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The Application of the Method of Horace to Get Number of Generators for an Ideal of s General Points in $\mathbf{P}^4$

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Abstract. Let S be a general set of s points in $\mathbf{P}^4$, and R the homogeneous coordinate ring of $\mathbf{P}^4$. Then the ideal of S, I_S has a minimal free resolution of the form:

$$0 \longrightarrow F_3 \longrightarrow F_2 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow I_S \longrightarrow 0$$

where $F_p = R(-d-p)^{a_{p-1}} \bigoplus R(-d-p-1)^{b_p}$, d being the smallest integer satifying $s \leq h^0(\mathbf{P}^4, \mathcal{O}_{\mathbf{P}^4}(d))$ and $a_p = h^0(\mathrm{T}_S \otimes \Omega^{p+1}_{\mathbf{P}^4}(d+p+1))$, $b_p = h^1(\mathrm{T}_S \otimes \Omega^{p+1}_{\mathbf{P}^4}(d+p+1))$ and $\binom{d+3}{4} < s \leq \binom{d+4}{4}$, with $0 \leq p \leq 3$ and when $p = 0$, we would have $a_{p-1} = \binom{d+4}{4} - s$ and when $p = 3$ then $b_p = s - \binom{d+3}{4}$. In this paper I prove that either $a_0 = 0$ or $b_0 = 0$ by proving maximal rank for the map:

$$H^0\big(\Omega_{\mathbf{P}^4}(d+1)\big) \longrightarrow \bigoplus_{i=1}^{s} \Omega_{\mathbf{P}^4}(d+1)_{|S_i}$$

by use of the methods of Horace to prove bijectivity for a specific number of fibres and then maximal rank for a general set.

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1. INTRODUCTION

The Minimal Resolution Conjecture (MRC) was first explicitly formulated by A Lorenzini in her PhD thesis [3] and it deals with the question of the form of the minimal free resolution for ideals of general points in projective spaces i.e. for a general set of points $\{P_1, \dots, P_s\} \in \mathbf{P}^n$, with $s \geq n+1$, then the homogeneous ideal of the sub-scheme of the union of these points, $I_S \subset R = \mathbf{k}[x_0, \dots, x_n]$, $\mathbf{k}$ an algebraically closed field and R the homogeneous coordinate ring of $\mathbf{P}^n$, has the following expected form:

$$0 \longrightarrow F_{n-1} \longrightarrow \cdots \longrightarrow F_p \longrightarrow \cdots \longrightarrow F_0 \longrightarrow I_S \longrightarrow 0,$$

$F_p = R(-d-p)^{a_{p-1}} \bigoplus R(-d-p-1)^{b_p}$,

d being the smallest integer satifying $s \leq h^0(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(d))$, thus

$a_p = \max\{0, \mathrm{rk}(\Omega^{p+1}_{\mathbf{P}^n})s - h^0(\mathbf{P}^n, \Omega^{p+1}_{\mathbf{P}^n}(d+p+1))\}$,

$b_p = \max\{0, h^0(\mathbf{P}^n, \Omega^{p+1}_{\mathbf{P}^n}(d+p+1)) - \mathrm{rk}(\Omega^{p+1}_{\mathbf{P}^n})s\}$ and

$$\binom{d+n-1}{n} < s \leq \binom{d+n}{n}.$$

The problem can be reduced to showing the following; for all $0 \leq p \leq n-1$ and non-negative integers l then existence of the above resolution is the same as saying the evaluation map below is of maximal rank i.e. it is surjective or injective or both.

$$H^0\big(\mathbf{P}^n, \Omega^{p+1}_{\mathbf{P}^n}(l)\big) \longrightarrow \bigoplus_{i=1}^{s} \Omega^{p+1}_{\mathbf{P}^n}(l)_{|P_i}.$$

C Walter [5] tackled the minimal free resolution for $\mathbf{P}^4$ in which his work yields many values but misses out the most difficult values. He gave bounds for the dimension of H^0 for which the homogeneous ideal I_S of s general points in $\mathbf{P}^4$ does not satisfy the MRC (i.e. $a_p b_p \neq 0$ for some p). In this paper, I prove that $a_p = 0$ or $b_p = 0$ for $p = 0$ which inturn implies that $a_p b_p = 0$. See the sequence below:

$$\cdots \longrightarrow R(-d-2)^{b_1} \bigoplus R(-d-1)^{a_0} \longrightarrow R(-d-1)^{b_0} \bigoplus R(-d)^{a_{-1}} \longrightarrow I_S \longrightarrow 0$$

which is deduced from the following from the proposition that is a particular case of the Minimal Resolution Conjecture[2]:

Proposition 1.1. *Let* $\boldsymbol{k}$ *be an algebraically closed field,* $\mathbf{P}^4$ *be a projective space over* $\boldsymbol{k}$ *and* $R = \boldsymbol{k}[X_0, X_1, X_2, X_3, X_4]$ *be the homogeneous coordinate ring of* $\mathbf{P}^4$*. If* $S = \{P_1, P_2, ..., P_s\}$ *is a general set of* s *points in* $\mathbf{P}^4$*, with* $s \geq 5$*, then the ideal,* I_S *has the expected minimal resolution if the map*

$$\mu : H^0\big(\mathbf{P}^4, \Omega_{\mathbf{P}^4}(d+1)\big) \longrightarrow \bigoplus_{i=1}^{s} \Omega_{\mathbf{P}^4}(d+1)_{|P_i}$$

is of maximal rank.

We wish to prove that μ is of maximal rank and as a consequence we have the following theorem.

Theorem 1.2. *Suppose we have a general set S, of s points in $\mathbf{P}^4$, $s \geq 5$ such that the map $\mu : H^0\big(\mathbf{P}^4, \Omega_{\mathbf{P}^4}(d+1)\big) \longrightarrow \bigoplus_{i=1}^{s} \Omega_{\mathbf{P}^3}(d+1)_{|S_i}$ is of maximal rank then the homogeneous ideal $I_S \subset \mathbf{k}[X_0, X_1, X_2, X_3, X_4]$ has $\big(\frac{1}{6}d(d+2)(d+3)(d+4) - 4m\big)_+$ number of minimal generators of degree $d+1$ and $\big(\frac{1}{6}d(d+2)(d+3)(d+4) - 4m\big)_-$ number of minimal relations of degree $d+1$, where $(x)_+ = \max(x, 0)$ and $(x)_- = \max(-x, 0)$.*

1.1. **Preliminaries.** Here we start by giving the maximal rank hypotheses or statements (the so called *Enonces*) as in [1] by Hirschowitz and Simpson.

Let X be a smooth projective variety and X' non-singular divisor of X. Let F be a locally free sheaf on X and

$$0 \longrightarrow \mathrm{F}'' \longrightarrow \mathrm{F}_{|X'} \longrightarrow \mathrm{F}' \longrightarrow 0$$

be a exact sequence of locally free sheaves on X'. The kernel E of $\mathrm{F} \longrightarrow \mathrm{F}'$ is a locally free sheaf on X and we have another exact sequence of locally free sheaves on X'

$$0 \longrightarrow \mathrm{F}'(-X') \longrightarrow \mathrm{E}_{|X'} \longrightarrow \mathrm{F}'' \longrightarrow 0$$

and as well exact sequences of coherent sheaves on X

$$0 \longrightarrow \mathrm{E} \longrightarrow \mathrm{F} \longrightarrow \mathrm{F}' \longrightarrow 0$$

and

$$0 \longrightarrow \mathrm{F}(-X) \longrightarrow \mathrm{E} \longrightarrow \mathrm{F}'' \longrightarrow 0.$$

Hypothesis 1.3. $\boldsymbol{R}(\mathrm{F}, \mathrm{F}', y; a, b, c)$
Let y, a, b and c be non-negative integers. The hypothesis $\boldsymbol{R}(\mathrm{F}, \mathrm{F}', y; a, b, c)$ asserts that there exists a points, $U_1, \ldots, U_a$, and b points, $V_1, \ldots, V_b \in X'$ such that for the quotients

$$\mathrm{F}'_{U_i} \longrightarrow A_i \longrightarrow 0,$$

$$\mathrm{F}_{V_i} \longrightarrow B_i \longrightarrow 0$$

there exists the points $W_1, \ldots, W_c$ such that for the quotients

$$\mathrm{F}_{W_i} \longrightarrow C_i \longrightarrow 0$$

with the kernel in $\ker(\mathrm{F}_{W_i} \longrightarrow \mathrm{F}'_{W_i})$ *then for a non-negative integer z, there exists y points, $Y_1, \ldots, Y_y$ in X and z points $Z_1, \ldots, Z_z$ in X' such that the map below is bijective.*

$$H^0\big(X, \mathrm{F}\big) \longrightarrow \bigoplus_{i=1}^{a} A_i \oplus \bigoplus_{i=1}^{b} B_i \oplus \bigoplus_{i=1}^{c} C_i \oplus \bigoplus_{i=1}^{y} \mathrm{F}'_{Y_i} \oplus \bigoplus_{i=1}^{z} \mathrm{F}_{Z_i}$$

Hypothesis 1.4. $\boldsymbol{RD}(\mathrm{F}, \mathrm{F}', y; a, b, c)$
Let y, a, b and c be non-negative integers. The hypothesis $\boldsymbol{R}(\mathrm{F}, \mathrm{F}', y; a, b, c)$ asserts that there exists a points, $U_1, \ldots, U_a$, and b points, $V_1, \ldots, V_b \in X'$ such that for the quotients

$$\mathrm{F}'_{U_i} \longrightarrow A_i \longrightarrow 0,$$

$$\mathrm{F}_{V_i} \longrightarrow B_i \longrightarrow 0$$

there exists the points $W_1, \ldots, W_c$ *such that for the quotients*

$$\gamma(Y) : \mathrm{F}_{W_i} \longrightarrow C_i(Y) \longrightarrow 0$$

with the kernel in $\ker(\mathrm{F}_{W_i} \longrightarrow \mathrm{F}'_{W_i})$ *then for a non-negative integer* z*, there exists* y *points,* $Y_1, \ldots, Y_y$ *in* X *and* z *points* $Z_1, \ldots, Z_z$ *in* X' *such that the map below is bijective.*

$$H^0\big(X, \mathrm{F}\big) \longrightarrow \bigoplus_{i=1}^{a} A_i \oplus \bigoplus_{i=1}^{b} B_i \oplus \bigoplus_{i=1}^{c} C_i(Y_1 \ldots Y_y) \oplus \bigoplus_{i=1}^{y} \mathrm{F}'_{Y_i} \oplus \bigoplus_{i=1}^{z} \mathrm{F}_{Z_i}$$

Hypothesis 1.5. $\boldsymbol{RD}(\mathrm{E}, \mathrm{F}'', y'; a', b', c')$
Let y', a', b' *and* c' *be non-negative integers. The hypothesis* $\boldsymbol{R}(\mathrm{E}, \mathrm{F}'', y'; a', b', c')$ *asserts that there exists* a' *points,* $U_1, \ldots, U'_a$*, and* b' *points,* $V_1, \ldots, V'_b \in X'$ *such that for the quotients*

$$\mathrm{F}''_{U_i} \longrightarrow A_i \longrightarrow 0,$$
$$\mathrm{E}_{V_i} \longrightarrow B_i \longrightarrow 0$$

there exists the points $W_1, \ldots, W'_c$ *such that for the quotients*

$$\gamma(Y) : \mathrm{E}_{W_i} \longrightarrow C_i(Y) \longrightarrow 0$$

with the kernel in $\ker(\mathrm{E}_{W_i} \longrightarrow \mathrm{F}''_{W_i})$ *then for a non-negative integer* z'*, there exists* y' *points,* $Y_1, \ldots, Y'_y$ *in* X *and* z' *points* $Z_1, \ldots, Z'_z$ *in* X' *such that the map below is bijective.*

$$H^0\big(X, \mathrm{E}\big) \longrightarrow \bigoplus_{i=1}^{a'} A_i \oplus \bigoplus_{i=1}^{b'} B_i \oplus \bigoplus_{i=1}^{c'} C_i(Y_1 \ldots Y'_y) \oplus \bigoplus_{i=1}^{y'} \mathrm{F}''_{Y_i} \oplus \bigoplus_{i=1}^{z'} \mathrm{E}_{Z_i}$$

1.2. **Notation.** Since we are talking about the MRC for projective spaces and the méthode d'Horace then we set

$X = \mathbf{P}^4$, $X' = \mathbf{P}^3$, $\mathrm{F} = \Omega_{\mathbf{P}^4}$, $\mathrm{F}' = \Omega_{\mathbf{P}^3}$, $\mathrm{E} = \mathcal{O}_{\mathbf{P}^4}^{\oplus 4}(-2)$, $\mathrm{F}'' = \mathcal{O}_{\mathbf{P}^3}(-1)$.
The exact sequences of the elementary transformations after twisting by $d+1$ are:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \Omega_{\mathbf{P}^4}(d) & = & \Omega_{\mathbf{P}^4}(d) & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \mathcal{O}_{\mathbf{P}^4}(d-1)^{\oplus 4} & \longrightarrow & \Omega_{\mathbf{P}^4}(d+1) & \longrightarrow & \Omega_{\mathbf{P}^3}(d+1) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \mathcal{O}_{\mathbf{P}^3}(d) & \longrightarrow & \Omega_{\mathbf{P}^4}|_{\mathbf{P}^3}(d+1) & \longrightarrow & \Omega_{\mathbf{P}^3}(d+1) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

From which we have the hypotheses:
$\boldsymbol{H}'_{\mathcal{O},4}(d+1;\alpha,\beta,\gamma) = \boldsymbol{H}(\Omega_{\mathbf{P}^4}(d+1), \Omega_{\mathbf{P}^3}(d+1), \alpha, \beta, \gamma)$ and
$\boldsymbol{H}'_{\mathcal{O},4}(d-1;\rho,\sigma,\tau) = \boldsymbol{H}(\mathcal{O}_{\mathbf{P}^4}(d-1)^{\oplus n}, \mathcal{O}_{\mathbf{P}^3}(d); \rho, \sigma, \tau)$ and
$\boldsymbol{H}''_{\mathcal{O},4}(d-1;\rho,\sigma,\tau) = \boldsymbol{H}(\mathcal{O}_{\mathbf{P}^4}(d-1)^{\oplus n}, \mathcal{O}_{\mathbf{P}^3}(d); \rho, \sigma, \tau)$.

Our method is to prove inductively certain statements $\boldsymbol{H}_{\Omega,4}(d+1)$ and $\boldsymbol{H}_{\mathcal{O},4}(d-1)$. The exact statements roughly speaking are:

Hypothesis 1.6. $\boldsymbol{H}'_{\Omega,4}(d+1;\alpha,\beta,\gamma)$
The hypothesis $\boldsymbol{H}'_{\Omega,4}(d+1;\alpha,\beta,\gamma)$ asserts that for non-negative integers α, β γ and ε satisfying the conditions:

$0 \leq \gamma \leq 1$, *and* $1 \leq \varepsilon \leq 2$,
$4\alpha + 3\beta + \varepsilon\gamma = h^0(\Omega_{\mathbf{P}^4}(d+1))$, *and*
$3\beta + \varepsilon\gamma \leq h^0(\Omega_{\mathbf{P}^3}(d+1))$ *having for* $\gamma = 1$ *a quotient* Γ' *then the map*

$$\eta : H^0\big(\mathbf{P}^4, \Omega_{\mathbf{P}^4}(d+1)\big) \longrightarrow \bigoplus_{i=1}^{\alpha} \Omega_{\mathbf{P}^4}(d+1)_{|A_i} \oplus \bigoplus_{j=1}^{\beta} \Omega_{\mathbf{P}^3}(d+1)_{|B_j} \oplus \Gamma'_{|C}$$

is bijective with $h^0(\Omega_{\mathbf{P}^4}(d+1)) = d\binom{d+4}{d+1}$ *and for* α *general points* $A_1 \dots A_\alpha \in \mathbf{P}^4$, $\beta+1$ *general points* $B_1 \dots B_\beta, C \in \mathbf{P}^3$.

Hypothesis 1.7. $\boldsymbol{H}_{\Omega,4}(d+1)$
The hypothesis $\boldsymbol{H}_{\Omega,4}(d+1)$ asserts that $\boldsymbol{H}'_{\Omega,4}(d+1;\alpha,\beta,\gamma)$ is true for all α, β and γ satisfying the conditions above.

Hypothesis 1.8. $\boldsymbol{H}'_{\mathcal{O},4}(d-1;\rho,\sigma,\tau)$
The hypothesis $\boldsymbol{H}'_{\mathcal{O},4}(d-1;\rho,\sigma,\tau)$ asserts that for non-negative integers ρ, σ, τ and θ satisfying the conditions:

$0 \leq \tau \leq 1$ *and* $2 \leq \theta \leq 3$,
$4\rho + \sigma + \theta\tau = h^0(\mathcal{O}_{\mathbf{P}^4}(d-1)^{\oplus 4})$, *and*
$\sigma + \theta\tau \leq h^0(\mathcal{O}_{\mathbf{P}^3}(d))$ *having for* $\tau = 1$ *a quotient* Γ *then the map*

$$\phi : H^0\big(\mathbf{P}^4, \mathcal{O}_{\mathbf{P}^4}(d-1)^{\oplus 4}\big) \longrightarrow \bigoplus_{i=1}^{\rho} \mathcal{O}_{\mathbf{P}^4}(d-1)^{\oplus 4}_{|R_i} \oplus \bigoplus_{j=1}^{\sigma} \mathcal{O}_{\mathbf{P}^3}(d)_{|S_j} \oplus \Gamma(S)_{|T}$$

is bijective with $h^0(\mathcal{O}_{\mathbf{P}^4}(d-1)^{\oplus 4}) = 4\binom{d+4-1}{d-1}$ *and for* ρ *general points* $R_1 \dots R_\rho \in \mathbf{P}^4$, $\sigma+1$ *general points* $S_1 \dots S_\sigma, T \in \mathbf{P}^3$.

Hypothesis 1.9. $\boldsymbol{H}_{\mathcal{O},4}(d-1)$
The hypothesis $\boldsymbol{H}_{\mathcal{O},4}(d-1)$ asserts that $\boldsymbol{H}'_{\mathcal{O},4}(d-1;\rho,\sigma,\tau)$ is true for any ρ, σ, and τ satisfying the conditions above.

Hypothesis 1.10. $\boldsymbol{H}''_{\mathcal{O},4}(d-1;\rho,\sigma,\tau)$
A variant version of the hypothesis $\boldsymbol{H}'_{\mathcal{O},4}(d-1;\rho,\sigma,\tau)$ with Γ independent of Γ' takes the form $\boldsymbol{H}''_{\mathcal{O},4}(d-1;\rho,\sigma,\tau)$ and it makes the same assertion as the hypothesis $\boldsymbol{H}'_{\mathcal{O},4}(d-1;\rho,\sigma,\tau)$ the only difference being quotient dependency.

1.3. **Méthodes d'Horace.** We will explain the méthodes d'Horace we use as we move on but here we look at one of them:

1.3.1. ***Hypercritical mèthode d'Horace.***

Lemma 1.11. *Consider $\boldsymbol{H}'_{\mathcal{O},n}(d-1;s_1,s_2,0)$ with $d \geq 1, s_1$, and s_2 being non-negative integers that satisfy: $ns_1 + s_2 = h^0(\mathcal{O}_{\mathbf{P}^n}(d-1)^{\oplus n})$ and $s_2 \leq h^0(\mathcal{O}_{\mathbf{P}^{n-1}}(d))$. Now suppose that the $H^0(\Omega_{\mathbf{P}^n}(d)) \longrightarrow H^0(\Omega_{\mathbf{P}^n}(d)|_{S_1})$ is injective and $H^0(\mathcal{O}_{\mathbf{P}^n}(d-1)^{\oplus n}) \longrightarrow H^0(\mathcal{O}_{\mathbf{P}^n}(d-1)^{\oplus n}|_{S_1})$ is surjective with a general $S_1 \subseteq \mathbf{P}^n$ then the hypothesis $\boldsymbol{H}'_{\mathcal{O},n}(d-1;s_1,s_2,0)$ is true.*

This Lemma is for when we have no quotient.

Proof. From the hypothesis $\boldsymbol{H}'_{\mathcal{O},n}(d-1;s_1,s_2,0)$ we have a set S_1 of s_1 general points in $\mathbf{P}^n$ and a set S_2 of s_2 general points in $\mathbf{P}^{n-1}$.

Consider the exact sequence:

$$0 \longrightarrow \Omega_{\mathbf{P}^n}(d) \longrightarrow \mathcal{O}_{\mathbf{P}^n}(d-1)^{\oplus n} \longrightarrow \mathcal{O}_{\mathbf{P}^{n-1}}(d) \longrightarrow 0$$

We take its global sections and evaluate at corresponding points and thus construct a diagram:

$$\begin{array}{ccccccccc}
 & & & & 0 & & & & \\
 & & & & \downarrow & & & & \\
 & & & & \ker\phi & \hookrightarrow & H^0(\mathcal{O}_{\mathbf{P}^{n-1}}(d)) & & \\
 & & & & \downarrow & & \| & & \\
0 & \longrightarrow & H^0(\Omega_{\mathbf{P}^n}(d)) & \longrightarrow & H^0(\mathcal{O}_{\mathbf{P}^n}(d-1)^{\oplus n}) & \longrightarrow & H^0(\mathcal{O}_{\mathbf{P}^{n-1}}(d)) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle\phi} & & & & \\
0 & \longrightarrow & H^0(\Omega_{\mathbf{P}^n}(d)|_{S_1}) & \longrightarrow & H^0(\mathcal{O}_{\mathbf{P}^n}(d-1)^{\oplus n}|_{S_1}) & \longrightarrow & 0 & & \\
 & & & & \downarrow & & & & \\
 & & & & 0 & & & &
\end{array}$$

That is $\ker\phi$ maps injectively on a subspace $V \subseteq H^0(\mathcal{O}_{\mathbf{P}^{n-1}}(d))$ i.e.

$$\begin{array}{ccccc}
\ker\phi & & \longrightarrow & & H^0(\mathcal{O}_{\mathbf{P}^{n-1}}(d)) \\
 & \searrow^{\alpha} & & \nearrow & \\
\downarrow{\scriptstyle\gamma} & & V & & \\
 & & \downarrow{\scriptstyle\beta} & & \\
H^0(\mathcal{O}_{\mathbf{P}^{n-1}}(d)|_{S_2}) & = & H^0(\mathcal{O}_{\mathbf{P}^{n-1}}(d)|_{S_2}) & &
\end{array}$$

The hypothesis $\boldsymbol{H}'_{\mathcal{O},n}(d-1;s_1,s_2,0)$ asserts that $s_2 = \dim V$ for $S_2 \subseteq \mathbf{P}^{n-1}$ general, then α is bijective and since β is bijective since $\mathcal{O}_{\mathbf{P}^{n-1}}(d)$ is a line bundle also, since V depends only on

S_1 but not S_2 then γ has no choice but to be bijective thus

$$H^0\big(\mathcal{O}_{\mathbf{P}^n}(d-1)^{\oplus n}\big) \longrightarrow \bigoplus_{i=1}^{s_1} \mathcal{O}_{\mathbf{P}^n}(d-1)^{\oplus n}_{|R_i} \oplus \bigoplus_{j=1}^{s_2} \mathcal{O}_{\mathbf{P}^{n-1}}(d)_{|S_j}$$

is bijective and the hypothesis $\boldsymbol{H}'_{\mathcal{O},n}(d-1;s_1,s_2,0)$ is true. □

Lemma 1.12. *Consider $\boldsymbol{H}'_{\mathcal{O},n}(d-1;s_1,s_2,1)$ where $d \geq 1, s_1,\ s_2$ and $2 \leq \theta \leq n-1$ are non-negative integers such that, $ns_1+s_2+\theta = h^0(\mathcal{O}_{\mathbf{P}^n}(d-1)^{\oplus n})$ and $s_2+\theta \leq h^0(\mathcal{O}_{\mathbf{P}^{n-1}}(d))$. Under the same hypotheses as Lemma 2.1 i.e. $H^0(\Omega_{\mathbf{P}^n}(d)) \longrightarrow H^0(\Omega_{\mathbf{P}^n}(d)|_{S_1})$ is injective and $H^0(\mathcal{O}_{\mathbf{P}^n}(d-1)^{\oplus n}) \longrightarrow H^0(\mathcal{O}_{\mathbf{P}^n}(d-1)^{\oplus n}|_{S_1})$ is surjective then the hypothesis $\boldsymbol{H}''_{\mathcal{O},n}(d-1;s_1,s_2,1)$ is true.*

Proof. The proof is identical to the previous Lemma since this was the hypothesis with a quotient $\mathcal{O}_{\mathbf{P}^n}(d-1)^{\oplus n} \longrightarrow \Gamma$ with Γ not depending on the S_js. □

2. STATEMENTS FOR THE THE INDUCTIVE STEPS

Hypothesis $\boldsymbol{H}'_{\Omega,4}(d+1;a,b,c)$. *There exists $A_1,\dots,A_a \in \mathbf{P}^4$, $B_1,\dots,B_b \in \mathbf{P}^3$, and a quotient $\Omega_{\mathbf{P}^3|C} \twoheadrightarrow \Gamma'_{|C}$ of dimension 1 or 2 if $c=1$ for a point $C \in \mathbf{P}^3$ such that the restriction map (1) is bijective.*

$$H^0\big(\mathbf{P}^4,\Omega_{\mathbf{P}^4}(d+1)\big) \longrightarrow \bigoplus_{i=1}^{a} \Omega_{\mathbf{P}^4}(d+1)_{|A_i} \oplus \bigoplus_{j=1}^{b} \Omega_{\mathbf{P}^3}(d+1)_{|B_j} \oplus \Gamma'_{|C} \tag{1}$$

Hypothesis $\boldsymbol{H}'_{\mathcal{O},4}(d-1;e,f,g)$. *For $\Gamma : (\mathbf{P}^3)^f \longrightarrow \boldsymbol{Gr}(1,\Omega_{\mathbf{P}^3|G}) \subseteq \boldsymbol{Gr}(2,\mathcal{O}^{\oplus 4}_{\mathbf{P}^4|G})$ or $\Gamma : (\mathbf{P}^3)^f \longrightarrow \boldsymbol{Gr}(2,\Omega_{\mathbf{P}^3}(d)_{|G}) \subseteq \boldsymbol{Gr}(3,\mathcal{O}_{\mathbf{P}^4}(d-1)^{\oplus 4}{}_{|G})$ for $g=1$ there exists $E_1,\dots,E_e \in \mathbf{P}^4$, $F_1,\dots,F_f,G \in \mathbf{P}^3$ such that the restriction map (2) is bijective.*

$$H^0\big(\mathbf{P}^4,\mathcal{O}_{\mathbf{P}^4}(d-1)^{\oplus 4}\big) \longrightarrow \bigoplus_{i=1}^{e} \mathcal{O}_{\mathbf{P}^4}(d-1)^{\oplus 4}_{|E_i} \oplus \bigoplus_{j=1}^{f} \mathcal{O}_{\mathbf{P}^3}(d)_{|F_j} \oplus \Gamma(F)_{|G} \tag{2}$$

Hypothesis $\boldsymbol{H}''_{\mathcal{O},4}(d-1;e,f,g)$. *For $\Gamma : (\mathbf{P}^3)^f \longrightarrow \boldsymbol{Gr}(1,\Omega_{\mathbf{P}^3|G}) \subseteq \boldsymbol{Gr}(2,\mathcal{O}^{\oplus 4}_{\mathbf{P}^4|G})$ or $\Gamma : (\mathbf{P}^3)^f \longrightarrow \boldsymbol{Gr}(2,\Omega_{\mathbf{P}^3}(d)_{|G}) \subseteq \boldsymbol{Gr}(3,\mathcal{O}_{\mathbf{P}^4}(d-1)^{\oplus 4}{}_{|G})$ for $g=1$ there exists $E_1,\dots,E_e \in \mathbf{P}^4$, $F_1,\dots,F_f,G \in \mathbf{P}^3$ such that the restriction map (3) is bijective.*

$$H^0\big(\mathbf{P}^4,\mathcal{O}_{\mathbf{P}^4}(d-1)^{\oplus 4}\big) \longrightarrow \bigoplus_{i=1}^{e} \mathcal{O}_{\mathbf{P}^4}(d-1)^{\oplus 4}_{|E_i} \oplus \bigoplus_{j=1}^{f} \mathcal{O}_{\mathbf{P}^3}(d)_{|F_j} \oplus \Gamma_{|G} \tag{3}$$

Lemma 2.1. (a) *If $\boldsymbol{H}'_{\Omega,4}(d+1;a,b,c)$ is true, then we have*

$$3b+\psi c \leq h^0\big(\Omega_{\mathbf{P}^3}(d+1)\big) = \frac{1}{2}d(d+2)(d+3), \tag{4a}$$

$$3b+\psi c \equiv h^0\big(\Omega_{\mathbf{P}^3}(d+1)\big) \pmod 4, \tag{4b}$$

$$a = \frac{1}{4}\left(h^0\big(\Omega_{\mathbf{P}^4}(d+1)\big) - 3b - \psi c\right) \tag{4c}$$

Where $\psi \in \{0,1,2\}$*, represents the dimension of the quotient*

(b) *If* d*,* b*, and* c *are non-negative integers verifying* (4a) *and* (4b)*, then the* a *defined by* (4c) *satisfies* $a \geq 0$*.*

Proof. (a) Suppose $\boldsymbol{H}'_{\Omega,4}(d+1;a,b,c)$ is true then:
In the sequences below, since α is surjective (and injective) and γ (and δ) is surjective, it follows that β is surjective and thus $3b+\psi c \leq h^0(\Omega_{\mathbf{P}^3}(d+1)$ thus (4a) is proven. Next due to α's bijectivity we have $4a+3b+\psi c = (h^0(\Omega_{\mathbf{P}^4}(d+1))$ hence (4c) follows. Also from $4a = \big(h^0\big(\Omega_{\mathbf{P}^4}(d+1)\big) - 3b - \psi c\big)$, a, a non-negative integer then $3b+\psi c \equiv h^0\big(\Omega_{\mathbf{P}^3}(d+1)\big) \pmod 4$ follows.

(b) Since α is injective (and bijective) and ϕ (and ψ)is injective then π has must be injective and thus a is bounded below by $h^0(\mathcal{O}_{\mathbf{P}^4}(d-1)^{\oplus 4} = \frac{1}{6}d(d+1)(d+2)(d+3) \geq 0$ for all $d \geq 0$.

$$
\begin{array}{ccc}
0 & & 0 \\
\uparrow & & \uparrow \\
H^\circ(\mathbf{P}^3, \Omega_{\mathbf{P}^3}(d+1)) & \xrightarrow[surj]{\beta} & \bigoplus_{j=1}^{b} \Omega_{\mathbf{P}^3|B_j} \oplus \Gamma'_{|C} \\
\gamma\uparrow & & \delta\uparrow \\
H^\circ(\mathbf{P}^4, \Omega_{\mathbf{P}^4}(d+1)) & \xrightarrow{\alpha} & \bigoplus_{i=1}^{a} \Omega_{\mathbf{P}^4|A_i} \oplus \bigoplus_{j=1}^{b} \Omega_{\mathbf{P}^3|B_j} \oplus \Gamma'_{|C} \\
\phi\uparrow & & \psi\uparrow \\
H^\circ(\mathbf{P}^4, \mathcal{O}_{\mathbf{P}^4}(d-1)^{\oplus 4}) & \xrightarrow[inj]{\pi} & \bigoplus_{i=1}^{a} \Omega_{\mathbf{P}^4|A_i} \\
\uparrow & & \uparrow \\
0 & & 0
\end{array}
$$

□

Lemma 2.2. (a) *If* $\boldsymbol{H}'_{\mathcal{O},4}(d-1;e,f,g)$ *is true, then we have*

$$f+g \leq h^0(\mathcal{O}_{\mathbf{P}^3}(d)) \text{ and } 4e+(\varepsilon-1)g \geq h^0(\Omega_{\mathbf{P}^4}(d)) \tag{5a}$$

$$f+\varepsilon g \equiv 0 \pmod 4 \tag{5b}$$

$$e = \tfrac{1}{4}(h^0\big(\mathcal{O}_{\mathbf{P}^4}(d-1)^{\oplus 4}\big) - f - \varepsilon g) \tag{5c}$$

Where $\varepsilon \in \{0,2,3\}$*, represents the dimension of the quotient*
(b) *If* $d \geq 1$*,* f *and* $0 \leq g \leq 1$ *are non-negative integers verifying* (5a) *and* (5b)*, then the* e *defined by* (5c) *satisfies* $e \geq 0$*.*

Proof. Consider the following sequences

$$\begin{array}{ccc}
0 & & 0 \\
\uparrow & & \uparrow \\
H^0(\mathbf{P}^3, \mathcal{O}_{\mathbf{P}^3}(d)) & \xrightarrow{\rho} & \bigoplus_{j=1}^{f} \mathcal{O}_{\mathbf{P}^3}(d)_{|F_j} \oplus \mathcal{O}_{\mathbf{P}^3}(d)_{|G} \\
\alpha\uparrow & & \gamma\uparrow \\
H^0(\mathbf{P}^4, \mathcal{O}_{\mathbf{P}^4}(d-1)^{\oplus 4}) & \xrightarrow{\beta} & \bigoplus_{i=1}^{e} \mathcal{O}_{\mathbf{P}^4}(d-1)^{\oplus 4}_{|E_i} \oplus \bigoplus_{j=1}^{f} \mathcal{O}_{\mathbf{P}^3}(d)_{|F_j} \oplus \Gamma(F)_{|G} \\
\eta\uparrow & & \varepsilon\uparrow \\
H^0(\mathbf{P}^4, \Omega_{\mathbf{P}^4}(d)) & \xrightarrow{\tau} & \bigoplus_{i=1}^{e} \mathcal{O}_{\mathbf{P}^4}(d-1)^{\oplus 4}_{|E_i} \oplus \Gamma'(F)_{|G} \\
\uparrow & & \uparrow \\
0 & & 0
\end{array}$$

(a) Since β is surjective (and injective), γ and α are also surjective, then ρ is left with no choice but to be surjective and thus $f + g \leq h^0(\mathcal{O}_{\mathbf{P}^3}(d))$.
Again since β is injective, η and ε are injective as well, then τ has to be injective and thus $4e + (\varepsilon - 1)g \geq h^0(\Omega_{\mathbf{P}^4}(d))$ having $\mathcal{O}_{\mathbf{P}^3}(d)_{|G} \oplus \Gamma'(F)_{|G} \cong \Gamma(F)_{|G}$ i.e. (5a) holds.
Since β is bijective then we have $4e + f + \varepsilon g = h^0(\big(\mathcal{O}_{\mathbf{P}^4}(d-1)^{\oplus 4})$ from which 4 divides $4e$ and $h^0(\big(\mathcal{O}_{\mathbf{P}^4}(d-1)^{\oplus 4})$ thus 4 divides $f + \varepsilon g$ i.e. $f + \varepsilon g \equiv 0 \pmod 4$ hence (5b) follows.
Finally, (5c) follows from bijectivity of β i.e. $4e + f + \varepsilon g = h^0(\big(\mathcal{O}_{\mathbf{P}^4}(d-1)^{\oplus 4})$.
(b) From (10a) we have $4e + (\varepsilon - 1)g \geq h^0(\Omega_{\mathbf{P}^4}(d))$ from which we have $4e \geq h^0(\Omega_{\mathbf{P}^4}(d)) - g + \varepsilon g$ and thus $e \geq \frac{1}{4}(h^0(\Omega_{\mathbf{P}^4}(d)) - g + \varepsilon g) \geq 0$ for all $d \geq 1$ since $\varepsilon g > g$, hence $e \geq 0$. □

3. THE GENERAL HYPOTHESES AND THE MAIN THEOREM

Hypothesis $\boldsymbol{H}_{\Omega,4}(d+1)$. *For all integers $b \geq 0$, $0 \leq c \leq 1$, and a verifying* (4a), (4b), *and* (4c), *the hypothesis $\boldsymbol{H}'_{\Omega,4}(d+1; a, b, c)$ is true.*

Hypothesis $\boldsymbol{H}_{\mathcal{O},4}(d-1)$. *For all integers $f \geq 0$, $0 \leq g \leq 1$, and e verifying* (5a), (5b), *and* (5c), *the hypothesis $\boldsymbol{H}'_{\mathcal{O},4}(d-1; e, f, g)$ is true.*

Goal. To prove $\boldsymbol{H}_{\Omega,4}(d+1)$ for $d \geq 2$ and $\boldsymbol{H}_{\mathcal{O},4}(d-1)$ for $d \geq 1$.

3.1. Main Theorem.

Theorem 3.1. *Suppose $\boldsymbol{H}_{\Omega,4}(d+1)$ is true. Then for any non-negative integer m, there exists a set, $S = \{P_1, P_2, \ldots, P_m\}$ of m points in $\mathbf{P}^4$ such that the evaluation map, μ, is of maximal rank.*

$$\mu : H^0\big(\mathbf{P}^4, \Omega_{\mathbf{P}^4}(d+1)\big) \longrightarrow \bigoplus_{i=1}^{m} \Omega_{\mathbf{P}^4|P_i}$$

Proof. Set $r = \frac{1}{4}\lfloor h^0(\Omega_{\mathbf{P}^4}(d+1))\rfloor$

(a) If $h^0(\Omega_{\mathbf{P}^4}(d+1)) \equiv 0 \pmod 4$ the r is the critical number of points needed for the bijectivity i.e. the map $H^0\big(\mathbf{P}^4, \Omega_{\mathbf{P}^4}(d+1)\big) \longrightarrow \bigoplus_{i=1}^{r} \Omega_{\mathbf{P}^4|P_i}$ is bijective and now consider the following cases:

(i) if $m = r$ then our map is bijective since we have the same number of points as the critical number i.e. the map α is bijective and γ an identity map and so μ is bijective see below:

$$\begin{array}{ccc} H^0(\mathbf{P}^4, \Omega_{\mathbf{P}^4}(d+1)) & \xrightarrow{\ \mu\ } & \bigoplus_{i=1}^{m} \Omega_{\mathbf{P}^4|P_i} \\ & \searrow^{\alpha} & \uparrow \gamma \\ & & \bigoplus_{i=1}^{n} \Omega_{\mathbf{P}^4|P_i} \oplus \bigoplus_{i=n+1}^{r} \Omega_{\mathbf{P}^4|P_i} \end{array}$$

(ii) if $m > r$ i.e. we have more points than the critical number and our map is injective i.e. since α is bijective and γ surjective then our map μ has to inject see below:

$$\begin{array}{ccc} H^0(\mathbf{P}^4, \Omega_{\mathbf{P}^4}) & \xrightarrow[\alpha]{\cong} & \bigoplus_{i=1}^{r} \Omega_{\mathbf{P}^4|P_i} \\ & \searrow^{\mu} & \uparrow\!\!\uparrow \gamma \\ & & \bigoplus_{i=1}^{r} \Omega_{\mathbf{P}^4|P_i} \oplus \bigoplus_{i=r+1}^{m} \Omega_{\mathbf{P}^4|P_i} \end{array}$$

(iii) if $m < r$ then we have the less points than the critical number thus our map surjects i.e. since α is bijective and γ surjective then our map μ is surjective see below:

$$\begin{array}{ccc} H^0(\mathbf{P}^4, \Omega_{\mathbf{P}^4}) & \xrightarrow{\ \mu\ } & \bigoplus_{i=1}^{m} \Omega_{\mathbf{P}^4|P_i} \\ & \searrow^{\alpha}_{\cong} & \uparrow\!\!\uparrow \gamma \\ & & \bigoplus_{i=1}^{m} \Omega_{\mathbf{P}^4|P_i} \oplus \bigoplus_{i=m+1}^{r} \Omega_{\mathbf{P}^4|P_i} \end{array}$$

(b) If $h^0(\Omega_{\mathbf{P}^4}(d+1)) \not\equiv 0 \pmod 4$ then $h^0(\Omega_{\mathbf{P}^4}(d+1)) \equiv \eta \pmod 4$ and η has 3 possiblities:

(i) When $\eta = 1$ i.e. $h^0(\Omega_{\mathbf{P}^4}(d+1)) \equiv 1 \pmod 4$ we have r general points $P_1, P_2, \ldots, P_r$ in $\mathbf{P}^4$ and a point B in $\mathbf{P}^3$ so that the map $H^0(\Omega_{\mathbf{P}^4}(d+1)) \longrightarrow \bigoplus_{i=1}^{r} \Omega_{\mathbf{P}^4}(d+1)|_{P_i} \oplus \Omega_{\mathbf{P}^3}(d+1)|_B$ is bijective.

If $m = r+1$ then $H^0(\Omega_{\mathbf{P}^4}(d+1)) \longrightarrow \bigoplus_{i=1}^{r} \Omega_{\mathbf{P}^4}(d+1)|_{P_i} \oplus \Omega_{\mathbf{P}^4}(d+1)|_B$ is injective

Next, if $m > r+1$ since the map $H^0(\Omega_{\mathbf{P}^4}(d+1)) \longrightarrow \bigoplus_{i=1}^{r} \Omega_{\mathbf{P}^4}(d+1)|_{P_i} \oplus \Omega_{\mathbf{P}^3}(d+1)|_S$ is bijective then $H^0(\Omega_{\mathbf{P}^4}(d+1)) \longrightarrow \bigoplus_{i=1}^{r} \Omega_{\mathbf{P}^4}(d+1)|_{P_i} \oplus \Omega_{\mathbf{P}^4}(d+1)|_B \oplus \bigoplus_{i=r+2}^{m} \Omega_{\mathbf{P}^4}(d+1)|_{P_i}$ is injective.

Finally, if $m < r+1$ then the map $H^0(\Omega_{\mathbf{P}^4}(d+1)) \longrightarrow \bigoplus_{i=1}^{m} \Omega_{\mathbf{P}^4}(d+1)|_{P_i}$ is surjective.

(ii) For the cases when $\eta = 2$ and $\eta = 3$ it means that we need a quotient of dimension 2 or 1 respectively. We have $h^0(\Omega_{\mathbf{P}^4}(d+1)) \equiv 2 \pmod 4$ or $h^0(\Omega_{\mathbf{P}^4}(d+1)) \equiv 3 \pmod 4$

meaning we have r general points $P_1, P_2, \ldots, P_r$ in $\mathbf{P}^4$ and a point C in $\mathbf{P}^3$ so that the map $H^0(\Omega_{\mathbf{P}^4}(d+1)) \longrightarrow \bigoplus_{i=1}^{r} \Omega_{\mathbf{P}^4}(d+1)|_{P_i} \oplus \Gamma'|_C$ is bijective.

If $m = r+1$ then map $H^0(\Omega_{\mathbf{P}^4}(d+1)) \longrightarrow \bigoplus_{i=1}^{r} \Omega_{\mathbf{P}^4}(d+1)|_{P_i} \oplus \Omega_{\mathbf{P}^4}(d+1)|_C$ is injective
Next, if $m > r+1$ since the map $H^0(\Omega_{\mathbf{P}^4}(d+1)) \longrightarrow \bigoplus_{i=1}^{r} \Omega_{\mathbf{P}^4}(d+1)|_{P_i} \oplus \Gamma'|_C$ is bijective then the map $H^0(\Omega_{\mathbf{P}^4}(d+1)) \longrightarrow \bigoplus_{i=1}^{r} \Omega_{\mathbf{P}^4}(d+1)|_{P_i} \oplus \Omega_{\mathbf{P}^4}(d+1)|_C \oplus \bigoplus_{i=r+2}^{m} \Omega_{\mathbf{P}^4}(d+1)|_{P_i}$ is injective.
Lastly, if $m < r+1$ then the map $H^0(\Omega_{\mathbf{P}^4}(d+1)) \longrightarrow \bigoplus_{i=1}^{m} \Omega_{\mathbf{P}^4}(d+1)|_{P_i}$ is surjective.

□

3.2. **The Main Methods.**

3.2.1. *The initial cases.*

Lemma 3.2. (a)$\boldsymbol{H}_{\Omega,4}(d+1)$ *is true when* $d=2$ *and*
(b)$\boldsymbol{H}_{\mathcal{O},4}(d-1)$ *is true when* $d=1$

Proof. (a) We prove that $\boldsymbol{H}_{\Omega,4}(3)$ is true by proving $\boldsymbol{H}'_{\Omega,4}(3; a, b, c)$.

The non-negative integers a, b and c satisfy the following:
$a \geq h^0(\mathcal{O}_{\mathbf{P}^4}(1)) = 5$
$4a + 3b + \psi c = 40 = h^0(\Omega_{\mathbf{P}^4}(3))$
$3b + \psi c \leq 20 = h^0(\Omega_{\mathbf{P}^3}(3))$
$c = 0$ or 1 and $\psi = 1$ or 2 and from these we have the following 6 possibilities for (a, b, c):

(i) $(10, 0, 0)$
(ii) $(9, 1, 1)$
(iii) $(8, 2, 1)$
(iv) $(7, 4, 0)$
(v) $(6, 5, 1)$
(vi) $(5, 6, 1)$

(i) The hypothesis $\boldsymbol{H}'_{\Omega,4}(3; 10, 0, 0)$ means we have 10 general points, $A_1, \cdots, A_{10}$ in $\mathbf{P}^4$.

We partition $S = \{A_1, \cdots, A_{10}\} \subseteq \mathbf{P}^4$ into $S = S_1 \cup S_2 \cup \{Q\}$ so that $|S_1| = 3$ with $S_1 \subset \mathbf{P}^4 \setminus \mathbf{P}^3$ and S_2, Q are in $\mathbf{P}^3$

$\frac{1}{4} h^0(\Omega_{\mathbf{P}^4}(4)) = 10 =\mid S \mid$

$\frac{1}{3} \lfloor h^0(\Omega_{\mathbf{P}^3}(4)) \rfloor = \lfloor \frac{20}{3} \rfloor = 6$

So of the 10 points, we specialize 7 points, $A_4, A_5, A_6, A_7, A_8, A_9, A_{10}$ to $\mathbf{P}^3$, the 7th point A_{10} is for a quotient (the fractional part) thus the sets are:

$S_1 = \{A_1, A_2, A_3, \}$

$S_2 = \{A_4, A_5, A_6, A_7, A_8, A_9\}$

$\{Q\} = \{A_{10}\}$

We have the following sequence for the quotient:

$$
\begin{array}{lccccl}
 & 0 & & 0 & & \\
 & \downarrow & & \downarrow & & \\
\dim 3 & \Omega_{\mathbf{P}^3}(2) & \twoheadrightarrow & D'_{|A_{10}} & & \dim 1 \\
 & \hookdownarrow & & \hookdownarrow & & \\
\dim 4 & \mathcal{O}_{\mathbf{P}^4}(1)^{\oplus 4}_{|A_{10}} & \twoheadrightarrow & D_{|A_{10}} & & \dim 2 \\
 & \twoheaddownarrow & & \twoheaddownarrow & & \\
\dim 1 & \mathcal{O}_{\mathbf{P}^3}(2)_{|A_{10}} & = & \mathcal{O}_{\mathbf{P}^3}(2)_{|A_{10}} & & \dim 1 \\
 & \downarrow & & \downarrow & & \\
 & 0 & & 0 & &
\end{array}
$$

We thus construct a diagram:

$$\begin{array}{ccccccccc}
 & & \dim 1 \text{ (depends only on } S_1) & & \dim 8 & & \dim 10 & & \\
0 & \longrightarrow & H^0(\mathrm{T}_{S_1}\otimes\Omega_{\mathbf{P}^4}(2)) & \longrightarrow & H^0(\mathrm{T}_{S_1,\mathbf{P}^4}(1)^{\oplus 4}) & \longrightarrow & H^0(\mathcal{O}_{\mathbf{P}^3}(2)) & & \\
 & \swarrow & \downarrow f & & \downarrow & \overset{\beta}{\underset{surj}{\searrow}} & \uparrow & & \\
H^0(\Omega_{\mathbf{P}^4}(2)|_{A_{10}}) & & & & & & V \dim 7 & \longrightarrow & 0 \\
 & \searrow & & & & & \alpha \downarrow \cong & & \\
0 & \longrightarrow & H^0(D'_{|A_{10}}) & \longrightarrow & H^0(\mathcal{O}_{\mathbf{P}^3}(2)_{|S_2}\oplus D_{|A_{10}}) & \longrightarrow & H^0(\mathcal{O}_{\mathbf{P}^3}(2)_{|S_2\cup\{A_{10}\}}) & \longrightarrow & 0 \\
 & & \dim 1 \text{ (depends only on } S_2) & & \dim 8 & & \dim 7 & &
\end{array}$$

From it we see that the map β is surjective by Proposition 3.13 and the map α is bijective since V does not depend on $S_2 \cup P_{10}$ it only depends on S_1.

What is $H^0(\mathrm{T}_{S_1} \otimes \Omega_{\mathbf{P}^4}(2))$???

The set $S_1 = \{A_1, A_2, A_3\}$ spans a $\mathbf{P}^2 = \{L_1 = L_2 = 0\}$

$$\longrightarrow R(-3)^8 \bigoplus R(-2) \longrightarrow R(-2)^3 \bigoplus R(-1)^2 \longrightarrow I|_{S_1} \longrightarrow 0$$

Thus $a_0 = h^0(\mathrm{T}_{S_1} \otimes \Omega_{\mathbf{P}^4}(2)) = 1$, and $b_0 = h^1(\mathrm{T}_{S_1} \otimes \Omega_{\mathbf{P}^4}(2)) = 3$ and so $H^0(\mathrm{T}_{S_1} \otimes \Omega_{\mathbf{P}^4}(2))$ is the Koszul relation between L_1 and L_2.

To show that f is bijective, we calculate.

f: sections in $H^0(\Omega_{\mathbf{P}^4}(2)) = \Lambda^2 W$ vanishing along $S_1 = \{A_1, A_2, A_3\}$

Where $W = H^0(\mathcal{O}_{\mathbf{P}^4}(1))$, linear forms.

Consider the exact sequence

$$0 \longrightarrow \Omega_{\mathbf{P}^4}(2) \longrightarrow W \otimes \mathcal{O}_{\mathbf{P}^4}(1) \longrightarrow \mathcal{O}_{\mathbf{P}^4}(2) \longrightarrow 0$$

Taking global sections yields:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H^0(\Omega_{\mathbf{P}^4}(2)) & \longrightarrow & W \otimes H^0(\mathcal{O}_{\mathbf{P}^4}(1)) & \longrightarrow & H^0(\mathcal{O}_{\mathbf{P}^4}(2)) & \longrightarrow & 0 \\
 & & \| & & \| & & \| & & \\
 & & \Lambda^2 W & & W \otimes W & & S^2 W & & \\
 & & & & & & \mathrm{Sym}^2 W & &
\end{array}$$

$$H^0(\Omega_{\mathbf{P}^4}(2)) \xrightarrow{rest\ to\ fiber} H^0(\Omega_{\mathbf{P}^4}(2)_{|A_{10}}) \xrightarrow{rk\ 1\ quot} H^0(D'_{A_{10}})$$

$$
\begin{array}{ccccccccc}
0 \longrightarrow & \Omega_{\mathbf{P}^4}(2)_{|A_{10}} & \longrightarrow & W \otimes \mathcal{O}_{\mathbf{P}^4}(1)_{|A_{10}} & \xrightarrow[\text{evaluation at } A_{10}]{(a_0:a_1:a_2:a_3:a_4)} & \mathcal{O}_{\mathbf{P}^4}(2) & \longrightarrow 0 \\
& \| & & \| & & & \\
& H^0(\mathrm{T}_{A_{10}}(1)) & & W & & & \\
& \| & & & & & \\
& (I_{A_{10}})_1 & & & & & \\
& \text{linear forms at} A_{10} & & & & &
\end{array}
$$

$L_1 \wedge L_2 \in \Lambda^2 W \longrightarrow L_1 \otimes L_2(A_{10}) - L_2 \otimes L_1(A_{10})$
Where $L_1,\ L_2 \in W$ and $L_1(A_{10}),\ L_2(A_{10})$ are forms in $\mathbf{P}^4$

If $L_1(A_{10}) \neq 0,\ L_2(A_{10}) = 0$ and $L_1 \wedge L_2 \longrightarrow L_1(Q) \cdot L_2 \in \Omega_{\mathbf{P}^4|Q},\ L_1 \wedge L_2$ vanishes at P where $L_1(P) = L_2(P) = 0$ so $f(L_1 \wedge L_2)$ spans the 1 dimensional subspace of linear forms vanishing at A_{10} composed of linear forms that vanish at $S \cup A_{10} = \{A_1, A_2, A_3, A_{10}\}$ so choose A_1, A_2, A_3 general so that this subspace $\subsetneqq \ker(\Omega_{\mathbf{P}^4|A_{10}} \longrightarrow D')$

(ii) The hypothesis $\boldsymbol{H}'_{\Omega,4}(3; 9, 1, 1)$ says that we have 9 general points, $A_1, \cdots, A_9$ in $\mathbf{P}^4$ and 2 points B, C in $\mathbf{P}^3$.
Consider the sequence;

$$0 \longrightarrow \mathcal{O}_{\mathbf{P}^4}(1)^{\oplus 4} \longrightarrow \Omega_{\mathbf{P}^4}(3) \longrightarrow \Omega_{\mathbf{P}^3}(3) \longrightarrow 0$$

On taking global sections for the sequence we have

$$
\begin{array}{ccccccc}
 & \dim 20 & & \dim 40 & & \dim 20 & \\
0 \longrightarrow & H^0(\mathbf{P}^4, \mathcal{O}_{\mathbf{P}^4}(1)^{\oplus 4}) & \longrightarrow & H^0(\mathbf{P}^4, \Omega_{\mathbf{P}^4}(3)) & \longrightarrow & H^0(\mathbf{P}^3, \Omega_{\mathbf{P}^3}(3)) & \longrightarrow 0
\end{array}
$$

Our fibres 9, 1 and 1 are of dimensions 4, 3 and 1 respectively giving us a total of 40 the $h^0(\mathbf{P}^4, \Omega_{\mathbf{P}^4}(3))$.

We invoke Lemma 3.4 with $d = 2$, $(a, b, c) = (9, 1, 1)$ and $(e, f, g) = (3, 5, 1)$
For the hypothesis $\boldsymbol{H}'_{\mathcal{O},4}(1; 3, 5, 1)$ invoke Lemma 1.12 with $d = 2$, $(s_1, s_2, \theta) = (3, 5, 1)$, $n = 4$
(iii) The hypothesis $\boldsymbol{H}'_{\Omega,4}(3; 8, 2, 1)$ says that we have 8 general points, $A_1, \ldots, A_8$ in $\mathbf{P}^4$ and are 3 general points, B_1, B_2, C in $\mathbf{P}^3$. We invoke Lemma 3.4 with $d = 2$, $(a, b, c) = (8, 2, 1)$ and $(e, f, g) = (4, 4, 0)$
We shall prove the hypothesis $\boldsymbol{H}'_{\mathcal{O},4}(1; 4, 4, 0)$ in Lemma 3.3 (ii) below.

(iv) The hypothesis $\boldsymbol{H}'_{\Omega,4}(3; 7, 4, 0)$ means that we have 7 general points, $A_1, \ldots, A_7$ in $\mathbf{P}^4$ and 4 general points, B_1, B_2, B_3, B_4in $\mathbf{P}^3$. We invoke Lemma 3.4 with $d = 2$, $(a, b, c) = (7, 4, 0)$ and $(e, f, g) = (4, 2, 1)$ and the hypothesis $\boldsymbol{H}'_{\mathcal{O},4}(1; 4, 1, 1)$ is proved in 3.3 (iv) below.

(v) In this case i.e. the hypothesis $\boldsymbol{H}'_{\Omega,4}(3; 6, 5, 1)$, we have 6 general points, $A_1, \cdots, A_6$ in $\mathbf{P}^4$ and 6 general points, $B_1, \cdots, B_5, C$ in $\mathbf{P}^3$ with a quotient at C. We invoke Lemma 3.4 with $d = 2$, $(a, b, c) = (6, 5, 1)$ and $(e, f, g) = (4, 1, 1)$

(vi) For the hypothesis $\boldsymbol{H}'_{\Omega,4}(3;5,6,1)$ we have 5 general points, say $A_1,\cdots,A_5\in\mathbf{P}^4$ and 7 general points, $B_1,\cdots,B_6,C\in\mathbf{P}^3$ with a quotient at C. We need to prove that the map below is bijective

$$H^0(\mathbf{P}^4,\Omega_{\mathbf{P}^4}(3))\longrightarrow\bigoplus_{i=1}^{5}\Omega_{\mathbf{P}^4}(3)_{|A_i}\oplus\bigoplus_{j=1}^{6}\Omega_{\mathbf{P}^3}(3)_{|B_j}\oplus\Gamma'_{|C}$$

We invoke Lemma 3.4 with $d=2$, $(a,b,c)=(5,6,1)$ and $(e,f,g)=(5,0,0)$. The hypothesis $\boldsymbol{H}'_{\mathcal{O},4}(1;5,0,0)$ is proved below in the next Lemma. □

From the above proofs several cases for the hypothesis $\boldsymbol{H}'_{\mathcal{O},4}(1;u,v,w)$ for specific u,v,w have arisen and they form part of the initial cases for (b). The hypotheses $\boldsymbol{H}'_{\mathcal{O},4}(1;u,v,w)$ are for specific u,v,w with $d=2$. It happens that certain of these hypotheses are false when a quotient depending badly on other points but we have:

Lemma 3.3. *The hypotheses* $\boldsymbol{H}'_{\mathcal{O},4}(1;3,5,1)$*,* $\boldsymbol{H}'_{\mathcal{O},4}(1;4,4,0)$*,* $\boldsymbol{H}'_{\mathcal{O},4}(1;4,2,1)$*,* $\boldsymbol{H}'_{\mathcal{O},4}(1;4,1,1)$ *and* $\boldsymbol{H}'_{\mathcal{O},4}(1;5,0,0)$ *are true.*

Proof. Lemma 4 (b)
(i) $(3,5,1)$ We have A_1,A_2,A_3 general points in $\mathbf{P}^4$ and $A_4,\cdots,A_8,A_9$ points in $\mathbf{P}^3$ we choose a hyperplane $H\subseteq\mathbf{P}^4$ disjoint from $A_4,\cdots,A_7$ in $\mathbf{P}^3$ with $H\cong\mathbf{P}^3$ since the points are general and the we construct an exact sequence:

$$\begin{array}{ccccccccc} & & 4 & & 20 & & 16 & & \\ 0 & \longrightarrow & \mathcal{O}_{\mathbf{P}^4}^{\oplus 4} & \longrightarrow & \mathcal{O}_{\mathbf{P}^4}(1)^{\oplus 4} & \longrightarrow & \mathcal{O}_H(1)^{\oplus 4} & \longrightarrow & 0 \\ & & (0,4,0) & & (3,5,1) & & (3,1,1) & & \end{array}$$

where the $(0,4,0)$ represents $A_4,\cdots,A_7\in\mathbf{P}^3$
$(3,5,1)$ represents $A_1,A_2,A_3\in\mathbf{P}^4$, $A_4,\cdots,A_8,A_9$ points in $\mathbf{P}^3$
$(3,1,1)$ represents $A_1,A_2,A_3\in\mathbf{P}^4$, A_8 and A_9 points in H

Thus taking global sections for the sequences above and evaluating at the corresponding points as listed above we have the following exact sequences:

$$\begin{array}{ccc} 0 & & 0 \\ \uparrow & & \uparrow \\ H^0(\mathcal{O}_H(1))^{\oplus 4} & \xrightarrow{\rho} & \bigoplus_{i=1}^{3}\mathcal{O}_H(1)^{\oplus 4}_{|A_i}\oplus\mathcal{O}_{\mathbf{P}^3}(2)_{|A_8}\oplus\Gamma'_{|A_9} \\ \uparrow & & \uparrow \\ H^0(\mathcal{O}_{\mathbf{P}^4}(1)^{\oplus 4} & \xrightarrow{\sigma} & \bigoplus_{i=1}^{3}\mathcal{O}_{\mathbf{P}^4}(1)^{\oplus 4}_{|A_i}\oplus\bigoplus_{i=4}^{8}\mathcal{O}_{\mathbf{P}^3}(2)_{|A_i}\oplus\Gamma'_{|A_9} \\ \uparrow & & \uparrow \\ H^0(\mathcal{O}_{\mathbf{P}^4}(1)^{\oplus 4}) & \xrightarrow{\tau} & \bigoplus_{i=1}^{7}\mathcal{O}_{\mathbf{P}^3}(1)_{|A_i} \\ \uparrow & & \uparrow \\ 0 & & 0. \end{array}$$

The quotient $\Gamma'|_{A_9}$ depends in principle on $A_4, \cdots, A_8$ but because we can move the 4 four points $A_4, \cdots, A_8$ without the others changing ρ, we can assume that it is a general quotient by lemme 5 in [1] dual and thus the map ρ is an isomorphism i.e. 3 general points, a line bundle and a $\dim 3$ quotient thus we have that the map τ implies σ giving us the hypothesis $\boldsymbol{H}'_{\mathcal{O},4}(0;0,4,0)$ implies $\boldsymbol{H}'_{\mathcal{O},4}(1;3,5,1)$ and we now prove $\boldsymbol{H}'_{\mathcal{O},4}(0;0,4,0)$ as follows:

We have $A_4, \cdots, A_7 \in \mathbf{P}^3$ and so the hypothesis $\boldsymbol{H}'_{\mathcal{O},4}(0;0,4,0)$ we show that the mapping $H^0(\mathbf{P}^4, \mathcal{O}_{\mathbf{P}^4}^{\oplus 4}) \longrightarrow \bigoplus_{i=4}^{7} \mathcal{O}_{\mathbf{P}^3}(1)_{|P_i}$ is an isomorphism. Consider the exact sequence

$$0 \longrightarrow \Omega_{\mathbf{P}^4}(1) \longrightarrow \mathcal{O}_{\mathbf{P}^4}^{\oplus 4} \longrightarrow \mathcal{O}_{\mathbf{P}^3}(1) \longrightarrow 0$$

now taking global sections and evaluating at the corresponding points we get

$$\begin{array}{ccccccccc} & & \dim 0 & & \dim 4 & & \dim 4 & & \\ 0 & \longrightarrow & H^0(\Omega_{\mathbf{P}^4}(1)) & \longrightarrow & H^0(\mathcal{O}_{\mathbf{P}^4}^{\oplus 4}) & \xrightarrow{\pi} & H^0(\mathcal{O}_{\mathbf{P}^3}(1)) & \longrightarrow & 0 \\ & & & & \downarrow \phi & & \rho \downarrow & & \\ & & & & \bigoplus_{i=4}^{7} \mathcal{O}_{\mathbf{P}^3}(1)_{|A_i} & = & \bigoplus_{i=4}^{7} \mathcal{O}_{\mathbf{P}^3}(1)_{|A_i} & & \end{array}$$

Thus π is bijective ρ is bijective also i.e. line bundles at four points and we have an identity map and ϕ is a composition map of ρ and the identity map thus ϕ must be bijective.
(ii) $(4,4,0)$
We have $A_1, A_2, A_3, A_4 \in \mathbf{P}^4$ and $A_5, A_6, A_7, A_8 \in \mathbf{P}^3$ we want to show that $\boldsymbol{H}'_{\mathcal{O},4}(1;4,4,0)$ is true.

We invoke the Plane Divisorial Method for $\mathbf{P}^4$ i.e. Lemma 3.8 with $d = 2$, $e = f = 4$, $e' = g = 0$ and we have the hypothesis $\boldsymbol{H}'_{\mathcal{O},4}(0;0,4,0)$ to prove but we proved immediately above.
(iii) (4, 1, 1)
Here we have 4 general points $A_1, A_2, A_3, A_4 \in \mathbf{P}^4$, and 2 general points, $A_5, A_6 \in \mathbf{P}^3$ and by Lemma 2.12 it is true.
(iv)(4,2,1)
We have general points, $S_1 = \{A_1, A_2, A_3, A_4\} \subset \mathbf{P}^4$, general points, $S_2 = \{A_5, A_6\}$ and A_7 in $\mathbf{P}^3$ and by Lemma 2.12 it is true. (iv)For the hypothesis $\boldsymbol{H}'_{\mathcal{O},4}(1;5,0,0)$ we invoke the Plane Divisorial Method for $\mathbf{P}^4$ i.e. Lemma 3.8 and we get the hypothesis $\boldsymbol{H}'_{\mathcal{O},4}(0;1,0,0)$ and this is true since the map $H^0(\mathcal{O}_{\mathbf{P}^4}^{\oplus 4}) \longrightarrow \mathcal{O}_{\mathbf{P}^4}^{\oplus 4}|_P$ is the map of constants evaluated at a point and is bijective. □

3.2.2. *The Inductive steps.* The inductive steps that we proceed to prove are;

a. Vectorial Method 1
b. Vectorial Method 2
c. Plane Divisorial Method
d. Hypercritical Method

Lemma 3.4. *Vectorial Method* 1
Suppose d, a, b, c *satisfy* (4a), (4b), *and* (4c). *Write* $h^0(\Omega_{\mathbf{P}^3}(d+1)) - 3b - \psi c = 3f + \theta g$ *with*

f, g, θ non-negative integers, $0 \le g \le 1$ and $0 \le \theta \le 2$. Set $e = a - f - g$. If e is a non-negative integer and $f + g \le h^0(\mathcal{O}_{\mathbf{P}^3}(d))$ then $\boldsymbol{H}'_{\mathcal{O},4}(d-1; e, f, g)$ implies $\boldsymbol{H}'_{\Omega,4}(d+1; a, b, c)$.

Proof. Consider the exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbf{P}^4}(d-1)^{\oplus 4} \longrightarrow \Omega_{\mathbf{P}^4}(d+1) \longrightarrow \Omega_{\mathbf{P}^3}(d+1) \longrightarrow 0$$

Taking global sections we have the following sequence with dimensions shown ;

$$0 \longrightarrow \overset{\frac{d(d+1)(d+2)(d+3)}{6}}{H^0(\mathbf{P}^4, \mathcal{O}_{\mathbf{P}^4}(d-1)^{\oplus 4})} \longrightarrow \overset{\frac{d(d+2)(d+3)(d+4)}{6}}{H^0(\mathbf{P}^4, \Omega_{\mathbf{P}^4}(d+1))} \longrightarrow \overset{\frac{d(d+2)(d+3)}{2}}{H^0(\mathbf{P}^3, \Omega_{\mathbf{P}^3}(d+1))} \longrightarrow 0$$

Let B and C be general sets of b and c points respectively in $\mathbf{P}^3$. We specialize A to $E \cup F \cup G$ with E a set of e general points in $\mathbf{P}^4$ and F, G sets of f, g general points respectively in $\mathbf{P}^3$. According to our work on $\mathbf{P}^3[4]$, the map

$$H^0(\Omega_{\mathbf{P}^3}(d+1)) \xrightarrow{\lambda} H^0(\Omega_{\mathbf{P}^3}(d+1)|_{B \cup F}) \oplus \Gamma'|_C$$

is surjective and

$$H^0(\Omega_{\mathbf{P}^3}(d+1)) \xrightarrow{\lambda} H^0(\Omega_{\mathbf{P}^3}(d+1)|_{B \cup F \cup C}$$

is injective. So by Lemma 3.12 there exists a quotient $\mathcal{O}_{\mathbf{P}^4}(d-1)^{\oplus 4} \xrightarrow{surj} \Gamma|_G(B, F)$ of dimension θ with the property that if,

$$H^0(\mathcal{O}_{\mathbf{P}^4}(d-1)^{\oplus 4}) \longrightarrow H^0(\mathcal{O}_{\mathbf{P}^4}(d-1)^{\oplus 4}|_E) \oplus H^0(\mathcal{O}_{\mathbf{P}^3}(d)|_F) \oplus \Gamma|_G(B, F)$$

is bijective then

$$H^0(\Omega_{\mathbf{P}^4}(d+1)) \longrightarrow H^0(\Omega_{\mathbf{P}^4}(d+1)|_{E \cup F \cup G = A} \oplus H^0(\Omega_{\mathbf{P}^3}(d+1)|_B) \oplus \Gamma'|_C$$

is bijective. But this is exactly $\boldsymbol{H}'_{\mathcal{O},4}(d-1; e, f, g)$ implies $\boldsymbol{H}'_{\Omega,4}(d+1; a, b, c)$. □

The hypothesis $\boldsymbol{H}'_{\mathcal{O},n}(d-1; e, f, g)$ with dependent quotient $\Gamma|_G(B, F)$ can be weakened to the hypothesis $\boldsymbol{H}''_{\mathcal{O},n}(d-1; e, f, g)$ with general quotient $\Gamma|_G$ in some cases.

Lemma 3.5. *Under the same hypotheses as the immediately above Lemma, if in addition $g = 0$ ($\theta = 0$) or $b \ge 2$, then $\boldsymbol{H}''_{\mathcal{O},n}(d-1; e, f, g)$ implies $\boldsymbol{H}'_{\Omega,n}(d+1; a, b, c)$.*

Proof. If $g = 0$ then $\Gamma|_G(B, F) = 0$ is independent of B, F.
If $b \ge 2$ apply [1] lemme 5 (dualized) to the map $\Psi(F) \longrightarrow \Gamma_G(B, F)$ with
$\Psi(F) = \ker(H^0(\mathcal{O}_{\mathbf{P}^4}(d-1)^{\oplus 4}) \longrightarrow H^0(\mathcal{O}_{\mathbf{P}^4}(d-1)^{\oplus 4}|_E) \oplus H^0(\mathcal{O}_{\mathbf{P}^3}(d)|_F))$ The relevant condition is $b \ge \dim \mathbf{Gr}(1, \Omega_{\mathbf{P}^3|_G}) = 2$ or $b \ge \dim \mathbf{Gr}(2, \Omega_{\mathbf{P}^3|_G}) = 2$ □

Lemma 3.6. *In the same circumstances as Lemma 3.4 we have $f + g \le h^0(\mathcal{O}_{\mathbf{P}^3})$ and $e = a - f - g \ge 0$.*

Proof. (i) We show $f + g \le h^0(\mathcal{O}_{\mathbf{P}^3}(d))$.
We have $h^0(\Omega_{\mathbf{P}^3}(d+1)) - 3b - \psi c = 3f + \theta g$ by the statement of the Lemma
This implies that $3f + \theta g \le h^0(\Omega_{\mathbf{P}^3}(d+1))$ i.e.

$$\begin{aligned} f + \tfrac{1}{3}\theta g &\le \tfrac{1}{3}(h^0(\Omega_{\mathbf{P}^3}(d+1))) \text{ i.e.} \\ f + \tfrac{1}{3}\theta g &\le \tfrac{1}{6}d(d+2)(d+3) \text{ i.e} \\ &< \tfrac{1}{6}d(d+2)(d+3) + \tfrac{1}{6}(d+2)(d+3) \text{ i.e} \end{aligned}$$

$$= \frac{1}{6}(d+1)(d+2)(d+3) = h^0(\mathcal{O}_{\mathbf{P}^3}(d)) \text{ i.e.}$$

Thus we have $f + \frac{1}{3}\theta g < h^0(\mathcal{O}_{\mathbf{P}^3}(d))$ and since $0 \leq \theta \leq 2$ setting $\theta = 3$ does no harm as long as we have $d \geq 0$ and thus $f + g \leq h^0(\mathcal{O}_{\mathbf{P}^3}(d))$ as required.

(ii) Next is $e = a - f - g \geq 0$?
We have just proved that $f + g \leq h^0(\mathcal{O}_{\mathbf{P}^3}(d))$ and from (4c) in Lemma 2.1 we know that $a = \frac{1}{4}(h^0(\Omega_{\mathbf{P}^4}(d+1)) - 3b - \psi c)$

Now we have, $e = a - f - g$

$$\begin{aligned} &= \tfrac{1}{4}(h^0(\Omega_{\mathbf{P}^4}(d+1)) - 3b - \psi c) - f - g \\ &= \tfrac{1}{4}(h^0(\Omega_{\mathbf{P}^4}(d+1)) - h^0(\Omega_{\mathbf{P}^3}(d+1))) \text{ since } 3b + \psi c \leq h^0(\Omega_{\mathbf{P}^3}(d+1)) \\ &= \tfrac{1}{4}(h^0(\mathcal{O}_{\mathbf{P}^4}(d-1)^{\oplus 4})) - f - g \\ &\geq \tfrac{1}{4}(h^0(\mathcal{O}_{\mathbf{P}^4}(d-1)^{\oplus 4})) - \tfrac{1}{4}h^0(\mathcal{O}_{\mathbf{P}^3}(d)) \\ &= h^0(\mathcal{O}_{\mathbf{P}^4}(d-1)) - \tfrac{1}{4}h^0(\mathcal{O}_{\mathbf{P}^3}(d)) \geq 0 \text{ for all } d \geq 1 \end{aligned}$$

Hence $e \geq 0$ for $d \geq 2$ as required. □

Lemma 3.7. *Vectorial Method* 2
Suppose d, e, f, g satisfy (5a), (5b), *and* (5c). *Write $h^0(\mathcal{O}_{\mathbf{P}^3}(d)) - f - g = \overline{b}$, where $\overline{b}$ is a non-negative integer. Set $\overline{a} = e - \overline{b}$, $\overline{c} = g$ and $\psi = \varepsilon - 1$. If $\overline{a} \geq 0$ and $3\overline{b} + \psi c \leq h^0(\Omega_{\mathbf{P}^3}(d))$, then $\boldsymbol{H}'_{\Omega,4}(d; \overline{a}, \overline{b}, \overline{c})$ implies $\boldsymbol{H}'_{\mathcal{O},4}(d-1; e, f, g)$.*

Proof. Consider the sequence;

$$0 \longrightarrow \Omega_{\mathbf{P}^4}(d) \longrightarrow \mathcal{O}_{\mathbf{P}^4}(d-1)^{\oplus 4} \longrightarrow \mathcal{O}_{\mathbf{P}^3}(d) \longrightarrow 0.$$

Taking global sections we have;

$$0 \longrightarrow \underset{H^0(\mathbf{P}^4, \Omega_{\mathbf{P}^4}(d))}{\dim \frac{(d-1)(d+1)(d+2)(d+3)}{6}} \longrightarrow \underset{H^0(\mathbf{P}^4, \mathcal{O}_{\mathbf{P}^4}(d-1)^{\oplus 4})}{\dim \frac{d(d+1)(d+2)(d+3)}{6}} \longrightarrow \underset{H^0(\mathbf{P}^3, \mathcal{O}_{\mathbf{P}^3}(d))}{\dim \frac{(d+1)(d+2)(d+3)}{6}} \longrightarrow 0$$

with the points $E_1, \cdots, E_e \in \mathbf{P}^4$, $F_1, \cdots, F_f, G \in \mathbf{P}^3$, we have $f + g \leq h^0(\mathcal{O}_{\mathbf{P}^3}(d))$.

Let F, G be general sets of f, g points respectively in $\mathbf{P}^3$, and specialize E to $\overline{A} \cup \overline{B}$ with $\overline{A}$ a general set of $\overline{a}$ general points in $\mathbf{P}^4$ and $\overline{B}$ a general set of $\overline{b}$ points in $\mathbf{P}^3$, let $\overline{C} = G$. The map $H^0(\mathcal{O}_{\mathbf{P}^3}(d)) \longrightarrow H^0(\mathcal{O}_{\mathbf{P}^3}(d)|_{F \cup G \cup \overline{B}})$is bijective, i.e. just specializing the points needed for bijectivity which works since $\mathcal{O}_{\mathbf{P}^3}(d)$ is a line bundle. We then construct the following diagram of exact sequences:

$$\begin{array}{ccc}
0 & & 0 \\
\uparrow & & \uparrow \\
H^0(\mathbf{P}^3, \mathcal{O}_{\mathbf{P}^3}(d)) & \xrightarrow{\rho} & \bigoplus_{i=1}^{f} \mathcal{O}_{\mathbf{P}^3}(d)_{|F_i} \oplus \mathcal{O}_{\mathbf{P}^3}(d)_{|G} \oplus \bigoplus_{j=1}^{\overline{b}} \mathcal{O}_{\mathbf{P}^3}(d)_{|B_j} \\
\uparrow & & \uparrow \\
H^0(\mathbf{P}^4, \mathcal{O}_{\mathbf{P}^4}(d-1)^{\oplus 4}) & \xrightarrow{\sigma} & \bigoplus_{i=1}^{e} \mathcal{O}_{\mathbf{P}^4}(d-1)^{\oplus 4}_{|E_i} \oplus \bigoplus_{i=1}^{f} \Omega_{\mathbf{P}^3}(d)_{|F_i} \oplus \Gamma|_G(F) \\
\uparrow & & \uparrow \\
H^0(\mathbf{P}^4, \Omega_{\mathbf{P}^4}(d)) & \xrightarrow{\tau} & \bigoplus_{i=1}^{\overline{a}} \Omega_{\mathbf{P}^4}(d)_{|A_i} \oplus \bigoplus_{i=1}^{\overline{b}} \Omega_{\mathbf{P}^3}(d)_{|B_i} \oplus \overline{\Gamma'}|_C(F) \\
\uparrow & & \uparrow \\
0 & & 0
\end{array}$$

The map ρ is bijective. The map τ is bijective by $\boldsymbol{H}'_{\Omega,4}(d;\overline{a},\overline{b},\overline{c})$ if $\Gamma'|_C(F)$ is general, which we may assume for F general (note that τ does not depend on F). So σ is also bijective, and $\boldsymbol{H}'_{\mathcal{O},4}(d-1;e,f,g)$ holds. The hypothesis $\boldsymbol{H}''_{\mathcal{O},4}(d-1;e,f,g)$ also holds because it is the same except with $\Gamma|_G$ and $\overline{\Gamma'}|_C$ not depending on F.

Note that:

For the hypothesis $\boldsymbol{H}'_{\Omega,4}(d;\overline{a},\overline{b},\overline{c})$ to be true, given e, f and g satisfy the conditions in Lemma 2.2, then $\overline{a}, \overline{b}$ and $\overline{c}$ must satisfy:

(i) $\overline{a} = e - \overline{b} \geq 0$ and

(ii) $3\overline{b} + \psi\overline{c} \leq h^0(\Omega_{\mathbf{P}^3}(d))$

(i) We investigate the condition $\overline{a} \geq 0$.

From (5c) in Lemma 2.2 we know $e = \frac{1}{4}(h^0\big(\mathcal{O}_{\mathbf{P}^4}(d-1)^{\oplus 4}) - f - \varepsilon g)$ and from the definition of this Lemma here we have $\overline{b} = h^0(\mathcal{O}_{\mathbf{P}^3}(d)) - f - g$.

We have,

$\overline{a} = e - \overline{b} = \frac{1}{4}(h^0\big(\mathcal{O}_{\mathbf{P}^4}(d-1)^{\oplus 4}) - f - \varepsilon g) - (h^0(\mathcal{O}_{\mathbf{P}^3}(d)) - f - g)$

$\qquad = h^0\big(\mathcal{O}_{\mathbf{P}^4}(d-1)\big) - h^0(\mathcal{O}_{\mathbf{P}^3}(d) + \frac{3}{4}f + g(1 - \frac{\varepsilon}{4})$. So we have

$\overline{a} \geq \frac{1}{4}(h^0\big(\mathcal{O}_{\mathbf{P}^4}(d-1)^{\oplus 4})) - h^0(\mathcal{O}_{\mathbf{P}^3}(d))$

$\quad = h^0\big(\mathcal{O}_{\mathbf{P}^4}(d-1)\big) - h^0(\mathcal{O}_{\mathbf{P}^3}(d)) \geq 0$ for all $d \geq 4$

$\quad = \binom{d+3}{3}(d-4)/4$ Thus $\overline{a} \geq 0$ as required for $d \geq 4$.

(ii) Now for $3\overline{b} + \psi\overline{c} \leq h^0(\Omega_{\mathbf{P}^3}(d))$ we have $h^0(\mathcal{O}_{\mathbf{P}^3}(d)) - f - g = \overline{b}$ from which we have 3 possibilities since $0 \leq \psi \leq 2$ and $\overline{c} = g$.

(a) $3\overline{b} = 3h^0(\mathcal{O}_{\mathbf{P}^3}(d)) - 3f - 3g$

(b) $3\overline{b} + \overline{c} = 3h^0(\mathcal{O}_{\mathbf{P}^3}(d)) - 3f - 2g$

(c) $3\overline{b} + 2\overline{c} = 3h^0(\mathcal{O}_{\mathbf{P}^3}(d)) - 3f - g$

Also since $h^0(\mathcal{O}_{\mathbf{P}^3}(d)) = \frac{1}{6}(d+3)(d+2)(d+1)$ and $h^0(\Omega_{\mathbf{P}^3}(d)) = \frac{1}{2}(d+2)(d+1)(d-1)$, We pose the question, **when is** $3\overline{b} + \psi\overline{c} \leq h^0(\Omega_{\mathbf{P}^3}(d))$**?**

We answer, **when** $3h^0(\mathcal{O}_{\mathbf{P}^3}(d)) - h^0(\Omega_{\mathbf{P}^3}(d)) = 2(d+1)(d+2)$ **is less than** $3f + \eta g$ **where** $1 \leq \eta \leq 3$ **for** $0 \leq \psi \leq 2$ **i.e.**
Hence $3\overline{b} + \psi\overline{c} \leq h^0(\Omega_{\mathbf{P}^3}(d))$ when $3f + \eta g \geq 2(d+1)(d+2)$ for $1 \leq \eta \leq 3$ for $0 \leq \psi \leq 2$. □

We have e general points, $E_1, \cdots, E_e$ in $\mathbf{P}^4$, and $F_1, \cdots, F_f, G$ in $\mathbf{P}^3$ and the number of fibers in $\mathbf{P}^3$ are few enough in comparison to d i.e. we use this method when the Vectorial Method 2 fails i.e. when none of the conditions relating d with f and g in the last Lemma fail specifically when we have:
(a) $f + g < \frac{2}{3}(d+1)(d+2)$ for $\psi = 0$
(b) $f + \frac{2}{3}g < \frac{2}{3}(d+1)(d+2)$ for $\psi = 1$
(c) $f + \frac{1}{3}g < \frac{2}{3}(d+1)(d+2)$ for $\psi = 2$

Lemma 3.8. *Plane Divisorial Method*
Suppose d, e, f, g are non-negative integers satisfying the conditions of Lemma 2.2. Set $e' = e - h^0(\mathcal{O}_{\mathbf{P}^3}(d-1)) = e - \frac{1}{6}(d+2)(d+1)d$. If we have $e' \geq 0$, and $f + g \leq h^0(\mathcal{O}_{\mathbf{P}^3}(d-1))$, then $\boldsymbol{H}'_{\mathcal{O},4}(d-2; e', f, g)$ implies $\boldsymbol{H}'_{\mathcal{O},4}(d-1; e, f, g)$ and similarly for $\boldsymbol{H}''_{\mathcal{O},4}(d-1; e, f, g)$.

Proof. We therefore choose a hyperplane $H \subseteq \mathbf{P}^4$ disjoint from $\{F_1, \cdots, F_f, G\}$ with $H \cong \mathbf{P}^3$ and send $y = \dim \mathcal{O}_H(d-1)$ points from $\mathbf{P}^4$ to H and we have exact sequence:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}_{\mathbf{P}^4}(d-2)^{\oplus 4} & \longrightarrow & \mathcal{O}_{\mathbf{P}^4}(d-1)^{\oplus 4} & \longrightarrow & \mathcal{O}_H(d-1)^{\oplus 4} & \longrightarrow & 0 \\ & & (e', f, g) & & (e, f, g) & & (h^0(\mathcal{O}_{\mathbf{P}^3}(d-1)), 0, 0) & & \end{array}$$

Taking global sections and evaluating at the corresponding points gives the sequence

$$\begin{array}{ccc} 0 & & 0 \\ \uparrow & & \uparrow \\ H^0(H, \mathcal{O}_H(d-1)^{\oplus 4}) & \xrightarrow[\cong]{\alpha} & \bigoplus_{i=1}^{y} \mathcal{O}_H(d-1)^{\oplus 4}_{|E_i} \\ \uparrow & & \uparrow \\ H^0(\mathbf{P}^4, \mathcal{O}_{\mathbf{P}^4}(d-1)^{\oplus 4}) & \xrightarrow{\beta} & \bigoplus_{i=1}^{e} \mathcal{O}_{\mathbf{P}^4}(d-1)^{\oplus 4}_{|E_i} \oplus \bigoplus_{i=1}^{f} \mathcal{O}_{\mathbf{P}^3}(d)_{|F_i} \oplus \Gamma_{|G} \\ \uparrow & & \uparrow \\ H^0(\mathbf{P}^4, \mathcal{O}_{\mathbf{P}^4}(d-2)^{\oplus 4}) & \xrightarrow{\gamma} & \bigoplus_{i=1}^{e'} \mathcal{O}_{\mathbf{P}^4}(d-2)^{\oplus 4}_{|E_i} \oplus \bigoplus_{i=1}^{f} \mathcal{O}_{\mathbf{P}^3}(d-1)_{|F_i} \oplus \Gamma_{|G} \\ \uparrow & & \uparrow \\ 0 & & 0 \end{array}$$

If γ is bijective then β is bijective giving us $\boldsymbol{H}'_{\mathcal{O},4}(d-2; e', f, g)$ implies $\boldsymbol{H}'_{\mathcal{O},4}(d-1; e, f, g)$.

Two conditions to be satisfied by e', f **and** g **are:**

(i) $e' \geq 0$ and
(ii) $f + g \leq h^0(\mathcal{O}_{\mathbf{P}^3}(d-1))$

(i) We have (i) $e' = e - y$
$$= \tfrac{1}{4}(h^0(\mathcal{O}_{\mathbf{P}^4}(d-1)^{\oplus 4}) - f - \varepsilon g) - h^0(\mathcal{O}_{\mathbf{P}^3}(d-1))$$
$$= \tfrac{1}{4}(h^0(\mathcal{O}_{\mathbf{P}^4}(d-1)^{\oplus 4}) - f - g + g - \varepsilon g) - h^0(\mathcal{O}_{\mathbf{P}^3}(d-1))$$
$$\geq \tfrac{1}{4}(h^0(\mathcal{O}_{\mathbf{P}^4}(d-1)^{\oplus 4}) - h^0(\mathcal{O}_{\mathbf{P}^3}(d)) + g(1-\varepsilon)) - h^0(\mathcal{O}_{\mathbf{P}^3}(d-1))$$
$$\geq \tfrac{1}{4}(h^0(\mathcal{O}_{\mathbf{P}^4}(d-1)^{\oplus 4}) - h^0(\mathcal{O}_{\mathbf{P}^3}(d)) - 4h^0(\mathcal{O}_{\mathbf{P}^3}(d-1)) + g(1-\varepsilon))$$
$= \frac{1}{4}(\frac{(d+2)(d+1)^2(d-3)}{6} + g(1-\varepsilon))$ since $g(1-\varepsilon) \in \{0, -1, -2\}$ for $d \geq 4$ thus $e' \geq 0$ for $d \geq 4$

(ii) In the cases where the Vectorial Method 2 does not work we have $f+g < [\frac{2}{3}(d+1)(d+2)+\frac{2}{3}]$
$= [\frac{4}{6}(d+1)(d+2) + \frac{2}{3}] \leq \frac{1}{6}d(d+1)(d+2)$ for $d \geq 4$.
For $d \geq 2$ follows the initial cases and hence we have $f + g \leq h^0(\mathcal{O}_{\mathbf{P}^3}(d-1))$.

□

Theorem 3.9. (a) *For* $d \geq 2, \boldsymbol{H}_{\mathcal{O},4}(d-1)$ *implies* $\boldsymbol{H}_{\Omega,4}(d+1)$
(b) *For* $d \geq 4, \boldsymbol{H}_{\Omega,4}(d)$ *and* $\boldsymbol{H}_{\mathcal{O},4}(d-2)$ *imply* $\boldsymbol{H}_{\mathcal{O},4}(d-1)$

Proof. (a) For any a, b, c satisfying the conditions of Lemma 2.1 we define e, f, g as in Lemma 3.4. By Lemma 3.6 they satisfy the conditions of Lemma 2.2. So since $\boldsymbol{H}_{\mathcal{O},4}(d-1)$ holds then, the hypothesis $\boldsymbol{H}'_{\mathcal{O},4}(d-1; e, f, g)$ holds. So by Lemma 3.4 the hypothesis $\boldsymbol{H}'_{\Omega,4}(d+1; a, b, c)$ holds this proves $\boldsymbol{H}_{\Omega,4}(d+1)$

(b) For any e, f, g verifying the conditions of Lemma 2.2 either we have $\boldsymbol{H}_{\Omega,4}(d; \overline{a}, \overline{b}, \overline{c})$ implying $\boldsymbol{H}'_{\mathcal{O},4}(d-1; e, f, g)$ or $\boldsymbol{H}'_{\mathcal{O},4}(d-2; e', f, g)$ implying $\boldsymbol{H}'_{\mathcal{O},4}(d-1; e, f, g)$.
In the first case, define $\overline{a}, \overline{b}, \overline{c}$ as in Lemma 4.9. If the numerical conditions hold then we have $\overline{a} \geq 0$ and $3\overline{b} + \psi\overline{c} \leq h^0(\Omega_{\mathbf{P}^3}(d))$ and so $\boldsymbol{H}'_{\Omega,4}(d; \overline{a}, \overline{b}, \overline{c})$ holds and so $\boldsymbol{H}_{\Omega,4}(d)$ holds.
If the numerical conditions do not hold the we have the second case and we define $e' = e - h^0(\mathcal{O}_{\mathbf{P}^3}(d-1))$ as in Lemma 3.8. We thus have $e' \geq 0$and $f + g \leq h^0(\mathcal{O}_{\mathbf{P}^3}(d-1))$ and so $\boldsymbol{H}'_{\mathcal{O},4}(d-2; e', f, g)$ holds. This proves $\boldsymbol{H}_{\mathcal{O},4}(d-1)$. □

This proves the goals we set ourselves to prove in the Subsection 4.3.

3.2.3. *Hypercritical mèthode d'Horace.*

Lemma 3.10. *Consider* $\boldsymbol{H}'_{\mathcal{O},4}(d-1; s_1, s_2, 0)$ *where* $d \geq 1, s_1,$ *and* s_2 *are non-negative integers and suppose that the map* $H^0(\Omega_{\mathbf{P}^4}(d)) \longrightarrow H^0(\Omega_{\mathbf{P}^4}(d)|_{S_1})$ *is injective and that the map* $H^0(\mathcal{O}_{\mathbf{P}^4}(d-1)^{\oplus 4}) \longrightarrow H^0(\mathcal{O}_{\mathbf{P}^4}(d-1)^{\oplus 4}|_{S_1})$ *is surjective with a general* $S_1 \subseteq \mathbf{P}^n$ *then the hypothesis* $\boldsymbol{H}'_{\mathcal{O},4}(d-1; s_1, s_2, 0)$ *is true.*

Proof. Follows from Lemma 1.11 □

3.3. Other Lemmas and Corollaries.

Lemma 3.11. *The following hypotheses are true.*

a. $\boldsymbol{H}'_{\mathcal{O},4}(2; 13, 8, 0)$

b. $\boldsymbol{H}'_{\mathcal{O},4}(2;12,12,0)$
c. $\boldsymbol{H}'_{\mathcal{O},4}(2;12,10,1)$
d. $\boldsymbol{H}'_{\mathcal{O},4}(2;12,9,1)$

Proof. For (a) and (b) we use Lemma 4.9. Set $s_1 = 13$ and $s_2 = 8$ for (a) and $s_1 = 12$ and $s_2 = 12$ for (b) and injectivity of the map $H^0(\Omega_{\mathbf{P}^4}(3)) \longrightarrow H^0(\Omega_{\mathbf{P}^4}(3)|_{S_1})$ will follow from Lemma 4.4 i.e. for 10 general points in $\mathbf{P}^4$ we have bijectivity and in this two cases we have 12 and 13 points thus it follows which in turn implies surjectivity of the map $H^0(\mathcal{O}^{\oplus 4}_{\mathbf{P}^4}(2)) \longrightarrow H^0(\mathcal{O}^{\oplus 4}_{\mathbf{P}^4}(2)|_{S_1})$

Now for (c) we proceed as follows:
For the hypothesis $\boldsymbol{H}'_{\mathcal{O},4}(2;12,10,1)$ we have 12 general points, $P_1 \cdots, P_{12}$ in $\mathbf{P}^4$, 10 general points, $Q_1, \cdots, Q_{10}$ in $\mathbf{P}^3$ and a quotient, $\Gamma(Q_1 \cdots, Q_{10})$ at C in $\mathbf{P}^3$.
Let $< F_1, F_2, F_3 >$ be the space of quadratic forms on $\mathbf{P}^4$ vanishing at $P_1 \cdots, P_{12}$. Identify $\mathcal{O}_{\mathbf{P}^4}(2)^{\oplus 4} = \mathcal{O}_{\mathbf{P}^4}(2) \otimes V$ with $V = H^0(\mathcal{O}_{\mathbf{P}^3}(1))$. Then we need to show that

$$< F_1, F_2, F_3 > \otimes V \longrightarrow \bigoplus_{i=1}^{10} \mathcal{O}_{\mathbf{P}^3}(3)|_{Q_i} \oplus \Gamma|_C(Q_1 \cdots, Q_{10})$$

is bijective for some $P_1 \cdots, P_{12}$, $Q_1, \cdots, Q_{10}$, C.
Now the 2 dimensional quotient

$$\mathcal{O}_{\mathbf{P}^4}(2) \otimes V \twoheadrightarrow \Gamma|_C(Q_1 \cdots, Q_{10})$$

has a kernel $< L_1(Q_1 \cdots, Q_{10}), L_2(Q_1 \cdots, Q_{10}) > \subseteq V$. The zero locus $L_1 = L_2 = 0$ is a line $D \subseteq \mathbf{P}^3$. Since $Q_1 \cdots, Q_{10}$ are in linear general position. So there exists two of the Q_i say Q_9, Q_{10} such that D, Q_9, Q_{10} span $\mathbf{P}^3$.
We claim that we can choose $P_1 \cdots, P_{12}$ and a basis F_1, F_2, F_3 of $H^0(\mathrm{T}_{P_1 \cdots, P_{12}}(2))$ such that F_1, F_2 vanish at C, Q_9, Q_{10} while F_3 does not vanish at C, Q_9, Q_{10}. In that case we have a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & < F_1, F_2 > \otimes V & \longrightarrow & < F_1, F_2, F_3 > \otimes V & \longrightarrow & F_3 \otimes V & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \bigoplus_{i=1}^{8} \mathcal{O}_{\mathbf{P}^3}(3)|_{Q_i} & \longrightarrow & \bigoplus_{i=1}^{10} \mathcal{O}_{\mathbf{P}^3}(3)|_{Q_i} \oplus \Gamma|_C & \longrightarrow & \bigoplus_{i=9}^{10} \mathcal{O}_{\mathbf{P}^3}(3)|_{Q_i} \oplus \Gamma|_C & \longrightarrow & 0 \end{array}$$

in which the first and third vertical arrows are isomorphisms. The space $< F_1, F_2 > \otimes V$ injects onto an 8 dimensional subspace $< \overline{F_1}, \overline{F_2} > \otimes V \subseteq H^0(\mathcal{O}_{\mathbf{P}^3}(3))$ because the points $H^0(\Omega_{\mathbf{P}^4}(3)) \longrightarrow \Omega_{\mathbf{P}^4}(3)|_{P_1, \cdots, P_{12}}$ is injective by Lemma 4.4. The first vertical arrow is then bijective for general $Q_1 \cdots, Q_8$ as is the middle row and we will be done.
To prove our claim, pick $P_1 \cdots, P_{10}$ so that vanishing at $P_1 \cdots, P_{10}, C, Q_9, Q_{10}$ impose impose 13 independent conditions on $H^0(\mathcal{O}_{\mathbf{P}^4}(2))$. Let $< F_1, F_2 >$ be the space of forms vanishing at those points. Let $J = H^0(\mathrm{T}_{P_1 \cdots, P_{10}}(2))$ it is 5 dimensional and contains $< F_1, F_2 >$.
Since vanishing at C, Q_9, Q_{10} impose non-trivial conditions on J, for general $P_{11}, P_{12} \in \mathbf{P}^4$ vanishing at P_{11}, P_{12}, C impose independent conditions on J, as do vanishing at P_{11}, P_{12}, Q_9 and P_{11}, P_{12}, Q_{10}. So $H^0(\mathrm{T}_{P_1 \cdots, P_{12}}(2)) =< F_1, F_2, F_3 >$ for an F_3 not vanishing at any of C, Q_9, Q_{10}. □

Lemma 3.12. *(A specific case of* [1] *Lemme 1) Suppose we are given a surjective morphism of vector spaces,*

$$\lambda : H^{\circ}(\mathbf{P}^3, \Omega_{\mathbf{P}^3}(d+1)) \twoheadrightarrow L$$

and suppose there exists a point Z' in $\mathbf{P}^3$ such that

$$H^{\circ}(\mathbf{P}^3, \Omega_{\mathbf{P}^3}(d+1)) \hookrightarrow L \oplus \Omega_{\mathbf{P}^3}(d+1)_{|Z'} \text{ and}$$

Suppose also that $H^1(\mathbf{P}^4, \mathcal{O}_{\mathbf{P}^4}(d-1)^{\oplus 4}) = 0$. Then there exists a quotient $\mathcal{O}_{\mathbf{P}^4}(d-1)^{\oplus 4}_{|Z'} \longrightarrow D(\lambda)$ with kernel contained in $\Omega_{\mathbf{P}^3}(d)_{|Z'}$ of dimension $\dim(D(\lambda)) = \operatorname{rank}(\Omega_{\mathbf{P}^4}(\mathrm{d}+1)) - \dim(\ker \lambda)$ *having the following property.*

Let $\mu : H^{\circ}(\mathbf{P}^4, \Omega_{\mathbf{P}^4}(d+1)) \longrightarrow M$ be a morphism of vector spaces then there exists Z in $\mathbf{P}^3$ such that if
$H^{\circ}(\mathbf{P}^4, \mathcal{O}_{\mathbf{P}^4}(d-1)^{\oplus 4}) \longrightarrow M \oplus D(\lambda)$ is of maximal rank then
$H^{\circ}(\mathbf{P}^4, \Omega_{\mathbf{P}^4}(d+1)) \longrightarrow M \oplus L \oplus \Omega_{\mathbf{P}^4}(d+1)|_Z$ is also of maximal rank.

Proposition 3.13. *For any $d \geq 1$ and any subspace $V \subseteq H^0(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(d))$ there exists $M_1, \ldots, M_m \in \mathbf{P}^n$ such that $V \longrightarrow \bigoplus_{i=1}^{m} \mathcal{O}_{\mathbf{P}^n}(d)|_{M_i}$ has maximal rank property.*

Proof. Consider the following maps, α, β and γ inter-vectorial spaces

$$\begin{array}{ccc} H^0\big(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(d)\big) & \xrightarrow[\beta]{} & V \\ & \searrow^{\alpha} & \downarrow \gamma \\ & & \bigoplus_{i=1}^{m} \mathcal{O}_{\mathbf{P}^n}(d)|_{M_i} \end{array}$$

If $h^0(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(d)) = m$ then α is bijective since it's an evaluation of line bundles at m points; β is surjective hence γ is injective.
If $h^0(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(d)) < m$ then α is injective; β is surjective and hence γ is injective.
If $h^0(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(d)) > m$ then α is surjective; β is surjective but then γ has 3 possiblities:
(a) if $m < \dim V$ then γ is surjective
(b) if $m = \dim V$ then γ is bijective
(c) finally if $m > \dim V$ then γ is injective
Hence γ is either injective, surjective or both (bijective) i.e. it is of maximal rank for as long as V is independent of the $M_1, \ldots, M_m \in \mathbf{P}^n$ □

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