Solutions of some visibility and contour problems in the visualisation of surfaces

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Abstract. We give a short survey of the main principles of our software for the visualisation and animation in mathematics and study the visibility and contour problems in the representation of surfaces, which we solve for generalised tubular surfaces, including their special cases, the envelopes of spheres, surfaces of rotation and ruled surfaces. We apply our results to visualise several results from differential geometry.

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1 Introduction

Visualisation and animation are of vital importance in modern mathematical education. They strongly support the understanding of mathematical concepts.

We developed our own software package [10, 3, 2, 5, 6, 7, 8, 9, 11] to provide the technical tools for the creation of graphics for the visualisations and animations mainly of the results from differential geometry. Our software package is intended as an alternative to most other conventional software packages.

Our graphics can be exported to BMP, PS, PLT, JVX and other formats (`http://www.javaview.de` for JavaView) or GCLC, the Geometry Constructions Language Converter ([1, 4] `http://www.matfbg.ac.yu/~janicic/gclc`).

We use line graphics, which means that we only draw curves, and represent surfaces by families of curves on them, normally by their parameter lines. We have chosen this approach, since it seems to be the most suitable one for many graphical representations in differential geometry. It also means that we do not need a special strategy for drawing surfaces, such as approximation by triangulation. Curves may be given by parametric representations or equations. They are approximated by polygons.

We developed an independent visibility check to analytically test the visibility of the vertices of the approximating polygons, immediately after the computation of their coordinates. Thus our graphics are generated in a geometrically natural way. The independence of our visibility check enables us to demonstrate, if necessary, desirable but geometrically unrealistic effects, or not to use any test at all for a fast first sketch. Two consecutive points are joined by a straight line segment if and only if both of

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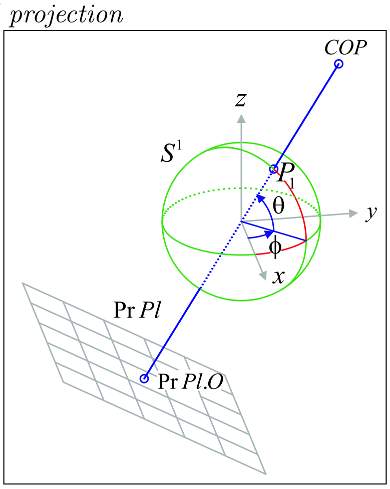
them are visible. Invisible parts of curves may either be dotted or not be drawn at all. To speed up the process, we use interpolation to close occasional gaps.

We use the central projection to create a two–dimensional image of our three–dimensional geometric configuration. This is the most general case. The central projection is uniquely defined by its centre of projection COP and the projection plane $PrPl$. The choices of COP and $PrPl$ are free with the obvious exceptions that COP must not be in $PrPl$, and the plane parallel to $PrPl$ through COP must not intersect the world interval. We decided to give the parameters of the projection in spherical coordiantes, since it seems to be easier to have an idea of the positions of the centre of projection and the projection plane with respect to spherical coordinates. Normally we choose the projection plane orthogonal to the direction of the centre of projection, and its origin antipodal to the centre of projection (Figure 1). This prevents distortions.

We emphasize that all the graphics in this paper were created by our own software and exported to PS files which then were converted to EPS files. The interested reader is referred to [3, 12] for more details.

Figure 1. *The definition of the central projection*

COP	*centre of projection in spherical coordinates* r, ϕ, Θ
$PrPl$	*projection plane (orthogonal to the direction of* COP*)*
$PrPl.O$	*origin of the projection plane (antipodal to* COP*)*
S^1	*unit sphere (to mark the angles* ϕ *and* Θ*)*
P_1	*point on* S^1 *with angles* ϕ *and* Θ



2 The visibility and contour problems

Here we outline the visibility and contour problems that have to be solved for the representation of surfaces.

2.1 The visibilty of points

First we study the visibility problems in the representation of surfaces. Since, in general, we may want to draw several surfaces in a configuration, every point on each surface has to be tested for its visibility. It has to be determined if the point is visible with respect to the surface it is on, and if the point is not hidden by any of the other surfaces. Consequently there are two procedures for the visibility check of a point, $Visibility$ and $NotHidden$.

Let P be a point of $\mathbb{R}^3$, S be a surface with a parametric representation $\vec{x}(u^i)$ ($(u^1, u^2) \in D \subset \mathbb{R}^2$), where D is a domain, and $C = COP$ be the centre of projection. We denote the position vector of P by $\vec{p}$, and write $\vec{v} = \overrightarrow{PC}$. Then P is hidden by S if and only if there exist a pair $(u^1, u^2) \in D$ and a positive real t such that

$$\vec{x}(u^1, u^2) = \vec{p} + t\vec{v}. \tag{2.1}$$

Thus we have to find the intersections of surfaces with straight lines.

2.2 The contour line of a surface

The use of line graphics has the effect that surfaces appear unfinished without their so–called contour lines.

Let S be a surface with a parametric representation $\vec{x}(u^i)$ and surface normal vectors $\vec{N}(u^i)$, $P \in S$ be a point and C be the centre of projection. Then we say that P is a *contour point of* S if and only if the following two conditions are satisfied.

(i) The projection ray to the point P is orthogonal to the surface normal vector $\vec{N}$ at P, that is

$$\overrightarrow{CP} \bullet \vec{N} = 0. \tag{2.2}$$

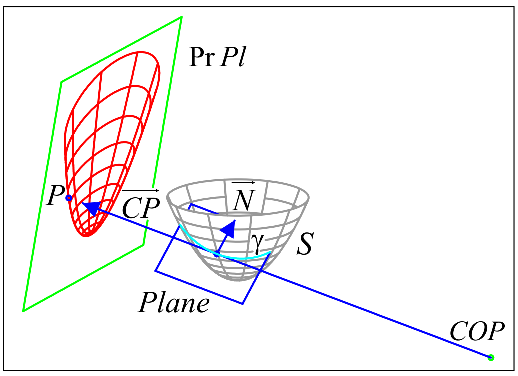
(ii) Let E be the plane through P spanned by the vectors $\overrightarrow{CP}$ and $\vec{N}$, and γ be the curve of intersection of S and E. Then there is a neighbourhood of P in which γ is completely on one side of the projection ray and has no points of intersection with it other than P (Figure 2).

The *contour line of a surface* is the set of all its contour points.

We only use the condition in (2.2) to draw contour lines, since checking the condition in (ii) would be very time consuming and only exclude rare cases (Figure 3). If they really appear the problem can normally be avoided by a slight change in the perspective.

Figure 2. *The definition of a contour point P*

COP centre of projection, $PrPl$ projection plane, S surface, P contour point, $\vec{N}$ surface normal vector, Plane spanned by $\overrightarrow{PC}$ and $\vec{N}$, γ intersection Plane $\cap S$

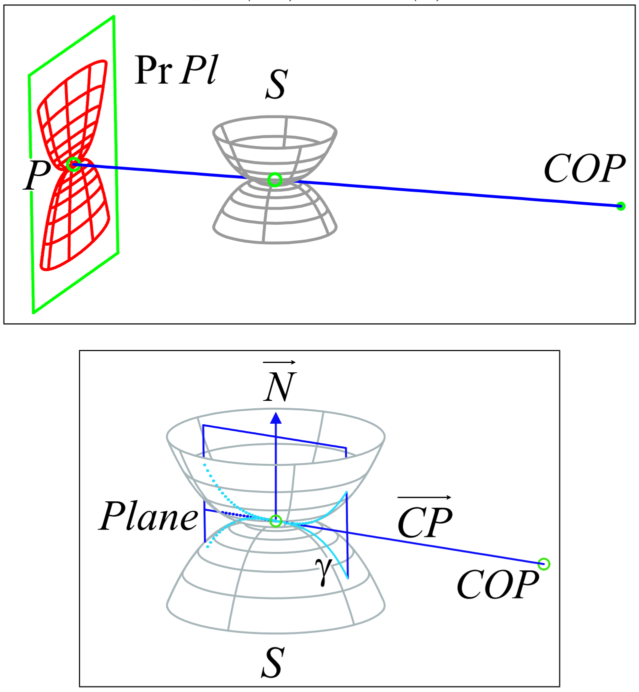


To find the contour line of a surface, we have to find the zeros of the function Ψ defined by

$$\Psi(u^1, u^2) = \vec{N}(u^1, u^1) \bullet \overrightarrow{CP}(u^1, u^2). \tag{2.3}$$

Methods to find the zeros of ϕ and draw the contour lines of surfaces in the general case, and their implementations can be found in [2, 3, 5].

Figure 3. *A point P that satisfies (2.2), but not (ii)*



3 Generalised tubular surfaces

Here we solve the visibility and contour problems for generalised tubular surfaces. They are surfaces generated by the movement of two vectors along a given curve γ with a parametric representation $\vec{y}(t)$ for t in some interval.

A *tubular surface* is the envelope of spheres of varying radii that move along γ. If $\vec{v}_2(t)$ and $\vec{v}_3(t)$ denote the principal normal and binormal vectors of γ at t, and $r(t)$ is the radius of the moving sphere at t, then, writing $u^1 = t$, we obtain a parametric representation for the tubular surface $TS(\gamma; r)$ (Figure 4)

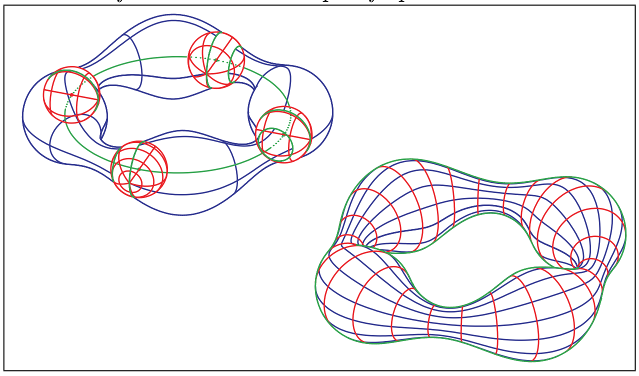
$$\vec{x}(u^i) = \vec{y}(u^1) + r(u^1)\left(\cos u^2 \vec{v}_2(u^1) + \sin u^2 \vec{v}_3(u^1)\right) \quad \left((u^1, u^2) \in I \times (0, 2\pi)\right). \tag{3.1}$$

We generalise as follows. Let γ be a curve with a parametric representation $\vec{y}(t)$ for t in some interval I_1, $\vec{w}_1(t_1)$ and $\vec{w}_2(t_1)$ be vectors with $\vec{w}_1(t_1) \bullet \vec{w}_2(t_1) = 0$ for all

$t \in I_1$, and $r_1, r_2 : I_1 \to \mathbb{R}$ and $h_1, h_2 : I_2 \to \mathbb{R}$ be functions. Then we consider *generalised tubular surfaces* with a parametric representation

(3.2)
$$\vec{x}(u^i) = \vec{y}(u^1) + r_1(u^1)h_1(u^2)\vec{w}_1(u^1) + r_2(u^1)h_2(u^1)\vec{w}_2(u^1) \quad \big((u^1, u^2) \in I_1 \times I_2\big).$$

Figure 4. *A tubular surface as the envelope of spheres*



The most important special cases are:

(i) tubular surfaces

Here we have $r = r_1 = r_2$, $r(t) > 0$ for all $t \in I_1$, $h_1 = \cos$, $h_2 = \sin$, $\vec{w}_1 = \vec{v}_2$ and $\vec{w}_2 = \vec{v}_3$, and obtain (3.1).

(ii) surfaces of rotation

Let $Q \in \mathbb{R}^3$ be a point, $\vec{q}$ its position vector, $\vec{y}(t) = h(t)\vec{d}$, where $h : I_1 \to \mathbb{R}$ is a function and $\vec{d}$ is a constant unit vector orthogonal to the constant unit vectors $\vec{w}_1$ and $\vec{w}_2$, $r = r_1 = r_2$ with $r(t) > 0$ on I_1, and $h_1(t_2) = \cos t_2$ and $h_2(t_2) = \sin t_2$. Then (3.2) reduces to

$$\text{(3.3)} \quad \vec{x}(u^i) = \vec{q} + h(u^1)\vec{d} + r(u^1)\left(\cos u^2\,\vec{w}_1 + \sin u^2\,\vec{w}_2\right) \quad \big((u^1, u^2) \in I_2 \times (0, \pi)\big);$$

this is a parametric representation of the surface of rotation that is generated by rotation the curve γ^* with a parametric representation $\vec{y}^*(t) = h(t)\vec{d} + r(t)\vec{w}_1$ ($t \in I_1$) in the plane through P, spanned by the vectors $\vec{d}$ and $\vec{w}_1$, about the axis through P in the direction of the vector $\vec{d}$ (Figure 5).

(iii) ruled surfaces

Ruled surfaces are generated by the movement of one vector $\vec{v}(t)$ along γ. They have a parametric representation

$$\text{(3.4)} \qquad \vec{x}(u^i) = \vec{y}(u^1) + u^2\vec{z}(u^1) \quad \big((u^1, u^2) \in I_1 \times I_2\big).$$

This is obtained from (3.2) by choosing $\vec{w}_1 = \vec{z}$, $r_1 \equiv 1$, $h_1 = id$ and $h_2 \equiv 0$.

Figure 5. *Generating a surface of rotation*

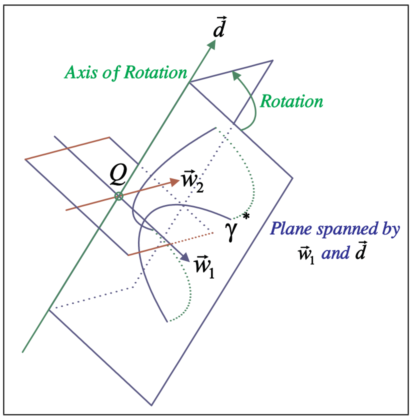


Figure 6. *A torus and a catenoid*

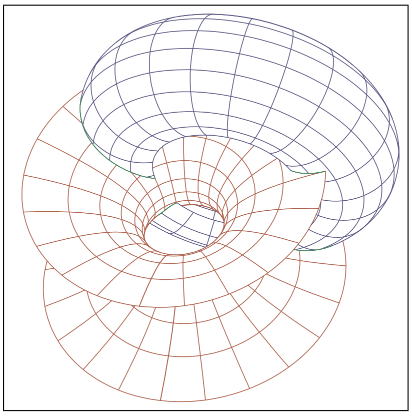


Figure 7. *A conoid* $\vec{y}(u^1) = h(u^1)\vec{e}^{\,3}$

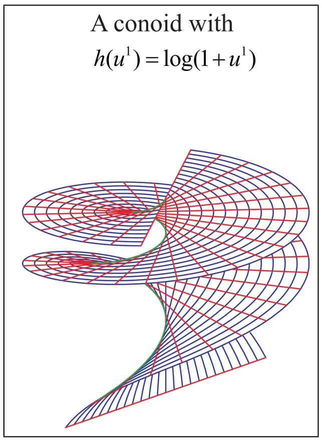
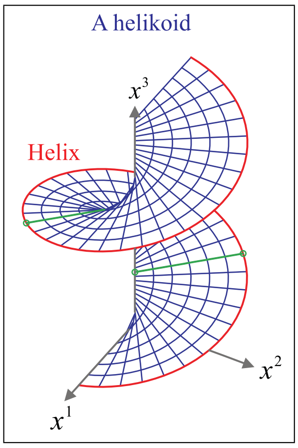


Figure 8. *A helikoid* $\vec{y}(u^1) = c_1 u^1 \vec{e}^{\,3}$



3.1 The visibility of some generalised tubular surfaces

Here we solve the visibilty problem for generalised tubular surfaces (3.2) when $r_1(u^1)$, $r_2(u^1) \neq 0$ for all $u^1 \in I_1$, $h_1(u^2) = \cos u^2$, $h_2(u^2) = \sin u^2$, and $\vec{w}_1(u^1)$ and $\vec{w}_2(u^1)$ are orthogonal unit vectors for all $u^2 \in I_2 \subset (0, 2\pi)$, that is we consider surfaces S with a parametric representation

$$\vec{x}(u^i) = \vec{y}(u^1) + r_1(u^1)\cos u^2 \vec{w}_1(u^1) + r_2(u^1)\sin u^2 \vec{w}_2(u^1) \quad \left((u^1, u^2) \in I_1 \times I_2\right). \tag{3.5}$$

Again we have to find the points of intersection of a straight line with S. We put $\vec{b}(u^1) = \vec{y}(u^1) - \vec{p}$. Now (2.1) becomes

$$t\vec{v} - \vec{b}(u^1) = r_1(u^1)\cos u^2 \vec{w}_1(u^1) + r_2(u^1)\sin u^2 \vec{w}_2(u^1). \tag{3.6}$$

This implies

$$\vec{w}_1(u^1) \bullet (t\vec{v} - \vec{b}(u^1)) = r_1(u^1)\cos u^2 \text{ and } \vec{w}_2(u^1) \bullet (t\vec{v} - \vec{b}(u^1)) = r_2(u^1)\sin u^2. \tag{3.7}$$

We put $\vec{w}(u^1) = \vec{w}_1(u^1) \times \vec{w}_2(u^1)$ and obtain from (3.6)

$$\vec{b}(u^1) \bullet \vec{w}(u^1) = t\vec{v} \bullet \vec{w}(u^1). \tag{3.8}$$

First we consider the case $\vec{v} \bullet \vec{w}(u^1) \neq 0$

Then we can solve (3.8) for t

$$t = t(u^1) = \frac{\vec{b}(u^1) \bullet \vec{w}(u^1)}{\vec{v} \bullet \vec{w}(u^1)}, \tag{3.9}$$

and it follows from (3.7) that

$$\begin{aligned} r_2^2(u^1)\left(\vec{w}_1(u^1) \bullet (t(u_1)\vec{v} - \vec{b}(u^1))\right)^2 + r_1^2(u^1)\left(\vec{w}_2(u^1) \bullet (t(u_1)\vec{v} - \vec{b}(u^1))\right)^2 \\ = r_1^2(u^1) r_2^2(u^1) \text{ with } t(u^1) \text{ from (3.9).} \end{aligned} \tag{3.10}$$

Putting $\vec{a}(u^1) = t(u^1)\vec{v} - \vec{b}(u^1)$ and $R(u^1) = r_1(u^1)r_2(u^1)$, we have to find the zeros $u_0^1 \in I_1$ of the function f with

$$f(u^1) = \left(r_2(u^1)\vec{w}_1(u^1) \bullet \vec{a}(u^1)\right)^2 + \left(r_1(u^1)\vec{w}_2(u^1) \bullet \vec{a}(u^1)\right)^2 - R^2(u^1). \tag{3.11}$$

Now we obtain from (3.9) the values $t_0 = t(u_0^1)$ that correspond to the zeros $u_0^1 \in I_1$ of f.

Now we consider the case $\vec{v} \bullet \vec{w}(u^1) = 0$

Then we have $\vec{b}(u^1) \bullet \vec{w}(u^1) = 0$ by (3.8), and we have to find the zeros $u_0^1 \in I_1$ of the function f with

$$f(u^1) = \vec{b}(u^1) \bullet \vec{w}(u^1). \tag{3.12}$$

We write

$$a_0 = r_2^2(u_0^1)\left(\vec{w}_1(u_0^1) \bullet \vec{v}\right)^2 + r_1^2(u_0^1)\left(\vec{w}_2(u_0^1) \bullet \vec{v}\right)^2,$$

$$\begin{aligned} b_0 = -\Big(r_2^2(u_0^1)\left(\vec{w}_1(u_0^1) \bullet \vec{v}\right)\left(\vec{w}_1(u_0^1) \bullet \vec{b}(u_0^1)\right) \\ + r_1^2(u_0^1)\left(\vec{w}_2(u_0^1) \bullet \vec{v}\right)\left(\vec{w}_2(u_0^1) \bullet \vec{b}(u_0^1)\right)\Big) \end{aligned}$$

and

$$c_0 = r_2^2(u_0^1)\left(\vec{w}_1(u_0^1) \bullet \vec{b}(u_0^1)\right)^2 + r_1^2(u_0^1)\left(\vec{w}_2(u_0^1) \bullet \vec{b}(u_0^1)\right)^2 - R^2(u_0^1),$$

and have to find the solutions $t(u_0)$ for every u_0 of (3.10) with u^1 and $t(u^1)$ replaced by u_0^1 and t, that is we have to solve the quadratic equation

$$a_0 t^2 + 2b_0 t + c_0 = 0. \tag{3.13}$$

Finally, in both cases, we have to determine the values $u_0^2 \in I_2$ corresponding to the zeros u_1^0 and t_0 from (3.7).

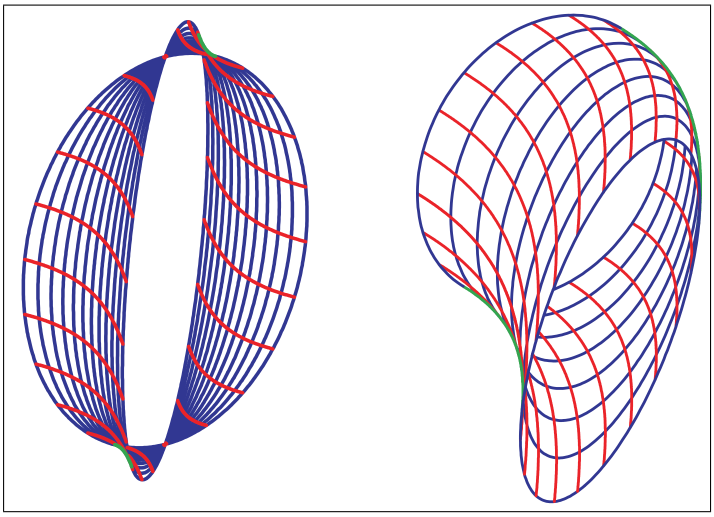
Remark 3.1. *The formulae reduce considerably in the special cases.*
(i) For tubular surfaces, we have $r = r_1 = r_2$, $\vec{w}_1 = \vec{v}_2$, $\vec{w}_2 = \vec{v}_3$. We obtain $\vec{w} = \vec{v}_1$, and from (3.6) $(t\vec{v} - \vec{b}(u^1))^2 = r^2(u^1)$ and $\vec{b}(u^1) \bullet \vec{v}_1 = t\vec{v} \bullet \vec{v}_1(u^1)$. If $\vec{v} \bullet \vec{v}_1(u^1) \neq 0$, then substituting $t = t(u^1) = (\vec{b}(u^1) \bullet \vec{v}_1)/(\vec{v} \bullet \vec{v}_1(u^1))$ in (3.10), we obtain

$$\left((\vec{b}(u^1) \bullet \vec{v}_1(u^1))\vec{v} - (\vec{v} \bullet \vec{v}_1(u^1))\vec{b}(u^1)\right)^2 = \left\|\vec{v}_1(u^1) \times (\vec{v} \times \vec{b}(u^1))\right\|^2 = r^2(u^1)\,(\vec{v}_1 \bullet \vec{v})^2$$

The function f in (3.11) reduces to $f(u^1) = \|\vec{v}_1(u^1) \times (\vec{v} \times \vec{b}(u^1))\|^2 - r^2(u^1)\,(\vec{v}_1 \bullet \vec{v})^2$. If $\vec{v} \bullet \vec{v}_1(u^1) = 0$, then the coefficients of the quadratic equation (3.13) reduce to $a_0 = \vec{v}^{\,2} = 1$, $b_0 = \vec{v} \bullet \vec{b}(u_0^1)$ and $c_0 = (\vec{b}(u_0^1))^2$.
(ii) For surfaces of rotation, we have $\vec{b}(u^1) = (\vec{q} - \vec{p}) + h(u^1)\vec{d}$ and we can choose $\vec{w} = \vec{d}$ in Part (i).

Remark 3.2. *The solution of the visibility problem given above can also be applied when the functions* cos *and* sin *are replaced by* cosh *and* sinh *(Figure 9).*

Figure 9. *Surfaces with $\vec{x}(u^i) = \{r(u^1)\cosh u^2, r(u^1)\sinh u^2, h(u^1)\}$*
Left $r(u^1) = \cosh u^1/4$, $h(u^1) = \cos u^1$ Right $r(u^1) = 4 + \cos u^1$, $h(u^1) = \sin u^1$



Remark 3.3. *The boundaries of neighbourhoods in $\mathbb{R}^3$ with respect to certain paranorms can be considered as generalised tubular surfaces (3.2) ([8, 9]). Let (p) =*

(p_1, p_2, p_3) with $p_k > 0$ $(k = 1, 2, 3)$ be given, $H = \max\{p_1, p_2, p_3\}$ and $M = \max\{1, H\}$. Then $g_{(p)}$ with $g_{(p)}(\vec{x}) = (\sum_{k=1}^{3} |x_k|^{p_k})^{1/M}$ for all $\vec{x} = \{x_1, x_2, x_3\}$ defines a paranorm. If X_0 is a given point in $I\!R^3$ with position vector $\vec{x}_0$, then the boundary of its neighbourhood $\partial B_{(p)}(X_0, r) = \{\vec{x} : g_{(p)}(\vec{x} - \vec{x}_0) = r\}$ $(r > 0)$ can be represented by a generalised tubular surface (3.2) with

$$\begin{aligned} \vec{y}(u^1) &= \vec{x}_0 + r^{M/p_3} \operatorname{sgn}(\sin u^1) |\sin u^1|^{2/p_3} \vec{e}^{\,3}, \\ \vec{w}_1 &= \vec{e}^{\,1}, \; \vec{w}_2 = \vec{e}^{\,2}, \\ r_1(u^1) &= r^{M/p_1} (\cos u^1)^{2/p_1}, \; h_1(u^2) = \operatorname{sgn}(\cos u^2) |\cos u^2|^{2/p_1}, \\ r_2(u^1) &= r^{M/p_2} (\cos u^1)^{2/p_2}, \; h_2(u^2) = \operatorname{sgn}(\cos u^2) |\cos u^2|^{2/p_2} \end{aligned}$$

for $(u^1, u^2) \in (-\pi/2, \pi/2) \times (0, 2\pi)$ (Figures 10 and 11).

3.2 The contour line

Here we consider the contour lines of generalised tubular surfaces (3.2) in some special cases when the function Ψ in (2.3) reduces considerably.

First we consider the surface generated by the osculating circles of a curve γ with non–vanishing curvature. Let γ be given by a parametric representation $\vec{y}(s)$ $(s \in I)$, where s is the arc length along γ, and $\rho(s) = 1/\kappa(s)$ be the radius of curvature of γ at s. The centre of curvature $\vec{y}_m(s)$ and the osculating circle of γ at s are given by

$$\vec{y}_m(s) = \vec{y}(s) + \rho(s)\vec{v}_2(s) \tag{3.14}$$

and

$$\vec{x}(s;t) = \vec{y}_m(s) + \rho(s)(\cos t\, \vec{v}_2(s) + \sin t\, \vec{v}_1(s)) \text{ for } t \in (0, 2\pi). \tag{3.15}$$

Figure 10. $\partial B_{(p)}(0,1)$ *for* $(p) = (1/2, 2, 1/4)$

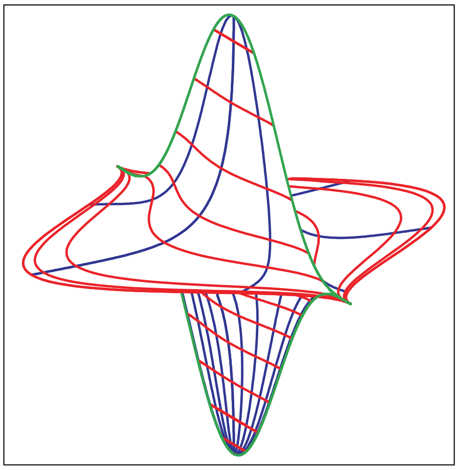
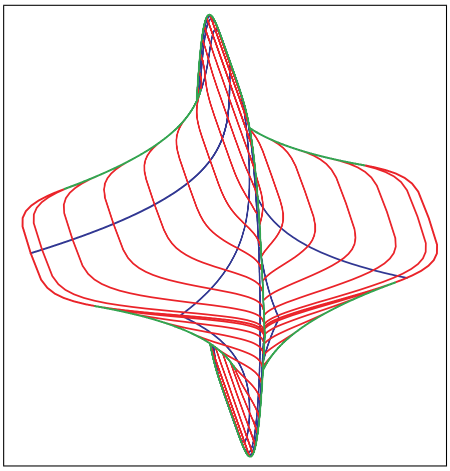


Figure 11. $\partial B_{(p)}(0,1)$ *for* $(p) = (1/4, 5, 1)$



Thus we consider the surface (3.5) with $\vec{y} = \vec{y}_m$, $r_1 = r_2 = \rho$, $\vec{w}_1 = \vec{v}_2$ and $\vec{w}_2 = \vec{v}_1$. Denoting the differentiation with respect to u^1 by $\cdot$, omitting the parameter u^1 and using Frenet's formulae, we obtain

$$\begin{aligned}\vec{x}_1 &= \vec{v}_1 + \dot\rho\cos u^2\,\vec{v}_2 + \dot\rho\sin u^2\,\vec{v}_1 + \rho(1+\cos u^2)\dot{\vec{v}}_2 + \rho\sin u^2\,\dot{\vec{v}}_1\\ &= \vec{v}_1 + \dot\rho\cos u^2\,\vec{v}_2 + \dot\rho\sin u^2\,\vec{v}_1 - (1+\cos u^2)\vec{v}_1 + \rho\tau(1+\cos u^2)\vec{v}_3 + \sin u^2\,\vec{v}_2\\ &= (\dot\rho\sin u^2 - \cos u^2)\vec{v}_1 + (\dot\rho\cos u^2 + \sin u^2)\vec{v}_2 + \rho\tau(1+\cos u^2)\vec{v}_3,\end{aligned}$$

$$\vec{x}_2 = -\rho\sin u^2\,\vec{v}_2 + \rho\cos u^2\,\vec{v}_1$$

and

$$\begin{aligned}\vec{n} = (1/\rho)(\vec{x}_1\times\vec{x}_2) &= -\sin u^2(\dot\rho\sin u^2 - \cos u^2)\vec{v}_3\\ &\quad - \cos u^2(\dot\rho\cos u^2 + \sin u^2)\vec{v}_3 + \rho\tau(1+\cos u^2)(\sin u^2\,\vec{v}_1 + \cos u^2\,\vec{v}_2)\\ &= \rho\tau(1+\cos u^2)(\sin u^2\,\vec{v}_1 + \cos u^2\,\vec{v}_2) - \dot\rho\vec{v}_3\end{aligned}$$

We write $\beta_k(u^1) = (\vec{y}_m(u^1) - \vec{c}) \bullet \vec{v}_k(u^1)$ for $k = 1,2,3$ and obtain for the function Ψ in (2.3)

$$\begin{aligned}\Psi(u^i) &= \vec{n}(u^i)\bullet(\vec{p}-\vec{c})\\ &= \vec{n}(u^i)\bullet(\vec{y}_m(u^1)-\vec{c}) + \rho(u^1)\vec{n}(u^i)\bullet\left(\cos u^2\,\vec{v}_2(u^1) + \sin u^2\,\vec{v}_1(u^1)\right)\\ &= (1+\cos u^2)\rho(u^1)\tau(u^1)\left(\beta_1(u^1)\sin u^2 + \beta_2(u^1)\cos u^2\right) - \dot\rho(u^1)\beta_3(u^1)\\ &\quad + \left(1+\cos u^2\right)\rho^2(u^1)\tau(u^1)\left(\sin^2 u^2 + \cos^2 u^2\right)\\ &= \left(1+\cos u^2\right)\rho(u^1)\tau(u^1)\left(\beta_1(u^1)\sin u^2 + \beta_2(u^1)\cos u^2 + \rho(u^1)\right) - \dot\rho(u^1)\beta_3(u^1).\end{aligned}$$

Now we consider the surface generated by the osculating spheres of a curve γ with non–vanishing curvature. Let γ be given by a parametric representation $\vec{y}(s)$ $(s\in I)$, where s is the arc length along γ. Then the centre and radius of the osculating sphere at s are given by

$$(3.16)\qquad \vec{m}(s) = \vec{y}(s) + \rho(s)\vec{v}_2(s) + \frac{\dot\rho(s)}{\tau(s)}\vec{v}_3(s) \text{ and } r(s) = \sqrt{\rho^2(s) + \left(\frac{\dot\rho(s)}{\tau(s)}\right)^2}.$$

We write $\phi(s) = \dot\rho(s)/\tau(s)$ and $\vec{z}(u^i) = r(u^1)(\cos u^2\,\vec{v}_2(u^1) + \sin u^2\,\vec{v}_3(u^1))$, and consider the surface given by

$$(3.17)\qquad \vec{x}(u^i) = \vec{m}(u^1) + \vec{z}(u^i) \text{ for } (u^1,u^2)\in I\times(0,2\pi).$$

It follows that

$$\dot{\vec{m}} = \vec{v}_1 + \dot\rho\vec{v}_2 + \rho\dot{\vec{v}}_2 + \dot\phi\vec{v}_3 + \phi\dot{\vec{v}}_3 = \vec{v}_1 + \dot\rho\vec{v}_2 - \vec{v}_1 + \rho\tau\vec{v}_3 + \dot\phi\vec{v}_3 - \dot\rho\vec{v}_2 = (\rho\tau+\dot\phi)\vec{v}_3,$$

$$\begin{aligned}\vec{z}_1 &= \dot r\left(\cos u^2\,\vec{v}_2 + \sin u^2\,\vec{v}_3\right) + r\left(\cos u^2\,\dot{\vec{v}}_2 + \sin u^2\,\dot{\vec{v}}_3\right)\\ &= \dot r\left(\cos u^2\,\vec{v}_2 + \sin u^2\,\vec{v}_3\right) + r\cos u^2\left(-\frac{1}{\rho}\vec{v}_1 + \tau\vec{v}_3\right) - r\tau\sin u^2\,\vec{v}_2\end{aligned}$$

$$= -\frac{r}{\rho}\cos u^2\,\vec{v}_1 + \left(\dot{r}\cos u^2 - r\tau\sin u^2\right)\vec{v}_2 + \left(\dot{r}\sin u^2 + r\tau\cos u^2\right)\vec{v}_3,$$

hence

$$\vec{x}_1 = -\frac{r}{\rho}\cos u^2\,\vec{v}_1 + \left(\dot{r}\cos u^2 - r\tau\sin u^2\right)\vec{v}_2 + \left(\dot{r}\sin u^2 + r\tau\cos u^2 + \rho\tau + \dot{\phi}\right)\vec{v}_3,$$

$$\vec{x}_2 = r\left(-\sin u^2\,\vec{v}_2 + \cos u^2\,\vec{v}_3\right)$$

and

$$\vec{n} = \frac{1}{r}(\vec{x}_1\times\vec{x}_2) = \frac{r}{\rho}\sin u^2\cos u^2\,\vec{v}_3 + \frac{r}{\rho}\cos^2 u^2\,\vec{v}_2 + \cos u^2\left(\dot{r}\cos u^2 - r\tau\sin u^2\right)\vec{v}_1$$
$$+\sin u^2\left(\dot{r}\sin u^2 + r\tau\cos u^2 + \rho\tau + \dot{\phi}\right)\vec{v}_1$$
$$= \left(\dot{r} + (\rho\tau + \dot{\phi})\sin u^2\right)\vec{v}_1 + \frac{\cos u^2}{\rho}\,\vec{z}(u^i).$$

We put $\beta_k(u^1) = (\vec{m} - \vec{c}) \bullet \vec{v}_k$ for $k = 1, 2, 3$, and obtain for the function Ψ in (2.3)

$$\Psi(u^i) = \vec{n}(u^i)\bullet(\vec{m}(u^1) - \vec{c}) + \vec{n}(u^i)\bullet\vec{z}(u^i)$$
$$= \left(\dot{r}(u^1) + (\rho(u^1)\tau(u^1) + \dot{\phi}(u^1))\sin u^2\right)\beta_1(u^1)$$
$$+\frac{r(u^1)}{\rho(u^1)}\cos u^2\left(\cos u^2\beta_2(u^1) + \sin u^2\beta_3(u^1)\right) + \frac{r^2(u^1)}{\rho(u^1)}\cos u^2$$
$$= \left(\dot{r}(u^1) + (\rho(u^1)\tau(u^1) + \dot{\phi}(u^1))\sin u^2\right)\beta_1(u^1)$$
$$+\frac{r(u^1)}{\rho(u^1)}\cos u^2\left(\cos u^2\beta_2(u^1) + \sin u^1\beta_3(u^1) + r(u^1)\right).$$

Remark 3.4. *The equation $\Psi(u^1, u^2) = 0$ can be solved for u^2 for surfaces of rotation and ruled surfaces, that is $u^2 = \psi(u^1)$. The functions ψ can be found in [10].*

3.3 Some applications in Differential Geometry

Here we apply the results of the previous two subsections to represent asymptotic and geodesic lines on a pseudo–sphere, the surface generated by the osculating circles and the tubular surface generated by the osculating spheres of an asymptotic line on a pseudo–sphere. We also represent the normal curvature along a curve γ on a torus on the ruled surface generated by γ and the surface normal vectors of the torus along γ.

We consider the pseudo–sphere PS with a parametric representation

$$\vec{x}(u^i) = \left\{\mathrm{e}^{-u^1}\cos u^2, \mathrm{e}^{-u^1}\sin u^2, \int\sqrt{1 - \mathrm{e}^{-2u^1}}\,du^1\right\} \tag{3.18}$$
$$\left((u^1, u^2) \in (0, \infty) \times (0, 2\pi)\right).$$

Proposition 3.5. *(a) The asymptotic lines on the pseudo–sphere PS are given by*

$$u^1(t) = t \text{ and } u^2_{\pm}(t) = \pm\log\left(e^t + \sqrt{e^{2t} - 1}\right) + c_{\pm} \text{ for all } t > 0, \tag{3.19}$$

where c_+ and c_- are constants (Figure 12). If we choose the upper sign and $c_+ = 0$ in (3.19) then the asymptotic line has a parametric representation with respect to its arc length

$$\vec{x}(s) = \left\{ \frac{\cos s}{\cosh s}, \frac{\sin s}{\cosh s}, s - \tanh s \right\} \text{ for all } s > 0. \tag{3.20}$$

(b) The vectors $\vec{v}_k$ $(k = 1, 2, 3)$ of the trihedra, the curvature κ and the torsion τ of the asymptotic line given by (3.20) are (Figure 13)

$$\vec{v}_1(s) = -\frac{\sinh s}{\cosh^2 s}\{\cos s, \sin s, -\sinh s\} + \frac{1}{\cosh s}\{-\sin s, \cos s, 0\}, \tag{3.21}$$

$$\vec{v}_2(s) = -\frac{1}{\cosh^2 s}\{\cos s, \sin s, -\sinh s\} - \tanh s\{-\sin s, \cos s, 0\}, \tag{3.22}$$

$$\vec{v}_3(s) = \frac{1}{\cosh s}\{\sinh s \cos s, \sinh s \sin s, 1\}, \tag{3.23}$$

$$\kappa(s) = \frac{2}{\cosh s} \quad \text{and} \quad \tau(s) = 1. \tag{3.24}$$

Figure 12. *Asymptotic lines on the pseudo–sphere; red for the upper sign, green for the lower sign*

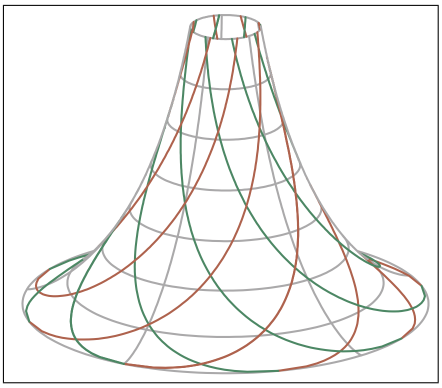
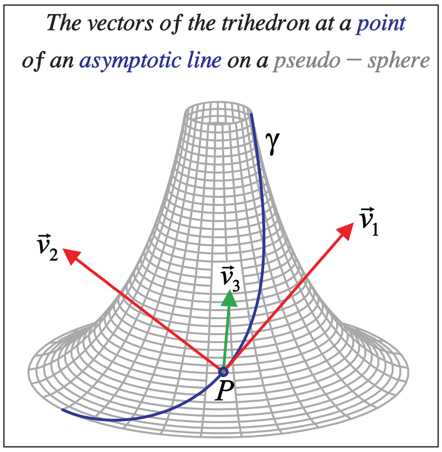


Figure 13. *The vectors of the trihedra along a curve on the pseudo–sphere*



Proof. (a) We recall that the first and second fundamental coefficients of a surface of rotation with a parametric representation

$$\vec{x}(u^i) = \{r(u^1)\cos u^2, r(u^1)\sin u^2, h(u^1)\}, \tag{3.25}$$

where $r(u^1) > 0$ and $|r'(u^1)| + |h'(u^1| > 0$ are given by

$$g_{11} = (r')^2 + (h')^2,\ g_{12} = 0,\ g_{22} = r^2, \tag{3.26}$$

$$L_{11} = \frac{h''r' - h'r''}{\sqrt{(r')^2 + (h')^2}},\ L_{12} = 0 \text{ and } L_{22} = \frac{h'r}{\sqrt{(r')^2 + (h')^2}}. \tag{3.27}$$

Since we have $(r')^2 + (h')^2 = 1$ for PS, the second fundamental coefficients reduce to

$$L_{11} = -\frac{r''}{h'} = \frac{r}{h'} \text{ and } L_{22} = rh'.$$

Since PS is a surface with constant Gaussian curvature $K \equiv -1$, the asymptotic lines exist everywhere; they are given by the solutions of the differential equation $L_{11}(du^1)^2 + L_{22}(du^2)^2 = 0$, that is

$$\text{(3.28)} \quad \frac{du^2}{du^1} = \pm\sqrt{-\frac{L_{11}(u^1)}{L_{22}(u^1)}} = \pm\sqrt{\frac{r''(u^1)}{r(u^1)(h'(u^1))^2}} = \pm\frac{1}{|h'(u^1)|} = \pm\frac{1}{\sqrt{1-\mathrm{e}^{-2u^1}}},$$

that is

$$u^2(u^1) = \pm\int\frac{du^1}{\sqrt{1-\mathrm{e}^{-2u^1}}} = \pm\int\frac{\mathrm{e}^{u^1}}{\sqrt{\mathrm{e}^{2u^1}-1}}\,du^1.$$

To solve the integral $I(u^1)$ in this identity, we substitute $t = \mathrm{e}^{u^1} > 1$, and obtain $du^1/dt = 1/t$ and

$$I(u^1) = \int\frac{dt}{\sqrt{t^2-1}} = \log\left(t+\sqrt{t^2-1}\right) = \log\left(\mathrm{e}^{u^1}+\sqrt{\mathrm{e}^{2u^1}-1}\right).$$

This yields the parametric representation (3.19) for the asymptotic lines on PS.
We obtain for the asymptotic line given by (3.19) with the upper sign and $c_+ = 0$

$$\|\vec{x}'(t)\|^2 = g_{11}(u^1(t))\left(\frac{du^1(t)}{dt}\right)^2 + g_{22}(u^1(t))\left(\frac{du^2(t)}{dt}\right)^2 = \frac{1}{\sqrt{1-\mathrm{e}^{-2t}}} = \frac{du^2(t)}{dt},$$

hence $s(t) = u^2(t)$, and then $\mathrm{e}^{-t} = 1/\cosh s$. Finally we have

$$\begin{aligned} h(s) = h(t(s)) &= \int\sqrt{1-\mathrm{e}^{-2t(s)}}\,\frac{dt(s)}{ds}\,ds = \int\sqrt{1-\frac{1}{\cosh^2 s}}\sqrt{1-\frac{1}{\cosh^2 s}}\,ds \\ &= \int\left(1-\frac{1}{\cosh^2 s}\right)ds = s - \tanh s. \end{aligned}$$

This yields the parametric representation in (3.20).
(b) We put $\phi(s) = 1/\cosh s$ and omit the argument s in ϕ and its derivatives $\dot{\phi}$ and $\ddot{\phi}$, and obtain $\dot{\phi} = -\phi^2\sinh s$, $(\dot{\phi})^2 = \phi^4(\cosh^2 s - 1) = \phi^2 - \phi^4$, $\ddot{\phi} = -\phi^2\cosh s - 2\dot{\phi}\phi\sinh s = -\phi + 2\phi^3\sinh^2 s = -\phi + 2\phi - 2\phi^3 = \phi - 2\phi^3$, $\ddot{\phi} - \phi = -2\phi^3$ and

$$\frac{d(\tanh(s))}{ds} = \frac{1}{\cosh^2 s} = \phi^2.$$

Now it follows that $\vec{v}_1 = \dot{\vec{x}}(s) = \{\dot{\phi}\cos s, \dot{\phi}\sin s, 1-\phi^2\} + \phi\{-\sin s, \cos s, 0\}$, which yields (3.21). Furthermore, we have

$$\text{(3.29)} \qquad \ddot{\vec{x}}(s) = \{(\ddot{\phi}-\phi)\cos s, (\ddot{\phi}-\phi)\sin s, -2\dot{\phi}\phi\} + 2\dot{\phi}\{-\sin s, \cos s, 0\}$$

and $\kappa^2(s) = \|\ddot{\vec{x}}(s)\|^2 = (\ddot{\phi}-\phi)^2 + 4(\dot{\phi}^2(\phi^2+1)) = 4(\phi^6 + \phi^2(1-\phi^2)(1+\phi^2)) = 4\phi^2$, and this and (3.29) yield (3.21) and $\kappa(s)$ in (3.23). Since $\kappa(s) \neq 0$, it follows that $\vec{v}_3(s) = \pm\vec{N}(u^i(s))$ for all s, where the surface normal vector $\vec{N}$ is given by

$$\text{(3.30)} \qquad \vec{N} = \frac{1}{\sqrt{(r')^2+(h')^2}}\{-h'\cos u^2, -h'\sin u^2, r'\}$$

with $(r')^2 + (h')^2 \equiv 1$, $h'(u^1(s)) = \tanh s$ and $r'(u^1(s)) = 1/\cosh s$, hence

$$\vec{N}(u^1(s)) = \frac{1}{\cosh s}\{-\sinh s \cos s, -\sinh s \sin s, 1\}.$$

Comparison of the third components of $\vec{N}(u^i(s))$ and $\vec{v}_1(s) \times \vec{v}_2(s)$ yields $\vec{v}_3(s) = -\vec{N}(u^i(s))$, hence (3.23). Finally, since the pseudo–sphere has constant Gaussian curvature $K(u^i) = -1$, it follows from the Beltrami–Enneper theorem that $|\tau(s)| = \sqrt{-K(u^i(s))} = 1$. □

Figure 14. *Osculating plane and circle at a point of the asymptotic line γ on the pseudo–sphere, and the curve γ_m of the centres of curvature of the asymptotic line*

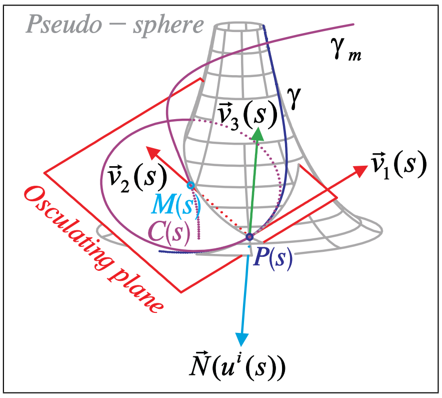
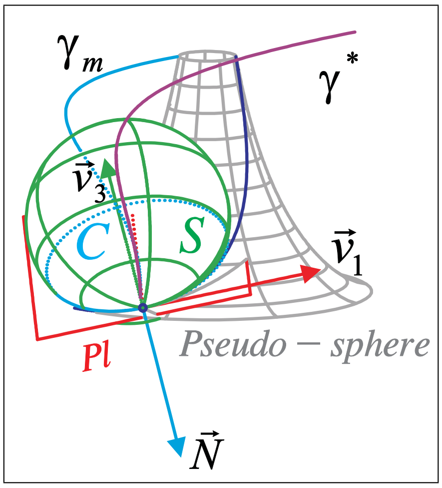


Figure 15. *Osculating plane and sphere at a point of the asymptotic line γ on the pseudo–sphere, and the curves γ_m and γ^* of the centres of curvature and the centres of the osculating spheres*



Example 3.6. *(a) We represent the generalised tubular surface generated by the osculating circles along the asymptotic line (3.20) on the pseudo–sphere (3.18) that is, by (3.15), the surface with a parametric representation*

$$\vec{x}(u^i) = \vec{y}_m(u^1) + \rho(u^1)(\cos u^2\, \vec{v}_2(u^1) + \sin t\, \vec{v}_1(u^1)) \text{ for } (u^1, u^2) \in I \times (0, 2\pi),$$

where $\vec{y}_m$, $\rho = 1/\kappa$, $\vec{v}_1$ and $\vec{v}_2$ are given by (3.14), (3.24) (3.21) and (3.22) with s replaced by u^1 (Figure 17).
(b) We represent the envelope of the osculating spheres along the asymptotic line (3.20) on the pseudo–sphere (3.18), that is the surface with a parametric representation (3.17) and $\vec{m}$, ρ, τ, r, $\vec{v}_2$ and $\vec{v}_3$ given by (3.16), (3.24), (3.22) and (3.23) (Figure 18).

Figure 16. *Osculating sphere at a point of the asymptotic line on the pseudo–sphere*

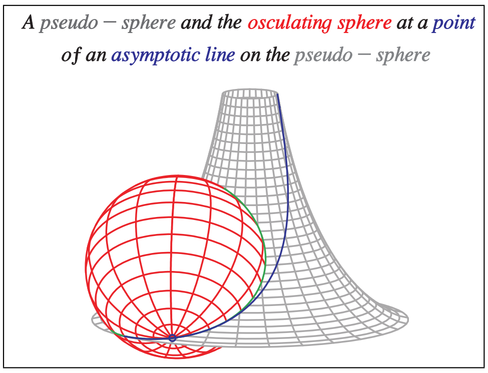


Figure 17. *The generalised tubular surface generated by the osculating circles along the asymptotic line on the pseudo–sphere*

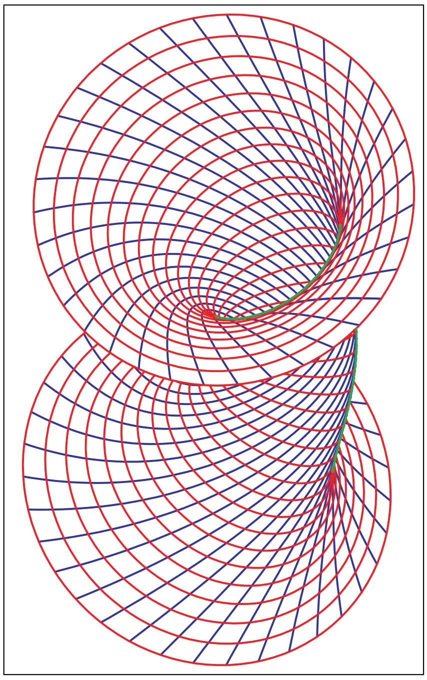
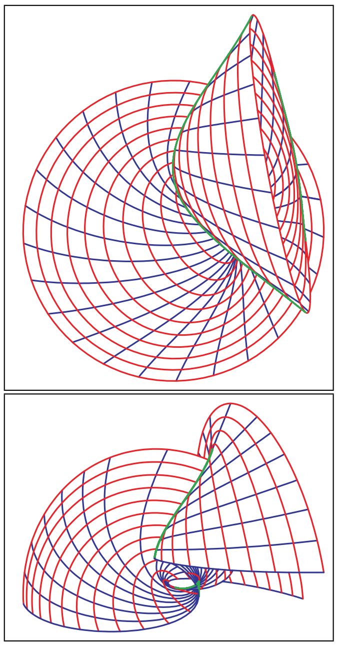


Figure 18. *Envelope of the osculating spheres along the asymptotic line on the pseudo–sphere*
Top: $(u^1, u^2) \in (-8, 8) \times (0, 2\pi)$
Bottom: $(u^1, u^2) \in (-8, 8) \times (0, \pi)$



Finally we visualise the normal curvature along a curve on a torus.

If S is a surface with a parametric representation $\vec{x}(u^i)$ $((u^1, u^2) \in D \subset \mathbb{R}^2)$ and γ is a curve on S with a parametric representation $\vec{x}(t) = \vec{x}(u^i(t))$ $(t \in I)$ and normal curvature $\kappa_n(t) = \kappa_n(u^i(t))$, then we may represent $\kappa_n(t)$ by the curve γ_n, given by a parametric representation $\vec{x}^{\,*}(t) = \vec{x}(t) + \kappa_n(t)\vec{N}(t)$ $(t \in I)$. Writing $u^{*1} = t$, we see that γ_n is a curve on the ruled surface RS that has a parametric representation

$$(3.31) \quad \vec{x}^{\,*}(u^{*i}) = \vec{y}(u^{*1}) + u^{*2}\vec{z}(u^{*1}) \; ((u^{*1}, u^{*2}) \in I \times \mathbb{R}) \\ \text{where } \vec{y}(u^{*1}) = \vec{x}(u^i(u^{*1})) \text{ and } \vec{z}(u^{*1}) = \vec{N}(u^i(u^{*1}))$$

and γ_n is given by putting $u^{*2} = \kappa_n(u^i(u^{*1}))$ in (3.31).

Example 3.7. *We consider the torus as a surface of rotation (3.25) with*

$$r(u^1) = r_1 + r_0 \cos u^1 \text{ and } h(u^1) = r_0 \sin u^1 \;\; ((u^1, u^2) \in I_1 \times I_2 \subset (0, 2\pi) \times (0, 2\pi)) \\ \text{where } r_1 \text{ and } r_0 \text{ are positive constants with } r_1 > r_0.$$

Since

$$r'(u^1) = -r_0 \sin u^1,\; r''(u^1) = -r_0 \cos u^1,\; h'(u^1) = r_0 \cos u^1 \text{ and } h''(u^1) = -r_0 \sin u^1,$$

it follows from (3.30), (3.26) and (3.27) that

$$\vec{N}(u^i) = \{-\cos u^1 \cos u^2, -\cos u^1 \sin u^2, -\sin u^1\}, \\ g_{11}(u^1) = r_0^2,\; g_{12}(u^1) = 0,\; g_{22}(u^1) = (r_1 + r_0 \cos u^1)^2,$$

$$L_{11}(u^1) = \frac{1}{r_0}(r_0^2 \sin^2 u^1 + r_0^2 \cos^2 u^1) = r_0,\; L_{12}(u^1) = 0$$

and

$$L_{22}(u^1) = \frac{(r_1 + r_0 \cos u^1) r_0 \cos u^1}{r_0} = (r_1 + r_0 \cos u^1) \cos u^1.$$

If we consider the curve γ on the torus, given by $u^1(t) = t$ and $u^2(t) = t^2$, then we obtain for the ruled surface in (3.31), writing $t = u^{*1}$,

$$\vec{y}(t) = \{(r_1 + r_0 \cos t) \cos t^2, (r_1 + r_0) \cos t \sin t^2, r_0 \sin t\}, \\ \vec{z}(t) = -\{\cos t \cos t^2, \cos t \sin t^2, \sin t\},$$

and for the normal curvature along the curve

$$\kappa_n(u^i(t)) = \frac{r_0 + 4t^2(r_1 + r_0 \cos t) \cos t}{r_0^2 + 4t^2(r_1 + r_0 \cos t)^2} \text{ (Figures 19 and 20).}$$

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Figure 19. *Representation of the normal curvature along the curve of Example 3.7*

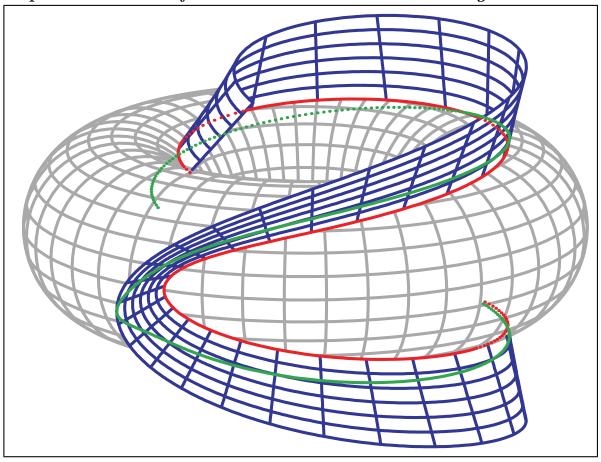
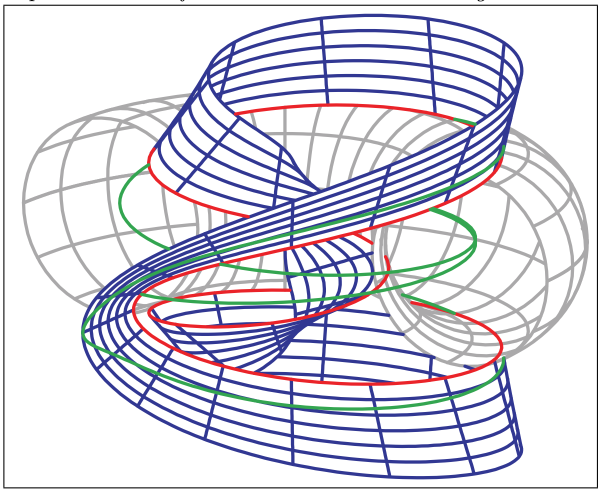


Figure 20. *Representation of the normal curvature along the curve of Example 3.7*



References

[1] M. Djorić, P. Janičić, *Constructions, instructions, teaching mathematics and its applications*, Qxford University Press, Vol. 23(2) (2004), 69–88. http://www.matfbg.ac.yu/∼janicic

[2] M. Failing, *Entwicklung numerischer Algorithmen zur computergrafischen Darstellung spezieller Probleme der Differentialgeometrie und Kristallograpie*, Ph.D. thesis, Giessen, Shaker Verlag Aachen (1996).

[3] M. Failing, E. Malkowsky, *Ein effizientes Nullstellenverfahren zur computergraphischen Darstellung spezieller Kurven und Flächen*, Mitt. Math. Sem. Giessen 229 (1996), 11–28.

[4] P. Janičić, I. Trajković, *WinGCLC–a workbench for formally describing figures*, Proceedings of the Spring Conference on Computer Graphics (SCCG 2003), April

24–26, 2003, ACM Press, New York, USA.
http://www.matfbg.ac.yu/∼janicic

[5] E. Malkowsky, *An open software in OOP for computer graphics and some applications in differential geometry*, Proceedings of the 20th South African Symposium on Numerical Mathematics (1994), 51–80.

[6] E. Malkowsky, *A software for the visualisation of differential geometry*, Visual Mathematics 4(1) (2002), Electronic publication.
http://www.mi.sanu.ac.yu/vismath/malkovsky/index.htm

[7] E. Malkowsky, *Visualisation and animation in mathematics and physics*, Proceedings of the Institute of Mathematics of NAS of Ukraine (50)(3) (2004), 1415–1422.
`http://www.imath.kiev.ua--snmp2003/Proceedings /Proceedings2003.html`

[8] E. Malkowsky, *FK spaces, their duals and the visualisation of meighbourhoods*, Applied Sciences (APPS), 8 No. 1 (2006), 112–127.

[9] E. Malkowsky, *The graphical representation of neighbourhoods in certain topologies*, Proceedings of the Cpnference Contemporary Geometry and Related Topics, Belgrade (2006), 305–324.

[10] E.Malkowsky, W. Nickel, *Computergrafik und Differentialgeometrie*, Vieweg–Verlag, Braunschweig, 1993.

[11] E. Malkowsky, V. Veličković, *Analytic transformations between surfaces with animations*, Proceedings of the Institute of Mathematics of NAS of Ukraine (50)(3) (2004), 1496–1501.
`http://www.imath.kiev.ua--snmp2003/Proceedings /Proceedings2003.html`

[12] V. Veličković, *The basic principles and concepts of a software package for visualisation of mathemarics*, BSG Proceedings 13, Geometry Balkan Press (2006), 192-203.

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