On the dual Bishop Darboux rotation axis of the dual space curve

Murat Kemal Karacan, Bahaddin Bukcu and Nural Yuksel

Abstract. In this paper, the Dual Bishop Darboux rotation axis for dual space curve in the dual space D^3 is decomposed in two simultaneous rotation. The axes of these simultaneous rotations are joined by a simple mechanism.

M.S.C. 2000: 53A04, 53A17, 53A25.
Key words: Dual space curve, Dual Bishop frame, Dual parallel transport frame, Dual Bishop Darboux vector, Dual Bishop Darboux rotation axis.

1 Preliminaries

The Bishop frame or parallel transport frame is an alternative approach to defining a moving frame that is well defined even when the curve has vanishing second derivative.

We can parallel transport an orthonormal frame along a curve simply by parallel transporting each component of the frame. The parallel transport frame is based on the observation that, while $T(s)$ for a given curve model is unique, we may choose any convenient arbitrary basis $(N_1(s), N_2(s))$ for the remainder of the frame, as long as it is in the normal plane perpendicular to $T(s)$ at each point. If the derivatives of $(N_1(s), N_2(s))$ depend only on $T(s)$ and not on each other, then we can make $N_1(s)$ and $N_2(s)$ vary smoothly throughout the path regardless of the curvature. Therefore, we have the alternative frame equations

$$\begin{bmatrix} T' \\ N_1' \\ N_2' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & k_2 \\ -k_1 & 0 & 0 \\ -k_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} T \\ N_1 \\ N_2 \end{bmatrix}$$

where

$$\kappa(s) = \sqrt{k_1^2 + k_2^2},\ \theta(s) = \arctan\left(\frac{k_2}{k_1}\right),\ \tau(s) = -\frac{d\theta(s)}{ds}$$

[6, 7, 2], so that k_1 and k_2 effectively correspond to a Cartesian coordinate system for the polar coordinates κ, θ with $\theta = -\int \tau(s)ds$. The orientation of the parallel transport frame includes the arbitrary choice of integration constant θ_0, which disappears from τ (and hence from the Frenet frame) due to the differentiation.

Applied Sciences, Vol.10, 2008, pp. 115-120.

2 Introduction

In the Euclidean 3-Space E^3, lines combined with one of their two directions can be represented by unit dual vectors over the the ring of dual numbers. The important properties of real vector analysis are valid for the dual vectors. The oriented lines E^3 are in one to one correspondence with the points of the dual unit sphere D^3.

A dual point on D^3 corresponds to a line in E^3, two different points of D^3 represents two skew lines in E^3.A differentiable curve on D^3represents a ruled surface E^3. If φ and φ^* are real numbers and $\varepsilon^2 = 0$ the combination $\widehat{\varphi} = \varphi + \varepsilon\varphi^*$ is called a dual number. The symbol ε designates the dual unit with the property $\varepsilon^2 = 0$. In analogy with the complex numbers W.K. Clifford defined the dual numbers and showed that they form an algebra, not a field. The our dual numbers are εa^*.

According to the definition, the pure dual numbers εa^* are zero divisors. No εa^* number has an inverse in this algebra. Later, E.Study introduced the dual angle subtended by two nonparallel lines E^3, and defined it as $\widehat{\varphi} = \varphi + \varepsilon\varphi^*$ in which φ and φ^* are, respectively, the projected angle and the shortest distance between the two lines.

The set

$$D = \{\widehat{x} = x + \varepsilon x^* \, | x, x^* \in R\}$$

of dual numbers is a commutative ring with respect to the operations

$$i) \quad (x + \varepsilon x^*) + (y + \varepsilon y^*) = (x + y) + \varepsilon(x^* + y^*)$$

$$ii) \quad (x + \varepsilon x^*).(y + \varepsilon y^*) = (x + y) + \varepsilon(xy^* + yx^*).$$

The division $\frac{\widehat{x}}{\widehat{y}}$ is possible and unambiguous if $\widehat{y} \neq 0$ and it can be easily seen that

$$\frac{\widehat{x}}{\widehat{y}} = \frac{x + \varepsilon x^*}{y + \varepsilon y^*} + \frac{x}{y} + \varepsilon\frac{x^* y - xy^*}{y^2}.$$

The set

$$D^3 = D \times D \times D = \left\{ \begin{array}{c} \widehat{x} \mid \widehat{x} = (x_1 + \varepsilon x_1^*, x_2 + \varepsilon x_2^*, x_3 + \varepsilon x_3^*) \\ = (x_1 + x_2 + x_3) + \varepsilon\left(x_1^* + x_2^* + x_3^*\right), \\ = x + \varepsilon x^*, x \in R^3, x^* \in R^3 \end{array} \right\}$$

is a module over the ring D. It is clear that any dual vector $\widehat{x}$ in D^3, consists of any two real vectors x and x^* in R^3, which are expressed in the natural right handed orthonormal frame in the 3-dimensional Euclidean space E^3. We call the elements of D^3, dual vectors. If $x \neq 0$, then the norm $\|\widehat{x}\|$ of $\widehat{x}$ is defined by

$$\|\widehat{x}\| = \sqrt{\langle \widehat{x}, \widehat{x} \rangle} = \|x\| + \varepsilon\frac{xx^*}{\|x\|}.$$

Let $\widehat{x} : I \to D^3$, $s \to \widehat{x}(s) = x(s) + \varepsilon x^*(s)$ be a C^4 curve with the arc-length parameter s of the indicatrix. Then

$$\frac{d\widehat{x}}{d\widehat{s}} = \frac{d\widehat{x}}{ds}\frac{ds}{d\widehat{s}} = \widehat{T}$$

is called the unit tangent vector of $\widehat{x}(s)$ [5]. Since $\widehat{T}$ has constant length 1, its differentiation with respect to $\widehat{s}$, which is given by

$$\frac{d\widehat{T}}{d\widehat{s}} = \frac{d\widehat{T}}{ds}\frac{ds}{d\widehat{s}} = \frac{d^2\widehat{x}}{d\widehat{s}} = \widehat{k_1}\widehat{N_1} + \widehat{k_2}\widehat{N_2},\ \widehat{k_1} = k_1 + \varepsilon k_1^*,\ \widehat{k_2} = k_2 + \varepsilon k_2^*$$

measures the way the curve is turning in D^3. We impose the restriction that the dual natural curvatures $\widehat{k_1},\ \widehat{k_2} : I \to D$ are never pure dual. We call the vectors $\widehat{T}$, $\widehat{N_1}$, $\widehat{N_2}$ dual Bishop trihedron of $\widehat{x}(s)$. Writing

$$\begin{aligned}
\frac{d\widehat{T}}{d\widehat{s}} &= a_{11}\widehat{T} + a_{12}\widehat{N_1} + a_{13}\widehat{N_2} \\
\frac{d\widehat{N_1}}{d\widehat{s}} &= a_{21}\widehat{T} + a_{22}\widehat{N_1} + a_{23}\widehat{N_2} \\
\frac{d\widehat{N_2}}{d\widehat{s}} &= a_{31}\widehat{T} + a_{32}\widehat{N_1} + a_{33}\widehat{N_2}
\end{aligned}$$

and using the properties of inner product and differentiations of the inner products $\widehat{T}$, $\widehat{N_1}$ and $\widehat{N_2}$, we may express the dual Bishop formulas of the dual Bishop trihedron in matrix form:

$$\begin{bmatrix} \widehat{T}' \\ \widehat{N}_1' \\ \widehat{N}_2' \end{bmatrix} = \begin{bmatrix} 0 & \widehat{k_1} & \widehat{k_2} \\ -\widehat{k_1} & 0 & 0 \\ -\widehat{k_2} & 0 & 0 \end{bmatrix} \begin{bmatrix} \widehat{T} \\ \widehat{N_1} \\ \widehat{N_2} \end{bmatrix}.$$

3 The dual Bishop Darboux rotation axis of the dual space curve

Let $\left\{\widehat{T}, \widehat{N_1}, \widehat{N_2}\right\}$ be the dual Bishop frame of the differentiable dual space curve in the dual space D. Then the dual Bishop frame equations are

$$\begin{aligned}
\widehat{T}' &= k_1N_1 + k_2N_2 + \varepsilon\left(k_1^*N_1 + k_1N_1^* + k_2^*N_2 + k_2N_2^*\right) \\
\widehat{N_1}' &= -k_1T - \varepsilon\left(k_1^*T + k_1T^*\right) \\
\widehat{N_2}' &= -k_2T - \varepsilon\left(k_2^*T + k_2T^*\right)
\end{aligned}$$

where $\widehat{k_1} = k_1 + \varepsilon k_1^*$ and $\widehat{k_2} = k_2 + \varepsilon k_2^*$ are nowhere pure dual natural curvatures and

$$\widehat{\kappa(t)} = \kappa + \varepsilon\kappa^* = \sqrt{k_1^2 + k_2^2 + 2\varepsilon\left(k_1k_1^* + k_2k_2^*\right)},$$

$$\widehat{\theta}(t) = \theta + \varepsilon\theta^* = \arctan\left(\frac{\widehat{k_2}}{\widehat{k_1}}\right) = \arctan\left(\frac{k_2}{k_1} + \varepsilon\frac{(k_1k_2^* - k_1^*k_2)}{k_1^2}\right),$$

$$\widehat{\tau}(t) = \frac{d\widehat{\theta}(t)}{dt}.$$

These equations form a dual Bishop rotation motion with dual Bishop Darboux vector,

$$\widehat{\varpi} = \varpi + \varepsilon\varpi^* = -\widehat{k_2}\widehat{N_1} + \widehat{k_1}\widehat{N_2} = -k_2N_1 + k_1N_2 + \varepsilon\left(k_1^*N_2 + k_1N_2^* - k_2^*N_1 - k_2N_1^*\right).$$

Also momentum dual rotation vector is expressed as follows:

$$\begin{aligned}\widehat{T}' &= \widehat{\varpi} \wedge \widehat{T} \\ \widehat{N_1}' &= \widehat{\varpi} \wedge \widehat{N_1} \\ \widehat{N_2}' &= \widehat{\varpi} \wedge \widehat{N_2}.\end{aligned}$$

The dual Bishop Darboux rotation of the dual Bishop frame can be separated into two rotation motions. The vector $\widehat{N_1}$ rotates with an angular speed $\widehat{k_1}$ about the vector $\widehat{N_2}$, that is

$$\widehat{N_1}' = \left(\widehat{k_1}\widehat{N_2}\right) \wedge \widehat{N_1} = \widehat{k_1}\left(\widehat{N_2} \wedge \widehat{N_1}\right) = -\widehat{k_1}\widehat{T}$$

and the vector $\widehat{N_2}$ rotates with a angular speed $-\widehat{k_2}$ about the vector $\widehat{N_1}$, that is

$$\widehat{N_2}' = \left(-\widehat{k_2}\widehat{N_1}\right) \wedge \widehat{N_2} = -\widehat{k_2}\left(\widehat{N_1} \wedge \widehat{N_2}\right) = -\widehat{k_2}\widehat{T}.$$

The separation of the rotation motion of the momentum dual Bishop Darboux axis into two rotation motions can be indicated as follows: The vector $\frac{\widehat{\varpi}}{\|\widehat{\varpi}\|}$ rotates with a angular speed

$$\widehat{W} = \frac{\widehat{k_1}'\widehat{k_2} - \widehat{k_1}\widehat{k_2}'}{\widehat{k_1}^2 + \widehat{k_2}^2}$$

about the tangent vector $\widehat{T}$, also

$$\left(\frac{\widehat{\varpi}}{\|\widehat{\varpi}\|}\right)' = \left(\widehat{W}.\widehat{T}\right) \wedge \frac{\widehat{\varpi}}{\|\widehat{\varpi}\|}$$

and the tangent vector $\widehat{T}$ rotates with a angular speed $\|\widehat{\varpi}\|$ about $\frac{\widehat{\varpi}}{\|\widehat{\varpi}\|}$. Dual Bishop Darboux axis, also

$$\widehat{T}' = \widehat{\varpi} \wedge \widehat{T}.$$

¿From now on we shall do a further study of momentum dual Bishop Darboux axis. We obtain the unit vector $\widehat{E}$:

$$\widehat{E} = \frac{\widehat{\varpi}}{\|\widehat{\varpi}\|} = \frac{-k_2N_1 + k_1N_2 + \varepsilon\left(k_1^*N_2 + k_1N_2^* - k_2^*N_1 - k_2N_1^*\right)}{\sqrt{k_1^2 + k_2^2 + 2\varepsilon\left(k_1k_1^* + k_2k_2^*\right)}}.$$

¿From the dual Bishop Darboux vector,

$$\begin{aligned}\widehat{\varpi}' &= -\widehat{k_2}'\widehat{N_1} - \widehat{k_2}\widehat{N_1}' + \widehat{k_1}'\widehat{N_2} + \widehat{k_1}\widehat{N_2}' \\ &= -\widehat{k_2}'\widehat{N_1} - \widehat{k_2}\left(-\widehat{k_1}\widehat{T}\right) + \widehat{k_1}'\widehat{N_2} + \widehat{k_1}\left(-\widehat{k_2}\widehat{T}\right) \\ &= -\widehat{k_2}'\widehat{N_1} + \widehat{k_1}'\widehat{N_2}.\end{aligned}$$

Differentiation of $\widehat{E}$,

$$E' = \left(\frac{\widehat{\varpi}}{\|\widehat{\varpi}\|}\right)' = \frac{\widehat{\varpi}'\|\widehat{\varpi}\| - \widehat{\varpi}\|\widehat{\varpi}\|'}{(\|\widehat{\varpi}\|)^2} = \frac{\left(\widehat{k_1}'\widehat{k_2} - \widehat{k_1}\widehat{k_2}'\right)}{\widehat{k_1}^2 + \widehat{k_2}^2}\frac{\left(\widehat{k_1}\widehat{N_1} + \widehat{k_2}\widehat{N_2}\right)}{\sqrt{k_1^2 + k_2^2 + 2\varepsilon\left(k_1k_1^* + k_2k_2^*\right)}}$$

is found. From this,

$$\widehat{E}' = \widehat{W}\left(\widehat{E} \wedge \widehat{T}\right)$$

is written. According to the dual Bishop frame,

$$\widehat{T}' = \|\widehat{\varpi}\|\left(\widehat{E} \wedge \widehat{T}\right)$$

and

$$\begin{aligned}\left(\widehat{E} \wedge \widehat{T}\right)' &= \widehat{E}' \wedge \widehat{T} + \widehat{E} \wedge \widehat{T}' \\ &= \widehat{W}.\widehat{T} \wedge \left(\widehat{E} \wedge \widehat{T}\right) + \|\widehat{\varpi}\|.\widehat{E}\left(\widehat{E} \wedge \widehat{T}\right) \\ &= \widehat{W}\left[\left\langle \widehat{T}, \widehat{T}\right\rangle \widehat{E} - \left\langle \widehat{T}, \widehat{E}\right\rangle \widehat{T}\right] + \|\widehat{\varpi}\|\left[\left\langle \widehat{E}, \widehat{T}\right\rangle \widehat{E} - \left\langle \widehat{E}, \widehat{E}\right\rangle \widehat{T}\right] \\ &= \widehat{W}.\widehat{E} - \|\widehat{\varpi}\|\,\widehat{T}\end{aligned}$$

are obtained. These three equations are in the form of dual Bishop frames that is

$$\begin{bmatrix} \widehat{T}' \\ \left(\widehat{E} \wedge \widehat{T}\right)' \\ \widehat{E}' \end{bmatrix} = \begin{bmatrix} 0 & \|\widehat{\varpi}\| & 0 \\ -\|\widehat{\varpi}\| & 0 & \widehat{W} \\ 0 & \widehat{W} & 0 \end{bmatrix} \begin{bmatrix} \widehat{T} \\ \left(\widehat{E} \wedge \widehat{T}\right) \\ \widehat{E} \end{bmatrix}$$

where the first coefficient $\|\widehat{\varpi}\|$ is nowhere pure dual and and second coefficient

$$\widehat{W} = \frac{\widehat{k_1}'\widehat{k_2} - \widehat{k_1}\widehat{k_2}'}{\widehat{k_1}^2 + \widehat{k_2}^2} = \frac{\left(\frac{\widehat{k_1}}{\widehat{k_2}}\right)'}{1 + \left(\frac{\widehat{k_1}}{\widehat{k_2}}\right)^2}$$

is related only to natural harmonic curvature $\frac{\widehat{k_1}}{\widehat{k_2}}$. Thus, the vectors $\left\{\widehat{T}, \left(\widehat{E} \wedge \widehat{T}\right), \widehat{E}\right\}$ define a dual rotation motion together with the dual rotation vector,

$$\widehat{\varpi_1} = -\widehat{W}.\widehat{T} + \|\widehat{\varpi}\|.\widehat{E} = -\widehat{W}.\widehat{T} + \widehat{\varpi}$$

As well, the momentum rotation vector is expressed as follows:

$$\begin{aligned}\widehat{T}' &= \widehat{\varpi_1} \wedge \widehat{T} \\ \left(\widehat{E} \wedge \widehat{T}\right)' &= \widehat{\varpi_1} \wedge \left(\widehat{E} \wedge \widehat{T}\right) \\ \widehat{E}' &= \widehat{\varpi_1} \wedge \widehat{E}.\end{aligned}$$

Remark. This dual rotation motion of dual Bishop Darboux axis can be separated into two dual rotation motions again. Here, the dual rotation vector $\widehat{\varpi_1}$ is the sum of the dual rotation vectors of the dual rotation motions. When continued in the similar way, the dual rotation motion of dual Bishop Darboux axis is done in a consecutive manner. In this way the series of dual Bishop Darboux vectors are obtained. That is $\widehat{\varpi_0} = \widehat{\varpi}, \widehat{\varpi_1} \ldots$, etc.

References

[1] W. Barthel, *Zum drehvorgang der Darboux-achse einer raumkurve*, J. Geometry 49, (1994), 46-49.

[2] L. R. Bishop, *There is more than one way to frame a curve*, Amer. Math. Monthly, 82 (1975), 246-251.

[3] O. Bottema and B. Roth, *Theoretical Kinematics*, North Holland Publ. Com., 1979.

[4] B. Bukcu and M. K. Karacan, *Special Bishop motion and Bishop Darboux rotation axis of the space curve in Euclidean 3 space*, submitted for publication.

[5] A. C. Coken and A. Görgülü, *On the dual Darboux rotation axis of dual space curve*, Demonstratio Mathematica, 37 (2004), 197-202.

[6] A. J. Hanson and H. Ma, *Parallel transport approach to curve framing*, Indiana University, Techreports-TR425, 11 (1995), 3-7.

[7] A. J. Hanson and H. Ma, *Quaternion frame approach to streamline visualization*, IEEE Transactions On Visualization and Computer Graphics, 1 (1995), 164-174.

[8] O. Munaux, *Cad Interface and Framework for Curve Optimisation. Applications*, Cranfield University, PhD Thesis, 2004.

[9] J. M. Selig, *Curves of stationary acceleration in $SE(3)$*, IMA Journal of Mathematical Control and Information, 7 (2006), 17-18.

Authors' addresses:

Murat Kemal Karacan
Usak University, Faculty of Science and Arts,
Department of Mathematics, Usak, Turkey.
E-mail: mkkaracan@yahoo.com , murat.karacan@usak.edu.tr

Bahaddin Bukcu
Gazi Osman Pasa University, Faculty of Science and Arts,
Department of Mathematics, Tokat, Turkey.
E-mail: mbukcu@gop.edu.tr, bbukcu@yahoo. com

Nural Yuksel
Erciyes University, Faculty of Science and Arts,
Department of Mathematics, Kayseri, Turkey.
E-mail: yukseln@erciyes.edu.tr