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**THE VERSATILITY OF THE
NEGATIVE BINOMIAL DISTRIBUTION
IN DESCRIBING DISPERSION**

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ABSTRACT

Probability distributions used to describe dispersion are considered. The negative binomial parameter k is considered as a measure of dispersion. The aim of this paper is to demonstrate analytically the versatility of the negative binomial distribution in describing dispersion. The versatility of the negative binomial distribution is established through reparameterization, truncation, generalizing, mixing and limiting. The bases for this establishment are index of patchiness and probability generating function. The results are summarized in a conceptual distributional framework.

KEYWORDS: Dispersion, overdispersion, randomness, negative binomial, index of patchiness, versatility

INTRODUCTION

Dispersion is the description of the pattern of distribution of organisms in space (Southwood, 1966) and is often referred to as spatial distribution¹. It is a characteristic ecological property. Probability distributions are used to quantify and classify the dispersion of organisms. In order to describe dispersion, we shall assume that organisms are confined to discrete habitable sites called units. We suppose further that a random variable X represents the number of individuals that a unit may contain. The spatial distributions are described in terms of the mean m and variance v of X . If the mean and variance are equal, the spatial distribution is said to be random and the population pattern is said to be random. If variance is greater than the mean, the spatial distribution is said to be contagious and the population pattern is said to be overdispersed or clumped or patchy or aggregated or clustered. If variance is less than the mean, spatial distribution is said to be regular and the population pattern is said to be underdispersed or uniform.

¹ Spatial distribution is a visual description and not a probability distribution.

The random, contagious and regular spatial distributions can be described by the Poisson, negative binomial and binomial distributions respectively. In ecological studies the variance is usually found to be larger than the mean and the Poisson distribution rarely fits observed frequency distribution of the number of individuals per unit. Many clumped pest populations that have been studied can adequately be described by the negative binomial distribution.

Fisher *et al.* (1943) developed the logarithmic series in an investigation of the frequency distribution of numbers of species of animals obtained in random samples. This investigation was based on practical application in bacterial counts of individual bacteria and colony counts. They found that the colony counts followed Poisson distribution whereas the numbers per colony were logarithmically distributed and consequently, the bacterial counts in random samples were distributed as negative binomial. Quenouille (1949) gave a relation between logarithmic, Poisson and negative binomial series.

Anscombe (1949) gave statistical analysis of insect counts based on the negative binomial distribution. The negative binomial distribution is described by the mean m and the positive exponent k . The Poisson distribution is obtained as the limit as $k \rightarrow \infty$, whereas, as $k \rightarrow 0$, it approaches the logarithmic distribution. The negative binomial parameter k is considered as a dispersion parameter. Small values of k ($k \rightarrow 0$) are associated with overdispersion whereas large values of k ($k \rightarrow \infty$) are associated with randomness. Bliss and Fisher (1953) fitted the negative binomial distribution to counts of red mites on apple leaves and gave approximate methods of estimating the parameter k .

Katti and Gurland (1961) considered the properties of the Poisson Pascal² distribution, which is a natural complement of the Poisson binomial distribution. They argued that elementary distributions such as the Poisson, the logarithmic and the binomial which can be formulated on the basis of simple models have been found to be inadequate in describing the situations which occur in a number of phenomena. The Neyman type A, negative binomial and Poisson binomial which combine two of the elementary distributions through the processes of mixing and generalizing have been found to fit with varying degrees of success to data from a number of biological populations.

Young and Young (1990) reviewed measures of aggregation, namely, variance to mean ratio, index of clumping, index of mean crowding and index of patchiness with respect to Poisson and negative binomial distributions. The four measures of aggregation revealed that decreasing values of k are associated with increasing measures of aggregation (departure from randomness). In particular, index of patchiness \bar{P} is defined as

$$\bar{P} = 1 + \frac{1}{m} \left(\frac{v}{m} - 1 \right) \quad (1)$$

so that $\bar{P} = 1$, $\bar{P} > 1$, and $\bar{P} < 1$ for random, contagious and regular spatial distributions respectively.

² The Pascal distribution is the NBD with k , a positive integer.

Kipchirchir (2010) demonstrated analytically that the negative binomial parameter k is a measure of dispersion by analysing equicorrelation matrix in relation to coefficient of determination, partial correlation and principal components with respect to k . The analysis demonstrated that small values of k are associated with overdispersion, whereas large values are associated with randomness. Further, the negative binomial parameter k and index of patchiness are shown to be inversely related.

Although the negative binomial is the most widely applicable of the distributions for overdispersion, several other distributions have been proposed, including, the Neyman Type A and the Polya-Aeppli which are also two-parameter distributions.

1. PROBABILITY DISTRIBUTIONS

We shall denote $P(X = x)$ by p_x , mean by m , variance by v , probability generating function by $g(s)$ and index of patchiness by \bar{P} . In order to describe dispersion we only require mean and variance of X . Thus, moments higher than second order are not required.

1.1. POISSON DISTRIBUTION (PD)

The PD with parameter λ is defined as

$$p_x = \frac{\lambda^x e^{-\lambda}}{x!}, \quad x = 0, 1, 2, \dots; \quad \lambda > 0 \quad (2)$$

so that

$$m = v = \lambda \quad (3)$$

$$g(s) = \exp\{\lambda(s - 1)\} \quad (4)$$

and

$$\bar{P} = 1 \quad (5)$$

implying PD describes a random distribution.

1.2. NEGATIVE BINOMIAL DISTRIBUTION (NBD)

The NBD with parameters k and p is defined as

$$p_x = \binom{k+x-1}{x} \left(\frac{p}{q}\right)^x \left(\frac{1}{q}\right)^k, \quad x = 0, 1, 2, \dots; \quad k, p > 0, \quad q = 1 + p \quad (6)$$

so that

$$m = kp, \quad v = kpq \quad (7)$$

$$g(s) = (q - ps)^{-k} \quad (8)$$

and

$$\bar{P} = 1 + \frac{1}{k} > 1 \quad (9)$$

implying NBD describes a contagious distribution.

We observe that

$$\lim_{k \rightarrow \infty} \bar{P} = \lim_{k \rightarrow \infty} \left(1 + \frac{1}{k}\right) = 1 \tag{10}$$

implying NBD³ with $k \rightarrow \infty$ describes a random distribution.

To verify using probability generating function we let $m = kp = \lambda$ (fixed), then $p = \frac{\lambda}{k}$ so that from Equation (8)

$$\lim_{k \rightarrow \infty} g(s) = \lim_{k \rightarrow \infty} \left(1 - \frac{\lambda(s-1)}{k}\right)^{-k} = \exp\{\lambda(s-1)\} \tag{11}$$

which is the probability generating function of a PD with parameter λ .

1.3. LOGARITHMIC DISTRIBUTION (LD)

The LD with parameter β is defined as

$$p_x = -\frac{\beta^x}{x \ln(1-\beta)}, \quad x = 1, 2, 3, \dots; \quad 0 < \beta < 1 \tag{12}$$

so that

$$m = -\frac{\beta}{(1-\beta)\ln(1-\beta)}, \tag{13}$$

$$v = -\frac{\beta}{((1-\beta)\ln(1-\beta))^2} (\ln(1-\beta) + \beta), \tag{14}$$

$$g(s) = \frac{\ln(1-\beta s)}{\ln(1-\beta)} \tag{15}$$

and

$$\bar{P} = -\ln(1-\beta). \tag{16}$$

Reparameterizing $\beta = \frac{p}{q}$, $q = 1 + p$, $p > 0$, Equation (16) becomes

$$\bar{P} = \ln q \tag{17}$$

so that $\bar{P} < 1$, $\bar{P} = 1$ and $\bar{P} > 1$ if $q < e$, $q = e$ and $q > e$ respectively.

³ Regular distributions ($\bar{P} < 1$) are not accounted for by the NBD.

1.4. NEYMAN TYPE A DISTRIBUTION (NTAD)

NTAD is a generalized⁴ distribution. Clusters per unit are distributed as Poisson (λ_1) and individuals per cluster are distributed as Poisson (λ_2). Consequently, individuals per unit are distributed as Poisson (λ_1) -Poisson (λ_2), which is called NTAD and defined by the recurrence formula

$$p_{x+1} = \frac{\lambda_1 \lambda_2 e^{-\lambda_2}}{x+1} \sum_{j=0}^x \frac{\lambda_2^j}{j!} p_{x-j}, \quad x = 0, 1, 2, \dots; \quad (18)$$

$$p_0 = e^{\gamma - \lambda_1}, \quad \gamma = \lambda_1 e^{-\lambda_2}, \quad \lambda_1 > 0, \quad \lambda_2 > 0$$

so that

$$m = \lambda_1 \lambda_2, \quad v = \lambda_1 \lambda_2 (1 + \lambda_2), \quad (19)$$

$$g(s) = \exp\{\lambda_1 (\exp\{\lambda_2 (s - 1)\} - 1)\} \quad (20)$$

and

$$\bar{P} = 1 + \frac{1}{\lambda_1} > 1 \quad (21)$$

implying NTAD describes a contagious distribution.

We observe that,

$$\lim_{\lambda_1 \rightarrow \infty} \bar{P} = \lim_{\lambda_1 \rightarrow \infty} \left(1 + \frac{1}{\lambda_1} \right) = 1 \quad (22)$$

which implies that as $\lambda_1 \rightarrow \infty$, NTAD describes a random distribution. This means that as the density of clusters increases, randomness prevails.

This result can be verified using probability generating function by letting $\lambda_1 \lambda_2 = m$ (fixed) and finding the limit $\lambda_1 \rightarrow \infty$ ($\lambda_2 \rightarrow 0$). Thus, from Equation (20)

$$\begin{aligned} \lim_{\lambda_1 \rightarrow \infty} g(s) &= \lim_{\lambda_2 \rightarrow 0} \exp \left\{ m(s-1) + \frac{m\lambda_2(s-1)^2}{2!} + \frac{m\lambda_2^2(s-1)^3}{3!} + \dots \right\} \\ &= \exp\{m(s-1)\} \end{aligned} \quad (23)$$

which is the probability generating function of PD with parameter m .

We also observe that if $\lambda_1 = k$, then index of patchiness for NTAD is the same as that of the NBD. Furthermore, if $\lambda_1 = k$ and $\lambda_2 = p$, then the NTAD which is Poisson (k)-Poisson (p) and the NBD have the same mean $m = kp$, and variance $v = kp(1 + p)$, hence

⁴ A generalized distribution has mean $m = m_1 m_2$, variance $v = m_1 v_2 + m_2^2 v_1$ and probability generating function $g(s) = g_1(g_2(s))$ where $m_1, v_1, g_1(s)$ refer to the distribution of clusters per unit and $m_2, v_2, g_2(s)$ refer to the distribution of individuals per cluster.

equivalent in describing dispersion. Reparameterizing $\beta = \frac{p}{1+p} = \frac{\lambda_2}{1+\lambda_2}$, $\lambda_2 \rightarrow 0$ ($p \rightarrow 0$ or $k \rightarrow \infty$ or $\beta \rightarrow 0$) imply a random distribution while $\lambda_2 \rightarrow \infty$ ($p \rightarrow \infty$ or $k \rightarrow 0$ or $\beta \rightarrow 1$) imply a contagious distribution. Thus, the parameter λ_2 can be used as a measure of dispersion with respect to NTAD.

1.5. POLYA-AEPPLI DISTRIBUTION (PAD)

PAD is defined by the recurrence formula

$$xp_x = (2\tau(x-1) + v(1-\tau))p_{x-1} - \tau^2(x-2)p_{x-2}, \quad x = 2,3,4, \dots; \tag{24}$$

$$p_1 = v(1-\tau)p_0, \quad p_0 = e^{-v}, \quad 0 < \tau < 1, \quad v > 0$$

so that

$$m = \frac{v}{1-\tau}, \quad v = \frac{v(1+\tau)}{(1-\tau)^2}, \tag{25}$$

$$g(s) = \exp\left\{\frac{v(s-1)}{(1-\tau s)}\right\} \tag{26}$$

and

$$\bar{p} = 1 + \frac{2\tau}{v} > 1 \tag{27}$$

implying PAD describes a contagious distribution.

We observe that if $\lambda_1 = \frac{v}{2\tau}$, then index of patchiness is the same as that of Poisson (λ_1) -Poisson (λ_2) (NTAD). Furthermore, if $\lambda_2 = \frac{2\tau}{1-\tau}$, then NTAD and PAD have the same mean $m = \lambda_1\lambda_2$ and variance $v = \lambda_1\lambda_2(1+\lambda_2)$, hence equivalent in describing dispersion.

With the reparameterization $\beta = \frac{\lambda_2}{1+\lambda_2} = \frac{2\tau}{1+\tau}$, $\tau \rightarrow 0$ ($\beta \rightarrow 0$), imply a random distribution while $\tau \rightarrow 1$ ($\beta \rightarrow 1$), imply a contagious distribution. In particular, as $\tau \rightarrow 0$, the probability generating function of PAD converges to that of PD with parameter v . Thus, τ can be used as a measure of dispersion with respect to PAD.

1.6. POISSON-LOGARITHMIC DISTRIBUTION (PLD)

Clusters per unit are distributed as Poisson (λ) and individuals per cluster are distributed as logarithmic with parameter β . Thus, from Equations (3), (13) and (14)

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$$m_1 = v_1 = \lambda \quad (28)$$

$$m_2 = -\frac{\beta}{(1-\beta)\ln(1-\beta)}, \quad v_2 = \frac{\lambda\beta}{(1-\beta)(\ln(1-\beta))^2}(\beta + \ln(1-\beta)) \quad (29)$$

and therefore for the distribution of individuals per unit,

$$m = \frac{\lambda\beta}{(1-\beta)\ln(1-\beta)}, \quad v = \frac{\lambda\beta}{(1-\beta)^2\ln(1-\beta)} \quad (30)$$

$$g(s) = \exp\left\{\frac{\ln(1-\beta s)}{\ln(1-\beta)} - 1\right\} \quad (31)$$

and

$$\bar{p} = 1 - \frac{(1-\beta)\ln(1-\beta)}{\beta} \frac{1}{\lambda} \left(\frac{1}{1-\beta} - 1\right) \quad (32)$$

Reparameterizing $\beta = \frac{p}{1+p} = \frac{p}{q}$, $p > 0$, Equation (32) becomes

$$\bar{p} = 1 + \frac{\ln q}{\lambda} \quad (33)$$

and if $k = \frac{\lambda}{\ln q}$, then index of patchiness is the same as that of NBD. Letting $\beta = \frac{p}{q}$ and

$k = \frac{\lambda}{\ln q}$ in Equation (31) we obtain

$$g(s) = (q - ps)^{-k} \quad (34)$$

which is the probability generating function of NBD and as $\beta \rightarrow 0$ ($p \rightarrow 0$ or $k \rightarrow \infty$) in Equation (31), we obtain

$$\lim_{\beta \rightarrow 0} g(s) = \lim_{\beta \rightarrow 0} \exp\left\{\lambda \left(\frac{s(1-\beta)}{(1-\beta s)} - 1\right)\right\} = \exp\{\lambda(s-1)\} \quad (35)$$

which is the probability generating function of PD with parameter λ . Thus, PLD and NBD are basically the same distribution.

1.7. POISSON-NBD (PNBD)

Clusters per unit are distributed as Poisson (λ) and individuals per cluster are distributed as negative binomial with parameters k and p . Thus, from Equations (3) and (7)

$$m_1 = v_1 = \lambda \quad (36)$$

$$m_2 = kp, \quad v_2 = kp(1+p) \quad (37)$$

and therefore for the distribution of individuals per unit,

$$m = \lambda kp, \quad v = \lambda kp(1 + p + kp), \quad (38)$$

$$g(s) = \exp\{\lambda(q - ps)^{-k} - 1\} \quad (39)$$

and

$$\bar{P} = 1 + \frac{1}{\lambda} + \frac{1}{\lambda k} > 1 + \frac{1}{\lambda} \quad (40)$$

which is greater than the index of patchiness for NTAD. This means that the pattern described by PNBD is more patchy than the pattern described by NTAD.

We observe that

$$\lim_{k \rightarrow \infty} \bar{P} = \lim_{k \rightarrow \infty} \left(1 + \frac{1}{\lambda} + \frac{1}{\lambda k} \right) = 1 + \frac{1}{\lambda} \quad (41)$$

which is the index of patchiness for NTAD.

To verify, let $kp = \lambda$ (fixed) and find the limit of probability generating function as $k \rightarrow \infty$ ($p \rightarrow 0$). Thus, from Equation (39)

$$\begin{aligned} \lim_{k \rightarrow \infty} g(s) &= \lim_{k \rightarrow \infty} \exp \left\{ \lambda \left(\left(1 - \frac{\lambda(s-1)}{k} \right)^{-k} - 1 \right) \right\} \\ &= \exp\{\lambda(\exp\{\lambda(s-1)\} - 1)\} \end{aligned} \quad (42)$$

which is the probability generating function of Poisson (λ)-Poisson (λ), that is, NTAD. Rewriting the index of patchiness of PNBD with $\lambda k = k'$ (fixed) we obtain

$$\bar{P} = 1 + \frac{k}{k'} + \frac{1}{k'} > 1 + \frac{1}{k'} \quad (43)$$

which is greater than index of patchiness for NBD with parameter k' . This means that the pattern described by PNBD with parameters λ , k , p is more patchy than the pattern described by NBD⁵ with parameters k' , p such that $k' > k$.

We observe that as $k \rightarrow 0$ ($\lambda \rightarrow \infty$) we obtain

$$\lim_{k \rightarrow \infty} \bar{P} = \lim_{k \rightarrow \infty} \left(1 + \frac{k}{k'} + \frac{1}{k'} \right) = 1 + \frac{1}{k'} \quad (44)$$

which is the index of patchiness for NBD with parameter k' .

To verify, let $\lambda k = k'$ (fixed) and find limit of probability generating function as $k \rightarrow 0$ ($\lambda \rightarrow \infty$). Thus, from Equation (39)

⁵ This is plausible since large values of k' ($k' > k$) result in reduced aggregation and ultimately randomness prevails ($k' \rightarrow \infty$)

$$\begin{aligned}
 \lim_{k \rightarrow 0} g(s) &= \lim_{k \rightarrow 0} \exp \left\{ \frac{k'}{k} \left((1 - p(s-1))^{-k} - 1 \right) \right\} \\
 &= \lim_{k \rightarrow 0} \exp \left\{ \frac{k'}{k} \left(\frac{kp(s-1)}{1!} + \frac{k(k+1)(p(s-1))^2}{2!} + \dots \right) \right\} \\
 &= \exp \left\{ \ln(1 + p(1-s))^{-k'} \right\} \\
 &= (q - ps)^{-k'}, \quad q = 1 + p
 \end{aligned} \tag{45}$$

which is the probability generating function of the NBD with parameter k' .

Letting $\lambda kp = \lambda'$ (fixed), index of patchiness for PNBD becomes

$$\bar{P} = 1 + \frac{1}{\lambda} + \frac{1}{\lambda k} = 1 + \frac{kp}{\lambda'} + \frac{p}{\lambda'} \tag{46}$$

and as $p \rightarrow 0$ ($k \rightarrow \infty$ or $\lambda \rightarrow \infty$) we obtain

$$\lim_{p \rightarrow 0} \bar{P} = \lim_{p \rightarrow 0} \left(1 + \frac{kp}{\lambda'} + \frac{p}{\lambda'} \right) = 1 \tag{47}$$

which is the index of patchiness of a random distribution.

To verify, let $\lambda kp = \lambda'$ (fixed) and find the limit of probability generating function as $p \rightarrow 0$ ($k \rightarrow \infty$ or $\lambda \rightarrow \infty$). Thus, from Equation (39)

$$\lim_{p \rightarrow 0} g(s) = \lim_{p \rightarrow 0} \exp \left\{ \lambda' \left((s-1) + \frac{(k+1)p(s-1)^2}{2!} + \dots \right) \right\} = e^{\lambda'(s-1)} \tag{48}$$

which is the probability generating function of PD with parameter λ' which describes a random distribution.

Letting $kp = \lambda'$ (fixed), index of patchiness for PNBD becomes

$$\bar{P} = 1 + \frac{1}{\lambda} + \frac{1}{\lambda k} = 1 + \frac{1}{\lambda} + \frac{p}{\lambda \lambda'} \tag{49}$$

and finding limit as $p \rightarrow 0$ ($k \rightarrow \infty$) we obtain

$$\lim_{p \rightarrow 0} \bar{P} = \lim_{p \rightarrow 0} \left(1 + \frac{1}{\lambda} + \frac{p}{\lambda \lambda'} \right) = 1 + \frac{1}{\lambda} \tag{50}$$

which is the index of patchiness for NTAD.

To verify, let $kp = \lambda'$ (fixed) and find the limit of probability generating function as $p \rightarrow 0$ ($k \rightarrow \infty$). Thus, from Equation (39),

$$\begin{aligned}
 \lim_{p \rightarrow 0} g(s) &= \lim_{k \rightarrow \infty} \exp \left\{ \lambda \left(1 - \frac{\lambda'(s-1)}{k} \right)^{-k} - 1 \right\} \\
 &= \exp \{ \lambda (\exp \{ \lambda'(s-1) \}) - 1 \}
 \end{aligned} \tag{51}$$

which is the probability generating function of Poisson (λ) -Poisson (λ') (NTAD).

1.8. TRUNCATED NBD (TNBD)

The NBD is defined on the non-negative integers. If we exclude or truncate zero, we then have the TNBD defined by

$$p_x = \frac{\Gamma(k+x)}{\Gamma(k)x!} \left(\frac{p}{1+p}\right)^x \frac{1}{(1+p)^k - 1}, \quad x = 1, 2, 3, \dots; \quad k, p > 0 \quad (52)$$

so that

$$m = \frac{kp}{1 - (1+p)^{-k}}, \quad v = m((k+1)p + 1 - m), \quad (53)$$

$$g(s) = \frac{(1-\beta s)^{-k} - 1}{(1-\beta)^{-k} - 1}, \quad \beta = \frac{p}{1+p} = \frac{p}{q} \quad (54)$$

and

$$\bar{p} = \frac{(k+1)p}{m}. \quad (55)$$

Noting that $m > kp$, we observe that

$$\bar{p} = \frac{(k+1)p}{m} < \frac{(k+1)p}{kp} = 1 + \frac{1}{k} \quad (56)$$

which implies that pattern described by TNBD is less patchy than pattern described by NBD. For fixed p ,

$$\lim_{k \rightarrow 0} \bar{p} = \lim_{k \rightarrow 0} \frac{1 - (1+p)^{-k}}{k} = \lim_{k \rightarrow 0} (1+p)^{-k} \ln(1+p) = \ln q \quad (57)$$

which is the index of patchiness for LD.

To verify using probability generating function, we have from Equation (54)

$$\lim_{k \rightarrow 0} g(s) = \lim_{k \rightarrow 0} \frac{-(1-\beta s)^{-k} \ln(1-\beta s)}{-(1-\beta)^{-k} \ln(1-\beta)} = \frac{\ln(1-\beta s)}{\ln(1-\beta)} \quad (58)$$

which is the probability generating function of LD.

1.9. POISSON-TNBD (PTNBD)

Clusters per unit are distributed as Poisson (λ) and individuals per cluster are distributed as truncated negative binomial with parameters k and m_2 . Thus, from Equations (3) and (53)

$$m_1 = v_1 = \lambda, \quad (59)$$

$$m_2 = \frac{kp}{1 - (1+p)^{-k}}, \quad v_2 = m_2((k+1)p + 1 - m_2) \quad (60)$$

and therefore for the distribution of individuals per unit,

$$m = \lambda m_2, \quad v = m((k+1)p + 1), \quad (61)$$

$$g(s) = \exp \left\{ \lambda \left(\frac{(1 - \beta s)^{-k} - 1}{(1 - \beta)^{-k} - 1} - 1 \right) \right\} \quad (62)$$

and

$$\bar{P} = 1 + \frac{k+1}{\lambda k} (1 - (1+p)^{-k}). \quad (63)$$

From Equations (40) and (63) we observe that

$$\bar{P} = 1 + \frac{k+1}{\lambda k} (1 - (1+p)^{-k}) < 1 + \frac{1}{\lambda} + \frac{p}{\lambda k} \quad (64)$$

which implies that pattern described by PTNBD is less patchy than the pattern described by PNBD.

For fixed p ,

$$\lim_{k \rightarrow 0} \bar{P} = \lim_{k \rightarrow 0} \left(1 + \frac{k+1}{\lambda} ((1+p)^{-k} \ln(1+p)) \right) = 1 + \frac{\ln p}{\lambda} \quad (65)$$

which is the index of patchiness of PLD which is equivalent to NBD.

To verify using probability generating function, we have from Equation (62)

$$\lim_{k \rightarrow 0} g(s) = \exp \left\{ \lambda \frac{\ln(1 - \beta s)}{\ln(1 - \beta)} - 1 \right\} \quad (66)$$

which is the probability generating function of the PLD.

Letting $\frac{\lambda k p}{1 - (1+p)^{-k}} = \lambda'$ (fixed), we get from Equation (63)

$$\lim_{p \rightarrow 0} \bar{P} = \lim_{p \rightarrow 0} \left(1 + \frac{(k+1)p}{\lambda'} \right) = 1 \quad (67)$$

which is the index of patchiness of a random distribution.

To verify using probability generating function, we note that

$$\lambda = \frac{\lambda'(1 - \beta)((1 - \beta)^{-k} - 1)}{k\beta(1 - \beta)^{-k}}, \quad \beta = \frac{p}{1 + p}$$

and finding limit as $\beta \rightarrow 0$ ($p \rightarrow 0$), we have from Equation (62)

$$\begin{aligned} \lim_{\beta \rightarrow 0} g(s) &= \lim_{\beta \rightarrow 0} \exp \left\{ \frac{\lambda'}{k} \left(\frac{(1 - \beta s)^{-k} - (1 - \beta)^{-k}}{\beta(1 - \beta)^{-k}} \right) \right\} - 0 \\ &= \lim_{\beta \rightarrow 0} \exp \left\{ \frac{\lambda'}{k} \left(\frac{ks(1 - \beta s)^{-k-1} - k(1 - \beta)^{-k-1}}{k\beta(1 - \beta)^{-k-1} + (1 - \beta)^{-k}} \right) \right\} \\ &= \exp\{\lambda'(s - 1)\} \end{aligned} \quad (68)$$

which is the probability generating function of PD with parameter λ' .

Letting $kp = \lambda'$ (fixed) and expressing index of patchiness in terms of k and finding limit as $k \rightarrow \infty$ ($p \rightarrow 0$), we obtain

$$\lim_{k \rightarrow \infty} \bar{P} = \lim_{k \rightarrow \infty} \left(1 + \frac{k+1}{\lambda k} \left(1 - \left(1 + \frac{\lambda'}{k} \right)^{-k} \right) \right) = 1 + \frac{1}{\lambda} (1 - e^{-\lambda'}) \quad (69)$$

which is the index of patchiness of NTAD of which clusters per unit are distributed as

$$\text{Poisson} \left(\frac{\lambda}{1 - e^{-\lambda'}} \right).$$

To verify, using probability generating function, we note that $\beta = \frac{p}{1+p}$ and finding limit as $k \rightarrow \infty$ ($p \rightarrow 0$), we have from Equation (62)

$$\begin{aligned} \lim_{k \rightarrow \infty} g(s) &= \lim_{k \rightarrow \infty} \exp \left\{ \lambda \left(\frac{1 - \frac{\lambda'}{k} (s-1)^{-k} - 1}{1 - \left(1 + \frac{\lambda'}{k} \right)^{-k}} \right) \right\} \\ &= \exp \left\{ \frac{\lambda}{1 - e^{-\lambda'}} (\exp\{\lambda'(s-1)\} - 1) \right\} \end{aligned} \quad (70)$$

which is the probability generating function of $\text{Poisson} \left(\frac{\lambda}{1 - e^{-\lambda'}} \right) - \text{Poisson}(\lambda')$ (NTAD).

We observe that

$$\bar{P} = 1 + \frac{1}{\lambda} (1 - e^{-\lambda'}) < 1 + \frac{1}{\lambda} \quad (71)$$

which implies that the pattern described by $\text{Poisson} \left(\frac{\lambda}{1 - e^{-\lambda'}} \right) - \text{Poisson}(\lambda')$ is less patchy than the pattern described by $\text{Poisson}(\lambda) - \text{Poisson}(\lambda')$, however, they are equivalent as $\lambda' \rightarrow \infty$, that is, as mean number of individuals per cluster increases.

1.10. POISSON-GAMMA DISTRIBUTION (PGD)

For given λ , let X have Poisson (λ), that is,

$$p_{x/\lambda} = \frac{\lambda^x e^{-\lambda}}{x!}, \quad x = 0, 1, 2, \dots; \quad \lambda > 0 \quad (72)$$

then $E(X/\lambda) = \text{Var}(X/\lambda) = \lambda$ and the pattern would be random.

Suppose that some units provide more favourable environment than others (units are dissimilar). Then λ , the expected number of individuals in a unit, varies from unit to unit,

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that is, the environment is heterogeneous resulting in contagion described by NBD obtained as a continuous mixture of Poisson distribution where gamma is the mixing distribution. More precisely, let λ be a realization of a random variable Λ having the Pearson Type III Distribution (gamma density)

$$g(\lambda) = \frac{\lambda^{k-1}}{\Gamma(k)p^k} e^{-\lambda/p}, \quad \lambda, k, p > 0 \tag{73}$$

then the marginal distribution of X (mixture distribution) is the NBD

$$p_x = \int_0^\infty p_{x/\lambda} g(\lambda) d\lambda$$

$$= \frac{\Gamma(k+x)}{\Gamma(k)x!} \left(\frac{p}{1+p}\right)^x \left(\frac{1}{1+p}\right)^k, \quad x = 0,1,2, \dots; \quad k, p > 0 \tag{74}$$

2. SUMMARY AND CONCLUSIONS

The foregoing analysis demonstrated that probability models corresponding to different biological phenomena lead to NBD. On the other hand, PD and LD are limiting distributions of the NBD and TNBD respectively. The conceptual distributional framework was arrived at by generalizing (G), mixing (M), reparameterizing (R), truncating (T) and limiting (L). The framework clearly depicts the versatility of the NBD in describing dispersion.

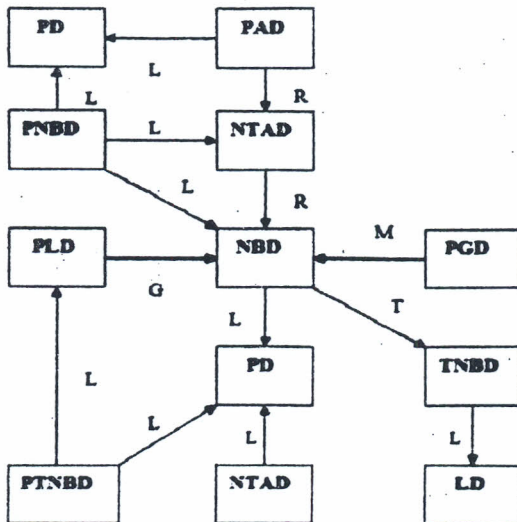


Figure 1: Conceptual Distributional Framework.

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