CURVATURE TENSORS' AND THEIR RELATIVISTICS SIGNIFICANCE

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(Received September 28, 1970)

Summary

In this paper we have defined the Curvature tensor and elaborated its various physical and geometric properties.

1. Introduction

In an n-dimensional space V_n, the tensors

$$C(X, Y, Z, T) = R(X, Y, Z, T) - \frac{R}{n(n-1)}[g(X, T)g(Y, Z) - g(Y, T)g(X, Z)], \tag{1.1}$$

$$L(X, Y, Z, T) = R(X, Y, Z, T) - \frac{1}{n-2}[g(Y, Z)\,\mathrm{Ric}\,(X, T) - g(X, Z)\,\mathrm{Ric}\,(Y, T) + g(X, T)\,\mathrm{Ric}\,(Y, Z) - g(Y, T)\,\mathrm{Ric}\,(X, Z)] \tag{1.2}$$

and

$$V(X, Y, Z, T) = R(X, Y, Z, T) - \frac{1}{n-2}\, g(X, T)\,\mathrm{Ric}\,(Y, Z) - g(Y, T)\,\mathrm{Ric}\,(X, Z) + g(Y, Z)\,\mathrm{Ric}\,(X, T) - g(X, Z)\,\mathrm{Ric}\,(Y, T)] + \frac{R}{(n-1)(n-2)}[g(X, T)g(Y, Z) - g(Y, T)g(X, Z)], \tag{1.3}$$

are called concircular curvature tensor, conharmonic curvature tensor and conformal curvature tensor respectively. These satisfy the symmetric and skew symmetric as well as the cyclic property possessed by curvature tensor $R(X, Y, Z, T)$.

The projective curvature tensor is given by:

$$W(X, Y, Z, T) = R(X, Y, Z, T) + \frac{1}{n-1}[g(X, Z)\,\mathrm{Ric}\,(Y, T) - g(X, T)\,\mathrm{Ric}\,(Y, Z)]. \tag{1.4}$$

We shall now define a tensor and obtain its properties.

2. Definition (2.1): We define a tensor

$$W_2(X, Y, Z, T) \overset{\text{def}}{=} R(X, Y, Z, T) + \frac{1}{n-1}[g(X, Z)\,\mathrm{Ric}(Y, T) - g(Y, Z)\,\mathrm{Ric}(X, T)]. \tag{2.1}$$

From equations (1.1) to (2.1), it is clear that for an empty gravitational field characterized by $\mathrm{Ric}(X, Y)=0$, these five fourth rank tensors are identical.

In the space V_n, from (1.1), (1.2) and (1.3), we have

$$V(X, Y, Z, T) = L(X, Y, Z, T) + \frac{n}{n-2}[R(X, Y, Z, T) - C(X, Y, Z, T)], \tag{2.2}$$

which in V_4 reduces to

$$V(X, Y, Z, T) = L(X, Y, Z, T) + 2R(X, Y, Z, T) - 2C(X, Y, Z, T). \tag{2.3}$$

Now we notice that tensor $W_2(X, Y, Z, T)$ is skew symmetric in X and Y and it also satisfies

$$W_2(X, Y, Z, T) + W_2(Y, Z, X, T) + W_2(Z, X, Y, T) = 0. \tag{2.4}$$

Breaking $W_2(X, Y, Z, T)$ ito two parts viz:

$$E(X, Y, Z, T) = 1/2\,[W_2(X, Y, Z, T) - W_2(X, Y, T, Z)]$$

and

$$F(X, Y, Z, T) = 1/2\,[W_2(X, Y, Z, T) + W_2(X, Y, T, Z)],$$

which are respectively skew-symmetric and symmetric in Z, T. From (2.1) it follows that

$$E(X, Y, Z, T) = R(X, Y, Z, T) + \frac{2}{2(n-1)}[g(X, Z)\,\mathrm{Ric}(Y, T) - g(Y, Z)\,\mathrm{Ric}(X, T) - g(X, T)\,\mathrm{Ric}(Y, Z) + g(Y, T)\,\mathrm{Ric}(X, Z)] \tag{2.5}$$

and

$$F(X, Y, Z, T) = \frac{1}{2(n-1)}[g(X, Z)\,\mathrm{Ric}(Y, T) - g(Y, Z)\,\mathrm{Ric}(X, T) + g(X, T)\,\mathrm{Ric}(Y, Z) - g(Y, T)\,\mathrm{Ric}(X, Z)]. \tag{2.6}$$

From (2.5) we see that $E(X, Y, Z, T)$ also possesses all the symmetric and skew symmetric properties of $R(X, Y, Z, T)$ as well as the cyclic property:

$$E(X, Y, Z, T) + E(Y, Z, X, T) + E(Z, X, Y, T) = 0. \tag{2.7}$$

From equations (1.3) and (2.5), we get

(2.8) $$E(X, Y, Z, T)=\frac{1}{2(n-1)}[nR(X, Y, Z, T)+(n-2)V(X, Y, Z, T) - \frac{R}{n-1}\{g(X, T)g(X, Z)-g(Y, T)g(X, Z)\}].$$

Which for electromagnetic field (or more generally in the case of space with vanishing scalar curvature) in V_4 becomes

(2.9) $$3E(X, Y, Z, T)=2R(X, Y, Z, T)+V(X, Y, Z, T),$$

also from equation (1.2) and (2.5), for V_4, we have

(2.10) $$3E(X, Y, Z, T)=2R(X, Y, Z, T)+L(X, Y, Z, T).$$

Thus equation (2.9) is the consequence of (2.10) for a space of vanishing scalar curvature.

We notice that $E(X, Y, Z, T)$ is identically equal to the skew symmetric part $P(X, Y, Z, T)$ [2] of the projective curvature tensor where as its symmetric part $Q(X, Y, Z, T)$ is different from $F(X, Y, Z, T)$.

On contracting W_{2hijk}, we get

(2.11) $$W_{2ij}=\frac{n}{n-1}\left(R_{ij}-\frac{R}{n}g_{ij}\right),$$

which vanishes in an Einstein space.

The scalar invariant

(2.12) $$W_2\equiv g^{ij}W_{2ij}=0,$$

identically. Now considering the scalar invariant of second degree in W_{2ij}, viz:

(2.13) $$(W_2)_{II}W_{2ij}W_2^{ij}=\left(\frac{n}{n-1}\right)^2\left(R_2-\frac{R^2}{n}\right),$$

where $R_2\equiv R_{ij}R^{ij}$.

From (2.11), we have

(2.14) $$W_{2ij}R^{ij}=\frac{n}{n-1}\left(R_2-\frac{R^2}{n}\right).$$

Hence

(2.15) $$W_{2ij}W_2^{ij}=\left(\frac{n}{n-1}\right)W_{2ij}R^{ij}.$$

From (2.5) we notice that contracted E_{ij} vanishes identically for Einstein space. This enables us to extend the Pirani formalism of gravitational waves to the Einstein space with the help of E_{hijk}.

For an Einstein space E_{hijk}, W_{2hijk}, W_{hijk} and V_{hijk} are identically equal.

We can show that the vanishing of the symmetric part F_{hijk} is the necessary and sufficient condition for a space to be an Einstein space.

The vector

$$Q_i = \frac{g_{ij}\epsilon^{jklm} R^p_k R_{pl;m}}{\sqrt{-g}\, R_{ab} R^{ab}}, \tag{2.16}$$

is called the complexion vector of a non null electromagnetic field with no matter by *Misner and Wheeler* [3] and its vanishing implies that field is purely electrical. A semi-colon stands for covariant differentiation.

It is seen that we can't get a purely electrical field with the help of W_{2hijk}.

Rainich [4] has shown that the necessary and sufficient conditions for the existence of the non-null electrovariance are

$$R=0\,, \tag{2.17}$$

$$R^i_j R^j_k = (1/4)\delta^i_k R_{ab} R^{ab}\,, \tag{2.18}$$

$$Q_{i;j} = Q_{j;i}\,. \tag{2.19}$$

In an electromagnetic field

$$W_{2ij} = (4/3) R_{ij}\,. \tag{2.20}$$

We can substitute W_{2ij} in place of R_{ij} in (2.16) and (2.18) such that the Rainich conditions so obtained are similar to those obtained with the help of W_{hijk}.

From the above discussion we conclude that except the vanishing of complexion vector and property of being identical in two spaces which are in geodesic correspondence, the tensor W_{2hijk} possesses the properties almost similar to W_{hijk}. Thus we can very well use W_{2hijk} in various physical and geometrical spheres in place of the projective curvature tensor.

REFERENCES

[1] Mishra, R. S. (1969): *H-Projective curvature tensor in Kahler manifold* (under publication) Proc. Nat. Inst. of Sci., New Delhi.

[2] K. P. Singh, L. Radhakrishna and R. Sharan: *Electromagnetic Fields and cylindrical symmetry*: Annals of Phys. Vol. 32, No. 1, pp. 46–68 (1965).

[3] C. W. Misner and J. A. Wheeler: Ann. Phys. (N. Y.) 2, 525 (1957).

[4] G. Y. Rainich: Trans. Am. Math. Soc. 27, 106 (1952)

[5] L. P. Eisenhart: *Riemannian Geometry,* Princeton University Press (1926).

[6] J. L. Synge: *Relativity, The general theory,* North Holland Amsterdam (1960).

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