bt)-1)\frac{\sigma^2}{1-\alpha}\right)^{\alpha-1} \\
&\cdot\exp\left(\frac{\alpha-1}{\alpha\sigma^2}\left[\left(1+\frac{A}{B}(\exp(Bt)-1)\frac{\sigma^2}{1-\alpha}\right)^{\alpha}-1\right]\right), \\
h(t) &= h_0(t)\left(1+\frac{\sigma^2}{1-\alpha}H_0(t)\right)^{\alpha-1} \\
&= A\exp(Bt)\left(1+\frac{\sigma^2}{1-\alpha}\cdot\frac{A}{B}(\exp(Bt)-1)\right)^{\alpha-1}. \qquad (4.34)
\end{aligned}
$$

Weibull-CPD Frailty Model
The Weibull function base force of mortality and cumulative intensity rate is given in Equations (4.13,4.14) respectively. Using the parametrization $\beta = \Theta$ leads to

$$
\begin{aligned}
S(t) &= \exp\left(\frac{-k}{\alpha}[(\beta+H_0(t))^{\alpha}-\beta^{\alpha}]\right) = \exp\left(\frac{-k}{\alpha}[(\beta+\Theta t^{\rho})^{\alpha}-\beta^{\alpha}]\right). \\
f(t) &= kh_0(t)(\beta+H_0(t))^{\alpha-1}\exp\left(\frac{-k}{\alpha}[(\beta+H_0(t))^{\alpha}-\beta^{\alpha}]\right), \\
&= k\Theta\rho t^{\rho-1}(\beta+\Theta t^{\rho})^{\alpha-1}\exp\left(\frac{-k}{\alpha}[(\beta+\Theta t^{\rho})^{\alpha}-\beta^{\alpha}]\right). \\
h(t) &= \frac{k\Theta\rho t^{\rho-1}(\beta+\Theta t^{\rho})^{\alpha-1}\exp\{\frac{-k}{\alpha}[(\beta+\Theta t^{\rho})^{\alpha}-\beta^{\alpha}]\}}{\exp\{\frac{-k}{\alpha}[(\beta+\Theta t^{\rho})^{\alpha}-\beta^{\alpha}]\}} \\
&= k\Theta\rho t^{\rho-1}(\beta+\Theta t^{\rho})^{\alpha-1}.
\end{aligned}
$$

For identifiability reasons we apply Laplace transform derived in Equation (3.33). The univariate marginal survivor, density and intensity rate for the Weibull-CPD mixture becomes:

$$
\begin{aligned}
S(t) &= \exp\left(\frac{\alpha-1}{\alpha\sigma^2}\left[\left(1+H_0(t)\frac{\sigma^2}{1-\alpha}\right)^{\alpha}-1\right]\right),\\
&= \exp\left(\frac{\alpha-1}{\alpha\sigma^2}\left[\left(1+\Theta t^{\rho}\frac{\sigma^2}{1-\alpha}\right)^{\alpha}-1\right]\right).\\
f(t) &= h_0(t)\cdot\left(1+H_0(t)\frac{\sigma^2}{1-\alpha}\right)^{\alpha-1}\cdot\exp\left(\frac{\alpha-1}{\alpha\sigma^2}\left[\left(1+H_0(t)\frac{\sigma^2}{1-\alpha}\right)^{\alpha}-1\right]\right),\\
&= \Theta\rho t^{\rho-1}\cdot\left(1+\Theta t^{\rho}\frac{\sigma^2}{1-\alpha}\right)^{\alpha-1}\cdot\exp\left(\frac{\alpha-1}{\alpha\sigma^2}\left[\left(1+\Theta t^{\rho}\frac{\sigma^2}{1-\alpha}\right)^{\alpha}-1\right]\right).\\
h(t) &= h_0(t)\left(1+\frac{\sigma^2}{1-\alpha}H_0(t)\right)^{\alpha-1} = \Theta\rho t^{\rho-1}\left(1+\frac{\sigma^2}{1-\alpha}\Theta t^{\rho}\right)^{\alpha-1}. \qquad (4.35)
\end{aligned}
$$

Generalized Weibull-CPD Frailty Model

The GW function base force of mortality and cumulative intensity rate is given in Equations (4.15,4.16) respectively.

$$
\begin{aligned}
S(t) &= \exp\left(\frac{-k}{\alpha}[(\beta+H_0(t))^{\alpha}-\beta^{\alpha}]\right),\\
&= \exp\left(\frac{-k}{\alpha}[(\beta-\log(1-[1-\exp(-\lambda t^{\rho})]^{b}))^{\alpha}-\beta^{\alpha}]\right).\\
f(t) &= kh_0(t)(\beta+H_0(t))^{\alpha-1}\exp\left(\frac{-k}{\alpha}[(\beta+H_0(t))^{\alpha}-\beta^{\alpha}]\right),\\
&= k\left(\frac{b(1-\exp(-\lambda t^{\rho}))^{b-1}\lambda\rho t^{\rho-1}\exp(-\lambda t^{\rho})}{1-[1-\exp(-\lambda t^{\rho})]^{b}}\right)\cdot\left(\beta-\log(1-[1-\exp(-\lambda t^{\rho})]^{b})\right)^{\alpha-1}\\
&\cdot\exp\left(\frac{-k}{\alpha}[(\beta-\log(1-[1-\exp(-\lambda t^{\rho})]^{b}))^{\alpha}-\beta^{\alpha}]\right).\\
h(t) &= \frac{k(\frac{b(1-\exp(-\lambda t^{\rho}))^{b-1}\lambda\rho t^{\rho-1}\exp(-\lambda t^{\rho})}{1-[1-\exp(-\lambda t^{\rho})]^{b}})(\beta-\log(1-[1-\exp(-\lambda t^{\rho})]^{b}))^{\alpha-1}}{\exp\{\frac{-k}{\alpha}[(\beta-\log(1-[1-\exp(-\lambda t^{\rho})]^{b}))^{\alpha}-\beta^{\alpha}]\}}\\
&\cdot\exp\left(\frac{-k}{\alpha}[(\beta-\log(1-[1-\exp(-\lambda t^{\rho})]^{b}))^{\alpha}-\beta^{\alpha}]\right).\\
&= k\left(\frac{b(1-\exp(-\lambda t^{\rho}))^{b-1}\lambda\rho t^{\rho-1}\exp(-\lambda t^{\rho})}{1-[1-\exp(-\lambda t^{\rho})]^{b}}\right)\cdot\left(\beta-\log(1-[1-\exp(-\lambda t^{\rho})]^{b})\right)^{\alpha-1}.
\end{aligned}
$$

For identifiability reasons we apply Laplace transform derived in Equation (3.33). The univariate marginal survivor, density and intensity rate for the generalized Weibull-CPD mixture becomes:

$$
\begin{aligned}
S(t) &= \exp\left(\frac{\alpha-1}{\alpha\sigma^2}\left[\left(1+H_0(t)\frac{\sigma^2}{1-\alpha}\right)^{\alpha}-1\right]\right),\\
&= \exp\left(\frac{\alpha-1}{\alpha\sigma^2}((1-\log(1-[1-\exp(-\lambda t^{\rho})]^{b})\frac{\sigma^2}{1-\alpha})^{\alpha}-1)\right).\\
f(t) &= h_0(t)\cdot\left(1+H_0(t)\frac{\sigma^2}{1-\alpha}\right)^{\alpha-1}\cdot\exp\left(\frac{\alpha-1}{\alpha\sigma^2}\left[\left(1+H_0(t)\frac{\sigma^2}{1-\alpha}\right)^{\alpha}-1\right]\right),\\
&= \left(\frac{b(1-\exp(-\lambda t^{\rho}))^{b-1}\lambda\rho t^{\rho-1}\exp(-\lambda t^{\rho})}{1-[1-\exp(-\lambda t^{\rho})]^{b}}\right)\cdot\left(1-\log(1-[1-\exp(-\lambda t^{\rho})]^{b})\frac{\sigma^2}{1-\alpha}\right)^{\alpha-1}\\
&\cdot\exp\left(\frac{\alpha-1}{\alpha\sigma^2}\left[\left(1-\log(1-[1-\exp(-\lambda t^{\rho})]^{b})\frac{\sigma^2}{1-\alpha}\right)^{\alpha}-1\right]\right).\\
h(t) &= h_0(t)(1+\frac{\sigma^2}{1-\alpha}H_0(t))^{\alpha-1},\\
&= \left(\frac{b(1-\exp(-\lambda t^{\rho}))^{b-1}\lambda\rho t^{\rho-1}\exp(-\lambda t^{\rho})}{1-[1-\exp(-\lambda t^{\rho})]^{b}}\right)\left(1-\frac{\sigma^2}{1-\alpha}\log(1-[1-\exp(-\lambda t^{\rho})]^{b})\right)^{\alpha-1}.
\end{aligned}
\tag{4.36}
$$

Exponential-CPD Frailty Model

The Exponential function is derived from the Weibull distribution when $\rho = 1$. Putting $\rho = 1$ in Equations (4.13,4.14) leads to

$$
\begin{aligned}
&h_0(t) = \beta.\\
&H_0(t) = \beta t.\\
&Using\ the\ parametrization\ \beta = \Theta\\
&h_0(t) = \Theta.\\
&H_0(t) = \Theta t.\\
&S(t) = \exp\left(\frac{-k}{\alpha}[(\beta+H_0(t))^{\alpha}-\beta^{\alpha}]\right) = \exp\left(\frac{-k}{\alpha}[(\beta+\Theta t)^{\alpha}-\beta^{\alpha}]\right).\\
&f(t) = kh_0(t)(\beta+H_0(t))^{\alpha-1}\exp\left(\frac{-k}{\alpha}[(\beta+H_0(t))^{\alpha}-\beta^{\alpha}]\right),\\
&\quad = k\Theta(\beta+\Theta t)^{\alpha-1}\exp\{\frac{-k}{\alpha}[(\beta+\Theta t)^{\alpha}-\beta^{\alpha}]\}.
\end{aligned}
$$

$$h(t) = \frac{k\Theta(\beta+\Theta t)^{\alpha-1}\exp\left(\frac{-k}{\alpha}[(\beta+\Theta t)^{\alpha}-\beta^{\alpha}]\right)}{\exp\{\frac{-k}{\alpha}[(\beta+\Theta t)^{\alpha}-\beta^{\alpha}]\}} = k\Theta(\beta+\Theta t)^{\alpha-1}.$$

For identifiability reasons we apply Laplace transform derived in Equation (3.33). The univariate marginal survivor, density and intensity rate for the exponential-CPD mixture becomes:

$$\begin{aligned}
S(t) &= \exp\left(\frac{\alpha-1}{\alpha\sigma^2}\left[\left(1+H_0(t)\frac{\sigma^2}{1-\alpha}\right)^{\alpha}-1\right]\right), \\
&= \exp\left(\frac{\alpha-1}{\alpha\sigma^2}\left[\left(1+\Theta t\frac{\sigma^2}{1-\alpha}\right)^{\alpha}-1\right]\right). \\
f(t) &= h_0(t)\cdot\left(1+H_0(t)\frac{\sigma^2}{1-\alpha}\right)^{\alpha-1}\cdot\exp\left(\frac{\alpha-1}{\alpha\sigma^2}\left[\left(1+H_0(t)\frac{\sigma^2}{1-\alpha}\right)^{\alpha}-1\right]\right), \\
&= \Theta\cdot\left(1+\Theta t\frac{\sigma^2}{1-\alpha}\right)^{\alpha-1}\cdot\exp\left(\frac{\alpha-1}{\alpha\sigma^2}\left[\left(1+\Theta t\frac{\sigma^2}{1-\alpha}\right)^{\alpha}-1\right]\right). \\
h(t) &= h_0(t)\left(1+\frac{\sigma^2}{1-\alpha}H_0(t)\right)^{\alpha-1} = \Theta\left(1+\frac{\sigma^2}{1-\alpha}\Theta t\right)^{\alpha-1}. \qquad (4.37)
\end{aligned}$$

Generalized Exponential-CPD Frailty Model

The GE base force of mortality and cumulative intensity rate is given in Equations (4.17,4.18) respectively.

$$\begin{aligned}
S(t) &= \exp\left(\frac{-k}{\alpha}[(\beta+H_0(t))^{\alpha}-\beta^{\alpha}]\right), \\
&= \exp\left(\frac{-k}{\alpha}[(\beta-\log(1-[1-\exp(-\lambda t)]^{b}))^{\alpha}-\beta^{\alpha}]\right). \\
f(t) &= kh_0(t)(\beta+H_0(t))^{\alpha-1}\exp\left(\frac{-k}{\alpha}[(\beta+H_0(t))^{\alpha}-\beta^{\alpha}]\right), \\
&= k\left(\frac{b\lambda(1-\exp(-\lambda t))^{b-1}\exp(-\lambda t)}{1-[1-\exp(-\lambda t)]^{b}}\right)\cdot(\beta-\log(1-[1-\exp(-\lambda t)]^{b}))^{\alpha-1} \\
&\quad\cdot\exp\left(\frac{-k}{\alpha}[(\beta-\log(1-[1-\exp(-\lambda t)]^{b}))^{\alpha}-\beta^{\alpha}]\right)
\end{aligned}$$

$$
\begin{aligned}
h(t) &= \frac{k(\frac{b\lambda(1-\exp(-\lambda t))^{b-1}\exp(-\lambda t)}{1-[1-\exp(-\lambda t)]^b})(\beta-\log(1-[1-\exp(-\lambda t)]^b))^{\alpha-1}}{\exp\{\frac{-k}{\alpha}[(\beta-\log(1-[1-\exp(-\lambda t)]^b))^\alpha-\beta^\alpha]\}}, \\
&\cdot \exp\left(\frac{-k}{\alpha}[(\beta-\log(1-[1-\exp(-\lambda t)]^b))^\alpha-\beta^\alpha]\right). \\
&= k\left(\frac{b\lambda(1-\exp(-\lambda t))^{b-1}\exp(-\lambda t)}{1-[1-\exp(-\lambda t)]^b}\right)\cdot(\beta-\log(1-[1-\exp(-\lambda t)]^b))^{\alpha-1}.
\end{aligned}
$$

For identifiability reasons we apply Laplace transform derived in Equation (3.33). The univariate marginal survivor, density and intensity rate for the generalized exponential-CPD mixture becomes:

$$
\begin{aligned}
S(t) &= \exp\left(\frac{\alpha-1}{\alpha\sigma^2}\left[\left(1+H_0(t)\frac{\sigma^2}{1-\alpha}\right)^\alpha-1\right]\right), \\
&= \exp\left(\frac{\alpha-1}{\alpha\sigma^2}\left[\left(1-\log(1-[1-\exp(-\lambda t)]^b)\frac{\sigma^2}{1-\alpha}\right)^\alpha-1\right]\right). \\
f(t) &= h_0(t)\cdot\left(1+H_0(t)\frac{\sigma^2}{1-\alpha}\right)^{\alpha-1}\cdot\exp\left(\frac{\alpha-1}{\alpha\sigma^2}\left[\left(1+H_0(t)\frac{\sigma^2}{1-\alpha}\right)^\alpha-1\right]\right), \\
&= \left(\frac{b\lambda(1-\exp(-\lambda t))^{b-1}\exp(-\lambda t)}{1-[1-\exp(-\lambda t)]^b}\right)\cdot\left(1-\log(1-[1-\exp(-\lambda t)]^b)\frac{\sigma^2}{1-\alpha}\right)^{\alpha-1} \\
&\cdot\exp\left(\frac{\alpha-1}{\alpha\sigma^2}\left[\left(1-\log(1-[1-\exp(-\lambda t)]^b)\frac{\sigma^2}{1-\alpha}\right)^\alpha-1\right]\right). \\
h(t) &= h_0(t)\left(1+\frac{\sigma^2}{1-\alpha}H_0(t)\right)^{\alpha-1}, \\
&= \left(\frac{b\lambda(1-\exp(-\lambda t))^{b-1}\exp(-\lambda t)}{1-[1-\exp(-\lambda t)]^b}\right)\cdot\left(1-\log(1-[1-\exp(-\lambda t)]^b)\frac{\sigma^2}{1-\alpha}\right)^{\alpha-1}.
\end{aligned}
\tag{4.38}
$$

Loglogistic-CPD Frailty Model
The loglogistic function base force of mortality and cumulative intensity rate is given in Equations (4.20,4.21) respectively.

$$
S(t) = \exp\left(\frac{-k}{\alpha}[(\beta+H_0(t))^\alpha-\beta^\alpha]\right) = \exp\left(\frac{-k}{\alpha}\left[\left(\beta+\log(1+(\frac{t}{\lambda})^\Theta)\right)^\alpha-\beta^\alpha\right]\right).
$$

$$
\begin{aligned}
f(t) &= kh_0(t)(\beta + H_0(t))^{\alpha-1}\exp\left(\frac{-k}{\alpha}\left[(\beta + H_0(t))^{\alpha} - \beta^{\alpha}\right]\right),\\
&= k\frac{(\Theta/\lambda)(t/\lambda)^{\Theta-1}}{1+(t/\lambda)^{\Theta}}\left(\beta + \log(1+(\frac{t}{\lambda})^{\Theta})\right)^{\alpha-1}\exp\left(\frac{-k}{\alpha}\left[\left(\beta + \log(1+(\frac{t}{\lambda})^{\Theta})\right)^{\alpha} - \beta^{\alpha}\right]\right).\\
h(t) &= \frac{k\frac{(\Theta/\lambda)(t/\lambda)^{\Theta-1}}{1+(t/\lambda)^{\Theta}}(\beta + \log(1+(\frac{t}{\lambda})^{\Theta}))^{\alpha-1}\exp\{\frac{-k}{\alpha}[(\beta+\log(1+(\frac{t}{\lambda})^{\Theta}))^{\alpha}-\beta^{\alpha}]\}}{\exp\{\frac{-k}{\alpha}[(\beta+\log(1+(\frac{t}{\lambda})^{\Theta}))^{\alpha}-\beta^{\alpha}]\}},\\
&= k\frac{(\Theta/\lambda)(t/\lambda)^{\Theta-1}}{1+(t/\lambda)^{\Theta}}\left(\beta + \log(1+(\frac{t}{\lambda})^{\Theta})\right)^{\alpha-1}.
\end{aligned}
$$

For identifiability reasons we apply Laplace transform derived in Equation (3.33). The univariate marginal survivor, density and intensity rate for the loglogistic-CPD mixture becomes:

$$
\begin{aligned}
S(t) &= \exp\left(\frac{\alpha-1}{\alpha\sigma^2}\left[\left(1 + H_0(t)\frac{\sigma^2}{1-\alpha}\right)^{\alpha} - 1\right]\right),\\
&= \exp\left(\frac{\alpha-1}{\alpha\sigma^2}\left[\left(1 + \log(1+(\frac{t}{\lambda})^{\Theta})\frac{\sigma^2}{1-\alpha}\right)^{\alpha} - 1\right]\right).\\
f(t) &= h_0(t)\cdot\left(1 + H_0(t)\frac{\sigma^2}{1-\alpha}\right)^{\alpha-1}\cdot\exp\left(\frac{\alpha-1}{\alpha\sigma^2}\left[\left(1 + H_0(t)\frac{\sigma^2}{1-\alpha}\right)^{\alpha} - 1\right]\right),\\
&= \frac{(\Theta/\lambda)(t/\lambda)^{\Theta-1}}{1+(t/\lambda)^{\Theta}}\cdot\left(1 + \log(1+(\frac{t}{\lambda})^{\Theta})\frac{\sigma^2}{1-\alpha}\right)^{\alpha-1}\\
&\cdot\exp\left(\frac{\alpha-1}{\alpha\sigma^2}\left[\left(1 + \log(1+(\frac{t}{\lambda})^{\Theta})\frac{\sigma^2}{1-\alpha}\right)^{\alpha} - 1\right]\right).\\
h(t) &= h_0(t)\left(1 + \frac{\sigma^2}{1-\alpha}H_0(t)\right)^{\alpha-1}\\
&= \frac{(\Theta/\lambda)(t/\lambda)^{\Theta-1}}{1+(t/\lambda)^{\Theta}}\left(1 + \frac{\sigma^2}{1-\alpha}\log(1+(\frac{t}{\lambda})^{\Theta})\right)^{\alpha-1}. \qquad (4.39)
\end{aligned}
$$

Lognormal-CPD Frailty Model
The lognormal function base force of mortality and cumulative intensity rate is given in Equations (4.23,4.24) respectively.

$$
\begin{aligned}
S(t) &= \exp\left(\frac{-k}{\alpha}[(\beta + H_0(t))^\alpha - \beta^\alpha]\right),\\
&= \exp\left(\frac{-k}{\alpha}\left[\left(\beta - \log[1-\Phi(\frac{\log t-\mu}{\sigma})]\right)^\alpha - \beta^\alpha\right]\right).\\
f(t) &= kh_0(t)(\beta + H_0(t))^{\alpha-1}\exp\left(\frac{-k}{\alpha}[(\beta + H_0(t))^\alpha - \beta^\alpha]\right),\\
&= k\frac{\phi(\frac{\log t-\mu}{\sigma})}{\sigma t[1-\Phi(\frac{\log t-\mu}{\sigma})]}\cdot\left(\beta - \log\left[1-\Phi(\frac{\log t-\mu}{\sigma})\right]\right)^{\alpha-1}\\
&\cdot\exp\left(\frac{-k}{\alpha}\left[\left(\beta - \log[1-\Phi(\frac{\log t-\mu}{\sigma})]\right)^\alpha - \beta^\alpha\right]\right).\\
h(t) &= \frac{k\frac{\phi(\frac{\log t-\mu}{\sigma})}{\sigma t[1-\Phi(\frac{\log t-\mu}{\sigma})]}(\beta - \log[1-\Phi(\frac{\log t-\mu}{\sigma})])^{\alpha-1}\exp\{\frac{-k}{\alpha}[(\beta-\log[1-\Phi(\frac{\log t-\mu}{\sigma})])^\alpha - \beta^\alpha]\}}{\exp\{\frac{-k}{\alpha}[(\beta-\log[1-\Phi(\frac{\log t-\mu}{\sigma})])^\alpha - \beta^\alpha]\}},\\
&= k\frac{\phi(\frac{\log t-\mu}{\sigma})}{\sigma t[1-\Phi(\frac{\log t-\mu}{\sigma})]}\cdot\left(\beta - \log\left[1-\Phi(\frac{\log t-\mu}{\sigma})\right]\right)^{\alpha-1}.
\end{aligned}
$$

For identifiability reasons we apply Laplace transform derived in Equation (3.33). The univariate marginal survivor, density and intensity rate for the lognormal-CPD mixture becomes:

$$
\begin{aligned}
S(t) &= \exp\left(\frac{\alpha-1}{\alpha\sigma^2}\left[\left(1 + H_0(t)\frac{\sigma^2}{1-\alpha}\right)^\alpha - 1\right]\right),\\
&= \exp\left(\frac{\alpha-1}{\alpha\sigma^2}\left[\left(1 - \log[1-\Phi(\frac{\log t-\mu}{\sigma})]\frac{\sigma^2}{1-\alpha}\right)^\alpha - 1\right]\right).\\
f(t) &= h_0(t)\cdot\left(1 + H_0(t)\frac{\sigma^2}{1-\alpha}\right)^{\alpha-1}\cdot\exp\left(\frac{\alpha-1}{\alpha\sigma^2}\left[\left(1 + H_0(t)\frac{\sigma^2}{1-\alpha}\right)^\alpha - 1\right]\right),\\
&= \frac{\phi(\frac{\log t-\mu}{\sigma})}{\sigma t[1-\Phi(\frac{\log t-\mu}{\sigma})]}\cdot\left(1 - \log\left[1-\Phi(\frac{\log t-\mu}{\sigma})\right]\frac{\sigma^2}{1-\alpha}\right)^{\alpha-1}\\
&\cdot\exp\left(\frac{\alpha-1}{\alpha\sigma^2}\left[\left(1 - \log[1-\Phi(\frac{\log t-\mu}{\sigma})]\frac{\sigma^2}{1-\alpha}\right)^\alpha - 1\right]\right).
\end{aligned}
$$

$$h(t) = h_0(t)\left(1+\frac{\sigma^2}{1-\alpha}H_0(t)\right)^{\alpha-1},$$
$$= \frac{\phi(\frac{\log t-\mu}{\sigma})}{\sigma t[1-\Phi(\frac{\log t-\mu}{\sigma})]}\left(1-\frac{\sigma^2}{1-\alpha}\log\left[1-\Phi(\frac{\log t-\mu}{\sigma})\right]\right)^{\alpha-1}. \quad (4.40)$$

Exponential-Power-CPD Frailty Model
The Exponential-Power function base force of mortality and cumulative intensity rate is given in Equations (4.26,4.27) respectively.

$$S(t) = \exp\left(\frac{-k}{\alpha}[(\beta+H_0(t))^{\alpha}-\beta^{\alpha}]\right) = \exp\left(\frac{-k}{\alpha}[(\beta+\exp{(rt^{\lambda})}-1)^{\alpha}-\beta^{\alpha}]\right).$$
$$f(t) = kh_0(t)(\beta+H_0(t))^{\alpha-1}\exp\left(\frac{-k}{\alpha}[(\beta+H_0(t))^{\alpha}-\beta^{\alpha}]\right),$$
$$= k\exp{(rt^{\lambda})}r\lambda t^{\lambda-1}(\beta+\exp{(rt^{\lambda})}-1)^{\alpha-1}\exp\left(\frac{-k}{\alpha}[(\beta+\exp{(rt^{\lambda})}-1)^{\alpha}-\beta^{\alpha}]\right).$$
$$h(t) = \frac{k\exp{(rt^{\lambda})}r\lambda t^{\lambda-1}(\beta+\exp{(rt^{\lambda})}-1)^{\alpha-1}\exp\{\frac{-k}{\alpha}[(\beta+\exp{(rt^{\lambda})}-1)^{\alpha}-\beta^{\alpha}]\}}{\exp\{\frac{-k}{\alpha}[(\beta+\exp{(rt^{\lambda})}-1)^{\alpha}-\beta^{\alpha}]\}},$$
$$= k\exp{(rt^{\lambda})}r\lambda t^{\lambda-1}(\beta+\exp{(rt^{\lambda})}-1)^{\alpha-1}.$$

For identifiability reasons we apply Laplace transform derived in Equation (3.33). The univariate marginal survivor, density and intensity rate for the exponential-power-CPD mixture becomes:

$$S(t) = \exp\left(\frac{\alpha-1}{\alpha\sigma^2}\left[\left(1+H_0(t)\frac{\sigma^2}{1-\alpha}\right)^{\alpha}-1\right]\right),$$
$$= \exp\left(\frac{\alpha-1}{\alpha\sigma^2}\left[\left(1+(\exp{(rt^{\lambda})}-1)\frac{\sigma^2}{1-\alpha}\right)^{\alpha}-1\right]\right).$$
$$f(t) = h_0(t)\cdot\left(1+H_0(t)\frac{\sigma^2}{1-\alpha}\right)^{\alpha-1}\cdot\exp\left(\frac{\alpha-1}{\alpha\sigma^2}\left[\left(1+H_0(t)\frac{\sigma^2}{1-\alpha}\right)^{\alpha}-1\right]\right),$$
$$= \exp{(rt^{\lambda})}r\lambda t^{\lambda-1}\cdot\left(1+(\exp{(rt^{\lambda})}-1)\frac{\sigma^2}{1-\alpha}\right)^{\alpha-1}$$
$$\cdot\exp\left(\frac{\alpha-1}{\alpha\sigma^2}\left[\left(1+(\exp{(rt^{\lambda})}-1)\frac{\sigma^2}{1-\alpha}\right)^{\alpha}-1\right]\right).$$

$$h(t) = h_0(t)\left(1 + \frac{\sigma^2}{1-\alpha}H_0(t)\right)^{\alpha-1}$$
$$= \exp(rt^\lambda)r\lambda t^{\lambda-1}\left(1 + \frac{\sigma^2}{1-\alpha}\exp(rt^\lambda) - 1\right)^{\alpha-1}. \quad (4.41)$$

Pareto-CPD Frailty Model
The Pareto function base force of mortality and cumulative intensity rate is given in Equations (4.29,4.30) respectively.

$$S(t) = \exp\left(\frac{-k}{\alpha}[(\beta + H_0(t))^\alpha - \beta^\alpha]\right) = \exp\left(\frac{-k}{\alpha}\left[\left(\beta + \log\left[\frac{t}{r}\right]^\vartheta\right)^\alpha - \beta^\alpha\right]\right).$$
$$f(t) = kh_0(t)(\beta + H_0(t))^{\alpha-1}\exp\left(\frac{-k}{\alpha}[(\beta + H_0(t))^\alpha - \beta^\alpha]\right),$$
$$= k\frac{\vartheta}{t}\left(\beta + \log\left[\frac{t}{r}\right]^\vartheta\right)^{\alpha-1}\exp\left(\frac{-k}{\alpha}\left[\left(\beta + \log\left[\frac{t}{r}\right]^\vartheta\right)^\alpha - \beta^\alpha\right]\right).$$
$$h(t) = \frac{k\frac{\vartheta}{t}(\beta + \log(\frac{t}{r})^\vartheta)^{\alpha-1}\exp\{\frac{-k}{\alpha}[(\beta + \log(\frac{t}{r})^\vartheta)^\alpha - \beta^\alpha]\}}{\exp\{\frac{-k}{\alpha}[(\beta + \log(\frac{t}{r})^\vartheta)^\alpha - \beta^\alpha]\}} = k\frac{\vartheta}{t}\left(\beta + \log\left[\frac{t}{r}\right]^\vartheta\right)^{\alpha-1}.$$

For identifiability reasons we apply Laplace transform derived in Equation (3.33). The univariate marginal survivor, density and intensity rate for the Pareto-CPD mixture becomes:

$$S(t) = \exp\left(\frac{\alpha-1}{\alpha\sigma^2}\left[\left(1 + H_0(t)\frac{\sigma^2}{1-\alpha}\right)^\alpha - 1\right]\right),$$
$$= \exp\left(\frac{\alpha-1}{\alpha\sigma^2}[\left(1 + \log\left[\frac{t}{r}\right]^\vartheta\frac{\sigma^2}{1-\alpha}\right)^\alpha - 1\right].$$
$$f(t) = h_0(t)\cdot\left(1 + H_0(t)\frac{\sigma^2}{1-\alpha}\right)^{\alpha-1}\cdot\exp\left(\frac{\alpha-1}{\alpha\sigma^2}\left[\left(1 + H_0(t)\frac{\sigma^2}{1-\alpha}\right)^\alpha - 1\right]\right),$$
$$= \frac{\vartheta}{t}\cdot\left(1 + \log\left[\frac{t}{r}\right]^\vartheta\frac{\sigma^2}{1-\alpha}\right)^{\alpha-1}\cdot\exp\left(\frac{\alpha-1}{\alpha\sigma^2}\left[\left(1 + \log\left[\frac{t}{r}\right]^\vartheta\frac{\sigma^2}{1-\alpha}\right)^\alpha - 1\right]\right).$$
$$h(t) = h_0(t)\left(1 + \frac{\sigma^2}{1-\alpha}H_0(t)\right)^{\alpha-1} = \frac{\vartheta}{t}\left(1 + \frac{\sigma^2}{1-\alpha}\log\left[\frac{t}{r}\right]^\vartheta\right)^{\alpha-1}. \quad (4.42)$$

4.6 Non-Central Gamma

The NCG is considered a special case of the CPD with gamma distributed weights.

Construction of Non-central Gamma Univariate Frailty Model
The NCG density with a being the non-centrality parameter and β the scale parameter is

$$f(u) = \sum_{j=0}^{\infty} \frac{\exp(-a\beta)(a\beta)^j}{j!} \cdot \left[\frac{u^{b+j-1} \exp(-\frac{u}{\beta})}{\Gamma(b+j)\beta^{b+j}} \right].$$

The Laplace transform derived in Equation (3.42) is

$$\begin{aligned}
\mathbb{L}_U(s) &= \exp\left(-\frac{sa\beta^2}{1+\beta s}\right), \\
\textit{Since} \quad S(t) &= \mathbb{L}_U(H_0(t)) = \exp\left(-\frac{H_0(t)a\beta^2}{1+\beta H_0(t)}\right). \\
f(t) &= -\mathbb{L}'_U(H_0(t)) = -\frac{d}{dt}\exp\left(-\frac{H_0(t)a\beta^2}{1+\beta H_0(t)}\right), \\
&= \frac{h_0(t)a\beta^2}{(1+\beta H_0(t))^2}\exp\left(-\frac{H_0(t)a\beta^2}{1+\beta H_0(t)}\right). \\
h(t) &= \frac{\frac{h_0(t)a\beta^2}{(1+\beta H_0(t))^2}\exp\{-\frac{H_0(t)a\beta^2}{1+\beta H_0(t)}\}}{\exp\{-\frac{H_0(t)a\beta^2}{1+\beta H_0(t)}\}} = \frac{h_0(t)a\beta^2}{(1+\beta H_0(t))^2}. \qquad (4.43)
\end{aligned}$$

Considering various choices of the base force of mortality function, we have the following NCG frailty models.

Gompertz-NCG Frailty Model
The Gompertz function base force of mortality and cumulative intensity rate is given in Equations (4.11,4.12) respectively.

$$S(t) = \exp\left(-\frac{H_0(t)a\beta^2}{1+\beta H_0(t)}\right) = \exp\left(-\frac{\frac{A}{B}(\exp(Bt)-1)a\beta^2}{1+\beta\frac{A}{B}(\exp(Bt)-1)}\right).$$

$$f(t) = \frac{h_0(t)a\beta^2}{(1+\beta H_0(t))^2}\exp\left(-\frac{H_0(t)a\beta^2}{1+\beta H_0(t)}\right),$$

$$= \frac{(A\exp(Bt))a\beta^2}{(1+\beta\frac{A}{B}(\exp(Bt)-1))^2}\exp\left(-\frac{\frac{A}{B}(\exp(Bt)-1)a\beta^2}{1+\beta\frac{A}{B}(\exp(Bt)-1)}\right).$$

$$h(t) = \frac{\frac{(A\exp(Bt))a\beta^2}{(1+\beta\frac{A}{B}(\exp(Bt)-1))^2}\exp\{-\frac{\frac{A}{B}(\exp(Bt)-1)a\beta^2}{1+\beta\frac{A}{B}(\exp(Bt)-1)}\}}{\exp\{-\frac{\frac{A}{B}(\exp(Bt)-1)a\beta^2}{1+\beta\frac{A}{B}(\exp(Bt)-1)}\}} = \frac{(A\exp(Bt))a\beta^2}{(1+\beta\frac{A}{B}(\exp(Bt)-1))^2}.$$

For identifiability reasons we apply Laplace transform derived in Equation (3.43). The univariate marginal survivor, density and intensity rate for the Gompertz-NCG mixture becomes:

$$S(t) = \exp\left(-\frac{H_0(t)}{1+\frac{1}{2}\sigma^2 H_0(t)}\right) = \exp\left(-\frac{\frac{A}{B}(\exp(Bt)-1)}{1+\frac{1}{2}\sigma^2\frac{A}{B}(\exp(Bt)-1)}\right).$$

$$f(t) = \left(1+\frac{1}{2}\sigma^2 H_0(t)\right)^{-2} h_0(t)\cdot\exp\left(-\frac{H_0(t)}{1+\frac{1}{2}\sigma^2 H_0(t)}\right),$$

$$= \left(1+\frac{1}{2}\sigma^2\frac{A}{B}(\exp(Bt)-1)\right)^{-2}(A\exp(Bt))\cdot\exp\left(-\frac{\frac{A}{B}(\exp(Bt)-1)}{1+\frac{1}{2}\sigma^2\frac{A}{B}(\exp(Bt)-1)}\right).$$

$$h(t) = \left(1+\frac{1}{2}\sigma^2 H_0(t)\right)^{-2}\cdot h_0(t)$$

$$= \left(1+\frac{1}{2}\sigma^2\frac{A}{B}(\exp(Bt)-1)\right)^{-2}\cdot A\exp(Bt). \tag{4.44}$$

Weibull-NCG Frailty Model
The Weibull function base force of mortality and cumulative intensity rate is given in Equations (4.13,4.14) respectively. Using the parametrization $\beta = \Theta$;

$$
\begin{aligned}
S(t) &= \exp\left(-\frac{H_0(t)a\beta^2}{1+\beta H_0(t)}\right) = \exp\left(-\frac{\Theta t^\rho a\beta^2}{1+\beta\Theta t^\rho}\right). \\
f(t) &= \frac{h_0(t)a\beta^2}{(1+\beta H_0(t))^2}\exp\left(-\frac{H_0(t)a\beta^2}{1+\beta H_0(t)}\right), \\
&= \frac{(\Theta\rho t^{\rho-1})a\beta^2}{(1+\beta\Theta t^\rho)^2}\exp\left(-\frac{\Theta t^\rho a\beta^2}{1+\beta(\Theta t^\rho)}\right). \\
h(t) &= \frac{\frac{(\Theta\rho t^{\rho-1})a\beta^2}{(1+\beta\Theta t^\rho)^2}\exp\{-\frac{\Theta t^\rho a\beta^2}{1+\beta(\Theta t^\rho)}\}}{\exp\{-\frac{\Theta t^\rho a\beta^2}{1+\beta\Theta t^\rho}\}} = \frac{(\Theta\rho t^{\rho-1})a\beta^2}{(1+\beta\Theta t^\rho)^2}.
\end{aligned}
$$

For identifiability reasons we apply Laplace transform derived in Equation (3.43). The univariate marginal survivor, density and intensity rate for the Weibull-NCG mixture becomes:

$$
\begin{aligned}
S(t) &= \exp\left(-\frac{H_0(t)}{1+\frac{1}{2}\sigma^2 H_0(t)}\right) = \exp\left(-\frac{\Theta t^\rho}{1+\frac{1}{2}\sigma^2(\Theta t^\rho)}\right). \\
f(t) &= \left(1+\frac{1}{2}\sigma^2 H_0(t)\right)^{-2} h_0(t)\cdot\exp\left(-\frac{H_0(t)}{1+\frac{1}{2}\sigma^2 H_0(t)}\right), \\
&= \left(1+\frac{1}{2}\sigma^2(\Theta t^\rho)\right)^{-2}(\Theta\rho t^{\rho-1})\cdot\exp\left(-\frac{\Theta t^\rho}{1+\frac{1}{2}\sigma^2(\Theta t^\rho)}\right). \\
h(t) &= \left(1+\frac{1}{2}\sigma^2 H_0(t)\right)^{-2} h_0(t) = \left(1+\frac{1}{2}\sigma^2(\Theta t^\rho)\right)^{-2}(\Theta\rho t^{\rho-1}). \qquad (4.45)
\end{aligned}
$$

Generalized Weibull-NCG Frailty Model
The GW function base force of mortality and cumulative intensity rate is given in Equations (4.15,4.16) respectively.

$$
\begin{aligned}
S(t) &= \exp\left(-\frac{H_0(t)a\beta^2}{1+\beta H_0(t)}\right) = \exp\left(\frac{\log(1-[1-\exp(-\lambda t^\rho)]^b)a\beta^2}{1-\log(1-[1-\exp(-\lambda t^\rho)]^b)\beta}\right). \\
f(t) &= \frac{h_0(t)a\beta^2}{(1+\beta H_0(t))^2}\exp\left(-\frac{H_0(t)a\beta^2}{1+\beta H_0(t)}\right), \\
&= \frac{(\frac{b(1-\exp(-\lambda t^\rho))^{b-1}\lambda\rho t^{\rho-1}\exp(-\lambda t^\rho)}{1-[1-\exp(-\lambda t^\rho)]^b})a\beta^2}{(1-\log(1-[1-\exp(-\lambda t^\rho)]^b)\beta)^2}\exp\left(\frac{\log(1-[1-\exp(-\lambda t^\rho)]^b)a\beta^2}{1-\log(1-[1-\exp(-\lambda t^\rho)]^b)\beta}\right).
\end{aligned}
$$

$$h(t) = \frac{\frac{\left(\frac{b(1-\exp(-\lambda t^{\rho}))^{b-1}\lambda\rho t^{\rho-1}\exp(-\lambda t^{\rho})}{1-[1-\exp(-\lambda t^{\rho})]^{b}}\right)a\beta^2}{(1-\log(1-[1-\exp(-\lambda t^{\rho})]^{b})\beta)^2}\exp\left\{\frac{\log(1-[1-\exp(-\lambda t^{\rho})]^{b})a\beta^2}{1-\log(1-[1-\exp(-\lambda t^{\rho})]^{b})\beta}\right\}}{\exp\left\{\frac{\log(1-[1-\exp(-\lambda t^{\rho})]^{b})a\beta^2}{1-\log(1-[1-\exp(-\lambda t^{\rho})]^{b})\beta}\right\}},$$

$$= \frac{\left(\frac{b(1-\exp(-\lambda t^{\rho}))^{b-1}\lambda\rho t^{\rho-1}\exp(-\lambda t^{\rho})}{1-[1-\exp(-\lambda t^{\rho})]^{b}}\right)a\beta^2}{(1-\log(1-[1-\exp(-\lambda t^{\rho})]^{b})\beta)^2}.$$

For identifiability reasons we apply Laplace transform derived in Equation (3.43). The univariate marginal survivor, density and intensity rate for the generalized Weibull-NCG mixture becomes:

$$\begin{aligned}
S(t) &= \exp\left(-\frac{H_0(t)}{1+\frac{1}{2}\sigma^2 H_0(t)}\right) = \exp\left(\frac{\log(1-[1-\exp(-\lambda t^{\rho})]^{b})}{1-\log(1-[1-\exp(-\lambda t^{\rho})]^{b})\frac{1}{2}\sigma^2}\right).\\
f(t) &= \left(1+\frac{1}{2}\sigma^2 H_0(t)\right)^{-2} h_0(t)\cdot\exp\left(-\frac{H_0(t)}{1+\frac{1}{2}\sigma^2 H_0(t)}\right),\\
&= \left(1-\log\left(1-[1-\exp(-\lambda t^{\rho})]^{b}\right)\frac{1}{2}\sigma^2\right)^{-2}\left(\frac{b(1-\exp(-\lambda t^{\rho}))^{b-1}\lambda\rho t^{\rho-1}\exp(-\lambda t^{\rho})}{1-[1-\exp(-\lambda t^{\rho})]^{b}}\right)\\
&\cdot\exp\left(\frac{\log(1-[1-\exp(-\lambda t^{\rho})]^{b})}{1-\log(1-[1-\exp(-\lambda t^{\rho})]^{b})\frac{1}{2}\sigma^2}\right).\\
h(t) &= \left(1+\frac{1}{2}\sigma^2 H_0(t)\right)^{-2} h_0(t),\\
&= \left(1-\log(1-[1-\exp(-\lambda t^{\rho})]^{b})\frac{1}{2}\sigma^2\right)^{-2}\left(\frac{b(1-\exp(-\lambda t^{\rho}))^{b-1}\lambda\rho t^{\rho-1}\exp(-\lambda t^{\rho})}{1-[1-\exp(-\lambda t^{\rho})]^{b}}\right).
\end{aligned} \tag{4.46}$$

Exponential-NCG Frailty Model

The Exponential function is derived from the Weibull distribution when $\rho = 1$. Putting $\rho = 1$ in Equations (4.13,4.14) leads to

$$h_0(t) = \beta.$$

$$H_0(t) = \beta t.$$

Using parametrization $\beta = \Theta$

$$\begin{aligned}
h_0(t) &= \Theta. \\
H_0(t) &= \Theta t. \\
S(t) &= \exp\left(-\frac{H_0(t)a\beta^2}{1+\beta H_0(t)}\right) = \exp\left(-\frac{\Theta t a\beta^2}{1+\beta\Theta t}\right).
\end{aligned}$$

$$\begin{aligned}
f(t) &= \frac{h_0(t)a\beta^2}{(1+\beta H_0(t))^2}\exp\left(-\frac{H_0(t)a\beta^2}{1+\beta H_0(t)}\right), \\
&= \frac{\Theta a\beta^2}{(1+\beta\Theta t)^2}\exp\left(-\frac{\Theta t a\beta^2}{1+\beta\Theta t}\right). \\
h(t) &= \frac{\frac{\Theta a\beta^2}{(1+\beta\Theta t)^2}\exp\{-\frac{\Theta t a\beta^2}{1+\beta\Theta t}\}}{\exp\{-\frac{\Theta t a\beta^2}{1+\beta\Theta t}\}} = \frac{\Theta a\beta^2}{(1+\beta\Theta t)^2}.
\end{aligned}$$

For identifiability reasons we apply Laplace transform derived in Equation (3.43). The univariate marginal survivor, density and intensity rate for the exponential-NCG mixture becomes:

$$\begin{aligned}
S(t) &= \exp\left(-\frac{H_0(t)}{1+\frac{1}{2}\sigma^2 H_0(t)}\right) = \exp\left(-\frac{\Theta t}{1+\frac{1}{2}\sigma^2\Theta t}\right). \\
f(t) &= \left(1+\frac{1}{2}\sigma^2 H_0(t)\right)^{-2} h_0(t)\cdot\exp\left(-\frac{H_0(t)}{1+\frac{1}{2}\sigma^2 H_0(t)}\right), \\
&= \left(1+\frac{1}{2}\sigma^2\Theta t\right)^{-2}\Theta\cdot\exp\left(-\frac{\Theta t}{1+\frac{1}{2}\sigma^2\Theta t}\right). \\
h(t) &= \left(1+\frac{1}{2}\sigma^2 H_0(t)\right)^{-2}\cdot h_0(t) = \left(1+\frac{1}{2}\sigma^2\Theta t\right)^{-2}\Theta. \qquad (4.47)
\end{aligned}$$

Generalized Exponential-NCG Frailty Model
The GE base force of mortality and cumulative intensity rate is given in Equations (4.17,4.18) respectively.

$$
\begin{aligned}
S(t) &= \exp\left(-\frac{H_0(t)a\beta^2}{1+\beta H_0(t)}\right) = \exp\left(\frac{\log(1-[1-\exp(-\lambda t)]^b)a\beta^2}{1-\log(1-[1-\exp(-\lambda t)]^b)\beta}\right). \\
f(t) &= \frac{h_0(t)a\beta^2}{(1+\beta H_0(t))^2}\exp\left(-\frac{H_0(t)a\beta^2}{1+\beta H_0(t)}\right), \\
&= \frac{\left(\frac{b\lambda(1-\exp(-\lambda t))^{b-1}\exp(-\lambda t)}{1-[1-\exp(-\lambda t)]^b}\right)a\beta^2}{(1-\log(1-[1-\exp(-\lambda t)]^b)\beta)^2}\exp\left(\frac{\log(1-[1-\exp(-\lambda t)]^b)a\beta^2}{1-\log(1-[1-\exp(-\lambda t)]^b)\beta}\right). \\
h(t) &= \frac{\frac{\left(\frac{b\lambda(1-\exp(-\lambda t))^{b-1}\exp(-\lambda t)}{1-[1-\exp(-\lambda t)]^b}\right)a\beta^2}{(1-\log(1-[1-\exp(-\lambda t)]^b)\beta)^2}\exp\left\{\frac{\log(1-[1-\exp(-\lambda t)]^b)a\beta^2}{1-\log(1-[1-\exp(-\lambda t)]^b)\beta}\right\}}{\exp\left\{\frac{\log(1-[1-\exp(-\lambda t)]^b)a\beta^2}{1-\log(1-[1-\exp(-\lambda t)]^b)\beta}\right\}}, \\
&= \frac{\left(\frac{b\lambda(1-\exp(-\lambda t))^{b-1}\exp(-\lambda t)}{1-[1-\exp(-\lambda t)]^b}\right)a\beta^2}{(1-\log(1-[1-\exp(-\lambda t)]^b)\beta)^2}
\end{aligned}
$$

For identifiability reasons we apply Laplace transform derived in Equation (3.43). The univariate marginal survivor, density and intensity rate for the generalized exponential-NCG mixture becomes:

$$
\begin{aligned}
S(t) &= \exp\left(-\frac{H_0(t)}{1+\frac{1}{2}\sigma^2 H_0(t)}\right) = \exp\left(\frac{\log(1-[1-\exp(-\lambda t)]^b)}{1-\log(1-[1-\exp(-\lambda t)]^b)\frac{1}{2}\sigma^2}\right). \\
f(t) &= \left(1+\frac{1}{2}\sigma^2 H_0(t)\right)^{-2} h_0(t)\cdot\exp\left(-\frac{H_0(t)}{1+\frac{1}{2}\sigma^2 H_0(t)}\right), \\
&= \left(1-\log\left(1-[1-\exp(-\lambda t)]^b\right)\frac{1}{2}\sigma^2\right)^{-2}\left(\frac{b\lambda(1-\exp(-\lambda t))^{b-1}\exp(-\lambda t)}{1-[1-\exp(-\lambda t)]^b}\right) \\
&\cdot\exp\left(\frac{\log(1-[1-\exp(-\lambda t)]^b)}{1-\log(1-[1-\exp(-\lambda t)]^b)\frac{1}{2}\sigma^2}\right). \\
h(t) &= \left(1+\frac{1}{2}\sigma^2 H_0(t)\right)^{-2} h_0(t), \\
&= \left(1-\log(1-[1-\exp(-\lambda t)]^b)\frac{1}{2}\sigma^2\right)^{-2}\left(\frac{b\lambda(1-\exp(-\lambda t))^{b-1}\exp(-\lambda t)}{1-[1-\exp(-\lambda t)]^b}\right).
\end{aligned}
\tag{4.48}
$$

Loglogistic-NCG Frailty Model
The loglogistic function base force of mortality and cumulative intensity rate is given in Equations (4.20,4.21) respectively.

$$
\begin{aligned}
S(t) &= \exp\left(-\frac{H_0(t)a\beta^2}{1+\beta H_0(t)}\right) = \exp\left(-\frac{\log(1+(\frac{t}{\lambda})^{\Theta})a\beta^2}{1+\beta\log(1+(\frac{t}{\lambda})^{\Theta})}\right). \\
f(t) &= \frac{h_0(t)a\beta^2}{(1+\beta H_0(t))^2}\exp\left(-\frac{H_0(t)a\beta^2}{1+\beta H_0(t)}\right), \\
&= \frac{\frac{(\Theta/\lambda)(t/\lambda)^{\Theta-1}}{1+(t/\lambda)^{\Theta}}a\beta^2}{(1+\beta\log(1+(\frac{t}{\lambda})^{\Theta}))^2}\exp\left(-\frac{\log(1+(\frac{t}{\lambda})^{\Theta})a\beta^2}{1+\beta\log(1+(\frac{t}{\lambda})^{\Theta})}\right). \\
h(t) &= \frac{\frac{\frac{(\Theta/\lambda)(t/\lambda)^{\Theta-1}}{1+(t/\lambda)^{\Theta}}a\beta^2}{(1+\beta\log(1+(\frac{t}{\lambda})^{\Theta}))^2}\exp\{-\frac{\log(1+(\frac{t}{\lambda})^{\Theta})a\beta^2}{1+\beta\log(1+(\frac{t}{\lambda})^{\Theta})}\}}{\exp\{-\frac{\log(1+(\frac{t}{\lambda})^{\Theta})a\beta^2}{1+\beta\log(1+(\frac{t}{\lambda})^{\Theta})}\}} \\
&= \frac{\frac{(\Theta/\lambda)(t/\lambda)^{\Theta-1}}{1+(t/\lambda)^{\Theta}}a\beta^2}{(1+\beta\log(1+(\frac{t}{\lambda})^{\Theta}))^2}.
\end{aligned}
$$

For identifiability reasons we apply Laplace transform derived in Equation (3.43). The univariate marginal survivor, density and intensity rate for the loglogistic-NCG mixture becomes:

$$
\begin{aligned}
S(t) &= \exp\left(-\frac{H_0(t)}{1+\frac{1}{2}\sigma^2 H_0(t)}\right) = \exp\left(-\frac{\log(1+(\frac{t}{\lambda})^{\Theta})}{1+\frac{1}{2}\sigma^2\log(1+(\frac{t}{\lambda})^{\Theta})}\right). \\
f(t) &= \left(1+\frac{1}{2}\sigma^2 H_0(t)\right)^{-2} h_0(t)\cdot\exp\left(-\frac{H_0(t)}{1+\frac{1}{2}\sigma^2 H_0(t)}\right), \\
&= \left(1+\frac{1}{2}\sigma^2\log(1+(\frac{t}{\lambda})^{\Theta})\right)^{-2}\frac{(\Theta/\lambda)(t/\lambda)^{\Theta-1}}{1+(t/\lambda)^{\Theta}}\cdot\exp\left(-\frac{\log(1+(\frac{t}{\lambda})^{\Theta})}{1+\frac{1}{2}\sigma^2\log(1+(\frac{t}{\lambda})^{\Theta})}\right). \\
h(t) &= \left(1+\frac{1}{2}\sigma^2 H_0(t)\right)^{-2}\cdot h_0(t) \\
&= \left(1+\frac{1}{2}\sigma^2\log\left[1+\left(\frac{t}{\lambda}\right)^{\Theta}\right]\right)^{-2}\frac{(\Theta/\lambda)(t/\lambda)^{\Theta-1}}{1+(t/\lambda)^{\Theta}}. \qquad (4.49)
\end{aligned}
$$

Lognormal-NCG Frailty Model
The Gompertz function base force of mortality and cumulative intensity rate is given in Equations (4.23,4.24) respectively.

$$
\begin{aligned}
S(t) &= \exp\left(-\frac{H_0(t)a\beta^2}{1+\beta H_0(t)}\right) = \exp\left(-\frac{(-\log[1-\Phi(\frac{\log t-\mu}{\sigma})])a\beta^2}{1-\beta\log[1-\Phi(\frac{\log t-\mu}{\sigma})]}\right).\\
f(t) &= \frac{h_0(t)a\beta^2}{(1+\beta H_0(t))^2}\exp\left(-\frac{H_0(t)a\beta^2}{1+\beta H_0(t)}\right),\\
&= \frac{(\frac{\phi(\frac{\log t-\mu}{\sigma})}{\sigma t[1-\Phi(\frac{\log t-\mu}{\sigma})]})a\beta^2}{(1-\beta\log[1-\Phi(\frac{\log t-\mu}{\sigma})])^2}\exp\left(-\frac{-\log[1-\Phi(\frac{\log t-\mu}{\sigma})]a\beta^2}{1-\beta\log[1-\Phi(\frac{\log t-\mu}{\sigma})]}\right).\\
h(t) &= \frac{\frac{(\frac{\phi(\frac{\log t-\mu}{\sigma})}{\sigma t[1-\Phi(\frac{\log t-\mu}{\sigma})]})a\beta^2}{(1-\beta\log[1-\Phi(\frac{\log t-\mu}{\sigma})])^2}\exp\{-\frac{-\log[1-\Phi(\frac{\log t-\mu}{\sigma})]a\beta^2}{1-\beta\log[1-\Phi(\frac{\log t-\mu}{\sigma})]}\}}{\exp\{-\frac{(-\log[1-\Phi(\frac{\log t-\mu}{\sigma})])a\beta^2}{1-\beta(\log[1-\Phi(\frac{\log t-\mu}{\sigma})])}\}}\\
&= \frac{(\frac{\phi(\frac{\log t-\mu}{\sigma})}{\sigma t[1-\Phi(\frac{\log t-\mu}{\sigma})]})a\beta^2}{(1-\beta\log[1-\Phi(\frac{\log t-\mu}{\sigma})])^2}.
\end{aligned}
$$

For identifiability reasons we apply Laplace transform derived in Equation (3.43). The univariate marginal survivor, density and intensity rate for the lognormal-NCG mixture becomes:

$$
\begin{aligned}
S(t) &= \exp\left(-\frac{H_0(t)}{1+\frac{1}{2}\sigma^2 H_0(t)}\right) = \exp\left(\frac{\log[1-\Phi(\frac{\log t-\mu}{\sigma})]}{1-\frac{1}{2}\sigma^2(\log[1-\Phi(\frac{\log t-\mu}{\sigma})])}\right).\\
f(t) &= \left(1+\frac{1}{2}\sigma^2 H_0(t)\right)^{-2} h_0(t)\cdot\exp\left(-\frac{H_0(t)}{1+\frac{1}{2}\sigma^2 H_0(t)}\right),\\
&= \left(1-\frac{1}{2}\sigma^2\left(\log\left[1-\Phi\left(\frac{\log t-\mu}{\sigma}\right)\right]\right)\right)^{-2}\left(\frac{\phi(\frac{\log t-\mu}{\sigma})}{\sigma t[1-\Phi(\frac{\log t-\mu}{\sigma})]}\right)\\
&\cdot\exp\left(\frac{\log[1-\Phi(\frac{\log t-\mu}{\sigma})]}{1-\frac{1}{2}\sigma^2(\log[1-\Phi(\frac{\log t-\mu}{\sigma})])}\right).
\end{aligned}
$$

$$
\begin{aligned}
h(t) &= \left(1+\frac{1}{2}\sigma^2 H_0(t)\right)^{-2} h_0(t),\\
&= \left(1-\frac{1}{2}\sigma^2\left(\log\left[1-\Phi\left(\frac{\log t-\mu}{\sigma}\right)\right]\right)\right)^{-2}\left(\frac{\phi(\frac{\log t-\mu}{\sigma})}{\sigma t[1-\Phi(\frac{\log t-\mu}{\sigma})]}\right).
\end{aligned}
\tag{4.50}
$$

Exponential-Power-NCG Frailty Model
The exponential-power function base force of mortality and cumulative intensity rate is given in Equations (4.26,4.27) respectively.

$$
\begin{aligned}
S(t) &= \exp\left(-\frac{H_0(t)a\beta^2}{1+\beta H_0(t)}\right) = \exp\left(-\frac{(\exp(rt^\lambda)-1)a\beta^2}{1+\beta(\exp(rt^\lambda)-1)}\right).\\
f(t) &= \frac{h_0(t)a\beta^2}{(1+\beta H_0(t))^2}\exp\left(-\frac{H_0(t)a\beta^2}{1+\beta H_0(t)}\right),\\
&= \frac{(\exp(rt^\lambda)r\lambda t^{\lambda-1})a\beta^2}{(1+\beta(\exp(rt^\lambda)-1))^2}\exp\left(-\frac{(\exp(rt^\lambda)-1)a\beta^2}{1+\beta(\exp(rt^\lambda)-1)}\right).\\
h(t) &= \frac{\frac{(\exp(rt^\lambda)r\lambda t^{\lambda-1})a\beta^2}{(1+\beta(\exp(rt^\lambda)-1))^2}\exp\{-\frac{(\exp(rt^\lambda)-1)a\beta^2}{1+\beta(\exp(rt^\lambda)-1)}\}}{\exp\{-\frac{(\exp(rt^\lambda)-1)a\beta^2}{1+\beta(\exp(rt^\lambda)-1)}\}} = \frac{(\exp(rt^\lambda)r\lambda t^{\lambda-1})a\beta^2}{(1+\beta(\exp(rt^\lambda)-1))^2}.
\end{aligned}
$$

For identifiability reasons we apply Laplace transform derived in Equation (3.43). The univariate marginal survivor, density and intensity rate for the exponential-power-NCG mixture becomes:

$$
\begin{aligned}
S(t) &= \exp\left(-\frac{H_0(t)}{1+\frac{1}{2}\sigma^2 H_0(t)}\right) = \exp\left(-\frac{(\exp(rt^\lambda)-1)}{1+\frac{1}{2}\sigma^2(\exp(rt^\lambda)-1)}\right).\\
f(t) &= \left(1+\frac{1}{2}\sigma^2 H_0(t)\right)^{-2} h_0(t)\cdot\exp\left(-\frac{H_0(t)}{1+\frac{1}{2}\sigma^2 H_0(t)}\right),\\
&= \left(1+\frac{1}{2}\sigma^2(\exp(rt^\lambda)-1)\right)^{-2}(\exp(rt^\lambda)r\lambda t^{\lambda-1})\cdot\exp\left(-\frac{\exp(rt^\lambda)-1}{1+\frac{1}{2}\sigma^2(\exp(rt^\lambda)-1)}\right).\\
h(t) &= \left(1+\frac{1}{2}\sigma^2 H_0(t)\right)^{-2}\cdot h_0(t) = \left(1+\frac{1}{2}\sigma^2(\exp(rt^\lambda)-1)\right)^{-2}\exp(rt^\lambda)r\lambda t^{\lambda-1}.
\end{aligned}
\tag{4.51}
$$

Pareto-NCG Frailty Model
The Pareto function base force of mortality and cumulative intensity rate is given in Equations (4.29,4.30) respectively.

$$
\begin{aligned}
S(t) &= \exp\left(-\frac{H_0(t)a\beta^2}{1+\beta H_0(t)}\right) = \exp\left(-\frac{(\log(\frac{t}{r})^{\vartheta})a\beta^2}{1+\beta(\log(\frac{t}{r})^{\vartheta})}\right).\\
f(t) &= \frac{h_0(t)a\beta^2}{(1+\beta H_0(t))^2}\exp\left(-\frac{H_0(t)a\beta^2}{1+\beta H_0(t)}\right),\\
&= \frac{\frac{\vartheta}{t}a\beta^2}{(1+\beta(\log(\frac{t}{r})^{\vartheta}))^2}\exp\left(-\frac{(\log(\frac{t}{r})^{\vartheta})a\beta^2}{1+\beta(\log(\frac{t}{r})^{\vartheta})}\right).\\
h(t) &= \frac{\frac{\frac{\vartheta}{t}a\beta^2}{(1+\beta(\log(\frac{t}{r})^{\vartheta}))^2}\exp\{-\frac{(\log(\frac{t}{r})^{\vartheta})a\beta^2}{1+\beta(\log(\frac{t}{r})^{\vartheta})}\}}{\exp\{-\frac{(\log(\frac{t}{r})^{\vartheta})a\beta^2}{1+\beta(\log(\frac{t}{r})^{\vartheta})}\}} = \frac{\frac{\vartheta}{t}a\beta^2}{(1+\beta(\log(\frac{t}{r})^{\vartheta}))^2}.
\end{aligned}
$$

For identifiability reasons we apply Laplace transform derived in Equation (3.43). The univariate marginal survivor, density and intensity rate for the Pareto-NCG mixture becomes:

$$
S(t) = \exp\left(-\frac{H_0(t)}{1+\frac{1}{2}\sigma^2 H_0(t)}\right) = \exp\left(-\frac{\log(\frac{t}{r})^{\vartheta}}{1+\frac{1}{2}\sigma^2\log(\frac{t}{r})^{\vartheta}}\right).
$$

$$
\begin{aligned}
f(t) &= \left(1+\frac{1}{2}\sigma^2 H_0(t)\right)^{-2} h_0(t)\cdot\exp\left(-\frac{H_0(t)}{1+\frac{1}{2}\sigma^2 H_0(t)}\right),\\
&= \left(1+\frac{1}{2}\sigma^2\log\left(\frac{t}{r}\right)^{\vartheta}\right)^{-2}\frac{\vartheta}{t}\cdot\exp\left(-\frac{\log(\frac{t}{r})^{\vartheta}}{1+\frac{1}{2}\sigma^2(\log(\frac{t}{r})^{\vartheta})}\right).\\
h(t) &= \left(1+\frac{1}{2}\sigma^2 H_0(t)\right)^{-2}\cdot h_0(t) = \left(1+\frac{1}{2}\sigma^2\left[\log\left(\frac{t}{r}\right)^{\vartheta}\right]\right)^{-2}\frac{\vartheta}{t}. \quad (4.52)
\end{aligned}
$$

4.7 Compound Negative Binomial Distribution

The CNBD has been applied by many authors in cases where there is possibility of zero susceptibility on the event of interest. Aalen & Tretli (1999) considers the CNBD to analyse testis cancer where $X \sim$ represents amounts of damages caused and $N \sim$ frequency of damages. The CNBD represents the number of damages until outset of cancer.

Construction of Compound Negative Binomial Univariate Frailty Model
The CNBD with $N \sim$ negative binomial and $X \sim$ gamma distributed is defined as

$$U = \begin{cases} X_1 + \ldots + X_N, & N > 0 \\ 0, & N = 0 \end{cases}$$

The Laplace transform derived in Equation (3.38) is

$$\mathbb{L}_U(s) = \left(\frac{p}{1 - q(1 + \frac{s}{\beta})^{-k}} \right)^{\alpha}.$$

$$\textit{Since} \quad S(t) = \mathbb{L}_U(H_0(t)) = \left(\frac{p}{1 - q(1 + \frac{H_0(t)}{\beta})^{-k}} \right)^{\alpha}.$$

$$\begin{aligned} f(t) &= -\frac{d}{dt} \left(\frac{p}{1 - q(1 + \frac{H_0(t)}{\beta})^{-k}} \right)^{\alpha}, \\ &= qk\alpha(p^{\alpha}) \left(1 - q \left[1 + \frac{H_0(t)}{\beta} \right]^{-k} \right)^{-\alpha-1} \left(1 + \frac{H_0(t)}{\beta} \right)^{-k-1} \frac{h_0(t)}{\beta}. \\ &= \frac{qk\alpha(p^{\alpha})(1 - q(1 + \frac{H_0(t)}{\beta})^{-k})^{-\alpha-1}(1 + \frac{H_0(t)}{\beta})^{-k-1}\frac{h_0(t)}{\beta}}{(\frac{p}{1-q(1+\frac{H_0(t)}{\beta})^{-k}})^{\alpha}}, \\ h(t) &= qk\alpha \left(1 - q \left[1 + \frac{H_0(t)}{\beta} \right]^{-k} \right)^{-1} \left(1 + \frac{H_0(t)}{\beta} \right)^{-k-1} \frac{h_0(t)}{\beta}. \end{aligned} \tag{4.53}$$

Considering various choices of $h_0(t)$ the base force of mortality function, we have the following CNBD frailty models.

Gompertz-CNBD Frailty Model
The Gompertz function base force of mortality and cumulative intensity rate given in Equations (4.11,4.12) respectively.

$$S(t) = \left(\frac{p}{1-q(1+\frac{H_0(t)}{\beta})^{-k}}\right)^{\alpha} = \left(\frac{p}{1-q(1+\frac{\frac{A}{B}(\exp{(Bt)}-1)}{\beta})^{-k}}\right)^{\alpha}.$$

$$f(t) = qk\alpha(p^{\alpha})\left(1-q\left(1+\frac{H_0(t)}{\beta}\right)^{-k}\right)^{-\alpha-1}\left(1+\frac{H_0(t)}{\beta}\right)^{-k-1}\frac{h_0(t)}{\beta},$$

$$= qk\alpha(p^{\alpha})\left(1-q\left(1+\frac{\frac{A}{B}(\exp{(Bt)}-1)}{\beta}\right)^{-k}\right)^{-\alpha-1}\left(1+\frac{\frac{A}{B}(\exp{(Bt)}-1)}{\beta}\right)^{-k-1}\frac{A\exp{(Bt)}}{\beta}$$

$$h(t) = qk\alpha\left(1-q\left(1+\frac{H_0(t)}{\beta}\right)^{-k}\right)^{-1}\left(1+\frac{H_0(t)}{\beta}\right)^{-k-1}\frac{h_0(t)}{\beta},$$

$$= qk\alpha\left(1-q\left(1+\frac{\frac{A}{B}(\exp{(Bt)}-1)}{\beta}\right)^{-k}\right)^{-1}\left(1+\frac{\frac{A}{B}(\exp{(Bt)}-1)}{\beta}\right)^{-k-1}\frac{A\exp{(Bt)}}{\beta}.$$

For identifiability reasons we apply Laplace transform derived in Equation (3.39). The univariate marginal survivor, density and intensity rate for the Gompertz-CNBD mixture becomes:

$$S(t) = \left(\frac{p}{1-q(1+\frac{H_0(t)\sigma^2 p}{p+k})^{-k}}\right)^{\alpha} = \left(\frac{p}{1-q(1+\frac{\frac{A}{B}(\exp{(Bt)}-1)\sigma^2 p}{p+k})^{-k}}\right)^{\alpha}.$$

$$f(t) = qk\alpha(p^{\alpha})\left(1-q\left(1+\frac{H_0(t)\sigma^2 p}{p+k}\right)^{-k}\right)^{-\alpha-1}\left(1+\frac{H_0(t)\sigma^2 p}{p+k}\right)^{-k-1}\frac{h_0(t)\sigma^2 p}{p+k},$$

$$= qk\alpha(p^{\alpha})\left(1-q\left(1+\frac{\frac{A}{B}(\exp{(Bt)}-1)\sigma^2 p}{p+k}\right)^{-k}\right)^{-\alpha-1}\left(1+\frac{\frac{A}{B}(\exp{(Bt)}-1)\sigma^2 p}{p+k}\right)^{-k-1}$$

$$\cdot\frac{A\exp{(Bt)}\sigma^2 p}{p+k}.$$

$$h(t) = qk\alpha \left(1 - q\left(1 + \frac{H_0(t)\sigma^2 p}{p+k}\right)^{-k}\right)^{-1} \left(1 + \frac{H_0(t)\sigma^2 p}{p+k}\right)^{-k-1} \frac{h_0(t)\sigma^2 p}{p+k},$$

$$= qk\alpha \left(1 - q\left(1 + \frac{\frac{A}{B}(\exp(Bt) - 1)\sigma^2 p}{p+k}\right)^{-k}\right)^{-1} \left(1 + \frac{\frac{A}{B}(\exp(Bt) - 1)\sigma^2 p}{p+k}\right)^{-k-1}$$

$$\cdot \frac{A\exp(Bt)\sigma^2 p}{p+k}.$$

Weibull-CNBD Frailty Model

The Weibull function base force of mortality and cumulative intensity rate given in Equations (4.13,4.14) respectively. Using the parametrizations $\beta = \Theta, \rho = \varpi$

$$S(t) = \left(\frac{p}{1 - q(1 + \frac{H_0(t)}{\beta})^{-k}}\right)^{\alpha} = \left(\frac{p}{1 - q(1 + \frac{\Theta t^{\varpi}}{\beta})^{-k}}\right)^{\alpha}.$$

$$f(t) = qk\alpha(p^{\alpha}) \left(1 - q\left(1 + \frac{H_0(t)}{\beta}\right)^{-k}\right)^{-\alpha-1} \left(1 + \frac{H_0(t)}{\beta}\right)^{-k-1} \frac{h_0(t)}{\beta},$$

$$= qk\alpha(p^{\alpha}) \left(1 - q\left(1 + \frac{\Theta t^{\varpi}}{\beta}\right)^{-k}\right)^{-\alpha-1} \left(1 + \frac{\Theta t^{\varpi}}{\beta}\right)^{-k-1} \frac{\Theta\varpi t^{\varpi-1}}{\beta}.$$

$$h(t) = qk\alpha \left(1 - q\left(1 + \frac{H_0(t)}{\beta}\right)^{-k}\right)^{-1} \left(1 + \frac{H_0(t)}{\beta}\right)^{-k-1} \frac{h_0(t)}{\beta},$$

$$= qk\alpha \left(1 - q\left(1 + \frac{\Theta t^{\varpi}}{\beta}\right)^{-k}\right)^{-1} \left(1 + \frac{\Theta t^{\varpi}}{\beta}\right)^{-k-1} \frac{\Theta\varpi t^{\varpi-1}}{\beta}.$$

For identifiability reasons we apply Laplace transform derived in Equation (3.39). The univariate marginal survivor, density and intensity rate for the Weibull-CNBD mixture becomes:

$$
\begin{aligned}
S(t) &= \left(\frac{p}{1-q(1+\frac{H_0(t)\sigma^2 p}{p+k})^{-k}}\right)^{\alpha} = \left(\frac{p}{1-q(1+\frac{\Theta t^{\varpi}\sigma^2 p}{p+k})^{-k}}\right)^{\alpha}. \\
f(t) &= qk\alpha(p^{\alpha})\left(1-q\left(1+\frac{H_0(t)\sigma^2 p}{p+k}\right)^{-k}\right)^{-\alpha-1}\left(1+\frac{H_0(t)\sigma^2 p}{p+k}\right)^{-k-1}\frac{h_0(t)\sigma^2 p}{p+k}, \\
&= qk\alpha(p^{\alpha})\left(1-q\left(1+\frac{\Theta t^{\varpi}\sigma^2 p}{p+k}\right)^{-k}\right)^{-\alpha-1}\left(1+\frac{\Theta t^{\varpi}\sigma^2 p}{p+k}\right)^{-k-1}\frac{\Theta\varpi t^{\varpi-1}\sigma^2 p}{p+k}. \\
h(t) &= qk\alpha\left(1-q\left(1+\frac{H_0(t)\sigma^2 p}{p+k}\right)^{-k}\right)^{-1}\left(1+\frac{H_0(t)\sigma^2 p}{p+k}\right)^{-k-1}\frac{h_0(t)\sigma^2 p}{p+k}, \\
&= qk\alpha\left(1-q\left(1+\frac{\Theta t^{p}\sigma^2 p}{p+k}\right)^{-k}\right)^{-1}\left(1+\frac{\Theta t^{p}\sigma^2 p}{p+k}\right)^{-k-1}\frac{\Theta p t^{p-1}\sigma^2 p}{p+k}.
\end{aligned}
\tag{4.54}
$$

Generalized Weibull-CNBD Frailty Model
The GW function base force of mortality and cumulative intensity rate given in Equations (4.15,4.16) respectively. Using the parametrization $\rho = \varpi$

$$
\begin{aligned}
S(t) &= \left(\frac{p}{1-q(1+\frac{H_0(t)}{\beta})^{-k}}\right)^{\alpha} = \left(\frac{p}{1-q(1-\frac{\log(1-[1-\exp(-\lambda t^{\varpi})]^{b})}{\beta})^{-k}}\right)^{\alpha}. \\
f(t) &= qk\alpha(p^{\alpha})\left(1-q\left(1+\frac{H_0(t)}{\beta}\right)^{-k}\right)^{-\alpha-1}\left(1+\frac{H_0(t)}{\beta}\right)^{-k-1}\frac{h_0(t)}{\beta}, \\
&= qk\alpha(p^{\alpha})\left(1-q\left(1-\frac{\log(1-[1-\exp(-\lambda t^{\varpi})]^{b})}{\beta}\right)^{-k}\right)^{-\alpha-1} \\
&\quad\cdot\left(1-\frac{\log(1-[1-\exp(-\lambda t^{\varpi})]^{b})}{\beta}\right)^{-k-1}\cdot\frac{\frac{b(1-\exp(-\lambda t^{\varpi}))^{b-1}\lambda\varpi t^{\varpi-1}\exp(-\lambda t^{\varpi})}{1-[1-\exp(-\lambda t^{\varpi})]^{b}}}{\beta}. \\
h(t) &= qk\alpha\left(1-q\left(1+\frac{H_0(t)}{\beta}\right)^{-k}\right)^{-1}\left(1+\frac{H_0(t)}{\beta}\right)^{-k-1}\frac{h_0(t)}{\beta},
\end{aligned}
$$

$$h(t) = qk\alpha \left(1 - q\left(1 - \frac{\log(1-[1-\exp(-\lambda t^{\varpi})]^b)}{\beta}\right)^{-k}\right)^{-1}$$
$$\cdot \left(1 - \frac{\log(1-[1-\exp(-\lambda t^{\varpi})]^b)}{\beta}\right)^{-k-1} \cdot \frac{\frac{b(1-\exp(-\lambda t^{\varpi}))^{b-1}\lambda\varpi t^{\varpi-1}\exp(-\lambda t^{\varpi})}{1-[1-\exp(-\lambda t^{\varpi})]^b}}{\beta}.$$

For identifiability reasons we apply Laplace transform derived in Equation (3.39). The univariate marginal survivor, density and intensity rate for the generalized Weibull-CNBD mixture becomes:

$$S(t) = \left(\frac{p}{1-q(1+\frac{H_0(t)\sigma^2 p}{p+k})^{-k}}\right)^{\alpha} = \left(\frac{p}{1-q(1-\frac{\log(1-[1-\exp(-\lambda t^{\varpi})]^b)\sigma^2 p}{p+k})^{-k}}\right)^{\alpha}.$$
$$f(t) = qk\alpha(p^{\alpha})\left(1 - q\left(1 + \frac{H_0(t)\sigma^2 p}{p+k}\right)^{-k}\right)^{-\alpha-1}\left(1 + \frac{H_0(t)\sigma^2 p}{p+k}\right)^{-k-1}\frac{h_0(t)\sigma^2 p}{p+k},$$
$$= qk\alpha(p^{\alpha})\left(1 - q\left(1 - \frac{\log(1-[1-\exp(-\lambda t^{\varpi})]^b)\sigma^2 p}{p+k}\right)^{-k}\right)^{-\alpha-1}$$
$$\cdot \left(1 - \frac{\log(1-[1-\exp(-\lambda t^{\varpi})]^b)\sigma^2 p}{p+k}\right)^{-k-1} \frac{(\frac{b(1-\exp(-\lambda t^{\varpi}))^{b-1}\lambda\varpi t^{\varpi-1}\exp(-\lambda t^{\varpi})}{1-[1-\exp(-\lambda t^{\varpi})]^b})\sigma^2 p}{p+k}.$$
$$h(t) = qk\alpha\left(1 - q\left(1 + \frac{H_0(t)\sigma^2 p}{p+k}\right)^{-k}\right)^{-1}(1 + \frac{H_0(t)\sigma^2 p}{p+k})^{-k-1}\frac{h_0(t)\sigma^2 p}{p+k},$$
$$= qk\alpha\left(1 - q\left(1 - \frac{\log(1-[1-\exp(-\lambda t^{\varpi})]^b)\sigma^2 p}{p+k}\right)^{-k}\right)^{-1}$$
$$\cdot \left(1 - \frac{\log(1-[1-\exp(-\lambda t^{\varpi})]^b)\sigma^2 p}{p+k}\right)^{-k-1} \frac{(\frac{b(1-\exp(-\lambda t^{\varpi}))^{b-1}\lambda\varpi t^{\varpi-1}\exp(-\lambda t^{\varpi})}{1-[1-\exp(-\lambda t^{\varpi})]^b})\sigma^2 p}{p+k}. \tag{4.55}$$

Exponential-CNBD Frailty Model

The Exponential function is derived from the Weibull distribution when $\rho = 1$. Putting $\rho = 1$ in Equations (4.13,4.14) leads to

$$h_0(t) = \beta.$$

$$H_0(t) = \beta t.$$

Using the parametrization $\beta = \Theta$

$$\begin{aligned}
h_0(t) &= \Theta. \\
H_0(t) &= \Theta t. \\
S(t) &= \left(\frac{p}{1 - q(1 + \frac{H_0(t)}{\beta})^{-k}}\right)^{\alpha} = \left(\frac{p}{1 - q(1 + \frac{\Theta t}{\beta})^{-k}}\right)^{\alpha}. \\
f(t) &= qk\alpha(p^{\alpha}) \left(1 - q\left(1 + \frac{H_0(t)}{\beta}\right)^{-k}\right)^{-\alpha-1} \left(1 + \frac{H_0(t)}{\beta}\right)^{-k-1} \frac{h_0(t)}{\beta}, \\
&= qk\alpha(p^{\alpha}) \left(1 - q\left(1 + \frac{\Theta t}{\beta}\right)^{-k}\right)^{-\alpha-1} \left(1 + \frac{\Theta t}{\beta}\right)^{-k-1} \frac{\Theta}{\beta}. \\
h(t) &= qk\alpha \left(1 - q\left(1 + \frac{H_0(t)}{\beta}\right)^{-k}\right)^{-1} \left(1 + \frac{H_0(t)}{\beta}\right)^{-k-1} \frac{h_0(t)}{\beta}, \\
&= qk\alpha \left(1 - q\left(1 + \frac{\Theta t}{\beta}\right)^{-k}\right)^{-1} \left(1 + \frac{\Theta t}{\beta}\right)^{-k-1} \frac{\Theta}{\beta}.
\end{aligned}$$

For identifiability reasons we apply Laplace transform derived in Equation (3.39). The univariate marginal survivor, density and intensity rate for the Exponential-CNBD mixture becomes:

$$\begin{aligned}
S(t) &= \left(\frac{p}{1 - q(1 + \frac{H_0(t)\sigma^2 p}{p+k})^{-k}}\right)^{\alpha} = \left(\frac{p}{1 - q(1 + \frac{\Theta t\sigma^2 p}{p+k})^{-k}}\right)^{\alpha}. \\
f(t) &= qk\alpha(p^{\alpha}) \left(1 - q\left(1 + \frac{H_0(t)\sigma^2 p}{p+k}\right)^{-k}\right)^{-\alpha-1} \left(1 + \frac{H_0(t)\sigma^2 p}{p+k}\right)^{-k-1} \frac{h_0(t)\sigma^2 p}{p+k}, \\
&= qk\alpha(p^{\alpha}) \left(1 - q\left(1 + \frac{\Theta t\sigma^2 p}{p+k}\right)^{-k}\right)^{-\alpha-1} \left(1 + \frac{\Theta t\sigma^2 p}{p+k}\right)^{-k-1} \frac{\Theta\sigma^2 p}{p+k}.
\end{aligned}$$

$$h(t) = qk\alpha \left(1 - q\left(1 + \frac{H_0(t)\sigma^2 p}{p+k}\right)^{-k}\right)^{-1} \left(1 + \frac{H_0(t)\sigma^2 p}{p+k}\right)^{-k-1} \frac{h_0(t)\sigma^2 p}{p+k},$$
$$= qk\alpha \left(1 - q\left(1 + \frac{\Theta t\sigma^2 p}{p+k}\right)^{-k}\right)^{-1} \left(1 + \frac{\Theta t\sigma^2 p}{p+k}\right)^{-k-1} \frac{\Theta\sigma^2 p}{p+k}. \qquad (4.56)$$

Generalized Exponential-CNBD Frailty Model
The GE base force of mortality and cumulative intensity rate is given in Equations (4.17,4.18) respectively.

$$S(t) = \left(\frac{p}{1 - q(1 + \frac{H_0(t)}{\beta})^{-k}}\right)^{\alpha} = \left(\frac{p}{1 - q(1 - \frac{\log(1-[1-\exp(-\lambda t)]^b)}{\beta})^{-k}}\right)^{\alpha}.$$

$$f(t) = qk\alpha(p^{\alpha}) \left(1 - q\left(1 + \frac{H_0(t)}{\beta}\right)^{-k}\right)^{-\alpha-1} \left(1 + \frac{H_0(t)}{\beta}\right)^{-k-1} \frac{h_0(t)}{\beta},$$
$$= qk\alpha(p^{\alpha}) \left(1 - q\left(1 - \frac{\log(1-[1-\exp(-\lambda t)]^b)}{\beta}\right)^{-k}\right)^{-\alpha-1}$$
$$\cdot \left(1 - \frac{\log(1-[1-\exp(-\lambda t)]^b)}{\beta}\right)^{-k-1} \cdot \frac{\frac{b\lambda(1-\exp(-\lambda t))^{b-1}\exp(-\lambda t)}{1-[1-\exp(-\lambda t)]^b}}{\beta}.$$

$$h(t) = qk\alpha \left(1 - q\left(1 + \frac{H_0(t)}{\beta}\right)^{-k}\right)^{-1} \left(1 + \frac{H_0(t)}{\beta}\right)^{-k-1} \frac{h_0(t)}{\beta},$$
$$= qk\alpha \left(1 - q\left(1 - \frac{\log(1-[1-\exp(-\lambda t)]^b)}{\beta}\right)^{-k}\right)^{-1} \left(1 - \frac{\log(1-[1-\exp(-\lambda t)]^b)}{\beta}\right)^{-k-1}$$
$$\cdot \frac{\frac{b\lambda(1-\exp(-\lambda t))^{b-1}\exp(-\lambda t)}{1-[1-\exp(-\lambda t)]^b}}{\beta}.$$

For identifiability reasons we apply Laplace transform derived in Equation (3.39). The univariate marginal survivor, density and intensity rate for the generalized exponential-CNBD mixture becomes:

$$S(t) = \left(\frac{p}{1 - q(1 + \frac{H_0(t)\sigma^2 p}{p+k})^{-k}}\right)^{\alpha} = \left(\frac{p}{1 - q(1 - \frac{\log(1-[1-\exp(-\lambda t)]^b)\sigma^2 p}{p+k})^{-k}}\right)^{\alpha}.$$

$$f(t) = qk\alpha(p^{\alpha})\left(1 - q\left(1 + \frac{H_0(t)\sigma^2 p}{p+k}\right)^{-k}\right)^{-\alpha-1}\left(1 + \frac{H_0(t)\sigma^2 p}{p+k}\right)^{-k-1}\frac{h_0(t)\sigma^2 p}{p+k},$$

$$= qk\alpha(p^{\alpha})\left(1 - q\left(1 - \frac{\log(1-[1-\exp(-\lambda t)]^b)\sigma^2 p}{p+k}\right)^{-k}\right)^{-\alpha-1}$$

$$\cdot\left(1 - \frac{\log(1-[1-\exp(-\lambda t)]^b)\sigma^2 p}{p+k}\right)^{-k-1}\frac{\frac{b\lambda(1-\exp(-\lambda t))^{b-1}\exp(-\lambda t)}{1-[1-\exp(-\lambda t)]^b}\sigma^2 p}{p+k}.$$

$$h(t) = qk\alpha\left(1 - q\left(1 + \frac{H_0(t)\sigma^2 p}{p+k}\right)^{-k}\right)^{-1}\left(1 + \frac{H_0(t)\sigma^2 p}{p+k}\right)^{-k-1}\frac{h_0(t)\sigma^2 p}{p+k},$$

$$= qk\alpha\left(1 - q\left(1 - \frac{\log(1-[1-\exp(-\lambda t)]^b)\sigma^2 p}{p+k}\right)^{-k}\right)^{-1}$$

$$\cdot\left(1 - \frac{\log(1-[1-\exp(-\lambda t)]^b)\sigma^2 p}{p+k}\right)^{-k-1}\cdot\frac{\frac{b\lambda(1-\exp(-\lambda t))^{b-1}\exp(-\lambda t)}{1-[1-\exp(-\lambda t)]^b}\sigma^2 p}{p+k}. \tag{4.57}$$

Loglogistic-CNBD Frailty Model
The loglogistic function base force of mortality and cumulative intensity rate is given in Equations (4.20,4.21) respectively.

$$S(t) = \left(\frac{p}{1-q(1+\frac{H_0(t)}{\beta})^{-k}}\right)^{\alpha} = \left(\frac{p}{1-q(1+\frac{\log(1+(\frac{t}{\lambda})^{\Theta})}{\beta})^{-k}}\right)^{\alpha}.$$

$$f(t) = qk\alpha(p^{\alpha})\left(1 - q\left(1 + \frac{H_0(t)}{\beta}\right)^{-k}\right)^{-\alpha-1}\left(1 + \frac{H_0(t)}{\beta}\right)^{-k-1}\frac{h_0(t)}{\beta},$$

$$= qk\alpha(p^{\alpha})\left(1 - q\left(1 + \frac{\log(1+(\frac{t}{\lambda})^{\Theta})}{\beta}\right)^{-k}\right)^{-\alpha-1}\left(1 + \frac{\log(1+(\frac{t}{\lambda})^{\Theta})}{\beta}\right)^{-k-1}\frac{(\Theta/\lambda)(t/\lambda)^{\Theta-1}}{\beta[1+(t/\lambda)^{\Theta}]}.$$

$$h(t) = qk\alpha\left(1 - q\left(1 + \frac{H_0(t)}{\beta}\right)^{-k}\right)^{-1}\left(1 + \frac{H_0(t)}{\beta}\right)^{-k-1}\frac{h_0(t)}{\beta},$$

$$= qk\alpha\left(1 - q\left(1 + \frac{\log(1+(\frac{t}{\lambda})^{\Theta})}{\beta}\right)^{-k}\right)^{-1}\left(1 + \frac{\log(1+(\frac{t}{\lambda})^{\Theta})}{\beta}\right)^{-k-1}\frac{(\Theta/\lambda)(t/\lambda)^{\Theta-1}}{\beta[1+(t/\lambda)^{\Theta}]}.$$

For identifiability reasons we apply Laplace transform derived in Equation (3.39). The univariate marginal survivor, density and intensity rate for the loglogistic-CNBD mixture becomes:

$$
\begin{aligned}
S(t) &= \left(\frac{p}{1-q(1+\frac{H_0(t)\sigma^2 p}{p+k})^{-k}}\right)^{\alpha} = \left(\frac{p}{1-q(1+\frac{\log(1+(\frac{t}{\lambda})^{\Theta})\sigma^2 p}{p+k})^{-k}}\right)^{\alpha}. \\
f(t) &= qk\alpha(p^{\alpha})\left(1-q\left(1+\frac{H_0(t)\sigma^2 p}{p+k}\right)^{-k}\right)^{-\alpha-1}\left(1+\frac{H_0(t)\sigma^2 p}{p+k}\right)^{-k-1}\frac{h_0(t)\sigma^2 p}{p+k}, \\
&= qk\alpha(p^{\alpha})\left(1-q\left(1+\frac{\log(1+(\frac{t}{\lambda})^{\Theta})\sigma^2 p}{p+k}\right)^{-k}\right)^{-\alpha-1}\left(1+\frac{\log(1+(\frac{t}{\lambda})^{\Theta})\sigma^2 p}{p+k}\right)^{-k-1} \\
&\cdot\frac{(\Theta/\lambda)(t/\lambda)^{\Theta-1}\sigma^2 p}{(p+k)[1+(t/\lambda)^{\Theta}]}. \\
h(t) &= qk\alpha\left(1-q\left(1+\frac{H_0(t)\sigma^2 p}{p+k}\right)^{-k}\right)^{-1}\left(1+\frac{H_0(t)\sigma^2 p}{p+k}\right)^{-k-1}\frac{h_0(t)\sigma^2 p}{p+k}, \\
&= qk\alpha\left(1-q\left(1+\frac{\log(1+(\frac{t}{\lambda})^{\Theta})\sigma^2 p}{p+k}\right)^{-k}\right)^{-1}\left(1+\frac{\log(1+(\frac{t}{\lambda})^{\Theta})\sigma^2 p}{p+k}\right)^{-k-1} \\
&\cdot\frac{(\Theta/\lambda)(t/\lambda)^{\Theta-1}\sigma^2 p}{(p+k)[1+(t/\lambda)^{\Theta}]}. \qquad (4.58)
\end{aligned}
$$

Lognormal-CNBD Frailty Model
The lognormal function base force of mortality and cumulative intensity rate is given in Equations (4.23,4.24) respectively.

$$
\begin{aligned}
S(t) &= \left(\frac{p}{1-q(1+\frac{H_0(t)}{\beta})^{-k}}\right)^{\alpha} = \left(\frac{p}{1-q(1-\frac{\log[1-\Phi(\frac{\log t-\mu}{\sigma})]}{\beta})^{-k}}\right)^{\alpha}. \\
f(t) &= qk\alpha(p^{\alpha})\left(1-q\left(1+\frac{H_0(t)}{\beta}\right)^{-k}\right)^{-\alpha-1}\left(1+\frac{H_0(t)}{\beta}\right)^{-k-1}\frac{h_0(t)}{\beta},
\end{aligned}
$$

$$f(t) = qk\alpha(p^{\alpha}) \left(1 - q\left(1 - \frac{\log[1-\Phi(\frac{\log t-\mu}{\sigma})]}{\beta}\right)^{-k}\right)^{-\alpha-1} \left(1 - \frac{\log[1-\Phi(\frac{\log t-\mu}{\sigma})]}{\beta}\right)^{-k-1}$$
$$\cdot \frac{\phi(\frac{\log t-\mu}{\sigma})}{\beta\sigma t[1-\Phi(\frac{\log t-\mu}{\sigma})]}.$$
$$h(t) = qk\alpha\left(1 - q\left(1+\frac{H_0(t)}{\beta}\right)^{-k}\right)^{-1}\left(1+\frac{H_0(t)}{\beta}\right)^{-k-1}\frac{h_0(t)}{\beta},$$
$$= qk\alpha \left(1 - q\left(1 - \frac{\log[1-\Phi(\frac{\log t-\mu}{\sigma})]}{\beta}\right)^{-k}\right)^{-1} \left(1 - \frac{\log[1-\Phi(\frac{\log t-\mu}{\sigma})]}{\beta}\right)^{-k-1}$$
$$\cdot \frac{\phi(\frac{\log t-\mu}{\sigma})}{\beta\sigma t[1-\Phi(\frac{\log t-\mu}{\sigma})]}.$$

For identifiability reasons we apply Laplace transform derived in Equation (3.39). The univariate marginal survivor, density and intensity rate for the lognormal-CNBD mixture becomes:

$$S(t) = \left(\frac{p}{1-q(1+\frac{H_0(t)\sigma^2 p}{p+k})^{-k}}\right)^{\alpha} = \left(\frac{p}{1-q(1-\frac{\log[1-\Phi(\frac{\log t-\mu}{\sigma})]\sigma^2 p}{p+k})^{-k}}\right)^{\alpha}.$$
$$f(t) = qk\alpha(p^{\alpha})\left(1-q\left(1+\frac{H_0(t)\sigma^2 p}{p+k}\right)^{-k}\right)^{-\alpha-1}\left(1+\frac{H_0(t)\sigma^2 p}{p+k}\right)^{-k-1}\frac{h_0(t)\sigma^2 p}{p+k},$$
$$= qk\alpha(p^{\alpha}) \left(1 - q\left(1 - \frac{\log[1-\Phi(\frac{\log t-\mu}{\sigma})]\sigma^2 p}{p+k}\right)^{-k}\right)^{-\alpha-1} \left(1 - \frac{\log[1-\Phi(\frac{\log t-\mu}{\sigma})]\sigma^2 p}{p+k}\right)^{-k-1}$$
$$\cdot \frac{\phi(\frac{\log t-\mu}{\sigma})}{(p+k)\sigma t[1-\Phi(\frac{\log t-\mu}{\sigma})]}\sigma^2 p.$$
$$h(t) = qk\alpha\left(1-q\left(1+\frac{H_0(t)\sigma^2 p}{p+k}\right)^{-k}\right)^{-1}\left(1+\frac{H_0(t)\sigma^2 p}{p+k}\right)^{-k-1}\frac{h_0(t)\sigma^2 p}{p+k},$$

$$h(t) = qk\alpha\left(1-q\left(1-\frac{\log[1-\Phi(\frac{\log t-\mu}{\sigma})]\sigma^2 p}{p+k}\right)^{-k}\right)^{-1}\left(1-\frac{\log[1-\Phi(\frac{\log t-\mu}{\sigma})]\sigma^2 p}{p+k}\right)^{-k-1}$$
$$\cdot\frac{\phi(\frac{\log t-\mu}{\sigma})}{(p+k)\sigma t[1-\Phi(\frac{\log t-\mu}{\sigma})]}\sigma^2 p. \quad (4.59)$$

Exponential-Power-CNBD Frailty Model

The Exponential-Power function base force of mortality and cumulative intensity rate is given in Equations (4.26,4.27) respectively.

$$S(t) = \left(\frac{p}{1-q(1+\frac{H_0(t)}{\beta})^{-k}}\right)^{\alpha} = \left(\frac{p}{1-q(1+\frac{(\exp(rt^{\lambda})-1)}{\beta})^{-k}}\right)^{\alpha}.$$

$$f(t) = qk\alpha(p^{\alpha})\left(1-q\left(1+\frac{H_0(t)}{\beta}\right)^{-k}\right)^{-\alpha-1}\left(1+\frac{H_0(t)}{\beta}\right)^{-k-1}\frac{h_0(t)}{\beta},$$

$$f(t) = qk\alpha(p^{\alpha})\left(1-q\left(1+\frac{(\exp(rt^{\lambda})-1)}{\beta}\right)^{-k}\right)^{-\alpha-1}\left(1+\frac{(\exp(rt^{\lambda})-1)}{\beta}\right)^{-k-1}$$
$$\cdot\frac{\exp(rt^{\lambda})r\lambda t^{\lambda-1}}{\beta}.$$

$$h(t) = qk\alpha\left(1-q\left(1+\frac{H_0(t)}{\beta}\right)^{-k}\right)^{-1}\left(1+\frac{H_0(t)}{\beta}\right)^{-k-1}\frac{h_0(t)}{\beta},$$
$$= qk\alpha\left(1-q\left(1+\frac{(\exp(rt^{\lambda})-1)}{\beta}\right)^{-k}\right)^{-1}\left(1+\frac{(\exp(rt^{\lambda})-1)}{\beta}\right)^{-k-1}$$
$$\cdot\frac{\exp(rt^{\lambda})r\lambda t^{\lambda-1}}{\beta}.$$

For identifiability reasons we apply Laplace transform derived in Equation (3.39). The univariate marginal survivor, density and intensity rate for the Exponential-power-CNBD mixture becomes:

$$S(t) = \left(\frac{p}{1-q(1+\frac{H_0(t)\sigma^2 p}{p+k})^{-k}}\right)^{\alpha} = \left(\frac{p}{1-q(1+\frac{(\exp(rt^{\lambda})-1)\sigma^2 p}{p+k})^{-k}}\right)^{\alpha}.$$

$$f(t) = qk\alpha(p^{\alpha})\left(1-q\left(1+\frac{H_0(t)\sigma^2 p}{p+k}\right)^{-k}\right)^{-\alpha-1}\left(1+\frac{H_0(t)\sigma^2 p}{p+k}\right)^{-k-1}\frac{h_0(t)\sigma^2 p}{p+k},$$

$$= qk\alpha(p^{\alpha})\left(1-q\left(1+\frac{(\exp(rt^{\lambda})-1)\sigma^2 p}{p+k}\right)^{-k}\right)^{-\alpha-1}\left(1+\frac{(\exp(rt^{\lambda})-1)\sigma^2 p}{p+k}\right)^{-k-1}$$

$$\cdot\frac{\exp(rt^{\lambda})r\lambda t^{\lambda-1}\sigma^2 p}{p+k}.$$

$$h(t) = qk\alpha\left(1-q\left(1+\frac{H_0(t)\sigma^2 p}{p+k}\right)^{-k}\right)^{-1}\left(1+\frac{H_0(t)\sigma^2 pp}{p+k}\right)^{-k-1}\frac{h_0(t)\sigma^2 p}{p+k},$$

$$= qk\alpha\left(1-q\left(1+\frac{(\exp(rt^{\lambda})-1)\sigma^2 p}{p+k}\right)^{-k}\right)^{-1}\left(1+\frac{(\exp(rt^{\lambda})-1)\sigma^2 p}{p+k}\right)^{-k-1}$$

$$\cdot\frac{\exp(rt^{\lambda})r\lambda t^{\lambda-1}\sigma^2 p}{p+k}. \tag{4.60}$$

Pareto-CNBD Frailty Model
The Pareto function base force of mortality and cumulative intensity rate is given in Equations (4.29,4.30) respectively.

$$S(t) = \left(\frac{p}{1-q(1+\frac{H_0(t)}{\beta})^{-k}}\right)^{\alpha} = \left(\frac{p}{1-q(1+\frac{\log(\frac{t}{r})^{\vartheta}}{\beta})^{-k}}\right)^{\alpha}.$$

$$f(t) = qk\alpha(p^{\alpha})\left(1-q\left(1+\frac{H_0(t)}{\beta}\right)^{-k}\right)^{-\alpha-1}\left(1+\frac{H_0(t)}{\beta}\right)^{-k-1}\frac{h_0(t)}{\beta},$$

$$= qk\alpha(p^{\alpha})\left(1-q\left(1+\frac{\log(\frac{t}{r})^{\vartheta}}{\beta}\right)^{-k}\right)^{-\alpha-1}\left(1+\frac{\log(\frac{t}{r})^{\vartheta}}{\beta}\right)^{-k-1}\frac{\vartheta}{t\beta}.$$

$$h(t) = qk\alpha\left(1-q\left(1+\frac{H_0(t)}{\beta}\right)^{-k}\right)^{-1}\left(1+\frac{H_0(t)}{\beta}\right)^{-k-1}\frac{h_0(t)}{\beta},$$

$$= qk\alpha\left(1-q\left(1+\frac{\log(\frac{t}{r})^{\vartheta}}{\beta}\right)^{-k}\right)^{-1}\left(1+\frac{\log(\frac{t}{r})^{\vartheta}}{\beta}\right)^{-k-1}\frac{\vartheta}{t\beta}.$$

For identifiability reasons we apply Laplace transform derived in Equation (3.39). The univariate marginal survivor, density and intensity rate for the Pareto-CNBD mixture becomes:

$$
\begin{aligned}
S(t) &= \left(\frac{p}{1-q(1+\frac{H_0(t)\sigma^2 p}{p+k})^{-k}}\right)^{\alpha} = \left(\frac{p}{1-q(1+\frac{\log(\frac{t}{r})^{\vartheta}\sigma^2 p}{p+k})^{-k}}\right)^{\alpha}. \\
f(t) &= qk\alpha(p^{\alpha})\left(1-q\left(1+\frac{H_0(t)\sigma^2 p}{p+k}\right)^{-k}\right)^{-\alpha-1}\left(1+\frac{H_0(t)\sigma^2 p}{p+k}\right)^{-k-1}\frac{h_0(t)\sigma^2 p}{p+k}, \\
&= qk\alpha(p^{\alpha})\left(1-q\left(1+\frac{\log(\frac{t}{r})^{\vartheta}\sigma^2 p}{p+k}\right)^{-k}\right)^{-\alpha-1}\left(1+\frac{\log(\frac{t}{r})^{\vartheta}\sigma^2 p}{p+k}\right)^{-k-1}\frac{\frac{\vartheta}{t}\sigma^2 p}{p+k}. \\
h(t) &= qk\alpha\left(1-q\left(1+\frac{H_0(t)\sigma^2 p}{p+k}\right)^{-k}\right)^{-1}\left(1+\frac{H_0(t)\sigma^2 p}{p+k}\right)^{-k-1}\frac{h_0(t)\sigma^2 p}{p+k}, \\
&= qk\alpha\left(1-q\left(1+\frac{\log(\frac{t}{r})^{\vartheta}\sigma^2 p}{p+k}\right)^{-k}\right)^{-1}\left(1+\frac{\log(\frac{t}{r})^{\vartheta}\sigma^2 p}{p+k}\right)^{-k-1}\frac{\frac{\vartheta}{t}\sigma^2 p}{p+k}.
\end{aligned}
\tag{4.61}
$$

4.8 Levy Distribution

Hakon *et al.*, (2003) generalized the standard frailty models by modeling risk as a weighted Levy process. Here, the individual risk is considered to be evolving over time.

Construction of Levy Univariate Frailty Model

The Levy distribution is derived from the inverse gamma whose PDF derived in Equation (3.27) is

$$f(u) = \sqrt{\frac{\mu^2/2}{u^3 2\pi}}\exp\left(\frac{-\mu^2}{2u}\right).$$

The Laplace of the Levy is

$$\mathbb{L}_U(s) = \exp(-\sqrt{2s\mu^2}).$$

$$
\begin{aligned}
Since \quad S(t) &= \mathbb{L}_U(H_0(t)) = \exp(-\sqrt{2H_0(t)\mu^2}). \\
f(t) &= -\frac{d}{dt}\exp(-\sqrt{2H_0(t)\mu^2}), \\
&= h_0(t)\mu^2(2H_0(t)\mu^2)^{-1/2}\cdot\exp(-\sqrt{2H_0(t)\mu^2}). \\
h(t) &= \frac{h_0(t)\mu^2(2H_0(t)\mu^2)^{-1/2}\cdot\exp(-\sqrt{2H_0(t)\mu^2})}{\exp(-\sqrt{2H_0(t)\mu^2})}, \\
&= h_0(t)\mu^2(2H_0(t)\mu^2)^{-1/2}. \qquad (4.62)
\end{aligned}
$$

Considering various choices of the base force of mortality function, we have the following Levy frailty models.

Gompertz-Levy Frailty Model
The Gompertz function base force of mortality and cumulative intensity rate is given in Equations (4.11,4.12) respectively.

$$
\begin{aligned}
S(t) &= \exp(-\sqrt{2H_0(t)\mu^2}) = \exp(-\sqrt{2\frac{A}{B}(\exp(Bt)-1)\mu^2}). \\
f(t) &= h_0(t)\mu^2(2H_0(t)\mu^2)^{-1/2}\cdot\exp(-\sqrt{2H_0(t)\mu^2}), \\
&= A\exp(Bt)\mu^2\left(2\frac{A}{B}(\exp(Bt)-1)\mu^2\right)^{-1/2}\cdot\exp\left(-\sqrt{2\frac{A}{B}(\exp(Bt)-1)\mu^2}\right). \\
h(t) &= h_0(t)\mu^2(2H_0(t)\mu^2)^{-1/2} = A\exp(Bt)\mu^2\left(2\frac{A}{B}(\exp(Bt)-1)\mu^2\right)^{-1/2}.
\end{aligned}
$$

For identifiability reasons we apply Laplace transform derived in Equation (3.29). The univariate marginal survivor, density and intensity rate for the Gompertz-Levy mixture is

$$
\begin{aligned}
S(t) &= \exp(-\sqrt{2(H_0(t))}) = \exp\left(-\sqrt{2(\frac{A}{B}(\exp(Bt)-1))}\right). \\
f(t) &= h_0(t)(2H_0(t))^{-1/2}\cdot\exp(-\sqrt{2H_0(t)}), \\
&= A\exp(Bt)\left(2\frac{A}{B}(\exp(Bt)-1)\right)^{-1/2}\cdot\exp\left(-\sqrt{2\frac{A}{B}(\exp(Bt)-1)}\right).
\end{aligned}
$$

$$h(t) = h_0(t)(2H_0(t))^{-1/2} = A\exp(Bt)\left(2\frac{A}{B}(\exp(Bt)-1)\right)^{-1/2}. \quad (4.63)$$

Weibull-Levy Frailty Model
The Weibull function base force of mortality and cumulative intensity rate is given in Equations (4.13,4.14) respectively.

$$\begin{aligned}
S(t) &= \exp(-\sqrt{2H_0(t)\mu^2}) = \exp(-\sqrt{2\beta t^\rho \mu^2}).\\
f(t) &= h_0(t)\mu^2(2H_0(t)\mu^2)^{-1/2}\cdot\exp(-\sqrt{2H_0(t)\mu^2}),\\
&= \beta\rho t^{\rho-1}\mu^2(2\beta t^\rho\mu^2)^{-1/2}\cdot\exp(-\sqrt{2\beta t^\rho\mu^2}).\\
h(t) &= h_0(t)\mu^2(2H_0(t)\mu^2)^{-1/2} = \beta\rho t^{\rho-1}\mu^2(2\beta t^\rho\mu^2)^{-1/2}.
\end{aligned}$$

For identifiability reasons we apply Laplace transform derived in Equation (3.29). The univariate marginal survivor, density and intensity rate for the Weibull-Levy mixture is

$$\begin{aligned}
S(t) &= \exp(-\sqrt{2(H_0(t))}) = \exp(-\sqrt{2(\beta t^\rho)}).\\
f(t) &= h_0(t)(2H_0(t))^{-1/2}\cdot\exp(-\sqrt{2H_0(t)}),\\
&= \beta\rho t^{\rho-1}(2\beta t^\rho)^{-1/2}\cdot\exp(-\sqrt{2\beta t^\rho}).\\
h(t) &= h_0(t)(2H_0(t))^{-1/2} = \beta\rho t^{\rho-1}(2\beta t^\rho)^{-1/2}. \quad (4.64)
\end{aligned}$$

Generalized Weibull-Levy Frailty Model
The GW function base force of mortality and cumulative intensity rate given in Equations (4.15,4.16) respectively.

$$\begin{aligned}
S(t) &= \exp(-\sqrt{2H_0(t)\mu^2}) = \exp(-\sqrt{-2\log(1-[1-\exp(-\lambda t^\rho)]^b)\mu^2}).\\
f(t) &= h_0(t)\mu^2(2H_0(t)\mu^2)^{-1/2}\cdot\exp(-\sqrt{2H_0(t)\mu^2}),\\
&= \frac{b(1-\exp(-\lambda t^\rho))^{b-1}\lambda\rho t^{\rho-1}\exp(-\lambda t^\rho)}{1-[1-\exp(-\lambda t^\rho)]^b}\mu^2(-2\log(1-[1-\exp(-\lambda t^\rho)]^b)\mu^2)^{-1/2}\\
&\quad\cdot\exp(-\sqrt{-2\log(1-[1-\exp(-\lambda t^\rho)]^b)\mu^2}).
\end{aligned}$$

$$
\begin{aligned}
h(t) &= h_0(t)\mu^2(2H_0(t)\mu^2)^{-1/2}, \\
&= \frac{b(1-\exp(-\lambda t^\rho))^{b-1}\lambda\rho t^{\rho-1}\exp(-\lambda t^\rho)}{1-[1-\exp(-\lambda t^\rho)]^b}\mu^2(-2\log(1-[1-\exp(-\lambda t^\rho)]^b)\mu^2)^{-1/2}.
\end{aligned}
$$

For identifiability reasons we apply Laplace transform derived in Equation (3.29). The univariate marginal survivor, density and intensity rate for the generalized Weibull-Levy mixture is

$$
\begin{aligned}
S(t) &= \exp(-\sqrt{2(H_0(t))}) = \exp(-\sqrt{-2(\log(1-[1-\exp(-\lambda t^\rho)]^b))}). \\
f(t) &= h_0(t)(2H_0(t))^{-1/2}\cdot\exp(-\sqrt{2H_0(t)}), \\
&= \frac{b(1-\exp(-\lambda t^\rho))^{b-1}\lambda\rho t^{\rho-1}\exp(-\lambda t^\rho)}{1-[1-\exp(-\lambda t^\rho)]^b}(-2\log(1-[1-\exp(-\lambda t^\rho)]^b))^{-1/2} \\
&\cdot\exp(-\sqrt{-2\log(1-[1-\exp(-\lambda t^\rho)]^b)}). \\
h(t) &= h_0(t)(2H_0(t))^{-1/2}, \\
&= \frac{b(1-\exp(-\lambda t^\rho))^{b-1}\lambda\rho t^{\rho-1}\exp(-\lambda t^\rho)}{1-[1-\exp(-\lambda t^\rho)]^b}(-2\log(1-[1-\exp(-\lambda t^\rho)]^b))^{-1/2}.
\end{aligned} \tag{4.65}
$$

Exponential-Levy Frailty Model
The Exponential function is derived from the Weibull distribution when $\rho = 1$. Putting $\rho = 1$ in Equations (4.13,4.14) leads to

$$
\begin{aligned}
h_0(t) &= \beta. \\
H_0(t) &= \beta t. \\
S(t) &= \exp(-\sqrt{2H_0(t)\mu^2}) = \exp(-\sqrt{2\beta t\mu^2}). \\
f(t) &= h_0(t)\mu^2(2H_0(t)\mu^2)^{-1/2}\cdot\exp(-\sqrt{2H_0(t)\mu^2}), \\
&= \beta\mu^2(2\beta t\mu^2)^{-1/2}\cdot\exp(-\sqrt{2\beta t\mu^2}). \\
h(t) &= h_0(t)\mu^2(2H_0(t)\mu^2)^{-1/2} = \beta\mu^2(2\beta t\mu^2)^{-1/2}.
\end{aligned}
$$

For identifiability reasons we apply Laplace transform derived in Equation (3.29). The univariate marginal survivor, density and intensity rate for the exponential-Levy mixture is

$$
\begin{aligned}
S(t) &= \exp{(-\sqrt{2(H_0(t))})} = \exp{(-\sqrt{2(\beta t)})}. \\
f(t) &= h_0(t)(2H_0(t))^{-1/2} \cdot \exp{(-\sqrt{2H_0(t)})}, \\
&= \beta(2\beta t)^{-1/2} \cdot \exp{(-\sqrt{2\beta t})}. \\
h(t) &= h_0(t)(2H_0(t))^{-1/2} = \beta(2\beta t)^{-1/2}. \qquad (4.66)
\end{aligned}
$$

Generalized Exponential-Levy Frailty Model

The GE base force of mortality and cumulative intensity rate is given in Equations (4.17,4.18) respectively.

$$
\begin{aligned}
S(t) &= \exp{(-\sqrt{2H_0(t)\mu^2})} = \exp{(-\sqrt{-2\log(1-[1-\exp{(-\lambda t)}]^b)\mu^2})}. \\
f(t) &= h_0(t)\mu^2(2H_0(t)\mu^2)^{-1/2} \cdot \exp{(-\sqrt{2H_0(t)\mu^2})}, \\
&= \frac{b\lambda(1-\exp{(-\lambda t)})^{b-1}\exp{(-\lambda t)}}{1-[1-\exp{(-\lambda t)}]^b}\mu^2(-2\log(1-[1-\exp{(-\lambda t)}]^b)\mu^2)^{-1/2} \\
&\cdot \exp{(-\sqrt{-2\log(1-[1-\exp{(-\lambda t)}]^b)\mu^2})}. \\
h(t) &= h_0(t)\mu^2(2H_0(t)\mu^2)^{-1/2}, \\
&= \frac{b\lambda(1-\exp{(-\lambda t)})^{b-1}\exp{(-\lambda t)}}{1-[1-\exp{(-\lambda t)}]^b}\mu^2(-2\log(1-[1-\exp{(-\lambda t)}]^b)\mu^2)^{-1/2}.
\end{aligned}
$$

For identifiability reasons we apply Laplace transform derived in Equation (3.29). The univariate marginal survivor, density and intensity rate for the generalized exponential-Levy mixture is

$$
\begin{aligned}
S(t) &= \exp{(-\sqrt{2(H_0(t))})} = \exp{(-\sqrt{-2(\log(1-[1-\exp{(-\lambda t)}]^b))})}. \\
f(t) &= h_0(t)(2H_0(t))^{-1/2} \cdot \exp{(-\sqrt{2H_0(t)})}, \\
&= \frac{b\lambda(1-\exp{(-\lambda t)})^{b-1}\exp{(-\lambda t)}}{1-[1-\exp{(-\lambda t)}]^b}(-2\log(1-[1-\exp{(-\lambda t)}]^b))^{-1/2} \\
&\cdot \exp{(-\sqrt{-2\log(1-[1-\exp{(-\lambda t)}]^b)})}. \\
h(t) &= h_0(t)(2H_0(t))^{-1/2}, \qquad (4.67) \\
&= \frac{b\lambda(1-\exp{(-\lambda t)})^{b-1}\exp{(-\lambda t)}}{1-[1-\exp{(-\lambda t)}]^b}(-2\log(1-[1-\exp{(-\lambda t)}]^b))^{-1/2}.
\end{aligned}
$$

Loglogistic-Levy Frailty Model
The loglogistic function base force of mortality and cumulative intensity rate is given in Equations (4.20,4.21) respectively.

$$S(t) = \exp\left(-\sqrt{2H_0(t)\mu^2}\right) = \exp\left(-\sqrt{2\log\left[1+\left(\frac{t}{\lambda}\right)^{\Theta}\right]\mu^2}\right).$$

$$f(t) = h_0(t)\mu^2(2H_0(t)\mu^2)^{-1/2}\cdot\exp\left(-\sqrt{2H_0(t)\mu^2}\right),$$

$$= \frac{(\Theta/\lambda)(t/\lambda)^{\Theta-1}}{1+(t/\lambda)^{\Theta}}\mu^2\left(2\log\left[1+\left(\frac{t}{\lambda}\right)^{\Theta}\right]\mu^2\right)^{-1/2}\cdot\exp\left(-\sqrt{2\log\left[1+\left(\frac{t}{\lambda}\right)^{\Theta}\right]\mu^2}\right).$$

$$h(t) = h_0(t)\mu^2(2H_0(t)\mu^2)^{-1/2} = \frac{(\Theta/\lambda)(t/\lambda)^{\Theta-1}}{1+(t/\lambda)^{\Theta}}\mu^2\left(2\log\left[1+\left(\frac{t}{\lambda}\right)^{\Theta}\right]\mu^2\right)^{-1/2}.$$

For identifiability reasons we apply Laplace transform derived in Equation (3.29). The univariate marginal survivor, density and intensity rate for the loglogistic-Levy mixture is

$$S(t) = \exp\left(-\sqrt{2(H_0(t))}\right) = \exp\left(-\sqrt{2\log\left[1+\left(\frac{t}{\lambda}\right)^{\Theta}\right]}\right).$$

$$f(t) = h_0(t)(2H_0(t))^{-1/2}\cdot\exp\left(-\sqrt{2H_0(t)}\right),$$

$$= \frac{(\Theta/\lambda)(t/\lambda)^{\Theta-1}}{1+(t/\lambda)^{\Theta}}\left(2\log\left[1+\left(\frac{t}{\lambda}\right)^{\Theta}\right]\right)^{-1/2}\cdot\exp\left(-\sqrt{2\log\left[1+\left(\frac{t}{\lambda}\right)^{\Theta}\right]}\right).$$

$$h(t) = h_0(t)(2H_0(t))^{-1/2},$$

$$= \frac{(\Theta/\lambda)(t/\lambda)^{\Theta-1}}{1+(t/\lambda)^{\Theta}}\left(2\log\left[1+\left(\frac{t}{\lambda}\right)^{\Theta}\right]\right)^{-1/2}. \tag{4.68}$$

Lognormal-Levy Frailty Model
The lognormal function base force of mortality and cumulative intensity rate is given in Equations (4.23,4.24) respectively.

$$S(t) = \exp(-\sqrt{2H_0(t)\mu^2}) = \exp\left(-\sqrt{-2\log\left[1-\Phi\left(\frac{\log t-\mu}{\sigma}\right)\right]\mu^2}\right).$$

$$f(t) = h_0(t)\mu^2(2H_0(t)\mu^2)^{-1/2}\cdot\exp(-\sqrt{2H_0(t)\mu^2}),$$

$$= \frac{\phi(\frac{\log t-\mu}{\sigma})}{\sigma t[1-\Phi(\frac{\log t-\mu}{\sigma})]}\mu^2\left(-2\log\left[1-\Phi\left(\frac{\log t-\mu}{\sigma}\right)\right]\mu^2\right)^{-1/2}$$

$$\cdot\exp\left(-\sqrt{-2\log\left[1-\Phi\left(\frac{\log t-\mu}{\sigma}\right)\right]\mu^2}\right).$$

$$h(t) = h_0(t)\mu^2(2H_0(t)\mu^2)^{-1/2} = \frac{\phi(\frac{\log t-\mu}{\sigma})}{\sigma t[1-\Phi(\frac{\log t-\mu}{\sigma})]}\mu^2\left(-2\log\left[1-\Phi\left(\frac{\log t-\mu}{\sigma}\right)\right]\mu^2\right)^{-1/2}.$$

For identifiability reasons we apply Laplace transform derived in Equation (3.29). The univariate marginal survivor, density and intensity rate for the lognormal-Levy mixture is:

$$S(t) = \exp(-\sqrt{2(H_0(t))}) = \exp\left(-\sqrt{-2\log\left[1-\Phi\left(\frac{\log t-\mu}{\sigma}\right)\right]}\right).$$

$$f(t) = h_0(t)(2H_0(t))^{-1/2}\cdot\exp(-\sqrt{2H_0(t)}),$$

$$= \frac{\phi(\frac{\log t-\mu}{\sigma})}{\sigma t[1-\Phi(\frac{\log t-\mu}{\sigma})]}\left(-2\log\left[1-\Phi\left(\frac{\log t-\mu}{\sigma}\right)\right]\right)^{-1/2}$$

$$\cdot\exp\left(-\sqrt{-2\log\left[1-\Phi\left(\frac{\log t-\mu}{\sigma}\right)\right]}\right).$$

$$h(t) = h_0(t)(2H_0(t))^{-1/2} = \frac{\phi(\frac{\log t-\mu}{\sigma})}{\sigma t[1-\Phi(\frac{\log t-\mu}{\sigma})]}\left(-2\log\left[1-\Phi\left(\frac{\log t-\mu}{\sigma}\right)\right]\right)^{-1/2}. \tag{4.69}$$

Exponential-Power-Levy Frailty Model

The exponential-power function base force of mortality and cumulative intensity rate is given in Equations (4.26,4.27) respectively.

$$
\begin{aligned}
S(t) &= \exp(-\sqrt{2H_0(t)\mu^2}) = \exp(-\sqrt{2(\exp(rt^\lambda) - 1)\mu^2}).\\
f(t) &= h_0(t)\mu^2(2H_0(t)\mu^2)^{-1/2} \cdot \exp(-\sqrt{2H_0(t)\mu^2}),\\
&= \exp(rt^\lambda)r\lambda t^{\lambda-1}\mu^2(2(\exp(rt^\lambda) - 1)\mu^2)^{-1/2} \cdot \exp(-\sqrt{2(\exp(rt^\lambda) - 1)\mu^2}).\\
h(t) &= h_0(t)\mu^2(2H_0(t)\mu^2)^{-1/2} = \exp(rt^\lambda)r\lambda t^{\lambda-1}\mu^2(2(\exp(rt^\lambda) - 1)\mu^2)^{-1/2}.
\end{aligned}
$$

For identifiability reasons we apply Laplace transform derived in Equation (3.29). The univariate marginal survivor, density and intensity rate for the Exponential-Power-Levy mixture is

$$
\begin{aligned}
S(t) &= \exp(-\sqrt{2(H_0(t))}) = \exp(-\sqrt{2(\exp(rt^\lambda) - 1)}).\\
f(t) &= h_0(t)(2H_0(t))^{-1/2} \cdot \exp(-\sqrt{2H_0(t)}),\\
&= \exp(rt^\lambda)r\lambda t^{\lambda-1}(2(\exp(rt^\lambda) - 1))^{-1/2} \cdot \exp(-\sqrt{2(\exp(rt^\lambda) - 1)}).\\
h(t) &= h_0(t)(2H_0(t))^{-1/2} = \exp(rt^\lambda)r\lambda t^{\lambda-1}(2(\exp(rt^\lambda) - 1))^{-1/2}. \qquad (4.70)
\end{aligned}
$$

Pareto-Levy Frailty Model
The Pareto function base force of mortality and cumulative intensity rate is given in Equations (4.29,4.30) respectively.

$$
\begin{aligned}
S(t) &= \exp(-\sqrt{2H_0(t)\mu^2}) = \exp\left(-\sqrt{2\log\left(\frac{t}{r}\right)^{\vartheta}\mu^2}\right).\\
f(t) &= h_0(t)\mu^2(2H_0(t)\mu^2)^{-1/2} \cdot \exp(-\sqrt{2H_0(t)\mu^2}),\\
&= \frac{\vartheta}{t}\mu^2\left(2\log\left(\frac{t}{r}\right)^{\vartheta}\mu^2\right)^{-1/2} \cdot \exp\left(-\sqrt{2\log\left(\frac{t}{r}\right)^{\vartheta}\mu^2}\right).\\
h(t) &= h_0(t)\mu^2(2H_0(t)\mu^2)^{-1/2} = \frac{\vartheta}{t}\mu^2\left(2\log\left(\frac{t}{r}\right)^{\vartheta}\mu^2\right)^{-1/2}.
\end{aligned}
$$

For identifiability reasons we apply Laplace transform derived in Equation (3.29). The univariate marginal survivor, density and intensity rate for the Pareto-Levy mixture is:

$$S(t) = \exp(-\sqrt{2(H_0(t))}) = \exp\left(-\sqrt{2\log\left(\frac{t}{r}\right)^{\vartheta}}\right).$$

$$f(t) = h_0(t)(2H_0(t))^{-1/2} \cdot \exp(-\sqrt{2H_0(t)}),$$

$$= \frac{\vartheta}{t}\left(2\log\left(\frac{t}{r}\right)^{\vartheta}\right)^{-1/2} \cdot \exp\left(-\sqrt{2\log\left(\frac{t}{r}\right)^{\vartheta}}\right).$$

$$h(t) = h_0(t)(2H_0(t))^{-1/2} = \frac{\vartheta}{t}\left(2\log\left(\frac{t}{r}\right)^{\vartheta}\right)^{-1/2}. \tag{4.71}$$

4.9 Bivariate Frailty

Bivariate frailty approach is applied to study dependence effects between two life spans with random effects. The selected frailty mixing distribution determines the association structure (Hougaard (2000)). In the construction of bivariate frailty mixtures the following assumptions are applied:
1. Conditional on the shared risk U_i, residual lifetimes of insureds in the i^{th} group are independent.
2. The frailty U_i is presumed to have a multiplicative effect on the insureds intensity rate: $h_{ij}(t|u) = u_i h_0(t)$, where j is an individual.

Proposition 4.1 *Suppose assumption 1 holds for the residual lifetimes under the shared risk. The bivariate marginal survivor function is expressed as*

$$S(t_{i1,}t_{i2}) = \mathbb{L}_{U_i}(H_0(t_{i1}) + H_0(t_{i2})). \tag{4.72}$$

Proof. Assuming independence, the bivariate conditional survivor function for a given shared frailty U_i is represented as:

$$S(t_{i1}, t_{i2}|u_i) = S(t_{i1}|u_i) \cdot S(t_{i2}|u_i),$$

applying

$$S(t|u) = \exp(-\int_0^t h_0(x|u)dx) = \exp(-uH_0(t)),$$

leads to

$$S(t_{i1}, t_{i2}|u_i) = \exp\left(-u_i[H_0(t_{i1}) + H_0(t_{i2})]\right).$$

Using expectation

$$S(t_{i1,}t_{i2}) = E[\exp\left(-u_i[H_0(t_{i1}) + H_0(t_{i2})]\right)],$$

this simplifies to

$$S(t_{i1,}t_{i2}) = \mathbb{L}_{U_i}(H_0(t_{i1}) + H_0(t_{i2})),$$

which is the Laplace transform for the shared risk solved at the sum of the cumulative base force of mortality hazards. □

The multivariate expression for say k observations is represented as:

$$S(t_{i1,\ldots,}t_{ik}) = E[\exp\{-u[H_0(t_{i1}) + \ldots + H_0(t_{ik})]\}] = \mathbb{L}_{U_i}(H_0(t_{i1}) + \ldots + H_0(t_{ik})). \tag{4.73}$$

4.10 Positive Stable Distribution

Using the positive stable Laplace transform allowing for identifiability derived in Equation (3.37), the marginal bivariate survivor function

$$S(t_{i1}, t_{i2}) = \mathbb{L}_U(H_0(t_{i1}) + H_0(t_{i2})),$$

is expressed as

$$S(t_{i1}, t_{i2}) = \exp\{-(H_0(t_{i1}) + H_0(t_{i2}))^{\alpha}\}. \tag{4.74}$$

The marginal bivariate density and intensity functions are:

$$\begin{aligned} f(t_{i1}, t_{i2}) &= \frac{\partial^2}{\partial t_{i1}\partial t_{i2}} S(t_{i1}, t_{i2}), \\ &= S(t_{i1}, t_{i2}) \cdot h_0(t_{i1})h_0(t_{i2}) \\ &\cdot [\alpha^2(H_0(t_{i1}) + H_0(t_{i2}))^{2\alpha-2} - \alpha(\alpha-1)(H_0(t_{i1}) + H_0(t_{i2}))^{\alpha-2}]. \\ h(t_{i1}, t_{i2}) &= \frac{f(t_{i1}, t_{i2})}{S(t_{i1}, t_{i2})} \\ &= h_0(t_{i1})h_0(t_{i2}) \cdot [\alpha^2(H_0(t_{i1}) + H_0(t_{i2}))^{2\alpha-2} - \alpha(\alpha-1)(H_0(t_{i1}) + H_0(t_{i2}))^{\alpha-2}]. \end{aligned} \tag{4.75}$$

The local measure of association defined as the relative risk $A(t_{i1}, t_{i2})$ (Sect. 6.4 in Wienke (2011)) for the first lifetime given that the second lifetime has died rather than surviving beyond a given lifetime is described as follows. Let $h(t_{i1}|T_{i2} = t_{i2})$ be the conditional intensity rate of T_{i1} at t_{i1} given that the second life died at t_{i2} and $h(t_{i1}|T_{i2} > t_{i2})$ be the conditional intensity rate of T_{i1} at t_{i1} given that the second life survived beyond t_{i2}. Then the relative risk is:

$$A(t_{i1}, t_{i2}) = \frac{h(t_{i1}|T_{i2} = t_{i2})}{h(t_{i1}|T_{i2} > t_{i2})}. \tag{4.76}$$

$$Since$$

$$h(t) = -\frac{\partial \log S(t)}{\partial t},$$

$$We \;\; have$$

$$\begin{aligned} A(t_{i1}, t_{i2}) &= \frac{-\frac{\partial}{\partial t_{i1}} \log(-\frac{\partial}{\partial t_{i2}} S(t_{i1}, t_{i2}))}{-\frac{\partial}{\partial t_{i1}} \log S(t_{i1}, t_{i2})}, \\ &= \frac{S(t_{i1}, t_{i2}) \frac{\partial^2}{\partial t_{i1} \partial t_{i2}} S(t_{i1}, t_{i2})}{\frac{\partial}{\partial t_{i1}} S(t_{i1}, t_{i2}) \frac{\partial}{\partial t_{i2}} S(t_{i1}, t_{i2})}. \end{aligned} \tag{4.77}$$

Thus, $A(t_{i1}, t_{i2}) > 1$ represents positive association, $A(t_{i1}, t_{i2}) < 1$ represents negative association and $A(t_{i1}, t_{i2}) = 1$ independence.

Theorem 4.1 *Using the positive stable frailty mixing distribution and integrated base force of mortality the measure of local association $A(t_{i1}, t_{i2})$ defined in Equation (4.77) is expressed as:*

$$A(t_{i1}, t_{i2}) = 1 - \left(1 - \frac{1}{\alpha}\right) (H_0(t_{i1}) + H_0(t_{i2}))^{-\alpha}. \tag{4.78}$$

Proof. From Equation (4.74) we have

$$\begin{aligned} \frac{\partial}{\partial t_{i1}} S(t_{i1}, t_{i2}) &= -\alpha (H_0(t_{i1}) + H_0(t_{i2}))^{\alpha-1} h_0(t_{i1}) \exp\{-(H_0(t_{i1}) + H_0(t_{i2}))^{\alpha}\}, \\ \frac{\partial}{\partial t_{i2}} S(t_{i1}, t_{i2}) &= -\alpha (H_0(t_{i1}) + H_0(t_{i2}))^{\alpha-1} h_0(t_{i2}) \exp\{-(H_0(t_{i1}) + H_0(t_{i2}))^{\alpha}\}, \\ \frac{\partial^2}{\partial_{t_{i1}} \partial t_{i2}} S(t_{i1}, t_{i2}) &= [\alpha^2 (H_0(t_{i1}) + H_0(t_{i2}))^{2\alpha-2} - \alpha(\alpha - 1)(H_0(t_{i1}) + H_0(t_{i2}))^{\alpha-2}] \\ &\quad \cdot h_0(t_{i1}) h_0(t_{i2}) \exp\{-(H_0(t_{i1}) + H_0(t_{i2}))^{\alpha}\}. \end{aligned}$$

$$
\begin{aligned}
A(t_{i1},t_{i2}) = & \frac{[\alpha^2(H_0(t_{i1})+H_0(t_{i2}))^{2\alpha-2} - \alpha(\alpha-1)(H_0(t_{i1})+H_0(t_{i2}))^{\alpha-2}]}{-\alpha(H_0(t_{i1})+H_0(t_{i2}))^{\alpha-1} h_0(t_{i1}) \exp\{-(H_0(t_{i1})+H_0(t_{i2}))^{\alpha}\}} \\
& \cdot \frac{h_0(t_{i1})h_0(t_{i2}) \exp\{-(H_0(t_{i1})+H_0(t_{i2}))^{\alpha}\} \exp\{-(H_0(t_{i1})+H_0(t_{i2}))^{\alpha}\}}{-\alpha(H_0(t_{i1})+H_0(t_{i2}))^{\alpha-1} h_0(t_{i2}) \exp\{-(H_0(t_{i1})+H_0(t_{i2}))^{\alpha}\}}, \\
& = 1 - \frac{\alpha(\alpha-1)(H_0(t_{i1})+H_0(t_{i2}))^{\alpha-2}}{\alpha^2(H_0(t_{i1})+H_0(t_{i2}))^{2\alpha-2}}, \\
& = 1 - \left(1 - \frac{1}{\alpha}\right)(H_0(t_{i1})+H_0(t_{i2}))^{-\alpha}.
\end{aligned}
$$

□

When α takes on values near zero high dependence is observed between T_{i1} and T_{i2}, while α near one indicates low dependence. $\alpha = 1$ corresponds to maximal independence and $\alpha = 0$ maximal dependence. α can be determined from a global association measure say τ using the simple form $\alpha = 1 - \tau$

4.11 Non-Central Gamma

Using the Laplace transform derived in Equation (3.43)

$$\mathbb{L}_U(s) = \exp\left(-\frac{s}{1+\frac{1}{2}\sigma^2 s}\right).$$

The marginal survivor, density and intensity rates are:

$$S(t_{i1},t_{i2}) = \mathbb{L}_{U_i}(H_0(t_{i1})+H_0(t_{i2})) = \exp\left(-\frac{H_0(t_{i1})+H_0(t_{i2})}{1+\frac{1}{2}\sigma^2(H_0(t_{i1})+H_0(t_{i2}))}\right), \tag{4.79}$$

$$
\begin{aligned}
f(t_{i1},t_{i2}) = & \exp\left(-\frac{H_0(t_{i1})+H_0(t_{i2})}{1+\frac{1}{2}\sigma^2 H_0(t_{i1})+H_0(t_{i2})}\right) \\
& \cdot \left(h_0(t_{i1})h_0(t_{i2})\left[\left(1+(H_0(t_{i1})+H_0(t_{i2}))\frac{\sigma^2}{2}\right)^{-4} + \sigma^2\left(1+(H_0(t_{i1})+H_0(t_{i2}))\frac{\sigma^2}{2}\right)^{-3}\right]\right),
\end{aligned}
$$

$$h(t_{i1},t_{i2}) = h_0(t_{i1})h_0(t_{i2}) \cdot \left[\left(1+(H_0(t_{i1})+H_0(t_{i2}))\frac{\sigma^2}{2}\right)^{-4} + \sigma^2\left(1+(H_0(t_{i1})+H_0(t_{i2}))\frac{\sigma^2}{2}\right)^{-3}\right]. \tag{4.80}$$

Theorem 4.2 *Using the NCG frailty mixing distribution and integrated base force of mortality $H_0(t)$, the measure of local association $A(t_{i1}, t_{i2})$ defined in Equation (4.77) is expressed as:*

$$A(t_{i1}, t_{i2}) = 1 + \sigma^2 \left(1 + \frac{\sigma^2}{2}(H_0(t_{i1}) + H_0(t_{i2}))\right). \quad (4.81)$$

Proof. From Equation (4.79) we have

$$\frac{\partial}{\partial t_{i1}} S(t_{i1}, t_{i2}) = -\frac{h_0(t_{i1})}{(1 + \frac{1}{2}\sigma^2(H_0(t_{i1}) + H_0(t_{i2})))^2} \exp\left(-\frac{H_0(t_{i1}) + H_0(t_{i2})}{1 + \frac{1}{2}\sigma^2(H_0(t_{i1}) + H_0(t_{i2}))}\right),$$
$$\frac{\partial}{\partial t_{i2}} S(t_{i1}, t_{i2}) = -\frac{h_0(t_{i2})}{(1 + \frac{1}{2}\sigma^2(H_0(t_{i1}) + H_0(t_{i2})))^2} \exp\left(-\frac{H_0(t_{i1}) + H_0(t_{i2})}{1 + \frac{1}{2}\sigma^2(H_0(t_{i1}) + H_0(t_{i2}))}\right),$$
$$\frac{\partial^2}{\partial_{t_{i1}} \partial t_{i2}} S(t_{i1}, t_{i2}) = \exp\left(-\frac{H_0(t_{i1}) + H_0(t_{i2})}{1 + \frac{1}{2}\sigma^2 H_0(t_{i1}) + H_0(t_{i2})}\right)$$
$$\cdot \left(h_0(t_{i1})h_0(t_{i2}) \left[\left(1 + (H_0(t_{i1}) + H_0(t_{i2}))\frac{\sigma^2}{2}\right)^{-4} + \sigma^2 \left(1 + (H_0(t_{i1}) + H_0(t_{i2}))\frac{\sigma^2}{2}\right)^{-3}\right]\right),$$

$$Since, \quad A(t_{i1}, t_{i2}) = \frac{S(t_{i1}, t_{i2}) \frac{\partial^2}{\partial t_{i1} \partial t_{i2}} S(t_{i1}, t_{i2})}{\frac{\partial}{\partial t_{i1}} S(t_{i1}, t_{i2}) \frac{\partial}{\partial t_{i2}} S(t_{i1}, t_{i2})},$$
$$we \quad get \quad A(t_{i1}, t_{i2}) = 1 + \sigma^2 \left(1 + \frac{\sigma^2}{2}(H_0(t_{i1}) + H_0(t_{i2}))\right).$$

□

4.12 Compound Poisson Distribution

Using the Laplace transform derived in Equation (3.43)

$$\mathbb{L}_U(s) = \exp\left(\frac{-k}{\alpha}((\beta + s)^\alpha - \beta^\alpha)\right).$$

Allowing for identifiability i.e. $E[U] = k\beta^{\alpha-1} = 1$ and $var[U] = \frac{1-\alpha}{\beta}$ the Laplace becomes

$$\mathbb{L}_U(s) = \exp\left(\frac{\alpha-1}{\alpha\sigma^2}\left[\left(1+s\frac{\sigma^2}{1-\alpha}\right)^{\alpha}-1\right]\right).$$

The marginal survivor, density and intensity rates are:

$$\begin{aligned}
S(t_{i1},t_{i2}) &= \mathbb{L}_U(H_0(t_{i1})+H_0(t_{i2})) = \exp\left(\frac{\alpha-1}{\alpha\sigma^2}\left[\left(1+(H_0(t_{i1})+H_0(t_{i2}))\frac{\sigma^2}{1-\alpha}\right)^{\alpha}-1\right]\right),\\
f(t_{i1},t_{i2}) &= \frac{\partial^2}{\partial t_{i1}\partial t_{i2}}S(t_{i1},t_{i2})\\
&= S(t_{i1},t_{i2})\cdot(h_0(t_{i1})h_0(t_{i2}))\left[\left(1+(H_0(t_{i1})+H_0(t_{i2}))\frac{\sigma^2}{1-\alpha}\right)^{2\alpha-2}\right.\\
&\quad\left.+\sigma^2\left(1+(H_0(t_{i1})+H_0(t_{i2}))\frac{\sigma^2}{1-\alpha}\right)^{\alpha-2}\right],\\
h(t_{i1},t_{i2}) &= (h_0(t_{i1})h_0(t_{i2}))\cdot\left[\left(1+(H_0(t_{i1})+H_0(t_{i2}))\frac{\sigma^2}{1-\alpha}\right)^{2\alpha-2}\right. \qquad (4.82)\\
&\quad\left.+\sigma^2\left(1+(H_0(t_{i1})+H_0(t_{i2}))\frac{\sigma^2}{1-\alpha}\right)^{\alpha-2}\right].
\end{aligned}$$

4.13 The Levy Distribution

The Levy is derived from the inverse gamma distribution. The Laplace is also derived from the IG Laplace in Equation (3.5).

$$\mathbb{L}_U(s) = \exp\{\sqrt{\varpi\vartheta}-\sqrt{(2s+\varpi)\vartheta}\},$$

when $\varpi = 0$,$\vartheta = \mu^2$.

$$\mathbb{L}_U(s) = \exp\{-\sqrt{2s\mu^2}\}.$$

For identifiability reasons assume the distribution of U has mean normalized to 1. The Laplace becomes

$$\mathbb{L}_U(s) = \exp\{-\sqrt{2s}\}.$$

The marginal survivor, density and intensity rates are:

$$
\begin{aligned}
S(t_{i1}, t_{i2}) &= \mathbb{L}_U(H_0(t_{i1}) + H_0(t_{i2})) = \exp\{-\sqrt{2(H_0(t_{i1}) + H_0(t_{i2}))}\}.\\
f(t_{i1}, t_{i2}) &= \frac{\partial^2}{\partial t_{i1}\partial t_{i2}} S(t_{i1}, t_{i2}),\\
&= \exp\{-\sqrt{2H_0(t_{i1}) + H_0(t_{i2}))}\} \cdot h_0(t_{i1})h_0(t_{i2}) \cdot \{(2(H_0(t_{i1}) + H_0(t_{i2})))^{-1}\\
&+ (2(H_0(t_{i1}) + H_0(t_{i2})))^{-3/2}\}.\\
h(t_{i1}, t_{i2}) &= h_0(t_{i1})h_0(t_{i2}) \cdot \{(2(H_0(t_{i1}) + H_0(t_{i2})))^{-1} + (2(H_0(t_{i1}) + H_0(t_{i2})))^{-3/2}\}.
\end{aligned}
\tag{4.83}
$$

Chapter 5

Application: Term Insurance Data Graduation Using Non-Central Gamma Frailty Mixture

In this chapter, we proposed an alternative to the commonly adopted gamma frailty mixture. The gamma is widely applied due to mathematical convenience but this distribution implies frailty level is constant with regard to time. The NCG is proposed as an alternative to the gamma which is a compound approach to time-varying frailty that is used to model real-life insurance data. The base force of mortality distributions considered are the GE and GW. Bayes inference based on Gibbs sampling is used to calibrate the base force of mortality variables. A comparison of the fit of the aforementioned models is done with the Deviance Information Criteria (DIC). Based on the results, the NCG-GW frailty model can effectively indicate the insurers' liability in the presence of heterogeneity. The implications of these findings on pricing term insurance products is discussed.

5.1 Introduction

The mortality model selected in valuation determines how term insurance and annuity products are priced (Batty *et al.,* (2010); Gildas *et al.,* (2018)). Life insurance-based frailty models measure population-level heterogeneity caused by unreported risks. On the other hand, heterogeneity caused by reported risk factors is determined during underwriting before issuing a policy to guarantee an optimal assignment of premium equivalent to insured risk for each contract. Excluding relevant factors or relying solely on age and sex may contribute to incorrectly priced assurance products. Term insurance contract is a policy in which a certain payment, say Ksh A per annum, is provided if the insured passes on within a stated period, say m years. This benefit is determined from the EPV as:

$$A \int_0^m v^t S_x(t) h(x+t) dt, \tag{5.1}$$

where: x is the insured's age, v^t the present value factor, $S_x(t)$ is the survivor probability and $h(x+t)$ the intensity function. The type of mortality model applied to $h(x+t)$ influences how the policy is priced. The expected time-to-death for the insured is represented as:

$$E[T_x] = \int_0^\infty S_x(t) dt. \tag{5.2}$$

For this exercise, we considered 732 term insurance contracts between calender years 2010-2015 from a large Kenyan insurance company. Demographic information of policyholders includes; age, date of signing the contract and mortality date. This data-set will be used to compute the times-to-death and crude intensity rates experienced by the insureds in the 22-64 age group as shown in Appendix **I**. Our aims of study are firstly to show that when the gamma is applied as a frailty distribution the intensity rates are overestimated at all ages compared to the NCG. Secondly, is to show the relevance of the NCG frailty mixture to graduate the insurance firm's crude intensity rates. For the study we assume that the force of mortality is pair-wise constant, assuming a fixed value across all ages consistent with Brouhns *et al.,* (2002) and Dodd *et al.,* (2018) assumption. We further assumed our frailty model has no observed covariates as only survival data was obtained for this analysis also in life insurance due to the underwriting procedures groups should be homogeneous with respect to observed covariates. Finally, and as given in the actual dataset we presumed that policyholders purchase term assurance policy at the age of 22-64 years.

5.2 The Proposed Model

Firstly, the GE distribution is proposed as the base force of mortality with the gamma frailty resulting in the gamma-GE frailty mixture intensity rate described explicitly as

$$h(t) = \frac{b\rho(1-\exp(-\rho t))^{b-1}\exp(-\rho t)}{1-[1-\exp(-\rho t)]^b} \cdot (1-\sigma^2\log(1-[1-\exp(-\rho t)]^b))^{-1}. \tag{5.3}$$

The NCG is further adapted as a frailty distribution resulting in the NCG-GE frailty intensity rate given by

$$h(t) = \frac{b\rho(1-\exp(-\rho t))^{b-1}\exp(-\rho t)}{1-[1-\exp(-\rho t)]^b} \cdot (1-0.5\sigma^2\log(1-[1-\exp(-\rho t)]^b))^{-2}. \tag{5.4}$$

The GE intensity rate is decreasing ($b < 1$), increasing ($b > 1$) or constant ($b = 1$).

Secondly, the GW distribution is proposed as the base force of mortality with the gamma frailty giving the gamma-GW frailty intensity rate expressed as

$$h(t) = \frac{b(1-\exp(-\lambda t^\rho))^{b-1}\lambda\rho t^{\rho-1}\exp(-\lambda t^\rho)}{1-[1-\exp(-\lambda t^\rho)]^b} \cdot (1-\sigma^2\log(1-[1-\exp(-\lambda t^\rho)]^b))^{-1}. \tag{5.5}$$

Similarly, the NCG-GW frailty intensity rate is described as

$$h(t) = \frac{b(1-\exp(-\lambda t^\rho))^{b-1}\lambda\rho t^{\rho-1}\exp(-\lambda t^\rho)}{1-[1-\exp(-\lambda t^\rho)]^b} \cdot (1-0.5\sigma^2\log(1-[1-\exp(-\lambda t^\rho)]^b))^{-2}. \tag{5.6}$$

The GW intensity curve is monotone decreasing if ($\rho \leq 1$) and ($b\rho \leq 1$); monotone increasing if ($\rho \geq 1$) and ($b\rho \geq 1$); unimodal if ($\rho < 1$) and ($b\rho > 1$) and bath-tub shaped if ($\rho > 1$) and ($b\rho < 1$).

5.3 Parameter Estimation

Butt & Haberman (2004) apply the frailty-based survival model to insurance. The authors first consider different choices of models and then apply them to two large life insurance mortality datasets. The results indicate a potential

range $\sigma^2 \approx (2.916, 14.444)$ in an insured population with $\sigma^2 = 14\%$ for the heterogeneous case. From the findings of the investigation by Butt & Haberman (2004), in this study we consider $\sigma^2 = 14\%$ for the heterogeneous case.
The base force parameters of mortality distributions are then determined by Bayes inference based on Gibbs sampling and Metropolis algorithm using intensity rates derived from actual insurance data. The posterior density derived from observed data likelihood corresponds to the likelihood-joint prior product, assuming all model parameters are independent.
For the GE distribution:

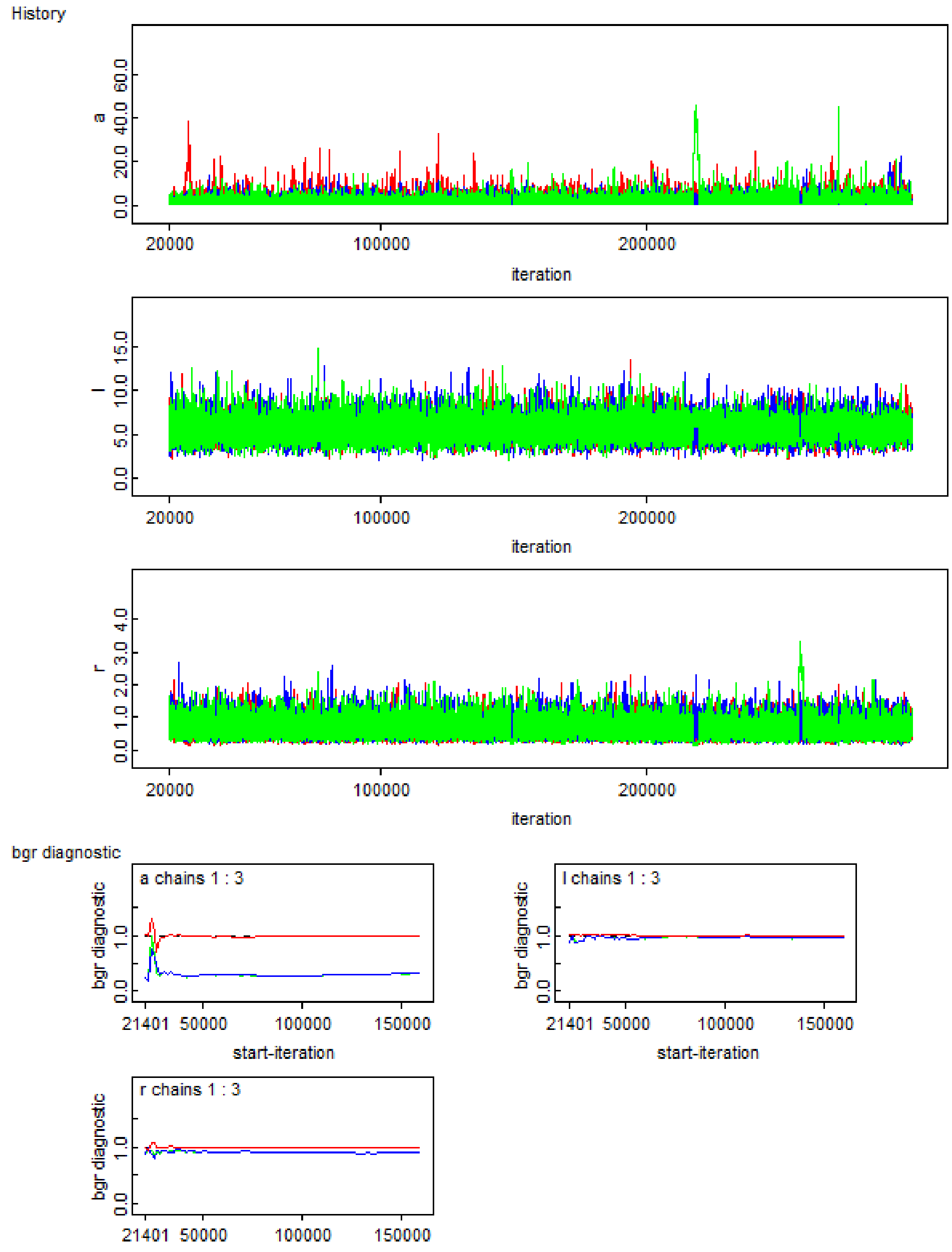
$$\mathbb{P}(\hat{\rho}, \hat{b}|Data) \propto \mathcal{L}(\hat{\rho}, \hat{b}|Data) \times \mathbb{P}(\hat{\rho}) \times \mathbb{P}(\hat{b}),$$

where: $\mathcal{L}(.)$ is the likelihood.
The priors: $\mathbb{P}(\hat{\rho}) = \mathbb{P}(\rho_1)\mathbb{P}(\rho_2|\rho_1)\ldots\mathbb{P}(\rho_h|\rho_1\ldots\rho_{h-1})$ and
$\mathbb{P}(\hat{b}) = \mathbb{P}(b_1)\mathbb{P}(b_2|b_1)\ldots\mathbb{P}(b_h|b_1...b_{h-1})$ where h denotes the time partitions.
Using Gibbs sampler, samples from posterior density $\mathbb{P}(\hat{\rho}, \hat{b}|Data)$ were derived indirectly, without having to calculate the density explicitly. From the starting points $\hat{\rho^0}, \hat{b^0}$ parameter samples were obtained recursively $\hat{\rho}, \hat{b}$ from their conditional posterior densities;
$\mathbb{P}(\rho_j|\hat{\rho_j}, Data) \propto f_1(\rho_j); j = 1, 2, \ldots, h$ and
$\mathbb{P}(b_j|\hat{b_j}, Data) \propto f_2(b_j); j = 1, 2, \ldots, h$.
Finally, the target (posterior) distribution is approximately converged from the prior distribution via the Metropolis-Hastings acceptance-rejection rule.
The algorithm is implemented as follows:

Step 1: The log-likelihood is derived from the GE(ρ, b) and GW(ρ, b, λ) densities i.e.
$\ell(\rho, b|\underline{t}) = k\log(\rho \times b) + (b-1)\sum_{i=1}^{k}\log(1-\exp(-\rho t_i)) - \rho\sum_{i=1}^{k} t_i$;
$\ell(\lambda, \rho, b|\underline{t}) = k\log(\lambda \times \rho \times b) + (b-1)\sum_{i=1}^{k}\log(1-\exp(-\lambda(t_i)^{\rho})) + (\rho - 1)\sum_{i=1}^{k}\log(t_i) - \rho\sum_{i=1}^{k}(t_i)^{\rho}$.
Step 2: Prior for all parameters is defined as:
$\rho \sim \Gamma(0.001, 0.001)$; $b \sim \Gamma(0.001, 0.001)$; $\lambda \sim \Gamma(0.001, 0.001))$.
We use non-informative priors because prior information about the base force of mortality parameters is lacking. The prior distributions picked fall within the range of parameter estimates.
Step 3: The initially chosen parameter estimates used for iteration represent the MLE estimates (obtained outside of Open Source Bayesian Inference using Gibbs Sampling (OpenBUGS)).

Step 4: Actual data values is specified as the times-to-death data obtained from the real-term assurance dataset.
Step 5: For posterior simulation, 2 Markov chains are considered in-order to monitor convergence.
Step 6: The computation of estimates is considered to take 100000 iterations.
Step 7: The burn-in level, that is, simulations to be rejected initially, is taken as 30000 (obtained from Brooks-Gelman-Rubin (BGR) plots).
All computations are performed with OpenBugs in **R** statistical package. The codes are shown in Appendix **IV**.

Figure 5.1: MCMC Trace Plots and BGR Diagnostics representing convergence for GW(ρ, b, λ) where $a = \rho, l = b, r = \lambda$



The MCMC trace plots displayed in Figure 5.1 is observed to be reverting around the mean and the chains appear to mix freely implying stationarity is attained. The BGR diagnostics is also presented in Figure 5.1. As the Markov chains iterations progresses, the total-sequence simulated values (green curvature) and average within-sequence interval width (blue curvature) values are monitored. Their ratio (red curvature) is observed to converge to 1 beyond 30000 simulations giving a probable burn-in period. The parameter estimates for the base force of mortality distributions are shown in Table 5.1.

Table 5.1: Base Force of Mortality Parameter Estimates.

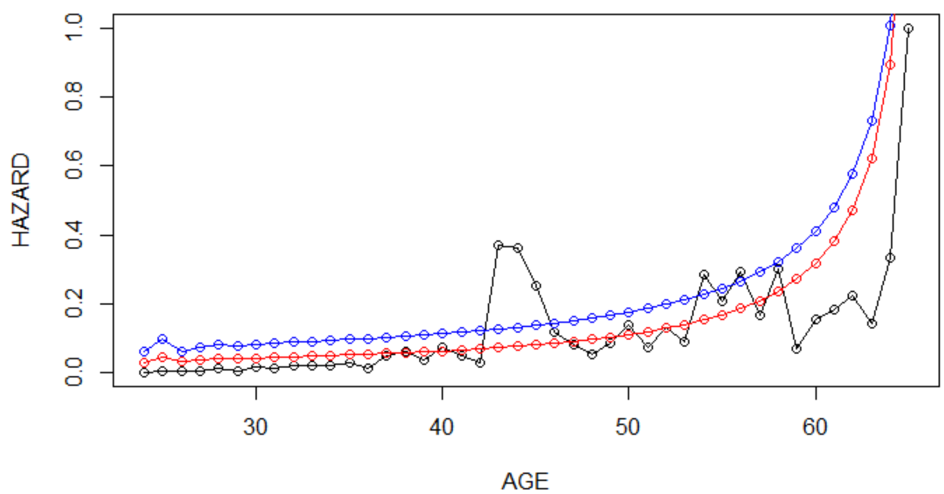
BASELINE MODEL	**PARAMETER ESTIMATES**	**DIC**
1. Generalized Exponential	$\rho = 0.1479, b = 319.7$	4995.0
2. Generalized Weibull	$\rho = 2.973, b = 4.98, \lambda = 0.00003135$	4895.0

Decision: OpenBUGS provides the DIC measure that penalises both excessive use of parameters and poor data fitting. From the above results, the least DIC suggests that the GW gives a better fit.

5.4 Results

The Gamma-GW frailty and NCG-GW frailty models given in Equations (5.5) and (5.6) respectively are as shown in Figure 5.2 where t is the time-to-death, $\sigma^2 = 0.14, \rho = 2.973, b = 4.98, \lambda = 0.00003135; h_0(t) \sim GW(2.973, 4.98, 0.00003135)$.

Figure 5.2: Crude intensity rates and frailty intensity functions



In Figure 5.2 graduation is done using the Gamma-GW (blue curve) and NCG-GW (red curve) frailty model both calibrated on the real term assurance times-to-death data. This is compared with the real term assurance intensity rates (black curve). As shown the Gamma-GW overestimates the intensity rate at all ages compared to the NCG-GW model. The NCG-GW is observed to fit well to the actual claims experience hazards. The chi-square test Table 5.2 and Kolmogorov-Smirnov (KS) hypothesis test Table 5.3 for overall goodness of fit is significant for NCG-GW. The chi-squared goodness-of-fit test has p-value greater than 0.01, indicating that the model fits well. Similarly, the KS goodness of fit test has p-value greater than 0.01, indicating that the distribution is a good fit.

Table 5.2: Chi-squared Goodness-of-fit of NCG-GW to the Crude Intensity Rates.

Name	**Value**
Chi-squared statistic	1722
Degree of freedom	1681
Chi-squared p-value	0.2379

Table 5.3: Goodness of fit using Kolmogorov-Smirnov test.

Name	**p-value**	**Test Statistic**
Kolmogorov-Smirnov test	0.06448	0.28571

5.5 Discussion

Our main goal in this chapter is to model time-varying frailties by applying NCG-GW frailty mixture. We applied our model to real term insurance times-to-death data. Using Bayesian inference the GW turns out to give a better fit since the DIC is smallest compared to the GE. As shown in Figure 5.2 the Gamma-GW model overestimates the intensity rates at all ages compared to the NCG-GW model since the intensity curve shifts upwards. The NCG-GW fits well to the insurers claims experience as shown in the chi-squared goodness of fit test Table 5.2 and KS hypothesis test Table 5.3 that is significant. The conclusion arrived at is that using the gamma as the frailty distribution may lead to inappropriate term assurance valuations resulting in high prices that negatively impacts marketability of term contracts. The gamma frailty index is time invariant and frailty remains constant throughout life. The NCG compound process represents time-varying frailty and is recommended for better term assurance valuations.

Chapter 6

Application: Life-Table Dependence Modeling Using Positive Stable Frailty

The positive stable frailty mixture is proposed to model dependence among insureds. A number of research in life insurance has focused on mortality dependence modeling (see, e.g., Luciano *et al.,* (2016); Yang (2017); D'Amato *et al.,* (2017); Gildas *et al.,* (2018); Arias & Cirillo (2021)). Focusing separately on either negative or positive effects of dependence. Frailty dependence modeling (Hougaard (2000); Fulla & Laurent (2008); Wienke (2011)) is a methodology that considers association in times to event of related individuals. Here, we assume conditional independence to explain both negative and positive effects of association.

The study adds to the existing body of knowledge in many ways. Firstly, we adopt the shared frailty methodology to life annuity risk determination and valuation outside the widely applied medical and bio-statistical fields. Secondly, lower-tail and upper-tail association are considered separately in most dependence models but here we apply a positively stable frailty model and assume conditional independence for an observed association measure to explain both upper-tail and lower-tail association. Thirdly, we apply an association measure through a positive-stable mixture to account for time-evolving common risk where the time-invariant gamma is mostly applied in literature. The implications of these findings on pricing joint-life insurance products is discussed.

6.1 Joint-life last survivor annuity

The proposed methodology is applicable to any type of joint-life annuity contracts. Its application to joint-life last survivor annuities is discussed. Conceptually, this refers to a policy that commences payment as long as two or more annuitants are alive and continues for the entire life of the last survivor. The goal is to guarantee steady income upon attaining the retirement age; hence, annuities are comparable to single-life pensions. We can express this as a series of payments beginning at time 1 for one single payment with an EPV of, for example, C annually:

$$\begin{aligned} EPV &= C \cdot a_{\overline{xy}}, \\ &= C \cdot \sum_{t=1}^{\infty} v^t S_{\overline{xy}}(t), \end{aligned} \tag{6.1}$$

where the time-to-death random variable $T = Max(t_{i1}, t_{i2})$. Assuming $i = 1, 2, ..., n$ represents the number of joint-life contracts in a portfolio of insureds.

Considering dependence we have

$$S_{\overline{xy}}(t) = S_x(t) + S_y(t) - S_{xy}(t), \tag{6.2}$$

under independence

$$S_{\overline{xy}}(t) = S_x(t) + S_y(t) - S_x(t) \cdot S_y(t). \tag{6.3}$$

The EPV in Equation (6.1) becomes

$$EPV = C \cdot \sum_{t=1}^{\infty} v^t [\exp\{-\int_0^t h_{x+s} ds\} + \exp\{-\int_0^t h_{y+s} ds\} - \exp\{-\int_0^t (h_{xy+s}) ds\}], \tag{6.4}$$

$$EPV = C \cdot \sum_{t=1}^{\infty} v^t [\exp\{-\int_0^t h_{x+s} ds\} + \exp\{-\int_0^t h_{y+s} ds\} - \exp\{-\int_0^t (h_{x+s} + h_{y+s}) ds\}], \tag{6.5}$$

respectively. Here to simplify the notation, we assume an infinite limiting age for our mortality table. Supposing an annuity buying price of B, then by applying the equivalence principle we get the series of payments C:

$$C = \frac{B}{a_{\overline{xy}}}. \tag{6.6}$$

Traditionally, due to simplicity in computations most insurers assume independence in valuation of joint-lives thereby adapting the EPV shown in Equation (6.5). Frailty dependence modeling accounts for both heterogeneity and dependence thus adapting the EPV as shown in Equation (6.4).

The Data
Major Kenyan insurance firm annuitants members data for 398 joint and last survivor contracts for the period 2001-2018 is applied to estimate the baseline parameters and to compute the insureds level of association. Demographic information of policyholders includes; gender, main life date of birth, spouse date of birth, effective date, main life term and spouse term. (see Appendix **III**)

6.2 The Positive Stable Mixture

The suggested positive stable frailty mixture has many merits. Firstly, it is easily implementable due to its simplified Laplace. Secondly, the positive stable variance is infinite. As a result, more heterogeneity can be accounted for than when a frailty mixture is used with fixed variance. Thirdly, the positive stable mixture permits the proportional-hazards to apply conditionally as well as unconditionally. In dependence frailty, it easy to define frailty distribution using the (Sect. 6.4 in Wienke (2011)) relative risk measure, describing how dependence of bivariate intensity rates changes with time. Two examples are presented below with specified frailty mixing densities to obtain the relative risk.
Example 1.
Using the positive stable frailty mixing distribution and integrated base force of mortality $H_0(.)$ the relative risk $A(t_{i1}, t_{i2})$ defined in Equation (4.77) is expressed as:

$$A(t_{i1}, t_{i2}) = 1 - (1 - \frac{1}{\alpha})(H_0(t_{i1}) + H_0(t_{i2}))^{-\alpha}. \tag{6.7}$$

(See Theorem **4.1** for proof.) This relative risk is time-varying and dependent on times-to-death. When α takes on values near zero high dependence is observed between T_{i1} and T_{i2}, while α near one indicates low dependence. $\alpha = 1$ and $\alpha = 0$ corresponds to maximal independence and dependence respectively. α is determined from a global dependence measure say Kendall's tau τ using the simple form $\alpha = 1 - \tau$.

Example 2.

Using the Gamma frailty mixing distribution and integrated base force of mortality $H_0(.)$, the relative risk $A(t_{i1}, t_{i2})$ is represented as:

$$A(t_{i1}, t_{i2}) = (1 + \sigma^2). \tag{6.8}$$

Where σ^2 is the frailty variance.

Using the identifiable gamma Laplace $\mathbb{L}_{U_i}(s) = (1 + s\sigma^2)^{-1/\sigma^2}$ and bivariate survivor $S(t_{i1}, t_{i2}) = (1 + \sigma^2(H_0(t_{i1}) + H_0(t_{i2})))^{-1/\sigma^2}$ we have From Equation (4.77):

$$\frac{\partial}{\partial t_{i1}} S(t_{i1}, t_{i2}) = h_0(t_{i1})(1 + \sigma^2(H_0(t_{i1}) + H_0(t_{i2})))^{-1/\sigma^2 - 1}$$

$$\frac{\partial}{\partial t_{i2}} S(t_{i1}, t_{i2}) = h_0(t_{i2})(1 + \sigma^2(H_0(t_{i1}) + H_0(t_{i2})))^{-1/\sigma^2 - 1}$$

$$\frac{\partial^2}{\partial t_{i1} \partial t_{i2}} S(t_{i1}, t_{i2}) = \frac{h_0(t_{i1}) h_0(t_{i2})(1 + \sigma^2)}{(1 + \sigma^2(H_0(t_{i1}) + H_0(t_{i2})))^{1/\sigma^2 + 2}}$$

$$\begin{aligned} A(t_{i1}, t_{i2}) &= \frac{(1 + \sigma^2(H_0(t_{i1}) + H_0(t_{i2})))^{-1/\sigma^2} \cdot \frac{h_0(t_{i1}) h_0(t_{i2})(1+\sigma^2)}{(1+\sigma^2(H_0(t_{i1}) + H_0(t_{i2})))^{1/\sigma^2 + 2}}}{h_0(t_{i1})(1 + \sigma^2(H_0(t_{i1}) + H_0(t_{i2})))^{-1/\sigma^2 - 1} \cdot h_0(t_{i2})(1 + \sigma^2(H_0(t_{i1}) + H_0(t_{i2})))^{-1/\sigma^2 - 1}} \\ &= (1 + \sigma^2) \end{aligned}$$

This relative risk is constant and independent of times-to-death.

Sensitivity Test

To test the importance of the dependence specification in our model, we compared the variation in relative risk under different estimates of α, say; $\alpha = 0.1$, $\alpha = 0.74$,$\alpha = 0.9$ and $\alpha = 1$. When the positive stable is adopted as frailty distribution, the Equation (4.77) is expressed as:

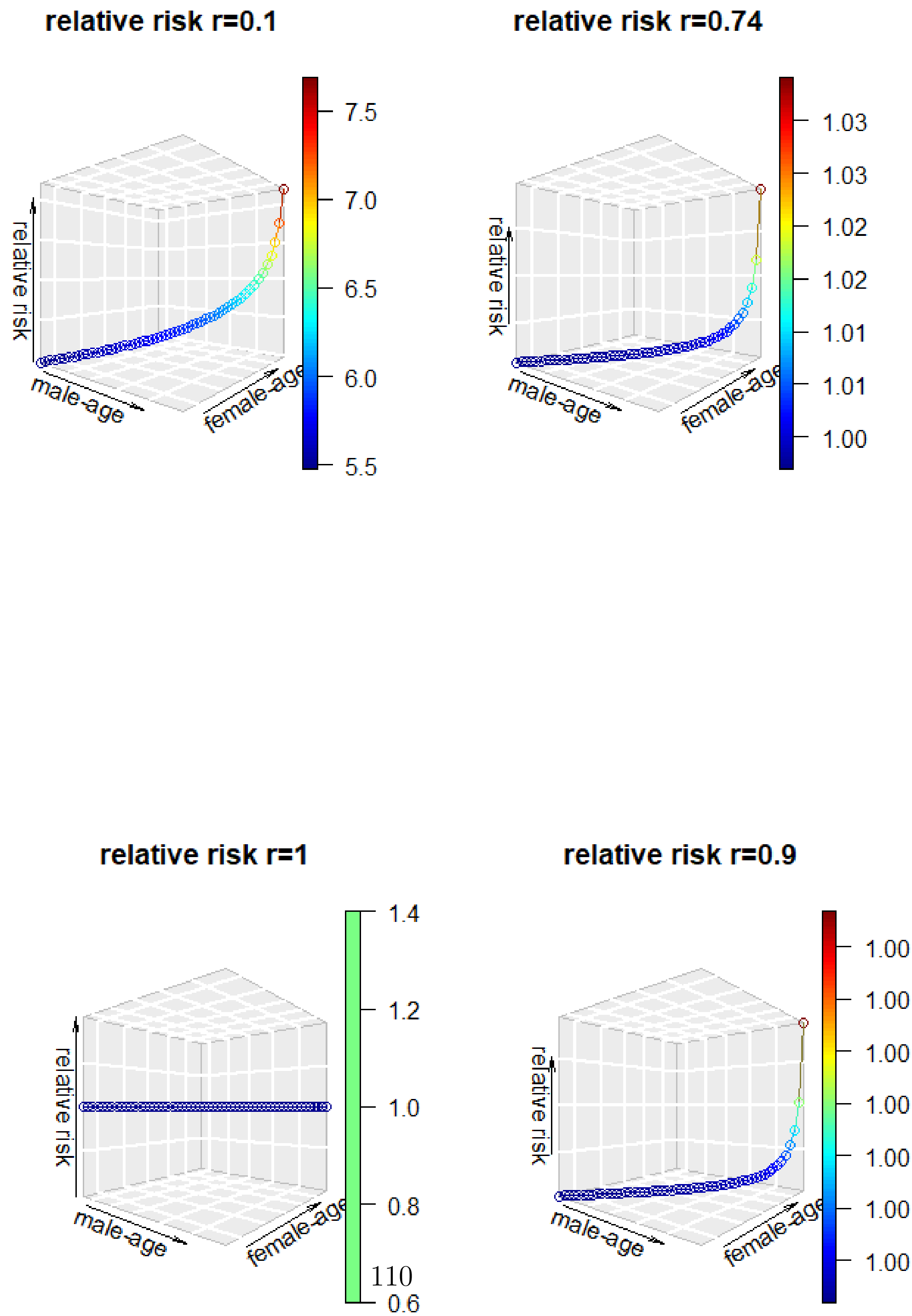
$$A(t_{i1}, t_{i2}) = 1 - (1 - \frac{1}{\alpha})(H_0(t_{i1}) + H_0(t_{i2}))^{-\alpha}.$$

If $h_0(t) \sim$ Weibull(ρ, ϖ)

$$A(t_{i1}, t_{i2}) = 1 - (1 - \frac{1}{\alpha})(\varpi_1 t_{i1}^{\rho_1} + \varpi_2 t_{i2}^{\rho_2})^{-\alpha},$$

where $\rho_1 = 0.67, \varpi_1 = 7.15; \rho_2 = 0.75, \varpi_2 = 10.17$

Figure 6.1: Sensitivity Test of Relative Risk $A(t_{i1}, t_{i2})$ Versus Male and Female Ages at Different Levels of Dependence $\alpha = r$.



From Figure 6.1, it is seen that lower α estimate closer to zero suggest that T_{i1} and T_{i2} are strongly associated, while α estimate nearer 1 indicate a weak association between them. $\alpha = 1$ corresponds to independence. If one of the paired lives has a high survival rate since dependence between them is a result of potential concurrent failure, this declines when survival for one of the paired lives increases.

6.3 The Model

If the base force of mortality follows the GE with pdf $f_0(t) = \varpi\rho(1 - \exp(-\varpi t))^{\rho-1}\exp(-\varpi t); t > 0, \rho, \varpi > 0$. Where ρ, ϖ represents the shape and scale specifications. Then the log-likelihood $\ell(\varpi, \rho)$ considering a given set of times-to-death data $\underline{t} = (t_1, t_2, \ldots, t_k)$ is expressed as

$$\ell(\varpi, \rho|\underline{t}) = k\log(\varpi\rho) + (\rho - 1)\sum_{i=1}^{k}\log(1 - \exp(-\varpi t_i)) - \varpi\sum_{i=1}^{k} t_i. \quad (6.9)$$

The estimates $\hat{\varpi}, \hat{\rho}$ can be derived from the non-linear equations $\frac{\partial\ell}{\partial\varpi} = 0$ and $\frac{\partial\ell}{\partial\rho} = 0$ using any iterative methods. In the study we apply OpenBUGS algorithms.

If the baseline follows the WE with pdf $f_0(t) = (1-\exp(-a\lambda t))\frac{1+\lambda}{\lambda}a\exp(-at); t > 0, \lambda, a > 0$. Where λ, a represents the shape and scale specifications (Gupta & Kundu (2009)). The WE is applied to model intensity rate that is rising, declining or unchanged. The survival, intensity and cumulative intensity functions are respectively;

$S_0(t) = \frac{1+\lambda}{\lambda}[\exp(-at) - \frac{\exp-(1+\lambda)at}{1+\lambda}]$,

$h_0(t) = \frac{(1-\exp(-a\lambda t))a\exp(-at)}{\exp(-at)-\frac{\exp\{-(1+\lambda)at\}}{1+\lambda}}$,

$H_0(t) = -\log(\frac{1+\lambda}{\lambda}[\exp(-at) - \frac{\exp\{-(1+\lambda)at\}}{1+\lambda}])$.

The log-likelihood $\ell(a, \lambda)$ considering a given set of times-to-death data $\underline{t} = (t_1, t_2, \ldots, t_k)$ is represented by

$$\ell(\lambda, a|\underline{t}) = k\log(\frac{a}{\lambda}(1 + \lambda)) + \sum_{i=1}^{k}\log(1 - \exp(-a\lambda t_i)) - a\sum_{i=1}^{k} t_i. \quad (6.10)$$

The estimates $\hat{a}, \hat{\lambda}$ can be derived from the non-linear equations $\frac{\partial\ell}{\partial a} = 0$ and $\frac{\partial\ell}{\partial\lambda} = 0$.

If $h_0(t)$ follows the WW with pdf $f_0(t) = abx^{b-1}\frac{1+\lambda^b}{\lambda^b}\exp\{-at^b\}(1-\exp\{-a\lambda x^b\}); t > 0, \lambda, a, b > 0$. Where a, b are the shape specifications and λ the scale specification (Roman, R. (2010)). The WW includes an additional shape specification that models bimodal data. The survival, intensity and cumulative intensity functions are respectively;

$S_0(t) = \frac{1+\lambda^b}{\lambda^b}[\exp\{-at^b\} - \frac{1}{1+\lambda^b}\exp\{-at^b(1+\lambda^b)\}]$,

$h_0(t) = \frac{abx^{b-1}\exp\{-at^b\}(1-\exp\{-a\lambda x^b\})}{\exp\{-at^b\}-\frac{1}{1+\lambda^b}\exp\{-at^b(1+\lambda^b)\}}$,

$H_0(t) = -\log(\frac{1+\lambda^b}{\lambda^b}[\exp\{-at^b\} - \frac{1}{1+\lambda^b}\exp\{-at^b(1+\lambda^b)\}])$.

The log-likelihood $\ell(a, b, \lambda)$ considering a given set of times-to-death data is given by

$$\ell(a,b,\lambda|\underline{t}) = k\log(\frac{ab(1+\lambda^b)}{\lambda^b}) - a\sum_{i=1}^{k} t_i^b + (b-1)\sum_{i=1}^{k}\log t_i + \sum_{i=1}^{k}\log(1-\exp(-a\lambda t_i{}^b)). \tag{6.11}$$

The estimates $\hat{a}, \hat{b}, \hat{\lambda}$ can be derived from the non-linear equations $\frac{\partial \ell}{\partial a} = 0, \frac{\partial \ell}{\partial b} = 0$ and $\frac{\partial \ell}{\partial \lambda} = 0$

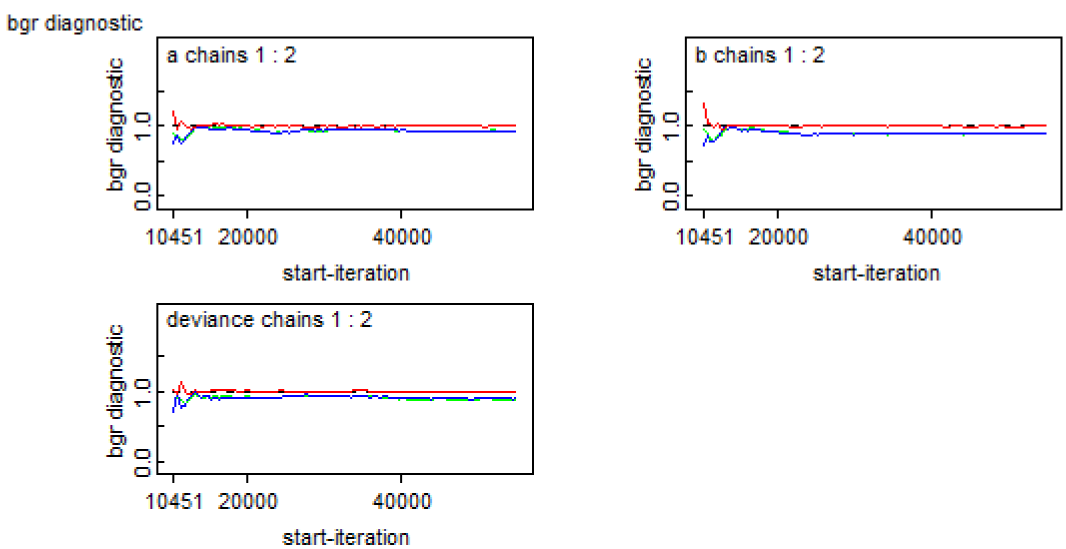
Parameter Estimation

In the Bayes technique, any unknown parameter is regarded as a varying quantity and its distribution is derived from what is known about them. This technique is used as estimation procedures in actuarial studies e.g. by Scollnik (1993) in analyzing concurrent mathematical statements for assurance pricing also by Rosenberg & Young (1999) in studying time-varying dependence when there exists shifts in variance estimation. The Bayes specification technique is executed in the following procedure using OpenBUGS. Firstly, we defined log-likelihood functions as shown in Equations (6.9,6.10 and 6.11) respectively. Since earlier information about the base force of mortality specifications is lacking, non-informative priors are selected and presumed to be flat. That is, $Gamma(0.001, 0.001)$ for the positive specifications see Hanagal (2020).

Actual data for males and female times-to-death is derived from the large Kenyan insurance firm last-survivor dataset. Model specifications will be obtained by considering the life terms from 39 exact through to 68 exact as given in the real dataset. The burn-in value is fixed at 30000 as shown in the Brooks-Gelman-Rubin Figure 5.1 this ensures posterior distributions sequences of draws have low auto-correlation and is obtained from the values of a run of Markov chain (Brooks & Gelman (1998)). Thereby diminishing

the effects of the initial density. We simulate 2 chains in parallel and thereafter stationarity will be monitored upon completion of 100000 replications. If convergence is achieved the average value of the posterior simulations is selected as a point estimate. Low DIC, Akaike Information Criteria (AIC) and Bayesian Information Criteria (BIC) would indicate a better model.
1. $DIC=\bar{A}+pA$ where $\bar{A} = E[-2\times\log\mathbf{L}]$ this is the posteria average indicating the goodness of fit quality of the proposed methodology. $\hat{A} = -2\times\log\mathbf{L}$ is the stochastic nodes posterior average and $pA = \bar{A} - \hat{A}$ measures the ultimate parameters specifications (see Spiegelhalter *et al.,* (2002)).
2. $AIC=\hat{A}+2\rho$ where; ρ is the aggregate specifications.
3. $BIC=\hat{A}+\rho\times\log(m)$; m represents sample size. The BIC is useful as it considers the BIC penalty for all parameters being estimated.
OpenBUGS algorithm applied to analyse the GE model is shown in the *Appendix B*.

Figure 6.2: BGR Diagnostics and Trace Graphs for $GE(\rho,\varpi)$ where $b = \rho, a = \varpi$.



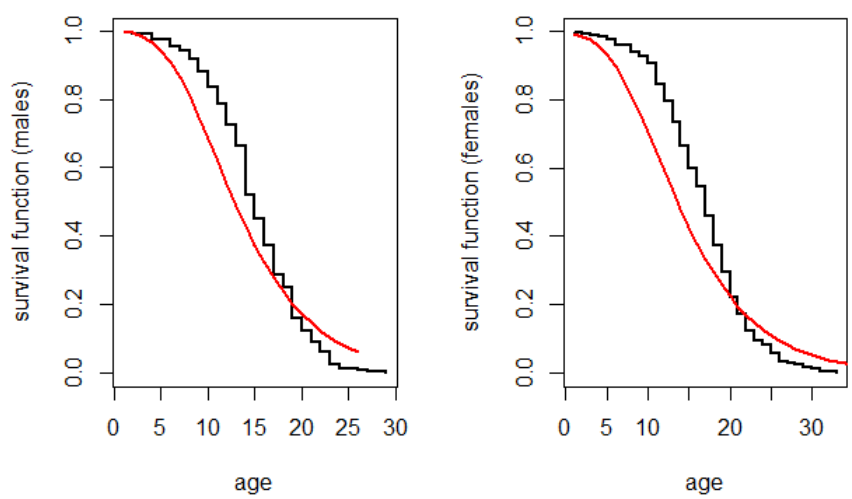
The diagnostic graphs for BGR nodes convergence examined is illustrated in Figure 6.2. As the simulation chains continues, the total-sequence simulated value (green curvature) and average within-sequence intervals (blue curvature) are examined. Their ratio (red curvature) is observed to merge to one after 30000 simulations hence providing a good burn-in period.

Table 6.1: Base Force of Mortality Parameter Estimates.

Baseline	**Parameters**	**AIC**	**DIC**	**BIC**
1. GE	$\rho_1 = 7050; \varpi_1 = 0.1816$	2483	2477	2491
	$\rho_2 = 4514; \varpi_2 = 0.1579$	2604	2602	2612
2. WE	$a_1 = 0.03896; l_1 = 0.006283$	3626	3623	3627
	$a_2 = 0.03571; l_2 = 0.009485$	3699	3697	3700
3. WW	$a_1 = 6.302E-6; b_1 = 3.074; l_1 = 2.642$	3037	3019	3036
	$a_2 = 1.49E-5; b_2 = 2.881; l_2 = 3.151$	3283	3038	3282

On the basis of Bayes inference based on Gibbs sampling the GE density is selected as the AIC, DIC and BIC values is lower in comparison with the other distributions. The model specifications applied in this study is displayed in Table 6.1 upon implementing the Bayes technique discussed previously.

Figure 6.3: Kaplan-Meier (black curve) versus GE (red curve) survivor curves.



Comparing Kaplan-Meier survival function plot (black curve) for the real dataset verses the GE model survivor functions for males and females (red curves) we visually observe a good fit (see Figure 6.3).

Figure 6.4: Q-Q Plot for GE model to the Kenyan last-survivor rates.

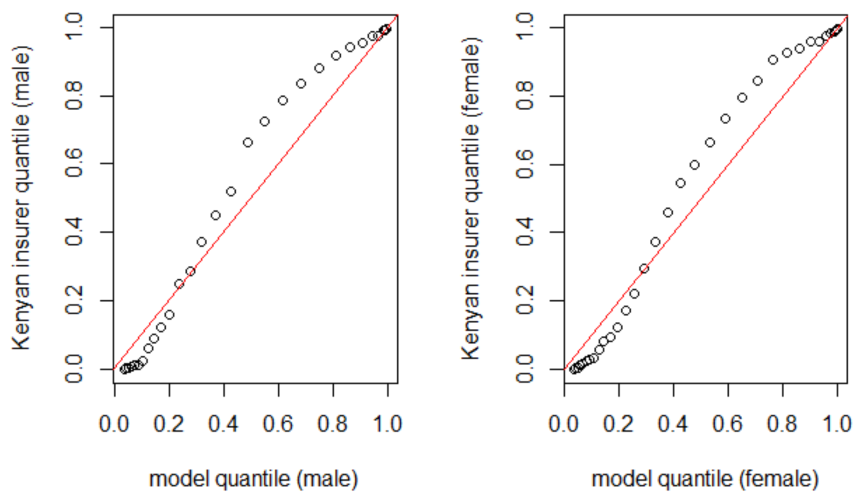


Table 6.2: Chi-square and Kolmogorov-Smirnov Goodness-of-fit of GE to the Kenyan Last-survivor Rates.

Name	Value (Males data)	Value (Females data)
Chi-squared statistic	812	1056
Degree of freedom	784	1024
Chi-squared p-value	0.2371	0.2374
Kolmogorov-Smirnov test statistic	0.19231	0.2222
Kolmogorov-Smirnov p-value	0.6069	0.3047

Further, a chi-square and Kolmogorov-Smirnov goodness of fit for the dataset to GE survivor rates is displayed in Table 6.2. As shown, the chi-square and Kolmogorov-Smirnov test p-values are $\geq 5\%$. Therefore, we do not reject the claim that Kenyan last-survivor rates can be effectively modeled using a GE survivor function at 5% significance level. The GE quantile-quantile (Q-Q) graph in Figure 6.4 displays a straight line through a majority of the quantiles further justifying the GE as a better fit. We can thus conclude that the GE best fits the data.

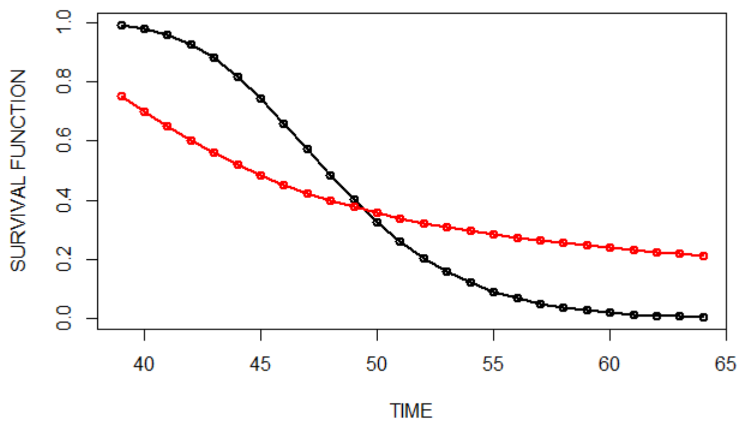
6.4 Results

The positive stable GE frailty bivariate survivor function is described explicitly as

$$S(t_{i1}, t_{i2}) = \exp\{-(-\log(1-[1-\exp(-\varpi_1 t_{i1})]^{\rho_1}))-\log(1-[1-\exp(-\varpi_2 t_{i2})]^{\rho_2}))^{\alpha}\}. \tag{6.12}$$

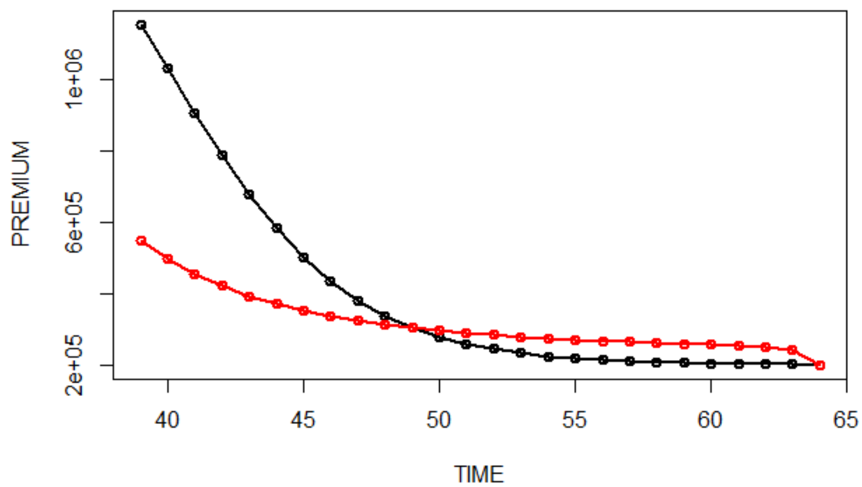
The positive stable GE dependence mixture is displayed in Figure 6.5 where (t_{i1}, t_{i2}) represents the male and female annuitants times-to-death. The baseline intensity rate parameters are $\rho_1 = 7050; \varpi_1 = 0.1816; \rho_2 = 4514; \varpi_2 = 0.1579$ computed from the Bayes inference technique.
The joint last-survivor local measure of association is determined from the large Kenyan insurance firm joint-life last survivor data-set. We consider 398 joint-life annuitants data in-force between 2001-2018. The dependence $S(t_{i1}, t_{i2})$ survivor rates is computed using Kendall's tau ($\tau = 0.7357$) obtained from the Kenyan joint lives dataset. Here $\alpha = 0.2643$ obtained using the simple relation $\alpha = 1 - \tau$.
The independence survivor rates $S(t_{i1}) \cdot S(t_{i2})$ is computed from Equation (6.12) when $\alpha = 1$. The EPVs and net single premiums are generated as discussed in Equation (6.1) at 7% interest rate (central bank of Kenya interest rate as at May 2022). Considering a case where the annuitants expect to receive Ksh 200000 annually.

Figure 6.5: Dependence (red curve) versus Independence (black curve) Survival Rates



As shown in Figure 6.5 the survivor function under independence is higher initially compared to the dependence assumption. This is explained by downside impacts of association incorporated in the frailty methodology (e.g., occurrence of a contagious disease or an accident). Thus incorporating short-term association that exists. Afterwards, there is an underestimation of survival rates in the independence approach in comparison with the dependence approach because of longevity risk. I.e the longer the joint lives survive beyond a given time period, the better their survival probabilities are. In this case, the long-term association is catered for. Therefore the independence approach under-values the survival risk at extreme advanced ages.

Figure 6.6: Dependence (red curve) versus Independence (black curve) Net Single Premium Rates



Moreover, in Figure 6.6 when the annuity net single premiums are compared it is observed that the independence approach leads to over-valuation of the insurance firm's product at the start of the policy and under-valuation later because mortality increase decelerates at extreme old ages. This can be explained by the fact that the insurance firm offers high prices when the survivor rates are high and vice versa because the benefits are paid for the entire life of the last survivor.

6.5 Discussion

The findings arrived at is that the GE baseline PDF gives a better fit to the Kenyan last-survivor data-set compared to the other baseline distributions following the models comparison criteria. Further, applying the positive stable GE frailty approach demonstrates that the relative risk is time-varying and dependent of lifetimes when compared to the independence approach.
The shared frailty shows a decrease in the expected obligation of the insurance firm at early annuitants ages (due to low survivor rates) but an increase in liability at extreme old ages (due to high survivor rates) when association

is considered. A good explanation for this trend is that the survivor rates for frail couples is assumed to be low in the initial stages of the policy, later increase in survivor rates at very old ages since high-risk couples have already died, emphasizing the importance of dependence modeling in collective valuation of annuity contracts. Thus assumptions of joint-life independence can result in biased annuity valuation.

Chapter 7

Conclusions and Recommendations

Although a number of research has been done in univariate frailty modeling to improve underwriting of single life insurance products, the widely applied frailty mixture is the gamma. One major drawback of the gamma is that it is time-invariant, thus in this research compound distributions are proposed to account for time-varying frailties. The NCG compound frailty mixture is proposed for single life contracts (Chapter 5) and positive stable mixture for joint-life contracts (Chapter 6). Dependence frailty modeling has been widely applied in medical and bio-statistical fields. Whereas, in actuarial literature the dependence models consider either only the negative or positive effects of dependence separately. In this thesis, our novel contribution is to apply the shared compound frailty approach in joint-life insurance valuation using the positive stable mixture to explain both negative and positive effects of association. The discussions below summarizes the main findings and recommendations.

The NCG-GW compound frailty mixture is proposed for single life contracts to account for time-varying risks. In particular, a term insurance data-set is fitted to the model. The findings shows that the Gamma-GW model overestimates the intensity rates at all ages compared to the NCG-GW model since the intensity curve shifts upwards. The NCG-GW fits well to the insurers claims experience. The conclusion arrived at is that using the gamma as the frailty distribution may lead to inappropriate term assurance valuations resulting in high prices that negatively impacts marketability of term contracts. The gamma frailty index is time invariant and frailty remains constant throughout life. The NCG shared process represents time-varying frailty and is recommended for better term assurance valuations.

The thesis further presents the positive stable frailty dependence approach calibrated on the Kenyan joint-life last-survivor dataset for both male and female lives. It is applied to construct joint lifetables and generate net single premiums for annuities. The methodology involves applying the conditional independence assumption to explain both positive and negative effects of association. The results shows a declined policyholder's annuity payments at early stages when dependence is incorporated. Later on in the contract the annuity payments increase. A good explanation for this trend is that we expect the frail couples to have died early and the less frail ones to survive to extreme ages. The so called "longevity risk". The positive stable frailty model is therefore recommended to account for dependence in modeling joint-life products. This will ensure accurate valuation of joint-life products.

Further Research
Despite the contributions in this thesis, a number of extensions exists in the work. First, in the positive stable shared frailty approach negative association, for instance, death of one couple leading to a positive effect on survival of the other (though not common in most applications) is not accounted for. This constraint comes in due to the positive stable index parameter (association parameter) that can only take positive values between zero and one. Second, the shared frailty approach may not be applicable in cases where risks within a group are correlated since the model accounts for dependence and heterogeneity by applying a single parameter. Future research is required to suggest advanced compound models that can overcome this limitations.

Similarities and differences between the shared frailty and the Archimeadean copula model widely applied in actuarial literature have been discussed in Section **2.4**. A comparison between the two methodologies can be studied in reference to mortality/longevity risk management applications.

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Appendices

Appendix I

This appendix shows the term assurance times-to-death data and construction of crude intensity rates from real term insurance data-set. The construction of the expected residual lifetimes is also shown.

Figure 7.2 columns description

$Age(x)$: is the age at death.

d_x: the insureds who died aged x.

E_x: the insureds.under observation.

$h(x) = \frac{dx}{E_x}$: the intensity function for the insurance firm, AKI 2010 data and NCG model respectively.

$S(x) = \exp(-h(x))$ is the survival function for the insurance firm and NCG model respectively.

$f(x) = h(x) \times S(x)$ density rate for insurance firm.

$e_x = \sum S(x)$ the expected residual lifetime for the insurance firm, AKI 2010 data and NCG model respectively.

Figure 7.1: Real Term Insurance Member Data from a Major Insurance Firm.

CONTRACTS INFORCE BETWEEN 2010-2020				
DATE_O_BIRTH	CONTRACT_START_DATE	AGE AT PURCHASE	DATE_O_DEATH	TIMES TO DEATH
07-Jan-89	19-Oct-11	22	18-Jul-13	24
03-Nov-88	30-Sep-12	23	30-Jul-13	24
12-Apr-90	20-Mar-14	23	05-Jan-15	24
06-Apr-88	09-Sep-13	25	14-Dec-13	25
10-Aug-86	01-Jun-11	24	15-Dec-12	26
28-Dec-85	01-Jan-12	26	16-Jul-12	26
30-Jan-86	03-Feb-12	26	03-Feb-12	26
26-Mar-87	03-Apr-13	26	07-Aug-13	26
22-Aug-88	02-Dec-13	25	30-Sep-14	26
25-Dec-83	08-Apr-11	27	25-Apr-11	27
19-Feb-86	01-Oct-13	27	02-Oct-13	27
23-Sep-86	01-Jan-14	27	30-Mar-14	27
16-Dec-86	01-Jan-14	27	12-Dec-14	27
20-Jun-86	27-Jan-14	27	15-Mar-14	27
...	...	...	...	...
...	...	...	...	...
10-Feb-57	17-Feb-14	57	21-Apr-14	57
01-Jan-55	20-Nov-12	57	17-Mar-13	58
01-Jan-55	20-Nov-12	57	19-May-13	58
05-May-54	01-Jan-13	58	03-Mar-13	58
21-Mar-56	26-May-14	58	07-Aug-14	58
01-Jan-56	26-Aug-14	58	23-Oct-14	58
01-Jan-56	17-Jan-15	59	04-Dec-14	58
26-Dec-55	03-Nov-14	58	21-Mar-15	59
01-Jan-53	16-Apr-13	60	14-May-13	60
23-Apr-50	01-Jan-11	60	15-Sep-11	61
11-Feb-51	01-Aug-11	60	07-Apr-12	61
25-Aug-53	28-Mar-14	60	24-Jun-15	61
11-Jul-50	01-Jan-12	61	11-Jul-12	62
24-Jun-50	17-Jun-13	62	01-Feb-14	63
01-Jan-51	12-May-14	63	20-Nov-14	63
01-Jan-50	20-Jul-14	64	04-Nov-14	64
17-Mar-48	20-Nov-12	64	16-Jul-13	65
01-Jan-48	01-Jul-13	65	05-Sep-13	65
01-Jan-50	01-May-14	64	11-Feb-15	65
01-Jan-50	18-Sep-14	64	02-Jun-15	65

Figure 7.2: Construction of Crude Intensity Rates from Real Term Insurance Dataset.

CONTRACTS INFORCE BETWEEN 2010-2020										
Age x	dx	Ex	s(x)	f(x)	INSURER_m(x)	NCG_m(x)	NCG_s(x)	INSURER_ex	NCG_ex	
24	3	729	0.99589	0.00410	0.00412	0.01004	0.99001	37.24	36.53	
25	1	726	0.99862	0.00138	0.00138	0.01052	0.98954	36.24	35.54	
26	5	725	0.99313	0.00685	0.00690	0.01104	0.98902	35.24	34.55	
27	6	720	0.99170	0.00826	0.00833	0.01159	0.98847	34.25	33.57	
28	8	714	0.98886	0.01108	0.01120	0.01219	0.98788	33.26	32.58	
29	8	706	0.98873	0.01120	0.01133	0.01284	0.98724	32.27	31.59	
30	7	698	0.99002	0.00993	0.01003	0.01354	0.98655	31.28	30.60	
31	9	691	0.98706	0.01286	0.01302	0.01429	0.98581	30.29	29.61	
32	15	682	0.97825	0.02152	0.02199	0.01512	0.98500	29.30	28.63	
33	11	667	0.98364	0.01622	0.01649	0.01601	0.98411	28.33	27.64	
34	17	656	0.97442	0.02525	0.02591	0.01699	0.98315	27.34	26.66	
35	15	639	0.97680	0.02293	0.02347	0.01806	0.98210	26.37	25.68	
36	10	624	0.98410	0.01577	0.01603	0.01924	0.98095	25.39	24.69	
37	33	614	0.94767	0.05093	0.05375	0.02053	0.97968	24.41	23.71	
38	34	581	0.94316	0.05519	0.05852	0.02196	0.97828	23.46	22.73	
39	27	547	0.95184	0.04698	0.04936	0.02354	0.97673	22.52	21.76	
40	34	520	0.93671	0.06125	0.06538	0.02530	0.97502	21.56	20.78	
41	24	486	0.95182	0.04700	0.04938	0.02726	0.97310	20.63	19.80	
42	14	462	0.97015	0.02940	0.03030	0.02947	0.97096	19.68	18.83	
43	170	448	0.68423	0.25964	0.37946	0.03195	0.96856	18.71	17.86	
44	99	278	0.70039	0.24942	0.35612	0.03475	0.96585	18.02	16.89	
45	45	179	0.77771	0.19551	0.25140	0.03794	0.96277	17.32	15.93	
46	18	134	0.87430	0.11744	0.13433	0.04160	0.95926	16.54	14.96	
47	8	116	0.93336	0.06437	0.06897	0.04580	0.95523	15.67	14.00	
48	6	108	0.94596	0.05255	0.05556	0.05068	0.95058	14.74	13.05	
49	10	102	0.90661	0.08888	0.09804	0.05638	0.94518	13.79	12.10	
50	14	92	0.85884	0.13069	0.15217	0.06309	0.93886	12.88	11.15	
51	4	78	0.95001	0.04872	0.05128	0.07108	0.93139	12.02	10.21	
52	10	74	0.87360	0.11805	0.13514	0.08069	0.92248	11.07	9.28	
53	11	64	0.84208	0.14473	0.17188	0.09238	0.91176	10.20	8.36	
54	13	53	0.78248	0.19193	0.24528	0.10681	0.89870	9.36	7.45	
55	9	40	0.79852	0.17967	0.22500	0.12489	0.88259	8.58	6.55	
56	9	31	0.74802	0.21717	0.29032	0.14799	0.86244	7.78	5.67	
57	3	22	0.87253	0.11898	0.13636	0.17812	0.83684	7.03	4.80	
58	6	19	0.72921	0.23028	0.31579	0.21848	0.80374	6.16	3.97	
59	1	13	0.92596	0.07123	0.07692	0.27426	0.76013	5.43	3.16	
60	1	12	0.92004	0.07667	0.08333	0.35445	0.70156	4.50	2.40	
61	3	11	0.76130	0.20763	0.27273	0.47565	0.62148	3.58	1.70	
62	1	8	0.88250	0.11031	0.12500	0.67142	0.51098	2.82	1.08	
63	2	7	0.75148	0.21471	0.28571	1.01818	0.36125	1.94	0.57	
64	1	5	0.81873	0.16375	0.20000	1.72244	0.17863	1.19	0.21	
65	4	4	0.36788	0.36788	1.00000	3.51489	0.02975	0.37	0.03	

Appendix II

In this appendix the construction of the joint-life life-table under the independence and dependence mortality assumptions is shown.

Figures 7.3 & 7.4 columns description

x, y: represents the male and female times-to-death
Using the independence mortality assumption described in chapter 7 the survivor rates are calculated as: $S(xy)_{ind} = S(x) \times S(y)$.

Consequently, using the frailty model obtained intensity rates the survivor rates is given by: $S(xy)_{frailty}$.

l_{xy}: is the total number of insureds with times-to-death (x,y) in the Kenyan joint lives dataset.

The number of deaths $d_{xy} = l_{xy} \times q_{xy}$

The joint probability of dying is $q_{xy} = 1 - S(xy)$.

The commutation functions: $D_{xy} = \frac{l_{xy}}{(1+i)^{(x+y)/2}}$, $N_{xy} = \sum D_{xy}$ assuming a deterministic interest rate of 7%.

The net single premium is then approximated by PRICE FOR 200000 p.a. $= 200000 \times \frac{N_{xy}}{D_{xy}}$

Figure 7.3: Constructions of Joint Life-table under the Independence Mortality Assumptions.

x	y	s(xy)_ind	lxy	dxy	qxy	Dxy	Nxy	PRICE FOR 200000peryear
39	43	0.991134	398	4	0.008866	24.83981	143.44324	1154947
40	44	0.979891	394	8	0.020109	23.00896	118.60344	1030933
41	45	0.959545	387	16	0.040455	21.07128	95.59448	907344
42	46	0.926867	371	27	0.073133	18.89612	74.52320	788767
43	47	0.879700	344	41	0.120300	16.36839	55.62708	679689
44	48	0.817797	302	55	0.182203	13.45727	39.25869	583457
45	49	0.743101	247	64	0.256899	10.28534	25.80143	501713
46	50	0.659386	184	63	0.340614	7.14303	15.51609	434440
47	51	0.571426	121	52	0.428574	4.40188	8.37306	380431
48	52	0.484055	69	36	0.515945	2.35080	3.97118	337858
49	53	0.401387	34	20	0.598613	1.06347	1.62038	304735
50	54	0.326383	13	9	0.673617	0.39894	0.55691	279197
51	55	0.260742	4	3	0.739258	0.12169	0.15797	259637
52	56	0.205045	1	1	0.794955	0.02965	0.03629	244730
53	57	0.159017	0	0	0.840983	0.00568	0.00663	233419
54	58	0.121830	0	0	0.878170	0.00084	0.00095	224871
55	59	0.092356	0	0	0.907644	0.00010	0.00011	218438
56	60	0.069374	0	0	0.930626	0.00001	0.00001	213617
57	61	0.051701	0	0	0.948299	0.00000	0.00000	210019
58	62	0.038270	0	0	0.961730	0.00000	0.00000	207345
59	63	0.028165	0	0	0.971835	0.00000	0.00000	205367
60	64	0.020625	0	0	0.979375	0.00000	0.00000	203910
61	65	0.015040	0	0	0.984960	0.00000	0.00000	202840
62	66	0.010928	0	0	0.989072	0.00000	0.00000	202058
63	67	0.007916	0	0	0.992084	0.00000	0.00000	201480
64	68	0.005719	0	0	0.994281	0.00000	0.00000	200000

Figure 7.4: Constructions of Joint Life-table Under the Dependence Mortality Assumptions.

x	y	s(xy)_frailty	lxy	dxy	qxy	Dxy	Nxy	PRICE FOR 200000peryear
39	43	0.750103	398	99	0.249897	24.83981	68.21739	549259
40	44	0.699433	299	90	0.300567	17.41347	43.37758	498207
41	45	0.649780	209	73	0.350220	11.38277	25.96411	456200
42	46	0.602690	136	54	0.397310	6.91243	14.58134	421888
43	47	0.559127	82	36	0.440873	3.89350	7.66891	393934
44	48	0.519534	46	22	0.480466	2.03454	3.77541	371131
45	49	0.483969	24	12	0.516031	0.98786	1.74087	352451
46	50	0.452245	11	6	0.547755	0.44682	0.75300	337051
47	51	0.424046	5	3	0.575954	0.18885	0.30618	324259
48	52	0.399001	2	1	0.600999	0.07484	0.11733	313545
49	53	0.376737	1	1	0.623263	0.02791	0.04249	304495
50	54	0.356902	0	0	0.643098	0.00983	0.01458	296784
51	55	0.339179	0	0	0.660821	0.00328	0.00476	290160
52	56	0.323290	0	0	0.676711	0.00104	0.00148	284424
53	57	0.308990	0	0	0.691010	0.00031	0.00044	279420
54	58	0.296072	0	0	0.703929	0.00009	0.00012	275024
55	59	0.284356	0	0	0.715644	0.00003	0.00003	271135
56	60	0.273691	0	0	0.726309	0.00001	0.00001	267675
57	61	0.263945	0	0	0.736055	0.00000	0.00000	264576
58	62	0.255008	0	0	0.744993	0.00000	0.00000	261784
59	63	0.246783	0	0	0.753217	0.00000	0.00000	259245
60	64	0.239189	0	0	0.760811	0.00000	0.00000	256873
61	65	0.232155	0	0	0.767845	0.00000	0.00000	254421
62	66	0.225621	0	0	0.774380	0.00000	0.00000	250825
63	67	0.219532	0	0	0.780468	0.00000	0.00000	241034
64	68	0.213845	0	0	0.786155	0.00000	0.00000	200000

Appendix III

In this appendix we show the annuitants member data from a large insurance data-set. Only in-force policies between 2001-2018 are considered.
Figure 7.5 columns description

Column 1. V-GENDER: This is the main annuitant (contributor) gender. I.e X(male) and Y(female)

Column 2. MAIN LIFE DOB: This is the main annuitant age at birth.

Column 3. SPOUSE DOB: This is the spouse age at birth.

Column 4. EFFECTIVE DATE: is the date when the contracted was effected.

Column 5. MAIN LIFE AGE AT PURCHASE: This is the main annuitant age when contract was bought.

Column 6. SPOUSE AGE AT PURCHASE: This is the spouse age when contract was bought.

Column 7. MAIN LIFE TERM: This is the main annuitant time-to-death.

Column 8. SPOUSE TERM: This is the spouse time-to-death.

Column 9. T-MAX(x,y): The maximum time-to-death of the main and spouse annuitants.

Column 10. AGE DIFFERENCE: is the age difference between the annuitant and spouse.

Figure 7.5: Annuitants Member Data from a Major Insurance Firm Dataset.

V_GENDER	MAIN LIFE DOB	SPOUSE DOB	EFFECTIVE DATE	MAIN LIFE AGE AT PURCHASE	SPOUSE AGE AT PURCHASE	MAIN LIFE TERM	SPOUSE TERM	T=MAX(x,y)	AGE DIFFERENCE
X	28-Apr-57	28-Apr-62	01-Jul-17	60	55	50	55	55	5
X	11-Nov-58	11-Nov-63	01-Jul-17	59	54	51	56	56	5
X	02-Apr-67	02-Apr-72	01-Jul-17	50	45	60	65	65	5
X	21-Jul-46	26-Nov-56	01-Sep-01	55	45	55	65	65	10
X	21-Jul-46	26-Nov-56	01-Sep-01	55	45	55	65	65	10
X	16-Apr-32	18-Jul-39	01-Oct-02	71	63	40	47	47	7
X	28-Aug-40	06-Jan-47	01-May-05	65	59	45	52	52	6
X	03-Jun-54	07-Jul-78	01-Jul-05	51	27	59	83	83	24
X	01-Jan-51	01-Jan-60	01-Sep-05	55	46	55	64	64	9
X	21-Feb-44	24-Jun-46	01-Dec-06	63	61	47	50	50	2
X	13-Mar-50	01-Jan-54	01-Jun-07	57	54	53	57	57	4
X	05-Jun-43	06-Nov-43	01-Jul-08	65	65	45	45	45	0
X	15-Jul-52	11-Sep-58	01-Sep-07	55	49	55	61	61	6
X	25-Jul-52	03-Jan-57	01-Aug-07	55	51	55	59	59	4
X	30-Jan-54	01-Jan-58	01-Oct-07	54	50	56	60	60	4
X	01-Nov-52	25-Sep-62	01-Dec-07	55	45	55	65	65	10
X	20-Oct-52	05-Jan-60	01-Nov-07	55	48	55	62	62	7
..	..	..	..	..	..	..	..	..	..
...	...	...	...	...	...	...	...	...	...
Y	28-Dec-57	09-Jul-57	25-Apr-14	57	57	54	53	54	0
Y	26-Dec-64	26-Dec-64	25-Apr-16	52	52	59	59	59	0
Y	18-Feb-62	01-Jan-57	25-Jul-16	55	60	56	50	56	5
Y	11-Apr-57	13-Dec-52	25-May-17	60	65	50	46	50	4
Y	13-Nov-55	30-May-53	25-Nov-15	60	63	50	48	50	2
Y	01-Oct-65	01-Jan-61	25-Aug-17	52	57	58	53	58	5
Y	01-Jan-59	10-Feb-54	25-Aug-17	59	64	51	46	51	5
Y	28-Mar-60	20-Feb-54	25-Oct-17	58	64	52	46	52	6
Y	23-Nov-67	21-Sep-62	25-Dec-17	50	55	60	55	60	5
Y	01-Jan-57	02-Feb-54	25-Dec-17	61	64	49	46	49	3
Y	30-Mar-67	23-Nov-65	25-Dec-17	51	52	59	58	59	1
Y	02-Apr-62	26-May-65	25-Apr-18	56	53	54	57	57	3
Y	13-Feb-47	18-Nov-43	25-Oct-18	72	75	38	35	38	3
Y	02-Dec-60	03-Oct-69	25-Aug-17	57	48	53	62	62	9
								AVERAGE	5

Appendix IV

R-Program

OpenBUGS GW distribution parameter estimation (Chapter 5)

```
MOD = function(){
for(j in 1:42){
simul[j]=0
simul[j] ~ dloglik(loglikelihood[j])
loglikelihood[j] = log(b * l * a) + (a - 1) * log(s[j]) + (b - 1)
*log(1 - exp(-l * pow(s[j], a))) - l * pow(s[j], a)}
a ~ dgamma(0.001, 0.001)
b ~ dgamma(0.001, 0.001)
l ~ dgamma(0.001, 0.001)}
write.model(MOD,"MOD.txt")
INIT=function() {
list(a=2.01,b=0.65,l=5.45)}
DATASET=list(s=times-to-death)
BUGS=bugs(inits=INIT,data=DATASET,parameters.to.save=c("b","l","a"),
n.chains=2,model.file="MOD.txt",n.iter=100000,
n.burnin=30000,debug=T)
```

OpenBUGS GE distribution parameter estimation (Chapter 6)

```
MOD = function(){
for(j in 1:398){
simul[j]=0
simul[j] ~ dloglik(loglikelihood[j])
loglikelihood[j] = log(b * a) + (b - 1) * log(1 - exp(-a * s[j])) - a * s[j]}
b ~ dgamma(0.001, 0.001)
a ~ dgamma(0.001, 0.001)}
write.model(MOD,"MOD.txt")
INIT=function() {
list(b=7040,a=0.1723)}
DATASET=list(s=X-lifetime)#for the females 's=Y-lifetime'
```

```
BUGS=bugs(data=inits=INIT,DATASET,parameters.to.save=c("b","a"),
n.chains=2,model.file="MOD.txt",n.iter=100000,
n.burnin=30000,debug=T)
```

Research Publications of Walter O. Onchere

1. Onchere, W., Weke, P., Otieno, J. and Ogutu, C. (2021a). Non-Central Gamma Frailty with application to life term assurance data: *Advances and Applications in Statistics*; Volume 67, Number 2, 2021, Pages 237-253.
http://dx.doi.org/10.17654/AS067020237

2. Onchere, W., Weke, P., Otieno, J. and Ogutu, C. (2021b). Shared Frailty Model with Application in Joint-life Annuity Insurance: *Advances and Applications in Statistics*; Volume 68, Number 1, 2021, Pages 23-42.
http://dx.doi.org/10.17654/AS068010023

3. Onchere, W., Weke, P., Otieno, J. and Ogutu, C. (2021d). Graduation of term assurance data using frailty approach: *Afrika Statistika*; Vol 16(3),2021, pp. 2807-2816.
http://dx.doi.org/10.16929/ajas/2021.2807.185

4. Onchere, W., Weke, P., Otieno, J. and Ogutu, C. (2022). Compound joint-life annuity frailty modeling (2022) *Afrika Statistika* (accepted)