APPLICATION OF SPERNER'S LEMMA IN FAIR DIVISION

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ABSTRACT

In this paper, we describe the approximate envy-free division using a mathematical technique from "game theory"; with help of Sperner's lemma. The examples of land distribution in families and other heterogeneous resources are given.

Key words: fair division, envy-free, triangulation, simplex, convex hull,

1. Introduction

Suppose you have visited your mathematics Professor and you've been invited to take coffee at the Professor's house. While you're busy sipping coffee and discussing the topic for your master project, his wife offers you a plate containing a ready-sliced home-made chocolate sponge cake. It looks gorgeous and, as a guest, you may choose the first slice! Unfortunately, you notice that it has not been divided equally. Maybe one slice is a little fatter, or contains an extra-large dollop of filling. Which do you go for?

In this particular social situation you may feel obliged to take the ***least*** desirable slice; maybe the professor will be thinking the same thing when it comes to his turn. To save the cringing embarrassment which the wrong choice could convey, you decide that it would be a better world if all slices were equally desirable to all parties, but that's not so easy to achieve. On the other hand, maybe you are in a cake-choosing situation where **selfishness** and **greed** are the rules of the game.

In this case, there's only one slice that will do, and that's the biggest. If you choose last, you get the smallest - it's unfair. Again, we need a way to divide our cake so that everyone is satisfied, but can it be done?

It's easy if there are only two of us: I cut and you choose. When I cut, I carefully control the position of the knife so that either half of the cake is equally desirable to me. You can choose whichever piece you like; we are both satisfied. Alternatively, you could cut and then I choose - it makes no difference. However, things get more complicated if there are three of us. Who cuts the cake, and who chooses first? What about the third person -do they have a say in things? In practice, there could be any number of cake eaters, so our method must reflect this. Now, we could, of course, appoint an **impartial referee** to supervise the cutting and distribution, according to a previously-agreed set of rules. Maybe they would cut as geometrically accurate a set of slices as they can, or maybe they would weigh the cake and then **trim** or **augment** each slice according to the calculated weight of an even portion. Either way, you get what you're given. You agreed the rules, so you can't complain; but, there's always the nagging feeling that the decision has been taken out of your hands, isn't there? And did you ever meet a **referee** that didn't like cake? Can you really trust them? We need a method which will work without external influences and rules. One cake, one knife and a set number of eaters.

You may be surprised to hear that it's a problem which has engaged mathematicians for many years. There we were, thinking that they were beavering away in their universities, developing the complex algebraic formulae which underpin new technological advances and engineering marvels. All the time they were sitting in the coffee lounge, eating cake. Steadily over the years, and now there are hundreds of research papers and a fair few books on the subject, too. We are not surprised!!

Over 50 years after H. Steinhaus, research papers are still using cake as the subject, but mathematicians may showed that this analysis is applicable to other real life problems of fair division and dispute resolution. Barbanel (1996) states that his method extends beyond cake, such that it applies to any 'divisible heterogeneous good'-indivisible parts make the theory much more complex. An example of this would be where a car and a motorcycle have to be shared. This is also an example of where the values may not add up nicely, as either can be used as transport. The use of money can make such problems much easier '.

In the real world of course people sometimes have a very accurate idea of how the other players value the goods and they may care very much about it. The case where they have complete knowledge of each others valuations can be modeled by game theory. Partial knowledge is very hard to model. A major part of the practical side of fair division is the devising and study of procedures that work well despite such partial knowledge or small mistakes. A fair division procedure lists actions to be performed by the players in terms of the visible data and their valuations. A valid procedure is one that guarantees a fair division for every player who acts rationally according to their valuation. In a situation where an action depends on a player's valuation the procedure is describing the strategy a rational player will follow. A player may act as if a piece had a different value but must be consistent. For instance if a procedure says the first player cuts the cake in two equal parts then the second player chooses a piece, then the first player cannot claim that the second player got more. Land division is certainly a candidate for this analysis, although you would presumably need a very large knife or other measuring gadgets, as well as deciding what to do with all the trimmings. However land being a problem in Kenya we use the **cake division algorithm** to illustrate how a simple mathematical idea can be used in conflict resolution-the thorny issue of land (**envy free land division)**. This algorithm, (Su 1999) presented a scheme to exploit the properties of Sperner's lemma to enable resource allocation among *n* players. This result is immediately applicable to multiagent/n-players resource allocation problems by changing some of the terminology.

However, some of the cake-cutting procedures that have been proposed are discrete, whereby players make cuts with a knife—usually in a sequence of steps—but the knife is not allowed to move continuously over the cake. Moving-knife procedures, on the other hand, permit such continuous movement and allow players to call "stop" at any point at which they want to make a cut or mark. While there are now about a dozen such procedures for dividing a cake among three players such that each player is assured of getting a largest or tied-for-largest piece (Brams, et al 1995)—and so will not envy another player (resulting in an *envy-free division*)—only one procedure (Stromquist, 1980) makes the envy-free division with only two cuts. This is the minimal number for three players; in general *n* - 1 cuts is the minimum number of cuts required to divide a cake into *n* pieces. A cake so cut ensures that each player gets a single connected piece, which is desirable in certain applications particularly for land division.

This paper is developed from the article "Rental Harmony: Sperner's Lemma in Fair Division" by Francis Su, (1999). Sperner's Lemma states that, given a triangulated triangle whose vertices have been tri-colored using a Sperner coloring, at least one inner triangle will have vertices of all three colors. This paper will focus on the applications which include the areas of land distribution in a family and the assets to be shared equitably between nations.

2. SPERNER'S LEMMA

We now make use of Sperner's lemma for triangles and describe the procedure on a triangle. The concept of n-simplex is the developed and proof of lemma is given.

FOR TRIANGLES

- Simple combinatorial lemma, due to Sperner in 1928.
- Simple but powerful as it can be used to give the proof (elementary) to Brouwer fixed point theorem or we can say Sperner's Lemma is equivalent to the Brouwer fixed point theorem.

The procedure (motivation)

Divide a triangle *T* into lots of baby triangles (also called elementary triangles), so that baby triangles only meet at a common edge or a common vertex. Label each main vertex of the whole triangle by 1, 2, or 3; then label vertices on the (12) side by either 1 or 2, on the (23) side by either 2 or 3, and the (13) side by either 1 or 3. Label the points in the interior by any of 1, 2, or 3. For instance,

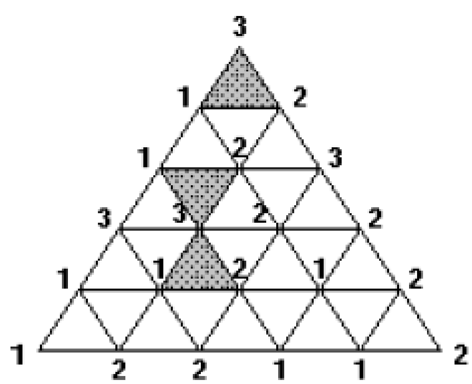

Figure 1

any such labelling must contain an baby (123) triangle! (In fact, there must be an *odd* number!)

The labelling we have chosen obeys two conditions:

(1) all of the main vertices of T have different labels, and
(2) the label of a vertex along any edge of T matches the label of one of the main vertices spanning that edge; labels in the interior of T are arbitrary.

Any labelled *triangulation* of T satisfying these conditions is called a **Sperner labelling.**

The claim: Sperner's Lemma for Triangles.
Any Sperner-labelled triangulation of T must contain an odd number of elementary triangles possessing all labels. In particular, there is at least one.

Remark. An analogous statement holds in any dimension.

The n-Dimensional Sperner's lemma

We need the concept of an n-simplex:

The standard triangle is called the 2-simplex. A 3-simplex would be a tetrahedron, whereas a 1-simplex would be a line. The "triangle" mentioned in the lemma can be generalized to any *n*-simplex. The *n*-simplex is defined as follows: "An *n*-simplex is the convex hull of *n+ 1 affinely independent point*." A simple example of "affinely independent points" are three points *u, v,* and *w,* which have non-zero *x, y,* and *z* coordinates. In vector notation:

$$u = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad v = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad w = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

The convex hull of *u, v,* and *w* therefore forms a two-dimensional triangle (i.e., a 2-simplex) embedded in three-dimensional Euclidean space, as shown in Figure 2. Similarly, a 3-simplex would be a tetrahedron embedded in 4-dimensional space. Thus one can embed an *n*-simplex in *n+1*-dimensional space. We use the 2-simplex (triangle) as a running example to explain various concepts of Sperner's lemma.

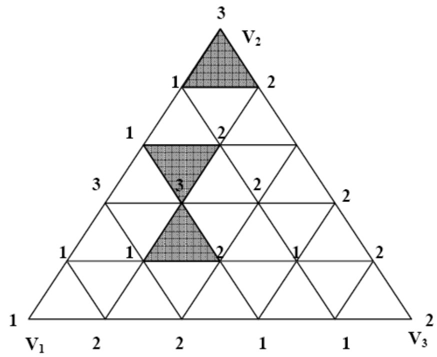

Figure 2

Triangulation

The interior of any *n*-simplex can be divided into smaller elementary *n*-simplices. The interior is to be covered exhaustively in this manner and elementary simplices should not intersect with each other except in a common face (if they are neighboring simplices). Such a set up is called the *triangulation* of the *n*-simplex. Figure 1 shows the triangulation of the 2-simplex into many smaller elementary simplices. An *n*-simplex can be built up inductively as an assembly of *n+1; n-1*-simplices, which form it's "facets." Thus a 4-simplex (tetrahedron) can be assembled from four 3-simplices as its facets. Each 3-simplex (triangle), in turn is built from three 2-simplices as its facets. Finally, each 2-simplex (line) is built from two 1-simplices (i.e., vertex points). The power of Sperner's Lemma derives from the fact that one can inductively transmit properties known to be true in lower dimensions upward to higher dimensions. Similarly, the process of triangulating an *n*-simplex automatically triangulates each *n-1*-simplex that forms its facet.

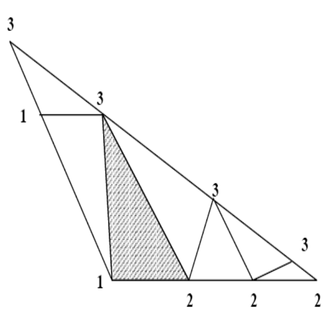

Figure 3

REMARK

Note that since Sperner's lemma is a topological result, the geometric shape of the triangles does not matter. Similarly, the elementary simplices that subdivide the main simplex can be triangles of arbitrary shape and size. Thus, any arbitrary triangulation will do. Figure 3 shows such an example, along with the associated Sperner labeling. The fully labeled elementary triangle is shaded.

Lemma 1 (Sperner's Lemma) Consider a triangulation of the simplex in R^n, whose subvertices are each labeled with one of 1, . . . , n + 1, subject to the restriction that each subvertex lying on the facet of the simplex with vertices $a_1, \ldots, a_k$ is itself labeled with one of $a_1, \ldots, a_k$. Then the number of sub simplices whose sub vertices are labeled 1, . . . , n + 1 is odd and thus nonzero.

(If counted with orientation, the number is equal to 1.)

Proof.

We proceed by induction on n. By the induction hypothesis, the face with vertices 1, . . . , n − 1 has been given a Sperner labeling and so contains an odd number of subsimplices labeled 1, . . . , n − 1. We refer to each subface labeled 1, . . . , n − 1 as a door. We have just seen that there are an odd number of doors on the boundary of the simplex. Moreover, each subsimplex contains 1 door if it is labeled 1, . . . , n, and 0 or 2 doors otherwise.

Now consider the graph on the subsimplices, which are considered adjacent if they share a common subface which is also a door. This graph is a disjoint set of paths of two types: those starting and ending on the boundary, and those starting on the boundary and ending at a subsimplex labeled 1, . . . , n. Since each path of the former type uses two boundary doors, there are an odd number of the second type, and so an odd number of subsimplices labeled 1, . . . , n.

Hence the lemma

Procedure for Locating a Fully Labeled Elementary Simplex

At least one fully labeled elementary simplex needs to be located. It is explained later in this paper how such an elementary simplex provides an approximate-fair solution to the problem of resource allocation in a multiagent system. The procedure is outlined through the example shown in Figure 1.

Formal descriptions of the procedure are provided by Kuhn (Kuhn 1968) and Cohen (Cohen 1967). The procedure for locating a fully labeled elementary triangle then can be applied as follows:

```
 get triangulated n-simplex Γ^n
for each fully labeled elementary 1-simplex τ^1 in Γ^1
      (Γ^1 is the 1-simplex labeled 1-2)
            set k to 2 (k denotes the dimension number)
            while k<n
            call compute-k-simplex with  τ^{k-1} returning τ^k
               (τ^{k-1} will have labels 1-2-…-k)
            if τ^k  is fully labeled (i.e., τ^k is labeled 1-2-…-k-k+1)
            set τ^{k-1}  to τ^k
            increment k
               else
             set τ^{k-1}  to the other facet of τ^k  with label 1-2-…-k
        endif
   endwhile
endfor
```

The procedure "*compute-k-simplex*" has the following pseudocode :

```
get k-1-simplex τ^{k-1} if τ^{k-1} lies on τ^{k-1}
           (τ^{k-1} is a facet of Γ^n and lies on its "surface")
     set τ^k  to null
endif
set τ^k  to the elementary k-simplex which has τ^{k-1} as its facet
set newVertex to the vertex in τ^k  but not in τ^{k-1}
if label of newVertex is not one of (1, 2, …, k)
     set τ^k  to null
endif
return τ^k
```

3. Applying Sperner's Lemma to Resource Allocation to n-players

(Su 1999) presented a scheme to exploit the properties of Sperner's lemma to enable resource allocation among *n* players. This result is immediately applicable to multiagent resource allocation problems by changing some of the terminology. Note that Su's solution was proposed for humans competing for **scarce resources**; one can easily modify the terms and apply it towards many problems real life. Specifically, the following assumptions made by Su are relevant to undertaken Land division and other resource distribution problems.

- People (players) are selfish.
- People (players) are autonomous.
- People (players) can misrepresent their true preferences (i.e. lying) or trying to be shrewdly diplomatic.

The resource is commonly modeled as rectangular cake which is infinitely divisible and recombinable. The resource is heterogeneous (for example: different parts of the cake have different icings) and agent preferences for these portions vary as we move along the cake. Mathematicians have worked on this problem from at least 1948, when Steinhaus (Steinhaus 1948) proposed the divide-and-choose method to apportion the cake between two people. Since then many procedures have been proposed to solve various facets of the resource allocation problem. Mathematicians have clubbed all this discussion together into the domain of "cake-cutting procedures". These procedures are tailored to fulfill various criteria that would create satisfactory portions for agents. One criterion that is commonly fulfilled by these procedures is *fairness.* A procedure is *fair* if every agent believes that it has received at least *1/n* of the total resource allocated. On the other hand, a procedure is envy-*free* if every agent believes that the portion it has received is at least as large as the largest piece allotted to any of the other agents.

Conventions and Assumptions

Before beginning a formal description of cake-cutting procedures using Sperner's lemma, we describe some conventions and assumptions. A knife – held parallel to one pair of edges of a rectangular cake – is moved slowly over it from the left edge of the cake to the right edge. The total size of the cake is 1 or 100% and the ***absolute measure*** of the i-th piece is x_i. By absolute measure we mean some metric like the "size," "area," or "length" of the cake that is universally agreed upon by agents/players as a way to determine the quantity of the resource. Thus: $x_1 + x_2 + ... + x_n = 1$ and each $x_i \geq 0$. Su's proposal therefore requires the resource to be **Lebesgue measurable**. Other assumptions are that the utility functions of the players be non-atomic, positive, and additive. Atomicity deals with the aspect that however, small a resource may be sliced, it must have some positive value for every player, i.e., there should be no "atoms" in the resource, where portions smaller than the "atom" are of zero value to the agent. We also assume that the utility function is positive at all points along the cake. This ensures that players prefer any finite-sized piece to an empty piece of the cake. Additivity for utility functions can be stated as:

$$v(A \bigcup B) = v(A) + v(B)$$

i.e., incrementally adding to a portion should increase its value by a proportional amount.

Approximate Envy-free Procedure for Multiagent Resource Allocation

The following procedure is due to Su (Su 1999). We now show how, given the assumptions and conventions about the agents, one can map the agent preferences into a correct Sperner labeling of the triangulation of an *n-1*-simplex. Once that is obtained, it will be shown how a fully labeled simplex represents an approximate envy-free allocation of the resource to various agents. An allocation is *approximate envy-free* if no agent thinks its portion **is smaller by *ε* than** the largest portion in the allocation.

The space of possible allocations with the constraint $x_1 + x_2 + ... + x_n = 1$ forms a standard *n-1*-simplex in *n*-dimensional Euclidean space. Each vertex's "ownership" is assigned to each of the *n* agents by using agent names as labels. Next, triangulate the simplex into many smaller elementary *n-1*-simplices. Label the interior vertices of the *n-1*-simplex by agent names and follow the rules of Sperner labeling. Although the labels in the interior of the *n-1*-simplex can be any of the ones on the main vertices of the simplex, the possibilities are further constrained in such a manner that every elementary simplex is completely labeled. For example, consider the case when *n=3* agents. They can be represented as the three vertices of the triangle (2-simplex). Figure 4 shows the necessary labeling of the vertices with A, B, and C being the agent names. Note that every elementary simplex carries all three labels.

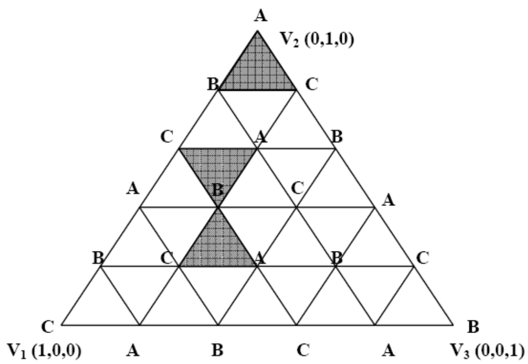

Figure 4

One can then achieve the auxiliary labeling of the elementary *n-1*-simplices in the following manner: ask the owner of each vertex which piece it would prefer out of the set of pieces of the resource corresponding to the location of the vertex. Thus for some vertex v^k, its co-ordinates will be $(v_1^k, v_2^k, \ldots, v_n^k)$ and piece *j* will be of size $x_j = v_j^k$. If the owner of the vertex, picks *j* as its preferred piece (because it views it as the largest) then that vertex is labeled *j.* Note that the auxiliary labels determined this way obeys the rules of Sperner labeling. Thus, each of the main vertices will be uniquely labeled by one of the *j=1,…,n* labels. The facets of the *n-1*-simplex would carry one of the labels of the *k* ($1 \le k \le n$) main vertices spanned by that facet. This procedure can be applicable to land distribution to successive generation through inheritance.

Figure 1 shows the example of such a Sperner labeling for *n=3* agents. By Sperner's lemma one is guaranteed that there exists at least one fully labeled elementary simplex in this triangulation. Each such elementary simplex represents an approximate envy-free allocation of portions to agents. In order to ensure that no agent thinks its portion is smaller by *ε* than the largest portion in the allocation, simply set the mesh size of the triangulation to be some small finite value, *δ>0,* such that if the absolute measure of a piece is less than *δ,* then each player values it less than *ε.*

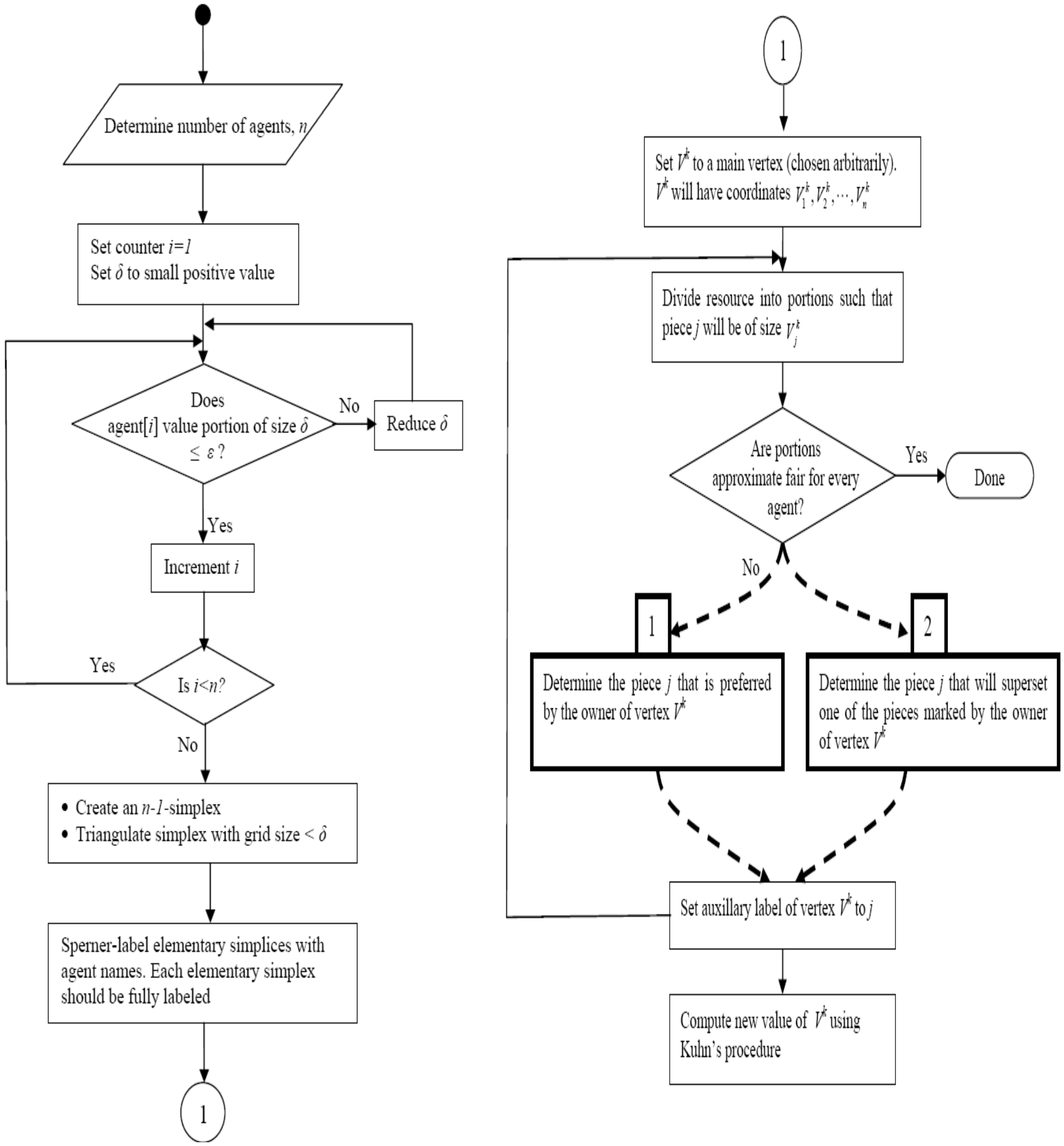
Determine number of agents, n
Set counter i=1
Set δ̂ to small positive value
Does agent[i] value portion of size δ̂ ≤ ε ?
No
Reduce δ̂
Yes
Increment i
Yes
Is i<n?
No
• Create an n-1-simplex
• Triangulate simplex with grid size < δ̂
Sperner-label elementary simplices with agent names. Each elementary simplex should be fully labeled
1
1
Set V^k to a main vertex (chosen arbitrarily). V^k will have coordinates V_1^k, V_2^k, ⋯, V_n^k
Divide resource into portions such that piece j will be of size V_j^k
Are portions approximate fair for every agent?
Yes
Done
No
1
2
Determine the piece j that is preferred by the owner of vertex V^k
Determine the piece j that will superset one of the pieces marked by the owner of vertex V^k
Set auxillary label of vertex V^k to j
Compute new value of V^k using Kuhn's procedure

Figure 6

Conclusion

In order to see the application of Sperner's lemma, we take the following two popular examples.

Example 1 Grandfather has 90 acres of land and has three sons. First son has 2 sons, second one has 3 sons and the third one has 4 sons. Using Sperner's lemma the third generation would have the following equitable distribution. At each generation nodes the choices would be given to the next generation that would be approximate envy free depending upon soil quality, crops grown, proximity to water source and infrastructure facilities.

Example 2 A just and caring [abstract] government wants to distribute 1,000,000 acres of land to 20,000 unemployed youths in a country, the provision for social amenities like schools, heath centers, social halls, players places can be curved out using Sperner's' lemma.

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