MODELING A HIERARCHICAL SYSTEM WITH A SINGLE ABSORBING STATE

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Abstract

The paper describes a Markov Chain transition model in which students in an education system, be they graduates or dropouts, are grouped together into a single absorbing state.

1 INTRODUCTION

Education systems are hierarchical in nature thus amiable to the use of discrete Markov Chains. Students graduate from the system, transit from one grade to the next higher grade, repeat the same grade or drop out of the system due to factors like lack of fees, sickness and poor academic performance. Thus, students finally enter permanent states, read absorbing states, as graduates or dropouts.

2 LITERATURE REVIEW

In the educational field, Gani (1963) proposed a Markovian model to forecast enrollment and degrees awarded in Australian Universities. Thonstad (1967) used stochastic models to study enrolments in the Norwegian educational system in his book on educational planning. Uche (1980) applied the Markovian model to the Nigerian educational system. Owino (1982) and Odhiambo and Owino (1985) considered several measures of academic survival for the Kenyan Primary education system. Odhiambo and Khogali (1986) studied the Kenyan Primary education system through a cohort analysis. Owino and Phillips (1988) compared the retention properties of the Kenyan Primary education system between 1964 and 1972 and also between 1972 and 1980. Owino and Odhiambo (1994) used a Markovian model to plan the Kenyan Primary education system by estimating several capital and human resource requirements for the system. Mbugua (2005) used the Markov Chain model to estimate the number of new entrants into the Kenyan Primary education system. Owino and Bodo (2005) derived the maintainable grade structures for an academic department.

In this paper, graduates and dropouts from an education system are grouped together into a single absorbing state.

3 THE MODEL

Consider a Markov Chain model with s non-absorbing states; 1, 2, 3...s corresponding to the grades of the system and r absorbing states corresponding to the various final qualifications. Here, $r + s = N$, thus N is the total number of possible states of the system.

An absorbing state is a state which becomes permanent once it has been entered hence transition probabilities between absorbing states should be represented by one, justifying the use of the identity matrix. Transition from an absorbing state to a non-absorbing state which is impossible, should be represented by zero, hence the matrix of zeroes. Transitions from non-absorbing states to absorbing states are possible, likewise transitions between non-absorbing states.

The transition probability matrix P of the Markov chain can then be represented in the following canonical form, assuming time homogeneity;

$$P = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 \\ . & . & . & \dots & . & . & . & . & \dots & . \\ 0 & 0 & 0 & \dots & 1 & 0 & 0 & 0 & \dots & 0 \\ r_{11} & r_{12} & r_{13} & \dots & r_{1r} & q_{11} & q_{12} & q_{13} & \dots & q_{1s} \\ r_{21} & r_{22} & r_{23} & \dots & r_{2r} & q_{21} & q_{22} & q_{23} & \dots & q_{2s} \\ . & . & . & \dots & . & . & . & . & \dots & . \\ r_{s1} & r_{s2} & r_{s3} & \dots & r_{sr} & q_{s1} & q_{s2} & q_{s3} & \dots & q_{ss} \end{pmatrix} \quad . \quad . \quad . \quad . \qquad (1)$$

that is

$$P = \begin{pmatrix} I & O \\ R & Q \end{pmatrix} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (2)$$

where

I is an r x r identity matrix which gives transition probabilities between absorbing states
O is an r x s matrix of zeroes which gives transition probabilities from an absorbing state to a non-absorbing state
R = (r_{ik}) is an s x r matrix, r_{ik} being the probability that a student in grade i at time (t-1) will graduate with final education k at time t, i=1, 2,...,s and k = 1, 2,...,r.
Q = (q_{ij}) is an s x s matrix, q_{ij} being the probability that a student who is in grade i at time (t-1) will be in grade j at time t; i, j = 1, 2,...,s.

By the Chapman-Kolmogorov result, the n-step transition probability matrix for the process in canonical form is given by

$$P^n = \begin{pmatrix} I & O \\ (I + Q + Q^2 + \dots + Q^{n-1})R & Q^n \end{pmatrix} \quad . \quad . \quad . \quad . \qquad (3)$$

that is

$$P^n = \begin{pmatrix} I & O \\ R^n & Q^n \end{pmatrix} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (4)$$

where

I is an r x r identity matrix which gives transition probabilities between absorbing states in n steps
O is an r x s matrix of zeroes which gives transition probabilities from absorbing states to non-absorbing states in n steps

$R^n = ((r_{ik}^{(n)})) = (I + Q + Q^2 + ... + Q^{n-1})R$ is an s x r matrix, which gives the probability that a student who is in grade i will graduate with final education k within n years, I = 1, 2,...,s and k = 1, 2,...,r. It is also called the completion ratio.
$Q^n = ((q_{ij}^{(n)}))$ is an s x s matrix which gives the probability that a student who is in grade i will be in grade j, n years later; i, j = 1, 2,...,s.

hence

$$\lim(n \to \infty) P^n = \begin{pmatrix} I & O \\ (I-Q)^{-1}R & O \end{pmatrix} \quad . \quad . \quad . \quad . \quad . \quad . \qquad (5)$$

The matrix L = $(I – Q)^{-1}$ is called the fundamental matrix of the absorbing Markov Chain.

4 APPLICATION OF THE MODEL

Initial Transition Matrix
Let the states of the system be denoted by integers 1, 2,..., N at times t = 0, 1, 2,...
Let P_{ij} denote the probability that a student in grade i at time (t-1) will be in grade j at time t, giving rise to transition matrix $P = ((P_{ij}))$; i.j = 1, 2, ...,N. Let $n_{ij}(t)$ represent the number of students in grade i at time (t-1) who will be in grade j at time t, also, let $n_i(t-1)$ represent the number of students in grade i at time (t-1), then assuming the multinomial distribution, the transition probabilities are estimated from

$$p_{ij} = n_{ij}(t)/n_i(t-1) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (6)$$

where i, j = 1, 2, ..., N. This is the proportion of students who were in grade i at time (t-1) who end up being in grade j at time t.

NOTE
In the single absorbing state model, k takes only value 1, denoting dropouts from the system. Hence the R component of matrix P is an s x 1 matrix.

Initial Transition Process with Single Absorbing State
The data for this study was extracted from Bachelor of Science, Actuarial Science in the School of Mathematics, University of Nairobi. Assuming time homogeneity, students enrolments in grades I, II, III and IV for the year 2004 and enrollments for the same students in grades II, III and IV for the year 2005 were as shown in the following Table 1.

Table 1 Student enrollments 2004/2005

G	E(2004)	PR(2005)	R(2005)	PD(2005)	D(2005)
I	75	67	2	4	2
II	85	77	1	4	3
III	49	42	1	4	2
IV	33	28	4	-	1

Key: G represents Grade, E represents Enrollment, PR represents Proceeded, R represents Repeated, PD represents Passed and Dropped out and D represents Discontinued.

The dropout proportions for students who were in grades I, II, III and IV were (6/75) = 0.0800, (7/85) = 0.0824, (6/49) = 0.1224 and (29/33) = 0.8788 respectively. This gives rise to the R component of the matrix P.
In the Q component of the matrix P, position 1,1 represents the proportion of students who repeated grade I (2/75) = 0.0267, position 1,2 represents the proportion of students who proceeded to grade II from grade I (67/75) = 0.8933. The same concept is applied to obtain the relevant proportions of students who were originally in grades II, III and IV for the remaining elements of the Q matrix.

Thus, the transition probability matrix with single absorbing state, P_1, assuming time homogeneity is;

$$P = \begin{pmatrix} 1.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ 0.0800 & 0.0267 & 0.8933 & 0.0000 & 0.0000 \\ 0.0824 & 0.0000 & 0.0118 & 0.9058 & 0.0000 \\ 0.1224 & 0.0000 & 0.0000 & 0.0205 & 0.8571 \\ 0.8788 & 0.0000 & 0.0000 & 0.0000 & 0.1212 \end{pmatrix} \quad . \quad . \quad . \qquad (7)$$

5 COMPLETION RATES

The dropout rate n years later from grade i is given by

$$r_{ik}^{(n)} = \sum_{j=1}^{s} q_{ij}^{(n-1)} r_{jk} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (8)$$

where i, j = 1, ..., s. Note that $q_{ij}^{(n-1)}$ is the probability that a student in grade i will be in grade j, (n-1) years later and r_{jk} is the probability that a student in grade j at time (t-1) graduates with final education k at time t. Actually, $r_{ik}^{(n)}$ is the $(i, k)^{th}$ element of the product $Q^{(n-1)}R$.

Hence, the cumulative dropout rate within x years from grade i is given

$$r_{ik}^{(x)} = \sum_{n=1}^{x} r_{ik}^{(n)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (9)$$

where, i = 1, ...,s and k = 1,...,r.

Again, $r_{ik}^{(x)}$ is the $(i, k)^{th}$ element of $(I+Q+Q^2+...+Q^{x-1})R$, the basis of computations in this work.

Completion rates under single absorbing state

In the single absorbing state model, dropouts from grades I, II, and III were identified. Moreover, as previously stated, dropouts and graduates from grade IV were lumped together. The completion rate is the $(i, k)^{th}$ element of $(I+Q+Q^2+...+Q^{x-1})R$. Table 2 is a summary of the completion rates within x years using a single absorbing state model

Table 2 Completion rates within x years using a single absorbing state model

Years(x)	I	II	III	IV
1	0.0800	0.0824	0.1224	0.8788
2	0.1577	0.1942	0.8781	0.9853
3	0.2577	0.8801	0.9849	0.9982
4	0.8731	0.9849	0.9982	0.9998
5	0.9831	0.9982	0.9998	1.0000
6	0.9979	0.9998	1.0000	1.0000
7	0.9997	1.0000	1.0000	1.0000
8	1.0000	1.0000	1.0000	1.0000
9	1.0000	1.0000	1.0000	1.0000
10	1.0000	1.0000	1.0000	1.0000

From Table 2, the completion rate for students who were in grade I within the first two years, by the year 2006, was 15.57 percent. The completion rate is expected to be 25.77 percent by the year 2007 since the dropouts from the system comprised only of the students who had not attained the maximum qualification. Moreover, by the year 2008, it is expected that 87.31 percent of the students will drop out of the system since most of the students who were in grade I are expected to graduate from the system four years later. Eventually, students who repeated particular grades are expected to continue dropping out of the system such that by the year 2012, all grade I students will have dropped out of the system.

Similarly, 88.01 percent of the students who were in grade II are expected to drop out of the system by 2007.Also, by 2006, 87.81 per cent of the students who were in grade III are expected to drop out of the system. Finally, 87.88 of the students who were in grade IV had dropped out of the system one year later.

6 ABSORBING RATES

If students remained in the system indefinitely, then the absorbing rate is given by

$$r_{i1}^{(\infty)} = \sum_{n=1}^{\infty} r_{i1}^{(n)} \qquad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (10)$$

$$= (I + Q + Q^2 + ...)\, R$$

$$= (I - Q)^{-1} R$$

The absorbing rate under single absorbing state is

$$\begin{pmatrix} 1.0000 \\ 1.0000 \\ 1.0000 \\ 1.0000 \end{pmatrix}$$

In Table 3.2, we considered proportions of students who dropped out of the system with time. Most of the students who were in grade I, II, III and IV dropped out within four, three, two and one year respectively.

After these time periods students who had repeated continued dropping out. In the long run, all the students are expected to drop out of the system, thus justifying the eventual invariant entry of 1 with time. The absorbing time gives us the eventual long run scenario thus under single absorbing state, all the entries are 1.

7 CONCLUSIONS

In the single absorbing state case, all the students who were discontinued, who dropped out with particular final grades and those who successfully graduated from the system were grouped together. It was impossible to ascertain the proportions of students who were grouped in each category. Hence, the major limitation of the single absorbing state model since it was not possible to determine whether most of the students successfully graduated from the system, as would be the main objective of the education system.

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