Class-Two Space-Time of Product Spaces in General Relativity.

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BERTOTTI [1] has considered the Riemannian fourfold corresponding the product of surfaces of constant curvature in the context of the electromagnetic field and it has been shown with the help of purely geometrical arguments that the fourfold represents a uniform nonnull electromagnetic field. In this paper we consider a product of two surfaces which are not necessarily of constant curvature to start with. The condition for a Riemannian fourfold to be decomposable into the product of two surfaces is obtained as

$$R_{hijk} = \tfrac{1}{2}(R_{hj}g_{ij} + R_{ik}g_{hj} - R_{ij}g_{hk} - R_{hk}g_{ij}), \tag{1.1}$$

which implies that the conharmonic curvature tensor

$$V_{hijk} = R_{hijk} + \frac{1}{n-2}(R_{hj}g_{ik} - g_{ij} + R_{ik}g_{hj} - R_{ij}g_{hk}) \tag{1.2}$$

vanishes in this case. Further such a space-time is incompatible with the perfect-fluid distribution and the material distribution is possible only for the eigen-values of the type $(\lambda, \lambda, \delta, \delta)$ as shown by SINGH and SHARAN [2]. In the second part we consider the group of motion and express the four vectors in terms of arbitrary functions. The relationship between these arbitrary functions and the functions and the metric potentials is obtained.

The metric for a Riemannian fourfold, which is the product, of two V_2's, can be written in the form

$$ds^2 = -A(dx^2 + dy^2) + B(dz^2 - dt^2), \tag{2.1}$$

where $A = A(x, y)$, $B = B(z, t)$ and x, y, z, t corresponds to

$$x^1,\ x^2,\ x^3,\ x^4 \text{ respectively}.$$

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(1) B. BERTOTTI: *Phys. Rev.*, **116**, 133 (1959).
(2) K. P. SINGH and R. SHARAN: *Nat. Inst. of Sci. India*, **31** A, No. 6, 584 (1965).

The class of each of the V_2's is one, thus the class of the metric (2.1) is two. The surviving components of the curvature tensor

$$R_{1212} = \frac{A_{11}+A_{22}}{2} - \frac{(A_1)^2+(A_2)^2}{2A}, \tag{2.2}$$

$$R_{3434} = \frac{B_{44}-B_{33}}{2} - \frac{(B_3)^2-(B_4)^2}{2B}. \tag{2.3}$$

The lower suffixes 1, 2, 3, 4, after a function indicate ordinary partial differentiation with respect to x, y, z, t, respectively.

The nonvanishing components of the Ricci tensor R_{ij} are given by

$$R_{11} = \frac{A_{11}+A_{22}}{2A} - \frac{(A_1)^2+(A_2)^2}{2A^2} = R_{22}, \tag{2.4}$$

$$R_{33} = \frac{B_{44}-B_{33}}{2B} - \frac{(B_3)^2-(B_4)^2}{2B^2} = R_{44}. \tag{2.5}$$

The scalar curvature R for the metric (2.1) is

$$R = \frac{A_{11}+A_{22}}{A^2} + \frac{(A_1)^2+(A_2)^2}{A^3} + \frac{B_{44}-B_{33}}{B^2} - \frac{(B_3)^2-(B_4)^2}{B^3}. \tag{2.6}$$

We now look into these equations and obtain the tensorial condition. From (2.2), (2.3) (2.4) and (2.5), we have

$$R_{hijk} = \tfrac{1}{2}(R_{hj}g_{ik} + R_{ik}g_{hj}) \tag{2.7}$$

as the necessary condition for a fourfold to be of the form $V_2 \times V_2$. We can write (2.7) in a more general form as

$$R_{hijk} = \tfrac{1}{2}(g_{hj}R_{ik} - g_{hk}R_{ij} + g_{ik}R_{hj} - g_{ij}R_{hk}). \tag{2.8}$$

We know that the conharmonic curvature tensor is defined as

$$V_{hijk} = R_{hijk} - \frac{1}{n-2}(g_{hj}R_{ik} - g_{hk}R_{ij} + g_{ik}R_{hj} - g_{ij}R_{hk}). \tag{2.9}$$

The conharmonic curvature tensor V_{hijk} vanishes in view of (2.8) and the space becomes conharmonically flat. Thus we have the following theorem:

THEOREM (2.1). The necessary condition that the Riemannian fourfold can be decomposed into the product of two surfaces of the form $V_2 \times V_2$ is that the space must be conharmonically flat.

It may be mentioned that our space-time is not conformally flat.

Now we explore the possibility of generation of groups of motion for the product space of class two.

We know that the group of motion is generated if the vectors ζ^a satisfy Killing's equations [3]:

$$\underset{\zeta}{\pounds}\, g_{ij} = g_{ij}\zeta^a_a + g_{aj}\zeta^a_i + g_{ia}\zeta^a_j = 0\,. \tag{3.1}$$

For the metric (2.1) we have the following set of Killing equations:

$$A_1\zeta^1 + A_2\zeta^2 + 2A\zeta^1_1 = 0\,, \tag{3.2}$$

$$A_1\zeta^1 + A_2\zeta^2 + 2A\zeta^2_2 = 0\,, \tag{3.3}$$

$$B_3\zeta^3 + B_4\zeta^4 + 2B\zeta^3_3 = 0\,, \tag{3.4}$$

$$B_3\zeta^3 + B_4\zeta^4 + 2B\zeta^4_4 = 0\,, \tag{3.5}$$

$$\zeta^1_2 + \zeta^2_1 = 0\,, \tag{3.6}$$

$$A\zeta^1_3 + B\zeta^3_1 = 0\,, \tag{3.7}$$

$$A\zeta^1_4 + B\zeta^4_1 = 0\,, \tag{3.8}$$

$$A\zeta^2_3 + B\zeta^3_2 = 0\,, \tag{3.9}$$

$$A\zeta^2_4 + B\zeta^4_2 = 0\,, \tag{3.10}$$

$$\zeta^3_4 + \zeta^4_3 = 0\,. \tag{3.11}$$

From (3.2) and (3.3), we have

$$\zeta^1_1 = \zeta^2_2\,. \tag{3.12}$$

Similarily from (3.4) and (3.5) we have

$$\zeta^3_3 = \zeta^4_4\,. \tag{3.13}$$

From (3.6) and (3.12) we get

$$\zeta^1 = \alpha(x^3, x^4)\beta(x^1 + x^2) + \gamma(x^3, x^4)\delta(x^2 - x^1) + p(x^3, x^4) \tag{3.14}$$

and

$$\zeta^2 = \alpha(x^3, x^4)\beta(x^2 - x^1) - \gamma(x^3, x^4)\delta(x^2 + x^1) + q(x^3, x^4)\,, \tag{3.15}$$

where α, β, γ, δ, p and q are arbitrary functions.

Similarily from (3.11) and (3.13) we get

$$\zeta^3 = v(x^1, x^2)\lambda(x^3 + x^4) + \sigma(x^2, x^1)\mu(x^3 - x^4) + \psi(x^1, x^2) \tag{3.16}$$

and

$$\zeta^4 = v(x^1, x^2)\lambda(x^3 + x^4) - \sigma(x^2, x^1)\mu(x^3 - x^4) + \chi(x^1, x^2)\,, \tag{3.17}$$

where v, σ, λ, μ, ψ and χ are also arbitrary functions.

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[3] K. YANO: *Differential Geometry on Complex and Almost Complex Spaces* (London, 1964), p. 17.

We now find out the relationship between these arbitrary functions and the functions A and B of the metric such that the group of motion is generated by the vectors ζ^a expressed by (3.14), (3.15), (3.16) and (3.17) for the metric (2.1).

From (3.7), (3.8), (3.9) and (3.14), (3.15), (3.16) and (3.17), we have the following relationship:

$$A(\alpha_3\beta + \gamma_3\delta + p_3) + B(v_1\lambda + \sigma_1\mu + \psi_1) = 0\,, \tag{3.18}$$

$$A(\alpha_4\beta + \gamma_4\delta + p_4) - B(v_1\lambda - \sigma_1\mu + \chi_1) = 0\,, \tag{3.19}$$

$$A(\alpha_3\beta - \gamma_3\delta + q_3) + B(v_2\lambda + \sigma_2\mu + \psi_2) = 0\,, \tag{3.20}$$

$$A(\alpha_4\beta - \gamma_4\delta + q_4) - B(v_2\lambda - \sigma_2\mu + \chi_2) = 0\,. \tag{3.21}$$

Thus the group of motion is generated by the vectors ζ^a expressed by (3.14), (3.15), (3.16) and (3.17) for the metric (2.1) of the product of two surfaces of class two and the arbitrary functions are related with the metric potentials by the equations (3.18), (3.19), (3.20) and (3.21).

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