# On the regular polygons in the Chinese checker plane

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**Abstract.** In this paper, we define Chinese checker regular polygons, and determine which Euclidean regular polygons are also Chinese checker regular, and which are not. Finally, we study the problem of existence or nonexistence of Chinese checker regular polygons.

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## 1 Introduction

In the game of Chinese checkers, checkers are allowed to move in the vertical (north and south), horizontal (east and west), and diagonal (northeast, northwest, southeast and southwest) directions. In [5], Krause asked how to develop a distance function that measures the lengths of ways mimicing the movements of the Chinese checkers in the Cartesian coordinate plane. Later, Chen [1] defined the distance and named it Chinese checker distance as follows:

Let $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ be two points in $\mathbb{R}^2$. The function $d_C : \mathbb{R}^2 \times \mathbb{R}^2 \to [0, \infty)$ defined by

$$d_C(P,Q) = \max\{|x_1 - x_2|, |y_1 - y_2|\} + (\sqrt{2} - 1)\min\{|x_1 - x_2|, |y_1 - y_2|\}$$

is called *Chinese checker* ($CC$) *distance* between $P$ and $Q$.

He then proved that the distance function $d_C$ is a metric. CC plane geometry is almost the same as Euclidean plane geometry; the points are the same, the lines are the same, and the angles are measured in the same way. However, the distance function is different. The CC distance between points $P$ and $Q$ is the Euclidean length of one of the shortest paths from $P$ to $Q$ composed of line segments, each parallel to one of the coordinate axes or one of angle bisectors of them. Since CC geometry has a distance function different from that in Euclidean geometry, it is interesting to study the CC analogues of topics that include the distance concept in Euclidean geometry (see [3], [2], [4], [7], [8], [6] for some works on this subject).

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## 2 CC regular polygons

As in the Euclidean plane, a *polygon* in the CC plane consists of three or more coplanar line segments; the line segments (*sides*) intersect only at endpoints; each endpoint (*vertex*) belongs to exactly two line segments; no two line segments with a common endpoint are collinear. If the number of sides of a polygon is $n$ ($n \geqslant 3, n \in \mathbb{N}$), then the polygon is called an *n-gon*. The following definitions for the polygons in the CC plane are given by means of the CC lengths instead of the Euclidean lengths:

**Definition 1** A polygon in the plane is called *CC equilateral* if the CC lengths of its sides are equal.

**Definition 2** A polygon in the plane is called *CC equiangular* if the measures of its interior angles are equal.

**Definition 3** A polygon in the plane is called *CC regular* if it is both CC equilateral and CC equiangular.

Definition 2 does not give a new equiangular concept because the CC and the Euclidean measure of an angle are the same. That is, every Euclidean equiangular polygon is also CC equiangular, and vice versa. However, since the CC plane has a different distance function, Definition 1 and therefore Definition 3 are new concepts. In this study, we determine which Euclidean regular polygons in the plane are also CC regular, and which are not. We also investigate the existence and nonexistence of CC regular polygons.

It is known that a rotation of $\theta$ radians around a point preserves the CC distance if and only if $\theta \in \left\{\frac{t\pi}{4} + 2k\pi : 0 \leq t \leq 7,\ t, k \in \mathbb{Z}\right\}$, and a reflection about the line $ax + by + c = 0$ preserves the CC distance if and only if $\frac{a}{b} \in \left\{0, \pm 1, \pm(\sqrt{2}-1), \pm(\sqrt{2}+1)\right\}$ or $b = 0$ (see [4]). Let us denote by $S$ the set of lines $x = 0$ , $y = 0$, $y = x$, $y = -x$, $y = (\sqrt{2}-1)x$, $y = -(\sqrt{2}-1)x$, $y = (\sqrt{2}+1)x$, and $y = -(\sqrt{2}+1)x$. Then one can immediately state following:

**Lemma 1.** *Let A, B, and C be three non-collinear points in the Cartesian plane such that* $d_E(A,B) = d_E(B,C)$. *Then* $d_C(A,B) = d_C(B,C)$ *if and only if the measure of the angle ABC is* $\pi/4$, $\pi/2$ *or* $3\pi/4$, *or A and C are symmetric about the line through B and parallel to a line in S.*

## 3 Euclidean regular polygons in the CC plane

Since every Euclidean regular polygon is also CC equiangular, it is obvious that a Euclidean regular polygon is CC regular if and only if it is CC equilateral. Therefore, to investigate the CC regularity of a Euclidean regular polygon, it is sufficient to determine whether it is CC equilateral or not. In doing so, we use following concepts: Any Euclidean regular polygon can be inscribed in a circle, and a circle can be circumscribed about any Euclidean regular polygon. A point is called the *center* of a Euclidean regular polygon if it is the center of the circle circumscribed about the polygon. A line $l$ is called *axis of symmetry* of a polygon if the polygon is symmetric about $l$, and in addition, if $l$ passes through a vertex of the polygon then $l$ is called *radial axis of symmetry* of the polygon. Clearly, every axis of symmetry of a Euclidean regular polygon passes through the center of the polygon.

Now, we are ready to investigate the CC regularity of Euclidean regular polygons. The following three propositions determine Euclidean regular polygons that are also CC regular.

**Proposition 1.** *Every Euclidean regular quadrilateral (Euclidean square) is CC regular.*

*Proof.* Since every side of the Euclidean square has the same Euclidean length and the angle between every two consecutive sides is a right angle (see Figure 1), by Lemma 1, every side has the same CC length. Therefore, every Euclidean square is CC equilateral, and therefore is CC regular. □

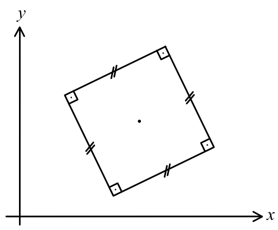


Figure 1

**Proposition 2.** *Every Euclidean regular octagon is CC regular.*

*Proof.* Since every side of the Euclidean regular octagon has the same Euclidean length and the measure of the angle between every two consecutive sides is $3\pi/4$ radians (see Figure 2), by Lemma 1, every side has the same CC length. Therefore, every Euclidean regular octagon is CC equilateral, and therefore is CC regular. □

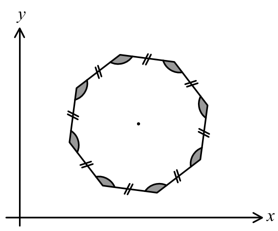


Figure 2

**Proposition 3.** *Every Euclidean regular 16-gon, one of whose radial axes of symmetry is parallel to a line in $S$, is CC regular.*

*Proof.* Clearly, every Euclidean regular 16-gon has eight radial axes of symmetry, and if one of them is parallel to a line in $S$, then each of the other radial axes of symmetry is parallel to a line in $S$, too (see Figure 3). Since every two consecutive sides of such a Euclidean regular 16-gon are symmetric about a line parallel to a line in $S$, and every side has the same Euclidean length, by Lemma 1, these sides have the same CC length. Thus a Euclidean regular 16-gon, one of whose radial axes of symmetry is parallel to a line in $S$, is CC equilateral, and therefore is CC regular. □

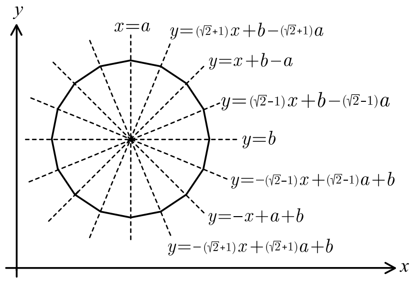


Figure 3

Following theorem states that there is no other Euclidean regular polygon that is also CC regular.

**Theorem 4.** *No Euclidean regular polygon, except the ones in Proposition 1, Proposition 2 and Proposition 3, is CC regular.*

*Proof.* Let us consider first a Euclidean regular $n$-gon such that $n \neq 4$, $n \neq 8$, and $n \neq 16$. Then the measure of the angle between any two consecutive sides of the $n$-gon, $\pi(n-2)/n$, is not $\pi/4$, $\pi/2$ or $3\pi/4$ radians. Clearly, the measure of the angle between two lines each of which is parallel to a line in $S$ is a multiple of $\pi/8$ radians. Since the measure of the angle between any two consecutive radial axes of symmetry of the $n$-gon, $2\pi/n$, is not a multiple of $\pi/8$ radians, one of those radial axes of symmetry is not parallel to any line in $S$. Therefore, there exist two consecutive sides which are symmetric to each other about a line not parallel to any line in $S$, and the angle between them is not $\pi/4$, $\pi/2$, or $3\pi/4$ radians. These sides have not the same CC length, by Lemma 1. Thus the $n$-gon is not CC regular. Let us consider now a Euclidean regular 16-gon, one of whose radial axes of symmetry is not parallel to any line in $S$. Clearly, the measure of the angle between two consecutive sides of the 16-gon is $7\pi/8$ radians, and none of the radial axes of symmetry is parallel to a line in $S$. Then there exist two consecutive sides which are symmetric to each other about a line not parallel to any line in $S$, and the angle between them is not $\pi/4$, $\pi/2$, or $3\pi/4$ radians. These sides have not the same CC length, by Lemma 1. Thus the 16-gon, one of whose radial axes of symmetry is not parallel to any line in $S$, is not CC regular. The proof is completed. □

## 4 Existence of CC regular $2n$-gons

We know now that which Euclidean regular polygons are CC regular, and which are not. Furthermore, we also know existence of some CC regular polygons. However, we do not have general knowledge about the existence of CC regular polygons. The following theorem shows the existence of CC regular $2n$-gons, using CC circles. Recall that a CC circle with center $A$ and radius $r$ is the set of all points whose CC distance to $A$ is $r$. This locus of points is a Euclidean regular octagon, one of whose radial axes of symmetry having slope 0. Just as for a Euclidean circle, the center $A$ and one point at a CC distance $r$ from $A$ completely determine the CC circle.

**Theorem 5.** *There exist two congruent CC regular $2n$-gons ($n \geq 2$) having given any line segment as a side.*

*Proof.* Clearly, the measure of each interior angle of an equiangular $2n$-gon ($n \geq 2$) is $\pi(n-1)/n$ radians. Let us consider now any given line segment $A_1A_2$ in the CC plane. It is obvious that $(n-1)$ line segments $A_iA_{i+1}$, $2 \leq i \leq n$, having the same CC length with $A_1A_2$ can be constructed using the CC circles with center $A_i$ and radius $d_C(A_1, A_2)$, such that the measure of the angle between every two consecutive segments is $\pi(n-1)/n$ radians (see Figure 4). Also it is not difficult to see that $\measuredangle A_2A_1A_{n+1} + \measuredangle A_nA_{n+1}A_1 = \pi(n-1)/n$ when these line segments are constructed.

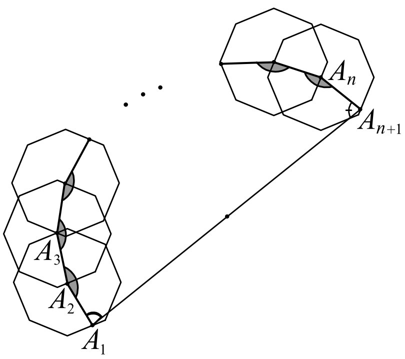


Figure 4

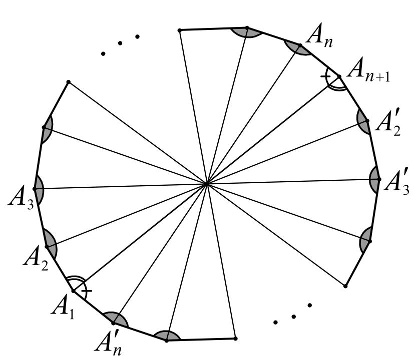


Figure 5

If we continue to construct line segments $A'_iA'_{i+1}$, $1 \leq i \leq n$, which are symmetric to $A_iA_{i+1}$ about the midpoint of $A_1A_{n+1}$, respectively, we get a $2n$-gon (see Figure 5). Since the symmetry about a point (rotation of $\pi$ radians around a point) preserves both the CC distances and the angle measures, we have $d_C(A_i, A_{i+1}) = d_C(A'_i, A'_{i+1}) = d_C(A_1, A_2)$ for $1 \leq i \leq n$, and $\measuredangle A_i = \measuredangle A'_i = \pi(n-1)/n$ for $2 \leq i \leq n$. Also it is easy to see that $\measuredangle A_1 = \measuredangle A_{n+1} = \pi(n-1)/n$. Thus, the constructed $2n$-gon is CC regular. Furthermore, on the other side of the line $A_1A_2$, one can construct another CC regular $2n$-gon, having the same line segment $A_1A_2$ as a side, using the same procedure (see Figure 6). However, it is easy to see that these two CC regular $2n$-gons are symmetric about the midpoint of the line segment $A_1A_2$, and congruent. □

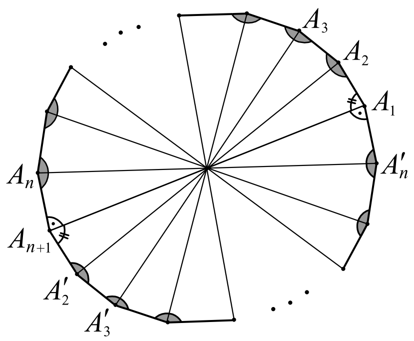


Figure 6

In any CC regular $2n$-gon, there are $n$ line segments joining the corresponding vertices of the $2n$-gon ($A_iA'_i$, $1 \leq i \leq n$, for the polygon in Figure 5 and Figure 6). We call each of these line segments *axis* of the polygon. Clearly, axes of any CC regular $2n$-gon intersect at one and only one point.

**Example** Using the procedure given in the proof of Theorem 5, one can easily construct CC regular $2n$-gons having given any line segment as a side. For example, we construct a CC regular quadrilateral (*CC square*), a CC regular hexagon and a CC regular octagon, having given line segment $A_1A_2$ as a side, in Figure 7, 8 and 9:

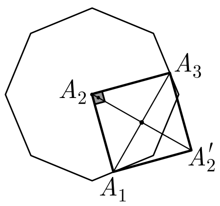


Figure 7

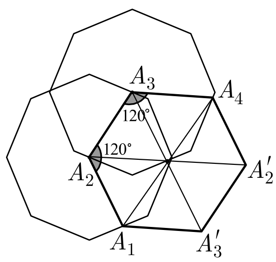


Figure 8

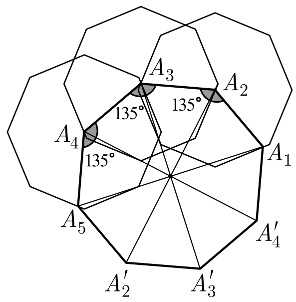


Figure 9

## 5 More about CC regular $2n$-gons

We know by Proposition 1 and Proposition 2 that every Euclidean square and Euclidean regular octagon are CC regular. By Proposition 6 and Proposition 7, we see that every CC square and CC regular octagon are also Euclidean regular. The following lemma is another statement of Lemma 1:

**Lemma 2.** *Let $A$, $B$, and $C$ be three non-collinear points in the Cartesian plane such that $d_C(A,B) = d_C(B,C)$. Then $d_E(A,B) = d_E(B,C)$ if and only if the measure of the angle $ABC$ is $\pi/4$, $\pi/2$ or $3\pi/4$, or $A$ and $C$ are symmetric about the line through $B$ and parallel to a line in $S$.*

The following three propositions determine CC regular polygons that are also Euclidean regular.

**Proposition 6.** *Every CC square is Euclidean regular.*

*Proof.* Since every side of the CC square has the same CC length and the angle between every two consecutive sides is a right angle, by Lemma 2, every side has the same Euclidean length. Therefore, every CC square is Euclidean equilateral, and therefore is Euclidean regular. □

**Proposition 7.** *Every CC regular octagon is Euclidean regular.*

*Proof.* Since every side of the CC regular octagon has the same CC length and the measure of the angle between every two consecutive sides is $3\pi/4$, by Lemma 2, every side has the same Euclidean length. Therefore, every CC regular octagon is Euclidean equilateral, and therefore is Euclidean regular. □

We need a new notion to prove the next proposition: An equiangular polygon with an even number of vertices is called *equiangular semi-regular* if sides have the same Euclidean length alternately. There is always a Euclidean circle passing through all vertices of an equiangular semi-regular polygon (see [9]).

**Proposition 8.** *Every CC regular 16-gon, one of whose axes is parallel to a line in $S$, is Euclidean regular.*

*Proof.* In every CC regular 16-gon, since the measure of the angle between any two alternate sides is $3\pi/4$ and sides have the same CC length, by Lemma 2, sides have the same Euclidean length alternately. Therefore, every CC regular 16-gon is equiangular semi-regular. It is obvious that if any two consecutive sides of an equiangular semi-regular polygon have the same Euclidean length, then the polygon is Euclidean regular. Let us consider a CC regular 16-gon, $A_1A_2...A_{16}$, one of whose axes, let us say $A_1A_9$, is parallel to $y = 0$, for one case (see Figure 10). Then there exist a Euclidean circle with diameter $A_1A_9$, passing through points $A_1, A_2, ..., A_{16}$, and there exist a CC circle with center $A_1$, passing through points $A_2$ and $A_{16}$. Since the Euclidean and the CC circles are both symmetric about the line $A_1A_9$, the intersection points of them, $A_2$ and $A_{16}$, are also symmetric about the same line. Then two consecutive sides $A_1A_2$ and $A_1A_{16}$ have the same Euclidean length. Therefore, the CC regular 16-gon, one of whose axes is parallel to the line $y = 0$, is Euclidean regular. The other cases are similar. □

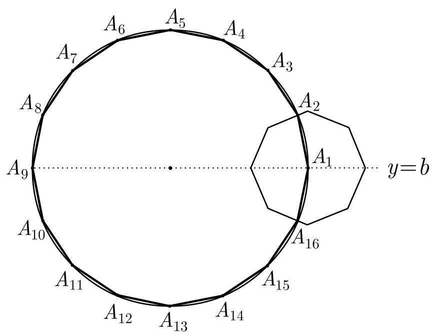


Figure 10

Following theorem states that there is no other CC regular polygon that is also Euclidean regular.

**Theorem 9.** *No CC regular polygon, except the ones in Proposition 6, Proposition 7 and Proposition 8, is Euclidean regular.*

*Proof.* Assume that there exists a CC regular polygon, except the ones in Proposition 6, Proposition 7 and Proposition 8, that is also Euclidean regular. Then there exists a Euclidean regular polygon, except the ones in Proposition 1, Proposition 2 and Proposition 3, that is also CC regular. But this is in contradiction with Theorem 4. Therefore, no CC regular polygon, except the ones in Proposition 6, Proposition 7 and Proposition 8, is Euclidean regular. □

Consequently, in the CC plane, squares and regular octagons have invariable shapes which are the same as those in the Euclidean plane, and they are the only polygons having this property.

## 6 On the nonexistence of CC $(2n-1)$-gons

The following proposition shows the nonexistence of CC regular triangle:

**Proposition 10.** *There is no CC regular triangle.*

*Proof.* Every CC equiangular triangle is Euclidean regular. Since no Euclidean regular triangle is CC regular by Theorem 4, no CC equiangular triangle is CC regular. Therefore, there is no CC regular triangle. □

Unfortunately, we could not reach any conclusion by reasoning about the existence or nonexistence of CC regular $(2n-1)$-gons for $n \geq 3$. However, we have seen that there is no CC regular 5-gon, 9-gon and 15-gon using a computer program called *C.a.R* (*Compass and Ruler*) (see [10]). Our conjecture is that there is no CC regular $(2n-1)$-gon. It seems interesting to study the open problem: "Does there exist any CC regular $(2n-1)$-gon?"

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