

ON INVARIANCE OF THE NUMERICAL RANGE
AND SOME CLASSES OF OPERATORS IN
HILBERT SPACES

By

Arthur Wanyonyi Wafula

A thesis submitted in fulfillment of the requirements for the award of the
Degree of Doctor of Philosophy in Pure Mathematics

School of Mathematics, University of Nairobi

15 April 2013

University of NAIROBI Library

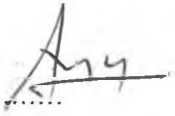


0430140 4

GRADUATE RESEARCH LIBRARY
UNIVERSITY OF NAIROBI

0.1 DECLARATION


This thesis is my original work and has not been presented for a degree in any other University

Signature.....

Date.....18/05/2012

Arthur Wanyonyi Wanambisi Wafula

This thesis has been submitted for examination with our approval as University supervisors;

Signature  Date 18/5/2012

Prof. J.M. Khalagai

Signature..... Date 18/5/2012

Prof. G.P. Pokhariyal

0.2 DEDICATION

This work is dedicated to my family ; namely, my wife Anne Atieno and my sons Davies Magudha and Mishael Makali.

0.3 ACKNOWLEDGEMENT

My humble acknowledgement go to the Almighty God Jehovah who endows us with the gift of the Human brain that enables us to be creative and imaginative.

I would also like to express my sincere gratitude to my Supervisor **Prof.J.M.Khalagai** for tirelessly guiding me in my research.I also wish to thank **Prof.G.P.Pokhariyal** for never ceasing to put me on my toes.

I sincerely wish to thank the Director of the School of Mathematics **DrJ.Were** for his invaluable support.

Furthermore,I am indebted to my colleagues Mrs.E.Muriuki, Dr.Mile, Dr.I.Kipchichir, Mr R.Ogik and Dr.N.Owuor who have been a source of inspiration.

I give special thanks to my family who have been with me and provided the emotional support. I thank my sister Eunice Kimaliro and my brothers Ken Walibora and Patrick Lumumba for particular interest they took in this work.

Finally, I acknowledge the University of Nairobi for giving me an opportunity to pursue my Ph.D. studies.

0.4 ABSTRACT

In this thesis the invariance of the numerical range under isometric similarity is examined. It is a well known fact that unitarily similar operators have the same numerical range.

Chapter one is devoted to basic definitions and some well known results on the numerical range.

In chapter two , some conditions are examined for operators that are isometrically equivalent to have the same numerical range .

In chapter three the norm properties of an operator with a norm attaining vector are examined. It is found that such an operator satisfies the generalized Daugavet equation.

Chapter four draws some conditions for two operator's to commute up to a scalar factor.

Also in this chapter the numerical range of a Hyponormal operator in finite dimensional spaces is shown to be polygonal in the complex plane. The vertices of the polygon are the eigenvalues of the operator.

0.5 LIST OF NOTATIONS

H : Complex Hilbert space

$B(H)$: Banach algebra of Linear operators on H

T : Linear operator

$\|T\|$: Norm of T

$W(T)$: Numerical range of T

$w(T)$: Numerical radius

$\sigma(T)$: Spectrum of T

$R_\sigma(T)$: Residual spectrum of T

$P_\sigma(T)$: Point spectrum of T

$A_\sigma(T)$: Approximate point spectrum

$r(T)$: Spectral radius

T^* : Adjoint of the operator T

Contents

0.1	DECLARATION	i
0.2	DEDICATION	ii
0.3	ACKNOWLEDGEMENT	iii
0.4	ABSTRACT	iv
0.5	LIST OF NOTATIONS	v
1	INTRODUCTION AND PRELIMINARIES	1
1.1	Introduction	1
1.2	Basic Notation and Definitions	5
1.3	Properties of the numerical range	7
1.4	The norm of an operator	7
1.5	Some classes of operators	8
1.6	The convex nature of numerical range	11
1.7	A note on projections	13
1.8	The spectrum of an operator	14
2	ON INVARIANCE OF THE NUMERICAL RANGE	17

2.1	Isometrically equivalent operators	17
2.2	On partial isometries	20
3	NORM PROPERTIES OF OPERATORS WHOSE NORMS ARE EIGEN- VALUES AND THEIR NUMERICAL RANGES	24
3.1	Norm attaining operators	25
3.2	On the bounds of a selfadjoint operator	34
3.3	On norm attaining operators and invariance of numerical range	39
4	THE NUMERICAL RANGE OF SOME CLASSES OF OPERATORS	41
4.1	Operators with a thin numerical range	41
4.2	λ - commuting operators	48
4.3	U-self adjoint operators	51
5	SUMMARY OF MAIN RESULTS	53
5.1	REFERENCES	56

Chapter 1

INTRODUCTION AND PRELIMINARIES

In this Chapter some well known results on classes of operators and the numerical range are reviewed. This provides a background for the proceeding development. No credit is due to the author for the results in this chapter. However, it does provide a necessary background on which subsequent chapters depend.

1.1 Introduction

A Hilbert space H is a linear space on which is defined a function (x, y) , called an inner product, that satisfies the following conditions :

- (i) $(x, y) = \overline{(y, x)}$ i.e. the inner product is conjugate symmetric in x and y .
- (ii) $(ax_1 + bx_2, y) = a(x_1, y) + b(x_2, y)$ i.e the inner product is linear in the first variable.

(iii) $(x, x) \geq 0$ i.e the inner product is a positive definite function .

The inner product establishes a geometry on a linear space quite similar to that of the Euclidean spaces .Inner product spaces are a generalization of Euclidean spaces to infinite dimensional spaces.

We thus define the norm of a vector by

$$\|x\| = (x, x)^{1/2}$$

and the distance between two vectors x and y in H to be

$$d(x, y) = \|x - y\|.$$

An operator A on a Hilbert space H is a mapping of H into H such that

$$A(\alpha x + \beta y) = \alpha A(x) + \beta A(y) \text{ for all points } x \text{ and } y \text{ in } H.$$

An operator A is said to be bounded if there exists a constant M such that $\|Ax\| \leq M\|x\|$

.The least constant for which this inequality holds is called the norm of the operator .

Hilbert spaces and operators were introduced in the early parts of the twentieth century.

They arose from the works of David Hilbert and Erhard Schmidt in the study of integral

equations. The mathematician Von Neuman coined the phrase 'abstract Hilbert spaces '

in his work on foundations of quantum mechanics and gave the first axiomatic treatment

of these spaces in 1929. The significance of Hilbert spaces came to the fore when it was

realized that they provided the best mathematical formulation of quantum mechanics.

Briefly the states in a quantum mechanical system are vector points in a Hilbert space

,the observable are self adjoint operators and the symmetries in the system are unitary

operators.

The numerical range of an operator T is denoted by $W(T)$ and is defined as the range of complex numbers (Tx, x) where x ranges among the unit vectors of a Hilbert space. On the other hand, the spectrum of an operator $\sigma(T)$ is the range of values λ for which $T - \lambda I$ is not invertible.

The numerical range of an operator has been an area of intense research. The motivation for the development arose from the classical theory of quadratic forms. The numerical range of an operator like the spectrum is a subset of the complex numbers, whose topological properties yield some vital information about the operator.

It is a well known fact that the numerical range of an operator is a subset of the real numbers if and only if the operator is self adjoint. However, if the spectrum of an operator is contained among the real numbers we can draw no meaningful conclusion about the operator. From the manner in which the numerical range is defined it bears both the algebraic and norm properties of an operator. For example, we have that $\sigma(T) \subset cl(W(T))$ and $\|T\| \leq 2w(T)$ are both algebraic and norm properties of the operator T .

The definition of the numerical range was first introduced by Toeplitz on finite dimensional spaces in 1918 [22]. This was motivated by the classical theory of quadratic forms. It became evident that the eigenvalues of an operator T on a finite dimensional Hilbert space are contained in the numerical range. Toeplitz [22] proved that the complement of $W(T)$ has a convex curve for its boundary. Then Hausdorff [14] proved that $W(T)$ is convex for finite dimensional Hilbert spaces. Finally Stone 1932 [21] proved

$W(T)$ is convex on arbitrary pre-Hilbert spaces.

In a more recent period of time several authors have written on the numerical range and its application to various problems. For example, in 1962 Berger proved the power inequality $w(T^n) \leq (w(T))^n$. This was a remarkable achievement as $w(AB) \leq w(A)w(B)$ is generally not true.

Some studies have examined the implication of the origin belonging to the numerical range or even on the boundary of the numerical range. For example Embry in 1970[9], analyzed the equation $AH = KA$ and obtained the result that if H and K are commuting normal operators then $H = K$ so long as zero does not belong to $W(A)$. In a study of partial isometries and the conditions of invertibility Khalaghai 2000[13] showed that if T is a partial isometry and zero is not in $W(T)$ or $W(T^2)$ then T is unitary. Bridges 2000[13] argued that the proof that $W(T)$ is convex as given in Halmos breaks down constructively. He then points out the flaws and reconstructs a new proof. Also, Shapiro in 2004[18] revisited the Toeplitz-Hausdorff theorem by first proving the result in the two dimension case then extended to general spaces through projections.

In 2003, Kittenah [15] revisited the power inequality of the numerical radius of an operator and proved that $w(T) \leq 1/2(\|T\| + \|T^2\|^{1/2})$. Then, in 2008 Albadawi[1] proved that $w(T) \leq 1/2(\|A\|^2 + \|A\|^{r/2})$ for values of r such that, $1 \leq r \leq 2$.

1.2 Basic Notation and Definitions

We employ capital letters A, B, T, K e.t.c for an operator on a Hilbert Space H . The symbol $B(H)$ will stand for the algebra of all bounded linear operators on the Hilbert space H .

Given an operator A on a Hilbert space H we denote by $W(A)$ the numerical range of A , where $W(A) = \{(Ax, x); \|x\| = 1\}$. The spectrum of an Operator T is denoted by $\sigma(T)$ and is the range of values λ for which $T - \lambda I$ is singular. We write $\|A\|, w(A), r(A)$ for the norm, numerical radius and spectral radius of A where $\|A\| = \sup\{\|Tx\|; \|x\| = 1\}$, $w(A) = \sup\{|\lambda|; \lambda \in W(T)\}$ and $r(A) = \sup\{|\lambda|; \lambda \in \sigma(A)\}$.

We say that T is a quasi-affinity if T is injective and has a dense range

An operator T is a quasi-affine transform of another operator B if there exists a quasi-affinity X such that $XA = BX$ and the operators A and B are quasi similar if they are quasi-affine transforms of each other.

An operator T is said to be:

Unitary if $T^*T = TT^* = I$,

Quasi-unitary if $TT^* = T^*T = T^* + T$,

An Isometry if $T^*T = I$,

A Co-Isometry if $TT^* = I$,

A Partial isometry if $T = TT^*T$,

A Contraction if $\|T\| \leq 1$,

A Normal operator if $T^*T = TT^*$,

A Quasi-normal operator if $T(T^*T) = (T^*T)T$,

A Subnormal operator if H has a normal extension i.e there exists a normal operator B on a Hilbert K such that $H \subset K$ and $Bx = Tx$, for all x belonging to H ,

A self adjoint operator if $T = T^*$,

A U -self adjoint operator if there exist a unitary operator U such that $UTU^* = T^*$,

A Positive operator if $(Tx, x) \geq 0$,

A Projection if $T^2 = T, T^* = T$,

The norm of an operator is defined by;

$$\|T\|^2 = \sup\{(Tx, Tx); \|x\| = 1\}$$

The numerical radius of an operator T is given by;

$$w(T) = \sup\{|\lambda|; \lambda \in W(T)\}$$

The spectrum of T is the range of values σ for which $T - \lambda I$ is not invertible

The spectral radius is given by; $r(T) = \sup\{|\lambda| : \lambda \in \sigma(T)\}$

A unit vector x is said to be a complete vector for an operator T if (Tx, x) is a real number such that $\|T\| = (Ix, x) = \|Tx\|$.

The following inclusions hold for some classes of operators:

Self adjoint \subset Normal \subset Quasi-normal \subset subnormal \subset Hyponormal

Anti-self adjoint \subset Normal \subset Quasi-normal \subset subnormal \subset Hyponormal

Also we have that

Unitary \subset Isometry \subset Partial isometry \subset contraction.

1.3 Properties of the numerical range

The following theorem provides well known properties of the numerical range.

Theorem 1.3.1 Shapiro,2004[19]

For all operators T on Hilbertspace H we have that

- (i) $W(\alpha I + \beta T) = \alpha + \beta W(T)$ for all $\alpha, \beta \in \mathbb{C}$
- (ii) $W(T^*) = \{\bar{\lambda} : \lambda \in W(T)\}$
- (iii) $w(T) \leq \|T\|$
- (iv) $|(Tx, x)| \leq w(T)\|x\|^2$
- (v) $W(U^*TU) = W(T)$ where U is a unitary operator

1.4 The norm of an operator

We now take note of the following often used properties of the norm of an operator

Theorem 1.4.1 Furuta 2002[11]

$$\|T\| = \sup\{\|Tx\| : \|x\| = 1\} = \sup\{\|Tx\| : \|x\| \leq 1\} = \sup\{|(Tx, y)| : \|x\| \leq 1, \|y\| \leq 1\}.$$

See Furuta, 2002[8].

Theorem 1.4.2 Furuta 2002,[11]

$$\|T\| = \|T^*\| \text{ for all operators } T \text{ on a Hilbert space } H.$$

Proof

Let x and y be contained in the unit disc then

$$|(Tx, y)| = |(x, T^*y)| \leq \|x\| \|T^*y\| \leq \|x\| \|T^*\| \|y\| \leq \|T^*\| \|x\| \|y\|.$$

When the supremum on the left hand side is taken we obtain the result $\|T\| \leq \|T^*\|$.

From this result we have

$$\|T^*\| \leq \|(T^*)^*\| = \|T\|$$

Theorem 1.4.3 Shilov, 1965 [20]

If T is any operator on a Hilbert space H then $\|T^*T\| = \|TT^*\| = \|T\|^2$

Now we can introduce a well known result that gives bounds on the numerical radius and the norm of an operator

Theorem 1.4.4[11] .

For all operators T we have that $\frac{\|T\|}{2} \leq w(T) \leq \|T\|$.

Theorem 1.4.5[11]

$\|AB\| \leq \|A\| \|B\|$ for all operators A and B on H .

1.5 Some classes of operators

In this section we review some results on self adjoint (Hermitian) operators. First, we need the following theorem which holds for general operators.

Theorem 1.5.1[3]

Let T be an operator on H , a complex Hilbert space, then if for all x in H we have

$(Tx, x) = 0$ then $T = \bar{0}$

corollary 1.5.2[3]

If $(T_1x, x) = (T_2x, x)$ for all x in H then $T_1 = T_2$.

Theorem 1.5.3[3]

$T^* = T$ if and only if $W(T) \subset \mathbb{R}$.

Theorem 1.5.4[19]

(i) If T_1 and T_2 are self adjoint operators then so are $\alpha T_1 + \beta T_2$ for all real values α, β .

(ii) If T_1 and T_2 are self adjoint operators then T_1T_2 is self adjoint if and only if $T_1T_2 = T_2T_1$.

Theorem 1.5.5[11].

If T is self adjoint then so are T^2, T^3, T^4, \dots self adjoint.

Theorem 1.5.6 [3]

If $T \geq 0$ then $\|Tx\|^2 \leq w(T)(Tx, x)$ for all x in H .

Theorem 1.5.7[11]

If T is a nonzero projection then $\|T\| = 1$.

Proof

Let $\lambda \in W(T)$ then for some unit vector x in H ;

$$\begin{aligned}\lambda &= (Tx, x) \\ &= (T^2x, x) \\ &= (Tx, T^*x) \\ &= (Tx, Tx) \\ &= \|Tx\|^2\end{aligned}\tag{1.1}$$

Thus we have that $w(T) \geq \|Tx\|^2$.

On taking supremum on the right hand side we have that $w(T) \geq \|T\|^2$.

However ,

$$\begin{aligned}\lambda &= (Tx, x) \\ &= (T^2x, x) \\ &\leq w(T^2) \\ &= \|T^2\| \\ &= \|T\|^2\end{aligned}\tag{1.2}$$

Again taking the supremum on the left hand side we obtain $w(T) \leq \|T\|^2$. Hence we have that $\|T\| = w(T) = \|T\|^2$ and so $\|T\|(1 - \|T\|) = 0$. Since T is a non zero operator we must have that $\|T\| = 1$.

1.6 The convex nature of numerical range

In this section we consider the Toeplitz-Hausdorff theorem that the numerical range is convex. We first provide a proof that the numerical range of a two by two Matrix is convex then, following the approach made by Shapiro ,we end with the general operator.

Lemma 1.6.1 [3]

If T is a two by two matrix with $\text{trace}(T) = 0$ then the eigen-values of T are of the form $a, -a$ where a is a complex number.

Proof. Let λ_1, λ_2 be eigenvalues of T where

$$B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

with $a + d = 0$ then the eigenvalues satisfy the equation:

$$\det(T - \lambda I) = 0 \Leftrightarrow (a - \lambda)(d - \lambda) - bc = 0 \Leftrightarrow \lambda^2 - (a + d)\lambda + ad - bc = 0.$$

Hence from the properties of roots of a quadratic equation we have that:

$$\lambda_1 + \lambda_2 = a + d = 0.$$

We will also require the following result called Schur's theorem;

Theorem 1.6.2 [17]

Any square matrix A is unitarily similar to an upper triangular matrix T .

Theorem 1.6.3[19]

Let T be a two by two matrix with eigenvalues λ_1 and λ_2 . The numerical range of T is

an elliptical disc with λ_1 and λ_2 as its foci .

Lemma 1.6.4 [19]

The numerical range of an operator T on H contains the numerical ranges of all its compressions.

Theorem 1.6.6[3]

If T is an operator on H then $W(T)$ is a connected subset of the complex plane.

Example;

Let $H = l^2$ the space of all square summable sequences and the operator T be defined by the matrix below

$$\begin{bmatrix} 1 & 0 & 0 & \dots \\ 0 & \frac{1}{2^r} & 0 & \dots \\ 0 & 0 & \frac{1}{3^r} & \dots \\ 0 & 0 & 0 & \dots \end{bmatrix}$$

where $r \geq 1$.

We observe that when x is a unit vector in H then

$$\|x\|^2 = (|x_1|^2 + |x_2|^2 + |x_3|^2 + \dots) = 1$$

Then

$$(Tx, x) = (|x_1|^2 + \frac{1}{2^r}|x_2|^2 + \frac{1}{3^r}|x_3|^2 + \dots) \leq (|x_1|^2 + |x_2|^2 + |x_3|^2 + \dots) = 1.$$

Note that $W(T) \subset \mathbb{R}$. Hence T is a self adjoint operator. Now set $x = (1, 0, 0, 0, \dots)$ then

x is a unit vector in l^2 and so $(Tx, x) = 1$ belongs to $W(T)$. Consequently $w(T) = 1$.

Further more, given $\epsilon > 0$ we have that there exists a unit vector x in H , such that

$(Tx, x) < \epsilon$. To see this choose n such that $\frac{1}{n} < \epsilon$, then with $x = (0, 0, 0, \dots, 0, 1, 0, 0, \dots)$,

where the one appears in the n -th position. We have that

$(Tx, x) = \frac{1}{n} < \epsilon$. By the convexity of the numerical range we have that $W(T) = (0, 1]$.

1.7 A note on projections

We now proceed to make some observations on projection operators .

Theorem 1.7.1

If T is a projection then $I - T$ is a projection.

Theorem 1.7.2 Khalagai 2000,[13]

If T is a contraction and T^2 is a partial isometry then $T(T^*)^2T$ is a projection.

For any two operators A and B the commutator of A and B is the operator $[A, B] = AB - BA$.

Definition 1.7.3 An operator T is said to be bi-normal if the commutator of T^*T and TT^* is zero i.e. $[T^*T, TT^*] = 0$.

Following the previous result Khalagai made an observation in this direction with different assumptions.

Theorem 1.7.4 Khalagai 2000,[13]

If a partial Isometry T is also a bi-normal operator then $T(T^*)^2T$ is a projection.

1.8 The spectrum of an operator

To each operator T we associate a compact subset $\sigma(T)$ called the spectrum of T . The spectrum $\sigma(T)$ is defined as the set of complex numbers λ for which $T - \lambda I$ is not invertible. We can identify the following categories that lead to $T - \lambda I$ to be singular;

(i) $T - \lambda I$ is not one to one. Thus $N(T - \lambda I) \neq \{0\}$. Hence there exists a nonzero vector x in H such that $(T - \lambda I)x = 0 \Leftrightarrow Tx = \lambda x$. We note in this λ is an eigenvalue of T .

The set of eigen-values is called the point spectrum of T and is denoted by $P_\sigma(T)$.

(ii) $T - \lambda I$ is one to one but is not onto .

Equivalently, $(T - \lambda I)H \neq H$ and $\text{closure}((T - \lambda I)H) = H$. The set of all such values of λ is called the continuous spectrum of T and is denoted by $C_\sigma(T)$.

(iii) $T - \lambda I$ is one to one but is not onto .

Equivalently, $(T - \lambda I)H \neq H$ and $\text{closure}((T - \lambda I)H) \neq H$. The set of such values of λ is called the residual spectrum of T and is denoted by $R_\sigma(T)$.

From the description we obtain the spectrum as

$$\sigma(T) = P_\sigma(T) \cup C_\sigma(T) \cup R_\sigma(T),$$

where the sets are pairwise disjoint.

Theorem 1.8.1[19]

If T is normal and $\lambda \in P_\sigma(T)$, then $\bar{\lambda} \in R_\sigma(T^*)$ and $N(T - \lambda I) = N(T^* - \bar{\lambda} I)$.

Theorem 1.8.2[19]

If $\lambda \in R_\sigma(T)$ then $\bar{\lambda} \in P_\sigma(T^*)$.

Corollary 1.8.3 [19]

If T is normal then $R_\sigma(T) = \Phi$.

A complex number λ is called an approximate proper value for T if there exists a sequence x_n of unit vectors in H Such that $(T - \lambda I)x_n \rightarrow 0$. The approximate spectrum $A_\sigma(T)$ is the set approximate proper values. Now the spectrum is made up of a disjoint union of the residual spectrum and the approximate spectrum that is, $\sigma(T) = A_\sigma(T) \cup \sigma(T)$.

Theorem 1.8.4[19]

$$\sigma(T) \subset \bar{W}(T)$$

Definition 1.8.5

The intersection of all convex sets of X that contain a subset K is called the convex hull of K and is denoted by $\text{conv}K$.

Theorem 1.8.6[19]

Let T be an operator on H , then $\text{conv}\sigma(T) \subset \bar{W}(T)$.

Definition 1.8.7

An operator T on H is said to be a translation invariant normaloid operator if $\|T + \lambda I\| = r(T + \lambda I)$

Theorem 1.8.8[19]

If T is invariant normaloid then $\text{conv}\sigma(T) = \bar{W}(T)$

We conclude this section by recalling the following often used result in the subsequent work.

Theorem 1.8.9 (Putnam-Fuglede, 1950)[10]

(i) If T is a normal operator and T commutes with any operator S then $T^*S = ST^*$

(ii) If A and B are normal operators such that $TA = BT$ then $TA^* = B^*T$, for all operators T on H .

Chapter 2

ON INVARIANCE OF THE NUMERICAL RANGE

In this chapter we compare the numerical ranges of isometrically similar operators and determine that under some conditions $W(A^*TA) = W(T)$.

2.1 Isometrically equivalent operators

It is well known that $W(A^*TA) = W(T)$, for all cases where A is a unitary operator.

However, it is not generally true that $W(A^*TA) = W(T)$ for all isometries A . Yet if we set some restrictions on A which we will now consider the result holds.

For example let A be a right shift operator, then A is an isometry since

$$(Ax, Ax) = ((0, x_1, x_2, \dots), (0, x_1, x_2, \dots)) = \|x\|^2.$$

Now for any unit vector e orthogonal to the range of A define $Tx = (x, e)$.

Then $TAx = (T(Ax), e) = \vec{0}$. Hence $A^*TA = \vec{0}$ and that means that $W(A^*TA) = W(\vec{0}) = \{0\}$.

But T is an orthogonal projection whose spectrum is $\{0, 1\}$ and hence, its numerical range is $[0, 1]$.

Theorem 2.1.1[23]

Let T be an operator in $B(H)$ and A an isometry then $W(A^*TA) \subset W(T)$.

Proof

Let $\lambda \in W(A^*TA)$, then we have :

$$\begin{aligned}\lambda &= (A^*TAx, x) \text{ for some unit vector } x \\ &= (TAx, Ax)\end{aligned}\tag{2.1}$$

. However,

$$\begin{aligned}\|Ax\|^2 &= (Ax, Ax) \\ &= (x, A^*Ax) \\ &= (x, x) = 1.\end{aligned}\tag{2.2}$$

Hence, λ belongs to $W(T)$.

Lemma 2.1.2 [23]

Let A be an isometry such that each unit vector is of the form Ax for some x in H then x must be a unit vector.

Proof

Let a unit vector y be of the form Ax then

$$\begin{aligned}(y, y) &= (Ax, Ax) \\ &= (x, A^*Ax) \\ &= (x, x) = 1\end{aligned}\tag{2.3}$$

Theorem 2.1.3[23]

If A is an Isometry and every unit vector of H is of the form Ax then $W(A^*TA) = W(T)$ for any bounded operators T on H .

Proof

Suppose that λ belongs to $W(T)$ then,

$$\begin{aligned}\lambda &= (Tx, x) \text{ for some unit vector } x \text{ in } H \\ &= (TAy, Ay) \text{ for some unit vector } y \text{ in } H \\ &= (A^*TAy, y) \in W(A^*TA).\end{aligned}\tag{2.4}$$

Thus, we obtain the result

$$W(T) \subset W(A^*TA).$$

Hence, from the previous theorem we have $W(T) = W(A^*TA)$.

Corollary 2.1.4[23]

If A is a Co-Isometry such that each unit vector is of the form Ax for some x in H then for any bounded operator T we have that $W(ATA^*) = W(T)$.

2.2 On partial isometries

Theorem 2.2.1 [23]

If a partial isometry A is either injective or has a dense range and each unit vector is of the form Ax then

$$W(A^*TA) = W(T) \text{ or } W(T) = W(ATA^*),$$

for all bounded operators T .

Proof

If A is a partial isometry then we have

$$A = AA^*A \Leftrightarrow A - AA^*A = 0 \Leftrightarrow A(I - A^*A) = 0 \Leftrightarrow (I - AA^*) = 0.$$

If A is injective then,

$$I - A^*A = 0 \Rightarrow A^*A = I.$$

Thus A is an isometry.

If A has a dense range then

$$I - AA^* = 0 \Rightarrow AA^* = I.$$

Thus A is a co-isometry. The result then follows from the previous corollary.

Remark 2.2.2

We note that in general the existence of a quasi-affine inverse does not imply invertibility of an operator A . However, for a partial isometry A , having a quasi-affine inverse implies that A is invertible and so is unitary (see [9]). In this case the following corollary to our main result is immediate .

Corollary 2.2.3[23]

If A is a partial isometry which has quasi-affine inverse then for any bounded operator T on H we have that $W(A^*TA) = W(T)$.

Proof

Since A has a quasi-affine inverse A is unitary .Hence $W(A^*TA) = W(T)$.

The following theorem on partial isometries proved by Khalagai will be needed in our next results.

Theorem 2.2.4[23]

Let A be a partial isometry on H .Then A is unitary under any of the following conditions:

- (i) A is a quasi-affinity.
- (ii) A^2 is a quasi-affinity.

Proof

(i) Since A is a partial isometry ,we have that $A = AA^*A$.

Thus $A - AA^*A = \bar{0}$. So

$$A(I - A^*A) = (I - AA^*)A = \bar{0}.$$

Since A is a quasiaffinity we obtain $A^*A = AA^* = I$. Hence, A is unitary .

(ii) Similarly using the fact that $A = AA^*A$ we have

$$A^2 = A^2A^*A = AA^*A^2$$

or equivalently,

$$A^2 - A^2A^*A = A^2 - AA^*A^2 = \bar{0}.$$

Consequently, we have

$$A^2(I - A^*A) = (I - AA^*)A^2 = \bar{0}.$$

Since A^2 is a quasi-affinity we have

$$I - A^*A = I - AA^* = \bar{0} \text{ i.e. } A^*A = AA^* = I.$$

Hence A is unitary.

The significance of this result is that for a partial isometry the concept of quasi-affinity is equivalent to invertibility. A more general theorem can be stated as follows;

Theorem 2.2.5[23]

Let T be a partial isometry on H . Then T is unitary if any of the following is satisfied:

- (i) T is a quasi-affinity.
- (ii) T^n is a quasi-affinity for some positive integer n .

Proof

(i) already proved in theorem 2.2.4

(ii) From $T = TT^*T$ by multiplying T ($n - 1$) times on the left and on the right we have

$$T^n = TT^*T^n = T^nT^*T.$$

Chapter 3

NORM PROPERTIES OF OPERATORS WHOSE NORMS ARE EIGEN-VALUES AND THEIR NUMERICAL RANGES

In this chapter we extend results obtained by Lin [2002][16] on properties of operators whose norms are eigen-values.

It is noteworthy that each compact operator on a Hilbert space has a norm attaining property(see [20]).Also, as noted by Donogue[7] if A is a normal operator and $W(A)$ is closed , then the extreme points of $W(A)$ are eigen values.

3.1 Norm attaining operators

The following remarkable result due to Lin[16] was a starting point in a paper he wrote on bounded operators whose norms are an eigen value. We provide a detailed alternative proof and proceed to make application of the lemma.

Lemma3.1.1 Lin,2002[16]

For a unit vector x the following statements are equivalent:

- (1) $\|A\| + \|B\|$ is an eigen value of $A + B$;
- (2) $1 + \|A\| + \|B\| = \|(I + A + B)x\| = ((I + A + B)x, x)$;
- (3) $\|A\|$ and $\|B\|$ are eigen values of A and B respectively;
- (4) $\|A\|$ and $\|B\|$ are in the numerical range of A and B ;
- (5) x is a complete vector for A and B respectively;
- (6) $\|A\| + \|B\|$ is in the numerical range of $A + B$;
- (7) $Ax = \|Ax\| = \|Ax\|x$ and $Bx = \|B\|x = \|Bx\|x$;

Proof

(i)(5 \rightarrow 3)

If $\|A\| = \|Ax\| = (x, Ax)$, then

$$\begin{aligned} & \|(\|A\|x - Ax)\|^2 = (\|A\|x - Ax, \|A\|x - Ax) \\ & = \|A\|^2(x, x) - \|A\|A\|(x, Ax) - \|A\|(Ax, x) + \|Ax\|^2 \end{aligned}$$

$$= \|A\|^2 - 2\|A\|^2 + \|A\|^2 = 0.$$

Hence, $Ax = \|A\|x$.

Similarly if x is a complete vector for B , then $\|B\|$ is an eigen value for B .

(ii) (3 \rightarrow 4).

$\|A\|$ an eigen-value for A corresponding to x implies that $Ax = \|A\|x$. Hence $(Ax, x) =$

$$(\|A\|x, x) = \|A\|.$$

Therefore $\|A\|$ is in the numerical range of A . A Similar argument shows that if $\|B\|$ is

an eigen-value for B then $\|B\|$ is in the numerical range of B .

(iii) (4 \rightarrow 7)

. Let $\|A\| = (Ax, x)$ then

$$\|Ax\| \leq \|A\| = (Ax, x) = |(Ax, x)| \leq \|Ax\| \|x\| = \|Ax\|.$$

Consequently, $\|A\| = (Ax, x) = \|Ax\|$.

Also,

$$\begin{aligned} \|Ax - \|A\|x\|^2 &= \|Ax\|^2 - \|A\|(Ax, x) - \|A\|(x, Ax) + \|A\|^2 \\ &= \|A\|^2 - 2\|A\|^2 + \|A\|^2 \\ &= 0 \end{aligned} \tag{3.1}$$

So we have that $\|Ax\| = \|A\|x$.

In the same way if $\|B\| = (Bx, x)$, then $Bx = \|B\|x = \|Bx\|x$.

(iv) (7 \rightarrow 5) .

Suppose that $Ax = \|Ax\|x = \|A\|x$ and $Bx = \|B\|x = \|Bx\|x$, then

$$(Ax, x) = \|Ax\|(x, x) = \|Ax\|$$

$$\text{Also, } (Ax, x) = \|A\|(x, x) = \|A\|.$$

$$\text{Hence } (Ax, x) = \|Ax\| = \|A\|.$$

Similarly,

$$(Bx, x) = \|Bx\| = \|B\|.$$

We move into another loop in the circular proof as follows

(v) $(1 \rightarrow 2)$. Let $(A + B)x = (\|A\| + \|B\|)x$ then consider the expansion below:

$$\begin{aligned} \|(I + A + B)x\|^2 &= ((x + Ax + Bx), (x + Ax + Bx)) \\ &= (x, x) + (x, (A + B)x) + ((A + B)x, x) + ((A + B)x, (A + B)x) \\ &= 1 + (\|A\| + \|B\|)(x, x) + (\|A\| + \|B\|)(x, x) + (\|A\| + \|B\|)^2(x, x) \\ &= 1 + 2(\|A\| + \|B\|) + (\|A\| + \|B\|)^2 \\ &= (1 + \|A\| + \|B\|)^2. \end{aligned}$$

The result follows on taking square roots.

$(2 \rightarrow 6)$ Let $1 + \|A\| + \|B\| = \|(I + A + B)x\| = ((I + A + B)x, x)$ then we have

$$\begin{aligned} \|(\|A\| + \|B\|)x - (A + B)x\|^2 &= \|(1 + \|A\| + \|B\|)x - (I + A + B)x\|^2 \\ &= ((1 + \|A\| + \|B\|)x - (I + A + B)x, (1 + \|A\| + \|B\|)x - (I + A + B)x) \\ &= (1 + \|A\| + \|B\|)^2 - ((I + A + B)x, x) - (x, (I + A + B)x) + \|(I + A + B)x\|^2 \\ &= (1 + \|A\| + \|B\|)^2 - 2(1 + \|A\| + \|B\|) + (\|1 + \|A\| + \|B\|)^2 \\ &= 0. \end{aligned}$$

(vi) $(6 \rightarrow 1)$ Now let $\|A\| + \|B\| = ((A + B)x, x)$ then note that

$$\begin{aligned}
\|A\| + \|B\| &= ((A + B)x, x) \\
&= |((A + B)x, x)| \\
&\leq \|(A + B)x\| \|x\| \leq \|A + B\| \|x\|^2 = \|A\| + \|B\|.
\end{aligned}$$

Hence $\|A\| + \|B\| = \|(A + B)x\|$.

Consequently,

$$\begin{aligned}
&\|(\|A\| + \|B\|)x - (A + B)x\|^2 = ((\|A\| + \|B\|)x - (A + B)x, (\|A\| + \|B\|)x - (A + B)x) \\
&= (\|A\| + \|B\|)^2 - (\|A\| + \|B\|)(x, (A + B)x) - (\|A\| + \|B\|)((A + B)x, x) + \|(A + B)x\|^2 \\
&= (\|A\| + \|B\|)^2 - 2(\|A\| + \|B\|)^2 + (\|A\| + \|B\|)^2 \\
&= 0.
\end{aligned}$$

(vii) 5 \rightarrow 6 Let x be a complete vector for both A and B then we have that

$$\|A\| = (Ax, x) = \|Ax\| \text{ and } \|B\| = (Bx, x) = \|Bx\|.$$

Hence,

$$(Ax, x) + (Bx, x) = \|A\| + \|B\| = ((A + B)x, x).$$

So $\|A\| + \|B\|$ is in the numerical range of $A + B$.

(viii) 6 \rightarrow 54 Let $\|A\| + \|B\| = ((A + B)x, x)$ then we have that

$$\|A\| + \|B\| = ((A + B)x, x) = |((A + B)x, x)| \leq \|A + B\| \leq \|A\| + \|B\|.$$

Thus $\|A\| + \|B\| = \|A + B\|$.

Hence there a positive number r such that $A = rB$. Hence from $\|A\| + \|rB\| = ((A + rB)x, x)$ we obtain the result $\|A\| = (Ax, x)$. Similarly $\|B\| = (Bx, x)$

After a number of corollaries Lin[16] proved the following result:

Theorem 3.1.2 Lin,2002[16]

Let x be a unit vector in H . Then the following are equivalent:

- (1) $\|T\|$ is an eigen-value of T , i.e. $Tx = \|T\|x$;
- (2) $1 + \|T\| = \|(I + T)x\|$.
- (3) $\|T\|$ is an eigen-value for T and $Tx = \|Tx\|x$, i.e. $Tx = \|T\|x$ and $Tx = \|Tx\|$;
- (4) $\|T\|$ is in the numerical range of T , i.e. $\|T\| = (Tx, x)$;
- (5) x is a complete vector for T , i.e. $(Tx, x) = \|T\|$;
- (6) $2\|T\|$ is an eigenvalue of $T^* + T$, i.e. $(T + T^*)x = 2\|T\|x$;
- (7) $\|T\|$ and $\|T\|^2$ are eigen-values of T and T^*T , respectively, with respect to x , i.e.
 $Tx = \|T\|x$ and $T^*Tx = \|T\|^2x$;
- (8) $(1 + \|T\|)\|T\|$ is an eigen-value of $(I + T^*)T$, i.e. $(I + T^*)x = (1 + \|T\|)\|T\|x$;
- (9) $\|T\|$ is a normal eigen-value for T , i.e. $Tx = \|T\|x = T^*x$;
- (10) x is a complete vector for T and T^* , i.e. $\|T\| = (Tx, x) = \|Tx\| = \|T^*x\|$;
- (11) x is a complete vector for T and T^*T , i.e. $\|T\| = (Tx, x) = \|Tx\|$ and $\|T\|^2 = \|Tx\|^2 = \|T^*Tx\|$;
- (12) $1 + \|T\| + \|T\|^2 = \|(I + T + T^*T)x\|$.

We make an extension of this result to a finite sum of powers of the norm of T as follows.

Lemma 3.1.3 Wafula and Khalaghai 2010,[24]

Let x be a unit vector in Hilbert space H and T an operator on H . Then the following are equivalent statements:

(i) x is an eigen-vector of T with eigenvalue $\|T\|$. Thus, $Tx = \|T\|x$

(ii) for any sequence $\alpha_1, \dots, \alpha_n$ of positive numbers

$$\sum_{k=0}^n \alpha_k \|T\|^k = \|(\sum_{k=0}^n \alpha_k T^k)x\|$$

Proof

To show that (i) implies (ii) we note that if $\|T\|$ is an eigen-value of T then

$$\|(\sum_{k=0}^n \alpha_k T^k)x\| = \|(\sum_{k=0}^n \alpha_k \|T\|^k)x\| = \|\sum_{k=0}^n \alpha_k \|T\|^k\| \|x\| = \sum_{k=0}^n \alpha_k \|T\|^k.$$

To prove that (ii) implies (i) we set $\alpha_1 = 1, \alpha_k = 0, k \neq 1$ to obtain $\|Tx\| = \|T\|$.

Also, from setting $\alpha_0 = \alpha_1 = 1, \alpha_k = 0, k \geq 2$ we obtain $1 + \|T\| = \|(I + T)x\|$.

Consequently we have

$$\begin{aligned} \|(I + T)x\|^2 &= ((I + T)x, (I + T)x) \\ &= (x, x) + (Tx, x) + (x, Tx) + (Tx, Tx) \\ &= 1 + (Tx, x) + (x, Tx) + \|T\|^2 \\ &= (1 + \|T\|)^2 \\ &= 1 + 2\|T\| + \|T\|^2 \end{aligned} \tag{3.2}$$

Hence, $(Tx, x) + (x, Tx) = 2\|T\|^2$

Now to show that $\|T\|^2$ is an eigen-value of T we consider the expansion below; -

$$\|(Tx - \|T\|^2x)\|^2 = (Tx - \|T\|^2x, Tx - \|T\|^2x) \quad (3.3)$$

$$= (Tx, Tx) - \|T\|^2(Tx, x) - \|T\|^2(x, Tx) + \|T\|^4(x, x) \quad (3.4)$$

$$= \|T\|^4 - \|T\|^2(2\|T\|^2) + \|T\|^4 = 0 \quad (3.5)$$

Hence, $Tx - \|T\|^2x = 0 \leftrightarrow Tx = \|T\|^2x$. So $\|T\|^2$ is an eigen-value of T .

Theorem 3.1.4[24]

If T is an operator on a Hilbert space H and x is a unit vector in H then the following are equivalent statements:

(i) any of the statements in theorem 3.1.2.

(ii) $\|(\sum_{k=0}^n \alpha_k T^k)x\| = \sum_{k=0}^n \|\alpha_k\| \|T\|^k$

Proof

This follows immediately from lemma 3.1.3.

In the same article Lin proved the following theorem.

Theorem 3.1.5 Lin 2002[11]

(1) x is a norm attaining vector for T , i.e $\|T\| = \|Tx\|$.

(2) $\|T\|^2$ is an eigen-value for T^*T , i.e. $T^*Tx = \|T\|^2x$

(3) $1 + \|T\|^2 = \|(I + T^*T)x\|$.

(4) x is a complete vector for T^*T , i.e. $\|T\|^2 = \|(T^*T)x\| = \|Tx\|^2$.

(5) $\|T\|^2$ is an eigen-value for T^*T and $T^*Tx = \|T^*Tx\|x$,

i.e. $T^*Tx = \|T\|^2x$ and $T^*T = \|T^*Tx\|x$.

(6) $\|T\|^2$ is in the numerical range of T^*T , i.e. $\|T\|^2 = (T^*Tx, x)$.

(7) $1 + \|T\|^2 + \|T\|^4 = \|I + T^*T + (T^*T)^2\|$.

(8) $\|T\|^2$ and $\|T\|^4$ are eigen-values of T^*T and $(T^*T)^2$ respectively with respect to x ,

i.e. $T^*Tx = \|T\|^2x$ and $(T^*T)^2x = \|T\|^4x$,

(9) $(1 + \|T\|^2)\|T\|^2$ is an eigen-value of $(I + T^*T)T^*T$

i.e. $(I + T^*T)T^*Tx = (1 + \|T\|^2)\|T\|^2x$.

(10) x is a complete vector for T^*T and $(T^*T)^2$,

i.e. $\|T\|^2 = \|Tx\|^2 = \|T^*T\|$ and $\|T\|^4 = \|T^*Tx\|^2 = \|(T^*T)^2x\|$.

We proceed to make an extension of this result by use of the following lemma.

Lemma 3.1.6[16]

Let T be an operator on a Hilbert space H and x be a unit vector in H then the following are equivalent statements

(i) x is an eigenvector of T^*T with eigen-value $\|T\|^2$, i.e. $T^*Tx = \|T\|^2x$.

(ii) for any enquence $\alpha_1, \dots, \alpha_n$ positive numbers we have that

$$\sum_{k=0}^n \alpha_k \|T\|^{2k} = \|(\sum_{k=0}^n \alpha_k (T^*T)^k)x\|$$

Proof

If we replace T with T^*T in theorem 3.1.4 we obtain;

$$\|(\sum_{k=0}^n \alpha_k (T^*T)^k)x\| = \sum_{k=0}^n \alpha_k \|T^*T\|^k$$

But, we have $\|T^*T\| = \|T\|^2$. Hence, the result follows.

We now can state the following theorem as a consequence of this lemma.

Theorem 3.1.7 [24]

Let T be an operator on a Hibert space H and x be a unit vector in H then the following are equivalent statements;

(i) any statement in theorem 3.1.5.

(ii)
$$\sum_{k=0}^n \alpha_k \|T\|^{2k} = \|(\sum_{k=0}^n \alpha_k (T^*T)^k)x\|$$

The following corollary shows that if $\|T\|$ is an eigen-value of T with respect to x then x is a norm attaining vector for T and T satisfies the Daugavet property.

Corollary 3.1.8[24]

Let x be a unit vector. Then any of the statements in Theorem 3.1.2 implies the following:

(1) any statement in theorem 3.1.7.

(2) T satisfies the Daugavet equation ,i.e. $1 + \|T\| = \|(I + T)\|$

(3) T and T^* satisfy the generalized Daugavet equation ,i.e. $\|I + T + T^*\| = 1 + 2\|T\|$.

- (4) T and T^*T satisfy the generalized Daugavet equation , i.e $\|I + T + T^*T\| = 1 + \|T\| + \|T\|^2$.
- (5) T is a normaloid operator,i.e. $w(T) = \|T\|$.
- (6) x is a norm attaining vector for $I + T$,i.e $\|I + T\| = \|(I + T)x\|$.
- (7) x is a norm attaining vector for $I + T + T^*$,i.e, $\|I + T + T^*\| = \|(I + T + T^*)x\|$.
- (8) x is a norm attaining vector for $I + T + T^*T$, i.e, $\|I + T + T^*T\| = \|(I + T + T^*T)x\|$

Proof

As in Lin [2002] with theorem 1 and 2 replaced with Theorem 3.1.4 and 3.1.7

3.2 On the bounds of a selfadjoint operator

In this section we consider the bounds of a self adjoint operator .We show that the norm of a self adjoint operator is the same as the numerical radius and make application of properties of norm attaining operators to obtain further results.

Theorem 3.2.1[3]

If T is a positive operator then $\|Tx\|^2 \leq w(T)(Tx, x)$ for all x in H .

Proof

We note that if we apply Cauchy-Schwartz inequality on the bilinear form (Tx, y) where we let $y = Tx$ we obtain

$$\begin{aligned}
\|Tx\|^4 &= (Tx, Tx)^2 \\
&= (Tx, y)^2 \\
&\leq (Tx, x)(Ty, y) \\
&\leq (Tx, x)w(T)\|y\|^2 \\
&= w(T)(Tx, x)\|Tx\|^2.
\end{aligned}$$

Thus, for all vectors x for which $Tx \neq 0$ we obtain $\|Tx\|^2 \leq w(T)(Tx, x)$. But, if $Tx = 0$ the result is obvious.

Theorem 3.2.2[3]:

If T is a self adjoint operator then $\|T\|^2 \leq w(T^2)$.

Proof

We have

$$\begin{aligned}
\|Tx\|^2 &= (Tx, Tx) \\
&= (T^*Tx, x) \\
&= (T^2x, x) \\
&\leq w(T^2).
\end{aligned} \tag{3.6}$$

Note that in this case T^2 is a positive operator. Furthermore, if we confine the x on the

unit sphere in H and take the supremum on the left hand side we obtain the result.

Theorem 3.2.3[3]

If T is self adjoint then $w(T^2) \leq (w(T))^2$.

Proof

Let $\lambda \in W(T^2)$ then by setting $y = Tx$ we obtain:

$$\begin{aligned}\lambda^2 &= (T^2x, x)^2 = (Tx, Tx)^2 \\ &= (Tx, y)^2 \\ &\leq (Tx, x)(Ty, y) \quad (\text{applying Cauchy-Schwarz inequality}) \\ &\leq w(T)(T(Tx), Tx) \\ &\leq w(T)^2 \|Tx\|^2 \\ &= w(T)^2 (Tx, Tx) \\ &= w(T)^2 (T^2x, x) \\ &= w(T)^2 \lambda\end{aligned}$$

Hence, for $\lambda > 0$ we have $\lambda \leq w(T)^2$.

Taking the supremum on the left hand side we obtain the result $w(T^2) \leq w(T)^2$

Corollary 3.2.4[3]

$\|T\| \leq w(T)$ for all self adjoint operators T .

Proof

From theorem 3.2.2 and 3.2.3 we have $\|T\|^2 \leq w(T^2) \leq (w(T))^2$.

On taking squareroots of the extremities of the inequality, we obtain the result.

Corollary 3.2.5[3]

$\|T\| = w(T)$ for all self adjoint operators T .

Proof

Note that

$$\begin{aligned} |(Tx, x)| &\leq \|Tx\| \|x\| \\ &\leq \|T\| \|x\|^2 \\ &= \|T\| \end{aligned} \tag{3.7}$$

for all unit vectors. Hence, when we take supremum on the left hand side we have that

$w(T) \leq \|T\|$. The reverse relation follows from corollary 3.2.4.

Of course the previous result is true for a larger class of operators, namely spectraloid operators.

corollary 3.2.6[3]

$\|T\|^2 = w(T^2) = (w(T))^2 = \|T^2\|$, for all self adjoint operators T

Proof

For T self adjoint we have from corollary 3.2.4 and 3.2.5

$$\begin{aligned} \|T\|^2 &\leq w(T^2) \\ &\leq (w(T))^2 \\ &\leq \|T\|^2 \end{aligned} \tag{3.8}$$

Hence, all the inequalities vanish. Also since T^2 is self adjoint $\|T^2\| = w(T^2)$

We now consider further results when the operator T is both self adjoint and compact.

In this case the operator has a norm attaining vector as shown by Shilov[20].

Theorem 3.2.7 Wafula and Khalaghai 2010,[24]:

If T is a compact self adjoint operator then $\|T^n\| = w(T^n) = (w(T))^n = \|T\|^n$,

Proof

Since T is self adjoint, T^n is also self adjoint and so the first equality follows from corollary 3.2.5. Also, if x is the norm attaining vector for T we will have ;

$$\begin{aligned}(T^n x, x) &= (T^{n-1} x, T x) \\ &= \|T\| (T^{n-1} x, x) \\ &= \|T\| (T^{n-2} x, T x) \\ &= \|T\|^2 (T^{n-2} x, x) \\ &= \|T\|^n \\ &= (w(T))^n.\end{aligned}\tag{3.9}$$

But we have $w(T^n) \geq (T^n x, x)$. Consequently, $w(T^n) \geq (w(T))^n$.

For the reverse inequality we note that $w(T^n) = \|T^n\| \leq \|T\|^n = (w(T))^n$

Corollary 3.2.8 Wafula and Khalaghai 2010[24]

If T is a compact self adjoint operator then

$$w(\sum_{k=1}^n \alpha_k T^k) = \|\sum_{k=1}^n \alpha_k T^k\| = \sum_{k=1}^n \alpha_k \|T\|^k = \sum_{k=1}^n \alpha_k (w(T))^k,$$

where α_k are non negative numbers.

Proof

The first equality follows from the fact that the sum of self adjoint operators is also self adjoint. The second follows from properties of norm attaining operators. A compact self adjoint operator therefore satisfies a generalized Daugavet equation[6].

3.3 On norm attaining operators and invariance of numerical range

Remark 3.3.1

We note that if $\|A\|$ is an eigen-value of A then $\|A\|$ is a normal eigen-value.

Thus, $Ax = \|A\|x = A^*x$. In this case we have the following result;

Theorem 3.3.2

Let T be any Operator on H and A be such that $\|A\|$ is an eigen value of A . Then We have $W(A^*TA) = \|A\|^2W(T)$.

Proof

Let $\lambda \in W(A^*TA)$ then we have

$$\lambda = (A^*TAx, x) = (TAx, Ax) = (T(\|A\|x), \|A\|x) = \|A\|^2(Tx, x) \in \|A\|^2W(T),$$

Conversely if $\lambda \in \|A\|^2W(T)$ then

$$\lambda = \|A\|^2(Tx, x) = (T(\|A\|x, \|A\|x) = (TAx, Ax) = (A^*TAx, x) \in W(A^*TA)$$

Corollary 3.3.3

Let $T \in B(H)$ be any operator and A be contraction such that $\|A\|$ is an eigenvalue of

A then we have

$$W(A^*TA) \subset W(T)$$

Proof

If A is contraction then $\|A\| \leq 1$. Hence, from Theorem 3.3.1 $W(A^*TA) \subset W(T)$.

Chapter 4

THE NUMERICAL RANGE OF SOME CLASSES OF OPERATORS

4.1 Operators with a thin numerical range

We now consider operators whose numerical range is a line segment in the complex plane. This collection includes all self adjoint operators

Theorem 4.1.1 Wafula and Khalaghai 2012 [26]

An operator T has its numerical range on a line segment if and only if $T = \alpha I + \beta A$ where A is a self adjoint operator.

Proof

Let $W(T)$ be a line segment then $W(T) = \{\alpha + \beta\lambda; \lambda \in R\}$.

Hence, $\beta^{-1}(\alpha + \beta\lambda) - \beta^{-1}\alpha$ is real. Now let $A = \beta^{-1}T - \beta^{-1}\alpha I$. Then for any unit vector x we have

$(Ax, x) = \beta^{-1}T(x, x) - \beta^{-1}\alpha I(x, x) = \beta^{-1}(\alpha + \beta\lambda) - \beta^{-1}\alpha$ is real.

Thus, A is a self adjoint operator and $T = \alpha I + \beta A$.

Conversely, if $T = \alpha I + \beta A$ where A is self adjoint then

$\lambda \in W(T)$ implies there exists a unit vector x such that

$$\lambda = (Tx, x) = ((\alpha I + \beta A)x, x) = \alpha(x, x) + \beta(Ax, x) = \alpha + \beta\lambda$$

where λ is real.

Hence, $W(T)$ is a line segment.

Example 4.1.2

Let

$$T = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

and $\lambda \in W(T)$ then $\lambda = (Tx, x)$ for some unit vector x . Thus

$$\lambda = \left(\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{pmatrix} w \\ z \end{pmatrix}, \begin{pmatrix} w \\ z \end{pmatrix} \right) = \left(\begin{pmatrix} -z \\ w \end{pmatrix}, \begin{pmatrix} w \\ z \end{pmatrix} \right) = z\bar{w} + w\bar{z} = 2\text{Im}(z\bar{w})$$

Evidently, the numerical range of T is pure imaginary set and it must then be a line segment. If we parameterize w and z so that $w = r_1 e^{i\theta_1}$ and $z = r_2 e^{i\theta_2}$ then we have

$$\lambda = -r_1 r_2 e^{\theta_2 - \theta_1} + r_1 r_2 e^{\theta_1 - \theta_2} = 2r_1(1 - r_1^2)^{\frac{1}{2}} \sin(\theta_2 - \theta_1) = 2r(1 - r^2)^{\frac{1}{2}} \sin\psi.$$

The supremum of the absolute value of the last term occurs, when $r = \frac{1}{\sqrt{2}}$ and $\sin\psi = 1$.

Hence, $w(T) = 1$

On the other hand, we have $\|Tx\|^2 = (Tx, Tx) = z\bar{z} + w\bar{w} = 1$, whenever x is a unit vector. Thus, $\|T\|^2 = 1$ leading to $\|T\| = 1 = w(T)$.

Theorem 4.1.3 Wafula and Khalaghai 2012,[26]

If $T = \beta A$, where A is a self adjoint operator and β is a non zero complex number, then

$$\|T\| = w(T).$$

Proof

$\beta^{-1}T = A$ is a self adjoint operator .

Hence $\|\beta^{-1}T\| = w(\beta^{-1}T)$.

Consequently,

$$|\beta^{-1}|\|T\| = |\beta^{-1}|w(T) \text{ and so}$$

$$\|T\| = w(T)$$

Theorem 4.1.4 [26]

If T is an operator whose numerical range is a line segment then T is normal.

Proof

Let $T = \alpha I + \beta A$ where A is self adjoint then

$$TT^* = (\alpha I + \beta A)(\bar{\alpha} I + \bar{\beta} A) = |\alpha|^2 + \alpha\bar{\beta}A + \beta\bar{\alpha}A + |\beta|^2.$$

On the other hand

$$T^*T = (\bar{\alpha} I + \bar{\beta} A)(\alpha I + \beta A) = |\alpha|^2 + \bar{\alpha}\beta A + \bar{\beta}\alpha A + |\beta|^2$$

Hence $TT^* = T^*T$.

Theorem 4.1.5 [11]

If T is normal and λ is an eigenvalue for T then $\bar{\lambda}$ is an eigenvalue for T^*

Let x be an eigen-vector corresponding to λ then $Tx = \lambda x$. Hence $(T - \lambda I)x = 0$.

Now $\|(T^* - \bar{\lambda}I)x\| = \|(T - \lambda I)x\| = 0$.

Hence, $T^*x = \bar{\lambda}x$.

Theorem 4.1.6[11]

If T is a normal operator then any eigen-vectors corresponding to distinct eigen-values of T are orthogonal.

Proof

Let x and y be eigen-vectors of T i.e. $Tx = \lambda x$ and $Ty = \beta y$, $\lambda \neq \beta$ then

$\lambda(x, y) = (\lambda x, y) = (Tx, y) = (x, T^*y) = (x, \bar{\beta}y) = \beta(x, y)$. Hence $(\lambda - \beta)(x, y) = 0$.

Consequently, $(x, y) = 0$ since $\lambda - \beta \neq 0$.

Theorem 4.1.7 Wafula and Khalaghai 2012[26]

If T is a normal operator on a Two dimensional Hilbert space H then $W(T)$ is a line segment

Proof

Let u and v be normalized eigen-vectors corresponding to distinct eigen-values α, β then by the previous theorem they are independent and hence form a basis for H . For each unit vector x in H we have $x = pu + qv$. So that

$$(Tx, x) = (p\alpha u + q\beta v, pu + qv) = |p|^2\alpha + |q|^2\beta$$

where $|p|^2 + |q|^2 = 1$.

If we set $t = |p|^2$, then we obtain

$$(Tx, x) = t\alpha + \sqrt{(1-t^2)}\beta.$$

Thus, $W(T)$ is a line segment from α to β .

Definition 4.1.8

An operator T is said to be Hypnormal if $\|T^*x\| \leq \|Tx\|$, for all x in H .

Theorem 4.1.9[11]

If T is a Hypnormal operator then eigen-vectors corresponding to distinct eigenvalues of T are orthogonal.

Theorem 4.1.10 Wafula and Khalaghai 2012 [26]

If T is a Hypnormal operator on a finite dimensional Hilbert space H then $W(T)$ is a polygon with n sides.

Proof

Let $\{u_n\}$ be normalized eigen-vectors corresponding to distinct eigenvalues $\{\alpha_n\}$.

Then, each unit vector x in H has the unique representation

$$\sum_{k=1}^{k=n} p_k u_k \text{ where } \sum_{k=1}^n |p_k|^2 = 1.$$

Therefore,

$$(Tx, x) = \left(T \left(\sum_{k=1}^n p_k u_k \right), \sum_{k=1}^n p_k u_k \right) \quad (4.1)$$

$$= \left(\sum_{k=1}^n p_k T(u_k), \sum_{k=1}^n p_k u_k \right) \quad (4.2)$$

$$= \left(\sum_{k=1}^n p_k \alpha_k u_k, \sum_{k=1}^n p_k u_k \right) \quad (4.3)$$

$$= \sum_{k=1}^n |p_k|^2 \alpha_k \quad (4.4)$$

Corollary 4.1.10 Wafula and Khalaghai 2012[26]

If an operator T is normal on a finite dimensional Hilbert space then $W(T)$ is a polygon with n sides

Proof

Each normal operator is hyponormal. The result follows from theorem 4.1.9.

Example Let

$$T = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

then T is a unitary operator. For eigen-values of T we must have that $\det(T - \lambda I) = 0$.

Hence,

$$\left| \begin{bmatrix} -\lambda & 0 & 1 \\ 1 & -\lambda & 0 \\ 0 & 1 & -\lambda \end{bmatrix} \right| = 0$$

This leads to the equation $-\lambda^3 + 1 = 0$. Hence, the eigen-values are the three cube roots of unity. From which we obtain the fact that the numerical range is the equilateral triangle that joins the vertices $1, w, w^2$

Applying Fuglede's theorem Rehder(1982) made some remarkable observations on the product of self adjoint operator as follows.

Theorem 4.1.11[18]

Let A and B be self adjoint operators and either A or B is positive. Then AB is self adjoint if and only if AB is normal

Proof

Obviously if AB is self adjoint then AB is normal. For the converse we note that

$$A(BA) = (AB)A$$

and so we invoke Fuglede's theorem to obtain

$$A(BA)^* = (AB)^*A \text{ i.e. } A^2B = BA^2.$$

Suppose that A is positive (otherwise interchanges roles with B) then A is the square root of A^2 hence A commutes with B and so

$$(AB)^* = BA = AB$$

Let T be a self adjoint operator then $T = AB$ where A is positive and B unitary; the polar decomposition of T . The previous theorem gives a converse in the class of normal operators. Thus if T is normal and $T = AB$ where A is positive and B self adjoint then T is self adjoint.

Corollary 4.1.11[17]

Let $T = A + iB$ be the canonical form of an operator whereby A and B are self adjoint

.If AB is normal and either A or B is positive then T is normal.

Proof

We have

$$TT^* = (A + iB)(A - iB) = A^2 + i(BA - AB) + B^2 = A^2 + B^2.$$

On the other hand,

$$T^*T = (A - iB)(A + iB) = A^2 + i(AB - BA) + B^2 = A^2 + B^2 = TT^*.$$

Corollary 4.1.12[17]

Let A and B be self adjoint, and either A or B be positive .If $AB - BA \neq 0$ then Also $AB + BA \neq 0$.

Proof

If $AB + BA = 0$ then $(AB)(AB)^* = (AB)(BA) = -(AB)^2$.

Similarly $(AB)^*(AB) = (BA)(AB) = -(AB)^2$.

Consequently , AB is normal and by theorem 1.5.7 AB is self adjoint, i.e $AB - AB = 0$.

4.2 λ - commuting operators

We consider operators that commute up to scalar multiples and make some application of the Putnam Fuglede property to provide an alternative proof of the following result obtained by Brooke [4]

Theorem 4.2.1Wafula and Khalaghai 2012 [25]

Let A, B be bounded operators on a Hilbert space H such that $AB = \lambda BA$ and $AB \neq 0$

where λ is a complex number. Then,

(i) If A or B is self adjoint we must have that λ is a real number

(ii) If A and B are Self adjoint then $\lambda \in \{1, -1\}$

(iii) If A and B are self adjoint and either A or B is positive then $\lambda = 1$.

Proof

(i) $AB = \lambda BA$ if and only if

$$B^*A^* = \bar{\lambda}A^*B^*.$$

But if say $A = A^*$ then

$$B^*A = \bar{\lambda}AB^*.$$

By Putnam Fuglede theorem $B^*A^* = \lambda A^*B^*$.

Hence, we have

$$\bar{\lambda}A^*B^* = \lambda A^*B^*.$$

Consequently, we have $(\bar{\lambda} - \lambda)A^*B^* = 0$.

Since $AB \neq 0$ we obtain the result $(\bar{\lambda} - \lambda) = 0$. So $\bar{\lambda} = \lambda$

(ii) If $A = A^*, B = B^*$ then $AB = \lambda BA$ and so $(AB)^* = \bar{\lambda}A^*B^*$. Hence, $BA = \bar{\lambda}AB =$

$$\bar{\lambda}\lambda BA = |\lambda|^2 BA.$$

Consequently, $(1 - |\lambda|^2)BA = 0$. So from (i) $\lambda = \pm 1$.

(iii) Let $AB = -BA$ then Consider the commutator of A and B namely $AB - BA =$

$AB + AB = 2AB \neq 0$. However, this leads to the anti commutator being non zero. i.e

$$AB + BA \neq 0.$$

This is a contradiction hence we must have $\lambda = 1$.

Corollary 4.2.2[25].

If A and B are self adjoint operators and $AB = \lambda BA$ then either $W(AB) \subset R$ or $W(AB) \subset iR$.

Proof

From the theorem $\lambda \in \{1, -1\}$. If $\lambda = 1$ the AB is self adjoint and its numerical range is a subset of the real numbers .

If $\lambda = -1$ then $(AB) + (AB)^* = AB + BA = AB - AB = 0$.

Therefore AB is an Anti-self adjoint operator, leading to $W(A) \subset iR$.

Definition 4.2.3

An operator A is called anti-self adjoint if $A + A^* = 0$.

Remark: If A is an anti-self adjoint operator then A is normal operator. To see this note $AA^* = -A^2 = A^*A$.

Theorem 4.2.4[25]

If an operator T is anti-self adjoint then $W(T)$ is a pure imaginary subset.

Proof

Note that $\lambda \in W(T) \Leftrightarrow \lambda = (Tx, x) = (x, T^*x) = -(x, Tx) = -\bar{\lambda}$.

Thus, $\lambda + \bar{\lambda} = 0$ and hence λ is a pure imaginary number.

Theorem 4.2.5 Wafula and Khalaghai 2012,[25]

Let $AB = \lambda BA$, $AB \neq 0$ and B be a normal operator . If A is anti-self adjoint then λ is a real number.

Proof

If $AB = \lambda BA$ then we take the adjoint on both sides to find $B^*A^* = \bar{\lambda}A^*B^*$.

Hence, applying the Putnam-Fuegleda theorem we have that

$$\begin{aligned} B^*A = \lambda AB^* &\Leftrightarrow -B^*A^* = -\lambda A^*B^* \Leftrightarrow (AB)^* = \lambda A^*B^* \Leftrightarrow (\lambda BA)^* = \lambda A^*B^* \\ &\Leftrightarrow \bar{\lambda} A^*B^* = \lambda A^*B^* \Leftrightarrow (\bar{\lambda} - \lambda)A^*B^* = 0. \text{Hence } \bar{\lambda} = \lambda. \end{aligned}$$

Corollary 4.2.6 Wafula and Khalaghai 2012,[25]

If A and B are anti-self adjoint operator such that $AB = \lambda BA$ and $AB \neq 0$ then $\lambda \in \{1, -1\}$.

Proof

Since A is normal we have λ is a real number. Furthermore, from $AB = \lambda BA$, we obtain

$$B^*A^* = \lambda A^*B^* \Leftrightarrow BA = \lambda AB = \lambda^2 BA \Leftrightarrow (1 - \lambda^2)BA = 0. \text{ Since } AB \neq 0 \text{ we obtain}$$

$$\lambda = \pm 1$$

Corollary 4.2.7

If A and B are anti-self adjoint operator such that $AB = \lambda BA$ and $AB \neq 0$, then either $W(AB)$ is a real or pure imaginary set.

Proof

If $\lambda = 1$, then AB is self adjoint and so $W(AB)$ is real. If $\lambda = -1$, then AB is anti-self adjoint and so $W(AB)$ is pure imaginary.

4.3 U-self adjoint operators

We make some observations on this class of operators in this section. First we note that the class of U -self adjoint operators is an algebra as seen in the following result.

Theorem 4.3.1

If A and B are U -self adjoint operators then so are the following operators;

(i) $\alpha A + \beta B$ for all complex numbers α, β .

(ii) AB .

Proof

(i) We have $UAU^* = A$ and $UBU^* = B$.

Hence,

$$U(\alpha A + \beta B)U^* = \alpha UAU^* + \beta UBU^* = \alpha A + \beta B.$$

(ii) Also, $UABU^* = UAU^*UBU^* = AB$.

Theorem 4.3.2

If A is U -self adjoint operator then $W(A) = W(A^*)$.

Proof

We have that for some unit vector x , $\lambda \in W(A)$ if and only if

$$\lambda = (Ax, x) = (x, A^*x) = (x, UAU^*x) = (U^*x, AU^*x) = (A^*U^*x, U^*x) = \lambda^* \in W(A^*).$$

The above result indicates that the numerical range of a U -self adjoint operator has mirror symmetry with the real axis.

Chapter 5

SUMMARY OF MAIN RESULTS

In the first chapter we have looked at the literature review, basic definitions and topological properties of the numerical range of operators. This includes boundedness, convexity and connectedness. We have also examined the location of the numerical range depending on the classification of the operator. For example, the numerical range of a self adjoint operator is real.

In chapter two we first note that it is a well known fact that the numerical range of T is invariant under unitary similarity. Indeed if T is any operator and A is unitary, then $W(A^*TA) = W(T)$. We have gone ahead to investigate $W(A^*TA) = W(T)$ where A is relaxed to the class of isometries.

We found that;

(i) If A is an isometry then $W(A^*TA) \subset W(T)$, with equality holding, if in addition each unit vector is of the form $y = Ax$.

(ii) If A is partial isometry which is either injective or has a dense range then $W(A^*TA) \subset$

$W(T)$.

A number of corollaries concerning co-isometries have also been deduced.

In chapter three, we considered some properties of norm attaining operators and the Daugavets equation. Lin proved among other results that $1 + \|T\| + \|T\|^2 = \|(I + T + T^*T)x\|$.

We established the fact that if an operator T has a norm attaining vector x then

(iii) for any sequence $\alpha_1, \dots, \alpha_n$ of positive numbers

$$\sum_{k=0}^n \alpha_k \|T\|^k = \|(\sum_{k=0}^n \alpha_k T^k)x\|.$$

$$(iv) \sum_{k=0}^n \alpha_k \|T\|^{2k} = \|(\sum_{k=0}^n \alpha_k (T^*T)^k)x\|.$$

The operator with a norm attaining property thus satisfies a generalized Daugavets equation.

We also showed the following results on invariance of numerical range for norm attaining operator A ;

(i) Let If A is a contraction such that $\|A\|$ is an eigen-value of A then we have that

$$W(A^*TA) \subset W(T)$$

(v) If A be such that $\|A\|$ is an eigen-value of A . Then We have that $W(A^*TA) =$

$$\|A\|^2 W(T)$$

In chapter four we considered the geometry of the numerical range of some operators on a finite dimensional Hilbert space. We established the fact that the numerical range of a hyponormal operator is polygonal in the complex plane. The vertices of the polygon are eigenvalues of the operator. In the two dimensional space the numerical range of an

operator is a line segment if and only if the operator is normal. We also consider some properties of operators that commute to a scale factor. We show that if $AB = \lambda BA$, $AB \neq 0$ and B be a normal operator and A is anti-self adjoint then λ is a real number.

5.1 REFERENCES

1. Albedawi H & Khalid S .Inequalities of the numerical radii,International Journal of mathematical analysis,vol2,2008,no 11,527-532.
2. De Bara G., Giles J.R.,& Sims B., On the numerical range of compact operators on Hilbert spaces, J,London Math.Soc.(2),5(1972),704-706.
3. Bonsal F.F. & Duncan J.,Numerical ranges of operators on normed AlgebrasLondon mathematical society lecture notes series(1971).
4. Bourhim A,Bounded point evaluations and local spectral theory,Dissertation,AICTP.
5. Brooke A.J. ,P.Busch & Pearson B,Commutativity up to a factor of bounded Operators in a complex Hilbert spaceDepartment of Mathematics and Statistics,University of Saskatchewan
6. DaugavetI.K.. A property of compact operators in the space C , UspekhiMat.Nauk,18(1961) 158.
7. Donogue W.F,On the numerical range of a bounded operator University of Kansas ,(1957),261-263.
- 8 Douglas B.& Robin H.,A constructive analysis that the numerical range is convex,London mathematical society,LMS J.Comp.Math.3(2000) 191-206
- 9 Embry,M.R.Similarities involving normal operators on Hilbert space,Pacific Journal of Mathematics vol.35 (1970)331-335

- 10 Fuegledé B, **A commutativity theorem for normal operators**, Proc.N.A.S.36(1950)35-40
11. Furuta T. **Invitation to linear operators**, Taylor and Francis, London 2002
- 12 Halmos P.R., **Hilbert space problem book**, Springer, New York-Heidelberg-Berlin, (1974)
13. Khalagai J.M. & Otieno M.O.E., **On quasiaffinity and quasi-affine inverses of partial isometries** Bulletin of the Allahabad mathematical society, vol.15(2000)
- 14 Hausdorff.F, **Der wertvrrat einer bilinear form**, Math z.vol.3(1919)pp314-316
- 15 Kittaneh, F. **A numerical radius inequality for certain operator sums**. Studia Math.158(2003), 17
16. Lin C.S., **On operators whose norm is an eigenvalue**, Bulletin of the institute of mathematics, vol.31(2002).
17. Nakos G. & Joyner D, **Linear Algebra with its applications**, Brooks/Cole Publishing Company (1998), 548
18. Rehder, W., **International journal of mathematics and science**, Vol.5, no.4(1982)
19. Shapiro J.H. , **Notes on the numerical range**, (2004) Michigan State University, East Lansing. MI 48824-1027, USA
20. Shilov, G.S., **Mathematical Analysis**, vol77, (1965), 221-224

21. Stone M.H. Linear transformations in Hilbert space, Amer. Math. soc. coll. Publ. XV (1932)
22. Toeplitz. O., Das algebraische Analogie zu einem satze von Fejer, Math. z. vol. 2 (1910) pp 187-197.
23. Wafula A.W. & Khalagai J.M., Invariance of Numerical range under isometric transformation, African Journal of Mathematics and Computer Science Vol 3(2) pp 036-037, (2010)
24. Wafula A.W. , Khalagai J.M. & Pokhariyal G.P. Norm properties of an operator whose norms are Eigenvalues, African Journal of Mathematics and Computer Science Vol 3(7) xxx-xxx, 2010
25. Wafula A.W. & Khalagai J.M., On the numerical range of λ -commuting Operators Global journal of Computer application and Technology (2012)- to appear
26. Wafula A.W. & Khalagai J.M. . The numerical range of some operators on a finite dimensional Hilbert space, African Journal of mathematics and Computer Science-submitted