THE IMPACT OF CONSTRUCTIVISM ON THE LEARNING OF GEOMETRY AMONG GIRLS IN KENYAN SECONDARY SCHOOLS

A thesis submitted in fulfillment of the requirements for the award of the degree of Doctor of Philosophy in the Department of Educational Communication and Technology, University of Nairobi

By

Japheth G. O. Origa

March 2000

DECLARATIONS

This thesis is my original work and has not been presented for a degree at any other University.

.... 24/10/01..............

Origa Japheth G. O, Date

The thesis has been submitted with our approval as University supervisors.

....

Prof. P. O. O. Digolo

Department of Educational Communication and Technology

University of Nairobi

....

Prof. P. K. Mutunga

Department of Educational Communication and Technology

University of Nairobi

TABLE OF CONTENTS

LIST OF TABLES

LIST OF DIAGRAMS

LIST OF FIGURES

DEDICATION

To my family:

Carren

And our children

Beverly Awuor

Awuono Enos Origa

Kenneth Martin Oduor

ACKNOWLEDGEMENTS

It is my pleasure to utilize this opportunity to extend my sincere gratitude to all the people whose contributions made it possible to complete the research whose findings are presented in this report. Foremost, I wish to thank my thesis supervisors; Professor P. O. O. Digolo of the Department of Educational Communication and Technology, University of Nairobi and Professor P. K. Mutunga of the Department of Educational Communication and Technology, Kenyatta University for the invaluable assistance during the entire period this work was in progress. It is also my desire to thank all the academic staff of the Department of Educational Communication and Technology, as well as the Faculty of Education postgraduate committee of the University of Nairobi for the initial directions they provided during the proposal writing.

It is also my wish to thank professors Hans Geog Weigand, Profke, and Franke all of Justus Liebeg University, Germany, for reading the draft copy of this report and for their positive and constructive comments. I am also grateful to all the staff of the institut der didaktik der mathematik, Justus Liebeg University for the facilities availed to me during the draft writing. My gratitude also extends to Silke Ruswich and Silke Thies for their briefing and demonstrations on computer usage.

Not to forget are the school heads and mathematics teachers for their consent of liberty to administer questionnaires and conduct teaching experiments in their respective schools. On the same note, I wish to extend my thanks and appreciation to all the research assistants and the students who participated in this study. Finally and whole-heartedly, I thank the German Academic Exchange Service (DAAD) and the government of Kenya for the financial support, without which, the accomplishment of this work would have been a distant dream.

ABSTRACT

THE IMPACT OF CONSTRUCTIVISM ON THE LEARNING OF GEOMETRY AMONG GIRLS IN KENYAN SECONDARY SCHOOLS

ORIGA JAPHETH G. O.

The study investigated the impact of constructivism (constructivist approach) on the learning of *three-dimensional geometry* among girls in Kenyan Secondary Schools. The testing form of teaching experiment that involved both bahaviourist and constructivist approaches was used to determine which, of the two approaches is superior in promoting the acquisition of geometry concepts. The experiment was designed to control the effects of pre-testing, the approach and materials used to teach *three-dimensional geometry*. Learner participants in the study were form three students (seventeen year olds) from sixty-two girls' secondary schools in Kenya.

The study was aimed at: determining the impact of constructivist approach on mastery of geometry concepts, determining the effect of manipulative materials on achievement in geometry, determining the effect of manipulative materials on transfer of geometry skills from concrete to abstract situations, exploring student errors and misconceptions in the learning of *three dimensional geometry (three-D)*, unearthing mathematical learning difficulties encountered during the learning of *three-D* and to uncover the factors that contribute to poor performance in geometry.

Data for the study were garnered from: questionnaires completed by practicing mathematics teachers, pretest scripts, posttest scripts, and from direct interviews with mathematics teachers. The data were analyzed both qualitatively and quantitatively.

Learners who used the constructivist approach attained superior mean scores and better ranks compared to their counterparts who used the bahaviourist approach irrespective of the materials used for instruction. Results of the analysis of the data from the study suggest that unsatisfactory performance in *three-D* tasks by Kenyan female students (at the secondary school level) is a direct consequence of multiple factors contributing singly and collectively. The factors include: learners' negative attitude, lack of prerequisite concepts, inability to apply prerequisite concepts, conceptual difficulties, abstract nature of *three-D*, lack of instructional resources, under utilization of instructional resources, deficiency in problem solving skills, language deficiency, over enrolment of learners and ineffective mode and manner of concept presentation.

The report includes recommendations on: workshops and seminars, mathematics, restructuring the mathematics syllabus, instructional approaches, goals of teaching mathematics, instructional resources, teacher training program, teaching load, evaluation of mathematics learning, basics for mathematics learning and research on mathematics education.

CHAPTER ONE: INTRODUCTION

1.00 Background to the study

Unsatisfactory performance in mathematics is not a problem that is unique to Kenya. The chairman of the African Mathematical Union, during a congress on mathematics held in Nairobi, urged African governments to device new strategies that would improve the teaching of mathematics in schools and colleges (Kuku, 1991). The call for ''new strategies'' to improve the teaching of mathematics (*and geometry*) points to the fact that the approaches that are being used to teach mathematics (*and geometry*) in schools and colleges have not been effective in the recent past. Consequently, achievement in mathematics (*and geometry*) at the secondary school level has lately been characterized by low levels and declining trends hence the need to change or revise the instructional strategies, approaches and designs that may have contributed to low achievement in mathematics (*and geometry*). Mathematics (*geometry*) is often presented in a formal way that circumvents the development of underlying concepts (Grant & Searl, 1997). Such a formal presentation may ignore the learners' background experiences as well as their role in acquiring new concepts and instead focus on the teacher's experience that may be too complex for the learners. Teachers have in fact been constructively criticized for adopting non-flexible models to represent mathematical (*geometry*) concepts (Bauersfeld, 1992). They usually drill learners in a rigid interpretation of the models. Learners' interpretations of such models are expected to conform to the teacher's version. Alternative views with pedagogically viable interpretations are seldom tolerated. Learning by contrasting is in fact missing in the current methods of teaching where meaning of mathematical (*geometry*) concepts is presented through positive instances only (Bauersfeld, 1992). Negative instances are never tolerated for fear that they may influence learners to acquire incorrect concepts.

Mathematics is comprised of subject matter that places different psychological and mathematical demands on the learner during the learning process and during problem solving episodes. *Three-dimensional geometry* and other mathematical topics may simply be regarded as ''mathematics'' without due consideration of the fact that they may differ in the effort required to achieve mastery. This down plays the fact that two tasks may appear to have roughly similar amounts of information but differ enormously in the effort required to achieve their mastery (Sweller, 1994). Mathematical (*geometry*) topics therefore need varying levels of abstraction to enhance concept acquisition hence different instructional approaches are necessary for successful instruction of different mathematical (*geometry*) topics. For instance, an approach that can be successfully used to teach algebra may not be successfully used to teach *three-dimensional geometry*. The method of geometry instruction has in fact, previously been challenged (Normadia, 1981) and the lack of success in achievement in geometry has been attributed to the level of instruction and ability level of the learner (Woodward, 1990). More often than not, learner's involvement in defining and explaining geometry concepts is ignored. Meaningful aspects (of geometry) are proposed to pupils without giving them any chance to grasp the reason for their significance (Marioti & Fischbein, 1997), a situation that provides fertile ground for instrumental learning. Even though the method of instruction can influence the level of achievement in mathematics learning in general and geometry learning in particular, other factors do play a significant role in determining the rate and level of success of geometry teaching. Poor motivation and failure to provide clear insight into the meaning and method of geometry have been blamed for high mortality in high school geometry (Butler & Wren, 1965).

Recently (1995), the International Commission on Mathematics Instruction (ICMI) registered its concern about persistent disagreements on the aims, content and methods of geometry teaching at various levels from primary to University. As a result of the disagreements, actual school practice in many countries has simply eliminated the more demanding sections of geometry from their syllabi. For instance, *three-dimensional geometry* has been reduced to a marginal role in the curricula in most countries (ICMI, 1995). In many countries, geometry does not appear in the syllabus before the secondary grades (Bauersfeld, 1992). In France, geometry is a part of mathematics that is reputed to be difficult among teachers as well as among students and the teaching of space geometry is not given a priority (Parzysz, 1988). The status of geometry instruction being as it is, creates the need for investigating the teaching and learning of geometry for the purpose of improving performance and possibly reversing the declining role of geometry in school curricula. An urgent need for an international study to exploit and implement new teaching methods has since been proposed (ICMI, 1995) following the declining role of geometry in the school curricula hence the study being reported.

Recent national and international surveys on mathematical knowledge of students reveal that geometry is either totally ignored or only very few items from geometry are included and performance in the geometry items included in the surveys is relatively poor (ICMI, 1995). The relatively poor performance in geometry has persisted for a duration extending beyond two decades. Prompted by unsatisfactory results obtained with angles in the fourth National Assessment of Educational Progress (NAEP) mathematics assessment, Carpenter et al (1975) suggested the provision of more experience with laboratory type activities for teaching geometry. It seems to be the case that laboratory type activities have been lacking or have not been optimally utilized in geometry instruction. This justifies the need to investigate the methodology and activities used in the teaching and learning of geometry. To achieve this goal (investigating the methodology and activities used in geometry instruction), the study being reported focused on the impact of constructivism on the learning of geometry. Constructivism, as used in this study, exposed experimental participants to an inquiry based learning where participants are free to manipulate concrete materials to investigate the properties of geometry objects, their components and relations between the components.

On the Kenyan situation, available evidence shows that performance in mathematics, a subject that incorporates geometry, has been dismal over a duration exceeding a decade. In particular, performance in mathematics (and geometry) as evidenced by mathematics results in the Kenya Certificate of Secondary Education (KCSE), has been poor at form four level in the Kenyan secondary schools.

Performance in mathematics (*and geometry*) at form four level registered a steady decline between 1979 and 1981. The failure rates were 62.3%, 72.7% and 75.1% for the years 1979, 1980 and 1981 respectively (Mwangi, 1983).

Apart from registering low performance levels, girls have consistently performed poorly in relation to boys. Evidence of this poor performance can be seen in table 1 which displays the mean raw scores (out of 200 for two mathematics papers) by sex as reported by the Kenya National Examinations Council (KNEC).

Table 1: Candidates' scores in mathematics, KCSE 1989 - 1993

	1989	1990	1991	1992	1993
F	51564	53482	55057	57722	598210
M	79365	78569	80412	79842	80616
MEAN					
F	17.9	20.5	26.3	18.5	22.6
M	27.1	31.0	38.6	28.8	33.9
TOTAL	130929	132051	135469	137564	140429

Source: Kenya National Examinations Council

For the five-year period, the data in table 1 translate to a maximum mean score of 13.15% for the female candidates and a maximum mean score of 19.3% for the male candidates in a given year. Because of the low performance levels in mathematics *(and geometry)* in the National Examinations at form four level, it was predicted that performance in *three-dimensional geometry* would similarly be low. Geometry was chosen for the study for two reasons. One, it was deemed necessary to focus the study on a limited domain of the subject matter so as to improve the reliability of the findings. Second, evidence of potential difficulties with *three-D* was noted on items testing *three-D* concepts in previous (1985, 1989, 1990, 1994) National Examinations.

During the period 1980 - 1985, KCSE mathematics results showed that approximately 68% of the candidates were not able to attain a passing grade. In the years 1982, 1983, 1984 and 1985, the failure rates were 72.9%, 67.2%, 68.3% and 66.1% respectively (Kiragu, 1986). The levels and trend of performance in mathematics (*and geometry*) during the period 1983 - 1995 can be seen in table 2.

Table 2: Level and trend of performance in KCSE (between 1983 and 1995)

YEAR	1983	1984	1985	1993	1994	1995
Candidates	105654	117413	122559	140762	141148	139754
Mean Raw Score	41.63	32.21	45.42	29.08	22.78	26.45

Source: Kenya National Examinations Council

The mean raw scores (out of 200 for two mathematics papers) translate to actual mean scores of 20.80%, 16.10%, 21.70%, 14.54%, 11.39%, and 13.29% for the years 1983, 1984, 1985, 1993, 1994 and 1995 respectively. Performance in mathematics (*and geometry*) at form four level as reflected in the overall mean scores, has taken a downward trend ten years down the road. There was a drop of

4.70% in the decade of 1984 to 1994 and a drop of 8.41% in the decade of 1985 to 1995.

Geometry appears to be a difficult topic for secondary school students in Kenya and the topic *three-dimensional geometry (three-D)*, as outlined in the Secondary Education Syllabus, is being avoided during the teaching of mathematics for the excuse of lack of time (Wanjala, 1984). Lack of time does not seem a convincing reason for not teaching *three-D*. Factors more fundamental than lack of time may be responsible for what seem to be a deliberate move by a significant section of mathematics teachers to avoid the teaching of *three-D*. In a survey (Mwangi, 1983) involving 723 students and 48 mathematics teachers selected from 22 schools nation-wide, geometry was rated second after probability as a difficult topic. In a pre-experimental survey by the researcher involving 54 mathematics teachers from 23 schools in Nyanza and Rift Valley provinces of Kenya, *three-dimensional geometry* was rated second after linear programming as a difficult topic. About 61% of the teachers who were interviewed said their students experience difficulties with *three-D*. Table 3 displays a summary of the teachers' responses following the survey interview.

Table 3: Topics in which learners experience difficulties

School Type	Number of Schools	Three-D	Navigation	Linear Programming	Probability
Girls'	6	5	6	3	3
Mixed	9	5	4	7	4
Boys	8	4	4	7	6
Total	23	14	14	17	13

Source: Field data

Data in table 3 tend to suggest that more of the girls' schools reported difficulties with *three-D* (83.3%) than mixed (55.6%) or boys' (50%) schools. A similar trend can be seen for the topic of navigation, which has elements of space relationships (latitudes and longitudes). This trend, coupled with the fact that girls seem to register a lower mean grade than boys in the KCSE mathematics examinations (see table 1), influenced the researcher's choice of female participants in the study. In a report on the 1985 mathematics examinations, it is indicated that most of the geometry questions were very unpopular, had the least mean mark and that most of the candidates who attempted geometry questions scored zero marks (KNEC, 1987). It was also observed in the same report that candidates are very poor in questions involving *three-D* and as a result, try to avoid them. In addition candidates were reportedly finding it difficult to differentiate between objects in two dimensions and those in three dimensions. This was evident in the 1985 mathematics examinations where candidates classified the net of a pyramid on a rectangular base as *three-dimensional*. The net of a pyramid is actually a plane object (*two-dimensional*).

Further evidence from KNEC reports tends to support the view that candidates have been experiencing some difficulties with *three-D*. For instance, the concept of angle between a line and a plain has been found to be difficult for learners who have not had any practical experience with models (KNEC, 1991). *Three-D* has always been unpopular and candidates do poorly (KNEC, 1991). In its report on the 1989 examinations, an item on *three-dimensional geometry* (frustum of a pyramid) was found to be difficult. In this case (frustum of a pyramid), candidates were not able to visualize the components (lines and planes) of the frustum from a plane diagram of the frustum. Many candidates in the 1989 examinations experienced difficulties in identifying the angle between a line and a plane (KNEC, 1991). In the 1990 examinations report, an item on *three-dimensional geometry* involving two spherical balls placed inside and in contact with a hemispherical bowl was found to be the most difficult. It was reported that 73% of the candidates who attempted the item scored zero marks. In the 1995 examination report, 71% of the candidates scored zero in an item that tested their knowledge of the volume of *three-dimensional* solids such as a prism.

The evidence given in the paragraphs above tends to suggest that there is a problem either with the teaching of *three-dimensional geometry* or with the learning of the same or both, in the Kenyan secondary schools. There was therefore a need to investigate, through research, the fundamental causes of learners' dismal performance in *three-D* at the secondary school level. Christiansen et al (1979) once stated that

> *...there are enormous gaps in our knowledge of pupils' difficulties in mathematics (and geometry), of sources of methodological misconceptions and of the natural thought which would be a starting point towards the formation of mathematical (geometrical) thought on the pupil [brackets added].*

The research being reported was necessary to expose gaps in our knowledge of learners' difficulties with *three-dimensional geometry* and to expose gaps between the learners' perception of plane and space relationships. It was also needed to review our methodological weaknesses so as to incorporate the learners' natural, spontaneous thoughts in the learning process. The decision to study the teaching and learning of geometry, at this point in time, may therefore be fulfilling the current research needs in mathematics education.

1.10 Justification of the study

Educators, researchers and scholars have consistently challenged the methodology of mathematics (*and geometry*) teaching in Kenyan secondary schools. Eshiwani (1984) criticized teachers for using outmoded teaching methods. He recommended research on methods that can be productive with crowded classrooms that are often witnessed in Kenyan schools. The Kenya Education Commission report (1964) criticized drill methods of teaching. It also criticized methods that ignore learner based activities and learner participation. The Kenya Education Commission report recommended teaching that focuses attention on the child and inculcates independent and constructive thinking in the Kenyan juveniles. The education Commission report (1964) further suggested that

students will not effectively break loose from the old bookish, rote methods until they have themselves shared in the exhilaration of autonomous learning.

It challenged teachers of academic subjects (*mathematics and geometry included*) to sponsor practical activities such as the making of models to illustrate the concepts they teach.

The Kenya government report by the National Committee on Educational Objectives and Policies (1976) decried a general deterioration in the quality of mathematics (*including geometry*) throughout the formal system of education and the dwindling numbers of students who are well qualified in science and mathematics particularly at form four and form six. The report also decried the persistent poor performance of students in public examinations. It recommended the promotion of a spirit of inquiry and innovation by encouraging the use of discovery methods in the secondary school curriculum.

Poor performance in mathematics (*and geometry*) in national examinations is worrying and does not augur well for the future (Kyungu, 1998). The poor performance is particularly bad for girls who seem to be consistently contributing more than half of the failing grades. In the 1995 KCSE mathematics examinations, girls contributed 53% of all grade "E''s and 21% of grade A. In 1996, girls contributed 57% of all grade "E''s and 24% of grade A. Kyungu attributes this poor performance to poor instructional techniques, lack of teaching materials, lack of text books and lack of teachers.

1.20 Statement of the problem

Curriculum designers in Kenya have not suggested preferential treatment of any of the mathematics (*and geometry*) topics that are listed in the syllabus for secondary education. The role of all the syllabus topics prepared by the Kenya Institute of Education (KIE) is collectively implied in the general objective of mathematics (*and geometry*) teaching. That is, producing people who are competent in appraising and utilizing mathematics and geometry skills in playing a positive role in the development of a modern (Kenyan) society (KIE, 1992). The role of *three-dimensional geometry* in producing competent citizens who are able to apply their knowledge (*geometry skills*) to the development of modern Kenya is strongly implied and acknowledged. In the sequencing of the syllabus topics, plane geometry (*two-dimensional*) precedes solid geometry (*three-D*). Learners encounter concepts (point, line, plane and angle) that are prerequisite to *three-dimensional geometry* before the same (*three-D*) is introduced. They also learn the names of common solids; sketching of solids, nets; and models of solids before proceeding to space concepts and relationships. Such a sequence is logical since learners are expected to have acquired basic knowledge they require in order to learn *three-D*. In such a scenario, learners would normally be expected to experience a smooth transition from plane geometry to *three-dimensional (space, solid) geometry*.

However, it has been observed that learners do experience difficulties with *three-dimensional geometry*. They experience difficulties in visualizing plane representations of *three-dimensional* objects and are unable to comprehend space relationships. Visualization is used in this report to mean that learners' are not able to form visual images (mental representations of spatial relationships) of *three-D* objects from plane diagrams of the objects. Consequently, they are not able to recognize elements (points, lines, and planes) of a *three-D* object from a plane representation of the object. As a result, they experience difficulties in understanding the relationship between the components of *three-dimensional* objects. Learners' difficulties already reported include inability to identify the angle between a line and a plane and the angle between two planes.

No explanation as to why Kenyan students perform poorly in tasks involving *three-dimensions* to the extent that as many as 73% of the candidates who attempt an item involving *three-D* score a mark of zero has been provided. Neither has any reason been given to explain the relatively poor performance of girls (in mathematics and geometry at form four level) in relation to that of boys. Worse still, no pedagogical reasons have been given to justify teachers' tendency to avoid the teaching of *three-D* nor has the learners' tendency to avoid tasks involving *three-D* been explained. The practice of skipping geometry with the excuse of lack of time (Wanjala, 1984) is therefore inconsistent with, and frustrating the goals of mathematics (and geometry) teaching. Learners' difficulties experienced with *three-D* and the subsequent unsatisfactory performance is also unwelcome if the ultimate goal of geometry teaching is to be realized. An investigation was therefore necessary to expose and explain the root causes of learners' conceptual difficulties with *three-D* and the persistent unsatisfactory performance that have been witnessed. This study therefore sought to provide solutions to the question:

What is the relationship between instructional design (instructional approach) and performance in three-dimensional geometry among girls at the secondary school level in Kenya?

1.21 Research questions

The study was designed and expected to provide solutions to the following questions in relation to the teaching and learning of *three-dimensional geometry* among girls in Kenyan secondary schools:

a) What are the effects of constructivism (the constructivist approach) on performance in geometry?

b) What are the effects of manipulative materials on performance in geometry?

c) What are the effects of manipulative materials on transfer of geometry skills from concrete to abstract situations?

d) What conceptual errors do learners make during the learning of geometry?

e) What mathematical learning difficulties are encountered during the learning of geometry?

f) What factors contribute to the poor performance in *three-dimensional geometry*?

1.30 Objectives of the study

The study sought to achieve the following objectives in relation to the teaching and learning of *three-dimensional geometry* among girls in Kenyan secondary schools:

a) To determine the impact of constructivism (the constructivist approach) on mastery of geometry concepts.

b) To determine the effect of manipulative materials on achievement in geometry.

c) To determine the effect of manipulative materials on the transfer of geometry skills from concrete to abstract (pencil and paper) situations.

d) To explore student errors and related misconceptions in the learning of geometry.

e) To uncover mathematical learning difficulties encountered during the learning of geometry.

f) To uncover the factors that contribute to poor performance in geometry.

1.40 Hypotheses

The hypotheses tested in an effort to answer the first three research questions were these:

a) There is no significant relationship between constructivism (the constructivist approach) and performance in geometry.

b) There is no significant relationship between use of manipulative materials and performance in geometry.

c) There is no significant relationship between use of manipulative materials and transfer of geometry skills.

1.50 Significance of the study

The constructivist approach was expected to expose practicing mathematics educators in Kenya to an alternative instructional approach if it proved to be superior to the conventional behaviourist approach in the learning of *three-D*. It was anticipated the approach would be productive for teaching geometry and other mathematical topics due to its flexibility in permitting the learner to progress from an intuitive state of mental activity to a more formal state through the use of manipulative materials, negotiation of meaning and reflection. The adoption of the approach for mathematics instruction was expected to boost performance in mathematics at the secondary school level and possibly lead to a reversal of the current level and trend of performance in mathematics. This, it was anticipated, would be a positive contribution towards achieving the goals of mathematics teaching.

The study emphasized analysis of errors as a way of exposing student difficulties encountered during the learning of *three-D*. The errors that were exposed and analyzed in this study provide an initial step in diagnosing learners' difficulties with *three-D*. Mathematical mistakes can turn a negative experience into a positive experience (Meyerson, 1976) if the mistakes are analyzed and the learners' faulty thought processes are corrected. Learners' mistakes should neither be seen in the light of poor cognition nor in the light of weak academic potential. They should be treated as information about each child's understanding (Resnick and Ford, 1981) and as indicators of underlying mental difficulties. Mistakes should in fact be viewed as vital symptoms of learners' difficulties with mathematical concepts. Identification and correction of learners' mistakes (errors) is a vital step towards improvement of performance in *three-D* in particular and mathematics in general. The diagnosis and exploration of errors will assist secondary school teachers in uncovering learners' conceptual difficulties encountered in the learning of *three-D* in particular and mathematics in general. They would then be in a position to design appropriate instructional interventions to overcome the uncovered difficulties. A successful instruction of *three-D* by the constructivist approach was expected to boost the learners' confidence in geometry and provide hope for improvement in performance in mathematics. A clear and stable understanding of *three-D* concepts would provide a base for exploring and gaining insight into relevant applications of the subject that are much needed in the contemporary society.

1.60 Limitations of the study

There were three constraints that could influence the results of this study. First, it may be difficult to control and confine human participants within the boundaries of experimental conditions. The participants' knowledge of their role as ''guinea pigs'' could have an impact on the results of the experiment. No disclosure was made to the student participants to the effect that they were taking part in an experiment. However, teacher participants were told precisely all the details and purpose of the study. It is therefore taken as an assumption that all the teachers who participated in the teaching experiment adhered to the instructional designs that were assigned to them. Second, entire class experiments that were witnessed in this study do not usually yield refined information as experiments with small groups or individuals. The large groups were used because they simulate the actual learning conditions in Kenyan Secondary schools. Finally, to ensure consistency with the instructional techniques designed for the study, each experimental design ought to have been implemented by one teacher. This was however not possible given the expanse of the geographical area covered by the study and the amount of time that would be required to accomplish this. Eight teachers were involved in the implementation of each design.

1.70 Delimitation of the study

The study covered twenty-seven districts in seven provinces in Kenya. North Eastern province was not included in the study. This was due to poor communication in the province and a relatively small number of students registered in secondary schools in the province. For instance, in 1990, there were only 3,400 secondary school students enrolled in North Eastern province compared to 119,000 enrolled in the Rift valley province. Insecurity in North Eastern Province rendered the province unsafe for research. Conducting research in the province would have called for the full involvement of government security agents at a prohibitive cost.

The study focused on form three female students. The female students were chosen because they consistently scored a lower mean grade in mathematics (*geometry*) at form four level during the period 1989 to 1993 (see table 1). Form threes were deemed appropriate for the study for three reasons. Foremost, they were expected to have acquired the prerequisite knowledge necessary for the learning of *three-D*. Second, the topic that was chosen to provide subject matter for the study is scheduled to be covered in form four hence no interference from the formal school instruction was anticipated. Lastly, form fours being in an examination class, were considered to be relatively busy preparing for their national examinations and either could have been uncooperative or the study could have been viewed to be interfering with their preparation for the examinations.

1.80 Basic assumptions

In the light of the limitations above, the assumptions that were made during the study were these:

a) Participants did not realize their role as "guinea pigs" in the study.

b) Mathematics teachers who participated in the teaching experiments adhered to the prescribed instructional techniques and experimental conditions.

1.90 Operational definitions

1.91 Behaviourist approach

Direct instruction by "telling" Learners are "told" the facts, rules and principles they are expected to practice and "learn"

1.92 Constructivist approach

A learning approach dominated by interactive learner activities that involve learning resources, other learners and the teacher.

1.93 Effectiveness

The degree to which a given instructional technique facilitates the acquisition of concepts as would be reflected in the level of the learners' performance in an achievement test on the concepts.

1.94 Secondary school

A learning institution that offers instruction to forms one, two, three, and four as is presently the case in Kenya.

1.95 Manipulative materials

Concrete learning resources used during the instructional process to reinforce the understanding of concepts.

1.96 Misconception

Incorrect application of a mathematical concept usually expressed as an error during a problem-solving episode.

1.97 Performance

Actual accomplishment showing the learner's ability as measured by a score on an achievement test.

1.98 Transfer of knowledge

Ability to solve *three-D* problems without referring to *three-D* models

1.99 Respondents and participants

"Respondents" is used in this report to mean teacher participants who answered the questionnaire for mathematics teachers. "Participants'' is used to refer to learners whose contribution involved one or more of the following: pretest, teaching experiment and the posttest.

CHAPTER TWO: LITERATURE REVIEW

2.10 Overview

Literature on previous research that was reviewed for this study is mainly on methodology used on mathematics (and geometry) instruction, on the learning of geometry, and on the effect of manipulative materials on the learning of mathematics.

2.20 Literature on methodology of mathematics instruction

Mbiriru (1983) investigated the problems of teaching junior secondary mathematics using a sample of twenty schools from three provinces in Kenya. It was reported that guided discovery is ignored and the lecture method is preferred. Teachers prefer the lecture method to discovery method because they believe in covering the syllabus before learners sit for the national examinations. The only way to achieve this goal (covering the syllabus) is to dominate the instructional process by lecture based approaches. Mbiriru (1983) also reported that most learners have developed the belief that mathematics is a difficult subject. Worse still, the learners also believe that their teachers are not competent to teach mathematics. Cobb et al (1992) used an instructional approach that is compatible with the constructivist view that mathematics learning is a process in which students reorganize their mathematical activity to resolve situations that they find problematic. Pair collaboration, class discussion, and interactive communications were developed in the teaching. It was reported that performance among project students was superior on items that tested the understanding of concepts and their applications. The participants in the study by Cobb et al (1992) were second graders learning arithmetic. Pirie and Kieran (1992) worked with students in a constructivist environment where learning was guided by discussion, use of manipulative materials, and the doctrine that the individual learner constructs knowledge. It was reported that children did show individual understanding of mathematics.

Wheatney (1992) investigated the role of reflection in mathematics learning with elementary school participants. He reported a superior performance in favour of classes in which constructivism was used for program development. (Cobb et al, 1991, Kamii and Lewis 1991, Nicholls et al, 1990 cited in Wheatney, 1992) reported that students engaged in problem-centred learning develop greater mathematical competence than students taught by the conventional explain-practice method. Their participants were students in the elementary school. Bednarz and Janvier (1988) investigated the effect of a constructivist approach to numeration in a study where the role of interactions and communications among children in the construction of knowledge was emphasized. It was reported that a little more than half of the children in the constructivist group were able to transfer skills to unfamiliar situations.

Burchert (1980) conducted a comparative study of traditional expository approach and a mathematization approach to informal geometry. It was reported that the mathematization approach emerged superior to the expository approach for students of all ability levels. Mathematization as an approach to the teaching and learning of mathematics is based upon the skills of model building. It involves using acquired mathematical knowledge in building a model for a given setting (Trelinski, 1983). Mathematization being exploratory in nature is similar to the constructivist approach that was used in this study. Participants in Burcherts' study were grade seven pupils. The researcher used students in their third year of secondary school education.

Kalmykova (1962), cited in Kantowski, 1979), successfully used the testing method of a teaching experiment in a study of mathematical applications in physics with two groups of learners. One group used an exploratory approach while the other used a heuristic approach. It was found that the heuristic approach was superior to the expository approach for the "weak" learners. Since the heuristic approach is similar to the constructivist approach, it is the researcher's assertion that the former would be productively used with weak female learners who appear to be less motivated to learn mathematics. Lacampagne (1979) found that male students rank mathematics significantly higher among favourite subjects than their female counterparts. The relatively low ranking of mathematics by female students would imply that geometry is similarly ranked. It therefore requires a teaching method that can arouse the learners' interest in geometry to improve the level and rate of concept acquisition.

Mayer (1978) cited in (Knupfer, 1993), provided evidence that discovery approach to learning may enhance meaningful learning since it encourages transfer of skills. Bell et al (1983) also reported that learning by discovery is superior to learning by exposition for long term retention, transfer of skills to new situations and for generalizations. In the study being reported, learners were expected to discover relations and patterns in space geometry and to construct meaning of *three-D* concepts. This was expected to improve their ability to transfer skills from concrete to abstract situations and to make generalizations. Hiatt (1979) made an effort to determine whether integrating the use of ten guided discovery lessons into a conventional didactic approach changes learners' attitudes and achievement in trigonometry. It was reported that students exhibited a strong preference for discovery learning laboratories; acquired problem solving skills and the attitude of female students was sustained in the positive direction. While the mathematical content for Hiatt's study was trigonometry, the study being reported involved a laboratory setting where learners manipulated concrete materials and models. It was expected that the laboratory type activities would boost the participants' ability to acquire concepts in space geometry.

A renowned Soviet psychologist, L. S. Vigotsky pioneered the use of individualized experiments in the early 1920's. The individualized experiments later developed into teaching experiments that were used in Soviet pedagogical research. Modifications have however been done to the original Soviet Teaching Experiments. Such modifications include the use of large group of participants (entire class) and the shift from longitudinal to short term studies. The modifications were first made by Kantowski in a problem solving research in geometry (Rachlin, 1979). The research being reported was a short-term study involving entire classes normally found in Kenyan schools (about 40 students per class). There are two widely used forms of the teaching experiment, the ''experiencing method'' and the ''testing method''. In the former, only one mode of instruction is employed to discover how it influences acquisition and mastery of information. In the latter, participants are split into groups and each group is assigned a specific method to discover which of the methods is superior to the other in promoting acquisition and mastery of information. In the study being reported, the testing form of the teaching experiment was used to compare the effectiveness of the behaviourist approach and the constructivist approach in promoting the acquisition and mastery of geometry concepts.

Butler & Wren (1965) blamed the unsatisfactory performance in geometry on the methods of geometry instruction. They expressed the opinion that geometry has not been taught in such a way as to arouse the learners' curiosity and to provide them with an intellectual challenge. They went further to assert that learners should approach geometry in the spirit of demonstration, formulating their own conjectures in order to discover solutions to *three-dimensional* problems and this, they say, can be realized through construction, comparison and examination of physical models. It would appear that demonstration, discovery, model construction, formulation of conjectures have not been integral components of geometry instruction hence the blame on the method of geometry instruction. Demonstration, manipulation of models and concrete materials, discovery and pattern generation were incorporated by the constructivist approach in the study being reported.

2.30 Literature on research in geometry

Mulindi (1979) reported that the Kenyan mathematics curriculum (which incorporates geometry) shows a poor horizontal articulation or integration into the immediate environment in the culture in which it operates. It appears therefore, that the Kenyan mathematics curriculum falls short of using real world applications of mathematics and application to other fields into its immediate environment. Geometry has a lot of applications and relevance to real life situations from which the Kenyan society can benefit. For instance, *three-dimensional geometry* would provide the Kenyan secondary school graduates with spatial skills and ability to interpret plane representations of space objects. Such skills are necessary if the learners' have to advance into fields like architecture and design. Learners with deficiency in basic skills are likely to register a low level of performance in the fields in which such basic skills are prerequisite. Spatial ability has been found to correlate significantly with success in geometry (Sherman, 1967 cited in Dyche et al, 1993). Learners with low spatial ability do experience difficulties in visualizing objects in space and are therefore more likely to experience difficulties during the learning of *three-D*.

Ben-Chan et al (1988) investigated the effect of spatial visualization training in particular concrete experiences (which included representation of concrete experiences in *two-dimensional* drawings and reading such drawings) on the learners' ability to communicate spatial information. It was reported that the training improves the learners' ability to successfully communicate spatial information regardless of their grade level or sex. Yerushalmy & Chazan (1990) used the supposer (a software that provides visual representations) to investigate visual obstacles in the learning of *three-dimensional geometry*. It was reported that students who used the supposer understood diagrams and their limitations better than students from the traditional classrooms. The supposer seems to be flexible and allows for student explorations of diagrams. The study being reported focused on *three-dimensional geometry* and incorporated its application to real life situations.

Moise (1975) exposed two difficulties that may be affecting the teaching and learning of Euclidean geometry in school mathematics. These are drilling of students in repertory of routines and teaching ideas that lead to processes for solving certain problems in a manner that the process tends to replace the problem in the mind of the students. It seems that in such circumstances, the learners are drilled to memorize the process (which may involve rules and theorems) of solving a problem at the expense of the mathematical and structural demands that could trigger cues for solving the problem. In addition, learners have expressed such feelings like mathematics is rule based, learning of mathematics is mostly memorizing (Brown et al, 1988; Gearhart, 1975). This suggests the need for an instructional approach that enables the learners to construct their own meaning of geometry concepts as opposed to an approach that involves drilling and requires them to memorize rules and concepts.

Gerhard (1975) reported that American high school teachers are particularly in favour of a concrete approach to geometry teaching. They seem to prefer an approach where models are used by learners to reinforce their understanding of geometry concepts. It appears to be the case that the approach previously used by American high school teachers to teach geometry was not effective in enhancing the understanding of geometry concepts. The constructivist approach used in this study involved the manipulation of concrete materials and models hence was expected to be productive in reinforcing the understanding of space geometry concepts.

Tindol (1979) investigated the effect of special reading instruction including the use of study guides and vocabulary builders with a group of 128 students in a high school plane geometry class. It was found that the mean score of the experimental group was significantly greater than that of the control group. Unlike Tindol's study that involved plane geometry, the study being reported involved *three-dimensional geometry* and a greater number of participants to determine the effect of constructivism on the learning of *three-D*.

Ernie (1979) reported a significant difference in mean achievement in favour of an algorithm treatment on high school geometry. In Eenie's study, the control group was not taught how to develop original algorithms. The algorithms were presented as lists of steps and flow charts. It appears that the experimental group was exposed to a logical procedure that ensured use of productive paths in problem solving. Emphasis on the process can be productive in the acquisition of geometry concepts especially if the process is initiated and developed independently by learners in an environment that is rich in models and other manipulative materials. Normadia (1981) found that activity-centred groups performed significantly better than the teacher-centred groups in the learning of introductory transformation geometry. Activities provide learners with an opportunity to construct their own knowledge and understanding at the same time making sense of the problems they pose or are given to solve (Grant & Searl, 1997). In the study being reported, half of the student participants were exposed to learner-centred activities.

Mayberry (1981), investigating the van Hiele levels of geometric thought with undergraduate pre-service teachers, revealed that 52% of the responses were below the second level. In the same study, Mayberry found that the pre-service teachers could not comprehend formal deduction as a means of developing geometric theory and logic. This seems to suggest that the prospective teachers could only recognize figures as "wholes" but could not distinguish their components and could neither classify geometry figures nor perceive relations between them. Given that prospective teachers (undergraduates) are at least a stage above high school students, it appears that a significant proportion of the latter are operating below level II of the van Hiele model of geometric understanding and hence would be expected to experience difficulties during the learning of geometry. Given also that prospective teachers of *three-dimensional geometry* at the high school level are potentially inept in geometry concepts, their students are much more likely to experience difficulties with the same. If the instruction offered is abstract in nature, the said difficulties would be compounded. The van Hiele model is however consistent with Piaget's stages of mental development in proposing learning that proceeds from concrete to abstract situations. The International Commission for Mathematics Instruction (1995) reported that the need for more teachers has caused, on average, a decline in their university preparation especially with the more demanding parts of mathematics, geometry in particular. The commission further argues that since younger teachers have learned mathematics under curricular that have ignored geometry, they lack a good background in this field. This fosters in them, the tendency to neglect the teaching of geometry to their pupils. This seems to imply that the pre-service teachers are not themselves adequately prepared to teach geometry. Buerger & Shaughnessy (1986) investigated the van Hiele levels of thought development in geometry among students who were taking or had taken secondary school geometry. It was reported that the use of formal deduction was nearly absent.

Carpenter et al (1975) exposed more evidence of learners' difficulties with geometry. In a report of the National Assessment on Educational Progress (NAEP) they reported that only 21% of the seventeen-year-old American students could recognize and apply Pythagorean theorem in a verbal problem involving distance. Brown et al (1988) cited in Woodward (1990) indicated in the results of the fourth NAEP mathematics assessment that less than half of grade eleven students who had received a formal instruction in geometry could apply Pythagorean theorem. Students who show deficiency in applying the Pythagorean theorem which is basic knowledge for calculating lengths and distances in three dimensions are for instance more likely to experience difficulties in calculating the slant height of a cone given the base radius and the altitude.

Triadafillidis (1992) investigated visual limitations on the study of shapes with high school students. It was reported that students faced difficulties not only in naming solids but also in identifying the number of edges, vertices and faces of the solids. The solids used in Triadafillidis's study were cube, cuboid, cylinder, square based pyramid, triangular prism, and tetrahedron. Learners are reported to have experienced more difficulties with the cube, prism and tetrahedron. The prism was named as a rectangular based pyramid the tetrahedron as a triangle. Hanna (1989) reported on an international study involving twenty countries that investigated mathematics achievement of boys and girls in grade eight. The report shows that performance was lowest in geometry among five mathematics areas that were tested. It was also reported that boys were more likely to be successful on three quarters of the items than girls. This was observed in the performance of the participants in an item that involved the projection of a cube on the horizontal. Boys registered a success rate of 69% against 47% for the girls.

2.40 Literature on the use of manipulative materials

Nyerere (1967) advocated for practical teaching to back theories taught and learnt in school. He stressed the near impossibility of integrating learners into the society if teaching remains theoretical. He regarded the Tanzanian curriculum and syllabus to be geared towards the examinations set. Teachers study past examination papers to predict the likely questions in future examinations. In Nyerere's view, examinations are used to assess a person's ability to learn facts and present the same on demand. This denies learners an opportunity to acquire practical skills that are much needed in real life situations. Plato (400 BC), Recorde (1510 - 1588), Montessori (1870 - 1952), Hadow committee (1931) and Cockroft report (1982) cited in Grant & Searl (1997) have advocated the utilization of practical activities in mathematics instruction for centuries. Dienes (1963) cited in Resnick & Ford (1981) proposed the creation of teaching materials that embody mathematical structures (patterns and relationships) and bring them within the realm of concrete experience. The persistence with which practical activities have been advocated conveys a message about their significance in mathematics (and geometry) instruction. It also reveals the fact that practical activities have been consistently missing in our teaching of mathematics. Grant & Searl (1997) reported that practical activities are seldom used in secondary schools. Mbiriru (1983) reported that project work, construction of models and activities that are useful in mathematics learning are seldom considered during mathematics instruction. The absence of practical activities in mathematics teaching seem to imply that use of manipulative materials that provide more opportunities for incorporating practical activities in mathematics lessons is a rare occurrence. Despite the apparent continued omission of manipulative materials in mathematics teaching, their instructional benefits may out number their limitations as instructional resources. Sourviney (1983) reported that highly perceptual models are useful in developing key measurement and classification concepts.

Ferrara-Mori and Morino-Abele (1961) cited in Dienes (1963) reported that pupils working in ''constructive situations'' which include handling of geometric forms acquire a better understanding of mathematical concepts and most of their possible logical implications and extensions. Participants in the cited research were fourth graders in the elementary school. Sowell (1989), used meta-analysis of the results of sixty studies to determine the effect of manipulative materials on mathematics instruction. It was found that mathematics achievement is improved through the long-term use of concrete instructional materials. Eshiwani (1981), pointed out that one of the reasons why many students dislike mathematics is the abstract approach employed by many teachers. He further says that in most mathematics lessons that take place in Kenyan schools, no reference is made to concrete examples or concrete materials and that there are hardly any teaching resources available. Concrete models and other manipulative materials would provide a rich learning environment for geometry instruction by the constructivist approach. This would provide an opportunity for gauging the effect of manipulative materials on concept acquisition and transfer of skills to abstract paper and pencil situations where learners work with diagrams.

Knupfer (1993), recommended research that is needed to determine the effects of visual learning on geometry and cognitive growth. This suggests the need to incorporate manipulative materials during the learning of geometry to provide the visual component of geometry (*three-D*) learning. Manipulative materials would boost the visual component of geometry learning by concretising concepts and hence be able to reduce the level of abstraction of the concepts. Concrete materials and models of geometric shapes if used in conjunction with an experiment-based approach, are likely to promote the transfer of geometry skills. The transfer is more likely to be realized if the approach encourages exploration of concepts. This creates the need to investigate the impact of manipulative materials on the acquisition and transfer of geometry skills.

2.50 Summary of the reviewed literature

The reviewed literature reveals that exploratory approach is superior to the expository approach, activity-centered learning promotes better acquisition of concepts than the teacher-centered approach; weak students benefit more from a heuristic approach where they are allowed to learn new concepts for themselves; discovery enhances meaningful learning, encourages transfer of skills and helps in sustaining a positive attitude of female students towards mathematics; students prefer discovery learning laboratories; mathematics achievement is improved through long term use of concrete materials and attitude towards mathematics is improved when instruction involves concrete materials.

Exploration, discovery, learner activity, learner independence are vital characteristics of the constructivist approach which were integrated with the use of manipulative materials to enrich the learning environment in the study being reported. It was expected that this would provide a conducive learning environment for learning geometry especially for girls who may seem to be less motivated to learn mathematics. None of the reviewed studies incorporated all these vital features of constructivism in a single study. In addition, the reviewed studies made no attempts to uncover mathematical learning difficulties and related misconceptions encountered during the teaching and learning of geometry through analysis of errors. Many of the studies reviewed involved elementary school mathematics. These provided a point of departure from the study reported here.

2.60 Theoretical framework

2.61 Overview

The study was based on a compromised position between the theories of behaviourism and the radical version of constructivism. In addition, it was guided by a model of geometric thought developed by van Hiele (1959) and the process of reflective thinking.

2.62 The van Hiele model

A salient factor that may appear to have been ignored by instructional approaches in geometry is the implication of van Hiele levels of thought development in geometry. The levels may have some significant impact on the acquisition of geometry (*three-D*) skills hence their incorporation in the study. According to the model, learners cannot operate adequately at a given level unless they have sufficient experiences on the previous levels. Each level has its own language, its own set of symbols and its own network of relations uniting these symbols. If the language of instruction does not conform to the symbolic language already developed by the learner, then no dialogue is likely to be established between the learner and the teacher. Consequently, no significant learning might occur. The five levels of geometric thought, as given in the more elaborate Russian post-experiment form by Physhkalo cited in (Mayberry, 1981) are discussed in the paragraphs that follow.

The first level (level 0) is characterized by the perception of geometric objects in their totality as entities. Geometric objects are globally recognized as shapes. The objects are judged according to their appearance. Learners are neither able to see parts of a figure nor relationships between its components. They are not able to see the relationships between different geometric objects. Learners reason about basic geometric concepts such as simple shapes, primarily by means of visual considerations of the concept as a whole without explicit regard to properties of its components. They can memorize the names of objects with relative ease, recognizing the objects with their shapes alone. At this level, they are not able to recognize the square as a rhombus or a rhombus as a parallelogram.

The second level (level 1) is characterized by analytic appreciation of the shapes' internal geometry. At this level, learners begin to discern the components of objects and are able to establish relationships between the components of an individual figure and between different objects. They reason about geometric concepts by means of an informal analysis of the component parts and attributes of an object. The necessary properties of a figure are established experimentally. The properties are described but they are not yet formally defined. Learners analyze objects through observations, measurements, drawings and modeling. They are at this level able to recognize objects with their properties. Thus a rectangle is recognized as having four right angles, equal diagonals and equal opposite sides.

At the third level (level 2), learners logically order the properties of concepts, form abstract definitions and can distinguish between the necessity and sufficiency of a set of properties in determining a concept. They are able to establish relations among the properties of a figure and among the objects themselves. They can perceive the possibility of one property following from another. The role of definitions becomes clear and deductive methods are developed in conjunction with experimentation. This permits other properties to be obtained by reasoning from experimentally determined properties. Thus a square can now be viewed as a rectangle and as a parallelogram.

At the fourth level (level 3), learners reason formally within the context of a mathematical concept. They are able to recognize the significance of deduction as a means of constructing and developing all geometric theory. The role of axioms, definitions, theorems and the logical structure of a proof become clear. Learners can now see the various possibilities for developing a theory proceeding from various premises. For instance, they can examine the whole system of properties and features of a parallelogram by using the textbook definition of a parallelogram- a quadrilateral in which the opposite sides are equal and parallel.

At the fifth and final level (level 4), learners attain abstraction from the concrete nature of objects and from the concrete meaning of relations connecting these objects. They can compare systems based on different axioms and can study various geometries in the absence of concrete models. They are capable of developing a theory without making any concrete interpretation i.e. in complete absence of concrete models.

In view of the above, the study recognized the Piagetian stages of mental development in addition to the van Hiele levels since they both seem to concur with the view adopted in this study that experience and experimentation should precede logical and abstract thought in the learning sequence. While presenting educational material, it is advantageous to proceed from familiar to the unfamiliar, using previously acquired knowledge and experience as a foundation for understanding and interpreting related new material that is less familiar (Ausubel, 1968). Models and other materials used in the course of experimentation present mathematical concepts in less abstract and therefore familiar forms in conformity to the learners' background experience. Teacher interventions and the learners' manipulation of concrete materials in an interactive learning environment were relied upon in this study. The reliance was intended to enhance the formation and adjustment of learners' mental structures to accommodate new geometry skills, concepts and principles. Acquisition of new knowledge is largely attributed to the interaction between the learner and the learning environment since it is during the process of interaction that the learner's existing cognitive structures are adjusted to accommodate new knowledge. Teaching experiments have been found to hold the possibility of assisting educators to comprehend not only how materials can be successfully presented to the learner, but also how the learner constructs meaning of the concepts introduced (Kieran, 1985).

2.63 Reflective thinking

Gagatsis & Patronis (1990) defined reflective thinking as a process through which the participant observes and conceives a procedure then tries to understand it and or explain it to others. Skemp (1971), cited in (Gagatsis & Patronis, 1990), distinguished two functional levels of intelligence, the intuitive and the reflective. At the intuitive level, people are aware through their senses, of data from external environment, which are automatically categorized and related to the data already known but they are not conscious of the cognitive processes involved in this activity. At the reflective level, the intervening mental operations are revealed to the cognizer through retrospection. Piaget (1971) cited in Gagatsis & Patronis (1990) pointed that children are not conscious of the operational structures of the intelligence although they (the structures) control the children's actions. For young children, intuition plays a significant role in the development of reflective thinking. Learners who have experienced problem-centred learning in which reflection is central are able to solve non-routine problems and to construct new knowledge (Wheatney, 1992). The learning of mathematics requires students to reflect consciously on their own mental structures and procedures by making observations, noticing things, and asking questions (Gagatsis & Patronis, 1990). This implies that the learning of mathematics (*three-dimensional geometry*) could become quite subtle for learners who largely operate at the intuitive level. The main stages of the process of reflective thinking in a mathematical activity, as elaborated by Gagatsis & Patronis (1990) are highlighted in the next paragraph.

The first stage includes learners' initial thoughts, primary intuitions on conceptions on subject matter for which mental images are to be formed. The initial thoughts are guided by intuitions. Learners make random observations that result in the formation of unstable mental images. The second stage involves learners' attempts to understand the subject matter. They organize the new experience to fit into their existing intuitive structures by classifying observations and analysing ''wholes'' into parts. Learners recall similar examples from their background information and use them to find counter examples and non-examples. They would then question their former beliefs about the subject and prior conceptions. Stage three is that of discovery and partial understanding. Learners find and or justify values obtained through a solution process. They can locate and provide explanations for possible errors committed during the solution process. Learners are also able to mentally reconstruct parts of dismantled ''wholes'' into new ''wholes''. The new parts are interpreted, partially reorganized according to previous structures and the construction of mental images is completed. The fourth stage involves introspection. Learners reflect on the process of the solution and their own mental structures while testing the results and conclusions obtained. They examine analogies and set up new questions. The whole situation is analysed again but at a higher level. At the final stage, learners become fully conscious of the situation and the solution process. They are at this juncture able to comprehend the underlying logic, become aware of their cognitive structures and processes. The existing structures are transformed and expanded into new structures. Radical reorganization of ideas takes place on new foundations and generalizations can be made. Theories can then be constructed and formulated.

Mathematics instruction that incorporates intuition, introspection, making observations, questioning, making discoveries, problem centred learning, generating patterns and making generalizations as its key elements can be said to involve reflective thinking. The instruction would be meaningful and productive if the intuitive process is reinforced by observation and manipulation of concrete representation of concepts. Instruction that has reflection as a primary component enables learners to construct robust mathematical relationships (Wheatney, 1992).

2.64 Behaviourism

Skinner (1975) cited in (Sahakian, 1984) defines behaviourism as the philosophy of the science of behaviour. Behaviourism owes its origin to the efforts of John B. Watson (1858-1958). Watson was merely interested in learners' overt behaviour rather than in their conscious experiences hence the term behaviourism. In response to mounting pressure to develop a psychology that was purely oriented towards objective behaviour, Watson (1913), published an article that triggered the psychological revolution that is now known as behaviourism.

Watson disposed of mentalism in favour of the objective science of behaviour. He was firm on his belief that no such a thing as mental activity exists (Anderson, 1980). It was his conviction that it was possible to dispense of consciousness and simply study behaviour - a thing he regarded as real, objective and practical (Sahakian, 1984). Watson also regarded learning as classical conditioning and explained it in terms of two principles - *frequency* and *recency*. The more frequently learners make a given response to a given stimulus, the more likely they are to make that response to the same stimulus again. The more recently learners made a given response, the more likely they are to make the response to the same stimulus again. The principle of *frequency* is comparable to the theory of drill and practice that support the view that learners learn better when they are drilled to practice a concept. The principle of *recency* seems to propagate the view that learners are more likely to recall concepts learnt recently than those learnt earlier. It appears, therefore, that learning under a behaviourist environment may not promote long-term retention of concepts.

Behaviourists regard learners to be passive recipients of new information. They view teaching as a process of presenting learners with new information and reinforcing them when they produce the desired response. They do not believe in the theory of internal structure and process. Their utmost interest is in the external behaviour of the learner. Proponents of behaviourism conceive of mathematics as a product to be transmitted to the learner. They believe that knowledge received by the learner from the teacher is in its final form that cannot be modified (Moreno-Armela, 1996) by the learner. Behaviourists therefore emphasize teaching rather than learning. Educators who belong to this school of thought regard learning as the transmission of knowledge from the authority (teacher) to passive recipients (students). Teaching under such an environment is usually characterized by a hierarchical presentation of concepts from simple to complex. More often than not, the teaching sequence take the form of a definition, explanation of concepts, presentation of examples followed by a drilling and practice exercise to reinforce memorization of facts, rules and theorems. The behaviorist approach as used in this study is an expository approach where the teacher is the authority who knows everything best. The approach emphasizes teaching to cover the syllabus rather than learning to develop the learners' potential. It usually regards the class as homogeneous in terms of learners' ability and involves minimal learner activities during the learning process. Teacher activities dominate the classroom proceedings that result in learning. Learners are regarded as passive recipients of the teacher's expert knowledge.

It may appear that mathematics instruction in Kenyan schools is predominantly behaviourist in nature. This is however subject to confirmation by an independent research.

Behaviourist notions were adequate in explaining the acquisition of simple information. However, the notions proved inadequate in explaining the development of complex phenomena and synthesis of information (Johnstone, 1987). Restricting learning to behavioural responses therefore denies educators, researchers and psychologists an opportunity to access, study and explain the internal processes that result in learning. Such inadequacies prompted cognitive psychologists to study internal processes that result in learning.

2.65 Constructivism

Unlike behaviourism, constructivism emphasizes learning rather than teaching. Proponents of constructivism as a theory of learning uphold the dictum that knowledge is actively constructed by the cognizing subject, not passively received from the environment (Lerman, 1989). They hold the theory that knowledge is constructed by the learner rather than transmitted by the teacher. They view learner activities as paramount in the learning process that begins with relevant experience, background knowledge and proceeds through experimentation. The radical version of constructivism regards learning of concepts as a process of spontaneous, unguided independent instruction (Cobb et al, 1992).

Teaching under a constructivist environment considers the learner to be engaged in a model construction process where prior knowledge is activated, combined, criticized and modified by the learner in order to form new knowledge structures (Clement, 1991; Michelson, 1984; cited in Craig et al, 1994). Piaget asserts that learners have to construct mental structures (tools for understanding concepts) but they do not construct concepts. For instance, learners would not be expected to construct the Pythagorean theorem. Instead, they would be expected to develop a relational understanding of the theorem by constructing mental structures (schema) to accommodate the relationship between the sides of a right-angled triangle. Mental structures are formed in view of some problem to be solved. The structures are always tentative, ready to be adjusted to accommodate abstract concepts.

Schematically learnt material has a better retention than material learnt by rote memorization (Skemp, 1971; cited in Origa, 1992). Schematic learning uses existing schemas (structures of related concepts) as tools for the acquisition of new knowledge (Skemp, 1972). In schematic learning, the learner constructs a lower level cognitive structure in which a new concept is assimilated. More abstract but related concepts are later learned when the structure is appropriately adjusted to accommodate them. This enhances the comprehension of the new concept. When a lower level cognitive structure cannot accommodate a higher order concept, the learner experiences some learning difficulty and no learning is registered. For instance, learners' conception of *dimension* is that *a point* is *zero-dimensional* because it has no length, width or height. A *line* is *one-dimensional* because it has only length. A *rectangle* is *two-dimensional* because it has both length and width. A *cuboid* is *three-dimensional* because it has length, width and height. The conception that *dimension* of an object is the number of sides (length, width, and height) breaks down when they need the *dimension* of a *circle* because a circle has no "length", "width" or "height". The structure already developed for the concept of *dimension* using the criteria of number of sides proves inadequate to handle the new situation. A modification in the learner's cognitive structure (accommodation) in respect of the concept of *dimension* is then necessary. Thus learners may classify geometric objects according to some criteria other than the number of sides. Geometric objects would then be categorized into points, lines, plane objects and solids. As a result of the classification, *a point* would be categorized as *zero-dimensional, a line* as *one-dimensional*, *plane* objects as *two-dimensional* and *solid* objects as *three-dimensional*. Under this new structure, all *plane* objects are classified as *two-dimensional* and therefore *a circle* is *two-dimensional* because it is *a plane* object. When this process of accommodation is complete, the new concept of dimension is assimilated in the new structure. Assimilation in this sense refers to the incorporation of an object of knowledge into the learner. Generalization of a concept would then occur when the structure acquires a new state of equilibrium.

Different versions of constructivism have been developed and may have varying implications for instructional development and designs. One version is the radical constructivism focuses mainly on the internal knowledge construction process and ignores the influence of the social environment in the process of knowledge construction (Glaserfeld, 1991; cited in Hwang, 1996). For radical constructivists, knowledge is a structure that is subjectively constructed by individual minds (Hwang, 1996) while instruction is regarded to be radically non-interventionist. To radical constructivists, cognition is adaptive and serves the organization of the experiential world (Glaserfeld, 1989). Coming to know is an adaptive process that organizes one's experiential world, it does not discover an independent pre-existing world outside the mind of the knower (Kilpatrick, 1987 cited in Lerman 1989). This view implies that learning should be natural and that teachers should not tell learners anything as they attempt to make sense of their environment. Mathematical objects are considered produced and constructed by the learners themselves through a continuous process of accommodation and assimilation that occur in the learners' cognitive structures (Moreno-Armela, 1996).

Another version is social constructivism. Social constructivists believe that knowledge is socially constructed. They emphasize the importance of the social environment in which learning takes place but do not pay attention to the internal cognitive learning process (Hwang, 1996). The focus is on generation of meaning as shaped by the social process. Our Knowledge of the world arises through our constructions of social reality (Berger & Luckman, 1967 cited in Hwang, 1996). The third version is eclectic constructivism that embraces both radical constructivism and social constructivism. This version regards learning as a constructive process in which learners build internal representations of knowledge as well as personal and social interpretations of such experiences (Bednar et al, 1991 cited in Hwang, 1996). Knowledge is constructed as individuals experience and interact in the physical and social worlds. The role of real world settings in nurturing learners' cognitive processes in relation to their personal and social experiences is acknowledged. Eclectic constructivists are of the view that learning episodes involve both the construction and transfer of knowledge. This is the view adapted in the compromised constructivism used in the study.

2.66 Compromised Constructivism

In practice, it is difficult to conceive of learning that occurs independently of the teacher and an interactive environment. All humans would develop certain structures of thinking (cognitive structures) as long as they maintained a normal interaction with the social and physical environment (Piaget, 1970 cited in Resnick and Ford, 1981). Children's learning depends to an important degree, on social environment and the opportunity it provides to interact with peers over intellectual tasks (Resnick and Ford, 1981). Social interaction has in fact been found to improve the development of basic concepts (Murray, 1972 cited in Resnick & Ford, 1981). The radical version of constructivism with its focus on individual construction of meaning therefore denies the social context of mathematics (Zeverbergen, 1996). Teachers have a responsibility to organize learning environments in such a way as to evoke the construction of knowledge that is objective and socially recognizable. Knowledge is objective and socially recognized if it is acceptable to others (experts) in the same knowledge domain. Individual construction of meaning without due regard to social interactions may often result in erroneous constructions or misconceptions. Misconceptions are socially unrecognisable knowledge that is attributed to erroneous constructions or ''misconstructions''. In mathematics education, misconceptions are regarded as incorrect general ideas representing the common attributes of a mathematical concept- usually influenced by an erroneous guiding rule. To circumvent ''misconstructions'', teachers' role in organizing learning environments to evoke the construction of objective knowledge is paramount. In mathematics (*three-D*) learning, discussing the construction of meaning using mathematical (geometry) language is essential for the construction of objective knowledge. Knowledge constructed by learners should be acceptable to experts, consistent with that used in the formal school system and recognized by the examination systems. This does not seal avenues for discovery of new knowledge. Completely new constructions or discoveries should have no problem fitting into the existing knowledge structures since they will be consistent with the existing knowledge. The new discoveries should be explainable in terms of existing objective knowledge and can be regarded as derivatives (or extensions) of existing knowledge. Radical constructivism, therefore, does not seem to recognize the fact that certain forms of knowledge are seen as legitimate and as a result being propagated by education systems through the process of schooling.

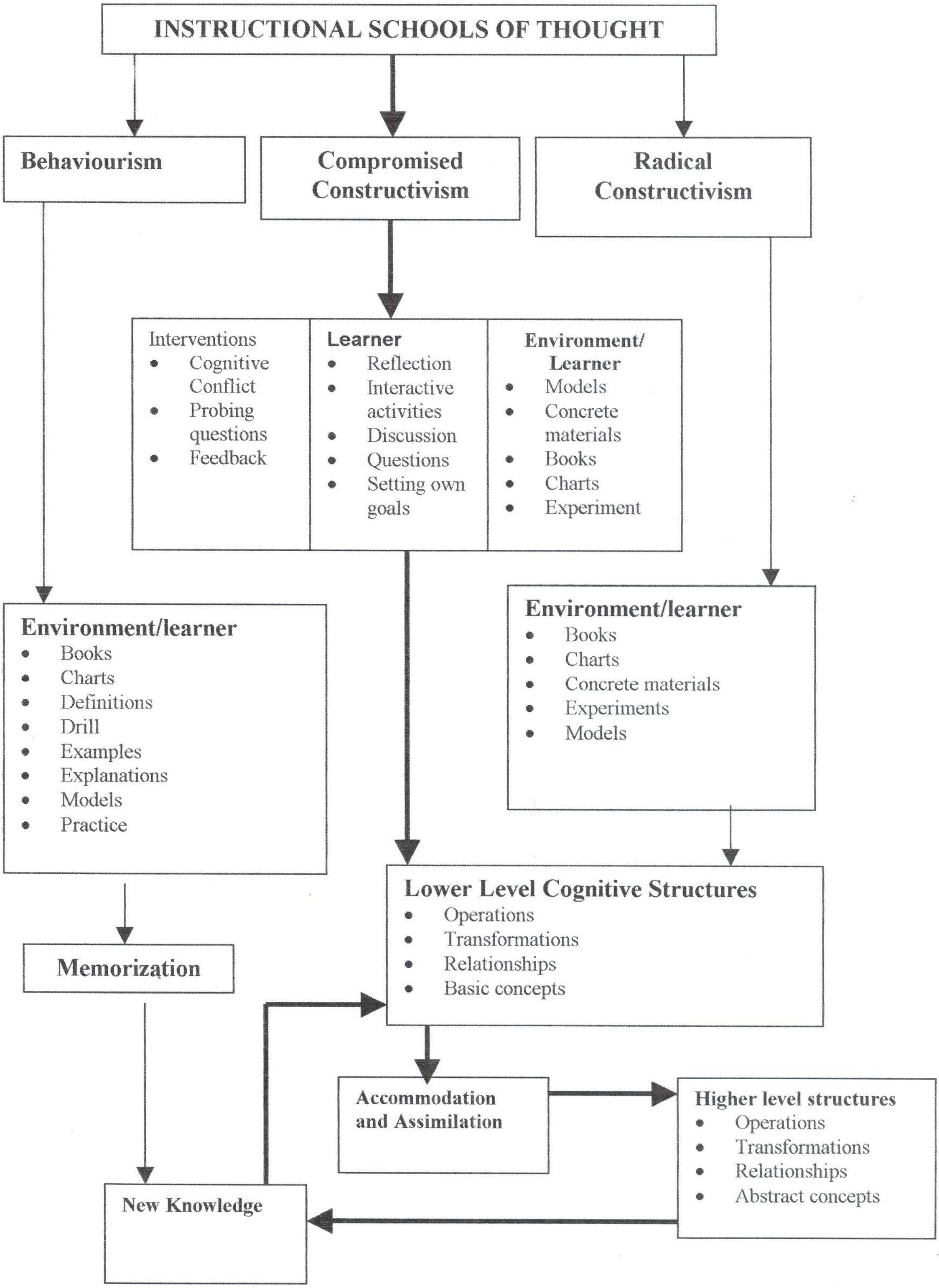
A possible weakness of the behaviourist approach to instruction is that learning may be viewed as a process of acquiring accurate internal representations of fixed mathematical structures and relationships that exist independently of the individual and collective activity. Teachers usually attempt to specify mathematical relationships for learners by using external representations. It is then expected that the mathematical structures and relationships represented externally by models, concrete and other manipulative materials would be transferred as a structure in the learners' mind. The external representations consist of elements (components) that are not likely to be well understood by learners if presented as a fixed single structure. More often than not, the instructional mode will not offer learners an opportunity to analyze and synthesize the relationship between the elements of the external representations. This denies learners a chance for a relational understanding of mathematical concepts since they are expected to take it as given by their teachers. External representations (models, diagrams and charts) alone may also be less helpful if the learners' social interaction with the environment is totally ignored. It is also important to note that the teachers' mental structures are already developed and may have been constructed in completely different environments from those of the learners. Teachers have already apprehended these external representations of mathematical (geometry) concepts while the learners are yet to do so. Therefore the learners' understanding of mathematical (geometry) concepts should not be used to determine their rate of knowledge acquisition. Similarly, teachers' understanding of mathematical (geometry) concepts should not be used to determine the rate and mode of concept presentation to learners.

It is against this background that the researcher integrated ideas from all the versions of constructivism and behaviourism. In the rest of this report constructivism is used to mean a compromised position between all the versions of constructivism on one hand and behaviourism on the other – a learning process that incorporates both knowledge transfer and construction. In the constructivist approach used in this study, learners were introduced to a given mathematical concept, guided by teacher interventions to discover and explore relations, generate patterns and generalize. Explorations were done in an environment that was rich in manipulative materials and encouraged learner-learner interactions, learner-teacher interactions and learner-environment interactions. The approach emphasizes learning to develop the learners' potential rather than teaching that results in rote memorization of facts, concepts and principles. The teacher's role is that of a guide, a collaborator and a team leader who introduces helpful interventions as learning progresses. The teacher is neither an authority nor a boss. The teacher fosters the learner's' engagement with facts instead of drawing their (learner's) attention to the facts Learners are taught to reflect on how they know a fact to be true. Teacher interventions were expected to include the Piagetian cognitive conflict, immediate feedback regarding the learners' responses and probing questions. Cognitive conflict is a situation (or motive) that triggers and provokes the learners' mental faculties to recognize and corrects poorly reasoned elements and principles of knowledge and promotes the learner's understanding from instrumental level to relational level.

Use of models and other manipulative materials were relied upon to reinforce the construction of meaning in conformity with an empirical approach to the learning of geometry. In the empirical approach, geometry concepts, relations and principles are discovered by observing their logical sequences as represented in models and concrete materials in the learning environment. Seeing and participating are essential components of learning (Smith, 1981 cited in Creig et al, 1994). Learners are much more capable of applying a principle where this had been learnt by experience of its instances than where it had been verbally communicated (Hendrix, 1947 cited in Bell et al, 1983).

In a constructivist environment, knowledge and understanding built by students is based on their primitive knowing (Pirie & Kieran, 1992). The teacher's task is to provide opportunities for validating learners' understanding through provocative challenges. It is the learners' response to the challenges rather than the challenges that determine their path to knowledge construction and understanding. A clear knowledge of the learners' path to concept acquisition enables teachers to correct learners' erroneous thought processes. In a constructivist classroom, the importance of correct answers is diminished and the importance of justifying any responses is increased (Foreman, 1987). In such learning situations, incorrect responses are more useful than correct responses because they are used as symptoms in the diagnosis of learners' conceptual difficulties. Teachers are guided by the principle that all people can learn given the relevant opportunity and should be willing to provide learners with a chance to initiate their own learning unimpeded by the teacher's helpful suggestions. The theoretical path to knowledge acquisition by the constructivist approach in relation to radical constructivism and behaviorism is shown in figure 1*.

Figure 1*: Path of knowledge construction and transfer



CHAPTER THREE: METHODOLOGY

3.10 Research Design

The researcher used an experimental design that is accredited to Solomon (1949) and incorporates a pretest, a teaching experiment and a posttest. The design consists of one experimental group and three control groups. In teaching experiments, more refined information is usually obtained with a small number of participants. For that reason and for efficient management and control of experimental conditions, it was necessary to use a small number of schools in the teaching experiment reported here. Eight schools were therefore used to provide experimental groups. According to the design, eight (8) experimental groups required twenty four control groups hence thirty two (32) schools equally distributed over four regional clusters were used in the study. Thirty (30) more schools were used for the pretest only. The administration of the pretest in the extra thirty schools was intended to improve the reliability of the pretest results and to confirm the same (pretest results) on a wider quantitative basis. In all, sixty-two (62) schools and 3429 seventeen year-old female participants were involved in the study. Two thousand one hundred and forty nine (2149) participated in the pretest only, six hundred and forty (640) took part in both pretest and the teaching experiment, a similar number (640) took part in the teaching experiment only. The thirty-two schools that participated in the teaching experiment were divided into two categories–experimental and control schools. In each of the four regional clusters, there were two experimental groups (E1 and E2) and six control groups (C1, C2, C3, C4, C5, and C6).

The participants were exposed to four treatments. The first treatment (D1) is a combination of behaviourist approach and use of materials that provide only non-plane representation of geometry concepts. The second treatment (T1) is a combination of the constructivist approach and use of materials that provide only non-plane representation of geometry concepts. The third treatment (D2) is a combination of behaviourist approach and use of materials that provide plane representation of geometry concepts in addition to those that provide non-plane representations. The fourth treatment (T2) is a combination of the constructivist approach and use of materials that provide plane representation of geometry concepts in addition to those that provide non-plane representations. The participants then sat for a posttest (P2). All participants in set one (treatments D1 and T1) used plane representations of *three-D* objects. Half of them used the constructivist approach and the other half the behaviourist approach. All participants assigned to set two (treatments D2 and T2) of the design used models together with plane representations. Half of them used the constructivist approach while the other half used the behaviourist approach. Table 4 and table 5 display a schematic presentation of the design used.

Table 4: Research design (set one)

Group	Pretest	Treatment	Post-test
C1	P1	D1	P2
E1	P1	T1	P2
C2	-	D1	P2
C3	-	T1	P2

C1 - Control group one.

E1 - Experimental group one.

C2 - Control group two.

C3 - Control group three.

P1 – Pre-test.

P2 - Post-test.

D1 - Treatment one.

T1 - Treatment two.

Groups E1 and C3 were exposed to the constructivist approach and only allowed to use plane representation of solid objects (charts, chalkboard and text book diagrams). Groups C1 and C2 were exposed to the behaviorist approach and only allowed to use plane representation of solid objects. Participants who were exposed to the treatments D1 and T1 did not use models.

Table 5: Research design (set two)

Group	Pretest	Treatment	posttest
C4	P1	D2	P2
E2	P1	T2	P2
C5	-	D2	P2
C6	-	T2	P2

C4 - Control group four.

E2 - Experimental group two.

C5 - Control group five.

C6 - Control group six.

D2 - Treatment three

T2 - Treatment four

P1 - Pre-test

P2 - Post-test

Groups E2 and C6 were exposed to the constructivist approach and used plane representation of *three-D* objects together with models of *three-D* objects. Groups C4 and C5 were exposed to the behaviourist approach and used models of *three-D* objects in addition to plane representation of the objects. Groups E1, E2, C1 and C4 took the test of prerequisite concepts. Pre-test was not administered to groups C2, C3, C5 and C6 as a control measure for the sensitization effect of the pre-test on participants. That is, to control the effect of instrument reactivity. A major advantage of the design is its ability to control extraneous variables such as history, maturation and pre-test treatment interaction. The effect of history was further controlled by two factors, a short time lapse between the pre-test and post-test, and ensuring that participants received no formal instruction (offered by the school as part of the normal teaching program) in the pre-test post test interlude. A small aspect of content was used to minimize fatigue and loss of motivation. The effects of statistical regression, differential selection of participants and experimental mortality were controlled through randomization. For the purpose of consistency, the same observers were used to administer both pre-test and post test. The researcher did the scoring of both pre-test and post test. The above controls were aimed at enhancing internal validity of the study.

3.20 Pre-test

Students who participated in the pretest were drawn from twenty-seven (27) districts in seven provinces in Kenya. The pretest was administered to 2789 participants from 30 schools. The pretest participants included 640 who eventually participated in the teaching experiment and took the posttest. The content covered in pretest was basic knowledge learners would normally be expected to have acquired before they learn *three-D* (appendix A.20). Learners who are equipped with prerequisite knowledge to a particular concept are considered to be mentally prepared to learn the concept. In this case, the pretest was expected to determine whether the participants had developed mental structures required to accommodate *three-D* concepts. It was designed to gauge the participants' cognitive potential to comprehend spatial relationships, their knowledge of geometrical components of solid objects and their ability to decode information from plane diagrams of solid objects.

The learning of *three-D* requires that the learner possess the ability to recognize the relationships between the components of space objects both from pictorial (diagrammatic) and concrete representations (models and real objects). The pre-test gauged the participants' knowledge of basic concepts like parallel and skew lines, intersecting lines, intersecting planes, perpendicular lines and perpendicular planes. According to the van Hiele model of geometric thought, students cannot learn a new concept in geometry unless the instruction is at the same level as the one at which they are mentally operating. The pre-test was therefore used to determine the level at which the participants were mentally operating so as to enhance the understanding of the new concepts they were expected to learn. The pre-test was typed and presented to the participants as a questionnaire. It was preferred that the title "questionnaire" be used instead of the title "pre-test" because it was anticipated that use of the former would minimize the participants' anxiety that is normally encountered in examination situations. A minimal level of anxiety ensured that participants responded with honesty, an event that would elicit information that validly represent the participants' knowledge of the tested concepts.

3.30 Teaching experiment

Teaching experiments were done in thirty-two schools and involved 1280 students. The subject matter covered in the experiment was restricted to the requirements of the Secondary Education Syllabus in Kenya (appendix A.70). The concepts covered in the experiment were geometric properties of common solids, projection of lines on planes, angle between two lines (including skew lines) angle between two planes, and calculating lengths of lines in three dimensions.

Although teaching experiments done with entire classes do not yield refined information as those done with individuals or small groups (Kieran, 1985), classroom-teaching experiments were used in this study. The entire class experimentation was used to simulate the actual school teaching conditions. Participants in the experiment were learners of mixed abilities. Grouping low ability students with average ability and high ability students might increase learning for low achievers while not being deleterious to average and high achievers (Knupfer, 1993). Each class comprised forty students, approximately the same number that is likely to be found in a typical mathematics classroom in Kenyan secondary schools. The participants were taught by their regular mathematics teachers mostly within the normal school program. The researcher held prior discussions and briefing with the teacher participants (respondents) on the purpose of the study and on the methods and resources they were to use to implement designated experimental treatments.

The respondents used instructional designs that were prepared by the researcher in advance. There were four such designs used by different groups of participants. Participants who received experimental treatment D1 used the first design. They were exposed to a teacher-centred instructional approach in which classroom communication was predominantly one way from the teacher to the learners. The teacher was regarded as the expert source of knowledge received by learners. Teaching sequence adopted the form of definition, examples followed by a practice exercise. Learning resources included books and charts but no manipulative materials. Participants who received treatment T1 used the second design. Participants were exposed to a learner centred instructional approach that was characterized by experimentation, explorations and inquiry. They were expected to interact with the teacher, fellow participants and learning resources. The learning resources used were restricted to textbooks and plain diagram representation of solid objects on books, charts and chalkboard. The teacher's role was that of a guide who introduced helpful interventions during the learning process. The interventions comprised probing questions, provision of feedback to learners' questions (and responses), and cognitive conflicts. The teacher introduced learners to the concept being taught, guided them to explore and discover relations, generate patterns and to generalize. The teacher encouraged three modes of interaction in the classroom environment: teacher-learner interaction, leaner-learner interaction, and learner-resources interaction. Learning activities included analysis of diagrams to decode information, measurements and sketching of solid objects. Learners were also involved in the naming of solids whose diagrams were presented, naming of lines, planes and calculating lengths and angles. Finally, learners were expected to establish relations between the components of solid objects. The third design was used with participants who received treatment D2. The instructional approach involved was similar to that of treatment D1 except for the learning resources used. For treatment D2, learning resources included manipulative materials in addition to books and charts.

Participants who received treatment T2 used the fourth design, which is similar to the design used by recipients of treatment T1 except for the resources used. Recipients of treatment T2 used models of solids in addition to plane diagram representation of *three-D* concepts (see appendix A.30 for instructional designs).

3.40 Post-test

The post-test was administered at least one week after instruction to all participants in both the experimental and control groups. The purpose of the post-test was to provide feedback on the extent to which participants were able to internalise the concepts taught and on the effectiveness of the experimental treatments on the acquisition of geometry concepts. The test was presented on pencil and paper situations and regulated by the normal examination conditions in Kenya except it was not timed. The time element was eliminated because the purpose of the post-test was to measure the participants' ability to solve *three-D* problems and not the speed at which they solved the problems. All the post-test tasks were curriculum based.

The test was constructed and graded by the researcher. Potential difficulties exposed by the questionnaire for mathematics teachers and by the pre-test were used as a guide to construct the post-test tasks.

3.50 Target population and the sample

The study targeted all the third form female students who were at the time of the study, enrolled in Kenyan secondary schools. There were seventy nine thousand four hundred and ninety six (79496) girls (in form three) in 1997. The study however involved one thousand two hundred and eighty girls. Thirty-two groups all of whom participated in entire class experiments were framed from the target population. The groups were framed from seven of the eight provinces in Kenya.

3.60 Sampling procedures

The participating schools were selected from 3028 girls' schools in four regional clusters by using a table of random numbers. The regional clusters were Nyanza and western, Rift valley, Nairobi and Central, Coast and Eastern. Eight girls' schools were selected by a random procedure from each of the four clusters. From each cluster, two randomly chosen schools provided experimental groups while the remaining six provided control groups. In a case where a selected school had two or more streams, the participating stream was randomly selected. The learners' experience in terms of difficulties was assumed to be the same. Any girls' school could therefore have been used for the study. This justified the random selection.

Experimental treatments to the randomly selected groups from the experimentally accessible population were then assigned at random. This is because it was practically not possible to assign students from different schools to one group. The random assignment of the experimental treatments was expected to enhance the external validity of the findings. The clusters ensured that rural, urban, private and public schools had a chance to participate in the study. This was expected to improve ecological (environmental) validity of the study.

3.70 Research instruments

Interviews, questionnaires and achievement tests were used as tools of research to elicit information. Questionnaires were answered by 557 mathematics teachers. The researcher interviewed 32 mathematics teachers from schools where teaching experiments were done. Effort was made to interview respondents who are not involved in the teaching experiment. This was a deliberate move to obtain independent information that was free from the influence of prior knowledge of the materials and procedures used in the experiment. The interviews were conducted on the day the materials for teaching were delivered to a participating school. The interviews were open ended and aimed at seeking clarification and confirmation on the responses to the teachers' questionnaires. Questionnaires for mathematics teachers were semi-structured to provide room for non-restricted responses. An unstructured pretest of prerequisite concepts was administered to participants to determine their state of mental readiness to learn *three-D*. The pretest items were curriculum based. Since one of the objectives of the study was to determine the effectiveness of instructional methods with achievement as a dependent variable, an achievement test (posttest) was used to gauge the impact of the methods and materials used in the teaching experiment. The posttest measured the participants' mastery of *three-D* concepts and their proficiency in the same.

3.71 Validity of the research instruments

The researcher established a rapport with the respondents during interviews and consultations to provide a cooperative atmosphere in which truthful information could be obtained. The administration of questionnaires and pretest to relatively large samples (n = 557 and 2789 for questionnaires and pretest respectively) enhanced population validity. Standard instructions given to participants in the tests and the administration of the tests under normal school conditions minimized extraneous factors such as personal appearance, mood and conduct of the researcher that would otherwise colour the results of the study. This improved the internal validity of the findings. In the pretest, participants were not under any obligation to write their names on the scripts. This improved the validity of the pretest since participants provided responses without fear.

To ensure content validity, the post-test was representative of the syllabus content (for *three-D*) and cognitive levels such as understanding of facts, interpretation, application, analysis, and evaluation. The post-test items covered all the aspects of *three-D* as outlined in the course syllabus. Factors such as speed and vocabulary were controlled in the study by using simple language and eliminating time element during the administration of the tests. The items were written in simple language so as to avoid ambiguity and to permit as many participants as possible to understand the items. Time element was eliminated from both pre-test and post-test. This allowed the participants adequate time to analyse the items for better understanding and to answer them to the best of their knowledge since there was no time pressure. Participants' behaviour may partly be influenced by their perception of the experiment and how they should respond to the experimental treatment (Hawthorne effect). Participants' knowledge of their involvement in an experiment may alter their responses to the experimental treatment. In this study, effort was made to conceal to the participants, the fact that they were taking part in an experiment. The experiments were conducted under the school's natural setting using their teachers and classrooms since the presence of observers unknown to the learners could induce anxious experiences and hence interfere with the natural and spontaneous responses of the participants. Such interference could easily affect the performance of the participants.

3.72 Reliability of the research instruments

In an effort to improve the reliability of the interviews with mathematics teachers, respondents were restricted to the topic *three-dimensional* geometry when expressing their views on the learning and teaching of the topic. For the same reason, instructions for the pretest and posttest were made explicitly clear to minimize misinterpretation of questions. Both pretest and posttest were made long enough to eliminate the element of chance in attaining the final grade. The pretest covered most, if not all, of the basic concepts learners would normally require to learn *three-D*. The posttest covered all the aspects of *three-D* listed in the syllabus. Heterogeneous groups (all ability levels) were used in the study to enhance reliability of the results. Finally a common scoring scheme was used with precision to centrally grade the posttest by the researcher to eliminate inter-marker differences in scoring the posttest.

3.80 Data collection procedures

Pre-test and post-test scripts were scrutinized to provide data on students' errors and potential conceptual difficulties. Participant scores on the post-test provided data for quantitative analysis. The data was used to measure the impact of the two approaches and to gauge the effect of the materials used in the experiment on the learning of *three-D*. The teachers' questionnaire was used to provide information on the resources used during the teaching and learning of *three-D*. Mathematics teachers were also interviewed to provide information on their experiences with the teaching and learning of *three-D*. Four textbooks popularly used by teachers were analysed to provide information on the manner in which they present and test geometry concepts. Qualitative data were obtained from diagrams, sketches and other records of participants' work on the scripts. Quantitative data was obtained from participants' scores on the post-test.

CHAPTER FOUR: ANALYSIS AND DISCUSSION OF RESULTS

4.00 Overview

The research elicited both qualitative and quantitative data. In a teaching experiment, qualitative data are generally reported using descriptive statistics. However, for the purpose of ascertaining the relative effectiveness of the approaches that were under investigation and the impact of non-plane representation of *three-D* concepts on the learning of space geometry (as measured by performance in the post-test), a statistical test of significance (the F-test) was used. In addition to the F-test, non-parametric test (Kruskal-Walis test) was used to provide more information about the ranks of the post test scores per treatment group and asymptotic significance (chi - square) of the scores. The pre-test was analysed item by item and the teachers' questionnaires by descriptive statistics.

4.10 Questionnaires

Questionnaires were completed by respondents and administered in twenty-seven districts from the seven provinces in which teaching experiments were conducted. The respondents were at the time of completing the questionnaires, practicing mathematics teachers in Kenyan secondary schools. A total of five hundred and fifty seven (557) questionnaires were analysed. The questionnaire sought information on the type of instructional materials and approaches teachers have used to teach *three-D* at the secondary school level in Kenya. In addition, it sought information on the status of topic coverage, the factors affecting the learning of *three-D* and the specific aspects of the topic that provide difficulties to learners. Results of the analysis of the questionnaires are presented in the sections that follow.

4.20 Instructional Materials

Instructional resources currently being used to reinforce the understanding of *three-D* concepts include concrete materials such as cartons, chalk boxes, classrooms as (cuboids), desks, milk packets etc. Also used are models made from manila papers, plane diagrams presented on chalkboard and textbooks, solid models (concrete and wooden blocks, glass prisms etc.) and skeleton models (made of wires, drinking straws, thread, plasticine and sticks). Teachers reported simultaneous use of different resources to represent *three-D* concepts. The frequency with which the resources are reportedly being used is shown in table 6.

Table 6: Use of instructional resources

Resource	% Not using resource	% Using resource
Chalkboard diagrams	91.39%	8.61%
Textbook diagrams	76.48%	23.52%
Manila paper models	72.71%	27.29%
Solid models	48.83%	51.17%
Real objects	33.39%	66.61%
Skeleton models	31.24%	68.76%

Source: field data

It seems to be the case that plane representation of space concepts is used more often than concrete representation of the same. This can be seen from the popularity of both Chalkboard and textbook diagrams in table 6.

4.30 Instructional methods

The questionnaires revealed that respondents have used three approaches to teach *three-D* in the Kenyan secondary schools. These are the learner-centred approach, teacher centred approach and empirical approach (based on experiments and observations). Some respondents reported having used different methods at different times. Others have integrated more than one approach in the same lesson. Respondents (68.4%) have used the learner-centred approach more than the teacher centred (49.19%) and the empirical (47.23%) approaches. The Success rate is highest among those who used the learner-centred approach (56.01%) followed by empirical approach (38.96%), teacher centred approach (26.03%) and a combination of approaches (1.62%) in that order.

4.40 Topic coverage

Those who normally cover the topic of *three-D* (41.29%) by the time the candidates are ready to sit for their national examinations that mark the end of secondary school education are out numbered by their counter parts who never cover the topic (58.71%) at the same point in time. It can be inferred that in a majority of Kenyan schools, the topic is never adequately covered. This therefore implies that learners are entering examination rooms without adequate preparation that would enable them to perform well in tasks involving space geometry. With such a poor background in *three-D* from the secondary school, mathematics teachers may only rely on their undergraduate studies and postgraduate experiences to prepare them for effective geometry teaching. Unfortunately, the Kenyan mathematics curriculum for pre-service secondary school teachers ignores the mathematics content for the Secondary schools. This coupled with the fact that seminars and workshops for mathematics teachers is a rare “gift” to mathematics teachers in Kenya, the teachers are ill prepared to handle *three-D*.

4.50 Factors affecting the learning of *three-D*

A number of factors were reported to be affecting the teaching and learning of *three-D*. Some respondents reported two or more factors. The reported factors that are likely to impact upon the learners' performance (in this case, negatively) are summarized in table 7.

Table 7: Factors affecting the learning of *three- D*

Factor	% reporting the factor
Lack of models	43.09%
Learners' attitude	30.88%
Topic is abstract	19.93%
Lack of basic skills	16.16%
Other learner characteristics	15.62%
Lack of time	12.75%
Poor instructional approach	11.31%
Text books	9.16%
Teacher characteristics	8.98%
Syllabus too broad	3.29%
Language deficiency	2.19%
Topic unpopular in exams	1.83%
Over enrolment	0.40%

Source: Field data

Respondents reported that learners find it difficult to pick up concepts in the absence of models. Lack of models and materials for model construction is one of the greatest setbacks in the struggle to improve performance in *three-D*. When a model does not accurately represent the concept it is purported to represent or when the model is used inappropriately or ineffectively, its role and power in reinforcing the understanding of concepts is compromised and diminished.

It was mentioned by some respondents in their response to item number eight in the questionnaire that modelling using Manila paper is difficult with skew lines. Others reported the difficulty of modelling certain concepts like projections, plans and elevations. The problem here could be one of rigidity or inaccessibility to alternative courses of action. The inadequacy of Manila paper models in representing skew lines should signal to the teacher, the use or exploration of other viable possibilities and alternatives. Skeleton models reported in responses to the questionnaire, as being used should be a suitable alternative for demonstrating the concept of skew lines. One way of doing this is to let learners themselves construct skeleton models of a cuboid (or any other regular solid) then allow them to explore the relationship between different pairs of the edges of the cuboid (or solid) that represent different categories of lines (perpendicular, parallel and skew) until all the edges are exhausted. They should be given time to discuss and explain their findings on the relationship between the different pairs of edges. Possibly, some of the learners will be able to identify both parallel and perpendicular lines but not skew lines. At this point, they can be asked to name pairs of intersecting lines from a plane drawing of the cuboid (or solid). Some learners may respond by naming skew lines as intersecting lines. A conflict can be generated by asking learners to identify the "*intersection*" of skew lines on the skeleton model and on the plane drawing. They can then continue to explore parallel and skew lines both on the diagram and the model by identifying and attempting definitions of both skew and parallel lines. In their definition, they are likely to regard parallel lines as lines that do no meet. They should then be probed further to explore and differentiate between parallel and skew lines by using their prior knowledge of parallel lines. They are likely to realize that skew lines do not fit their descriptions of parallel lines. The conflict is likely to shed light on the definition of parallel lines as lines that have a constant distance between them. After a discussion of whether such lines exist or not, the concept of skew lines can formally be defined by the teacher and the learners together. More insight into the learners' knowledge of skew lines can be presented by asking them to identify several other pairs of edges that represent lines that are neither meeting nor parallel.

Learners' attitude was rated second by the sample of respondents as a stumbling block in the learning of *three-D*. Learners hold belief systems, which seem to be firmly rooted in their thoughts. Apart from having a phobia for mathematics, female learners believe that mathematics is a masculine subject and therefore a preserve of the boys. One of the respondents whom the researcher interviewed about her learners' problems with mathematics reported that her female students tell her that mathematics is for boys and "mad women" (sic). A mad person is one whose mental faculties are impaired and under normal circumstances, would not be able to learn mathematics (or any other subject) unless the insanity is medically corrected. Girls who excel in mathematics are considered to be "abnormal" in some way by their peers. This not only demonstrates the extent to which girls believe mathematics is difficult but also discourages potential high achievers in mathematics among the female students. Girls also do believe that they can only perform better in languages and humanities but not in mathematics hence may be putting insufficient effort in their struggle to learn mathematics.

Among other learner characteristics that were reported to be affecting the learning of *three-D* are lack of interest, low motivation, nature of the learner (low ability levels and slow learners), inability to apply *three-D* concepts to real life situations, inability to comprehend concepts, incorrect interpretation of examination questions and inability to transfer geometry skills to unfamiliar situations. One of the respondents reported that *learners are not able to apply concepts and skills learned with the use of models to answer questions in the absence of models*. This confirms learners' difficulties or their inability to transfer and apply geometry skills and concepts in situations that are different from those under which the skills and concepts were learned.

Learners are reported to be deficient in prerequisite concepts. Those who posses the basic concepts are not able to apply them in a problem situation. They are not able to recall and apply a combination of basic skills that are prerequisite to the solution of a problem. Those who apply them do so inappropriately with little or no success. The basic skills that learners are either lacking or are unable to apply include trigonometry, logarithms, similarity and enlargement (as in frustums of pyramids and cones), sketching of solids, nets, points, lines, and Pythagorean theorem. With such a background, effective teaching, subsequent learning and good performance may be difficult to realize. For meaningful and successful instruction, it is important for the teacher to recognize the deficiency, act upon it by bridging the gap between learners who are mentally ready to learn the new concept and those who are deficient in basic knowledge and skills. This should be done before the lesson can be taught otherwise the learners would be operating without a relevant foundation upon which to build new concepts. In this case they would be lacking a foundation upon which they can develop *three-D* concepts and spatial relationships.

It seems to be the case that time is a factor whose negative contribution to the performance of *three-dimensional geometry* in particular and mathematics in general may not be ignored. Mathematics teachers lack adequate time to plan and effectively implement instructional designs during the presentation of the subject. Time is reportedly not enough for them to be able to adequately cover the prescribed curriculum content for mathematics. Consequently, they rush learners through the mathematics (and geometry) syllabus in order to have a touch of all the topics before learners are ‘‘ready’’ to sit for their examinations that mark the end of secondary education. The rush to cover the mathematics syllabus implies that learners are similarly rushed through the geometry content of the syllabus. As a result, not much constructive learning (learning with understanding) can be realized. Some respondents attributed the time factor to overcrowding in the curriculum that results to a relatively heavy teaching load hence teachers do not have enough time for their lesson preparation and model construction. For the same reason, they are not able to attend to learners who need special attention. Worse still, little marking of learners' assignments that would expose their errors and other weaknesses is done.

The instructional approach used has a share of the blame for learners' difficulties with *three-D* and the unsatisfactory performance that has been witnessed. Some of the respondents (11.31%) were of the view that the approach adopted by teachers when teaching the topic is not appropriate and lacks coordination in the sense that there is no cooperative teaching. Some teachers over use the teacher-centred approach. Others use introduction that has no link with the concept to be taught and do not consider the prerequisite knowledge that learners require at the beginning of the lesson in order to understand the concept to be developed in the lesson. Because of the desire to complete the syllabus, learners are rushed through lessons and are usually minimally involved in the lesson development since most activities are teacher dominated. Some lessons are taught without models citing lack of time for model construction. Consequently, teachers normally depend and rely on chalkboard diagrams for illustrating space concepts. In such situations, lessons end up being more theoretical than practical and no real life applications of the concepts taught are emphasized. Experimentation is rare to come by and examples cited in the lesson are limited and restricted to those given in the textbooks. Creativity and reflection are not encouraged since learners neither set their own goals nor pose their own questions during the learning process.

Availability and adequacy of textbooks in terms of depth and scope of content coverage is another factor that affects the learning of space geometry. It appears that available textbooks do not treat *three-dimensional geometry* to the satisfaction of teachers in regard to the examination requirements. The books lack a number of important pedagogical qualities such as logical sequencing, presentation and organization of subject matter. Respondents reported that in some books, the introduction lacks clarity, is not logical and not related to the main concepts in the text. Other books provide a limited number of solved examples for learners' private study. Other weaknesses of the texts being used include poor quality diagrams (i.e. lack clear illustrations), shallow depth of content coverage, inadequate practice exercises, and a poor linkage between plane geometry and solid geometry. Class texts and reference books are generally few and therefore cannot cater for the interest of all the learners and teachers. About one tenth of respondents (9.16%) indicated that the textbooks they are using are faulty in some way. Overall, 39.32% of the respondents reported good coverage of *three-D* by the textbooks they are using. Some (40.22%) were not certain while 18.13% reported with confidence that none of the available textbooks adequately covers *three-dimensional geometry*. The rest (2.33%) did not respond to this item.

A review by the researcher, of four most popularly used textbooks in teaching *three-D*, confirmed some of the weaknesses reported by the respondents. Three of the reviewed books do not link the prerequisite concepts (needed to learn *three-D*) to the main concepts developed in the chapter on *three-D*. The concepts that are used to develop and discuss the relationships among the elements of *three-D* objects include point, line, plane, perpendicularity, parallelism, angle, intersection, rotation and translation. These concepts are therefore necessary for the learning and understanding of *three-D* concepts. Despite this, most of the books do not review them (prerequisites) or remind readers (most of whom may be encountering the concepts presented in the book for the first time) of their role in acquiring the concepts being developed.

All the four books used the concept of projection to define the angle between a line and a plane. The angle between a line and a plane is the angle between a line and its projection on the plane (Channon et al, 1994; K. I. E., 1994; Patel, 1994). This definition requires learners to be conversant with the concepts of angle, line, plane and projection. Unfortunately, none of the four books addressed these concepts before using them to develop the concept of angle between a line and a plane. Projection deserves a separate treatment as a sub-topic whose presentation should precede the presentation of angle between a line and a plane. All the four books discussed it simultaneously with the concept of angle between a line and a plane. This is likely to cause confusion to learners. Three of the four books did not have answers (to selected items) for learners' private practice.

The idea of rotation is used to explain the angle between two planes as the angle through which one plane moves to fit on to the other (Patel, 1994). The notion of translation is used to explain the concept of skew lines yet it (translation as a transformation) is not discussed anywhere in the book. Translation does not even appear in the syllabus. This makes it absolutely necessary to define or explain with illustrations, at the beginning of the presentation, technical terms used to develop a concept. In the absence of any explanation, the reader should be referred elsewhere for acquaintance with the basic concepts. One of the books discussed four concepts (projection, angle between a line and a plane, angle between two planes and angle between skew lines) without presenting a practice exercise. This could be detrimental in the sense that too much is presented within a short span of time to the extent that it becomes difficult for the novice reader to follow and comprehend the concepts presented. Tasks testing each concept should be presented at the end of the presentation of that particular concept. This would allow learners to test (through practice exercises) their understanding of *three-D* concepts in a more orderly fashion. A revision exercise that test all the concepts developed in the chapter can be presented at the end of the chapter.

About twenty different textbooks are being used to teach *three-D*. Respondents reported the use of several books simultaneously. Of these, the four most commonly used (in decreasing order of popularity) are Secondary Mathematics (58.35%), Mathematics for Kenya Schools (43.81%), Form Three Mathematics (34.65%) and General Mathematics for Secondary schools (17.04%). The reported ranking of the textbooks used does not necessarily mean that the most popularly used book is the best or vice versa. Participants reported what is available in their schools and what is available is not necessarily the best. It was however observed that no single textbook provides in-depth coverage of the topic. Teachers therefore resort to use a variety of books at their disposal to obtain enough material for their lessons.

The nature of learners and lack of material resources have been blamed for the unsatisfactory results exhibited by learners in *three-dimensional geometry*. However, teachers also have a share of the blame for the poor performance. They are reported to be having certain characteristics that would not promote effective learning. These include among others, a phobia for *three-dimensional geometry* exhibited by some teachers who either teach it poorly or avoid teaching it altogether. Some teachers lack motivation, commitment, and devotion to their teaching duties hence do not prepare well for their lessons. Others appear to be uninterested or precisely put, inept in handling the topic and cannot teach it effectively and therefore skip it most of the time. Another teacher characteristic that was reported to be responsible for the observed poor performance in *three-D* is lack of team teaching and cooperation. There seem to be minimal or no cooperation at all among mathematics teachers. There is also no cooperation between mathematics teachers and their counterparts who are handling related subjects like technical drawing and industrial education. Teachers handling industrial education and technical drawing or those who have a good background in technical education from their school days are likely to have superior skills in plane representation of *three-D* objects and a good command of spatial properties. They are therefore likely to be good resource persons for their colleagues with weaker backgrounds in *three-D* if there is cooperation among them.

The syllabus being used may also be contributing to the learners' dismal performance in one way or another. It is reported to be too wide and presents the topic too late in forms three and four. This causes a lot of anxiety among learners because they view the topic as "difficult". Respondents also mentioned inappropriate sequencing of *three-D* sub-topics in the syllabus. For instance, it would be logical for the sub topic "common solids" to immediately precede *three-D* for continuity and better flow of concepts. That is however not the case. The former is taught in form one and the latter in form four. The situation is worsened if, as was reported by some respondents, the sub-topic "*common solids*" is not taught in form one. The arrangement of syllabus topics therefore results in poor linkage between *three-D* and other geometry topics such as modelling. Inconsistency was also reported between the syllabus and the Kenya Institute of Education (K.I.E) textbooks that appear to be more popular in most schools than other books. The syllabus presents the topic in form four while the revised edition (1994) of the K.I.E. textbook presents it in form three. This implies that the topic is taught to form fours when the book is with form threes, a situation that denies the form fours optimum use of the book during the learning of *three-D*.

Language deficiency is another factor that was reported to be affecting the learning of *three-D*. Language and terminology used in some textbooks, examination questions and during instruction put some learners off. The mathematical concept of "*frustum*" seems to be an unknown word to some learners hence questions involving "*frustum*" are normally not well understood by learners. The phrases "*line of greatest slope*" and "*lines perpendicular to the line of intersection*" were reported to have caused difficulties to learners because they do not comprehend the phrases. Learning would normally be promoted by an interactive mode of instruction where the learner and the instructor both have a chance of discussing ideas of pedagogical significance. Dialogue with other people is an extremely useful tool for refining concepts (Lockhead, 1992). Exchange of ideas may be witnessed to a lesser degree or not at all in a learning environment where communication between the source of information and the receiver are operating at different levels. As such, dialogue between the learner and the source of information can only thrive when the language used is at the level of the learner. When learners are not able to understand the word *frustum*, then it becomes an up hill task for them to construct a schema for and refine the concept of *frustum* hence may not be able to comprehend space concepts related to the word "*frustum*". Consequently they cannot perform well in items that test concepts related to "frustum".

It seems to be the case that the national examinations do influence the content taught and the method(s) used to teach such content. Teachers rush to complete the syllabus in readiness for the examinations and for a silent competition among the schools to secure prestigious positions in the ladder of academic excellence. The rush to complete the syllabus implies that long but more productive methods and approaches of teaching are sacrificed in favour of those that are deemed to be less time consuming. In such situations, creative and innovative instruction would be least witnessed. The rush to complete the syllabus, use of short cut methods that are not necessarily productive, lack of creativity and innovation would all yield unsatisfactory results. The frequency with which a particular mathematics topic appears in the national examinations was found to influence the teaching of the topic. During their teaching teachers give preference to topics that are more frequent in the examinations at the expense of those that are rarely tested. This preferential selection of topics renders the teaching of mathematics a gambling adventure influenced by chance. Questions on *three-D* rarely appear on the national examinations and if they do, then only a small section on one of the two papers. Because of the rare testing of *three-D* concepts in the national examinations, some teachers do not find it necessary to cover it in their teaching. These partly explain why performance in *three-D* items by Kenyan students has been dismal and declining during the past decade.

The last factor reported in this study to be affecting the learning of *three-D* is the student teacher ratio. In some schools there is an over enrolment of learners. This makes it difficult for the teacher to interact with all students and attend to those who may be in need of special attention. This implies that the much needed teacher interventions to let learners off their difficulties might be given too late or never at all. Respondents were also of the opinion that too many calculations involved in the solution of *three-D* problems contribute to poor performance. This is in fact lack of computational skills.

4.60 Learners' difficulties with *three-dimensional geometry*

Some aspects of *three-D* were identified by respondents to be responsible for learners' difficulties with the topic. Top on the list is visualization, which was identified by 63.55% of the respondents. Learners lacked the ability to mentally view and identify the components of a *three-dimensional* object. Learners experienced difficulties with decoding (reading or extracting) information from plane diagrams, skeleton models and from solid models. Learners were not able to extract information from plane representation of space objects. They find it difficult to recognize how the components of space objects fit together by reading plane diagram representations (chalkboard diagrams, text book diagrams, and manila paper diagrams) of the objects. Learners were reported to be confining their thinking to two dimensions only. This difficulty could be explained in two ways. First is because plane diagrams ''hide'' the third dimension. Some (if not all) of the features of an object that make it three-dimensional (depth and height) are "quashed" and "concealed" in a plane drawing of the object. Learners therefore require a special skill in reading information from the diagrams in order to understand the mathematical relations presented in the diagrams. Second, diagrams distort some features of the components (lengths, angles, shapes of planes etc.) of *three-D* objects. Some lengths appear shorter on a diagram than they actually are on the real object while others appear longer. Right angles may appear acute or obtuse. Rectangles and squares appear as if they have been transformed by a shear to acquire the shape of parallelograms with acute and obtuse interior angles. Skew lines appear on a diagram as if they intersect. Learners therefore need a formal training on how to recognize and interpret the features of *three-D* objects as they appear on plane diagrams. They need to be familiar with the conventions used in plane diagram drawings. For instance they need to be acquainted with the convention to use dotted and continuous lines to represent "hidden" and "visible" lines respectively on plane diagrams of *three- D* objects.

Skeleton models do not completely concretise the concept of plane. The spaces between the frames representing edges of the solid are seen to be "empty" and not as planes. An attempt to explain to learners that the "empty spaces" actually represent planes creates a conflict in their mind since they can neither see nor touch the planes. Components of a solid object (points, lines and planes) that are not on the surface of a solid remain abstract and learners have to imagine their existence. Such components are not easy to recognize from a solid model. For instance when a ball is used to represent the concept of sphere, learners have to use their imagination to recognize the angle subtended by an arc of a great circle on the surface of the ball at the centre of the ball. Learners who are weak at imagination therefore find it difficult to recognize the angle since they can neither see the centre of the ball nor the lines that form the angle at the centre of the ball. The difficulties are illustrated in diagrams 1 and 2.

Diagram 1 Visualizing the components of a cuboid from a plane drawing

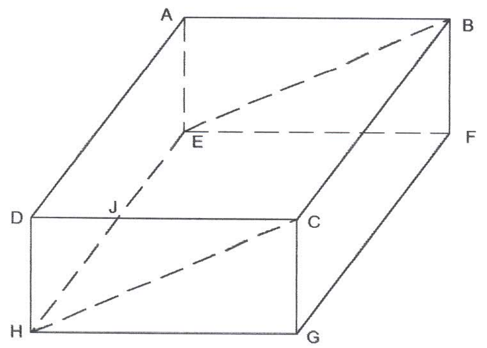


Diagram 1

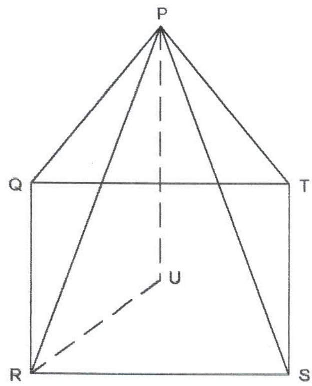


Diagram 2

Diagram 2 Visualizing the components of a pyramid from a plane drawing

In diagram 1, the planes BCHE and BCGF appear as "parallelograms" with acute and obtuse interior angles when they are actually rectangles. Angles DHE and CHE appear to be acute when they are right angles. Angles BEH and HGF on the cuboid both appear to be obtuse when they are actually right angles. Angle PQR on the pyramid appears to be obtuse yet it is acute. Measuring the angles practically would confirm the learners' incorrect thoughts and visual perceptions of the angles. This creates a conflict in their minds because the angles obtained by measurement would differ from what they theoretically know about the angles. It can also be observed from diagram 2 of a pyramid that line PQ appears to be shorter than line PR yet they are edges of the same pyramid on a square base and therefore its edges should be the same. Lines DC and HE on the cuboid and lines QT and PR on the pyramid appear to be meeting yet they are skew lines. On the diagram of the cuboid, there are four triangular ''planes'', four trapezoid ''planes'' and one plane in the shape of a quadrilateral that learners may recognize yet the planes (if they exist) do not posses such regular shapes that are portrayed by the diagram. The vertices A, D, E and the "intersection" of the lines DC and HE ("point JJ) define one of the trapezoid planes. Such ''planes'' could interfere with the learners and distract them from recognizing the regular planes of the cuboid.

Learners were reported to be experiencing difficulties in identifying and calculating the angle between two planes especially in prisms, pyramids and tetrahedrons. This was reported by 45.24% of the respondents. Others have difficulties in identifying and calculating the angle between a line and a plane especially lines and angles that do not involve the edges of the solid. Identification of skew lines and calculating the angle between them is also a problem to learners. This difficulty could be attributed to the definition of skew lines as ''lines that do not meet''. The definition and the angle to be calculated presents a conflicting situation since learners know that an angle is formed at an intersection of two lines and its size is determined by the position or orientation of the lines in relation to the point of intersection. They therefore experience difficulties conceptualising an angle formed between two lines that do not meet. When asked to calculate lengths and angles in three dimensions, learners tend to measure instead of calculating the angles. Learners also experience difficulties with nets of solids. The process of "mentally" folding the net to identify the solid whose net is presented become quite elusive. They are not able to analyse the net and understand the relationship between the net and the solid. The net is viewed in isolation and considered unrelated to the solid. Identification of vertical and perpendicular planes from a plane diagram of a *three-D* object also presents difficulties to learners. They find it difficult to recognize lines that are part of a diagram but are not shown. It was also reported that learners experience difficulties with drawing diagrams of *three-D* objects on two-dimensional planes. A summary of these difficulties together with percentage of the respondents reporting the difficulty is given in table 8. Each respondent reported more than one difficulty.

Table 8: Learners difficulties with *three-D*

Difficulty	Respondents reporting
Visualization	63.55%
Angle between two planes	45.24%
Angle between a line and a plane	23.88%
Projection of lines on planes	15.98%
Angle between skew lines	10.98%
Calculating lengths in *three-D*	8.04%
Sketching diagrams of *three-D* objects	3.77%
Drawing plans and elevations	3.30%
Identifying vertical and perpendicular planes	3.10%
Angle between two lines	2.74%
Lines not shown on a diagram	0.90%
Calculating surface area of common solids	0.90%
Volume of a frustum of a solid	0.18%

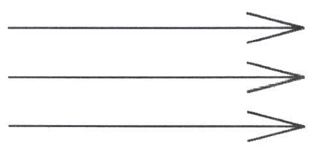
Source: Field data

4.70 Item analysis

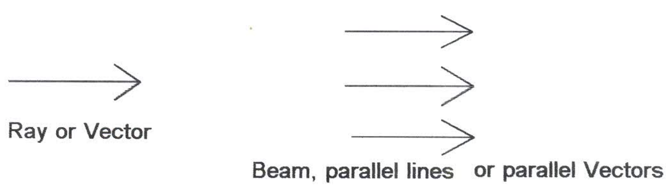
4.71 Pre-test

The purpose of the pre-test was to gauge the participants' knowledge of basic skills, which are prerequisite to the learning of *three-D*. The pre-test was initially intended for participants in the teaching experiment only but was extended to non participants when preliminary observations and analysis of the six hundred and forty scripts indicated potential difficulties with certain basic concepts. The extension was to confirm the prevalence and consistency of the preliminary observations made from the responses of the initial 640 pretext participants to a wider sample. The pre-test items are given in appendix A.20.

Item one tested the participants' knowledge of parallel lines, perpendicular lines, intersecting lines and their ability to distinguish between parallel and perpendicular lines. It was anticipated that participants would identify parallel and perpendicular lines by the respective symbols used to denote them. It was observed that 64.83% of the participants exhibited a clear knowledge of parallel and perpendicular lines and were able to distinguish between the two types of lines. However, it was evident that some participants were not conversant with the symbol for parallel lines. They gave different interpretations of this set of parallel lines.



Some participants referred to the set of lines as "*a beam*", apparently in reference to a beam of light in physics. Others referred to it as *horizontal lines*. Other descriptions were *vectors, arrows, east lines* and *lines facing east*. The message inherent in these responses is that learners posses a rich experience with divergent understandings of mathematical concepts, notations, rules and theorems. Their interpretations of mathematical symbols used to represent mathematical concepts are also different. Mathematics teachers should therefore capitalize on this rich and divergent background to generate pedagogically healthy discussions. Such discussions should be aimed at refining concepts from the same knowledge domain (e.g. mathematics) and harmonizing concepts emanating from different knowledge domains (e.g. mathematics and biology). The harmonization should clearly show the meaning(s) of the same symbol used in more than one subject discipline. In mathematics, the notation for a vector is a line with an "arrow" either at the end of the line or somewhere near the middle of the line.



The same symbol is used in physics to represent a ray of light. Two or more of such lines with constant distances apart are used in geometry to represent parallel lines and in physics to represent a beam of light (a collection of rays). The symbol (word) "vector" has different interpretations in mathematics and biology. In the former, a vector means a directed line. In the latter, a vector is an organism that carries and transmits a parasite. The different interpretations underscore the need for instruction to highlight or pay attention to different meanings attached to the same symbol in different subject disciplines. This would avert possible misinterpretations of concepts and eventual incorrect application (misapplication) of such concepts. For the same reason, teachers need to test or seek the learners' understanding (prior experiences) of concepts before instruction commences.

Item two tested the participants' knowledge of skew and parallel lines. Less than one third of the participants (30.62%) were able to recognize parallel and skew lines as lines that do not meet at all. The rest of the participants (69.38%) were not able to recognize parallel and skew lines as lines that never meet. By implication, 69.38% of the participants may not be able to distinguish between parallel and skew lines.

Item three consisted of four nets. Two were nets of an open cube; one of a closed cube and one was a net that cannot be folded to form a cube. Participants were required to identify the nets according to whether they would form an open cube, a closed cube or no cube at all. Nearly one third of the participants (32.48%) correctly identified all the four nets, close to half of the participants (49.16%) identified two or three of the nets correctly, while 18.36% identified one or non of the nets.

Item four tested the participants' knowledge of the Pythagorean theorem (relationship between the sides of a right-angled triangle) and their ability to apply the theorem. Nearly nine tenths (87.38%) of the participants had a clear concept of the theorem and were able to apply it to calculate the third side of a right-angled triangle given the hypotenuse and the base. The rest (12.62%) did not exhibit a clear understanding of the Pythagorean theorem. When stating the theorem, they expressed the square of the opposite side as the sum of the square of the hypotenuse and the square of the base ($opp^2 = hyp^2 + base^2$). The second part of item four required the participants to calculate the size of one of the acute angles of a right-angled triangle given the hypotenuse and the base. One of the mathematical demands of the item was the recall and use of trigonometric ratios to calculate the size of an angle. More than half of the participants (60.16%) could not apply the ratios to calculate the size of an acute angle.

Item five required the participants to match tangible solid objects with their diagrams. The objects presented were a tetrahedron, a pyramid, a sphere, a cone, and a cylinder. Less than four tenths (39.83%) of the participants were able to match all the five objects presented with their shapes, 27.07% identified three or four objects. One third (33.1%) could only identify two and below objects correctly.

Item six required participants to identify solids from their drawings on plane diagrams. The diagrams presented were those of a cone, a sphere, a pyramid, a prism, and a cylinder. The results showed that 24.96% were able to identify all the five solids from their diagrams, 69.59% were able to identify three or four solids, while 5.45% identified two or less of the solids from their plane diagrams. It also came to be known that as many as 69.13% were not able to identify a prism from its drawing on a plane. They referred to a prism as a pyramid and to a pyramid as a prism. Some thought it is a tetrahedron. Other popular responses in reference to a diagram of a prism were a cube, a cuboid and a trapezium. This seems to suggest that participants had difficulties with names and identities of regular solids.

Item seven tested the participants' ability to recognize components of a composite solid from a plane drawing of the solid. A pictorial arrangement of four cubes was presented as shown in diagram 3. Majority of the participants (82.65%) were able to identify the number of cubes (four) in the drawing while 17.35% could not recognize the fourth cube that is directly below the cube numbered 1 in diagram 3. They simply counted the visible cubes (3). The second part of the item required the participants to obtain the number of extra similar cubes that would be required to form a cube of volume $2 \times 2 \times 2$ (diagram 4).

Diagram 3 Composite solid of four identical cubes

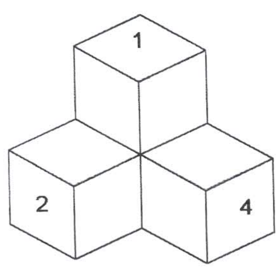


diagram 3

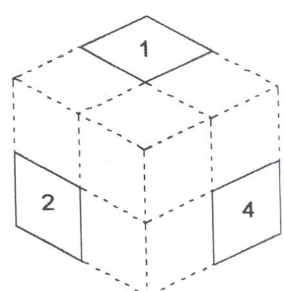


diagram 4

Diagram 4 Composite solid of eight cubes

Less than half (41.52%) were able to obtain the number of extra cubes that were required to complete diagram 3 to the form presented in diagram 4. None of the participants who obtained the correct number of extra cubes (4) needed to transform diagram 3 into diagram 4 used a visual strategy. They all used calculations to obtain the result despite the fact that diagram 4 could be "mentally constructed" and the extra cubes counted without any calculations. The participants may have preferred a calculation strategy to a visual strategy possibly because of their inability to mentally manipulate visual information. The choice for a calculation strategy could also be a direct result of learners' effort to indicate all the details of the solution process.

Item eight consisted of four parts that required participants to identify the components (points, lines, planes, and angles) of a cuboid from a plane drawing of the cuboid. Participants experienced difficulties in using vertices to describe lines and planes. They used one vertex to describe a line and two vertices to describe a plane. This demonstrates lack of understanding of the concepts of "line" and "plane". Less than one third of the participants (30.62%) were able to identify perpendicular lines from a plane diagram of a cuboid. A very surprising response for this item is the fact that 37.29% of the participants named parallel lines for perpendicular lines. The most popular pair of parallel lines that was given for a perpendicular line is AC and HF (see diagram 5 on page 76). The act of giving parallel lines for perpendicular lines was surprising because in item number one, the participants had demonstrated a better understanding of parallel and perpendicular lines (64.83% were able to identify and differentiate between parallel and perpendicular lines). The discrepancy is an indicator of the participants' difficulties with decoding information from plane diagrams of solid objects. The second part of the item tested the participants' knowledge of skew lines. In particular, the learners' ability to identify skew lines from a plane diagram of a cuboid. Only 10.9% were able to identify correct pairs of skew lines. This was however not surprising because the concept of skew lines had not been formally taught to the participants as at the time of answering the pre-test items.

The third part of the item tested the participants' ability to identify acute angles from a plane diagram of a cuboid. Slightly more than one third (37.29%) were able to recognize acute angles from a plane diagram of a cuboid. This and other learners' difficulties with decoding information from plane diagrams can be attributed to the fact that plane diagrams distort information about the object represented by the diagram. Many attributes and characteristics of the object are not retained in a plane diagram of the object. For instance, equality of lengths and angles is not retained. Perpendicularity of lines may be distorted in the sense that an angle of 90° may appear to be acute or obtuse. In diagram 5 (page 76), the angles DCF, DGF, DCE, ABE, DHE, HAD, AHE, CFE, ABF, CBE, and ACF were seen by participants to be acute. Visually, the angles HAD, DGH and CBE in diagram 5 appear to be acute while angles AHG, ADG, BEF and BCF appear to be obtuse yet they are all 90° in the actual cuboid. The curved surface of a cone was perceived as a triangle due to the fact that it takes a triangular shape when drawn on a plane diagram. Part four of item eight tested the participants' knowledge of intersecting planes. It tested their ability to identify planes that posses a common line or a common point of intersection from the diagram of a cuboid. This proved to be a difficult task for the participants as only 14.95% could identify intersecting planes.

While it may be argued that learners are expected to be conversant with concepts such as acute angles, lines and planes during their third year of secondary education, the participants' responses to item eight reveal two things. First, the responses reveal that learners are deficient in basic concepts that are prerequisite for the learning of *three-D*. Second, the integration of basic concepts in a plane diagram of a solid object seem to present a rather complex scenario for learners. They are not able to recognize the basics that they know from the complex situation. Alternatively, the distortions arising from the plane diagram representation of a solid object presents a conflict with the learners' knowledge of the basic concepts. This disorganizes and incapacitates their ability to apply basic knowledge in situations that are less similar to the instances under which the basics were learned. It is, therefore, necessary and appropriate to reconcile the learners' conflicts and inconsistencies between their prior and current experiences in order to limit or eliminate the obstacles encountered in the learning of *three-D*. The conflicts and inconsistencies may largely be attributed to the distortion of some of the characteristics of solid objects by plane diagrams.

Learners' prior experience of perpendicular lines is that they are lines that meet at 90°. They also associate an angle of 90° between a pair of lines with perpendicularity of the lines and they can verify this from a real cuboid or a concrete model of a cuboid. Their new experience is provided in the distortion of these prior experiences such that an angle of 90° between two perpendicular edges of the cuboid appears obtuse (or acute) and the edges ‘‘cease’’ to be perpendicular on a plane diagram of the cuboid and in the mind of the learner. An attempt by the learners to physically measure the distorted angles just confirms the distortion to be a real and true phenomenon since the results of the measurements will be consistent with their incorrect visual perceptions of the edges as viewed from the plane diagram. Skew lines are usually defined to learners as ‘‘lines that are not parallel yet they do not meet’’. The learners can verify this prior experience or description of the concept of skew lines when they manipulate a concrete model of a cuboid. Skew lines as presented by distortions in the new experience, appear as intersecting lines (AD and HE) in diagram 5. The pairs of skew lines DC and EF, HE and AD were seen by participants to be perpendicular since they appear to be intersecting at 90° in diagram 5.

Diagram 5 Distortion of lines, angles and planes

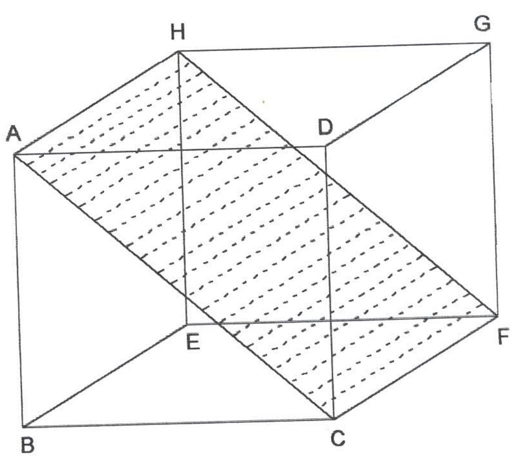


Diagram 5

Recognition of planes from a plane diagram of a solid places a greater cognitive load on the learners. The load results from ''overlapping planes'' that present irregular and unfamiliar planes in the form of some familiar regular shape. One such plane is the ''plane'' whose vertices are B C E and the intersection of lines AC and EF (diagram 5). This ''trapezoid'' plane is created by visual overlap between the planes ACFH and BCFE.

Item nine tested the participants' ability to internalise the transformation of a plane object (a circle) into a space object (a sphere) through a mathematical process (a rotation of 360° about the diameter of the circle). It also tested the participants' ability to recognize the product of the transformation and to synthesize spatial information. Just over one third of the participants (35.14%) were able to identify the solid of revolution. Some of the participants could neither internalise the transformation nor handle spatial information involved in the item. This could be inferred from their response that the ''solid'' formed by the transformation is a circle, an incorrect classification of a plane object as a solid object. This particular response also reveals that learners do not reflect back on their solution to problems by evaluating the practicality and reasonability of their responses. Some participants described the solid formed as ''a nautical mile''.

Item ten tested the participants' ability to state differences and similarities between objects of different dimensions. Responses to this item revealed the participants' inadequate knowledge of the concept of dimension. Only 0.25% has the knowledge of zero and one dimensions. A relatively bigger proportion (41.45%) has the knowledge of both second and third dimensions while 58.30% exhibited no knowledge of the concept of dimension. Participants' responses to the item revealed that they used "sides" to mean faces of a solid. Thus a cone has two "sides" while a pyramid has five "sides". Both pyramid and cone were described as "void".

Participants had three conceptions of dimension. One lot used "dimension" to mean the number of sides (or edges) of an object thus both square and rectangle are *four-dimensional*. Another group conceived of dimension as the number of faces hence a square is *one-dimensional*, a pyramid is *five dimensional* and a cube is *six dimensional*. The third group had a correct version of dimension as the number of "lengths" used to calculate area or volume. Thus a rectangle is two-dimensional because it has length and width. A cube is *three-dimensional* because it has length, width and height. The item also revealed weaknesses in the participants' descriptions of the concepts of rectangle, square and cube. Their descriptions of a rectangle were these:

- *three-dimensional*
- *four-dimensional*
- *one-dimensional.*
- *has volume calculated by* $L \times W$.
- *has three sides.*
- *a solid.*
- *has two sides.*

The response that a rectangle is *four-dimensional* may have been based on the use of "dimension" to mean "sides". The responses seem to suggest that some learners in their third year of secondary education can neither define a rectangle nor classify it according to its dimension.

It is also evident that some learners do have incorrect conceptions of the square as a geometrical object. This can be inferred from these descriptions and references to a square:

- *when calculating a square you use* $\frac{1}{2}$bh
- *the formula for finding a square is* $L \times W$
- *has two sides (L and W)*
- *one dimensional*
- *is three dimensional*
- *has four dimensions*
- *is a solid figure*
- *is a triangle*
- *a solid*
- *has volume*
- *has five faces*
- *has no dimension*
- *one sided figure*
- *has three corners*
- *has four faces*

The response that a square has one dimension was evident in more than half of the pre-test schools. It may be based on their definition of the concept of dimension *as the number of faces*. Participants' conception of a square as having *three corners* or as *a triangle* is based on their recognition of a square as an instrument used in geometrical constructions (set square) and not as a geometrical object. "calculating a square" seems to have been used to mean calculating the area of the instrument set square. The instrument is triangular, has three corners (vertices), therefore its area is calculated as half the product of base and height hence the response "*when* *calculating a square you use* $\frac{1}{2}$bh ''. Their reference to a square as *a solid*, as having *volume* or as being *three-dimensional* implies that participants incorrectly classify a square as a space object and not as a plane object.

The participants' descriptions of a cube were these:

- *has three sides (length, width and height)*
- *the third power of a number or variable*
- *the formula for finding a cube* **is L × B × H**
- *can have one side bigger than the other*
- *a solid body having six equal square sides*
- *area of a cube is* **L × B × H**
- *is five dimensional*
- *is a plane figure*
- *has four dimensions*
- *has two sides*
- *has no sides*
- *has six sides*
- *has six dimensions*
- *has four surfaces*
- *has eight sides*
- *has six corners*
- *has three faces*
- *is two-dimensional*

The description of a cube as *a solid body having six equal square sides* and as having *six sides* suggests that participants used "sides" to mean "faces". This was evident in nearly all pre-test schools (twenty-eight out of thirty). The notion that a cube *has no sides* may have been influenced by the recognition of a cube as an index and not as a geometrical object. From the response that *a cube can have one side bigger than the other*, it can be inferred that some of the participants could not differentiate between a cube and a cuboid. Evidence of some participants' inability to differentiate between a square and a cube can be seen in these responses:

- *a square has squared sides (cm^2) while a cube has tripled sides (cm^3)*
- *a cube is to a power of 3 and a square to a power of 2*
- *no difference between a square and a cube*
- *a square is a figure but a cube is an object*
- *squares are shorter in height than cubes*
- *a square is bigger in size than a cube*
- *a square has a larger volume than a cube*
- *a square is straight while a cube is curved*

Use of the words "*square*" and "*cube*" as verbs as in $a \times a = a^2$ and $a \times a \times a = a^3$ to refer to the product of mathematical operation × (multiplication) appear to have influenced some participants to think of the objects ***square*** and ***cube*** as indices.

4.72 Post-test

The post-test was administered to 1280 participants in 32 districts from 7 provinces in Kenya. The purpose of the post-test was to gauge the effect of instruction on learners who had been exposed to experimental treatments during the learning of three D. The post-test was taken about one week after instruction. The test comprised five items that tested eleven concepts and was administered under normal examination conditions in Kenya except it was not timed. The participants were allowed adequate time to be able to attempt all the items to their satisfaction or hand in their scripts when they felt they had done all they could and were unable to continue. However, they all made attempts to answer all the items that were presented to them on typed question papers. The reason for eliminating the time element was because the test was intended to measure the participants' ability to solve the problems rather than the speed at which they solved them. Enough working spaces were provided after each item but the participants were free to ask for extra papers if and whenever they needed them. The items tested six out of thirteen concepts that had previously been reported by the respondents to be causing difficulties to their learners. Post-test items are presented in appendix A.40.

The first part of item one tested the participants' ability to differentiate between a point and a line. Learners performed well in this item as 44.61% of them were able to distinguish clearly between a point and a line. Slightly more than one third (35.39%) were able to clearly explain either the concept of point or the concept of line but not both. It emerged that twenty percent of the participants could not differentiate between a point and a line. Participants were considered to posses a clear understanding of both concepts if they were able to recognize a point as zero-dimensional and a line as one-dimensional. Their recognition of a line as a set of points (or as the distance between two points) and a point as an intersection of lines would similarly demonstrate a clear understanding the concepts. Among the participants' responses that was accepted as a clear understanding the concept of concept of point is their description of a point as having ''one value of x and one value of y. Participants' description of a line as having several values of **x** and **y** on the Cartesian plane'' similarly illustrated their understanding of the concept of line. This demonstrates the potential of the Cartesian plane in building space concepts when the third (**z**) axis is included.

In their response to this item, some learners gave non-geometric descriptions of the concept of point such as:

- *a pointed tip of something.*
- *an area where an object is standing*
- *a position from which something is viewed*
- *what is placed between fractional decimals*
- *a letter used to label on a figure*
- *the direction where an object is facing*
- *the end of a sentence*
- *a dot*
- *a full stop*
- *a sharp end*
- *a certain place*
- *any dot used in writing*

These responses appear to have been influenced by participants' experiences outside the classroom and by symbols used in writing as seen in their description of a point as the "*end of a sentence*" or as ''*a letter used to label on a figure*''. Use of the word ***point*** in real life experiences seems to have been instrumental in influencing the description of a ***point*** as a position from where something is viewed. This response is triggered by learners' prior encounter with road signs directing tourists to ''points'' (sites) where tourist attractions can be viewed. For instance ''hippo point'' is used in Kenyan national parks to refer to the location of hippos in the park. The description of a point as ''*the direction where an object is facing*'' may have been influenced by use of the word point as a verb. A compass needle ''pointing'' north could be said to be ''facing'' north. The participants' description of a point as ''*what is placed between fractional decimals*'' regards the geometric point as a symbol used to combine integers with fractional decimals. Some participants incorrectly classified a point as *one-dimensional*. Others classified it as *a two-dimensional* object while a section of participants regarded ''a point'' as a static object. This is evident in their descriptions of a point "*as a stationary dot*", "*stagnant*", "*constant*", "*fixed*", and as having "*no movement*". Some learners may be lacking appropriate mathematical language to describe the concept of point as can be seen in these descriptions of a point:

- *a place that separates a line from another angle*
- *a place where two angles meet*
- *an angle where two lines meet*
- *a point is an area of contact of different edges or lines*
- *a point forms an angle*
- *a point has one letter*

These responses demonstrate that the participants lack the language with which they can verbalize concepts. Participants described a *line* as "*a place that separates a line from another* and as *an area of contact of different edges or lines"*. This seems to suggest that participants lack the technical term "***intersection***" that they could use to describe the concept of point as the intersection of two lines. The term "***intersection***" used to be introduced into learners' vocabulary by elementary set theory. Unfortunately, elementary set theory is excluded from the current secondary school mathematics syllabus. The description of a point as "*having one letter"* suggests that participants think of a point as a vertex of a geometric object, which is correct, but the letter is used as a property of the point. Again this is an indication of lack of language with which to express the concept.

Other participants described a point as having some mathematical property that is either incorrect or trivial as can be seen in these cases:

- *a point has an area*
- *a point has 360° at its centre*
- *a point has two sides joining together*
- *a point is a small circle without a radius*
- *a point is a small circle without space inside*
- *a point is an edge*
- *a point has direction*
- *a point has volume*
- *a point is folded*

Participants had a correct intuitive notion that a set of points makes up a line. However, majority of them lacked the language to express their thoughts with precision. This is evident in these descriptions of a line:

- *a continuous joint of dots*
- *a combination of a point*
- *a continuous connection of dots*
- *a mark made by drawing the point*
- *made up of two points*
- *an extension of two points*
- *a combination of a dot*
- *an extension of two points*
- *an extension of dots*
- *a series of dots joined together*
- *a group of dots joined together*
- *the distance covered by a prolonged point*
- *a chain of dots*
- *jointed dots*
- *jointed dotted lines*
- *a combination of dots*
- *an elongation of dots*
- *a progression of dots*
- *an enlarged point*
- *a continuous dot*
- *a constructed point*
- *two points in a joint*
- *a joint in two points*
- *two points in a joint*

The word "dot" appears to have been used to mean a "***point***". Participants seem to associate "***a line***" with "motion" as can be seen from these descriptions of a line:

- *a line starts from a place and moves*
- *a mark made by a moving point*
- *the flowing of continuous dots*
- *a series of moving dots*
- *a measurement of a movement*
- *a dot which is moving in a specific direction*
- *a continuous marking which moves in one direction*
- *a series of moving dots*
- *a moving dot*
- *a continuous motion*
- *a straight moving mark*
- *a continuous movement*

Other participants have conceptualized only the straight form of a line and possibly have generalized that all lines must be straight. They described a line as:

- *a straight figure running diagonally or horizontally*
- *continuous moving dots usually in a straight line*
- *jointed dots that have been extended to be straight*
- *the distance of an object which is straight*
- *a long straight drawing*
- *dots in a straight manner*
- *a straight edge*
- *a line is straight*
- *a line goes "straightly"*

The participants' thinking is restricted to the straight form of a line. This could be the effect of an instructional process that is mainly characterized by giving the definition of a concept followed by an example of the concept but does not consider various forms of a concept (in this case straight lines and curves), non examples of a concept and counter examples. Hence learners may not conceive of a curve or an arc as linear. Counter examples provide evidence that a statement is not correct. For instance a curve is a counter example of the statement "all lines are straight".

The second part of item one tested the participants' ability to differentiate between a rectangle and a cuboid. It was expected that they would recognize a rectangle as a plane object and a cuboid as a space (*three-D*) object. Responses that differentiated between the two objects in terms of the number of ''surfaces'' (phases) were accepted as a clear understanding of the identity of the objects. That means they could recognize a rectangle as having one phase and a cuboid as having five or six phases depending on whether it is open or closed. Participants who differentiated between the objects by stating the number of sides of a rectangle (four) and the number of edges of a cuboid (twelve) were also regarded to exhibit a clear understanding of the identities of both rectangle and cuboid. Another response that was accepted is the number of vertices of the two objects, four for rectangle and eight for cuboid. More than half (60.86%) of the participants were able to clearly differentiate between a rectangle and a cuboid. About one fifth (21.33%) were not able to distinguish between a rectangle and a cuboid while 17.81% had a clear concept of only one of the objects.

Participants exhibited a better understanding of the concepts of rectangle and cuboid than the concepts of point and line. This could be due to the fact that both plane (rectangle) and solid (cuboid) can be more accurately represented concretely than zero dimensional point and *one-dimensional* line. It could also be attributed to the fact that not much is said about the concepts of point and line during instruction. The latter case is supported by the revelation that participants used only one vertex of a cuboid to describe a line. They referred to vertex A (say) as line A. It may appear that little effort is made by the instructional process to break down a solid object into its constituent elements (points, lines and planes). Such analysis of solids would enable learners to consider vertices and edges of solid objects as points and lines respectively. Some participants were found to have developed incorrect conceptions of the object rectangle as presented in these descriptions of a rectangle:

- *a figure with a flat base, height and length*
- *a solid body which has four sides*
- *a rectangle has base and height*
- *a cube with sides which are unequal*
- *a solid which has two dimensions*
- *a three sided figure and has three angles*
- *a figure with six surfaces of different length*
- *a rectangle does not have 90°*
- *a solid body that has two sides*
- *a solid object*
- *a rectangle is two sided*
- *a rectangle has one side*
- *a scalar figure*
- *a solid with four sides*
- *an angle with faces*
- *a rectangle cannot be felt*
- *a three dimensional figure*
- *a rectangle has three edges*
- *an object with six sides of which only two sides opposite each other are equal*
- *a three sided figure with three angles that add up to 180°*
- *a figure which has three sides; length, width and height*
- *a rectangle has 14 vertices, 6 faces, and 8 edges*
- *an angle whose sides are opposite to each other*
- *a rectangle cannot exist in reality*
- *a rectangle has length and weight*

The response that *''a rectangle cannot be felt''* seem to suggest that participants were not able to relate the concept of ''rectangle'' to any tangible object that exists in real life. Possibly, they were not able to think of a rectangle as an object that can be found in real life situations. To them, a rectangle may be regarded as an abstract object of learning that is restricted to the classroom situation. Descriptions of a rectangle such as "a figure with six surfaces of different lengths" seem to suggest that participants thought of a rectangle as a cuboid.

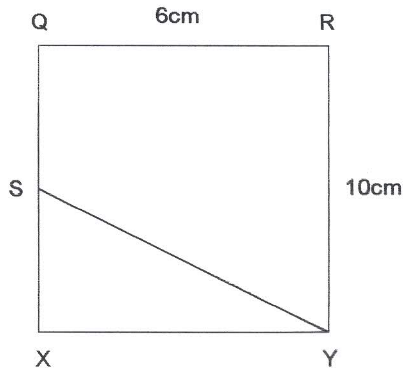
Participants' responses also showed deficiency in the understanding of the concept of cuboid. This was evident in their description of a cuboid as:

- *a rectangular figure with six sides*
- *a four sided figure with volume*
- *a cuboid is cubical in shape*
- *a rectangle in three dimensions*
- *a six-sided figure with both ends closed*
- *a six sided figure*
- *a four sided figure*
- *a cuboid has eight sides*
- *a cuboid has three sides*
- *a cuboid has three faces*
- *a cuboid has length weight and height and exist in reality*
- *an object made of six sides, all of which are equal*
- *a figure with six surfaces of equal length*
- *a shape made by joining two rectangles*

The response that a cuboid is *"a six sided figure"* seem to suggest that the participants are using ''sides'' to mean ''faces''. It also emerged to be the case that some participants were not able to differentiate between a rectangle and a trapezium. In diagram 6a, they treated line SY as equal to 6cm. They regarded lines QR and SY as opposite sides of a rectangle.

Diagram 6a   Difficulties in differentiating between a rectangle and a trapezium

Diagram 6a



Item two tested the participants' knowledge of three concepts. The first part tested their ability to name the angle between two parallel planes from a plane drawing of a space object (the top of a desk). Participants found it rather difficult to identify the angle between parallel planes. Many of them (84.69%) scored zero in this part of the item with some referring to the angle as "a skew angle". They were completely unable to recognize the parallel position of the planes. Only 15.31% were able to recognize the two planes as parallel and hence the angle between them is 0° or 180°. More than half of the participants regarded the parallel planes as perpendicular and gave the angle between the planes to be 90°. It appears there is a link between this response and the response in item eight of the pre-test where 37.29% of the pre-test candidates named parallel lines when they were asked for perpendicular lines. It seem to be the case that some learners are not able to distinguish between the terms "parallel" and "perpendicular" and therefore use "parallel" to mean "perpendicular" and vice versa. It may also be the case that, to the participants, parallel means "perpendicular" and perpendicular means "parallel".

The second part of item two tested the participants' understanding of the concept of projection. They were required to calculate the projection of a line on a horizontal plane. Approximately one third of the participants (32.82%) were able to calculate the projection while 42.74% were not. Slightly less than one quarter (24.4%) used a correct solution strategy but were not able to complete the solution due to their inability to apply the Pythagorean theorem. Participants who used the behaviourist approach and plane representations only found it difficult to understand the concept of projection. Instead of calculating the projection of a line (WY in item two) some calculated the area of a triangle (WXY), some calculated lengths of lines (such as XY and MY) that are not the required projection. Others treated WY as its own projection. The third part of item two required participants to calculate the angle between a line and a plane. Nearly one third (31.10%) were able to identify and calculate the angle while 58.83% could neither calculate nor identify the angle between a line and a plane. About one tenth (10.07%) identified the required angle, used a correct solution strategy but could not apply trigonometry ratios to complete the solution. Participants treated lines WV and VQ (page 197) to be equal in length despite the fact that the two lines form the hypotenuse and base of a right angled triangle respectively. The height of the triangle was clearly given as 10cm. Participants also found it difficult to recognize lines such as QY, which do not represent the edges of an object.

Item three required participants to calculate the angle between two lines from a plane drawing of a *three-D* object. Just over one quarter of the participants (26.10%)

were able to calculate the angle between two lines while 30.16% could not even identify the angle between the two lines. The rest (43.74%) had a correct notion of what was to be done but could not complete the calculation because of their inability to apply Pythagorean theorem and trigonometric ratios especially the sine ratio. Participants who failed to identify the angle between the two lines experienced difficulties in correctly interpreting the problem situation. As a result, they treated the hypotenuse and the base of a right-angled triangle as having the same length. The error resulted in incorrect lengths used to calculate the required angle. Many participants regarded Lines FK and KG to be equal in length. This could be attributed to the participants' inability to accurately read information from a plane diagram of a *three-D* object. Line FK being the hypotenuse of a right angled triangle and meeting the base (KG) at an angle of 30° cannot be equal to the base of the same triangle (KFG). Participants neither recognized the angle between the two lines nor did they recognize their relationship as base and hypotenuse of triangle KFG.

Item four consisted of three parts. The first part required participants to calculate the angle between skew lines. Participants had difficulties recognizing the angle between skew lines WY and QR as the angle formed between line WY and the projection of line QR on the plane WXYZ. Consequently, they used inappropriate triangles to calculate the required angle. Instead of using triangle PQR (or WXY), to calculate the angle between the lines WY and QR, some participants used triangles QRW and QRY. Others responded by saying that there is no angle between the lines because the lines do not meet or because the lines do not have a common point. The responses seem to suggest that participants had difficulties in understanding the concept of "*projection*". The *angle* between "lines that do not meet" present a conflicting situation to the learners. About one third (32.74%) could neither calculate nor identify the angle between skew lines while 38.83% were able to calculate the angle. The rest (28.42%) of the participants used productive solution paths but were not able to apply the Pythagorean theorem and trigonometric ratios hence did not complete the solution. In general, participants found it difficult to recognize skew lines from a diagram of a *three-D* object because the diagrams present the lines as intersecting.

The second part of item four required participants to calculate the angle between a line and a plane. The item proved to be the most difficult for the participants. Only 5.81% were able to calculate the angle successfully. Majority of the participants (86.10%) could neither calculate nor identify the angle. Others (8.09%) were not able to complete the solution because they were unable to apply Pythagorean theorem and the tangent ratio. The third part of item four required participants to calculate the angle between two planes. Only 14.46% were able to successfully complete the solution while 82.66% could neither calculate nor identify the angle between two planes. A small percentage (2.88%) of the participants were able to identify the required angle but could not complete the solution due to their inability to apply trigonometric ratios.

Item five tested the participants' knowledge of nets. They were presented with a drawing of a net of a tetrahedron. The first part required them to use the net to count the number of edges of the tetrahedron. More than half (65.55%) obtained the correct number of edges of the tetrahedron from its net. About a third (34.46%) could not obtain the correct number of edges of the tetrahedron from its net. Those who were not able to obtain the correct number of edges counted and gave the number of lines on the net (9) for the number of edges of the tetrahedron. The second part of item five required the participants to use the net of a tetrahedron to obtain the number of vertices of the tetrahedron. More than half (63.75%) were able to accomplish the task while 36.25% could not. Those who did not obtain the correct number of vertices of the tetrahedron counted and gave the number of vertices of the net for the vertices of the solid. Participants who gave incorrect responses for the number of edges and the number of vertices were not able to distinguish clearly between the two concepts. Nearly half of the incorrect responses involved swapping of the responses for edges and vertices. Part three of item five required participants to calculate the length of one of the edges of a tetrahedron from its net. Less than half (41.57%) were able to use the net to calculate the length of the edge. Slightly more than half (52.03%) were completely unable to use the net to calculate the length of an edge of the net. They were also unable to identify from the net, the required edge of the solid. Some (6.4%) used a correct solution path but could not reach the end of the solution because of their inability to calculate the square root of 164. In their attempt to calculate the length of the solid from its net, some participants applied Pythagorean theorem where it does not apply. They used the theorem in a triangle in which all the interior angles are less than 90°. They also used the theorem in a triangle in which two interior angles are acute and the third one is obtuse. On the net, all the lines and vertices of the solid are represented on the same plane. This may have influenced some participants to think that the tetrahedron whose net was presented in the problem is *two-dimensional.*

For the participants who used plane representations only (treatments T1 and D1), the constructivist approach emerged to be superior to the behaviourist approach. Participants who received treatment T1 did better than those who received treatment D1 in eight out of twelve tasks. The constructivist group performed better in tasks that required participants to: differentiate between a point and a line; differentiate between plane objects and three D objects; name the angle between parallel planes; calculate; the projection of a line, the angle between a line and a plane, the angle between two lines and the angle between two planes. For participants who used concrete representation of solid objects in addition to plane representations (treatments T2 and D2), constructivist approach was superior in all the twelve tasks. In all, participants who used the constructivist approach consistently did better than those who used the behaviourist approach in eight out of twelve tasks irrespective of the materials used. Tasks involving the angle between a line and a plane, angle between parallel planes and angle between two intersecting planes proved to be the most difficult. More than 82% of the participants (table 9) scored zero in such tasks.

A summary of the proportion of participants who scored zero in each of the post-test tasks is presented in table 9.

Table 9: Proportion of participants with zero scores in post-test tasks

Concept tested	D1	T1	D2	T2	Overall %
Angle between a line and a plane	90.94	83.44	86.56	83.44	**86.10**
Angle between parallel planes	91.56	82.19	87.19	77.81	**84.69**
Angle between two planes	90.00	78.75	80.62	81.25	**82.66**
Angle between a line and a plane	77.19	45.94	58.13	54.06	**58.83**
Lengths from a net of a solid	52.81	56.25	52.81	46.25	**46.02**
Projection of a line on a plane	50.31	45.94	38.13	36.56	**42.74**
Angle between skew lines	48.75	30.94	42.50	25.63	**36.96**
Net of a solid (Vertices)	40.00	43.44	31.25	30.31	**36.25**
Net of a solid (edges)	34.44	42.19	35.00	27.19	**34.71**
Angle between two lines	27.50	66.56	16.25	10.31	**30.16**
Differentiating 2D and 3D object	26.25	23.14	20.00	15.94	**21.33**
Differentiating 0D and 1D object	19.37	16.24	26.25	15.94	**21.33**

Source: Field data

It can be observed from table 9 that from the group that received treatment T2, fewer participants scored zero compared to those that received treatments T1, D1 and D2. This can be attributed to the superiority of the combined effect of the constructivist approach and use of manipulative materials to represent space concepts that resulted in a better performance. It is also evident from table 9 that the concept of *angle* seems to be quite subtle for learners if a plane is involved. More than half of the participants scored zero in tasks that required them to calculate the angle between a line and a plane or between two planes.

4.80 Statistical analysis

Descriptive statistics, statistical tests of significance (the F-test) and non-parametric tests were used to analyse and explain the post-test scores. The scores were sorted into two categories; the pre-test (P) and non-pre-test (NP) then analysed using each of the three techniques.

4.81 Descriptive statistics

Comparisons were made on the mean posttest scores to determine the effect of the experimental treatments, D1, T1, D2 and T2 on both pretest and non-pretest categories of participants. The mean percentage scores and the difference in mean scores between the two categories are displayed in table 10.

Table 10: Mean Post-test Scores

Treatment	Pre-test (P)	Non-Pre-test (NP	Differences (NP-P
D1	30.33%	36.41%	6.08%
T1	38.38%	36.28%	-2.10%
D2	39.85%	43.64%	3.79%
T2	45.01%	52.67%	7.66%

Source: Field data

Experimental groups that received treatment T2 had superior mean scores (45.0% and 52.7% for the pretest and non-pretest categories respectively) compared to the corresponding control groups that received treatment D2 (39.8% and 43.4% for the pretest and non-pretest category respectively). Except for the experimental groups that received treatment T1, the pretest categories had lower mean scores than the non-pretest category. In the pretest category, the experimental group that received treatment T1 had a better mean score (38.38%) than the corresponding control group that received treatment D1 (30.34%) in the pretest category. There was however no significant difference in the mean scores (36.28% and 36.44%) between non-pretest groups that received treatments T1 and D1. The results provide evidence that learners who used the constructivist approach obtained superior mean scores than their counterparts who used the traditional behaviourist approach. The results also provide evidence that participants, who used non-plane representation of space concepts in addition to plane representation, had a better performance than participants who used only plane representations. The pretest seems to have had a negative impact on the performance of the participants who were treated to it. This was observed in three out of four treatment groups in which participants who sat the pretest attained lower mean marks compared to the groups that did not sit for the pretest. The difference ranged between 3.79 and 7.66 percentage points.

4.82 Test of significance (the F-test)

The F-test and the SPSS computer software were used to measure the main effects of two independent variables, instruction and materials and the main effects of their interaction. The analysis was done separately for the pretest and non-pretest participants. The two independent variables were used to describe the computations for analysis of variance with a factorial design. The analysis involved a 2 x 2 factorial design with different participants in each group. Each of the independent variables, instruction and materials, consisted of two levels. Instruction had two levels: the behaviourist approach and the constructivist approach. Similarly, materials had two levels: manipulative materials and plane representations. Manipulative materials level involved both concrete representations and plane representations. The factorial design permitted the evaluation of three effects. The main effect of the type of instruction, the main effect of the type of material and the interaction of the type of instruction and type of materials. The main effect of instruction was whether the constructivist approach is superior to the behaviourist approach. The main effect of materials was whether the participants who used manipulative materials scored differently from those who used non-manipulative materials only. The interaction of the two variables examined whether the effect of the method of instruction was different depending on the type of material used. Analysis of the results for the pretest category of participants is summarized in table 11. Variables in table 11 are the method of instruction and type of materials.

Table 11: Summary of F-test Analysis (Pre-test category)

Source of Variation	Sum of Squares	Degrees o Freedom	Mean Square	F
Method	6976.2	1	6976.2	14.58
Materials	10312.1	1	10312.1	21.55
Method × Material	309.4	1	309.4	0.65
Error	304295.7	636	478.5	
Total	321893.4	639		

Source: Field data

The critical value of F for 1 and 636 degrees of freedom at 0.05 is 3.84. The calculated values of F are 14.58, 21.55 and 0.65 for method, material and interaction respectively. Since the calculated F is larger than the critical F for the main effects, the main effects were significant at the 0.05 level. The constructivist approach therefore emerged superior to the behaviourist approach and the candidates who used both concrete materials and plane representations had a better performance than those who used plane representations only. The interaction effect was however insignificant in the case of the pretest category of participants. A summary for the analysis of the results of the candidates who were not pre-tested is presented in table 12.

Table 12: Summary of F-test Analysis (NP category)

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F
Method	3404.025	1	3404.025	8.07
Materials	21925.806	1	21925.025	51.99
Method × Material	3600.506	1	3600.506	8.54
Error	268216.638	636	421.724	
Total	297146.975	639		

Source: Field data

The critical value of F for 1 and 636 degrees of freedom at 0.05 is 3.84. As was the case with the pretest candidates, the calculated values of F are all larger than the critical F, hence the main and interaction effects are significant at the 0.05 level. The constructivist approach emerged superior to the behaviourist approach and the participants who used both concrete representations and plane representations had a better performance than those who used plane representations only. The contribution of materials used appears to have been more pronounced than the contribution of the methods used.

4.83 Non-parametric test (Kruskal-Wallis)

The Kruskal-Wallis test, a non-parametric extension of the Mann-Whitney-Wilcoxin test to K mutually independent random samples was used to analyze the posttest scores. The test revealed that the NP category of participants had better mean ranks for treatment groups D1, D2 and T2. Judging performance by the mean ranks, the NP category displayed a superior performance (in comparison to the P category) in three out of four treatment groups. For the NP category, there was a significant difference in performance between participants who used models together with plane representations irrespective of the approach used for instruction. The constructivist approach was superior to the behaviourist approach for participants who used models and plane representations. However, for participants who used plane representations only, there was insignificant difference in mean ranks between treatment groups that used the behaviourist and constructivist approaches. Mean ranks for the NP category are shown in table 13a.

Table 13a: Mean Ranks (NP category)

Treatment	N	Mean Rank
D1	160	274.60
T1	160	267.19
D2	160	327.63
T2	160	412.58
Total	640	

Source: Field data

Differences in mean ranks attributed to type of materials used were significant in both cases. Differences attributed to type of approach used was significant in the case of participants who used concrete representations (T2 and D2) and insignificant in the case of participants who used plane representations only (T1 and D1). That suggests that the contribution of materials in enhancing the understanding of *three-D* concepts was greater than the contribution of the approach used. Differences in mean ranks and the variation source are shown in table 13b.

Table 13b: Differences in Mean Ranks (NP category)

Treatment	Difference	Source of Variation
D2 - D1	53.03	Material
T2 - T1	145.39	Material
T1 – D1	-7.41	Approach
T2 – D2	84.95	Approach

Source: Field data

For the pretest category, participants treated to the constructivist approach did better than those who were treated to the behaviourist approach. Participants who used concrete representations together with plane representations also did better than those who used plane representations only. Mean ranks for the P category are displayed in table 14a.

Table 14a: Mean Ranks (Pre-test category)

Treatment	N	Mean Rank
D1	160	254.50
T1	160	301.70
D2	160	345.02
T2	160	380.78
Total	640	

Source: Field data

Differences in mean ranks attributed to type of materials used were significant in both cases and were better than the differences attributed to the type of approach used for instruction. Differences attributed to type of approach used were also significant in both cases but were inferior to the differences attributed to the type of material used. Thus the contribution of materials in enhancing the understanding of *three-D* concepts was clearly greater than the contribution of the approach used. Differences in mean ranks and the variation source are shown in table 14b.

Table 14b: Differences in Mean Ranks (Pre-test category)

Treatment	Difference	Source of Variation
D2 - D1	90.52	Material
T2 - T1	79.08	Material
T1 – D1	47.20	Approach
T2 – D2	35.76	Approach

Source: Field data

The results of analysis by Kruskal-Walis test are summarized in table 15.

Table 15: Results of Non-Parametric Test (Kruskal-Wallis test)

Category	Chi-Square (χ^2)	Degrees of Freedom	Asymptotic Significance
Pre-test (P)	41.88	3	0.00
Non-Pre-test (NP)	63.11	3	0.00

Source: Field data

The critical value of $\chi^2(.95, 3)$ is 7.81. Calculated values of $\chi^2(.95, 3)$ for the pretest and non-pretest categories are 41.882 and 63.109 respectively. The null hypothesis that the samples (treatment groups) have the same median is rejected and the alternative hypothesis that the medians of the two samples are significantly different is accepted. On the basis of the results, two conclusions can be drawn. One, a combination of concrete and plane representations of *three-D* concepts led to a better performance than plane representations only. Two, participants who used the constructivist approach did better than those who used the behaviourist approach.

4.90 Conceptual Difficulties

The conceptual difficulties reported and discussed in this section include those that involve the concepts of dimension, Pythagorean theorem, angle, trigonometry, square roots, squares and logarithms. A part from the concepts of dimension and angle, the difficulties are encountered with basic concepts that learners require to able to calculate distances and angles involving *three-D*. Basic concepts are concepts that are prerequisite to the learning of other concepts. In this case, they are concepts required for the learning of *three-D* concepts. Learners build on basic concepts to acquire higher order concepts. Basic concepts constitute a significant part of the mathematical demands required to learn *three-D* concepts and to solve problems involving *three-D*. Concepts required for the learning of *three-D* comprise lower order geometry concepts and other skills and concepts that are not necessarily geometry, but are required for the learning of *three-D*.

4.91 Dimension

It emerged to be the case that participants lack a relational understanding of the concept of dimension. They were not able to classify certain geometric objects according to their dimensions. Some of the classifications were correct but based on invalid premises. For instance a cuboid was classified as *three-dimensional* because "*it has three planes*''. While it is true that a cuboid is three dimensional, it is not *three-dimensional* because it has three planes. It actually has more than three planes. Incorrect classifications such as ''*a rectangle is one dimensional structure thus it has length and width*''; a point is where two lines meet and is "*two dimensional*'' were however based on correct geometrical properties of the objects. Table 16 shows some of the incorrect classifications.

Table 16: Classification of geometric objects

Dimension	1D	2D	3D
Object	Point Rectangle Cuboid	Point Line Cube Cuboid	Point Rectangle

Source: Field data

The participants' reasons for the incorrect classifications of objects reveal their poor conception of dimension. Their difficulties with the concept of dimension could partly be blamed on an inconsistent manner in which the concept is treated in the syllabus. The term dimension appears late i.e. when learners begin to learn about *three-D* but not earlier when they are dealing with lines and planes. The syllabus is silent about the dimension of a point, a line and a plane yet it refers to solid objects as *three-dimensional*. This creates a gap in the learners' understanding of the concept of dimension. The inconsistency means that learners may not be aware that a plane is *two-dimensional*, a line, *one-dimensional* and a point, *zero dimensional*. They may incorrectly conclude that zero, one and two dimensions do not exist-especially if instruction also ignores or does not mention the existence of the dimensions. As if that is not enough, the syllabus presents topics dealing with points, lines and planes (e.g. angles, plane figures and surface area of regular solids) before the topic ''common solids''. It does not seem pedagogically advisable to teach the surface area of a pyramid before learners are conversant with the pyramid itself. It is also worth noting that from the learner's experience, the solid is encountered before the point, line and plane. It would seem logical for instructional sequence to present "common solids" before presenting topics that deal with planes lines and points.

4.92 Pythagorean theorem

Some participants experienced difficulties that suggest they were not certain of what the Pythagorean theorem is. They could neither state and or apply the theorem correctly. The theorem was used in situations where it does not apply. It was used in triangles that are not right angled. Others stated and used incorrect relationships between the sides of right angled triangles such as:

$$base^2 = hypotenuse^2 - base^2.$$

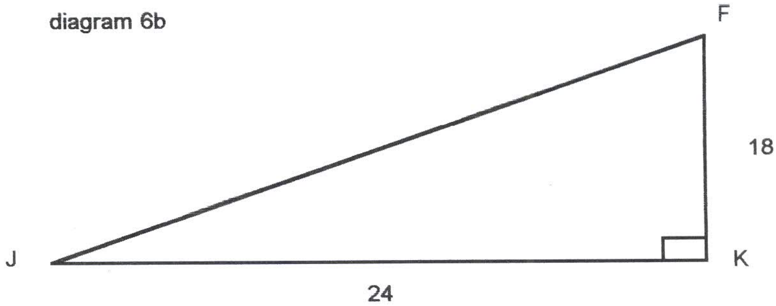
$$hypotenuse^2 = base^2 - height^2.$$

$$base = hypotenuse^2 + height^2.$$

$$base^2 = height^2 - hypotenuse^2.$$

In reference to the right angled triangle in diagram 6b, a participant stated and used the theorem incorrectly as follows:

Diagram 6b Use of the Pythagorean theorem where it does not apply



$$JF^2 = FK + KJ$$
$$= 18 + 24$$
$$= \frac{42}{2}$$
$$JF = 21.$$

This participant expressed the square of the hypotenuse as a sum of the height and the base instead of the sum of the square of the height and the square of the base. To her, getting the square root means division by two as seen in the division of 42 by 2. Others appeared mixed up between the concept of angle and the Pythagorean theorem and used "equations" such as; *angle FJG* = 18^2+24^2 in their effort to calculate the angle between two lines FJ and JG (item three of the posttest).

4.93 Angle

Participants' difficulties with angles include inability to identify the angle between two planes and the angle between a line and a plane. Some participants were also unable to use vertices to name an angle. They used two vertices instead of three, to name an angle. Statements such as *angle WY*, *angle VX*, $\tan YW = \frac{8}{6}$, *angle x* =

sin50° = *130°* were used to describe an angle or a trigonometry ratio. In the last case (and other similar ones), participants seem to have been using the unit circle to obtain angle x but experienced a hitch due to the inverse relation between sin50° and angle x. They did not recognize the impossibility of angle x being equal to both sin50° and 130°. As was the case with many others, the participant experienced difficulties in expressing an angle in terms of its sine, cosine or tangent. They have difficulties understanding and using the inverse form of the function $\sin\theta° = a$. Due to difficulties in identifying angles, participants were not able to extract the relevant triangles they require to calculate angles between planes, between lines and between lines and planes. For the same reason, they found it difficult to calculate the distance between two points on a three dimensional object. Teacher participants who used plane representations only reported difficulties in practically demonstrating to their learners, the angle between two planes in the absence of models. This experience supports the view that concrete representations should be used to present mathematical concepts to learners.

4.94 Trigonometry

Participants' difficulties with trigonometry involve definition and application of trigonometric ratios, use of inverse trigonometric relations, recall and application of sine and cosine rules. In some cases they were unable to distinguish between the sine ratio and the sine rule on one hand and between the cosine ratio and the cosine rule on the other. The rules were used in situations where the ratios ought to have been used. Given the sine, cosine or tangent of an angle, participants were often unable to read the value of the angle from the corresponding table of the ratio. A few cases extracted from the participants' posttest scripts are these:

$\tan\theta^\circ = 1.33$	$\tan\theta^\circ = 1.33$	$\tan\theta^\circ = 1.33$	$\tan\theta^\circ = 0.31$	$\tan\theta^\circ = 1.33$
$\theta^\circ = 18.3^\circ$	$\theta^\circ = 45.9^\circ$	$\theta^\circ = 36.9^\circ$	$\theta^\circ = 16.71^\circ$	$\theta^\circ = 18.43^\circ$
$\tan\theta^\circ = 0.2$	$\tan\theta^\circ = 1.2$	$\tan\theta^\circ = 0.2$	$\tan\theta^\circ = 1.2^\circ$	$\tan\theta^\circ = 0.333$
$\theta^\circ = 1.12^\circ$	$\theta^\circ = 48.3^\circ$	$\theta^\circ = 14.47^\circ$	$\theta^\circ = 52.15^\circ$	$\theta^\circ = 1.9^\circ$
$\tan\theta^\circ = 1.33$	$\tan\theta^\circ = 0.2$	$\tan\theta^\circ = 0.2$	$\tan\theta^\circ = 0.2$	$\tan\theta^\circ = 0.2$
$\theta^\circ = 87^\circ 18'$	$\theta^\circ = 1.13^\circ$	$\theta^\circ = 1.15^\circ$	$\theta^\circ = 50.19^\circ$	$\theta^\circ = 1.14^\circ$
$\tan\theta^\circ = 1.334$	$\tan\theta^\circ = 1.333$	$\tan\theta^\circ = 1.2$		
$\theta^\circ = 56.14^\circ$	$\theta^\circ = 87.3^\circ$	$\theta^\circ = 50.3^\circ$		

Some participants were not able to transform equations of the form $\tan\theta^\circ = y$ into the form $\theta^\circ = \tan^{-1}y$. These were seen in responses such as $\tan\theta^\circ = 37^\circ$ where participants accurately calculated a numerical value of the ratio but were not able to proceed and obtain the angle that corresponds to the calculated ratio. Despite accurate calculations of $\tan\theta^\circ$, participants were not able to write values of θ for $\tan\theta^\circ = 0.8333$, $\tan\theta^\circ = 1.3333$, and $\tan\theta^\circ = 0.6360$. This suggests an underlying difficulty in rewriting a given equation in terms of the independent variable. Either the problem is that of algebra where participants are not able to make θ the subject of equations like $\tan\theta^\circ = y$ or the relation between θ° and $\tan\theta^\circ$ in equations of the form $\tan\theta^\circ = y$ is not clear to them.

Evidence of participants' use of incorrect definitions of the tangent ratio can be seen in these cases:

$$\text{Tan}\theta = \frac{\text{opposite}}{\text{hypotenuse}} \qquad \text{Tan}\theta = \frac{\text{adjacent}}{\text{opposite}} \qquad \text{Tan}\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$$

$$\text{Tan}\theta = \frac{\text{opposite}}{\text{hypotenuse}} \qquad \text{Tan}\theta = \frac{\text{hypotenuse}}{\text{opposite}}$$

Many of the incorrect definitions expressed the tangent as a ratio of the opposite side to the hypotenuse.

Difficulties with the sine ratio could mainly be attributed to the definition of the ratio. Participants did not exhibit a clear understanding of the ratio as reflected in these definitions:

$$\text{Sin}\theta = \frac{\text{opposite}}{\text{adjacent}} \qquad \text{Sin}30° = x^2+18^2$$

$$\text{Tan}\theta = \frac{\text{adjacent}}{\text{hypotenuse}} \qquad \text{Sin}\theta = \frac{\text{hypotenuse}}{\text{adjacent}}$$

The incorrect definitions led to unrealistic values such as $sin\theta = 1.5$, and $sin90° = 1.8$. Some participants stated the sine rule correctly but did not make the correct substitutions. They substituted the angle instead of the sine. Evidence of incorrect substitutions into the sine rule can be seen in this case that was popular among many participants:

$$\frac{k}{\text{SinK}} = \frac{g}{\text{SinG}}$$

$$\frac{k}{30} = \frac{30}{90}$$

The error in the substitutions leads to an impossible interpretation that sinK° = 30 and SinG° = 90. It also emerged that participants had difficulties defining the cosine ratio. They defined the ratio as:

$$\text{Cos}\theta = \frac{\text{opposite}}{\text{adjacent}}, \qquad \text{Cos}\theta = \frac{\text{opposite}}{\text{hypotenuse}}, \qquad \text{Cos}\theta = \frac{\text{hypotenuse}}{\text{opposite}}$$

Computational tedium proved to be a problem for many participants who attempted to use the cosine rule. They stated the rule correctly, made the correct substitutions but were unable to cope with computational drudgery that was to follow and gave up after making a number of computational errors.

Learners' difficulties with trigonometric ratios could be attributed to the situations under which the definitions were learned. It may be the case that definitions are presented to learners in a final form without a significant role in the definition process played by the learners. It also appears to be the case that the definition is presented using a right-angled triangle that is in a specific (fixed) position. The understanding of the ratios is then reinforced by examples that assume the same position as the instances where the definitions are given. Confusion reigns in the learners' minds when a problem situation requires the use of a right-angled triangle in a different orientation from that used in the definition and examples. The change in orientation results in lose of track of the identity of the sides of the triangle. Consequently, learners are not able to identify the sides of the triangle as *adjacent*, *opposite* or *hypotenuse*. Participants defined the sine of angle FJG in triangle JGF (diagram 7a) that is right angled at vertex G as $\frac{\text{JG}}{\text{JF}}$. This is the sine of the angle at F and is clearly seen when the triangle is as in position J′G′F′ (diagram 7b) or J”G”F” (diagram 7c). The learning of the ratios does not appear to have been generalized.

Diagram 7 Difficulties with the sine ratio

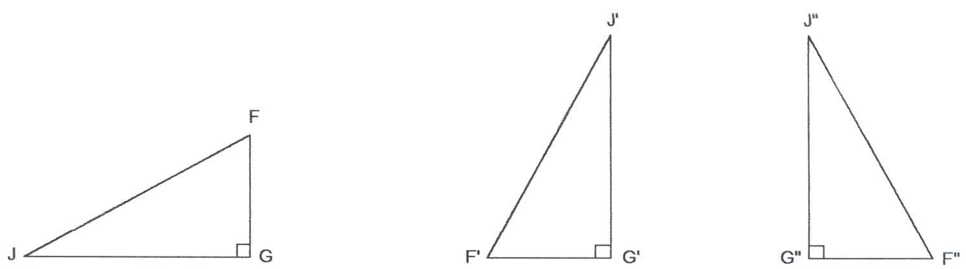


Diagram 7a Diagram 7b Diagram 7c

In triangle ABC (diagram 8), participants defined the cosine of the angle at C as $\frac{\text{AB}}{\text{BC}}$. This definition seems to suggest that participants are of the view that the cosine of an angle is the ratio of the side (*adjacent* or *opposite*) of the triangle that is horizontal, to the *hypotenuse*.

Diagram 8   Difficulties with the cosine ratio

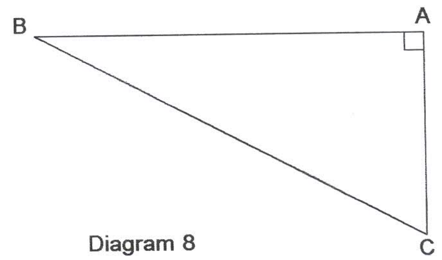


Diagram 8

It may also be the case that the terms *adjacent*, *opposite* and *hypotenuse* as they refer to a right-angled triangle is not clear to the learners. A clear understanding of the ratios may still be elusive to many participants who gave values such as $\sin\theta = \frac{10}{8}$. Values larger than one were given for both sine and cosine ratios.

4.95 Square roots

Participants' difficulties with square roots can be placed in three categories. One category experienced difficulties with reading tables of square root values. Two tables for square roots are available to learners in the mathematical tables. One table displays the square roots for numbers from 1 to 10 while the other table gives square roots of numbers from 10 to100. Some participants read the table for square roots of numbers from 1 to 10 when they needed the roots of numbers between 10 and 100. Others read the table for numbers from 10 to 100 when they needed the roots of numbers between 1 and 10. Others used the table of squares instead of the table of square roots. Use of inappropriate tables lead to results such as

$\sqrt{164}$ =2.69 where the participant read the square of 1.64 and wrote it for the square root of 164.

Below is an extract from the script of a participant who was calculating the length of a line.

$$JF^2 = 24^2+18^2$$

$$= 1.55+1.34$$

$$= 2.89$$

$$JF = \sqrt{2.89}$$

$$= 5.4$$

In this solution, the participant summed the square roots of 2.4 and 1.8 and obtained 2.89 instead of squaring 24 and 18. In third line of the solution, a table for the roots of numbers from 10 to 100 was used to obtain the square root of 2.89. The result (5.376), which is the square root of 28.9, was then rounded up to give 5.4. This demonstrates lack of a clear understanding of the operations ''square root'' and ''square''. Participants do not seem to know when to square and when to obtain the square root. It also demonstrates their inability to use tables of squares and square roots. Other similar cases that demonstrate the participants' difficulties with using tables in their effort to calculate square roots are these:

$\sqrt{100}$ = 3.162 read the root of 10. $\sqrt{136}$ = 3.688 read the root of 13.6.

$\sqrt{109}$ = *3.302* read the root of 10.9. $\sqrt{164}$ = *4.05* read the root of 16.4.

$\sqrt{164} = 1.281$ read the root of 1.64. $\sqrt{164} = 2.69$ read the square of 1.64.

$\sqrt{360} = 6$ read the root of 36. $\sqrt{576} = 7.6$ read the root of 57.6.

$\sqrt{324} = 5.7$ read the root of 32.4 $\sqrt{899} = 9.5$ read the root of 89.9.

$\sqrt{250} = 5$ read the root of 25. $\sqrt{144} = 3.75$ read the root of 14.4.

$\sqrt{150} = 3.873$ read the root of 15. $\sqrt{900} = 9.487$ read the root of 90.

$\sqrt{9000} = 9.487$ read the root of 90. $\sqrt{900} = 3.000$ read the root of 9.

$\sqrt{1300} = 3.6$ read the root of 13. $\sqrt{2500} = 5$ read the root of 25.

$\sqrt{1500} = 3.873$ read the root of 15 $\sqrt{800} = 8.944$ read the root of 80.

$\sqrt{27.01} = 1.6$ read the root of 2.701 $\sqrt{91} = 3.017$ read the root of 9.1.

$\sqrt{164} = 12.77$ read the root of 1.63 then multiplied by 10.

$\sqrt{500} = 7.071$ read the square root of 50.

It seems to be the case that participants had difficulties using the factor method to obtain square roots of three digit numbers. Their difficulties with square roots were compounded by the fact that they lacked a clear understanding of the process involved in the factor method. The method requires that they express the number (**N**) whose second roots are to be calculated as a product of two factors **a** and **b** where **a** > 0 and **b** is a positive integer multiple of 10. They should then obtain the roots of both **a** and **b** and multiply the two roots together to obtain the root of **N**.

However, participants obtained the root of a only then multiplied the result by **b** suggesting that the problem with the factor method is the inability to apply the exponential law $\sqrt[n]{(a \times b)} = \sqrt[n]{a} \times \sqrt[n]{b}$ involving rational exponents. They interpret the law as $\sqrt[n]{(a \times b)} = \mathbf{b} \times \sqrt[n]{a}$ Evidence of these difficulties from extracts of the participants' posttest scripts are given in table 17 together with possible explanations of the solvers' actions that resulted in the error.

Table 17 Square roots and the factor method

Participants' Responses	Possible explanations
$\sqrt{136} = 36.88$	$\sqrt{136} = \sqrt{(13.6 \times 10)} = 3.668 \times 10 = 36.88$
$\sqrt{243} = 49.30$	$\sqrt{243} = \sqrt{(2.43 \times 10)} = 4.93 \times 10 = 49.3$
$\sqrt{900} = 94.87$	$\sqrt{900} = \sqrt{(90 \times 10)} = 9.487 \times 10 = 94.87$
$\sqrt{900} = 300$	$\sqrt{900} = \sqrt{(9 \times 10^2)} = 3 \times 10^2 = 300$
$\sqrt{9000} = 300$	$\sqrt{9000} = \sqrt{(9 \times 10^3)} = 3 \times 1000 = 3000$
$\sqrt{90} = 30$	$\sqrt{90} = \sqrt{(9 \times 10)} = 3 \times 10 = 30$
$\sqrt{719} = 84.8$	$\sqrt{719} = \sqrt{(71.9 \times 10)} = 8.479 \times 10 = 84.79 \approx 84.8$
$\sqrt{164} = 40.50$	$\sqrt{164} = \sqrt{(16.4 \times 10)} = 4.05 \times 10 = 40.5$
$\sqrt{164} = 128.1$	$\sqrt{164} = \sqrt{(1.64 \times 10^2)} = 1.281 \times 10^2 = 128.1$
$\sqrt{89} = 29.3$	$\sqrt{89} = \sqrt{(8.9 \times 10)} = 2.983 \times 10 = 29.83$
$\sqrt{250} = 50$	$\sqrt{250} = \sqrt{(25 \times 10)} = 5 \times 10 = 50$
$\sqrt{2500} = 500$	$\sqrt{2500} = \sqrt{(25 \times 100)} = 5 \times 100 = 500$

Source: field data

Use of logarithms to calculate square roots also proved to be an uphill task for a section of the participants. Their difficulties with using logarithms to calculate square roots can be seen in this solution that shows a participant's effort to calculate the length of a line:

$$\sqrt{1200} = WY$$

$$1.2 \times 10^3 \Rightarrow 3.0792$$

$$\times \quad 2$$

$$6.1584$$

$$= 6.16$$

The solution exposes lack of understanding of the process involved in use of logarithms to compute square roots. The participant correctly obtained the logarithm of 1200 but multiplied it by 2 instead of dividing by 2. She then corrected the result of the multiplication by 2 to two places of decimal and presented that as the $\sqrt{1200}$. She did not complete the solution process by reading the antilogarithm to obtain the desired root. Further evidence of the participants' difficulties with square roots can be seen in this solution also taken from a participant's script.

$$3^2 + 6^2 = \sqrt{y^2}$$

$$9 + 36 = \sqrt{y^2}$$

$$45 = \sqrt{y}$$

There were two errors in this solution. One, the solver did not take the root of 45. Two, the second root of y^2 is taken as $\sqrt{y}$, which is the fourth root of y^2. The final step of the solution ought to have been $\sqrt{45} = y$. More evidence from posttest scripts that illustrate lack of understanding of the process involved in the computation of square roots can be seen in this solution:

$$\sqrt{900} = \sqrt{(90 \times 10)}$$

$$= \frac{9.4868}{2}$$

$$= 4.7434 \times 10$$

$$= 47.$$

The initial step of the solution, which involves the factor method, is perfect. In the second step, the participant divided the root of 90 by 2, an action that suggests use of logarithms in computing square roots. She ought to have multiplied the root of 90 by root 10 to obtain the root of 900 but was mixed up between using logarithms (the division by two) and using the factor method.

Other participants were not successful in calculating square roots due to incorrect order of operations applied as can be seen in this example.

$$\sqrt{(576 + 324)} = 7.6 + 5.7$$

$$= 13.3$$

The correct order of the mathematical operations $\sqrt{}$ and + were not followed. She ought to have simplified the bracket by adding 576 to 324 before computing the square root. She just did the opposite. In addition, she used the square root tables inappropriately by reading the roots of 57.6 and 32.4 from a table of square root values instead of the roots of 576 and 324. Other participants had an incorrect conception of square root as *division by two* as was evident in the case below and other similar ones.

$$AB^2 = 12$$

$$AB = \frac{12}{2}$$

$$= 6$$

It seems to be the case that the two methods the participants used to calculate square roots (logarithm method and factor method) were not clearly understood by them. It is in situations such as this that marking of learners' work is likely to benefit them if constructive comments exposing incorrect procedures and errors are given alongside the usual "ticks", "crosses" and numerical scores. More often than not, actual classroom practice on marking is to award correct responses by ticks and numerical scores. Incorrect responses are "rewarded" by crosses and a zero score. The sum of all the numerical scores is usually qualified by a value judgment such as “fair”, “pass”, “unsatisfactory” etc. At the end of the school term or year, the comments take an even more general form such as “more effort is necessary for mathematics”. While it is true that such general comments could help motivate some learners, the comments fall short of letting learners know their specific problems with specific mathematical concepts. Constructive comments detailing learners’ difficulties would therefore assist learners in knowing which areas of mathematics they should "put more effort" aimed at improving performance.

4.96 Squares

Three sources of participants’ difficulties with squares were identified. They were not able to use the factor method. They expressed the number (M) to be squared as a factor of two numbers c and d where d is 10 or a positive integer multiple of 10. M^2 can be obtained by using the expression $M^2 = \mathbf{(cd)^2 = c^2d^2}$. The value of c^2 can be read from tables and multiplied by d^2. However, participants’ responses such as $30^2 = 90$, $40^2 = 160$ seem to suggest they experienced difficulties with the factor method when computing squares. Possible solution paths used to arrive at $30^2 = 90$ and $40^2 = 160$ are these:

$$30^2 = (3\times10)^2 = 9\times10 = 90$$

$$40^2 = (4\times10)^2 = 16\times10 = 160$$

Ten (10) is not squared. The problem is similar to that experienced with using the factor method to calculate square roots. They have problems expanding brackets when squares and square roots are involved. The second source of participant's difficulty with squares is an incorrect conception that squaring a number (P) has the same effect as multiplying the number by two. That is, $P^2 = 2P$. Evidence of this conception can be seen in these solutions from the participants' scripts:

$5^2 = 10$ explained as $5^2 = 5\times2 = 10$

$30^2 = 600$ explained as $30^2 = (3\times10)^2 = 3^2\times10^2 = 6\times100 = 600$.

$40^2 = 800$ explained as $40^2 = (4\times10)^2 = 4^2\times10^2 = 8\times100 = 800$.

Other responses such as $8^2 = 80$, $30^2 = 300$ and $40^2 = 400$ seem to have been based on an erroneous rule that squaring a number means multiplying the number by ten. The third source of participants' difficulties with squares is their inability to identify the relevant table to use. In the responses $24^2 = 1.55$ and $18^2 = 1.34$, they read the table for square roots and corrected the readings to two decimal places instead of reading the table for squares. Other errors such as $18^2 = 344$ and $24^2 = 376$ may have been random misreads in which 324 was read as 344 and 576 was read as 376. There were many such misreads.

It also seem to be the case that participants have not developed a stable schema for the concept "square". This view is supported by the fact that many participants gave the square of 15.59 as 2430. They did not realize that the square of 15.59, cannot exceed the square of 20 (say) or the square of 30 since 15.59 is less than both 20 and 30.

4.97 Logarithms

The participants' difficulties with logarithms can be traced to arithmetic and procedural errors, lack of understanding, and use of inappropriate tables. It appears they are quite uncertain of when to use logarithms of sines, logarithms of cosines and logarithms of tangents. Arithmetic errors as a source of difficulties with logarithms can be seen in three extracts taken from the participants' scripts. In an effort to calculate the length of a line segment, a participant used logarithms to multiply 18 by sin30° as shown below:

		No	*Log*
		18	1.**2**553
		S*in*30°	$\bar{1}$.**6**990
x = 22.61	←	2.261	1.3543

The bold numbers in the participant's solution indicate the location of the errors committed. She computed 1 + 2 + 6 and obtained 13, wrote 3 down and "carried" 1 to the leftmost column where she computed 1+1+(-1) and obtained 1. She did not evaluate the result to confirm whether it is realistic or not. The product of 18 and sin30° (in base ten) cannot exceed 18 since the maximum value of sin30° is 1. This seems to suggest that learners neither estimate the results of their solutions before computation begins nor evaluate the end result of their computations. It also appears that either they do not recall what they know about a mathematical task (mathematical demands of the task), or they are not able to apply much of what they know to assist them in solving mathematical tasks.

Errors committed by participants when using logarithms in computations seem to be a direct consequence of difficulties with logarithm tables. The difficulties can be attributed to inaccurate reading of logarithm tables and inability to interpret the readings. Difficulties in reading logarithm tables were manifested in errors such as $15.59^2 = 2430$ and other similar cases. In this case they used logarithms correctly to multiply 15.59 by 15.59 but were not able to interpret 2.3856, the logarithm of 243. From tables, participants read 2430 for the antilogarithm of 2.3856. However, they were not able to interpret it as 2.43×10^2 to obtain 243 and just wrote down 2430 as the square of 15.59 despite the fact that 2430 looks too large to be the square of 15.59.

Participants made consistent arithmetic errors that thwarted their effort to successfully use logarithms for computations. Many of them could not calculate the value θ from the equation $\tan\theta = \frac{6}{9.434}$ because of arithmetic errors. An example of such errors taken from a participants' script is this:

No		log
6		0.**7**782
9.434		0.9747
26.8	←	$\bar{1}$.**7**035

She obtained 26.8 by reading the table of logarithms of tangents for the angle whose logarithm of tangent is $\bar{1}.7035$.

The error (17 – 9 = 7) in the solution above leads to the incorrect value of 26.8 for θ instead of 32.5. The mathematical procedure used to obtain θ by the participants who committed this particular error was however correct. Another arithmetic error that was committed by many participants can be seen in this attempt to solve for θ in the equation $\sin\theta° = \frac{9}{30}$.

		NO		log
		9		0.9542
0.293		30		1.4771
11.33°	←	2.932×10^{-1}	←	$\bar{1}.4671$

In addition to the arithmetic error 14 – 7 = 6, the participant read the angle whose logarithm of sine is $\bar{1}.2932$ from a table of logarithm of sines and obtained 11.33° instead of 17.05°. She ought to have read the table of sines for the angle whose sine is 0.2932 (or the table of logarithm of sines for the angle whose logarithm of sine is $\bar{1}.4671$.

Arithmetic errors and use of inappropriate tables resulted to incorrect solution products. It is also important to note that $\frac{9}{30}$ could have been simplified without using logarithms but nearly all participants used logarithms. This seems to suggest that learners do not consider alternative solution paths available to them before they tackle a problem. More evidence of participants' use of inappropriate tables can be seen in their attempt to solve for θ in the equation $\cos\theta = \frac{28.629}{30}$.

$\underline{No}$		$\underline{\log}$
28.62		1.4567
$\underline{30}$		$\underline{1.4771}$
72.6^0	←	$\bar{1}.9796$

To obtain 72.6°, participants read the table of logarithms of sines (instead of a table of logarithms of cosines) for the angle whose logarithm of sine is $\bar{1}.9796$

When solving for θ in the equation $\tan\theta = \frac{9}{28.62}$, some participants used logarithms correctly and obtained $\bar{1}.4975$ for the logarithm of $\frac{9}{28.62}$ but read the table of logarithms of tangents for the angle whose logarithm of tangent is 0.4975 instead of $\bar{1}.4975$. This suggests difficulties in handling the characteristic of logarithms when, as in this case, the characteristic is negative and the mantissa is positive. When calculating the length of a line, a participant did the following:

Opp = 18sinθ^0	$\underline{No}$		$\underline{\log}$
	18		1.2553
= 69.3°	$\underline{\text{Sin}\theta^\circ}$		$\underline{\bar{1}.6990}$
	$\underline{0.8980}$	←	$\underline{0.9543}$

The procedure used in this solution is correct except the solver read the antilogarithm of 0.9533 instead of 0.9543 and obtained 0.8980 instead of 9.001×10^0. She then proceeded to read the table of sines for the angle whose sine is 0.8980 and obtained 63.9°. The incorrect antilogarithm (0.8980) of 0.9543 should have been written as 8.980×10^0. This participant and others who committed similar mistakes had difficulties writing antilogarithms. They also exhibited lack of understanding on the use of sines and logarithms. The problem was to calculate the length of a line and not the size of an angle yet she expressed the results of her computations in degrees. As was the case with many others, she did not reflect back on the product of her solution to check whether it was realistic. Since the maximum value of the sine of an angle is 1, the product of 18 and sin30°cannot exceed 18.

In their attempts to simplify $\frac{9}{28.63}$, many candidates committed the same error when reading the table of antilogarithms. They wrote 3.143 for the antilogarithm of 1.4973 instead of 0.3143. They did not multiply 3.143 by 10^{-1}. In the case of $\frac{9}{28.63}$, the expected result is a fraction and cannot be greater than 1. Dividing a number n by another number N where $N > n$ results in a fraction and not an integer. It therefore appears that participants neither estimate the results of their calculations nor reflect back on their solutions to evaluate the reasonability of the solutions.

Participants made procedural errors that frustrated their efforts to obtain correct solution products with logarithms. An example of such errors taken from one of the participants' scripts is this:

$$\sin\theta = \frac{12}{13}$$

$$= 1.0792$$

$$\underline{1.1139}$$

$$2.1931$$

$$10^2 \times 1.560 = 156^\circ.$$

The participant added the logs instead of subtracting. The error resulted in an unrealistic value of sineθ (156°). It could also be possible that participants who made such errors were not certain of how to obtain θ once the sine is calculated. Mistakes arising from use of tables that are discussed here indicate that participants' skills in using tables are not yet refined especially with respect to trigonometric ratios.

4.100 Errors

Consistent errors committed by many participants in different schools were notably those of arithmetic and trigonometry.

4.101 Arithmetic errors

Errors of arithmetic nature include those of addition of whole numbers and simplification of fractions. When adding or subtracting whole numbers, participants did not take into consideration, numbers "carried" or "borrowed" from other columns. For instance, participants obtained 990 as the sum of 576 and 324 instead of 1000. In this case, 1 "carried" from the unit column is not added to 7 and 2 in the tens column. When simplifying fractions, participants reduced only one component of the fraction (the numerator or denominator) by a factor and completely ignored the other component. For example, when simplifying the fraction 9/30, participants reduced 9 by a factor of three in a two-step reduction but did not remember to reduce 30 by the same factor. This action resulted to 1/10 as the final form of the reduced fraction. Similar errors and their possible sources were these:

Error	**Possible explanation.**
576 + 324 = 890	*1 "carried" from the unit column is not added.*
576 + 324 = 1000	*1 + 5 + 3 = 10.*
576 + 324 = 1600	*multiplied 3 by 5 then added 1.*
576 + 324 = 990	*1 "carried" from the unit column is not added.*
576 + 324 = 899	*4 + 6 = 9*
576 + 324 = 903	*6 + 4 = 13*
576 + 324 = 600	*3 in the hundreds column is not added.*
576 + 324 = 800	*1 "carried" from the tens column is not added.*
576 – 324 = 900	*+ was recorded as -.*
324 – 81 = 143	*3 – 1 = 1 or 2 – 0 = 1*
100 – 9 = 81	*1 "borrowed" twice ∴ 2 is subtracted from 10.*
184 – 36 = 156	*14 – 6 = 6, 1 "borrowed" is not accounted for*
$\frac{10}{8} = 0.8$	*divided 8 by 10.*
$\frac{9}{30} = \frac{1}{10}$	$\frac{9}{30} = \frac{3}{30} = \frac{1}{10}$ *did not divide 30 by 3.*
$\frac{4}{3} = 0.75$	*divided 3 by 4 instead of 4 by 3*
$\frac{8}{6} = 0.75$	*divided 6 by 8*
$\frac{10}{5} = 0.2$	*divided 1 by 5*
$18\sin 30° = 3.6$	$18(\frac{1}{2}) = 18(0.2) = 3.6$, $\frac{1}{2} = 0.2$
$\frac{10}{8} = 0.25$	*recorded 0 instead of 1.*

4.102 Errors on trigonometry

Participants consistently used rules and theorems (cosine rule, tangent ratio and Pythagorean theorem) where the rules and theorems do not apply. Sometimes, the rules and theorems were stated incorrectly and used. In other cases, the rules and theorems were stated correctly but incorrect substitutions were made resulting in incorrect solution products. Such errors are symptoms of difficulties with trigonometry. They used cosine rule in situations where it does not apply. For instance, the rule was used to calculate the sides of an object that is not a regular plane object and is not a triangle. The rule was incorrectly stated as $PS^2 = PQ + QX^2 - 2PQ(QX)\cos Q$ in reference to the object shown in diagram 9.

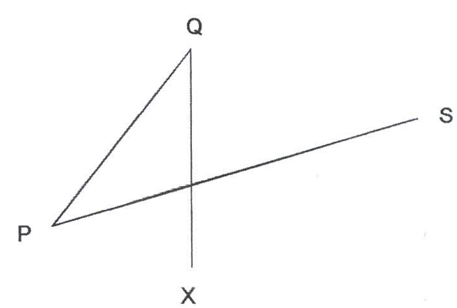
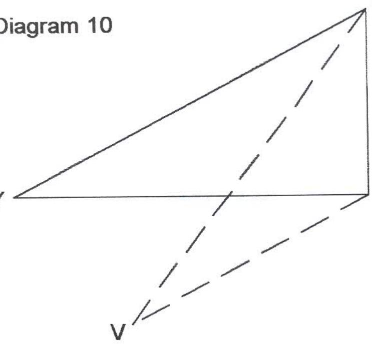


Diagram 9

The tangent ratio was also used in situations where it does not apply. In diagram 10, the tangent of the angle at vertex Y was equated to $\frac{WV}{VY}$.



Other participants stated the rules correctly but made incorrect substitutions and consequently obtained inaccurate and sometimes impractical cosine values. An example of errors arising from difficulties with trigonometry can be seen in this calculation:

$$\tan\theta = \frac{4}{3}$$

$$= 0.75$$

$$= 36.87°$$

The division of 3 by 4 instead of 4 by 3 resulted in an incorrect solution product but use of the ratio was correct. Other errors were committed while reading tables of trigonometry ratios. For instance, in many instances, participants calculated the cosine of an angle correctly but were not able to read tables for the corresponding value of the angle. Similar errors were made with tangent and sine ratios. Some of the errors arising from incorrect reading of tables are these:

$\cos\theta = 0.6$	$\text{sine}\theta = 0.3$	$\sin\theta = 0.3$
$\theta = 33.1°$	$\theta = 1.7°$	$\theta = 2.8°$
$\text{sine}\theta = 0.3$	$\text{sine}\theta = 0.8$	$\sin\theta = 0.9232$
$\theta = 1.75°$	$\theta = 38.66°$	$\theta = 69.33°$

Some errors were due to the reading of inappropriate tables. For instance, $\theta = 38.66°$ was obtained as a result of reading the table of tangents instead of reading sine tables for θ when $\text{sine}\theta = 0.8$.

CHAPTER FIVE: CONCLUSIONS AND RECOMMENDATIONS

5.10 Conclusions

From the teachers' experience, learner centered approach was found to be more productive in teaching *three-D* than the teacher centered approach. In the study, groups that used the constructivist approach (treatments T2 and T1) attained superior mean scores and better ranks (Kruskal-Walis test) compared to corresponding groups (treatments D2 and D1 respectively) that used the behaviourist approach irrespective of the materials used for instruction. Analysis of the posttest scores by the F-test showed that the results were significant (α = 0.05) for both pretest and non-pretest participants. However the interaction effect between the instructional approach and the materials used to reinforce the comprehension of concepts was significant in the case of non-pretest participants but insignificant in the case of pretest participants. The pretest seems to have had a negative impact on the participants' who sat for it.

Teachers rely more on plane diagram representation of *three-D* concepts (charts, chalkboard and text book diagrams) for teaching *three-D* but rarely prepare and use charts citing lack of materials and time as reasons for not preparing the charts. However, unlike their counterparts who used models, participants who used plane representations only found it difficult to pick up concepts in the absence of models. Learners who used models in addition to plane representation of *three-D* concepts did better than those who used plane representations only. A combination of plane representation and concrete representation led to a better performance. This seems to suggest that concrete materials were effectively used in reinforcing the understanding of *three-D* concepts. Some participants were not able to transfer geometry concepts and skills to situations that appear different from instances under which the concepts were learnt. For instance, they were not able to apply concepts and skills learned with the use of models to answer questions in the absence of models. This seems to suggest that learners who participated in this study were not able to transfer geometry kills from concrete to abstract situations. Learners however showed a lot of interest in the charts that were used to reinforce the understanding of *three-D* concepts.

Surprising but consistent and popular arithmetic errors committed by participants of the study were those of addition of whole numbers and simplification of fractions. When adding whole numbers, participants did not include in the operation, "carried" or "borrowed" numbers. When simplifying fractions, participants reduced only one component of the fraction (the numerator or the denominator) by a factor and ignored the other. They also divided the denominator by the numerator instead of dividing the former by the latter. Rules and theorems were used where they do not apply, stated incorrectly or stated correctly but followed with incorrect substitutions. Errors committed when using mathematical tables were misreading the appropriate table or reading a table that does not apply.

Learners' conceptual difficulties exposed by this study involve the concept of dimension, the concept of angle, and prerequisite concepts such as Pythagorean theorem, trigonometry, square roots, squares and logarithms. The concept of angle seems to be quite subtle for learners. The magnitude of the subtlety increases when a plane is involved. Learners' difficulties with angles may largely be attributed to their inability to decode information from plane diagrams of *three-D* objects especially identifying and calculating angles that involve a plane and or skew lines. They experience difficulties in recognizing angles that do not involve the edges of a solid and lines that are part of the solid but are not shown on a diagram of the solid. The difficulties are more pronounced when the angle to be calculated involves a cuboid, a prism, a pyramid, a tetrahedron and composite objects that comprise multiple shapes of regular solids. Learners' inability to decode information from plane representation of *three-D* objects could be attributed to the fact that plane diagrams distort information about the features (angles and distances) of the solid object represented in a plane drawing. Learners' inability to identify required angles frustrates and thwarts their effort to extract relevant triangles they require to calculate angles between planes and between lines and planes.

Learners also experience difficulties in applying trigonometric ratios, Pythagorean theorem, square roots, logarithms and squares to calculate angles and distances. Pythagorean theorem was often stated incorrectly and used in situations where it does not apply. They have difficulties defining and applying trigonometric ratios; and stating and applying the cosine and sine rules. They experience more difficulties in applying the inverse trigonometric relations to obtain the size of an angle whose ratio (trigonometric) has already been calculated. Their difficulties with squares and square roots are due to inability to use mathematical tables correctly and inadequate comprehension of the methods used (factor and logarithm methods). Generally, learners preferred to solve mathematical tasks by using strategies that had been taught even in cases where there were other viable alternatives with shorter solution paths.

The study elicited evidence that about 20% of learners in their third year of secondary education can neither define rectangle or cuboid nor classify the relevant objects according to their dimensions. They are not able to distinguish between plane objects and *three-D* objects. Others are not able to name some regular solids from their drawings or concrete presentations. For instance, 69.13% could not identify a prism from a plane drawing of the same. Some mistook it for a pyramid while others said it was a tetrahedron. This seems to suggest that the participants are operating below level 2 of the van Hiele model of geometric understanding.

A number of factors were found to impact negatively upon the learning of *three-D*. These include lack of models, negative learners' attitude and the abstract nature of *three-D*. The factors also include lack of and or inability to apply basic skills. Others are lack of interest, low motivation, incorrect interpretation of *three-D* questions (including examination questions), poor instructional approach, lack of time, unavailability and inadequacy of text books, teacher characteristics (phobia of *three-D* and lack of motivation), poor syllabus coverage, rare testing of *three-D* concepts in the national examinations, over enrolment of learners and language deficiencies. More than half of the teachers involved in the study confirmed that they only partially cover the syllabus by the time the learners are supposedly ready to sit for the national examinations that mark the end of secondary school education. Participants were found to be lacking appropriate mathematical language (including technical terms) with which to express their thoughts and describe mathematical concepts with precision. They used incorrect tenses and expressions with unclear semantics. For instance, they referred to ''faces'' of a solid object as ''sides'', talked of ''calculating an object" (e.g. a cube) when they meant calculating the area (or volume) of the object. In their attempts to differentiate between a rectangle and a cuboid, participants said that ''a rectangle is measured by L × W while a cuboid is measured by L × W × H" when they meant that a rectangle has length and width while a cuboid has length width and height.

There is evidence to suggest that unsatisfactory performance in *three-D* tasks by Kenyan female students is a direct consequence of multiple factors contributing both singly and collectively. The factors include: lack of and or inability to apply prerequisite concepts; conceptual difficulties; lack of and or under utilization of instructional resources; deficiency in problem solving skills; and ineffective mode and manner of concept presentation.

5.20 Recommendations for implementation

a) Workshops and Seminars

Teachers need to be in-serviced on new and various innovations in mathematics education that can update their instructional skills and approaches. Poor instructional approaches and teacher characteristics such as phobia for *three-D* can be overcome through workshops and seminars on mathematics teaching and learning. Seminars and workshops would provide a forum for mathematics teachers to freely exchange professional skills on mathematics teaching and learning This would provide an opportunity for teachers who are not competent in handling specific areas of mathematics content a chance to build their competence and confidence in handling such areas.

b) Restructuring the mathematics syllabus.

i) In the learners' experience, solids (*three-D*) are encountered before points, lines and planes. Teaching angles, plane figures and area of regular solids before common solids is taught as is the case in the current syllabus does not seem to be logical. The mathematics syllabus should therefore be restructured with a view to introducing the topic *three-dimensional* geometry earlier (preferably in form two) during the four-year high school education in Kenya. This is likely to give learners more time for private practice and will reduce the time lapse between the learning of shapes and solids in form one and that of three-dimensional geometry in forms four. It would also provide learners with a more relevant background experience upon which spatial relationships can be developed.

ii) The mathematics syllabus should be split into two alternative parts and students are given a choice to make between the two alternatives. One alternative being for continuing students and the other for terminal students. The alternative for continuing students should comprise all the prerequisite concepts for the mathematics studied at the tertiary institutions in Kenya and the basics for real life applications. The other alternative for terminal students should comprise only the basics for real life applications. Problem solving should be incorporated in the current mathematics syllabus to expose learners to solution of non-routine problems and generation of alternative paths to problem solutions.

c) Instructional approach

The teaching of three dimensional geometry in particular and mathematics in general should involve constructive learning whose strategies include; cooperative learning projects, demonstrations, discovery oriented methodologies, and interactive approaches that incorporate up to date technology.

d) Goals of teaching

Mathematics teaching should focus on developing the learner's potential in mathematical skills, concepts and principles and their applications to real life situations. Currently, mathematics teaching seems to be examination oriented (probably unofficially). Schools are competing for prestigious positions in the "ladder of academic excellence''. The ''competition'' means that learners are rushed through the syllabus so as to complete the same with the hope (which may not be realized) of performing well in the national examinations.

e) Resources

The ministry of education, through the Kenya Institute of Education, should prepare a standard set of teaching models (for three dimensional geometry in particular and mathematics in general) and encourage all schools to buy them in the same way the schools buy laboratory chemicals and equipment for science subjects. This should eventually be aimed at establishing mathematics resource rooms in Kenyan schools to be equipped with among other things, models, completed project work, mathematics reference books, calculators and computers.

f) Teacher training program

An evaluation of the current teacher-training program is needed to determine its efficiency and effectiveness. Actual school practice should shift from drill and rote memorization of knowledge to interactive learning dominated by learner activity and construction of knowledge.

g) Teaching load

A review of mathematics teaching load should be done with a view to reducing the number of lessons per week so as to provide teachers with adequate time for preparation of their lessons to incorporate necessary and appropriate teaching resources. A reduced teaching load would provide time for teachers to pay specialized attention to learners with mathematical difficulties. The time for rendering this service should officially be indicated on the school timetable as an office hour. The time created should also allow teachers to provide a more qualitative assessment of learners' assignments, tests and examinations.

h) Sketching diagrams.

More emphasis should be placed on sketching of solids and decoding information from plane diagram representation of *three-D* solids.

i) Basics

Emphasis is needed in the mathematics content for forms one and two that provide the basics for secondary education mathematics. Preferably, teaching of mathematics at the lower secondary classes should be done by experienced teachers so as to provide a good foundation on which to build more demanding concepts. More often than not, these lower classes are usually ignored when a school has problems of understaffing.

j) Evaluation

Effort should be made to include constructive remarks that focus on learners' specific conceptual difficulties. This would assist learners (and teachers) to focus their private study (and remediation) on the identified difficulty with an aim of overcoming the difficulty.

k) Research on mathematics education

Research on the teaching and learning of mathematics should include both qualitative and quantitative aspects. This is likely to expose and explain learners' conceptual difficulties with mathematics.

l) Technical terms

Meaning of technical mathematical terms should be emphasized during the teaching and learning of mathematics. Distinction should be made between the technical meaning and daily language use of all mathematical terms. Consultations can be made between the mathematics and language departments to facilitate this. Teacher interventions are necessary for learners who cannot verbalize their mathematical thoughts because of language deficiencies.

5.30 Recommendations for research

A research is needed to:

a. Investigate teacher's fear of *three-D*.

b. Investigate a possible link between performance in the language of instruction and performance in geometry.

c. Investigate learners' strategies of solving *three-D* problems.

d. Investigate the content validity of mathematics examinations at form four level and the reasons for the rare testing of *three-D* concepts in such examinations.

e. Investigate the impact of calculators on minimizing computational errors encountered in the learning of *three-D*.

f. Investigate the impact of computers on the learning of *three-D* concepts.

REFERENCES

Ale, S. O. (1981). Difficulties facing mathematics teachers in developing countries: a case study of Nigeria, in Bishop, A. J.(ed.), *Educational Studies in Mathematics 12, 479 - 488*. Dordrecht: Reidel publishing company.

Anderson, J. R. (1980). *Cognitive Psychology and its Implications p. 235*. San Francisco: W. H. Freeman and company.

Anthony, G. (1996). Active Learning in a Constructive Framework, in Hanna, G. and Streefland, L. (eds.), *Educational Studies in Mathematics 31(4), 349-366*. Dordrecht: Kluwer Academic Publishers.

Artzt, A. F. (1983). The comparative effect of the student team-method of instruction and the traditional teacher-centred method of instruction upon student achievement, attitude and social interaction in high school mathematics courses, in Colling, P. (ed.), *Dissertation Abstracts International 44 (12) , 3619-A*. Ann Arbor: University Microfilms International.

Ary, D., Cheser, J. and Razavieh, (1979). *Introduction to Research in Education (2nd edn.), 237 - 260*. New York: Holt, Rinehart and Winston.

Ausubel, D. P. (1968). *Educational Psychology: a cognitive view, P.339*. New York: Holt, Rinehart and Winston.

Bauersfeld, H. (1992). Classroom cultures from a social constructivist perspective, in Glaserfeld, E. V. (ed.), *Educational Studies in Mathematics 23 (5), 467 - 481*. Dordrecht: Kluwer Academic Publishers.

Benders, N. & Janeiro, B. (1988). A constructivist approach to numeration in primary school: results of a three year intervention with the same group of children, in Bishop, A. J. (ed.), *Educational Studies in Mathematics 19 (3), 299 - 330*. Dordrecht: Kluwer Academic Publishers.

Bell, A. W., Costello, J. & Kuchman, D. (1983). *A review of Research in Mathematics Education: Research on Learning and Teaching, 172*. Windsor: NFER-NELSON.

Ben-Cham, D. et al (189). Adolescents' ability to communicate spatial information: analysing and effecting students' performance, in Bishop, A. J. (ed.), *Educational Studies in Mathematics 20 (2), 120 - 143*. Dordrecht: Kluwer Academic Publishers.

Borg, R. W. (1981). *Applying Educational Research. New York*: Longman.

Borg, R. W. & Gall, D. M. (1989). *Educational Research: An introduction (5th edn.), 705 - 706*. New York: Longman.

Buckwell, G. D. & Githua, S. W. (1989). *Gold medal mathematics pp. 166-167*. London: Macmillan press Ltd.

Buerger, F. W. & Shaughnessy, J. M. (1986). Characterizing the van Hiele Levels of Development in geometry, in Kilpatrick, J. (ed.), *Journal for Research in Mathematics Education 17 (1), 31 - 47*. Reston: NCTM.

Butler, C. H. & Wren, F. L. (1965). *The teaching of Secondary Mathematics (4th. edn.), 455 - 457, 494 - 495*. New York: McGraw Hill book company.

Campbell, W. G., Ballou, S. V., & Slade, C. (1990). *Form and style: Theses, reports, term papers (8th edn.)*, Boston: Houghton Mifflin.

Carpenter, T. P. et al (1975). Results and implications of the NAEP Mathematics Assessment for Secondary Schools, in Fey, J. T. et al (eds.), *The Mathematics Teacher 68 (6), 465 - 467*. Reston: NCTM inc.

Central Bureau of Statistics, (Kenya) Ministry of Planning and National Development. *Statistical Abstract 1991, 185 - 188*. Nairobi: Government printers.

Channon, J. B. et al (1993). *General Mathematics for Secondary Schools book 3, pp. 154 – 165*. Essex: Longman.

Channon, J. B. et al (1993). *General Mathematics for Secondary Schools book 4, pp. 181 – 182*. Essex: Longman.

Christiansen, B. et al (eds.), (1979). *New trends in mathematics teaching IV, 44 - 45*. Paris: UNESCO.

Clement, J. (1991). Constructivism in the classroom - A review of Transforming Children's Mathematics Education: International Perspectives, in Carpenter, T. P. et al (eds.), *Journal for Research in Mathematics Education 22 (5), 339, 422*. Reston: NCTM.

Clements, D. H. & Battista, M. T. (1989). Learning of geometric concepts in a logo environment, in Carpenter et al (eds.), *Journal for Research in Mathematics Education 20 (5), 450*. Reston: NCTM.

Cobb, P. et al (1992). A constructivist Alternative to the Representational view of the mind in mathematics Education, in Carpenter, T. P. et al (eds.), *Journal for Research in Mathematics Education 23 (1), 27 - 28*. Reston: NCTM.

Cohen, L. (1989). *Research Methods in Education pp. 38 - 40, 292 - 293*. Palo Alto: Mayfield publishing company.

Craig, T. M. (1994). Pre-service Teachers' Reactions to an interactive Approach to English Language Arts course work, in Ducharme, R. E and Ducharme, M. K. (eds.), *Journal of Policy, Practice and Research in Teacher Education 45 (29), 96 - 97*. Washington: American Association of colleges for Teacher Education.

Cuevas, G. J. (1984). Mathematics Learning in English as a second Language, in Kilpatrick, J. and Reyes, L. H. (eds.), *Journal for Research in Mathematics Education, 15 (2), 133*. Reston: NCTM.

Dirkstra, S. (1997). The integration of instructional systems design models and Constructivistic design principles, in Goodyear et al (eds.), *Instructional Science: an International Journal of Learning and Cognition 25 (1), 1 – 13*. Dordrecht: Kluwer Academic Publishers.

Dolgos, K. A. and Elias, J. S. (1996). New directions in the teaching of mathematics, science, and technology, in Walker, D. (ed.), *International Journal of Mathematical Education in Science and Technology 27 (5), 725 – 729*. London: Taylor and Francis.

Dugast-Portes, F. (1997). Contents and methods in Secondary education, in Kallen, D. (ed.), *European Journal of Education: research, development and policies 32 (1), 33 – 43*. Abingdon: Carfax Publishing Company.

Dyche, S. et al (1993). Questions and conjectures concerning models, misconceptions and spatial ability, in Enochs, L. G. (ed.), *School Science and Mathematics 93 (4), 195*. Bowling Green: School Science and Mathematics Association Inc.

Ernie, K. T. K. (1979). The contribution of formal study of algorithms on deductive reasoning ability of high school geometry students, in colling, P. (ed.), *Dissertation Abstracts International 49 (9), 4939-A*. Ann Arbor: University Microfilms International.

Eshiwani, G. S. (1981). The Death of New Mathematics, in Nyambala, P. (ed.), *The Kenya Teacher 31, 11*. Nairobi: Nairobi Bookmen.

Eshiwani, G. S. (1984). *Research in Education: the Kenya Register, occasional paper number 3050 p. 12*. Nairobi: Bureau of Educational Research, Kenyatta University.

Fennema, E. &. Tartre, L. A. (1985). Use of spatial Visualization, in Kilpatrick, J. (ed.), *Journal for Research in Mathematics Education 16 (3), 184-205*. Reston: NCTM.

Ferara-Mori, F. & Morino-Abelle, F. (1963). Perception of geometrical gestalt learning of mathematics concepts, in Dienes, Z. P. (ed.), *An Experimental Study of Mathematical Learning p. 200*. London: Hutchinson and company.

Foreman, G. (1987). The constructivist perspective, in Roopraine, J. L. & Johnson, J. E. (eds) .... pp 71-82. Columbus: Merril Publishing Company.

Gagatsis, A. and Pazronis, T. (1990). Using geometrical models in a process of reflective thinking in learning and teaching mathematics, in Hanna, G. and Streefland, L. (eds.), *Educational Studies in Mathematics 21 (1), 29 - 52*. Dordrecht: Kluwer Academic Publishers.

Gagne, R. M. et al (1992). *Principles of Instructional design, 4th edn. pp. 106 – 110.* Forth Worth: Harcourt Brace Jovanovich.

Gary, P. (1967). *The Dictionary of the Biological Sciences p. 561*. New York: Reinhold Publishing Corporation.

Gearhart, G, (1975). What do mathematics teachers think about high school geometry controversy?, in Fey, J. T. et al (eds.), *The mathematics teacher 68 (6), 490 - 492.* Reston: NCTM.

Gerdes, P. (1966). How to recognize hidden geometrical thinking: a contribution to the development of anthropological mathematics, in Wheeler, D (ed.), *For the Learning of Mathematics 6(2), 10-17.* Montreal: FLM Publishing Association.

Gibbons, J. D. (1985). *Non-parametric Statistical Inference pp. 198-201.* New York: Marcel dekker, inc.

Good, C. V. (ed.)(1959). *Dictionary of Education p. 173.* New York: McGraw-Hill Book Company.

Government of Kenya, (1964). *Kenyan Education Commission Report pp.62, 117, 166.* Nairobi: Government printers.

Grant, F. & Searl, J. (1997). Practical Activities in the Mathematics Classroom, in Bradshaw, J. R. et al (eds.), *Mathematics in School 26 (4), 27 - 28.* London: Pitman Publishing.

Hanna, G. (1989). Mathematics Achievement of Girls and Boys in Grade Eight: Results from Twenty Countries, in Bishop, A. J. (ed.), *Educational Studies in Mathematics 20 (2), 225 - 231.* Dordrecht: Kluwer Academic Publishers.

Harying, L. L. & Lounsburs, J. F. (1983). *Introduction to scientific geographic research (3rd edn.) pp. 16 - 19.* Dubuque: WMC Brown Company.

Hill, W. F. (1977). *Learning: a Survey of Psychological Interpretations (third edn.) pp. 30-35*. New York: Thomas Y. Crowell Company Inc.

Hiatt, J.C. (1959). Achievement and attitude in trigonometry: the effect of guided discovery lessons in high school trigonometry, in Colling, P. (ed.), *Dissertation Abstracts International 40 (7), 3846-A*. Ann Arbor: University Microfilms International.

HollingDale, S. (1989). *Makers of Mathematics pp. 34 - 36, 130 –1 39*. London: Penguin Books.

Hwang, A. (1996). Positivist and constructivist Persuasions in Instructional Development, in Goodyear, P. et al (eds.), *Instructional Science: An International Journal of learning and Cognition 24 (5), 343 - 356.* Dordrecht: Kluwer Academic Publishers.

Johnson, R. & Bhatta C. G. (1985). *Statistics principles and methods pp. 325-329*. New York: John Wiley and sons.

Johnstone, J. (1987). *Electronic Learning p. 5*. Hillsdale: Laurence Erlbaum Associates.

Kantowski, M. G. (1978). *The Teaching Experiment and Soviet Studies of Problem solving*. Unpublished paper, University of Florida.

Kenya Government, (1976). *Report of the National Committee on Educational objectives and Policies pp. 58, 67, 69-70.* Nairobi: Government Printers.

Kenya Institute of Education, (1992). *Secondary Education Syllabus volume 7 pp. 1 - 22*. Nairobi: Kenya literature bureau.

Kenya Institute of Education (1994). *Secondary Mathematics pp. 263 – 288.* Nairobi: Kenya Literature Bureau.

Kenya, Republic of (1998). *Economic Survey p. 199.* Nairobi: Government printers.

Kenya, Republic of, (1988). *Presidential Working party on Education and Manpower Training for the next decade and beyond p. 23*. Nairobi: Government Printers.

Kerlinger, F. N. (1973). *Foundations of Behavioural Research, pp. 16 - 20.* London: Halt, Rinehart and Winston.

Kieran, C. (1985). The Soviet Teaching Experiment, in Romerg, T. A. (ed.), *Research Methods for Studies in Mathematics Education: Some considerations and alternatives, 71-28*. Madison: Winscosin Education Resource Centre.

Kiragu, F. W. (1986). Achievement in Mathematics: An investigation of the factors that contribute to mass failure in K.C.E Mathematics p. 1. Bureau of Educational Research, Kenyatta University.

Knupfer, N. M.(1993). Logo and Transfer of Geometry Knowledge: Evaluating the Effects of Student Ability Grouping, in Underhill, R. G. (ed.), *School Science and Mathematics 93 (7), 360-366*. Bloomsbury: school Science and Mathematics Association Inc.

Koul, L. (1984). *Methodology of Educational Research pp. 463 - 455*. Delhi: Vani Educational Books.

Kuku, A. O. (1991). Congress on mathematics. *East African Standard number 24030 of august 29th 1991, p. 5.*

Kyungu, S.(1998). *Mathematics decline in schools. Daily Nation number 11616 of April 7th. 1998, p. 5.*

Lacampagne, C. B. (1979). An Evaluation of Women and Mathematics Program and Associated Sex-related differences in the Teaching, Learning and Counselling of Mathematics, in Colling, P. (ed.), *Dissertation Abstracts International 40 (9), 4939-A*. Ann Arbor: University Microfilms International.

Lean, G. & Clements, M. A. (1981). Spatial Ability, Visual Imagery, and Mathematical performance, in Bishop, A. J. (ed.), *Educational Studies in Mathematics 12, 267 - 296.* Dordrecht: Reidel publishing company.

Lerman, S. (1989). Constructivism, Mathematics and Mathematics Education, in Bishop, A. J. (ed.), *Educational Studies in Mathematics 20 (2), 211 - 223.* Dordrecht: Kluwer Academic Publishers.

Leslie, P. S. & Wiegel, H. G. (1992). On reforming Practice in Mathematics Education, in Glasersfeld, E. V. (ed.), *Educational Studies in Mathematics 23 (5), 445 - 463.* Dordrecht: Kluwer Academic Publishers.

Lockhead, J. (1992). Knowing down the building blocks of learning: Constructivism and the Ventures program, in Glaserfeld, E. V. (ed.), *Educational Studies in Mathematics 23 (5), 543 - 552*. Dordrecht: Kluwer Academic Publishers.

Marioti, M. A. & Fischbein, E. (1997). Defining in Classroom Activities, in Streefland, L. et al (eds.), *Educational Studies in Mathematics 34 (3), 219 - 248*. Dordrecht: Kluwer Academic Publishers.

Mayberry, J. W. (1981). An investigation of the van Hiele levels of geometric thought in undergraduate pre-service teachers, in Colling, P. (ed.), *Dissertation Abstracts International 42 (5), 2008-A*. Ann Arbor: University Microfilms International.

Mbiriru, M. W. N. K. (1983). Problems of Teaching Junior Secondary Mathematics in Kenya, in Sachsenmeier, P. & Kuansah, K. (eds), *African Studies in Curriculum and Evaluation project number 107*. Nairobi: GTZ.

Mevarech, Z. R. (1983). A deep structure model of students' statistical misconceptions, in Bishop, A. J. (ed.), *Educational Studies in Mathematics 14, 415 - 426*. Dordrecht: Reidel publishing company.

Meyerson, L. S. (1976). Mathematical mistakes, in Fielker, D. S. (ed.), *Mathematics Teaching 76, 39*. Nelson: Association of Teachers of Mathematics.

Mitchelmore, C. M. (1983). Geometry and Spatial Learning: some lessons from Jamaican experience, in Wheeler, D. (ed.), *For the Learning of Mathematics 3 (3), 2*. Montreal: FLM Publishing Association.

Moise, E. (1975). The learning of Euclidean Geometry in School Mathematics, in Fey, J. T. et al (eds.), *The Mathematics Teacher 68 (6), 473*. Reston: NCTM.

Moreno-Armela, L. E. (1996). Mathematics: a historical and didactic perspective, in Walker, D. (ed.), *International Journal of Mathematical Education in Science and Technology 27 (5), 633 - 639*. London: Taylor and Francis.

Mulindi, H. A. (1979). Guidelines for pure and applied modern Mathematics Curriculum Developments for Secondary schools in Kenya, in Colling, P. (ed.), *Dissertation Abstracts International 40 (9), 4940-A*. Ann Arbor: University Microfilms International.

Mwangi, D. T.(1983). Factors influencing the performance and learning of mathematics among students in Kenya. Bureau of Educational Research, Kenyatta University.

Nesher, P. (1987). Towards an Instructional Theory: The role of students' misconceptions, in Wheeler, D. (ed.), *For the Learning of Mathematics 7 (3), 33 - 38*. Montreal: FLM publishing Association.

Neter, J. and Wasserman, W. (1974). *Applied Linear Statistical models pp 419-635*. Homewood: Richard D. Irwin, inc.

Nemivovsky, R. and Noble, T. (1997). On mathematical visualization and the place where we live, in Ruthven, K. (ed.), *Educational Studies in Mathematics 33 (2), 99 - 131*. Dordrecht: Kluwer Academic Publishers.

Normadia, B. R. (1981). The relationship between cognitive level and modes of instruction, teacher-centered and activity-centered, to the learning of introductory transformational geometry, in patricia, A. K. et al (eds.), *Dissertation Abstracts International 43 (1), 102-A*. Ann Arbor: University Microfilms International.

Nyerere, J. K. (1967). Education for Self Reliance, in Hinzen, H. and Hundsdorfer, V. W.(eds), *Education for Liberation and Development: the Tanzanian Experience pp. 17-32*. London: Evans Brothers Limited.

Origa, J. G. O. (1992). *Mathematical Learning Difficulties encountered in the Teaching and Learning of Linear Programming: Masters Thesis p. 50,* Concordia University, Montreal.

Parzysz, B. (1988). "Knowing" vs. "Seeing". Problems of the Plane Representation of Space Geometry Figures, in Bishop, A. J. (ed.), *Educational Studies in Mathematics19 (1), 79 - 91*. Dordrecht: Reidel publishing company.

Parzysz, B. (1991). Representation of Space and Students' Conceptions at high school, in Hanna, G., and Streefland, L. (eds.), *Educational Studies in Mathematics 22 (6), 575 - 592.* Dordrecht: Kluwer Academic Publishers.

Patel, N. M. (1994). *Mathematics for Kenya Schools, book 4 pp. 125 – 141.* Nairobi: Mwalimu Publications.

Pirie, S. & Kieran, T. (1992). Creating constructivist environments and constructing creative mathematics, in Glaserfeld, E. V. (ed.), *Educational Studies in Mathematics 23 (5), 505 - 527.* Dordrecht: Kluwer Academic Publishers.

Rachlin. S. (1979). *Clinical Approaches to the Study of Mathematical Abilities.* University of Calgary: Paper presented at the symposium on mathematical ability, San Francisco.

Rainer, J. D. & Guyton, E. (1994). Developing a Constructivist Teacher Education Program: The Policy Making Stage, in Ducharme R. E. & Ducharme, M. K. (eds.), *Journal of Policy, Practice and Research in Teacher Education 45 (2), 141.* Washington: American Association of Colleges for Teacher Education.

Resnic, L. B. & Ford, W. W. (1981). *The Psychology of mathematics for Instruction pp. 97 - 193.* Hillsdale: Laurence Erlbaum Associates, Publishers.

Robitaille, D. & Dirks, M. (1982). Models for the mathematics curriculum, in Wheeler, D. (ed.), *For the Learning of Mathematics 2 (3), 7*. Montreal: FLM Publishing Association.

Sahakian, . W. (1984. *Introduction to the Psychology of Learning (2nd edn) pp. 1-4*. ITASCA: F. E. PEACOCK Publishing inc.

Servais, W. & Varga, T. (1971). *Teaching School Mathematics p. 22*. Middlesex: Penguin Books.

Simon, M. A. & Schifter, D. (1991). Towards a constructive perspective: an intervention study of mathematics teacher development, in Hanna, G. and Streefland, L. (eds.), *Educational Studies in Mathematics 22 (4), 309-330*. Dordrecht: Kluwer Academic Publishers.

Skemp, R. R. (1971). *The Psychology of Learning Mathematics p. 42*. Middlesex: Penguin Books.

Skemp, R. R. (1972). Schematic Learning, in Champman, L. R. (ed.), *The process of learning mathematics pp.183 – 204*. Oxford: Pergamon Press.

Souviney, R. J. (1983). Mathematics achievement: classroom practices in Papua New Guinea, in Bishop, A. J. (ed.), *Educational Studies in Mathematics 14, 183-212*. Dordrecht: Reidel publishing company.

Sowell, E. J. (1989). Effect of manipulative materials on mathematics instruction, in carpenter et al (eds.), *Journal for Research in Mathematics Education 20 (5), 498 - 504*. Reston: NCTM.

Sweller, J. (1994). Cognitive load theory, learning difficulty, and instructional design, in Säljö, R. (ed.), *Learning and Instruction: The Journal of the European Association for Research in Learning and Instruction 4 (4), 295 - 311*. Oxford: Pergamon.

The International Commission on Mathematics Instruction, (1995). Perspectives on the teaching of geometry for the 21st century, in Hanna, G. and Streefland, L. (eds.), *Educational Studies in Mathematics 28 (1), 91-98*. Dordrecht: Kluwer Academic Publishers.

The Kenya National Examinations Council (1987). *1985 K. C. E. Mathematics report*. Nairobi: Kenya National Examinations Council.

The Kenya National Examinations Council (1991). *1989 K.C.S.E. Examination Report pp 49-62*. Nairobi: Kenya National Examinations Council.

The Kenya National Examinations Council (1993). *1990 K.C.E. Mathematics report*. Nairobi: Kenya National Examinations Council.

The Kenya National Examinations Council (1997). *1995 K.C.S.E. Examination Report pp21-36*. Nairobi: Kenya National Examinations Council.

Tindol, N. R. (1979). The effect of Special Reading Instruction of Students' Achievement in Plane Geometry Teaching, in Colling, P. (ed), *Dissertations Abstracts International 40 (7), 3852-A*. Ann Arbor: University Microfilms International.

Trelinski, G. (1983). Spontaneous Mathematization of situations outside mathematics, in Bishop, A. J. (ed), *Educational Studies in Mathematics 14, 275*. Dordrecht: Reidel publishing company.

Triadafillidis, T. A. (1992). Circumventing Visual Limitations in Teaching the Geometry of Shapes, in Hanna, G. and Streefland, L. (eds.), *Educational Studies in Mathematics 29 (3), 225 - 234*. Dordrecht: Kluwer Academic Publishers.

Wadsworth, H. M. (1990). *Handbook of Statistical Methods for Engineers and Scientists pp. 11.6 - 11.11.* New York: McGraw-Hill publishing company.

Wanjala, W. (1986). Problems faced by teachers in teaching mathematics in different categories of secondary schools, in Eshiwani, G. S. (ed.), *Thesis and dissertation abstracts on education in Kenya: Occasional paper number 3041*. Bureau of Educational Research, Kenyatta University.

Wheatly, G. H. (1992). The role of reflection in mathematics learning, in Glaserfeld, E. V. (ed.), *Educational Studies in Mathematics 23 (5), 529 - 540*. Dordrecht: Kluwer Academic Publishers.

Williams, G. (1981). *Mathematics with Applications in the Management, Natural and Social Sciences pp. 297 - 299*. Boston: Allyn and Bacon, Inc.

Wise, A. et al (1986). *Intermediate Algebra with Applications p. 411*. San Diego: Harcourt Braco Jovanovic.

Wonnacott, T. H. & Wonnacott, R. J. (1977). *Introductory Statistics, third edition pp. 277 - 303*. New York: John Wiley & sons.

Woodward, E. (1990). High School Geometry should be a laboratory course, in Stiff, L. V. et al (eds.), *The mathematics Teacher 83 (1), 4 - 5*. Reston: NCTM.

Yerushalmy, M. & Chazan, D. (1990). Overcoming visual obstacles with the aid of the supposer, in Hanna, G. & Streefland, L. (eds.), *Educational Studies in Mathematics 21 (3), 199 - 216*. Dordrecht: Kluwer Academic Publishers.

Zevenbergen, R. (1996). Constructivism as a liberal bourgeois discourse, in Hanna, G. and Streefland, L. (eds.), *Educational Studies in Mathematics 31 (1 - 2), 95 - 112*. Dordrecht: Kluwer Academic Publishers.

A.00 APPENDICES

A.10QUESTIONNAIRE FOR MATHEMATICS TEACHERS

1. Put a tick mark (√) in the blank provided before each item used in your school to teach *three-dimensional geometry*.

....Solid models

....Models made from folded manila paper

....Illustrative diagrams on chalkboard

....Text book diagrams.

b). Other than the items in 1a) above list any other items you may have used with success to teach *three-dimensional geometry*.

....

....

....

2a) Put a tick mark (√) in the blank provided before each approach you have used to teach *three-dimensional geometry*.

..........Empirical approach (based on experiments and observations).

............Teacher centred approach.

.... Learner centred.

............Other (Specify).

....

....

....

....

....

3. From your experience, the condition of topic coverage may be described as:

....................always adequately covered.

....................sometimes fully covered.

....................always partially covered.

....................not usually covered.

4. List some of the factors that in your view negatively affect the learning of *three-dimensional geometry*.

....

....

....

....

....

....

.....

5. List the aspects of *three-dimensional geometry* that have presented difficulties to your students.

....

....

....

....

....

....

.....

6. Do the text books available in your school provide in-depth coverage of *three-dimensional geometry*?

...................Yes.

...................No.

...................Somehow.

7. List two text books which, from your experience, provide adequate coverage of *three-dimensional geometry*.

Author	Book title.
....	....
....	....
....	....
....	....

8. Any other comments concerning the teaching of *three-dimensional geometry*?

....

....

....

....

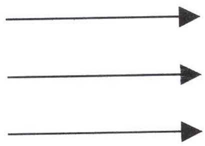
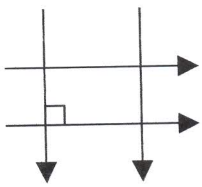
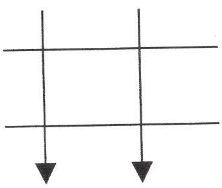
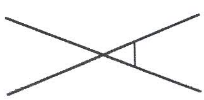
....

....

A.20 PRETEST OF PREREQUISITE CONCEPTS

Answer all items as far as you can.

1. Use one word to describe each set of lines given below:

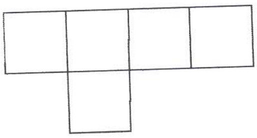
Set of lines	Description
a). 	....
b). 	....
c). 	....
d). 	....

2. From the list given below, mark with a tick (√), the type of lines that do NOT meet at all.

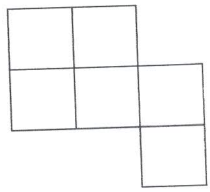
Skew lines	[ ]
Line segments	[ ]
Perpendicular lines	[ ]
Parallel lines	[ ]

3. Below is a set of four nets:

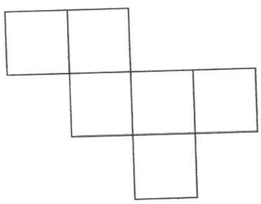
(i)



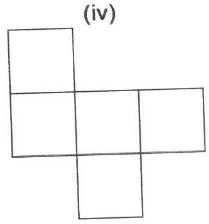
(ii)



(iii)



(iv)



a). Which of the nets will NOT be folded to form a cube?

....

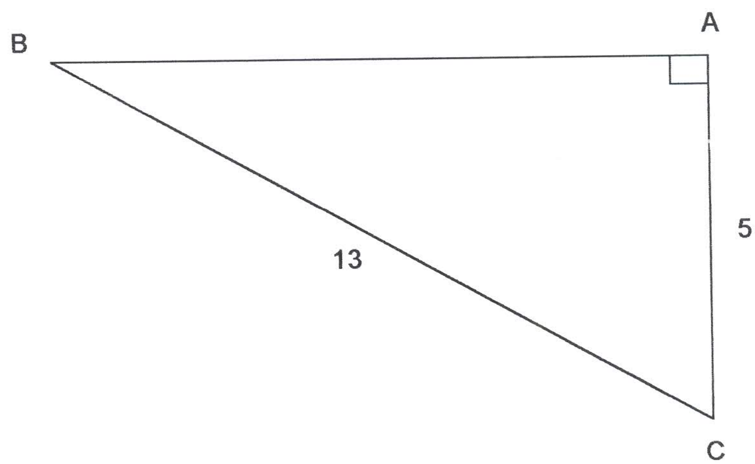
b). Which of the nets will be folded to form a cube with an open top?

....

c). Which of the nets will be folded to form a closed cube?

....

4. Given that the sides of the triangle below are in cm,



i). Calculate the length of the side AB.

ii). Calculate the size of angle ACB.

5. From the objects presented, identify the ones with the following shapes.

a). Tetrahedral.

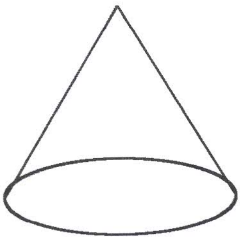
b). Pyramidal.

c). Spherical.

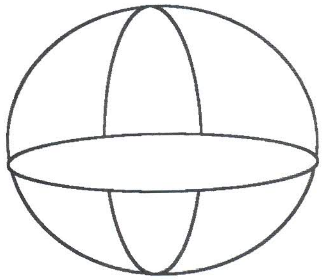
d). Conical.

e). Cylindrical.

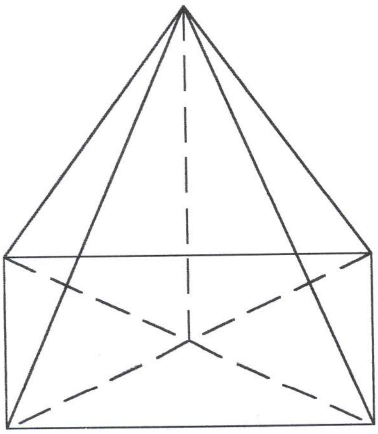
6. Name the shapes drawn below.



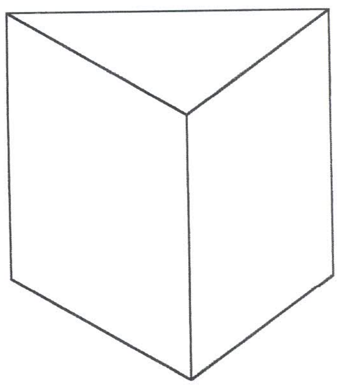
....



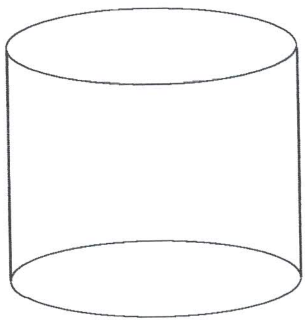
....



....

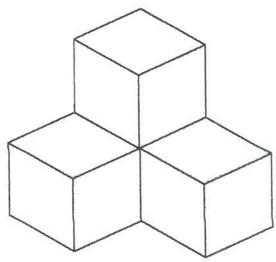


....



....

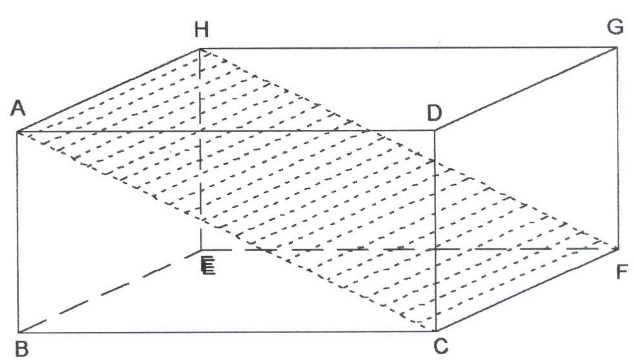
7. How many cubes are there in the diagram below?



....

b). If the volume of each cube is 1x1x1, how many more similar cubes would be required to form a cube of volume 2x2x2?

8.



Using the drawing above, name:

i). any two perpendicular lines.

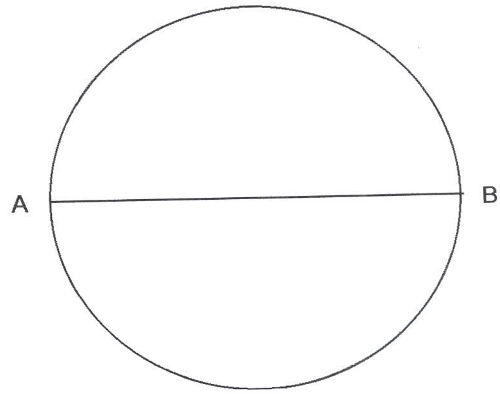
ii). any two skew lines.

iii). any two lines that meet at an angle less than 90°.

iv). any three planes that intersect at a point.

v). a line segment where three planes meet.

9. The circle shown below is allowed to turn on AB as an axis. Name the solid figure that would be formed after a complete revolution (a turn of 360°). AB passes through the centre of the circle.



10. State any differences or similarities that may exist between the following objects:

a). A point and a line;

Differences Similarities.

....

....

....

....

....

b). A rectangle and a square;

Differences Similarities.

....

....

....

....

....

....

....

c). A square and a cube;

Differences. Similarities.

....

.

....

.

....

.

....

.

....

d). A cone and a pyramid;

Differences. Similarities.

....

.

....

.

....

.

....

.

....

.

A.30 INSTRUCTIONAL DESIGNS

A.31 Treatment D1

The approach that was used with the group of participants who received experimental treatment D1 was teacher centered and characterized by knowledge transmission and a hierarchical presentation of concepts from simple to complex. Classroom communication was predominantly one way from the teacher to the students. Consequently, teacher-learner interaction mode dominated the proceedings. Learning activities were mainly teacher definitions, teacher demonstrations and practice by learners. The teacher was regarded as the expert source of information while learners were mostly passive recipients of the teacher's expert knowledge. Teaching sequence adopted the form of definition of concepts, presentation of examples followed by a practice exercise. Learning resources included books and charts. The treatment was expected to promote an instrumental understanding of geometry concepts.

LESSON ONE: geometric properties of common solids

1. Show learners diagrams of the following solids: Cuboid, cylinder, cone, pyramid, sphere, tetrahedron, frustum of a pyramid and frustum of a cone.

2. Describe the diagrams of the following solids to learners.

   Cuboid, cylinder, cone, pyramid, sphere tetrahedron frustum of a pyramid and frustum of a cone.

3 Some of the properties of common solids are the following:

i. they have volume (they occupy space, they are *three-dimensional*).

ii. they have surfaces (planes).

iii. they have vertices except the cylinder, sphere, and frustum of a cone.

iv. they have edges except the sphere.

Exercise 1

1. Name two solids that have no vertices.

2. State two differences between plane objects and solid objects.

3. List examples of geometric figures of the following dimensions:

   i. one dimension.

   ii. two dimensions.

   iii. three dimensions.

LESSON TWO: Projection of lines on planes

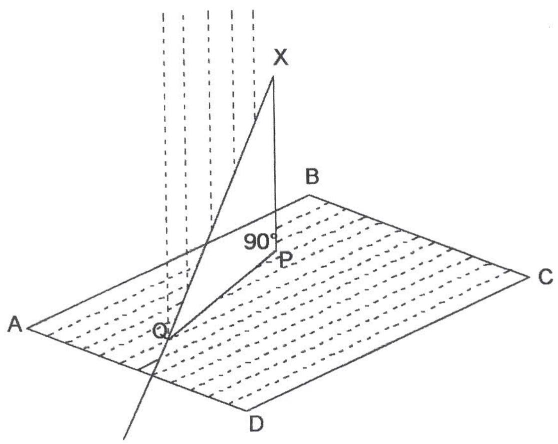


Figure 1.

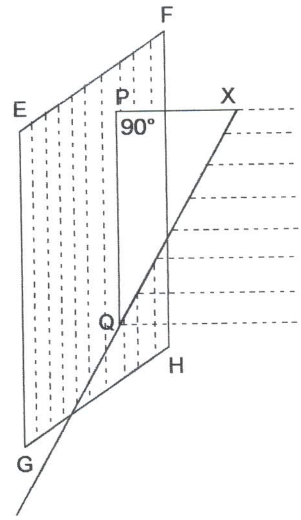


Figure 2.

1. Figure 1 shows the shadow, QP, of a chalkboard ruler (QX) when a torch is shone vertically above a horizontal floor.

   i. Angle QPX is 90° because line PX is perpendicular to the horizontal floor.

   ii. The angle between the ruler and the ground is angle PQX.

   iii. When the torch is shone on the ruler QP becomes the shadow of the ruler (QX).

   iv. Line QP is called the projection of line QX on the horizontal plane ABCD.

   v. Teacher to demonstrate to the learners, the formation of the shadow using a spot light and chalkboard ruler.

2. In figure 2:

   i. the shadow of the chalkboard ruler (QX) when the source of light is horizontal against a vertical wall (EFHG), is QP.

   ii. the angle formed between the ruler and the vertical plane EFHG is PQX.

   iii. the projection of line QX on the plane EFHG is line PQ.

3. Teacher to demonstrate the formation of the shadow using a spot light and chalkboard ruler in the two positions shown in figures 1 and 2.

Exercise 2

1. Awuono who is 1 meter tall stands 60 meters away from the foot of a tree and 20 meters away from the end of the shadow of the tree. He is looking at the top of the tree at an angle of elevation of 25.8°, calculate:

   i the height of the tree.

   ii the length of Awuono's shadow on the ground if he is seeing the sun at the same angle of elevation as the top of the tree.

LESSON THREE: Angle between two lines

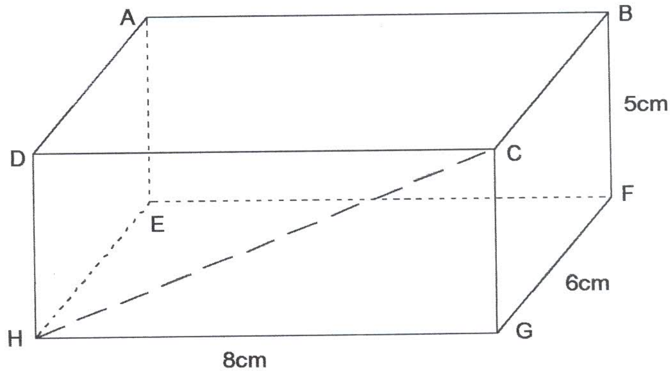


Figure 3

1. In figure 3, lines AB and DC, CH and BE are parallel. Lines CH and DH intersect at point H.

2. Lines AB and GC are NOT parallel yet they do not meet. They are called skew lines.

3. Lines AE and DC is another example of skew lines.

4. Using the diagram of a cuboid on a manila paper:

   i) Place a ruler or any straight edge to coincide with line AD.

   ii) Slide the straight edge in a parallel manner to line AD until it coincides with line BC.

   iii) The angle formed between the image of line AD after translation (BC) and GB is angle GBC.

   iv) The angle between the skew lines AD and GB is the angle between GB and BC (angle GBC).

5. Show learners the angle between the skew lines AD and HB.

6. Calculate on chalkboard, the angle between skew lines AH and BC.

7. Calculate on chalkboard, the angle between lines AG and GE.

Exercise 3

1. Calculate the angle between skew lines GE and AB.

2. Calculate the angle between lines AB and BE.

LESSON FOUR: Angle between a line and a plane

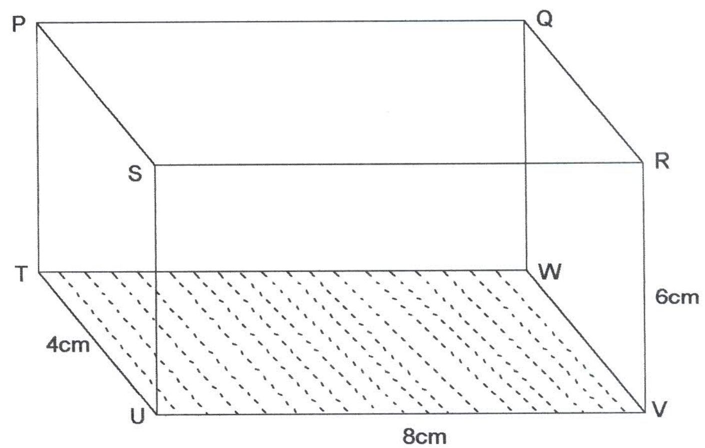


Figure 4.

1. Figure 4 is a diagram of a cuboid.

   i. P and V are two vertices of the cuboid.

   ii. TWVU and PSVW are two planes of the cuboid.

   iii. PT and VU are two edges of the cuboid.

   iv. angles QRS and RVW are right angles.

   v. angles PQT and QSR are acute angles (less than 90°).

   vi. there is no angle shown in figure 4 that has a value greater than 90°.

   vii. the projection of line SW on the plane TWVU is WU.

   viii. the projection of line SW on the plane PSUT is line TS.

   ix the angle between line PV and the plane TWVU is angle PVT.

2. On chalkboard, sketch and indicate the length of the sides of triangle PTV then show learners how to calculate the angle between line PV and the plane TWVU.

Exercise 4

1. Calculate the angle between line WU and the plane PQRS.
2. A cone has a diameter of 6cm and a height of 7cm. Calculate the angle between the slant height and the base of the cone.

LESSON FIVE: Angle between two planes

1. Instruct learners to observe the drawing of an open book shown in figure 5.

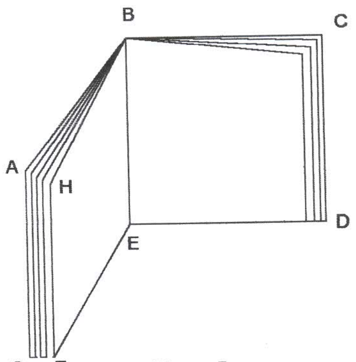


Figure 5

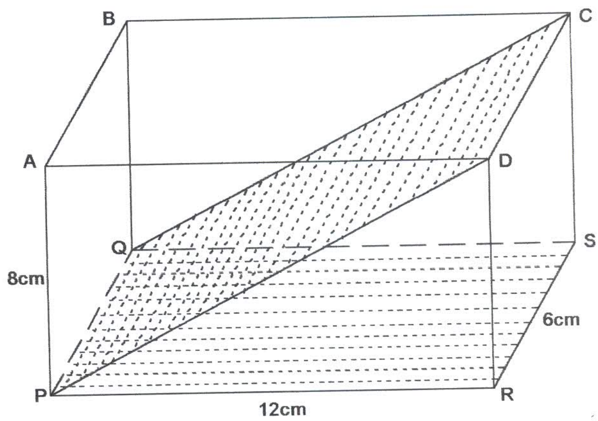


Figure 6.

Using figure five, name and show learners:

i two planes.

ii a line where two planes intersect.

iii an angle between two planes.

2. In figure 6, name and calculate on chalkboard:

i. the angle formed between the shaded planes.

ii. the angle between the planes ABQP and PQCD.

Exercise 5

1. A door two meters high by 1.5 meters wide is opened to an angle of 50° as shown in figure 7.

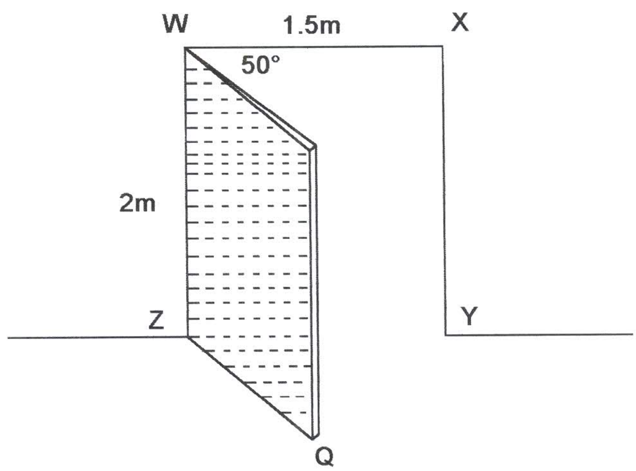


Figure 7

Calculate the angle between the planes YZQ and WQY.

LESSON SIX: Calculating lengths and angles in solids

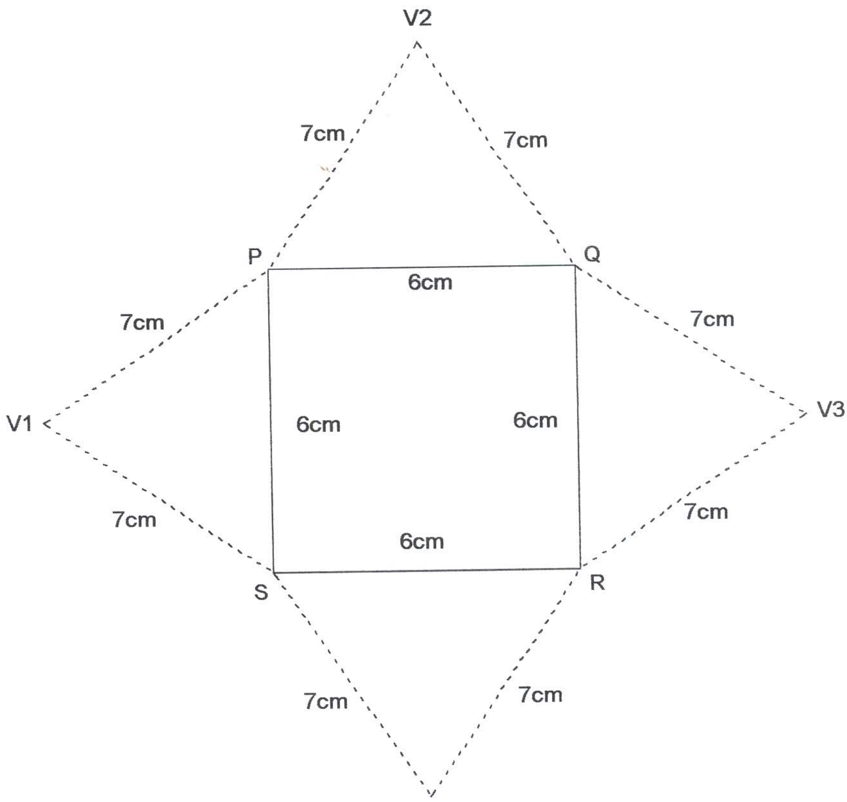


Figure 8

2. Using figure 8, do the following for learners:

    i. state a suitable title for the drawing in figure 8.

    ii redraw figure 8 accurately on a sheet of manila paper.

    iii. name the solid that would be formed when you fold along the continuous lines such that V1, V2, V3, and V4 all coincide at one point V.

    iv. calculate the height of the solid object formed.

    v. sketch the solid object.

    vi. calculate the angle between line VP and the base PQRS.

    vii. calculate the angle between the planes VQR and PQRS.

2. Using figure 9, calculate:

i. the length of line MK.

ii. the angle between line OK and the plane JKLM.

iii. the angle between the planes KON and JKLM.

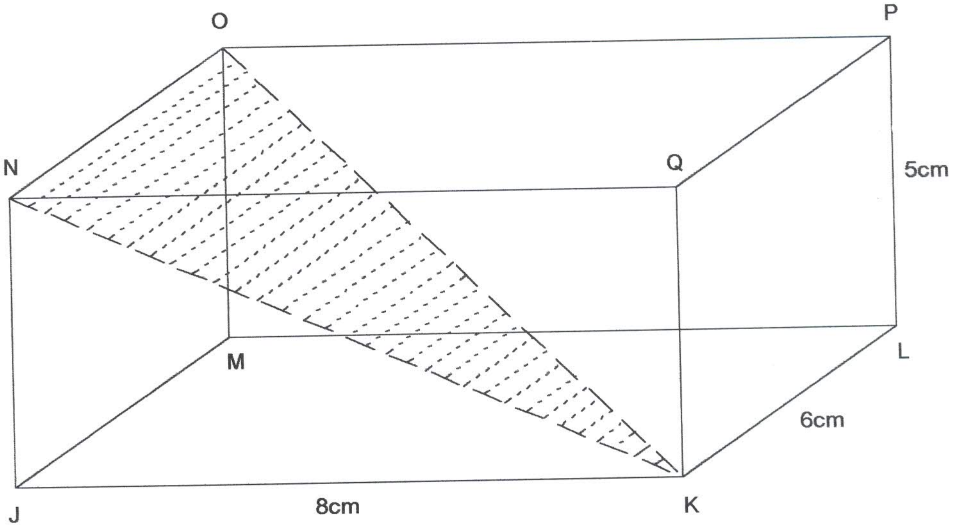


Figure 9

Exercise 6

1. Figure 10 is a cuboid with a square base. B1B3 = 15cm, T3B3 = 12cm and K is the mid point of B1B3. Calculate:

   i. the length of B1T3.

   ii. the area of triangle T3B1K.

   iii. the length of line T1B2.

   iv. the volume of the cuboid.

   v. The angle between the plane T1T2B3B2 and the base of the cuboid.

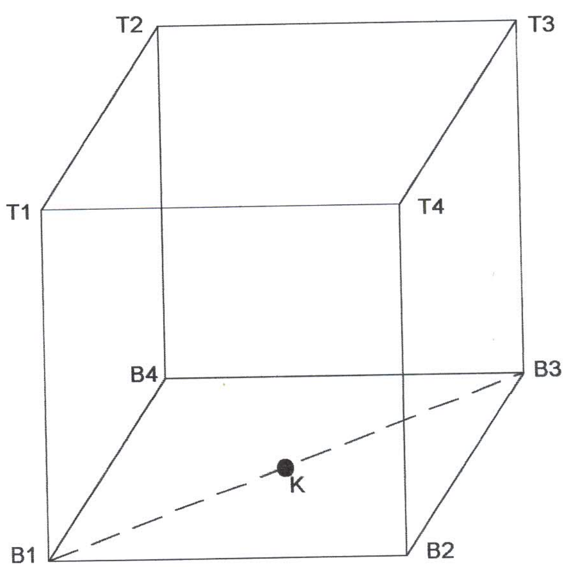


Figure 10

A.32 Treatment D2

The approach that was used with the group of participants who received experimental treatment D2 was teacher centred and characterized by knowledge transmission and a hierarchical presentation of concepts from simple to complex. Classroom communication was predominantly one way from the teacher to the students. Consequently, teacher-learner interaction mode dominated the proceedings. Learning activities were mainly teacher definitions, teacher demonstrations and practice by learners. The teacher was regarded as the expert source of information while learners were mostly passive recipients of the teacher's expert knowledge. Teaching sequence adopted the form of definition of concepts, presentation of examples followed by a practice exercise. Learning resources included books, charts and models of *three-D* objects. Treatment D2 was expected to promote an instrumental understanding of geometry concepts.

LESSON ONE: geometric properties of common solids

1. Show learners models of the following solids: Cuboid, cylinder, cone, pyramid, sphere, tetrahedron, frustum of a pyramid and frustum of a cone.
2. Describe and show learners diagrams of the following solids:

   Cuboid, cylinder, cone, pyramid, sphere, tetrahedron, frustum of a pyramid and frustum of a cone.

3 Some of the properties of common solids are the following:

i. they have volume (they occupy space, they are *three-dimensional*).

i. they have surfaces (planes).

ii. they have vertices except the cylinder, sphere, and frustum of a cone.

iii. they have edges except the sphere.

Exercise 1

1. Name two solids that have no vertices.

2. State two differences between plane objects and solid objects.

3. List examples of geometric figures of the following dimensions:

   i one dimension.

   ii. two dimensions.

   iii. three dimensions.

LESSON TWO: Projection of lines on planes

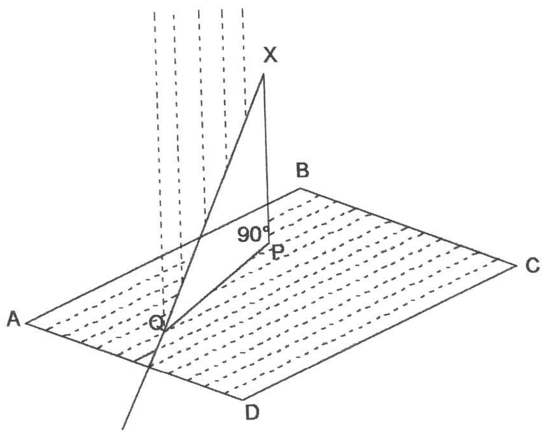


Figure 1.

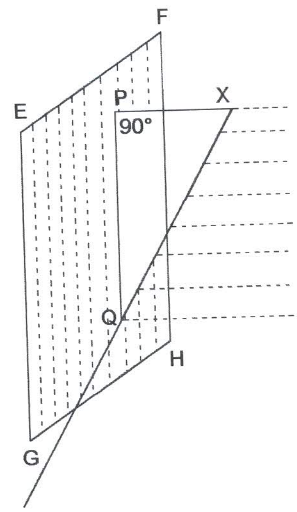


Figure 2.

1. Figure 1 shows the shadow, QP, of a chalkboard ruler (QX) when a torch is shone vertically above a horizontal floor:

   i. angle QPX is 90° because line PX is perpendicular to the horizontal floor (ABCD).

   ii. the angle between the ruler and the ground is angle PQX.

   iii. when the torch is shone on the ruler, QP becomes the shadow of the ruler (QX).

   iv. line QP is called the projection of line QX on the horizontal plane ABCD.

   v. teacher to demonstrate to learners, the formation of the shadow using a spot light and chalkboard ruler.

2. In figure 2:

i. the shadow of the chalkboard ruler (QX) when the source of light is horizontal is QP.

ii. the angle formed between the ruler and the vertical plane EFHG is PQX.

iii. the projection of line QX on the plane EFHG is line PQ.

3. Teacher to demonstrate the formation of the shadow using a spot light and chalkboard ruler in the two positions shown in figures 1 and 2.

Exercise 2

1. Awuono who is 1 meter tall stands 60 meters away from the foot of a tree and 20 meters away from the end of the shadow of the tree. He is looking at the top of the tree at an angle of elevation of 25.8°, calculate:

i the height of the tree.

ii the length of Awuono's shadow on the ground if he is seeing the sun at the same angle of elevation as the top of the tree.

LESSON THREE: Angle between two lines

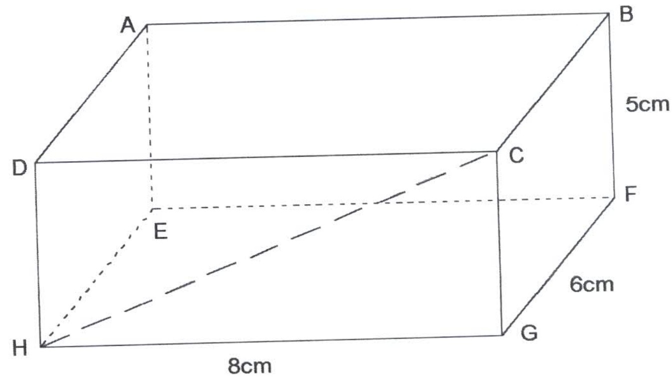


Figure 3

1. In figure 3, lines AB and DC, CH and BE are parallel. Lines CH and DH intersect at point H.

1. Lines AB and GC are NOT parallel yet they do not meet. They are called skew lines.

2. Lines AE and DC is another pair of skew lines.

4. i. Using a skeleton model of a cuboid, place a ruler or any straight edge to coincide with line AD.

   ii. Slide the straight edge in a parallel manner to line AD until it coincides with line BC.

   iii. The angle formed between the image of line AD after translation (BC) and GB is angle GBC.

   iv. The angle between the skew lines AD and GB is the angle between GB and BC (angle GBC).

5. Show learners the angle between the skew lines AD and HB on a diagram and on a skeleton model.

6. Calculate on chalkboard, the angle between skew lines AH and BC.

7. Calculate on chalkboard, the angle between lines AG and GE.

Exercise 3

1. Calculate the angle between skew lines GE and AB.

2. Calculate the angle between lines AB and BE.

LESSON FOUR: Angle between a line and a plane

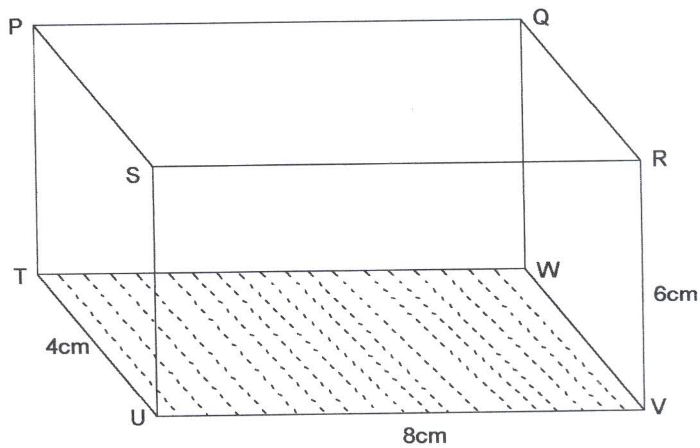


Figure 4.

1. Figure 4 is a diagram of a cuboid:

i. P and V are two vertices of the cuboid.

ii. TWVU and PSVW are two planes of the cuboid.

iii PT and VU are two edges of the cuboid.

iv angles QRS and RVW are right angles.

v angles PQT and QSR are acute angles (less than 90°).

vi no angle shown on figure 4 is greater than 90°.

vii. the projection of line SW on the plane TWVU is WU.

viii. the projection of line SW on the plane PSUT is line TS.

ix the angle between line PV and the plane TWVU is angle PVT.

2. On chalkboard, sketch and indicate the length of the sides of triangle PTV then show learners how to calculate the angle between line PV and the plane TWVU.

Exercise 4

1. Calculate the angle between line WU and the plane PQRS.
2. A cone has a diameter of 6cm and a height of 7cm. Calculate the angle between the slant height and the base of the cone.

LESSON FIVE: Angle between two planes

1. Instruct learners to observe the drawing of an open book shown in figure 5.

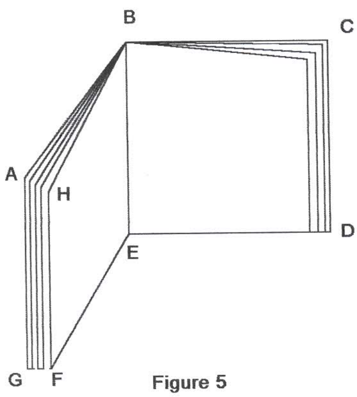


Figure 5

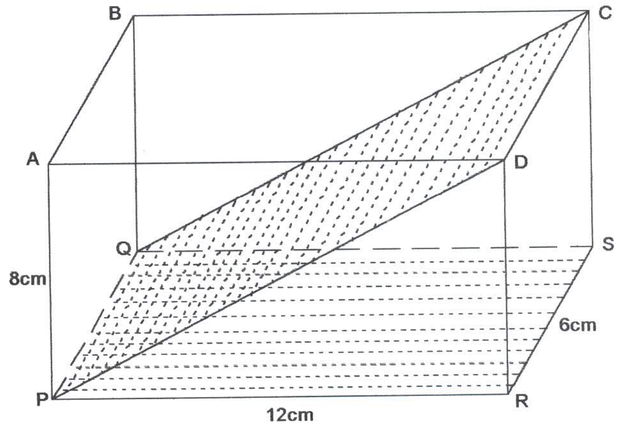


Figure 6.

Using figure five, name and show learners:

i. two planes.

ii. a line where two planes intersect.

iii. an angle between two planes.

2. In figure 6, name and calculate on chalkboard:

i. the angle formed between the shaded planes.

ii. the angle between the planes ABQP and PQCD.

Exercise 5

1. A door two meters high by 1.5 meters wide is opened to an angle of 50° as shown in figure 7.

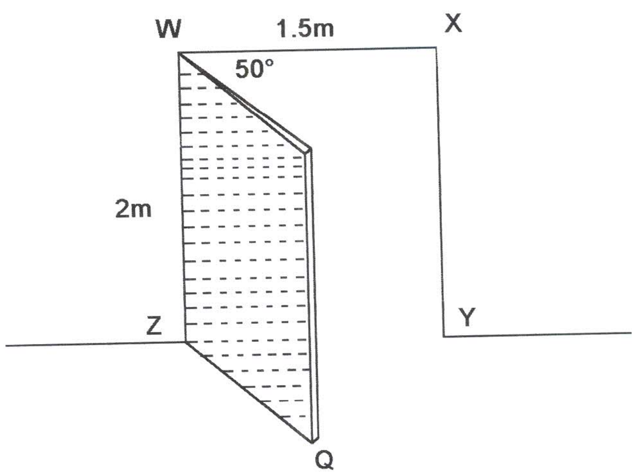


Figure 7

Calculate the angle between the planes YZQ and WQY.

LESSON SIX: Calculating lengths and angles in solids

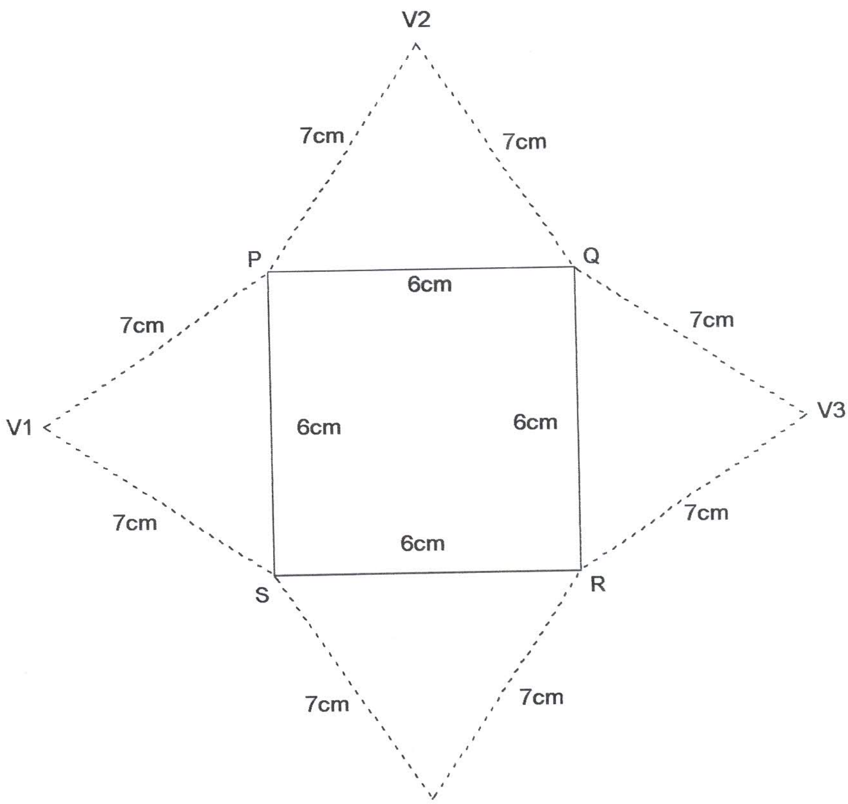


Figure 8

2. Using figure 8, do the following for learners:

i. state a suitable title for the drawing in figure 8.

ii. redraw figure 8 accurately on a sheet of manila paper.

iii. name the solid that would be formed when you fold along the continuous lines such that V1, V2, V3, and V4 all coincide at one point V.

iv. calculate the height of the solid object formed.

v. sketch the solid object.

vi. calculate the angle between line VP and the base PQRS.

vii. calculate the angle between the planes VQR and PQRS.

2. Using figure 9, calculate:

i. the length of line MK.

ii. the angle between line OK and the plane JKLM.

iii. the angle between the planes KON and JKLM.

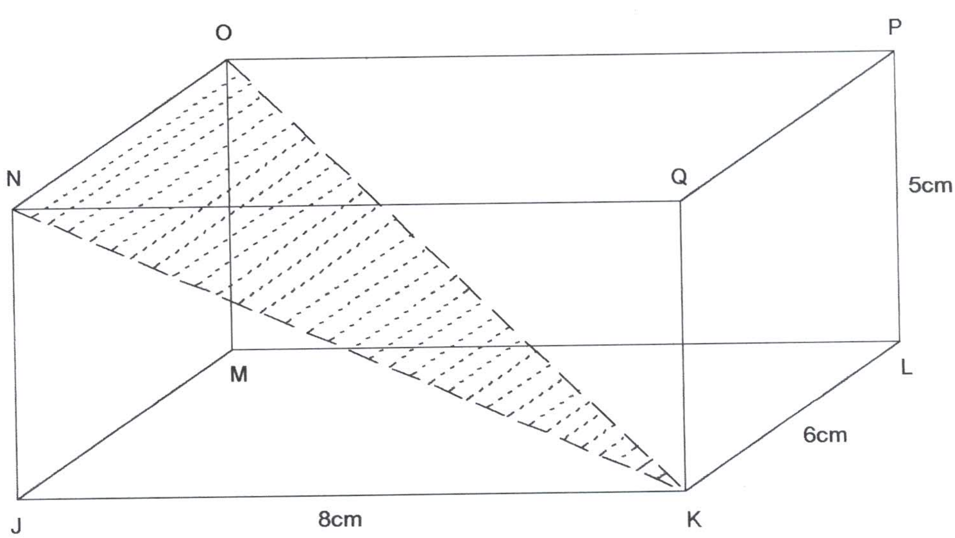


Figure 9

Exercise 6

1. Figure 10 is a cuboid with a square base. B1B3 = 15cm, T3B3 = 12cm and K is the mid point of B1B3. Calculate:

   i. the length of B1T3.

   ii. the area of triangle T3B1K.

   iii. the length of line T1B2.

   iv. the volume of the cuboid.

   v. the angle between the plane T1T2B3B2 and the base of the cuboid.

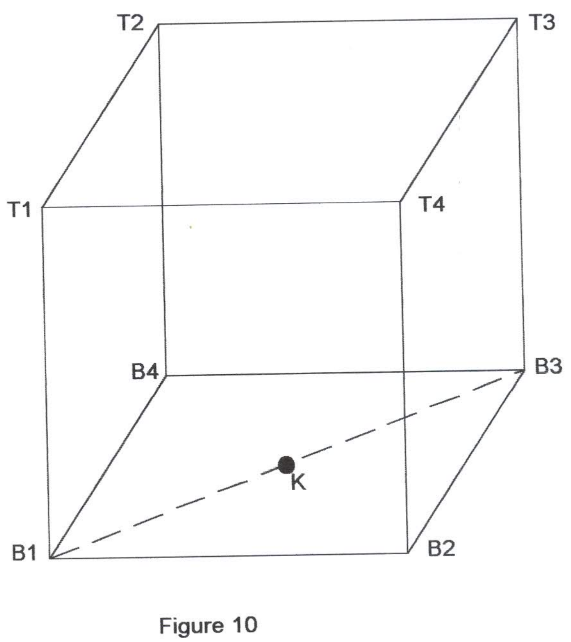


Figure 10

A.33 TREATMENT T1

The approach that was used with the group of participants who received experimental treatment T1 was learner centred and characterized by experimentation, interaction with the learners' environment and inquiry. Learners were expected to interact freely with their environment, which included the teacher, fellow students, charts, diagrams and textbooks. The diagrams used were those of *three-dimensional* objects. The teachers' role was that of a guide who introduced helpful interventions during the learning process. The teacher introduced learners to the concept that was to be learned, guided them to explore and discover relations, generate patterns and to generalize. The teacher was expected to encourage three modes of interaction in the classroom namely the teacher-learner interaction, learner-learner interaction, and learner-environment interaction. Teacher interventions were to be given in the form of probing questions and provision of feedback to learners' questions and responses. Learning activities included analysis of diagrams through observations, and sketching. Learners were also expected to name *three-D* objects from their diagrams; lines, planes and angles. They were also expected to calculate lengths and sizes of angles and establish relations among the components of *three-D* objects.

LESSON ONE: geometric properties of common solids

1. Present diagrams of the following solids to the learners:

   Cuboid, cylinder, cone, pyramid, sphere, tetrahedron, frustum of a pyramid and frustum of a cone.

2. Learners should then copy and complete the table below after observing the diagrams presented.

Solid	Number of surfaces Flat	Curved	Number of edges	Number of vertices
Cuboid				
Cylinder				
Cone				
Pyramid				
Sphere				
Tetrahedron				
Frustum of a pyramid				
Frustum of a cone				

3. Ask learners to state five properties of solids.

Exercise 1

1. Name two solids that have no vertices

2. State two differences between plane objects and solid objects.

3. List examples of geometric figures of the following dimensions:

    i. one dimension.

    ii. two dimensions.

    iii. three dimensions.

LESSON TWO: Projection of lines on planes

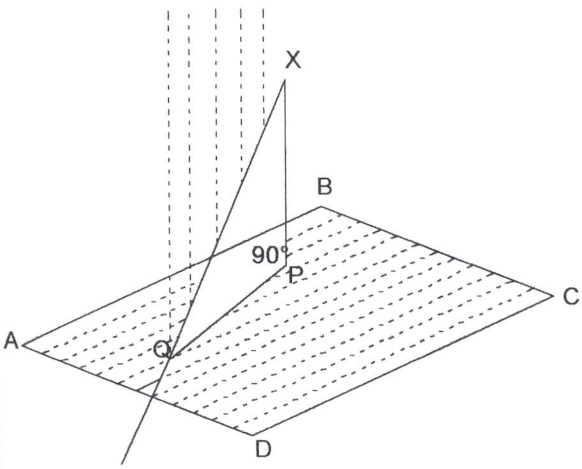


Figure 1.

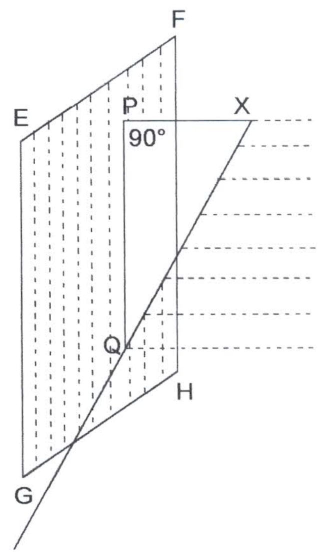


Figure 2.

1. Figure 1 shows the shadow, QP, of a chalkboard ruler (QX) when a torch is shone vertically above the ruler.

   i. Justify the value shown for the size of angle QPX.

   ii. Name the angle between the ruler and the ground (ABCD).

   iii. As light shines on the ruler, (QX), how does line QX relate to line QP?

   iv. Line QP is the projection of line QX on the horizontal plane ABCD.

2. In figure 2, name:

i. the shadow of the chalkboard ruler (QX) when the source of light is horizontal against a vertical wall, EFHG..

ii. the angle formed between the ruler and the vertical plane EFHG.

iii. the projection of line QX on the plane EFHG.

Exercise 2

1 Awuono who is 1 meter tall stands 60 meters away from the foot of a tree and 20 meters away from the end of the shadow of the tree. He is looking at the top of the tree at an angle of elevation of 25.8°, calculate:

i. the height of the tree.

ii. the length of Awuono's shadow on the ground if he is seeing the sun at the same angle of elevation as the top of the tree.

LESSON THREE: Angle between two lines

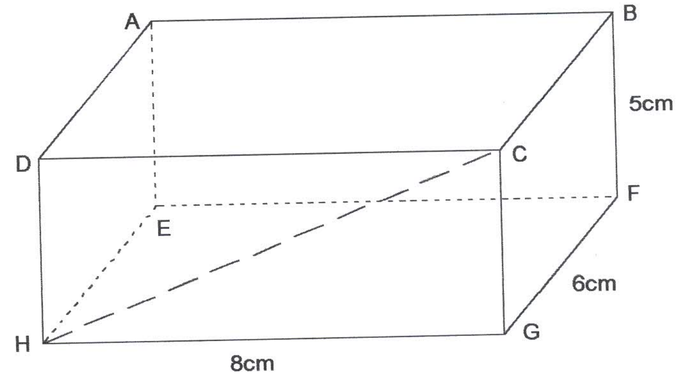


Figure 3

1. Using figure 3, ask learners to name:

   i. any two parallel lines.

   ii. any two intersecting lines.

2. Lines AB and GC are NOT parallel yet they do not meet. They are called?

3. From figure 3, name another pair of lines that are not parallel yet do not meet.

4. Using the diagram of a cuboid:

   i. Place a ruler or any straight edge to coincide with line AD.

   ii. Slide the straight edge in a parallel manner to line AD until it coincides with line BC.

   iii. Name the angle formed between the image of line AD after translation (BC) and GB.

   iv. The angle between the skew lines AD and GB is the angle between GB and BC.

5 Lines AD and HB are neither intersecting nor parallel. Name the angle between them.

5. Calculate the angle between skew lines AH and BC.

6. Calculate the angle between lines AG and GE.

Exercise 3

1. Calculate the angle between skew lines GE and AB.

2. Calculate the angle between lines AB and BE.

LESSON FOUR: Angle between a line and a plane

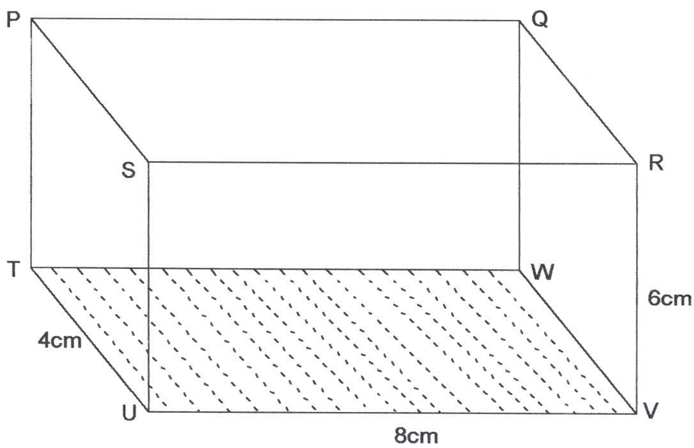


Figure 4.

1. Figure 4 is a diagram of a cuboid. Use it to name:

   i. any two vertices of the cuboid.

   ii any two planes of the cuboid.

   iii any two edges of the cuboid.

   iv any two right angles.

   v any two angles whose sizes are less than 90°

   vi any angle whose size is greater than 90°

   vii the projection of line SW on the plane TWVU.

   viii the projection of line SW on the plane PSUT.

   ix the angle between line PV and the plane TWVU.

2. i. Sketch and indicate the length of the sides of triangle PTV.

ii Calculate the angle between line PV and the plane TWVU.

Exercise 4

1. Calculate the angle between line WU and the plane PQRS.

2. A cone has a diameter of 6cm and a height of 7cm. Calculate the angle between the slant height and the base of the cone.

LESSON FIVE: Angle between two planes

1 Instruct learners to open their mathematics exercise books as shown in figure 5.

Using figure five and the opened book, name:

i two planes that you can observe.

ii a line where two planes intersect.

iii an angle between two planes.

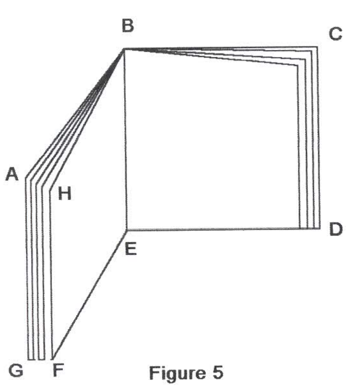


Figure 5

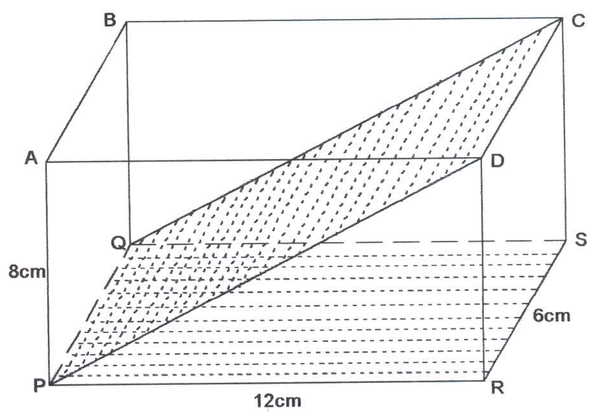


Figure 6.

2. In figure 6:

i name and calculate the angle formed between the shaded planes.

ii calculate the angle between the planes ABQP and PQCD:

Exercise 5

1. A door two meters high by 1.5 meters wide is opened to an angle of 50° as shown in figure 7.

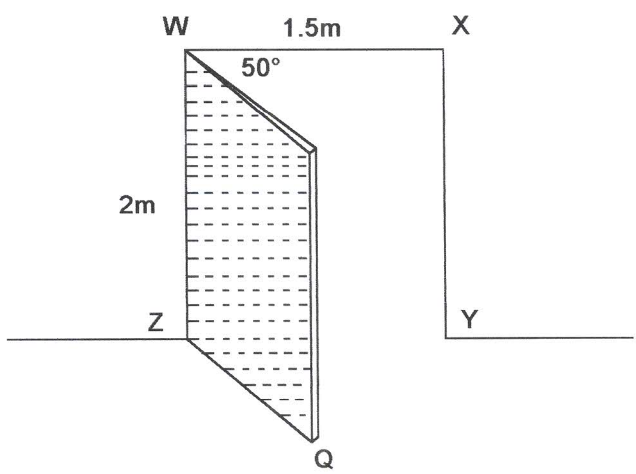


Figure 7

Calculate the angle between the planes YZQ and WQY.

LESSON SIX: Calculating lengths and angles in solids

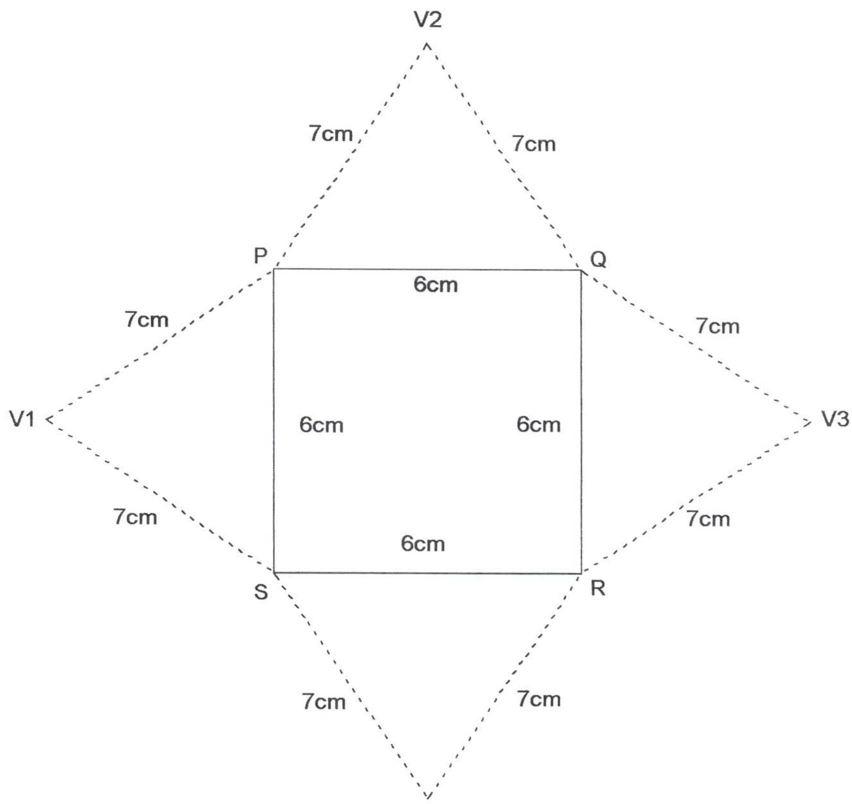


Figure 8

1. Using figure 8:

   i. state a suitable title for the drawing in figure 8.

   ii. redraw figure 8 accurately on a sheet of manila paper.

   iii. use a razor blade or a pair of scissors to cut along the continuous lines leaving flaps for folding.

   iv. name the solid formed when you fold along the dotted lines such that V1, V2, V3, and V4 all coincide at one point V.

   v. calculate the height of the solid object formed.

vi. sketch the solid object.

vii. calculate the angle between line VP and the base PQRS.

viii. calculate the angle between the planes VQR and PQRS.

2. Using figure 9, calculate:

i. the length of line MK.

ii. the angle between line OK and the plane JKLM.

iii. the angle between the planes KON and JKLM.

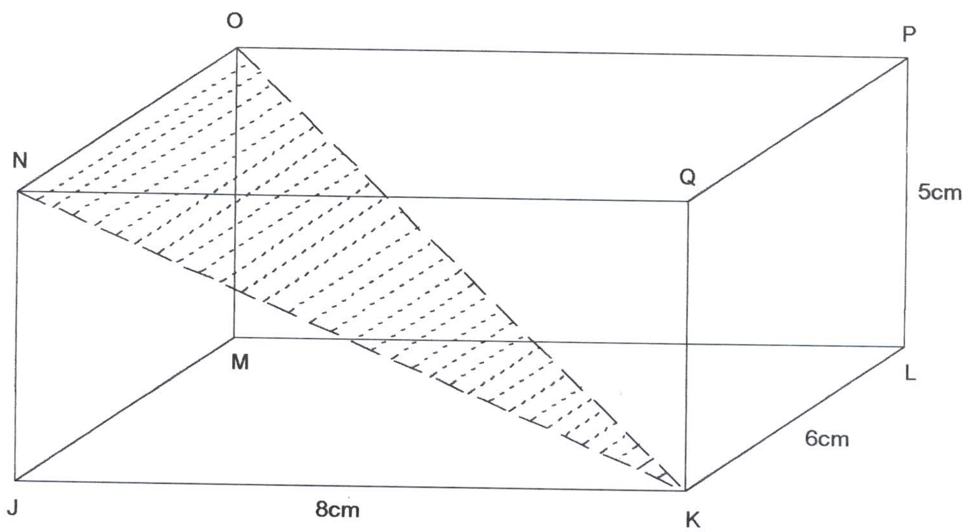


Figure 9

Exercise 6

1. Figure 10 is a cuboid with a square base. B1B3=15cm, T3B3=12cm and K is the mid point of B1B3 Calculate:

i. the length of line B1T3.

ii. the area of triangle T3B1K.

iii. the length of line T1B2.

iv. the volume of the cuboid.

v. the angle between the plane T1T2B3B2 and the base of the cuboid.

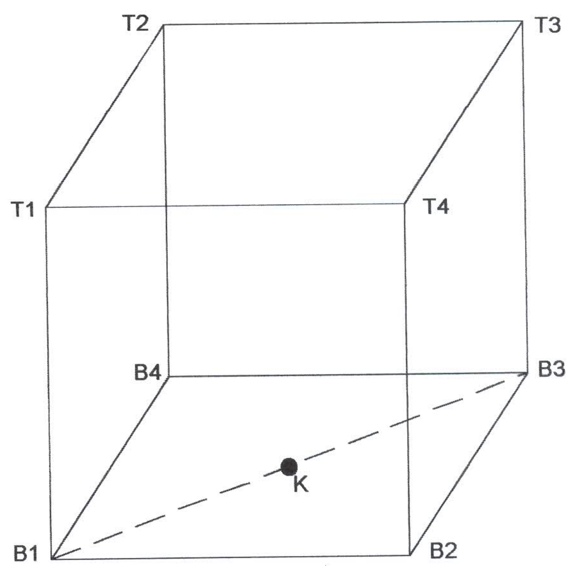


Figure 10

A.34 Treatment T2

The approach that was used with the group of participants who received experimental treatment T2 was learner centred and characterized by experimentation, interaction with the learners' environment and inquiry. Learners were expected to interact freely with their environment, which included the teacher, fellow students, charts, models and textbooks. The models used were those of *three-dimensional* objects. The teachers' role was that of a guide who introduced helpful interventions during the learning process. The teacher introduced learners to the concept that was to be learned, guided them to explore and discover relations, generate patterns and to generalize. The teacher was expected to encourage three modes of interaction in the classroom namely the teacher-learner interaction, learner-learner interaction, and learner-environment interaction. Teacher interventions were to be given in the form of probing questions and provision of feedback to learners' questions and responses. Learning activities included analysis of objects through observations, measurements and modelling. Learners were also expected to name thee D objects, lines, planes and angles. They were also expected to calculate lengths and sizes of angles and establish relations among the components of *three-D* objects.

LESSON ONE: geometric properties of common solids

1. Present models of the following solids to the learners:

   Cuboid, cylinder, cone, pyramid, sphere, tetrahedron, frustum of a pyramid and frustum of a cone.

2. Learners should then copy and complete the table below after observing the models presented.

Solid	Number of surfaces		Number of edges	Number of vertices
	Flat	Curved		
Cuboid				
Cylinder				
Cone				
Pyramid				
Sphere				
Tetrahedron				
Frustum of a pyramid				
Frustum of a cone				

3. Learners to state five properties of solids.

Exercise 1

1. Name two solids that have no vertices.

   ....

2. State two differences between plane objects and solid objects.

   ....

   .

   ....

   .

   ....

   .

3. List examples of geometric figures of the following dimensions:

   i. one dimension. ....

   ....

   ii. two dimensions. ....

   ....

   iii. three dimensions. ....

   ....

LESSON TWO: Projection of lines on planes

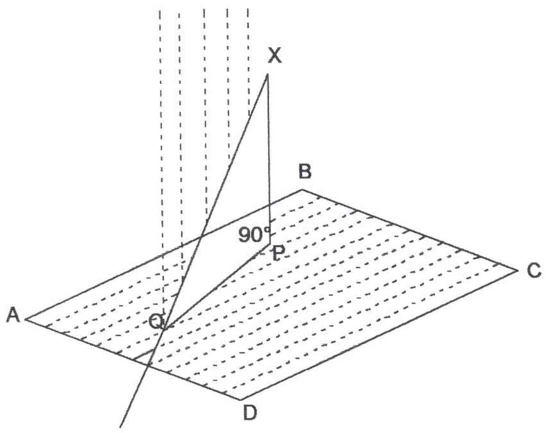


Figure 1.

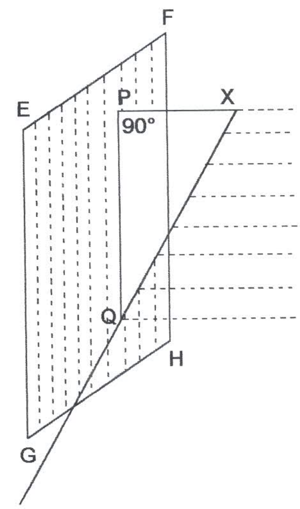


Figure 2.

1. Figure 1 shows the shadow, QP, of a chalkboard ruler (QX) when a torch is shone vertically above a horizontal floor.

   i. Justify the value shown for the size of angle QPX.

   ii. Name the angle between the ruler and the ground (ABCD).

   iii. As light shines on the ruler, (QX), how does line QX relate to line QP?

   iv. Line QP is the projection of line QX on the horizontal plane ABCD.

2. In figure 2, name:

   i. the shadow of the chalkboard ruler (QX) when the source of light is horizontal against a horizontal wall, EFHG.

   ii. the angle formed between the ruler and the vertical plane EFHG.

   iii. the projection of line QX on the plane EFHG.

Exercise 2

1. Awuono who is 1 meter tall stands 60 meters away from the foot of a tree and 20 meters away from the end of the shadow of the tree. He is looking at the top of the tree at an angle of elevation of 25.8°, calculate:

   i. the height of the tree.

   ii. the length of Awuono's shadow on the ground if he is seeing the sun at the same angle of elevation as the top of the tree.

LESSON THREE: Angle between two lines

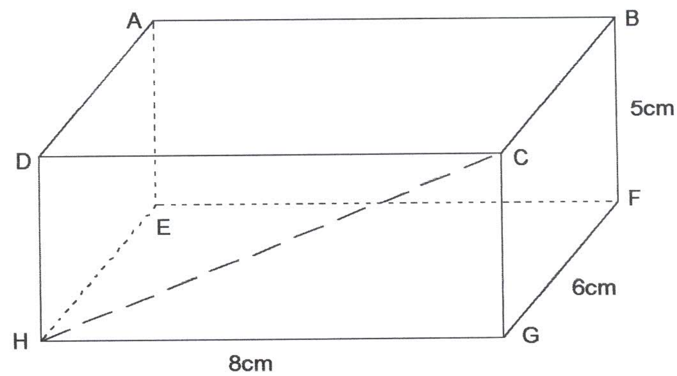


Figure 3

1. Using figure 3, name:

   i. any two parallel lines.

   ii. any two intersecting lines.

2. Lines AB and GC are NOT parallel yet they do not meet. They are called?

3. From figure 3, name another pair of lines that are not parallel yet do not meet.

4 Using the skeleton model of a cuboid:

   i Place a ruler or any straight edge to coincide with line AD.

   ii. Slide the straight edge in a parallel manner to line AD until it coincides with line BC.

   iii. Name the angle formed between the image of line AD after translation (BC) and GB.

   iv. The angle between the skew lines AD and GB is the angle between GB and BC.

5. Lines AD and HB are neither intersecting nor parallel. Name the angle between them.

6. Calculate the angle between skew lines AH and BC.

7. Calculate the angle between lines AG and GE.

Exercise 3

1. Calculate the angle between skew lines GE and AB.

2. Calculate the angle between lines AB and BE.

LESSON FOUR: Angle between a line and a plane

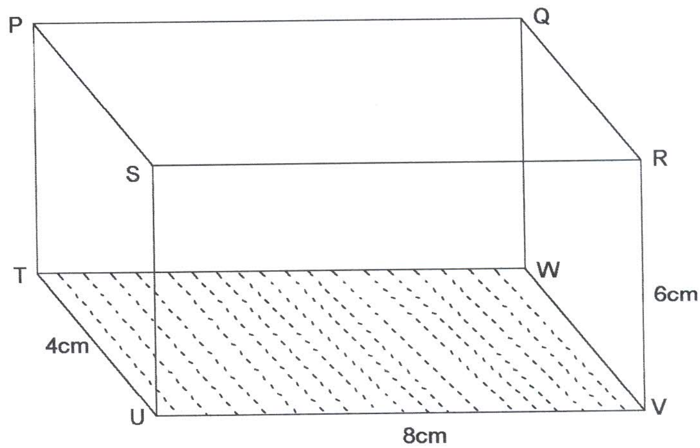


Figure 4.

1. Figure 4 is a diagram of a cuboid. Use it to name:

   i. any two vertices of the cuboid.

   ii. any two planes of the cuboid.

   iii. any two edges of the cuboid.

   iv. any two right angles.

   v. any two angles whose sizes are less than 90°

   vi. any angle whose size is greater than 90°

   vii. the projection of line SW on the plane TWVU.

   viii. the projection of line SW on the plane PSUT.

   ix. the angle between line PV and the plane TWVU.

2 a. Sketch and indicate the length of the sides of triangle PTV.

b. Calculate the angle between line PV and the plane TWVU.

3. Locate triangle PTV on the skeleton model of a cuboid.

Exercise 4

1 Calculate the angle between line WU and the plane PQRS.

2 A cone has a diameter of 6cm and a height of 7cm. Calculate the angle between the slant height and the base of the cone.

LESSON FIVE: Angle between two planes

1. Instruct learners to open their mathematics exercise books as shown in figure 5.

   Using figure five and the opened book, name:

   i. two planes that you can observe.

   ii. a line where two planes intersect.

   iii. an angle between two planes.

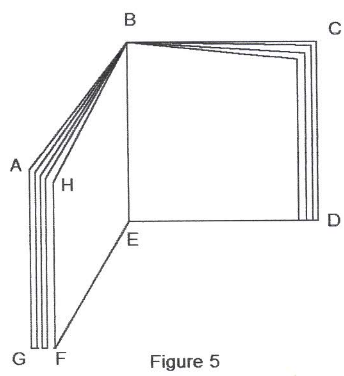


Figure 5

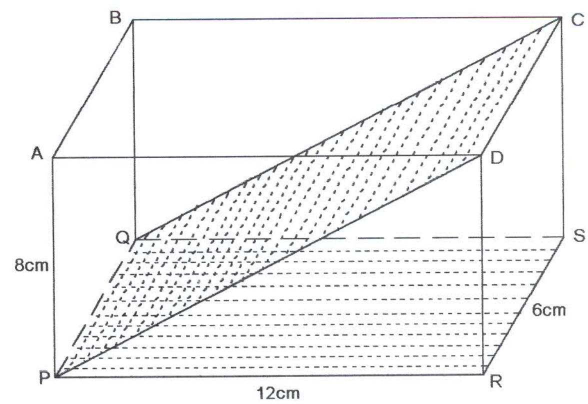


Figure 6.

Exercise 5

1 A door two meters high by 1.5 meters wide is opened to an angle of 50° as shown in figure 7. Calculate the angle between the planes YZQ and WQY.

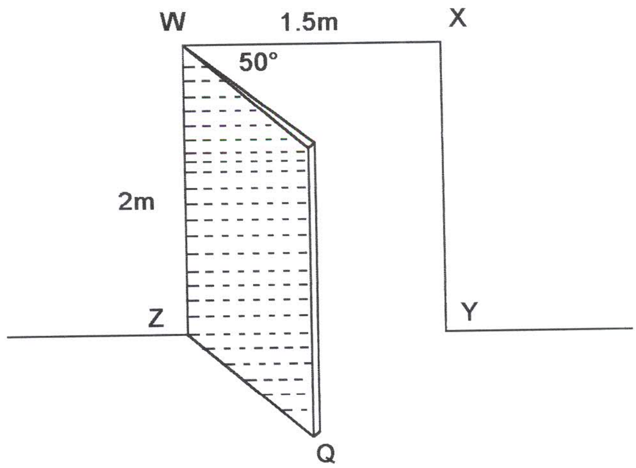


Figure 7

LESSON SIX: Calculating lengths and angles in solids

1. Using figure 8:

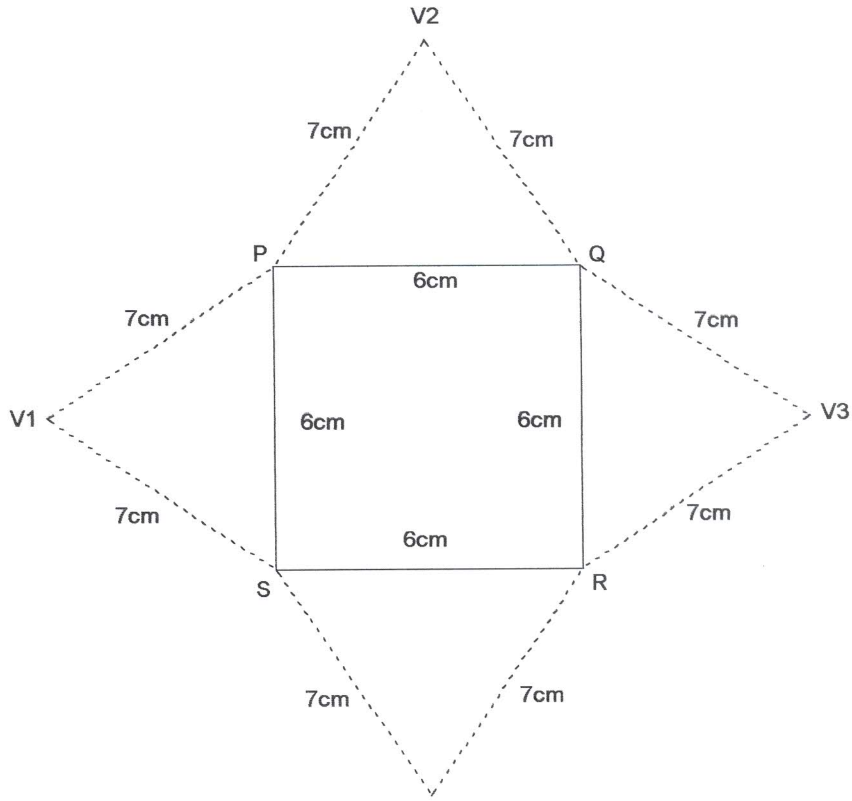


Figure 8

i. state a suitable title for the drawing in figure 8.

ii. redraw figure 8 accurately on a sheet of manila paper.

iii. use a razor blade or a pair of scissors to cut along the continuous lines leaving flaps for folding.

iv. name the solid formed when you fold along the dotted lines such that V1, V2, V3, and V4 all coincide at one point V.

v. calculate the height of the solid object formed.

vi. sketch the solid object.

vii. calculate the angle between line VP and the base PQRS.

viii. calculate the angle between the planes VQR and PQRS.

2. Using figure 9, calculate:

   i. the length of line MK.

   ii. the angle between line OK and the plane JKLM.

   iii. the angle between the planes KON and JKLM.

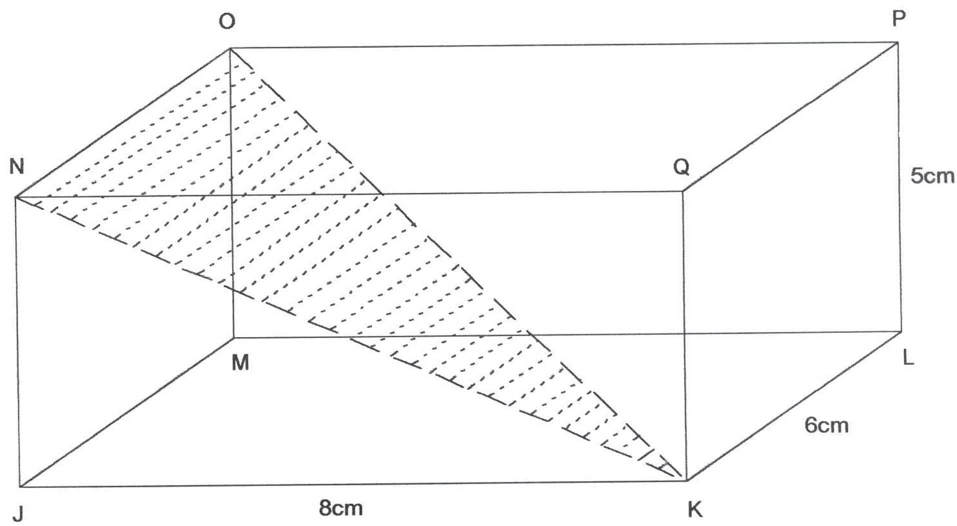


Figure 9

Exercise 6

1. Figure 10 is a diagram of a cuboid with a square base. B1B3=15cm, T3B3=12cm and K is the mid point of B1B3. Calculate:

   i. the length of B1T3

   ii. the area of triangle T3B1K

   iii. the length of line T1B2

   iv. the volume of the cuboid.

   v. the angle between the plane T1T2B3 B2 and the base of the cuboid.

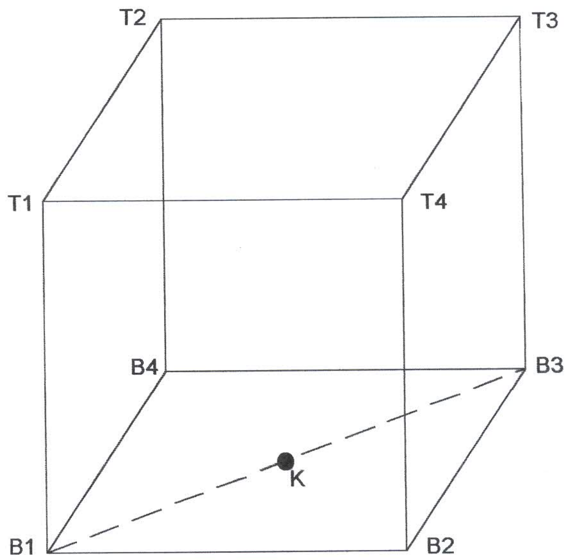


Figure 10

A.40 POSTTEST

Name ....

School ....

Instructions:

1. Attempt all questions in the spaces provided after each question.

2. Show all your working.

Q1. State the difference(s) between the following objects:

i) A point and a line.

....

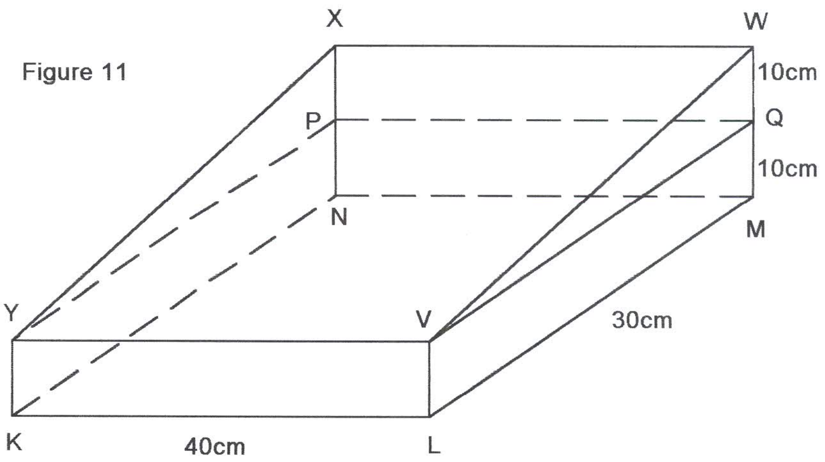
....

ii) A rectangle and a cuboid.

....

....

Q2. Figure 11 below shows the top of a desk. The edges KY, LV, MW, and NX are perpendicular to the base. The base KLMN is a rectangle measuring 40cm by 30cm. VWXY is a rectangle. KY = LV= 10cm, NX = MW = 20cm, QW = 10cm.



a). Name the angle between the planes KLMN and PQVY.

b). Calculate:

i. the projection of line WY on the horizontal.

ii. the angle between line WY and the plane PQVY.

Q3. A rectangular sloping ground is 24m wide, 18m long and slopes at an angle of 30° to the horizontal as shown in figure 12. Calculate the angle between line FJ and line GJ.

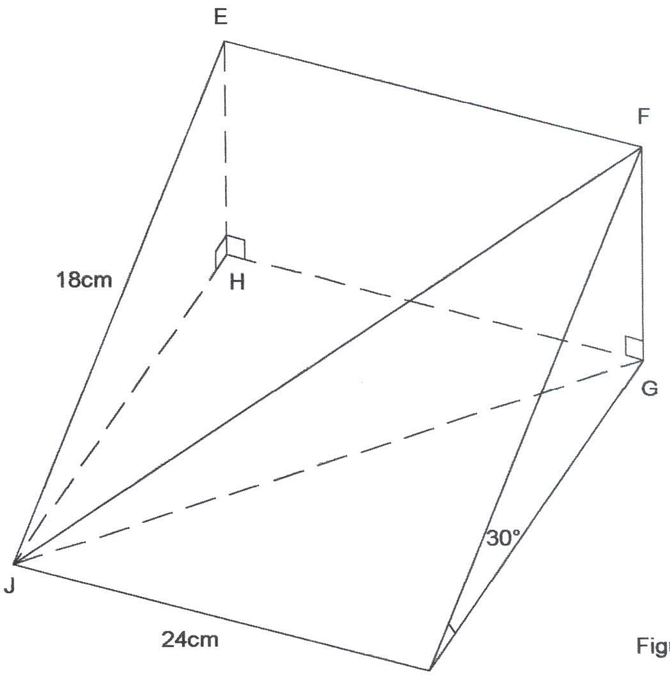


Figure 12

Q4. Figure 13 shows the diagram of a cuboid in which PQ = 8cm, PS = 6cm, and RY = 10cm. T is the mid point of PW.

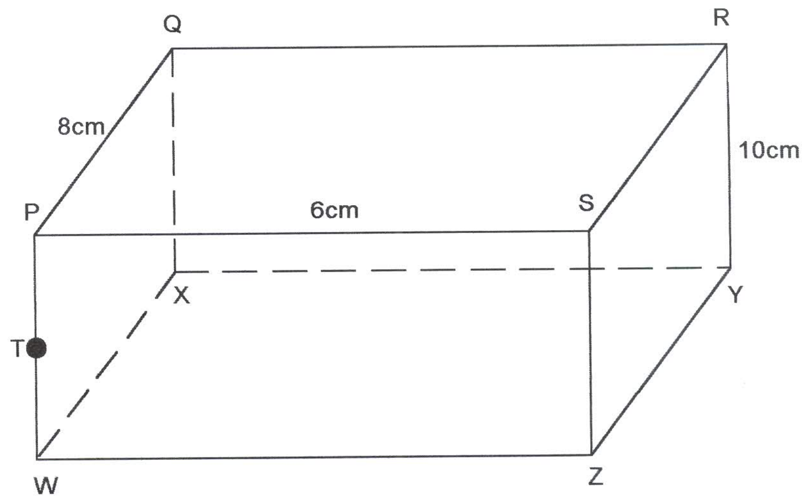


Figure 13

Calculate:

a. the angle between skew lines WY and QR.

b. the angle between line TY and the plane PQWX.

c. the angle between the planes TYZ and WXYZ:

Q5. Figure 14 shows the net of a solid. GE = 10cm, GF = 8cm and GK = 6cm.

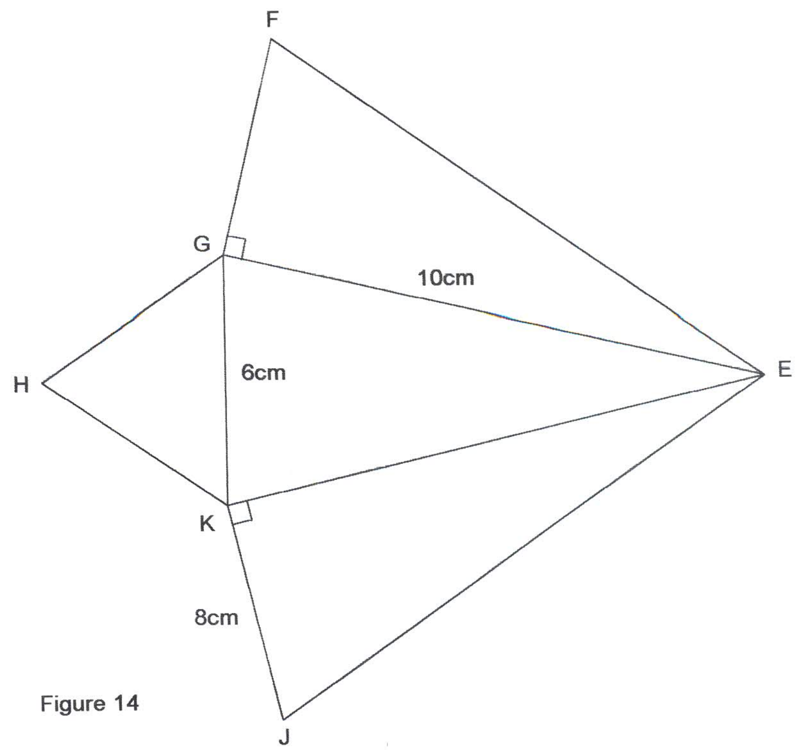


Figure 14

a. i. Count the number of edges of the solid.

....

....

ii. Count the number of vertices of the solid.

....

....

b. Calculate the length EH after the net is folded to form the solid.

A.50 POST TEST GRADING KEY

Item	Suggested Response	Remarks
1 i	- A point is zero dimensional while a line is one dimensional. - A set of points constitute a line (which may be straight or curved).	1 mark for any one correct response
ii	- A rectangle is a plane object while a cuboid is a space object. - A rectangle is two dimensional while a cuboid is three-dimensional. - A rectangle has one face while a cuboid has five faces when open and six faces when closed. - A cuboid has intersecting planes while a rectangle has none.	1 mark for any one correct response. Max. 2 marks.
2 a	- The two planes are parallel - The angle between them is o° or 180°	1 mark for either.
b i	The projection of line WY on the horizontal is line QY. $QY^2 = QV^2 + VY^2$ $QY^2 = 30^2 + 40^2$ $= 2500$ $QY = 50$ cm.	1 mark. 1 mark 1 mark
ii	The angle between line WY and the plane PQVY is WYQ. From triangle WYQ, tan WYQ = 10/50 = 0.200. angle WYQ = 11.31°.	1 mark. 1 mark 1 mark. Max. 7 marks.
3	From triangle JFK, $JF^2 = 18^2 + 24^2$ $= 900$ JF = 30 m. From triangle GFK, FG = 18sin30° FG = 9 m. Angle FJG = $\sin^{-1}(9/30)$ $= \sin^{-1}(0.3)$ = 17.46°	1 mark 1 mark 1 mark 2.marks 1 mark. 1 mark. 1 mark.

4 a.	The angle between skew lines WY and QR is angle XYW or angle QRP From triangle WXY (QRP), $WY^2=WX^2+XY^2$ (or $PR^2=PQ^2+QR^2$). $WY^2=PR^2=8^2+6^2$. $=64+36$ $=100$ WR=PR=10cm Sin XYW=SinQRP =8/10 =0.8 $\sin^{-(0.8)}=53.13^0$. AngleXYW=QRP=53.13^0.	1 mark 1 mark 1 mark 1 mark
4 b.	The angle between line TY and the plane PQWX is XTY. In triangle TYX, $TX^2 = TW^2 + WX^2$ $= 5^2 + 8^2$ $= 89$ $TX = \sqrt{89}$. Angle XTY = $\tan^{-1}(6/\sqrt{89})$. = 32.46°	1 mark. 1 mark 1 mark 1 mark 1 mark.
4 c	The angle between the planes TYZ and WXYZ is angle TZW. Angle TZW = $\tan^{-1}(5/6)$ = 39.81°	1 mark 1 mark 1 mark.
5 a i.	The solid has 6 edges	1 mark.
ii.	The solid has 4 vertices	1 mark.
b	The required distance is EF. In triangle EFG, $EF^2= GF^2 + GE^2$ $=8^2 + 10^2$ $= 64 + 100$ $= 164$ EF = 12.81 cm.	1 mark 1 mark 1 mark. Max. 5 marks

A.60 POSTTEST SCORES

PRETESTED PARTICIPANTS WHO RECEIVED TREATMENT D1

83	71	65	58	56	54	53	47	44	43	42	38	50	29	28	26
25	25	24	24	22	21	19	18	17	15	15	14	14	11	11	11
10	10	10	8	8	8	7	6	54	54	51	50	47	44	44	42
42	40	40	40	40	40	39	39	38	38	35	35	31	31	26	25
24	15	14	14	14	13	13	8	6	4	3	3	1	1	0	0
78	72	67	58	53	53	53	51	44	44	42	40	35	32	26	25
25	25	24	22	22	21	19	18	17	14	14	14	13	11	11	11
10	10	8	8	8	7	7	19	78	72	63	60	58	58	56	53
53	50	46	46	40	40	39	38	38	36	36	36	36	35	33	33
32	31	29	26	26	24	22	22	22	22	17	17	17	15	14	14

UNPRETESTED PARTICIPANTS WHO RECEIVED TREATMENT D1

72	69	69	69	67	61	58	58	58	56	56	56	56	56	54	53
51	51	47	44	44	44	42	39	33	33	31	31	31	25	25	22
17	11	11	11	8	8	3	3	46	44	44	43	42	42	40	40
39	38	33	32	31	31	31	24	22	18	18	17	17	17	15	15
13	13	13	11	11	10	10	10	10	10	8	8	7	7	6	4
89	86	86	82	81	78	76	72	71	67	67	67	65	65	64	63
63	61	60	58	58	56	54	53	53	51	51	50	49	47	47	44
43	43	43	39	38	35	32	28	61	60	56	56	56	56	47	47
47	39	36	36	35	33	32	22	21	21	21	19	18	15	14	14
11	11	11	8	8	8	7	7	7	7	6	4	4	3	3	0

PRETESTED PARTICIPANTS WHO RECEIVED TREATMENT D2

85	76	75	74	74	53	72	72	72	72	67	64	63	61	61	61
61	58	58	56	53	50	50	50	49	47	44	43	42	42	42	39
38	36	36	33	29	25	25	18	79	67	67	65	61	53	51	47
46	44	43	42	36	36	32	31	28	28	25	25	22	22	22	21
19	18	15	14	14	13	13	10	10	8	7	6	6	6	3	0
58	46	44	44	43	42	39	38	38	35	33	33	33	32	32	32
31	31	29	28	28	26	26	24	22	21	19	19	18	18	18	15
15	15	15	14	14	14	14	13	63	58	57	57	57	57	57	56
56	56	56	54	54	54	54	53	53	53	53	53	53	53	51	51
51	50	50	50	50	49	49	46	44	43	42	40	38	28	19	18

UN-PRETESTED PARTICIPANTS WHO RECEIVED TREATMENT D2

85	71	53	53	50	49	47	46	46	42	36	35	35	33	32	28
25	25	24	22	22	21	19	18	18	17	15	15	14	14	13	11
10	8	8	7	6	6	4	3	99	99	93	93	90	90	85	83
83	83	82	79	79	78	74	74	74	74	71	69	68	65	64	64
63	63	58	56	54	50	50	47	46	46	40	39	32	32	26	25
85	83	65	64	58	56	51	51	51	49	47	46	46	46	43	42
42	40	40	40	40	40	38	35	35	35	33	32	32	29	29	29
29	28	28	26	26	25	22	21	79	71	68	65	65	65	65	64
57	57	53	53	51	51	50	47	46	46	43	40	39	38	38	38
38	33	33	31	31	31	29	24	18	18	11	3	3	3	3	1

PRETESTED PARTICIPANTS WHO RECEIVED TREATMENT T1

97	96	96	94	94	93	92	92	92	86	72	72	72	69	69	67
56	53	50	50	47	40	39	36	36	33	33	28	25	24	22	21
19	18	17	14	11	10	10	8	53	50	44	44	44	43	43	40
40	38	36	36	35	33	33	33	32	32	31	29	29	28	28	26
26	25	24	24	24	22	22	22	21	19	18	18	18	17	17	15
69	63	57	49	47	46	36	33	33	28	28	28	28	26	26	25
25	25	22	22	18	17	15	14	14	14	14	14	14	14	14	13
13	13	13	13	13	13	13	11	97	97	97	97	97	94	94	92
89	75	72	69	69	64	56	51	50	49	43	38	36	36	36	33
31	28	28	26	25	19	17	14	11	11	11	8	7	6	6	4

UN-PRETESTED PARTICIPANTS WHO RECEIVED TREATMENT T1

89	75	75	75	69	69	69	67	64	61	58	58	58	56	53	50
50	47	44	44	44	44	42	39	39	39	39	36	36	36	36	36
33	33	29	29	28	28	25	22	81	67	65	65	64	61	60	58
57	56	53	47	47	47	42	42	36	35	35	32	31	28	25	24
22	21	21	19	18	17	15	15	15	14	14	14	14	14	11	10
64	63	51	47	47	47	44	40	36	35	33	31	28	28	26	26
26	25	22	22	22	21	21	21	19	18	17	17	17	17	14	14
14	14	13	13	11	11	11	11	78	75	63	61	53	47	47	47
43	42	40	40	39	39	36	36	35	33	33	32	32	32	31	31
29	28	28	28	26	26	25	22	22	21	21	19	18	18	18	17

PRETESTED PARTICIPANTS WHO RECEIVED TREATMENT T2

82	76	75	67	63	61	61	58	58	57	56	56	51	44	44	43
40	39	39	36	36	35	33	32	31	31	28	28	25	25	25	24
22	22	22	21	19	19	19	18	92	86	86	83	79	78	76	75
65	64	63	58	57	57	50	50	50	50	47	44	44	44	44	43
42	42	42	40	40	39	39	38	33	28	28	25	22	22	19	17
90	89	85	83	81	76	75	75	75	71	69	68	67	64	63	61
58	58	57	54	47	46	44	43	40	39	38	36	35	33	33	32
32	32	32	31	29	29	28	26	81	72	69	67	67	64	64	63
61	56	53	50	44	44	42	38	36	36	36	33	32	31	31	28
24	22	21	19	17	17	17	17	17	17	15	15	11	11	10	8

UN-PRETESTED PARTICIPANTS WHO RECEIVED TREATMENT T2

96	72	69	60	64	58	54	54	53	47	47	46	46	44	43	43
43	42	42	42	40	40	40	39	39	39	39	38	36	36	35	33
33	32	31	29	28	28	25	25	78	72	72	67	61	60	56	54
54	53	51	51	50	50	49	47	47	46	42	42	40	40	39	39
39	39	36	35	33	33	31	29	28	25	25	24	24	22	22	19
99	94	93	89	89	86	83	82	82	75	75	75	71	69	68	68
67	65	65	64	64	61	61	61	61	61	60	60	60	58	57	57
56	54	53	53	53	53	51	51	85	81	78	74	72	71	69	69
68	67	65	64	64	63	61	61	58	57	56	56	56	54	54	54
53	53	53	51	50	47	46	46	44	43	42	38	35	32	31	28

A.70 The syllabus

Specific objectives

At the end of this topic, the learner should be able to:

a). state geometric properties of common solids.

b). identify projection of lines on planes.

c). identify and calculate;

  i). the angle between a line and a line (including skew lines).

  ii). the angle between a line and a plane.

  iii). the angle between two planes.

d). calculate length of lines in three dimensions.

Content.

i Geometric properties of common solids.

ii. Projection of a line on a plane.

iii. The angle between a line and a line (including skew lines).

iv. The angle between a line and a plane.

v. The angle between two planes.

vi. The skew lines.

vii. Angles between skew lines.