African Journal of Mathematics and Computer Science Research Vol. 3(7), pp. xxx-xxx, Juiy, 2010
Available online at http://www.academicjournals.org/AJMCSR ISSN 2006-9731

Full Length Research Paper

Norm properties of operators who's norms are Eigenvalues

A.W. Wafula, J.M. Khalagai and G.Pokhariyal

School of Mathematics, University of Nairobi, P.O.Box 30197 Nairobi, Kenya.

Accepted 19 May 2010

In this paper we present properties of a norm-attaining operator on a Hilbert space and there implications. We show that if T has norm attaining vector then $\left\|(\sum_{k=0}^{n}\alpha_k T^k)x\right\| = \sum_{k=0}^{n}\alpha_k \|T\|^k$ where the scalars are nonnegative numbers. Thus T satisfies a generalized Daugavet condition.

Keywords: Numerical range, eigenvalue, normalloid operator, Daugavet property.

INTRODUCTION

In this article we extend results obtained by C.S Lin on properties of an operator whose norm is an Eigen value.
It is noteworthy that each compact operator on a Hilbert space (Halmos, 1794) has a norm-attaining vector (Shilov, 1965). Thus these properties are characteristics of compact operators. We will denote operators on a Hilbert space by capital letters. The numerical range of an operator T is the convex set of complex numbers defined by $W(T)=\{(Tx,x):\|x\|=1, x\in H\}$. We shall denote by T^* the ad joint of T. We say that T satisfy the Daugavet (1963) equation if $\|I+T\|=1+\|T\|$
A unit vector x in H is a norm attaining vector for T if $\|T\|=\|Tx\|$. Lin (2002) wrote a paper on a bounded operator on a Hilbert space whose norm is an eigenvalue and established the following theorem namely that

THEORY

Theorem 1

Let x be a unit vector in H. Then the following are equivalent.

1. $\|T\|$ is an eigenvalue of T, that is, $Tx=\|T\|x$.
2. $1+\|T\|=\|(I+T)x\|$.
3. $\|T\|$ is an eigenvalue of T and $Tx=\|Tx\|x$ i.e $Tx=\|T\|x$ and $Tx=\|Tx\|x$.
4. $\|T\|$ is in the numerical range of T that is. $\|T\|=(Tx,x)$.
5. x is a complete vector for T,that is, $\|T\|=(Tx,x)=\|Tx\|$
6. 2 $\|T\|$ is an eigen-value of $T+T^*$,that is $(T+T^*)x=2\|T\|x$.
7. $\|T\|$ and $\|T\|^2$ are eigen-values of T and T^*T, respectively , with respect to x ,that is, $Tx=\|T\|x$ and $T^*Tx=\|T\|^2x$.
8. $(1+\|T\|)\|T\|$ is an eigen- value of $(I+T^*)T$,i.e. $(I+T^*)Tx=(1+\|T\|)\|T\|x$.
9. $\|T\|$ is a normal eigenvalue for T,i.e. $Tx=\|T\|x=T^*x$.
10. x is a complete vector for T and T^*, i.e. $Tx=(Tx,x)=\|Tx\|=\|T^*x\|$.
11. x is a complete vector for T and

Corresponding author email:arthurwanyonyi@yahoo.com

T^*T ,i.e. $\|T\| = (Tx,x) = \|Tx\|$ and $\|T\|^2 = \|Tx\|^2 = \|T^*Tx\|$

12. $1+\|T\|+\|T\|^2 = \|(I+T+T^*T)x\|$.

Now we make a natural extension of part 2 of theorem 1 as follows

Lemma 2

Let x be an operator on a Hilbert space H and x be a unit vector in H. Then the following statements are equivalent;

(i) X is an eigenvector of T with eigenvalue $\|T\|$, that is, $Tx = \|T\|x$.

(ii) For any sequence $\alpha_1, \ldots, \alpha_n$ of positive numbers

$$\left\|(\sum_{k=0}^{n}\alpha_k T^k)x\right\| = \sum_{k=0}^{n}\alpha_k\|T\|^k$$

Proof. $(i) \to (ii)$ If $\|T\|$ is an eigenvalue of T then it follows that

$$\sum_{k=0}^{n}\alpha_k\|T\|^{2k} = \left\|\left(\sum_{k=0}^{n}\alpha_k(T^*T)^k\right)x\right\|$$

$$\left\|(\sum_{k=0}^{n}\alpha_k T^k)x\right\| =$$

$$\left\|\sum_{k=0}^{n}\alpha_k\|T\|^k x\right\| = \sum_{k=0}^{n}\alpha_k\|T\|^k\|x\| = \sum_{k=0}^{n}\alpha_k\|T\|^k$$

$(ii) \to (i)$ Now set $\alpha_k = 0, \alpha_1 = 1, k \neq 1$ to obtain $\|Tx\| = \|T\|$.

Also from $\alpha_0 = \alpha_1 = 1, \alpha_k = 0, k \succ 1$ we obtain $1+\|T\| = \|(I+T)x\|$.

Hence $\|(I+T)x\|^2 = (1+\|T\|)^2$.

Consequently $\|(I+T)x\|^2 = ((I+T)x,(I+T)x)$

$= (x,x)+(Tx,x)+(x,Tx)+(Tx,Tx)$

$= 1+(Tx,x)+(x,Tx)+\|T\|^2 = 1+2\|T\|+\|T\|^2$

This leads to the result $(Tx,x)+(x,Tx) = 2\|T\|$. To show that $\|T\|$ is an eigenvalue of T we consider the following expansion.

$$\|Tx-\|T\|x\|^2 = (Tx-\|T\|x, Tx-\|T\|x)$$

$= (Tx,Tx)-\|T\|(Tx,x)-\|T\|(x,Tx)+\|T\|^2(x,x)$

$= \|T\|^2-\|T\|[(Tx,x)+(x,Tx)]+\|T\|^2$

$= \|T\|^2-\|T\|(2\|T\|)+\|T\|$

$= 0$

Hence $Tx-\|T\|x = 0 \Leftrightarrow Tx = \|T\|x$ and so $\|T\|$ is an eigenvalue of T.

In the same article Lin proved the theorem below which enumerates the properties of an operator a with norm attaining vector

Theorem 3

If T is an operator on a Hilbert space H and x is norm one vector in H then the following are equivalent statements; any of the statements in theorem 1

(ii) $\left\|(\sum_{k=0}^{n}\alpha_k T^k)x\right\| = \sum_{k=0}^{n}\alpha_k\|T\|^k$

Proof

It follows immediately from lemma 1

In the same article Lin proved the following

Theorem 4

Let x be a unit vector. Then the following are equivalent.

1. x is a norm attaining vector for T that is, $\|T\| = \|Tx\|$.
2. $\|T\|^2$ is an eigenvalue for T i.e. $T^*Tx = \|T\|^2 x$.
3. $1+\|T\|^2 = \|(I+T^*T)x\|$.
4. x is a complete vector

T^*T ,i.e. $\|T\|^2 = \|(T^*T)x\| = \|Tx\|^2$

5. $\|T\|^2$ is an eigenvalue for T^*T ,and $T^*Tx = \|(T^*T)x\| x$,i.e. $T^*Tx = \|T\|^2 x$ and $T^*Tx = \|(T^*T)x\| x$.

6. $\|T\|^2$ is in the numerical range of T^*T ,i.e. $\|T\|^2 = ((T^*T)x, x)$.

7. $1 + \|T\|^2 + \|T\|^4 = \|I + T + T^*T + (T^*T)^2\|$.

8. $\|T\|^2$ and $\|T\|^4$ are eigenvalues for T^*T and $(T^*T)^2$,respectively with respect to x, that is, $T^*Tx = \|T\|^2 x$ and $(T^*T)^2 x = \|T\|^4 x$.

9.
$$1 + \|(I + T + T^*T\| = 1 + \|T\| + \|T\|^2 + \|T\|^2 = \|(I + T + T^*T\| = 1 + \|(I + T + T^*T\| \|I + T + T^*T\| = \|(I + T + T^*T)x\|$$

$(1 + \|T\|^2)\|T\|^2$ is an eigenvalue of $(I + T^*T)T^*T$, that is, $(I + T^*T)T^*T$ x= $(1 + \|T\|^2)\|T\|^2$ x.

10. x is a complete vector for T^*T and $(T^*T)^2$, that is, $\|T\|^2 = \|Tx\|^2 = \|T^*Tx\|$ and $\|T\|^4 = \|T^*Tx\|^2 = \|(T^*T)^2 x\|$.

We now prove a general result to the above in the following lemma

Lemma 5

Let T be an operator on a Hilbert space H and let x be a unit vector in H then the following are equivalent statements

(i) X is an eigenvector of T^*T with Eigen value $\|T\|^2$ that is, $T^*Tx = \|T\|^2 x$

(ii) For any sequence $\alpha_1, \ldots, \alpha_n$ of positive numbers

$$\sum_{k=0}^{n} \alpha_k \|T\|^{2k} = \left\| \left(\sum_{k=0}^{n} \alpha_k (T^*T)^k \right) x \right\|$$

Proof

If we replace T with T^*T then we obtain the (i) if and only if

$$\sum_{k=0}^{n} \alpha_k \|T^*T\|^k = \left\| \left(\sum_{k=0}^{n} \alpha_k (T^*T)^k \right) x \right\|.$$

But we have that $\|T^*T\| = \|T\|^2$. Hence we obtain the result.

Theorem 6

Let T be an operator on a Hilbert space H and x be a unit vector then the following are equivalent statements;
Any statement in theorem 6

For any $\alpha_1, \ldots, \alpha_n$ positive numbers

$$\sum_{k=0}^{n} \alpha_k \|T\|^{2k} = \left\| \left(\sum_{k=0}^{n} \alpha_k (T^*T)^k \right) x \right\|$$

Proof

The result follows from lemma
The following corollary which shows that if $\|T\|$ is an eigenvalue of T with respect to x, then x is a norm attaining vector for T and satisfies the Daugavet property that is,

Corollary 7

Let x be a unit vector. Then any statement in theorem 2 implies the following;
Any statement in theorem 4
T satisfy the Daugavet equation, that is, $1 + \|T\| = \|(I + T)x\|$.

T and T^* satisfy the generalized Daugavet equation $\|(I + T + T^*\| = 1 + 2\|T\|$.

T and T^*T satisfy the generalized Daugavet equation $\|(I + T + T^*T\| = 1 + \|T\| + \|T\|^2$.

T is a normaloid operator, that is, $r(T) = \|T\|$.

X is a norm attaining vector forI+T, that is, $\|I + T\| = \|(I + T)x\|$.

X is a norm attaining vector for I+T+ T^* , that is, $\|I + T + T^*\| = \|(I + T + T^*)x\|$.

X is a norm attaining vector for $I + T + T^*T$, that is, $\|I + T + T^*T\| = \|(I + T + T^*T)x\|$

Proof,

As in Lin with Theorem 1 and 2 now replaced with 3 and 4

We now consider further results when the operator T is both self ad joint and compact. In this case the operator has a norm attaining vector as shown by Shilov

Theorem 8

If T is a compact self ad joint operator then $||T^n|| = w(T^n) = (w(T))^n = ||T||^n$

Proof

Since T is self ad joint T^n is also self ad joint and so the first equality follows from corollary

2. Also, if x is the norm attaining vector for T we have ;

$(T^n x;x) = (T^{n-1}x;Tx) = ||T|| (T^{n-1}x;x) = ||T||(T^{n-2}x;Tx) = ||T||^2(T^{n-2}x;x) = ||T||^n = (w(T))^n$.

But we have $w(T^n) \geq (T^n x;x)$. Consequently $w(T^n) \geq (w(T))^n$.

For the reverse inequality we note that $w(T^n) = ||T^n|| \leq ||T||^n = (w(T))^n$ corollary 9

If T is a compact self adjoint operator then

$$w(\sum_{k=1}^{n} \alpha_k T^n) = || \sum_{k=1}^{n} \alpha_k T^n || = \sum_{k=1}^{n} \alpha_k || T ||^n = \sum_{k=1}^{n} \alpha_k (w(T))^n$$

k=0 k=0 k=0 k=0

Where α_k are non negative numbers.

Proof

The first equality follows from the fact that the sum of self ad joint operators is also self ad joint. The second follows from properties of norm attaining operators. A compact self-ad joint operator therefore satisfies a generalized Daugavet equation (Lin, 2002).

Conclusion

The main result here is that if an operator satisfies a Daugavet condition then it also satisfies a generalized Daugavet condition with nonnegative scalars that is, $\left\|(\sum_{k=0}^{n} \alpha_k T^k)x\right\| = \sum_{k=0}^{n} \alpha_k \|T\|^k$ .It would be of interest if this result can be extended to an infinite sum.

REFERENCES

Lin CS (2002). On 0perator whose norm is an eigenvalue, Bulletin of the institute of mathematics. pp: 31

Daugavet IK (1963). A property of compact operators in the space C, Uspekhi Mat. Nauk, 18: 157-158(Russian).

*Halmos PR (1974). Hilbert space problem book, Springer, New York-Heidelberg-Berlin (The name and year did not correspond in the work)

Shilov GS (1965). Mathematical Analysis, International series of monographs in pure and applied matyhematics. pp: 77