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MULTI-TYPE STEP-WISE GROUP SCREENING DESIGNS WITH UNEQUAL A-PRIORI PROBABILITIES

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Abstract. In this paper, we discuss multi-type step-wise group screening in which group-factors contain differing number of factors. We describe a procedure for grouping the factors in the absence of concrete prior information, so that the relative testing cost is minimal. We shall derive an expression for the expected number of runs in an r-type step-wise group screening design with unequal group sizes and obtain values of the group sizes that minimize the expected number of runs.

Key Words and Phrases. Group Screening, Group factors, multi-type step-wise group screening, expected number of runs, Optimum group screening designs.

1. **Introduction.** There are investigations where a large number of factors need to be examined. In such a situation, we have to run an experiment to identify the influential factors. Once these have been isolated, future experimentation can then study them in greater detail. By reducing the size of the experiment at the screening, one can conserve resources and more efficiently study the important factors.

The method of group testing was first introduced by Dorfman (1943), who proposed that instead of testing each blood sample individually for the presence of a rare disease, blood samples be pooled and analysed together.

Watson (1961) introduced the two-stage group screening procedure. This method was generalized to more than two stages by Li (1962) and Patel (1962). In particular, Patel discussed multistage group screening designs in which all the factors had the same prior probability of being defective.

The notion of step-wise group screening designs was introduced by Patel and Manene (1987). Manene et al. (2002). extended step-wise group screening to multi-type step-wise group screening designs. They considered the case when all factors, have the same prior probability of being defective.

The device of using differing group-sizes when prior probabilities differ has been discussed by Watson (1961). In their article, Otieno and Patel extended the idea of two stage group screening with unequal prior probabilities to include situations where no prior information is available, so that no natural partitioning can be assumed. Odhiambo and Patel (1986) extended the work done by however that if a negative test is performed on the $(w-1)^{th}$ individual whilst searching for the first defective in a defective group, we can infer that the w^{th} individual is the defective one. On the other hand, if a positive test is performed on the $(w-1)^{th}$ individual, the remainder consists only of the w^{th} individual and only one further test is required.

4. **Screening With One Type of Search Steps.** Suppose there are f factors to be tested for their effect. The f factors are divided into a fixed number of first order group-factors so that the ith group-factor contain k_{1i} factors

$i = 1,2,\ldots, g \left(\sum_{i=1}^{g} k_{1i} = f \right)$. The first order group-factors are then tested for

their effects. The non-defective group-factors are set aside keeping the defective ones separate. Let $\underline{n}_1$ be the number of first order group-factors found to be defective in the initial step. In type one search steps, factors within each defective first order group-factor are classified as defective or non-defective using the step-wise group screening procedure.

Let p_i be the a-priori probability that a factor in the ith first order group-factor is defective and let p_{1i}^{*} be the probability that the ith first order group-factor is defective. Then

$$p_{1i}^{*} = \sum_{s_{1i}=1}^{k_{1i}} \binom{k_{1i}}{s_{1i}} p_i^{s_{1i}} (1-p_i)^{k_{1i} - s_{1i}}$$

$$= 1 - q_i^{k_{1i}} \qquad (4.1)$$

where $q_i = 1 - p_i$.

In the initial step, we shall require

$$R_I = 1 + g \qquad (4.2)$$

runs to test the g first order group-factors orthogonally.

Define a random variable U_{1i} as follows:-

$$\begin{cases} U_{1i} = 1 & \text{if the } i^{th} \text{ first order group-factor is defective} \\ 0 & \text{otherwise} \end{cases} \qquad (4.3)$$

Then

$$E(U_{1i}) = p_{1i}^{*}, \quad \underline{n}_1 = \sum_{i=1}^{g} U_{1i} \qquad (4.4)$$

5. **Screening With Two Types.** When screening with two types of search steps, we first divide the f factors into g first order group-factors of unequal sizes, so that the ith group-factor contains k_{1i} factors $\left(f = \sum_{i=1}^{g} k_{1i}\right)$. In the initial step, the g first order group-factors are tested for their effects in an experiment. Let $\underline{n}_1$ be the number of first order group-factors found to be defective. If the ith first order group-factor $(i = 1, 2, \ldots, g)$ is found to be defective, then it is divided into g_{2i} second order group-factors each containing k_{2i} factors so that $k_{1i} = k_{2i}\, g_{2i}$. In type one search steps, the g_{2i} second order group-factors within a defective ith first order group-factor are classified as defective or non-defective using the step-wise group screening procedure. In this case, each second order group-factor is considered as a unit. Type one search steps are performed for all first order group-factors found to be defective in the initial step.

Suppose $\underline{n}_{2i}$ of the g_{2i} second order group-factors within the ith first order group-factor are found to be defective at the end of type one search steps, then the total number of second order group-factors found to be defective from all the g first order group-factors is equal. $\sum_{i=1}^{g} U_{1i}\underline{n}_{2i}$.

Factors within each of the $\sum_{i=1}^{g} U_{1i}\underline{n}_{2i}$ second order group-factors are then classified as defective or non-defective in type two search steps using the step-wise group screening procedure.

Let p_i be the probability that a factor in the ith first order group-factor is defective and p_{1i}^* be the probability that the ith first order group-factor is defective. Further let p_{2i}^* denote the probability that a second order group-factor belonging to the ith defective first order group-factor is defective then

$$p_{1i}^* = 1 - q_i^{k_{1i}} \tag{5.1}$$

and

$$p_{2i}^* = 1 - q_i^{k_{2i}} \tag{5.2}$$

Denote by $p_{2i/1i}^*$ the probability that a second order group-factor is defective given that it is within the ith first order group-factor which is known to be defective, then

Let $E_{g_{2i}}\left(R_{j_1}\right)$ be the number of runs required to classify as defective or non defective using the step-wise procedure all the second order group-factors within the ith defective first order group-factor if it contains exactly j_1 defective second order group-factors then

$$E_{g_{2i}}\left(R_{j_1}\right) = \frac{j_1 g_{2i}}{j_1 + 1} + j_1 + \frac{j_1}{j_1 + 1} - \frac{2 j_1}{g_{2i}} \qquad (5.9)$$

$$j_1 = 1, 2, \ldots, g_{2i}$$

Further let $E_{k_{2i}}\left(R_{j_2}\right)$ be the number of runs required to classify as defective or non-defective using the step-wise procedure all k_{2i} factors within a second order group-factor which is known to be defective if it contains exactly j_2 defective factors, then

$$E_{k_{2i}}\left(R_{j_2}\right) = \frac{j_2 k_{2i}}{j_2 + 1} + j_2 + \frac{j_2}{j_2 + 1} - \frac{2 j_2}{k_{2i}} \qquad (5.10)$$

$$where\ j_2 = 1, 2, \ldots, k_{2i}$$

Denote the number of runs required to classify as defective or non-defective all the g_{2i} second order group-factors within the ith first order group-factor which is known to be defective by $R_{t_{1i}}$, then

$$E\left(R_{t_{1i}}\right) = \sum_{j_1=1}^{g_{2i}} E_{g_{2i}}\left(R_{j_1}\right) P_{g_{2i}}\left(j_1\right)$$

$$= \left(1-q^{k_{1i}}\right)^{-1}\left[\left(g_{2i}+1\right)+g_{2i}\,p^*_{2i} - 2p^*_{2i} - \frac{1}{p^*_{2i}}\left(1-q^{*\,g_{2i}+1}_{2i}\right)\right] \qquad (5.11)$$

using (5.7) and (5.9) noting $q^*_{2i} = 1 - p^*_{2i}$. The number of runs R_{t_1} required to classify as defective or non-defective all the $\sum_{i=1}^{g} U_{1i}\, g_{2i}$ second order group-factors found to be defective in the initial step is given by

$$R_{t_1} = \sum_{i=1}^{g} U_{1i}\, E\left(R_{t_{1i}}\right) \qquad (5.12)$$

6. **Screening With r-Types of Search Steps.** In the general case we have the initial step and r types of search steps. In the initial step, the f factors are partioned into a fixed number g of first order group-factors, the ith first order group-factor containing k_{1i} factors $(i = 1,2, \ldots, g)$ $\left(f = \sum_{i=1}^{g} k_{1i}\right)$. These first order group-factors are then tested for their effects. Let $\underline{n}_1$ be the number of first order group-factors found to be defective in the initial step. The ith ($i=1,2, \ldots, \underline{n}_1$) first order group factor found to be defective in the initial step is divided into g_{2i} second order group-factors each of size k_{2i} $(k_{1i} = k_{2i}\, g_{2i})$. In type one search steps, the g_{2i} second order group-factors are classified as defective or non-defective using the step-wise screening procedure. The type one search steps are performed for each of the $\underline{n}_1$ defective first order group-factors.

Suppose $\underline{n}_{2i}$ second order group-factors from the ith first order group-factor are found to be defective at the end of type one-search steps. Each of these $\underline{n}_{2i}$ second order group-factors is divided into g_{3i} third order group-factors each containing k_{3i} factors $(k_{2i} = k_{3i}\, g_{3i})$. Thus we have $\underline{n}_{2i}\, g_{3i}$ third order group-factors from the ith defective first order group-factor.

In type two search steps, the g_{3i} third order group-factors within a defective second order group-factor from the ith defective first order group-factor are classified as defective or non-defective using the step-wise screening procedure. This process is repeated for each of the $\underline{n}_{2i}$ defective second order group-factor within the ith defective first order group-factor.

Generally suppose that $\underline{n}_{(r-1)i}$ $(r-1)^{th}$ order group-factors from the $n_{(r-2)i}\, g_{(r-1)i}$ $(r-1)^{th}$ order group-factors originating from the ith defective first order group factor are found to be defective at the end of type (r-2) search step(s) (r = 2,3, ... identifying type zero search step as the initial step). Each of the $\underline{n}_{(r-i)i}$ defective $(r-1)^{th}$ order group-factors is further divided into g_{ri} rth order group-factors of size k_{ri} each $\left(k_{(r-1)i} = k_{ri}\, g_{ri}\right)$. In type r-1 search steps, the g_{ri} rth order group-factors within a defective $(r-1)^{th}$ order group-factor originating from the ith defective first order group-factor are classified as defective or non-defective using the step-wise group-screening procedure. This

$$= \binom{n_{mi}\, g_{(m+1)i}}{n_{(m+1)i}} p^{*\, n_{(m+1)i}}_{(m+1)i/mi} \left(1 - p^{*}_{(m+1)i}\right)^{n_{mi}\, g_{(m+1)i} - n_{(m+1)i}} \tag{6.4}$$

where m = 1,2,...,r -1 i = 1,2,...,g_1 and $n_{(m+1)i} = 0, 1, 2, \ldots, n_{mi}\, g_{(m+1)i}$

Thus

$$E\left(\underline{n}_{(m+1)i} / \underline{n}_{mi}, U_{1i} = 1\right) = \underline{n}_{mi}\, g_{(m+1)i}\, p^{*}_{(m+1)i/mi} \tag{6.5}$$

and

$$E\left(\underline{n}_{(m+1)i}\right) = g_{(m+1)i}\, p^{*}_{(m+1)i/mi}\, E\left[E\left(\underline{n}_{mi} / \underline{n}_{(m-1)i}, U_{1i} = 1\right)\right] \tag{6.6}$$

Using (6.3) and (6.4) recursively in (6.6) we obtain

$$E\left(\underline{n}_{(m+1)i}\right) = p^{*}_{(m+1)i} \prod_{j=2}^{m+1} g_{ji} \tag{6.7}$$

Let $P_{g_{(m+1)i}}(j_m)$ $(m = 1, 2, \ldots, r-1)$ be the probability that an m[th] order group-factor coming from the i[th] defective first order group-factor contains exactly j_m defective (m + 1)[th] order group-factors, then

$$P_{g_{(m+1)i}}(j_m) = \left(1 - q_i^{k_{mi}}\right)^{-1} \binom{g_{(m+1)i}}{j_m} p^{*\, j_m}_{(m+1)i} \left(1 - p^{*}_{(m+1)i}\right)^{g_{(m+1)i} - j_m} \tag{6.8}$$

$$j = 1, 2, \ldots, g_{(m+1)i} \qquad m = 1, 2, \ldots, r-1.$$

Further let $P_{k_{ri}}(j_r)$ be the probability that a defective r[th] order group-factor coming from the i[th] defective first order group-factor contains exactly j_r defective factors, then

$$P_{k_{ri}}(j_r) = \left(1 - q_i^{k_{ri}}\right)^{-1} \binom{k_{ri}}{j_r} p_i^{j_r} \left(1 - p_i\right)^{k_{ri} - j_r} \tag{6.9}$$

where $j_r = 1, 2, \ldots, k_{ri}$

Denote by $E_{g_{(m+1)i}}\left(R_{j_m}\right)$

the expected number of runs required to classify as defective or non-defective all the (m + 1)[th] order group-factors within a defective m[th] (m = 1,2, ..., r-1) order group-factor originating from the i[th] defective first order group-factor, then

$$E_{g_{(m+1)i}}\left(R_{j_m}\right) = \frac{j_m g_{(m+1)i}}{j_m + 1} + j_m + \frac{j_m}{j_m + 1} - \frac{2 j_m}{g_{(m+1)i}} \tag{6.10}$$

$$E\left(R_{t_{ri}}\right) = \sum_{j_r=1}^{k_{ri}} E_{k_{ri}}\left(R_{j_r}\right) P_{k_{ri}}\left(j_r\right) \tag{6.14}$$

$$= \left(1-q_i^{k_{ri}}\right)^{-1}\left[\left(k_{ri}+1\right) + k_{ri}p_i - 2p_i - p_i^{-1}\left(1-q_i^{k_{ri}+1}\right)\right]$$

using (6.9) and (6.11).

If R_{t_r} denotes the number of runs required to classify as defective or non-defective all the $\sum_{i=1}^{g} U_{1i}\, \underline{n}_{ri}\, k_{ri}$ factors within the $\sum_{i=1}^{g} U_{1i}\, \underline{n}_{ri}$ r^{th} order group-factors found to be defective at the end of type r search steps then

$$R_{t_r} = \sum_{i=1}^{g} U_{1i}\, \underline{n}_{ri}\, E\left(R_{t_{ri}}\right) \tag{6.15}$$

Theorem 6.1. Let R be the total number of runs required to isolate all the defective factors in an r-type step-wise group screening experiment, then

$$E(R) = 1 + f + g + \sum_{i=1}^{g} \sum_{m=1}^{r-1(r\geq 2)} \frac{k_{1i}}{k_{(m+1)i}}\left(2 - q_i^{k_{(m+1)i}}\right)$$

$$+ \sum_{i=1}^{g} \sum_{m=1}^{r-1(r\geq 2)} \frac{k_{1i}}{k_{mi}}\left[2q_i^{k_{(m+1)i}} - \left(1-q_i^{k_{(m+1)i}}\right)^{-1}\left(1-q_i^{k_{mi}+k_{(m+1)i}}\right) - 1\right]$$

$$+ \sum_{i=1}^{g} \frac{k_{1i}}{k_{ri}}\left[1 + k_{ri}\, p_i - 2p_i - \frac{1}{p_i}\left(1 - q_i^{k_{ri}+1}\right)\right] \tag{6.16}$$

where k_{mi} and k_{ri} are the sizes of an m[th] order group-factor (m = 1,2, …, r-1) and an r[th] order group-factor respectively.

Proof. The expected total number of runs is given by

$$E(R) = R_I + \sum_{m=1}^{r-1(r\geq 2)} E\left(R_{t_m}\right) + E\left(R_{t_r}\right)$$

$$= 1 + g + E\left[\sum_{i=1}^{g} \sum_{m=1}^{r-1(r\geq 2)} U_{1i}\, \underline{n}_{mi}\, E\left(R_{t_{mi}}\right)\right]$$

$$+ E\left[\sum_{i=1}^{g} U_{1i}\, \underline{n}_{ri}\, E\left(R_{t_{ri}}\right)\right] \tag{6.17}$$

Using (4.2), (6.2), (6.7), (6.8) and (6.14) and simplifying, equation (6.17) yields equation (6.16). This completes the proof.

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Theorem 6.2. The values of $k_{1i}, k_{2i}, \ldots, k_{ri}$ which minimize the expected number of runs in an r-type step-wise group screening design in which the i^{th} first order group-factor contains factors with same probability p_i of being effective, are approximately given by

$$k_{1i} = \frac{f}{\left(\frac{p_i}{2}\right)^r \sum_{i=1}^{g} \left(2/p_i\right)^r}$$

and

$$k_{mi} = \frac{f^{(r+1-m)/r}}{\left(\frac{p_i}{2}\right)^{(r+1-m)} \left\{ \sum_{i=1}^{g} \left(\frac{2}{p_i}\right)^r \right\}^{(r+1-m)/r}} \tag{6.20}$$

where $i = 1, 2, \ldots, g; m = 2, 3, \ldots r$ assuming that p_i's are small.

Proof. For small values of p_i's the expected number of runs is as given in equation (6.18). We wish to determine $k_{1i}, k_{2i}, \ldots, k_{ri}$ which minimize E(R) in equation (6.18) subject to the condition that $f = \sum_{i=1}^{g} k_{1i}$. Using this condition in the formular for E(R) we obtain

$$\begin{aligned}
E(R) \approx\ & 1 + g + \sum_{i=1}^{g} k_{2i}\, p_i + \sum_{i=1}^{g-1} \left(\frac{3}{2}(r-1) k_{1i}\, p_i + \frac{k_{1i}^2\, p_i}{2k_{2i}} \right) \\
& + \frac{3}{2}(r-1)\left(f - \sum_{i=1}^{g-1} k_{1i} \right) p_g + \left(\frac{f - \sum_{i=1}^{g-1} k_{1i}}{2k_{2i}} \right)^2 p_g \\
& + \sum_{i=1}^{g-1} \sum_{m=2}^{r-1} k_{1i}\, p_i \left[\frac{k_{mi}}{2k_{(m+1)i}} - \frac{2\,k_{(m+1)i}}{k_{mi}} \right] \\
& + \sum_{m=2}^{r-1} \left(f - \sum_{i=1}^{g-1} k_{1i} \right) p_g \left[\frac{k_{mi}}{2k_{(m+1)i}} - \frac{2k_{(m+1)i}}{k_{mi}} \right] \\
& + \sum_{i=1}^{g-1} k_{1i}\, p_i \left[\frac{3}{2} - \frac{2}{k_{ri}} + \frac{k_{ri}}{2} \right]
\end{aligned}$$

7. **Examples of Screening Plans.** In this section we give a few examples of possible partitions of $f = 100$ factors into unequal group sizes in an r-type step-wise group screening design, for $r = 1,2,3$ and 4. We base our calculations on corollary 6.1 and theorem 6.2.

We shall use the same number of groups and the same p_i's as used by Odhiambo and Patel (1986) and compare the expected number of runs required for the current procedure with the expected number of runs obtained using their multistage procedure. We shall also compare the expected number of runs obtained under the current procedure with the expected number of runs obtained using multi-type step-wise group screening designs with equal group sizes as given by Manene et al. (2002). c.f. (6.19).

In our calculations we shall assume that $p_1, p_2, \ldots, p_g$ are such that $p = \max(p_1, p_2, p_3, \ldots, p_g)$. In the following tables, we give for illustrative purposes only, the group sizes corresponding to given values of r, f, g and p. We also give the corresponding expected number of runs. We shall use the notation below to mean as indicated;

r – Tstwse r – type step-wise group screening with equal group sizes.
s – SGSu s-stage group screening with unequal group sizes.
S – SGESe s-stage group screening with equal group sizes.

Table 7.1

One – type step-wise group screening design for $f = 100$, $p = 0.035$ and $g = 13$. Here $r = 1$.

i	p_i	k_{1i}
1	0.008	15.5623
2	0.009	13.8331
3	0.010	12.4498
4	0.013	9.5768
5	0.015	8.2999
6	0.017	7.3234
7	0.020	6.2249
8	0.022	5.6590
9	0.025	4.9799
10	0.027	4.6110
11	0.030	4.1499
12	0.033	3.7727
13	0.035	3.5571

E(R)=22.12.

1 – TStwse E(R) = 30.69
2 – SGSu E(R) = 26.45
2 – SGSe E(R) = 36.91

From tables 7.1 to 7.4 we notice that we can generally achieve a substantial saving by resorting to group screening with unequal group sizes. We also notice that a multi-type step-wise group screening design out performs a corresponding multistage design provided the selected probability interval is as proposed by Patel (1962) and also given by Odhiambo and Patel (1986). For all practical purposes, the values of k_{mi}'s given in the tables are to be rounded to integers. The partitions illustrated in tables 7.1– 7.4 are those used by Odhiambo and Patel (1986), and are not unique. Generally speaking, the p_i's should be selected such that the grouping at each type of the multi-type experiment is as uniform as possible. We stress the fact that theorem 6.2 is used only as a guide in partitioning the factors into feasible groups.

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