# PROBABILITY DISTRIBUTIONS BASED ON DIFFERENCE DIFFERENTIAL EQUATIONS FOR PURE BIRTH PROCESSES

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## DECLARATION

I declare that this is my original work and has not been presented for an award of a degree in any other university

**Signature ………………………… Date ………………..**

**Charles Kamunge Karanja**

This thesis is submitted for examination with my approval as the university supervisor

**Signature ………………………………… Date……………………………..**

**Prof J.A.M Ottieno**

## DEDICATION

***To all those who believe God can raise them up from the ashes.***

## EXECUTIVE SUMMARY

The main objective of this work is to identify probability distributions emerging by solving difference- differential equations of a pure birth process given by $p'_0(t) = -\lambda_0 p_0(t)$ for n = 0 and $p'_n(t) = -\lambda_n p_n(t) + \lambda_{n-1} p_{n-1}(t)$, for $n \geq 1$. The special cases are: Poisson Process $(\lambda_n = \lambda)$, Simple Birth process $(\lambda_n = n\lambda)$, in Simple Birth process with immigration $(\lambda_n = n\lambda + v)$ and the Polya process $\left(\lambda_n = \left(\frac{1 + an}{1 + \lambda at}\right)\lambda\right)$.

Four methods have been applied in solving the difference – differential equations are:

(1) the iterative technique

(2) the Laplace Method

(3) the Langranges Method

(4) the generator Matrix technique.

The results through the four Methods are similar. The means and Variances were obtained by definition, the pgf technique and by the method of moments:

The results are:

From the Poisson process, we obtain Poisson distribution with parameter $\lambda t$ both when the initial condition is $X(0) = 0$ and also when $X(0) = n_0$. i.e. $p_n(t) = \frac{e^{-\lambda t}(\lambda t)^n}{n!}$ for n = 0, 1, 2, .

$p_n(t) = \frac{e^{-\lambda t}(\lambda t)^k}{k!}$ for $n = n_0 + k$ and $k = 0, 1, 2, ....,$

The mean is $E[X(t)] = \lambda t$ and the variance is $Var[X(t)] = \lambda t$.

From the Simple Birth process we obtain a geometric distribution, $p_n(t) = e^{-\lambda t}\left(1 - e^{-\lambda t}\right)^{n-1}$, when the initial condition is $X(0) = 1$ and a negative binomial distribution,

$p_{n_0+k}(t) = \binom{k + n_0 - 1}{k} \times \left(1 - e^{-\lambda t}\right)^k \times \left(e^{-\lambda t}\right)^{n_0}, k = 0, 1, 2, ...,$ when the initial condition is $X(0) = n_0$.

The mean is $E(X) = n_0 e^{-\lambda t}$ and variance $n_0 e^{2\lambda t}\left(1 - e^{-\lambda t}\right)$.

From the Simple Birth Process with immigration, we obtain a negative binomial distribution

$$p_{n_0+j}(t) = \binom{n_0 + \frac{v}{\lambda} + j - 1}{j}\left(e^{-\lambda t}\right)^{n_0+\frac{v}{\lambda}}\left(1 - e^{-\lambda t}\right)^{j} \qquad j = 0, 1, 2, \ldots$$

The mean $E\left[X(t)\right] = \dfrac{n_0 + \frac{v}{\lambda}\left(1 - e^{-\lambda t}\right)}{e^{-\lambda t}}$ and the variance $Var\left[X(t)\right] = \left(n_0 + \dfrac{v}{\lambda}\right)e^{\lambda t}\left(e^{\lambda t} - 1\right)$.

From the Polya process, we obtain a negative binomial distribution

$$p_{n_0+k}(t) = \binom{n_0 + \frac{1}{a} + k - 1}{k}\times\left(\frac{1}{1 + \lambda at}\right)^{n_0+\frac{1}{a}}\times\left(1 - \frac{1}{(1 + at)^{\lambda}}\right)^{k} \qquad k = 0, 1, 2, \ldots$$

The mean is $\lambda t$ and the variance $Var\left[X(t)\right] = \lambda t(1 + a\lambda t)$.

**Exceptions**

Laplace Method did not work for the Polya process

Generator matrix could not work for the Poisson process.

## ACKNOWLEDGEMENT

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# CHAPTER ONE

## GENERAL INTRODUCTION

### 1.1 Background Information

When a Bank opens, the customers enter to be served. This involves queuing or waiting for service. After being served, one leaves. This process involves coming in, service and going out until the time of closure. At closure time, no entry is allowed. There is only exit. This process is known as a birth and death process. At the opening time, we have entry only which is analogous to pure birth process. At closure time, there is exit only which is analogous to the pure death process. Time in between, we have both getting in and going out which is analogous to the birth and death process. In situations like the emission of electrons in physics, we have only pure Birth process.

Such processes can be described mathematically through what is called the Stochastic Processes.

### 1.2 Stochastic Processes

### Definition and classifications

A stochastic process is a family of random variables $\{X(t);\, t > 0\}$ on a probability space with t ranging over a suitable parameter set T (t often represents time). The **state space** of the process is a set S in which possible values of each $X(t)$ lie. This $X(t)$ can be either discrete or continuous. The set of t is called **parameter space** T. The parameter t can also be either discrete or continuous. The parameter space is said to be discrete if the set T is countable otherwise it is continuous.

Thus, we have the following four classifications of stochastic processes.

**1** Discrete Parameter and Discrete state

**2** Discrete Parameter and Continuous state

**3** Continuous Parameter and Discrete state

**4** Continuous Parameter and Continuous state

Further, stochastic processes are broadly described according to the nature of dependence relationships existing among the members of the family. Some of the relationships are characterized by

(i) Stationary process

A process $\{X(t), t \in T\}$ is said to be a stationary if different observations on time intervals of the same length have the same distribution.

i.e. For any $s, t \in T, X(t+s) - X(t)$ has the same distribution as $X(s) - X(0)$.

(ii) Markov Processes

A process $X(t)$ is a **Markov process** if given the event $X(t)$ at some time t, the future event $X(s)$ for $s > t$, depend only on the immediate past and not the remote past. This property is referred to as the Markov or the memoryless property.

A **Markov chain** is a discrete time space Markov process with discrete state space. A **Markov jump process** is a continuous time space Markov process with discrete state space.

(iii) Processes with independent increments

A process $X(t)$ is called a **process with independent increments** (or an additive Process) if for $t' > t, X(t') - X(t)$ is independent of $X(t)$. It is a Markov Process and is said to be homogeneous if the distribution of $X(t') - X(t)$ depends only on $t' - t$.

(iv) Martingales process

A stochastic process $\{X(t), t \geq 0\}$ with finite means is said to be a (continuous Parameter) Martingale if the conditional expectation of $X(t_{n+1})$, given the values $X(t_1), X(t_2), \ldots, X(t_n)$ is equal to the most recently observed value $X(t_n)$.

i.e. If for any set of times

$t_1 < t_2 < \ldots < t_n$, $E\left[X(t_{n+1}) \mid X(t_1), X(t_2), \ldots, X(t_n)\right] = X(t_n)$.

A stochastic process $\{X_n, n = 1, 2, \ldots\}$ with finite means is said to be a (discrete Parameter) Martingale if for any integer n, $E\left[X_{n+1} \mid X_1, X_2, \ldots, X_n\right] = X_n$.

(v) Point Process

When we consider events as occurring in continuous one dimensional time, we consider a process as a point process when interest is concentrated on the individual occurrences of the events themselves, events being distinguished only by their positions in time, rather than concentrating on a group of individuals.

### 1.3 Pure Birth Processes

Pure birth process is a continuous time, discrete state Markov process. Specifically, we deal with a family of random variables $\{X(t); 0 < t < \infty\}$ where the possible values of $X(t)$ are non negative integers. $X(t)$ represents the population size at time t and the transitions are limited to birth. When a birth occurs, the process goes from state $n$ to state $n+1$. If no birth occurs, the process remains at the current state. The process cannot move from higher state to a lower state since there is no death. The birth process is characterized by the birth rate $\lambda_n$ which varies according to the state n of the system.

### 1.4 Literature Review

The negative binomial arises from several stochastic processes. The time - homogeneous birth and immigration process with zero initial population was first obtained by McKendrick (1914). The non - homogeneous process with zero initial population known as the Polya process was developed by Lundberg (1940) in the context of risk theory.

Other stochastic processes that lead to negative binomial distribution include the simple birth process with non – zero initial population size (Yule, 1925; Furry, 1937). Simple birth process was originally introduced by Yule (1924) to model new species evolution and by Furry (1937) to model particle creation.

Kendall (1948) considered non homogeneous birth – and – death process with zero death rate. He also worked on the simple birth – death – and immigration process with zero initial population (Kendall 1949).

A remarkable new derivation as the solution of the simple birth – and – immigration process was given by Mckendrick(1914).

Kendall (1949) formed Lagrange’s equation from the differential difference equations for a distribution of the population, and via auxiliary equation, obtained a complete solution of the equations governing the generalized birth and death process in which the birth rate and death rates may be any specified function of time.

Karlin and Mcgregor (1958) expressed transitional probabilities of birth and death processs (BDPs) in terms of a sequence of orthogonal polynomials and spectra Measures. Birth rate and death rate uniquely determine the unique measure on the real axis with respect to the sequence of orthogonal polynomials. Their work gave valuable insights about the existence of unique solution of a given process.

Gani and swift (2008) attempted to derive equations for the probability generating function of the Poisson, pure birth and death process subject to mass movement immigration and emigration. He considered mass movement immigration and emigration as positive and negative mass movements. The resulting probability generating functions turned out to be a product of the probability generating functions of the original processes modified by immigration process.

Janardan (1994) developed a stochastic model to study the number of children born to a couple up to time t. The model was constructed under the assumption that the rate at which a couple already having two children goes for subsequent child is smaller than the rate at which the first and the second child are born.

Janardan (2003) considered pure birth process starting with no individual, with birth rates $\lambda_n = \lambda$ for $n = 0, 1, \ldots, m-1$ and $\lambda_n = \mu$ for $n > m$. Using integral representation, he obtained the associated distribution. This was an extension of the work he had done in 1980 while analyzing data on the hyper distributed eggs laid by a weevil on mung beans.

Trobaugh, D.E. et al (1969), applying pure birth processes and directly relating accident rates to the total number of accidents, presented a method predicting aircraft accidents.

## 1.5 Problem Statement and Objectives of the Study

### Statement of the Problem

Only a few researchers and textbook authors have presented alternative approaches for solving pure birth processes basic difference differential equations and even then only sketch them in outline or present details in different sections.

### Objectives

(a) To solve pure birth basic difference differential equations for different processes using four different approaches.

The four different approaches are the iteration Method, the Laplace approach, the probability Generating Function Method and the Generator Matrix Method.

(b) To critically review and put together the various work done in this area by other researchers.

## 1.6 Areas of Application

Pure birth and death processes play a fundamental role in the theory and applications that embrace population growth. Examples include the spread of new infections in cases of a disease where each new infection is considered as a birth.

Pure birth and death processes have a lot of application in the following areas.

(a) Radioactivity

Radioactive atoms are unstable and disintegrate stochastically. Each of the new atoms is also unstable. By the emission of radioactive particles these new atoms pass through a number of physical states with specified decay rates from one state to the adjacent. Thus radioactive transformation can be modeled as birth process.

(b) Communication

Suppose that calls arrive at a single channel telephone exchange such that successive calls arrivals are independent exponential random variables. Suppose that a connection is realized if the incoming call finds an idle channel. If the channel is busy, then the incoming call joins the queue. When the caller is through, the next caller is connected. Assuming that the successive service times are independent exponential variables, the number of callers in the system at time t is described by a birth and death process.

(c) Biological field

Theory of birth – and death processes provide a natural mathematical framework for modeling a variety of biological processes. Examples of these biological processes include population dynamics such as the spreading of infectious diseases, somatic evolution of cancers among others.

(d) Industry

Suppose that a number of automatic machines are serviced by an operator. Owing to random mistakes, the machines may break down and call for service. If we assume that the machines work independently and that the operator is busy if there is a machine in the waiting line and that the service times are identical and independent random variables. Furthermore, if we suppose that the service times are identically distributed, independent random variables with a known distribution function, then such a case can also be modeled as a birth process.

# CHAPTER TWO

## GENERAL BIRTH PROCESS

### 2.1 Introduction

In this Chapter, we will derive the basic difference differential equations from the first principles and also state the assumptions that are made in the derivation. In addition, we will highlight on some of the Mathematical tools which are a prerequisite before venturing into the solving of these equations. These mathematical tools will be highlighted for each of the four methods that will be used in solving these equations. In addition, the key steps to be followed while using each of the methods to solve the basic difference differential equations are also given. These methods are the Iteration Method, the Laplace method, the probability generating function method and the generator Matrix Method.

### 2.2 Derivation of the Basic Difference Differential Equations for the General Pure Birth process

The Main Objective of this sub topic is to derive what is called the basic differential equations.

**Definitions**

Let $X(t) =$ the population at time t

Let $p_n(t) = \text{Prob}\left[X(t) = n\right]$

$\therefore \quad p_n(t + \Delta t) = \text{Prob}\left[X(t + \Delta t) = n\right]$

We wish to find $p'_n(t) = \lim_{\Delta t \to 0} \left\{ \dfrac{p_n(t + \Delta t) - p_n(t)}{\Delta t} \right\}$ from the first principles of calculus.

**Assumptions**

The event is birth

(i) The probability of having a birth between time $t$ and $t + \Delta t$ when $X(t) = n$ is given by

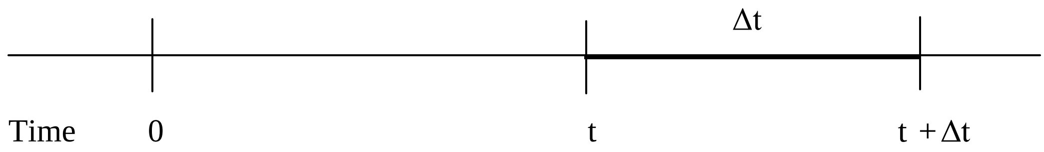
$\lambda_n \Delta t + o(\Delta t)$ where $o(\Delta t)$ is of order $\Delta t$

A function is of order $\Delta t$ if it tends to 0 (zero) faster than $\Delta t$

(ii) The probability of having no birth within the interval $\Delta t$ is $1 - \left[\lambda_n \Delta t + o(\Delta t)\right]$

(iii) The probability of two or more births within the interval $\Delta t$ is $o(\Delta t)$ i.e. negligible.

Diagrammatically




Size $\quad X(0) \qquad X(t) \qquad X(t+\Delta t)$

If $\;X(t+\Delta t) = n\;$ what is X(t)?

$$X(t+\Delta t) = \begin{cases} X(t) = n & \text{if no birth within the interval } \Delta t \\ X(t) = n-1 & \text{if a birth occurs within the interval } \Delta t \end{cases}$$

Therefore,

$$p_n(t+\Delta t) = \text{Prob}\left[X(t+\Delta t) = n\right]$$

$$= \text{Prob}\left[X(t+\Delta t) = n,\, X(t) = n\right] + \text{Prob}\left[X(t+\Delta t) = n,\, X(t) = n-1\right]$$

Further,

$$p_n(t+\Delta t) = \text{Prob}\left[X(t+\Delta t) = n / X(t) = n\right]\text{Prob}\left[X(t) = n\right] +$$

$$\text{Prob}\left[X(t+\Delta t) = n / X(t) = n-1\right]\text{Prob}\left[X(t) = n-1\right]$$

$$= \left\{\text{Prob}\left[\text{of no birth in the interval}\right] p_n(t)\right\} + \left\{\text{Prob}\left[\text{of a birth in the interval}\right] p_{n-1}(t)\right\}$$

$$= \left\{1 - \left[\lambda_n \Delta t + o(\Delta t)\right]\right\} p_n(t) + \left[\lambda_n \Delta t + o(\Delta t)\right] p_{n-1}(t)$$

Therefore,

$$p_n(t+\Delta t) - p_n(t) = \left\{1 - \lambda_n \Delta t - o(\Delta t)\right\} p_n(t) + \left[\lambda_{n-1}\Delta t + o(\Delta t)\right] p_{n-1}(t) - p_n(t)$$

$$= p_n(t) - \lambda_n \Delta t\, p_n(t) - o(\Delta t) p_n(t) + \lambda_{n-1}\Delta t\, p_{n-1}(t) + o(\Delta t) p_{n-1}(t) - p_n(t)$$

$$= -\lambda_n \Delta t\, p_n(t) - o(\Delta t) p_n(t) + \lambda_{n-1}\Delta t\, p_{n-1}(t) + o(\Delta t) p_{n-1}(t)$$

Dividing both sides by $\Delta t$

$$\frac{p_n(t+\Delta t) - p_n(t)}{\Delta t} = -\lambda_n\, p_n(t) - \frac{o(\Delta t)}{\Delta t} P_n(t) + \lambda_{n-1} p_{n-1}(t) + \frac{o(\Delta t)}{\Delta t} p_{n-1}(t)$$

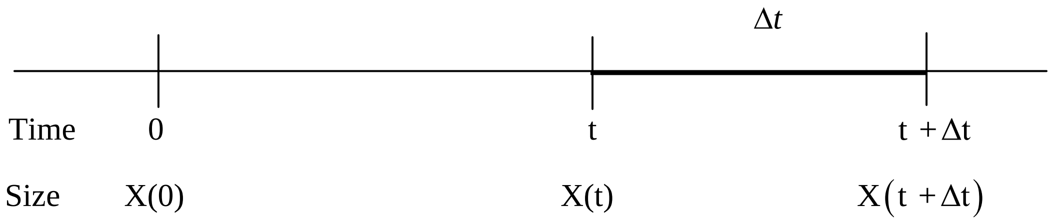
But since $\lim\limits_{\Delta t \to 0} \dfrac{o(\Delta t)}{\Delta t} = 0$

$$\lim_{\Delta t \to 0} \frac{p_n(t+\Delta t) - p_n(t)}{\Delta t} = -\lambda_n\, p_n(t) + \lambda_{n-1} p_{n-1}(t), \quad n \ge 1$$

Therefore

$$p_n'(t) = -\lambda_n\, p_n(t) + \lambda_{n-1} p_{n-1}(t)$$

For $n = 0$




$X(t+\Delta t) = 0 \;\Rightarrow\; X(t) = 0 \;\Rightarrow\;$ No birth

$$p_0(t+\Delta t) = \text{Prob}\left[X(t+\Delta t) = 0\right]$$

$$= \text{Prob}\left[X(t+\Delta t) = 0,\, X(t) = 0\right]$$

$$= \text{Prob}\left[X(t+\Delta t) = 0 / X(t) = 0\right]\text{Prob}\left[X(t) = 0\right]$$

$$= \left\{1 - \left[\lambda_0 \Delta t + o(\Delta t)\right]\right\} p_0(t)$$

$\Rightarrow$

$$p_0(t+\Delta t) - p_0(t) = -\left[\lambda_0 \Delta t + o(\Delta t)\right] p_0(t)$$

Therefore,

$$\frac{p_0(t+\Delta t) - p_0(t)}{\Delta t} = -\left[\lambda_0 + \frac{o(\Delta t)}{\Delta t}\right] p_0(t)$$

Therefore

$$\lim_{\Delta t \to 0} \frac{p_0(t+\Delta t) - p_0(t)}{\Delta t} = -\left[\lambda_0 + \lim_{\Delta t \to 0} \frac{o(\Delta t)}{\Delta t}\right] p_0(t)$$

And since $\lim_{\Delta t \to 0} \dfrac{o(\Delta t)}{\Delta t} = 0$

Therefore

$$p_0'(t) = -\lambda_0\, p_0(t)$$

In summary, the basic difference Differential equations for the general birth process are

$$\mathbf{p_0'(t) = -\lambda_0\, p_0(t)} \qquad \textbf{(2.1)}$$

and

$$\mathbf{p_n'(t) = -\lambda_n\, p_n(t) + \lambda_{n-1} p_{n-1}(t), \quad n \geq 1} \qquad \textbf{(2.2)}$$

### 2.3 Iteration

- Solve the first of the two basic difference equations. You will get $p_0(t)$.
- Put n = 1 in the second differential equation to generate a recursive relation. Solve the differential equation using the integrating factor method. You will get $p_1(t)$.
- Repeat for some more higher values of n.
- Generalize using induction.

### 2.4 Solution of Linear Partial Differential Equations

Let P, Q and R be functions of x, y and z. suppose we have an equation of the form

$$P\frac{dz}{dx} + Q\frac{dz}{dy} = R \qquad \textbf{(2.3)}$$

subject to some appropriate boundary conditions. Such an equation is called a linear partial differential equation.

The procedure for solving the equation takes the following steps;

**STEP I:** Form subsidiary equations given by

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \qquad \textbf{(2.4)}$$

The subsidiary equations are also called the AUXILIARY equations. Note that there are 3 subsidiary equations from **(2.4)**, namely,

$$\frac{dx}{P} = \frac{dy}{Q} \qquad (i)$$

$$\frac{dx}{P} = \frac{dz}{R} \qquad (ii)$$

$$\frac{dy}{Q} = \frac{dz}{R} \qquad (iii)$$

**STEP II:** Consider any two equations and solve them

**STEP III:** Solutions of the two considered subsidiary equations are in the form

$$U(x, y, z) = \text{Constant} \quad \textbf{(2.5)}$$

and

$$V(x, y, z) = \text{Constant} \quad \textbf{(2.6)}$$

**STEP IV:** The most general solution of **(1.3)** is now given by

$$u = \psi(v)$$

where $\psi$ is an arbitrary function. The precise form of this function is determined when the boundary conditions have been inserted.

**Procedure**

- Define $G(s,t) = \sum_{n=0}^{\infty} p_n(t)s^n$. Consequently define $\frac{d}{ds}G(s,t)$ and $\frac{d}{dt}G(s,t)$.
- Multiply both sides of the second differential equation by $s^n$ and sum over n, and taking advantage of the initial conditions, write the resulting equation in terms of the definitions above. You will get an equation of the form of the equation **(2.3).**
- Summarize the results by a single Lagrange Partial differential equation for a generation function and then solving the resulting equation by means of auxiliary equations as in the procedure shown above. You will get $G(s,t)$.
- $p_n(t)$ is the coefficient of $s^n$ in the expansion of $G(s,t)$.

**2.5 Laplace Transforms**

**Definition:**

Let $f(t)$ be a function of a positive real variable t. Then the Laplace transform (L.T) of $f(t)$ is defined by $\bar{f}(s) = \int_0^{\infty} e^{-st} f(t)dt$ for the range of values of s for which the integral exists. It is also written in the form $L(f(t))$.

**Laplace transform of $\mathbf{f'(t)}$**

$$L[f'(t)] = \int_0^{\infty} e^{-st} f'(t)\, dt \quad \textbf{(2.7)}$$

Using integration by parts

$$\int v\,du = uv - \int u\,dv \qquad \textbf{(2.8)}$$

Let $v = e^{-st} \Rightarrow \frac{dv}{dt} = -se^{-st}$

Also, let $du = f'(t) \quad \Rightarrow \quad u = f(t)$

Substituting in equation **(2.8)**

$$\int_0^\infty e^{-st} f'(t)\,dt = \left[f(t).e^{-st}\right]_0^\infty - \int_0^\infty f(t).-se^{-st}\,dt$$

$$= \left[f(t).e^{-st}\right]_0^\infty + s\int_0^\infty f(t).\,e^{-st}\,dt$$

$$= \left[(0) - (f(0))\right] + s\int_0^\infty f(t).\,e^{-st}\,dt$$

$$= s\int_0^\infty f(t).\,e^{-st}\,dt - f(0)$$

Thus

$$L\left[f'(t)\right] = sL\left[f(t)\right] - f(0)$$

Substituting $f(t)$ with $p_n(t)$, the equation above becomes

$$L\left[p_n'(t)\right] = sL\left[p_n(t)\right] - p_n(0) \qquad \textbf{(2.9)}$$

**Procedure**

- Take the Laplace transform of the second of the two basic difference equations.
- Apply the relation $L\left[p_n'(t)\right] = sL\left[p_n(t)\right] - p_n(0)$ to replace $L\left[p_n'(t)\right]$ and simplify leaving $L\left[p_n(t)\right]$ as the subject of the formula.
- Starting with the conditions at $t = 0$, generate a recursive relation and use it to generalize for $L\left[p_n(t)\right]$.
- Find the Laplace inverse of the $L\left[p_n(t)\right]$ so got.

### 2.6 Generator Matrix

- Replace $\lambda_n$ in the equation $p_{i+k}(t) = \left\{\prod_{j=0}^{k-1}\lambda_{i+j}\right\}\left\{\sum_{r=0}^{k}\frac{e^{-\lambda_{i+r}t}}{\prod_{\substack{j=0\\j\neq r}}^{k-1}\left(\lambda_{i+j}-\lambda_{i+r}\right)}\right\}$ and simplify.

# CHAPTER THREE

# POISSON PROCESS

## 3.1 Introduction

A Poisson process is a pure birth process when the rate is constant i.e. when $\lambda_n = \lambda$. The formal definition is given as follows.

**Definition 1**

A counting process $\{X(t),\ t \geq 0\}$ is said to be a Poisson process with rate $\lambda > 0$ if

(1) $X(0) = 0$

(2) $X(t)$ is a process with independent increments

(3) The number of events in any interval of length t is Poisson distributed with rate $\lambda t$; i.e.

for all $s,\ t \geq 0,\ \text{Prob}\left[X(t+s) - X(s) = x\right] = \dfrac{e^{-\lambda t}(\lambda t)^x}{x!};\ x = 0, 1, 2, \ldots$ **(3.1)**

**Definition 2**

A counting process $\{X(t),\ t \geq 0\}$ is said to be a Poisson process with rate $\lambda > 0$ if

(1) X(0) = 0

(2) X(t) is a process with independent and stationary increments

(3) (i) $\text{Prob}\{X(t+h) - X(t) = 1\} = \lambda h + o(h)$ **(3.2)**

(ii) $\text{Prob}\{X(t+h) - X(t) \geq 2\} = o(h)$ **(3.3)**

where a function f(x) is said to be of order o(h) if $\lim\limits_{h \to 0} \dfrac{f(h)}{h} = 0$.

The objective of this Chapter is to solve the basic difference differential equations given in Chapter 2 (refer to equations **(2.1)** and **(2.2)**) when $\lambda_n = \lambda$. We shall specifically look at three methods namely the iterative method, the Laplace transform and the Lagranges method. We shall look at each of these methods when the initial conditions are (i) X(0) = 0 and (ii) $X(0) = n_0$

When $\lambda_n = \lambda\ \forall\ n$ the basic differential equations now become

$$p_0'(t) = -\lambda\, p_0(t) \quad \textbf{(3.4)}$$

$$p_n'(t) = -\lambda\, p_n(t) + \lambda\, p_{n-1}(t), \quad n \geq 1 \quad \textbf{(3.5)}$$

### 3.2 Iteration Method

### 3.2.1 Deriving $p_n(t)$ Iteration Method

***Initial conditions: When t = 0, X(0) = 0***

From equation **(3.4)**

$$p_0'(t) = -\lambda\, p_0(t)$$

$$\frac{p_0'(t)}{p_0(t)} = -\lambda$$

$$\frac{d}{dt}\left[\ln p_0(t)\right] = -\lambda$$

Integrating both sides with respect to *t*, we have

$$\int \frac{d}{dt}\left(\ln p_0(t)\right) dt = -\int \lambda\, dt$$

$$\ln p_0(t) = -\lambda t + c$$

$$p_0(t) = ke^{-\lambda t}$$

From the initial conditions, when t = 0, $X(0) = 0$. This implies that $p_0(0)=1$ and

$p_n(0)=0\ \forall\ n \neq 0.$ Thus,

$$p_0(0) = k \times e^{-\lambda \times 0}$$

$$1 = k \times 1$$

$$k = 1$$

Therefore

$$p_0(t) = e^{-\lambda t} \tag{3.6}$$

We can use the second difference differential equation to obtain recursive relation when $n \geq 1$.

Substituting $n = 1$ in equation **(3.5)**, we have

$$p_1'(t) = -\lambda\, p_1(t) + \lambda\, p_0(t)$$

$$= -\lambda\, p_1(t) + \lambda \times e^{-\lambda t}$$

Rearranging, we have

$$p_1'(t) + \lambda\, p_1(t) = \lambda \times e^{-\lambda t}$$

Next, we integrate the above equation by integrating factor method.

The integrating factor $= e^{\int \lambda dt} = e^{\lambda t}$

Multiplying the equation above by the integrating factor and simplifying, we have

$$e^{\lambda t} \times p_1'(t) + e^{\lambda t} \times \lambda\, p_1(t) = e^{\lambda t} \times \lambda \times e^{-\lambda t}$$

$$\frac{d}{dt}\left[e^{\lambda t} p_1(t)\right] = \lambda$$

Integrating both sides with respect to t

$$\int d\left(e^{\lambda t} p_1(t)\right) dt = \int \lambda\, dt$$

$$e^{\lambda t} p_1(t) = \lambda t + c_1$$

Therefore

$$p_1(t) = \left(\lambda t + c_1\right) e^{-\lambda t}$$

From the initial conditions when t = 0, $X(0) = 0$ .This implies that. $p_0(0)=1.$ Therefore

$$p_1(0) = \left(\lambda \times 0 + c_1\right) e^{-\lambda \times 0}$$

$$0 = \left(0 + c_1\right)$$

$$c_1 = 0$$

Thus,

$$p_1(t) = \lambda t e^{-\lambda t} \qquad \textbf{(3.7)}$$

For $n = 2$

$$p_2'(t) = -\lambda\, p_2(t) + \lambda\, p_1(t)$$

$$p_2'(t) = -\lambda\, p_2(t) + \lambda \times \lambda t e^{-\lambda t}$$

Re arranging,

$$p_2'(t) + \lambda\, P_2(t) = \lambda^2 t e^{-\lambda t}$$

Next, we integrate the above equation by use of integrating factor method.

The integrating factor is $e^{\int \lambda dt} = e^{\lambda t}$

Thus

$$e^{\lambda t} \times p_2'(t) + e^{\lambda t} \times \lambda\, p_2(t) = e^{\lambda t} \times \lambda^2 t e^{-\lambda t}$$

$$\frac{d}{dt}\left[e^{\lambda t} p_2(t)\right] = \lambda^2 t$$

Integrating both sides with respect to t

$$\int d\left(e^{\lambda t} p_2(t)\right) dt = \int \lambda^2 t \, dt$$

$$e^{\lambda t} p_2(t) = \frac{\lambda^2 t^2}{2} + c_2$$

Equivalently

$$p_2(t) = \left(\frac{\lambda^2 t^2}{2} + c_2\right) e^{-\lambda t}$$

From the initial conditions, when $t = 0$, $P_2(0) = 0.$ The equation above becomes

$$p_2(0) = (0 + c_2) e^{-\lambda \times 0}$$
$$\Rightarrow$$
$$c_2 = 0$$

Therefore,

$p_2(t) = \left(\frac{\lambda^2 t^2}{2}\right) e^{-\lambda t}$ which can also be written as

$$p_2(t) = \frac{(\lambda t)^2}{2} e^{-\lambda t} \qquad \textbf{(3.8)}$$

For $n = 3$, equation **(3.5)** becomes

$$p_3'(t) = -\lambda \, p_3(t) + \lambda \, p_2(t)$$

$$p_3'(t) = -\lambda \, p_3(t) + \lambda \times \left(\frac{\lambda^2 t^2}{2}\right) e^{-\lambda t}$$

Rearranging

$$p_3'(t) + \lambda \, p_3(t) = \lambda \times \left(\frac{\lambda^2 t^2}{2}\right) e^{-\lambda t}.$$

Next, we integrate the above equation by use of integrating factor method.

Integrating factor $= e^{\int \lambda dt} = e^{\lambda t}$. Multiplying the equation above by the integrating factor, we have

$$e^{\lambda t} \times p_3'(t) + e^{\lambda t} \times \lambda \, p_3(t) = e^{\lambda t} \times \lambda \times \left(\frac{\lambda^2 t^2}{2}\right) e^{-\lambda t}$$

$$\frac{d\left[e^{\lambda t} p_3(t)\right]}{dt} = \frac{\lambda (\lambda t)^2}{2} = \frac{\lambda^3 t^2}{2}$$

Integrating both sides with respect to t, we have

$$\int d\left(e^{\lambda t}p_3(t)\right)dt = \int \frac{\lambda^3 t^2}{2}dt$$

$$e^{\lambda t}p_3(t) = \frac{\lambda^3}{2}\times\frac{t^3}{3} + c_3$$

Therefore,

$$p_3(t) = \left(\frac{\lambda^3}{2}\times\frac{t^3}{3} + c_3\right)e^{-\lambda t}$$

But from the initial conditions, when $t = 0,\ p_3(0) = 0.$ Thus

$$p_3(0) = \left(\frac{\lambda^3}{2}\times\frac{0^3}{3} + c_3\right)e^{-\lambda\times 0}$$

$$c_3 = 0$$

Therefore

$p_3(t) = \left(\frac{\lambda^3}{2}\times\frac{t^3}{3}\right)e^{-\lambda t}$. This equation can also be written as

$$p_3(t) = \frac{(\lambda t)^3}{3!}e^{-\lambda t} \qquad \textbf{(3.9)}$$

Listing down our solutions from above, we obtain a pattern.

$$p_0(t) = e^{-\lambda t}$$

$$p_1(t) = \lambda t e^{-\lambda t}$$

$$p_2(t) = \frac{(\lambda t)^2}{2}e^{-\lambda t}$$

$$p_3(t) = \frac{(\lambda t)^3}{3!}e^{-\lambda t}$$

By induction, assume for $n = k - 1$

$$p_{k-1}(t) = \frac{(\lambda t)^{k-1}}{(k-1)!}e^{-\lambda t} \text{ is true.}$$

We want to find $p_k(t)$.

Putting $n = k$ in the equation **(3.5)** we have

$$p'_k(t) = -\lambda\, p_k(t) + \lambda\, p_{k-1}(t)$$

$$p'_k(t) = -\lambda\, p_k(t) + \lambda \times \frac{(\lambda t)^{k-1}}{(k-1)!} e^{-\lambda t}.$$

Re arranging

$$p'_k(t) + \lambda\, p_k(t) = \frac{\lambda^k t^{k-1} e^{-\lambda t}}{(k-1)!}$$

Next, we integrate the above equation by use of integrating factor method.

Integrating factor $= e^{\int \lambda dt} = e^{\lambda t}$. Multiplying both sides of the equation by the integrating factor, we get

$$e^{\lambda t} \times p'_k(t) + e^{\lambda t} \times \lambda\, p_k(t) = e^{\lambda t} \times \frac{\lambda^k t^{k-1} e^{-\lambda t}}{(k-1)!}$$

$$e^{\lambda t} \times p'_k(t) + e^{\lambda t} \times \lambda\, p_k(t) = \frac{\lambda^k t^{k-1}}{(k-1)!}$$

$$\frac{d\left[e^{\lambda t} p_k(t)\right]}{dt} = \frac{\lambda^k t^{k-1}}{(k-1)!}$$

Integrating both sides with respect to t, we have

$$\int d\left(e^{\lambda t} p_k(t)\right) dt = \frac{\lambda^k}{(k-1)!} \int t^{k-1} dt$$

$$\Rightarrow$$

$$\left[e^{\lambda t} p_k(t)\right] = \frac{\lambda^k t^k}{k(k-1)!} + c_k$$

$$e^{\lambda t} p_k(t) = \frac{\lambda^k t^k}{k(k-1)!} + c_k$$

Thus

$$p_k(t) = \left(\frac{\lambda^k t^k}{k(k-1)!} + c_k\right) e^{-\lambda t}$$

From the initial conditions, when $t = 0$, $p_k(0) = 0, k \neq 0.$ The equation above becomes

$p_k(0) = \left(\frac{\lambda^k \times 0^k}{k(k-1)!} + c_k\right) e^{-\lambda \times 0}$ which implies that $c_k = 0.$

Thus,

$p_k(t) = \left(\frac{\lambda^k t^k}{k(k-1)!}\right) e^{-\lambda t}$ which can also be written in the form,

$$\mathbf{p_k(t) = \frac{e^{-\lambda t}(\lambda t)^k}{k!}} \qquad k = 0, 1, 2, 3, \ldots \qquad \textbf{(3.10)}$$

which is Poisson probability density function with parameter $\lambda t$.

***Initial Condition: When t = 0, $X(0) = n_0$ $(n_0 \geq 1)$***

This implies that $p_{n_0}(0) = 1$, $p_n(t) = 0 \; \forall \; n \neq n_0$ and $p_n(t) = 0 \; \forall \; n < n_0$.

Substituting $n = n_0$, equation **(3.5)** becomes

$$p'_{n_0}(t) = -\lambda p_{n_0}(t) + \lambda p_{n_0 - 1}(t)$$

But from the initial conditions, $p_{n_0 - 1}(t) = 0$.

Thus

$$p'_{n_0}(t) = -\lambda p_{n_0}(t)$$

$$\frac{p'_{n_0}(t)}{p_{n_0}(t)} = -\lambda$$

$$\frac{d}{dt}\left(\log_e p_{n_0}(t)\right) = -\lambda$$

Integrating both sides with respect to t, we have

$$\int d\left(\ln p_{n_0}(t)\right) = \int -\lambda \, dt$$

$$\Rightarrow$$

$$\ln p_{n_0}(t) = -\lambda t + c$$

Taking the exponential of both sides, we have

$$p_{n_0}(t) = e^{-\lambda t + c} = e^{-\lambda t} e^{c} = k \, e^{-\lambda t}$$

From the initial conditions; When $t = 0$, $p_{n_0}(0) = 1$

This implies that $k = 1$

Therefore,

$$p_{n_0}(t) = e^{-\lambda t} \qquad \textbf{(3.11)}$$

Next let $n = n_0 + 1$

Equation **(3.5)** now becomes

$$p'_{n_0+1}(t) = -\lambda p_{n_0+1}(t) + \lambda p_{n_0}(t)$$

$$p'_{n_0+1}(t) = -\lambda p_{n_0+1}(t) + \lambda \times e^{-\lambda t}$$

Rearranging

$$p'_{n_0+1}(t) + \lambda p_{n_0+1}(t) = \lambda \times e^{-\lambda t}$$

Next, we integrate the above equation by use of integrating factor method.

Integrating factor for the equation above $= e^{\int \lambda dt} = e^{\lambda t}$

$$e^{\lambda t} \times p'_{n_0+1}(t) + e^{\lambda t} \times \lambda p_{n_0+1}(t) = e^{\lambda t} \times \lambda \times e^{-\lambda t}$$

$$\frac{d}{dt}\left[e^{\lambda t} \times p_{n_0+1}(t)\right] = \lambda$$

Integrating both sides with respect to t

$$\int d\left(e^{\lambda t} p_{n_0+1}(t)\right)dt = \int \lambda dt$$

$$e^{\lambda t} p_{n_0+1}(t) = \lambda t + c_1$$

Equivalently,

$$p_{n_0+1}(t) = (\lambda t + c_1)e^{-\lambda t}$$

The initial conditions are: When t = 0, $X(0) = n_0$. This implies that $p_{n_0+1}(0) = 0$. Therefore, at t = 0,

$$p_{n_0+1}(0) = (\lambda \times 0 + c_1)e^{-\lambda \times 0}$$

$$c_1 = 0$$

Therefore,

$$p_{n_0+1}(t) = (\lambda t)e^{-\lambda t} \qquad \textbf{(3.12)}$$

For $n = n_0 + 2$, the second differential equation become

$$p'_{n_0+2}(t) = -\lambda p_{n_0+2}(t) + \lambda p_{n_0+1}(t)$$

$$p'_{n_0+2}(t) = -\lambda p_{n_0+2}(t) + \lambda \times \lambda t e^{-\lambda t}$$

Re arranging,

$$p'_{n_0+2}(t) + \lambda p_{n_0+2}(t) = \lambda^2 t e^{-\lambda t}$$

Next, we integrate the above equation by use of integrating factor method.

The integrating factor is $e^{\int \lambda dt} = e^{\lambda t}$

Thus

$$e^{\lambda t} \times p'_{n_0+2}(t) + e^{\lambda t} \times \lambda\, p_{n_0+2}(t) = e^{\lambda t} \times \lambda^2 t e^{-\lambda t}$$

$$\frac{d\left[e^{\lambda t} p_{n_0+2}(t)\right]}{dt} = \lambda^2 t$$

Integrating both sides with respect to *t*

$$\int d\left[e^{\lambda t} p_{n_0+2}(t)\right] dt = \int \lambda^2 t\, dt$$

$$e^{\lambda t} p_{n_0+2}(t) = \frac{\lambda^2 t^2}{2} + c_2$$

Equivalently

$$p_{n_0+2}(t) = \left(\frac{\lambda^2 t^2}{2} + c_2\right) e^{-\lambda t}$$

From the initial conditions, when $t = 0$, $p_{n_0+2}(0) = 0$. Thus

$$p_{n_0+2}(0) = (0 + c_2) e^{-\lambda \times 0}$$

$$c_2 = 0$$

Therefore,

$$p_{n_0+2}(t) = \left(\frac{\lambda^2 t^2}{2}\right) e^{-\lambda t}$$ which can also be written as

$$p_{n_0+2}(t) = \frac{(\lambda t)^2}{2} e^{-\lambda t} \qquad \textbf{(3.13)}$$

For $n = n_0 + 3$, equation **(3.5)** becomes

$$p'_{n_0+3}(t) = -\lambda\, p_{n_0+3}(t) + \lambda\, p_{n_0+2}(t)$$

$$p'_{n_0+3}(t) = -\lambda\, p_{n_0+3}(t) + \lambda \times \left(\frac{\lambda^2 t^2}{2}\right) e^{-\lambda t}$$

Rearranging

$$p'_{n_0+3}(t) + \lambda\, p_{n_0+3}(t) = \lambda \times \left(\frac{\lambda^2 t^2}{2}\right) e^{-\lambda t}.$$

Next, we integrate the above equation by use of integrating factor method

Integrating factor $= e^{\int \lambda dt} = e^{\lambda t}$.

Multiplying the equation above by the integrating factor, we have

$$e^{\lambda t} \times p'_{n_0+3}(t) + e^{\lambda t} \times \lambda\, p_{n_0+3}(t) = e^{\lambda t} \times \lambda \times \left(\frac{\lambda^2 t^2}{2}\right) e^{-\lambda t}$$

$$\frac{d}{dt}\left[e^{\lambda t} p_{n_0+3}(t)\right] = \frac{\lambda(\lambda t)^2}{2} = \frac{\lambda^3 t^2}{2}$$

Integrating both sides with respect to *t*, we have

$$\int d\left[e^{\lambda t} p_{n_0+3}(t)\right] dt = \int \frac{\lambda^3 t^2}{2} dt$$

$$e^{\lambda t} p_{n_0+3}(t) = \frac{\lambda^3}{2} \times \frac{t^3}{3} + c_3$$

Equivalently,

$$p_{n_0+3}(t) = \left(\frac{\lambda^3}{2} \times \frac{t^3}{3} + c_3\right) e^{-\lambda t}$$

From the initial conditions, when t = 0, $p_{n_0+3}(0) = 0$. Thus at $t = 0$,

$$p_{n_0+3}(0) = \left(\frac{\lambda^3}{2} \times \frac{0^3}{3} + c_3\right) e^{-\lambda \times 0}$$

$$c_3 = 0$$

Therefore,

$p_{n_0+3}(t) = \left(\frac{\lambda^3}{2} \times \frac{t^3}{3}\right) e^{-\lambda t}$ which can be written as

$$p_{n_0+3}(t) = \frac{(\lambda t)^3}{3!} e^{-\lambda t} \qquad \textbf{(3.14)}$$

Listing down our solutions from above, we obtain a pattern.

$$p_{n_0}(t) = e^{-\lambda t}$$

$$p_{n_0+1}(t) = \lambda t e^{-\lambda t}$$

$$p_{n_0+2}(t) = \frac{(\lambda t)^2}{2} e^{-\lambda t}$$

$$p_{n_0+3}(t) = \frac{(\lambda t)^3}{3!} e^{-\lambda t}$$

By induction, assume that for $n = n_0 + k - 1$, $p_{n_0 + k - 1}(t) = \frac{(\lambda t)^{k-1}}{(k-1)!} e^{-\lambda t}$ is true.

Then, equation **(3.5)** becomes

$$p'_{n_0 + k}(t) = -\lambda\, p_{n_0 + k}(t) + \lambda\, p_{n_0 + k - 1}(t)$$

$$p'_{n_0 + k}(t) = -\lambda\, p_{n_0 + k}(t) + \lambda \times \frac{(\lambda t)^{k-1}}{(k-1)!} e^{-\lambda t}.$$

Re arranging

$$p'_{n_0 + k}(t) + \lambda\, p_{n_0 + k}(t) = \frac{\lambda^k t^{k-1} e^{-\lambda t}}{(k-1)!}$$

Next, we integrate the above equation by use of integrating factor method.

Integrating factor $= e^{\int \lambda dt} = e^{\lambda t}$.

Multiplying both sides of the equation by the integrating factor, we get

$$e^{\lambda t} \times p'_{n_0 + k}(t) + e^{\lambda t} \times \lambda\, p_{n_0 + k}(t) = e^{\lambda t} \times \frac{\lambda^k t^{k-1} e^{-\lambda t}}{(k-1)!}$$

$$= \frac{\lambda^k t^{k-1}}{(k-1)!}$$

This equation can now be written in the form

$$\frac{d}{dt}\left[e^{\lambda t} p_{n_0 + k}(t)\right] = \frac{\lambda^k t^{k-1}}{(k-1)!}$$

Integrating both sides with respect to t, we have

$$\int d\left[e^{\lambda t} p_{n_0 + k}(t)\right] dt = \frac{\lambda^k}{(k-1)!} \int t^{k-1} dt$$

$$\left[e^{\lambda t} p_{n_0 + k}(t)\right] = \frac{\lambda^k t^k}{k(k-1)!} + c_k$$

$$e^{\lambda t} p_{n_0 + k}(t) = \frac{\lambda^k t^k}{k(k-1)!} + c_k$$

Equivalently,

$$p_{n_0 + k}(t) = \left(\frac{\lambda^k t^k}{k(k-1)!} + c_k\right) e^{-\lambda t}$$

From the initial conditions, when $t = 0$, $P_{n_0 + k}(0) = 0$. The above equation becomes

$$p_{n_0 + k}(0) = \left( \frac{\lambda^k \times 0^k}{k(k-1)!} + c_k \right) e^{-\lambda \times 0}$$

$$c_k = 0$$

Thus,

$p_{n_0 + k}(t) = \left( \frac{\lambda^k t^k}{k(k-1)!} \right) e^{-\lambda t}$ which can also be written in the form

$$\mathbf{p_n(t) = \frac{e^{-\lambda t} (\lambda t)^k}{k!}}, \quad \text{for } n = n_0 + k \text{ and } k = 0, 1, 2, .... \qquad \textbf{(3.15)}$$

which is Poisson p.d.f with parameter $\lambda t$.

### 3.2.2 Mean and Variance by Definition

(i) **Mean**

$$p_k(t) = \frac{(\lambda t)^k e^{-\lambda t}}{k!}$$

$$E[X(t) = k] = \sum_{k=1}^{\infty} k \frac{(\lambda t)^k e^{-\lambda t}}{k!}$$

$$= e^{-\lambda t} \sum_{k=1}^{\infty} \frac{(\lambda t)^k}{(k-1)!}$$

Therefore,

$$E[X(t) = k] = \lambda t \, e^{-\lambda t} \sum_{k=1}^{\infty} \frac{(\lambda t)^{k-1}}{(k-1)!}$$

But $e^{\lambda t} = \sum_{k=0}^{\infty} \frac{(\lambda t)^k}{k!}$.

$$E[X(t) = k] = \lambda t e^{-\lambda t}. e^{\lambda t} = \lambda t$$

Thus

$$E[X(t)] = \lambda t \qquad \textbf{(3.16)}$$

(ii) **Variance**

$$E\left[\left(X(t)=k\right)^2\right] = \sum_{k=0}^{\infty} k^2 \frac{(\lambda t)^k e^{-\lambda t}}{k!} = \sum_{k=0}^{\infty} k \frac{(\lambda t)^k e^{-\lambda t}}{(k-1)!} = e^{-\lambda t}\lambda t \sum_{k=0}^{\infty} k \frac{(\lambda t)^{k-1}}{(k-1)!}.$$

But

$$\frac{d}{dt}\left(\lambda t\, e^{\lambda t}\right) = \frac{d}{dt}\left[\lambda t \sum_{k=0}^{\infty} \frac{(\lambda t)^k}{k!}\right] = \frac{d}{dt}\left[\sum_{k=0}^{\infty} \frac{(\lambda t)^{k+1}}{k!}\right] = \lambda \sum_{k=0}^{\infty} \frac{(k+1)(\lambda t)^k}{k!}$$

Alternatively,

$$\frac{d}{dt}\left[\lambda t\, e^{\lambda t}\right] = \lambda e^{\lambda t} + \lambda^2 t e^{\lambda t} = \lambda\left[e^{\lambda t} + \lambda t e^{\lambda t}\right]$$

Therefore,

$$e^{\lambda t} + \lambda t e^{\lambda t} = \sum_{k=0}^{\infty} \frac{(k+1)(\lambda t)^k}{k!}$$

$$e^{\lambda t}[1+\lambda t] = \sum_{k=0}^{\infty} \frac{(k+1)(\lambda t)^k}{k!}$$

$$E\left[\left(X(t)\right)^2\right] = e^{-\lambda t}\lambda t\left[e^{\lambda t}(1+\lambda t)\right]$$

$$E\left[\left(X(t)\right)^2\right] = \lambda t[1+\lambda t] = \lambda t + (\lambda t)^2$$

$$\mathrm{Var}\left[X(t)\right] = E\left[X(t)\right]^2 - \left[E\left(X(t)\right)\right]^2$$

$$= \lambda t + (\lambda t)^2 - (\lambda t)^2 = \lambda t$$

$$\mathrm{Var}\left[X(t)\right] = \lambda t \qquad \textbf{(3.17)}$$

### 3.3 Determining $p_n(t)$ by Laplace Method

***Initial conditions: When t = 0, $X(0) = n_0$.***

This implies that $p_{n_0}(0) = 1$. Further, $p_n(0) = 0 \ \forall \ n \neq n_0$ and $p_n(t) = 0 \ \forall \ n < n_0$.

Taking the Laplace transform of both sides of equation **(3.5)**, we have

$$L[p'_n(t)] = -\lambda L[p_n(t)] + \lambda L[p_{n-1}(t)], \quad n \geq 1$$

Taking advantage of rule **(2.9)** $L[p'_n(t)] = sL[p_n(t)] - p_n(0)$ which was derived in section **(2.5)** of Chapter 2, we have

$$sL[p_n(t)] - p_n(0) = -\lambda L[p_n(t)] + \lambda L[p_{n-1}(t)]$$

$$\Rightarrow$$

$$\{s + \lambda\} L[p_n(t)] = \lambda L[p_{n-1}(t)] + p_n(0)$$

Equivalently,

$$L[p_n(t)] = \frac{\lambda}{s+\lambda} L[p_{n-1}(t)] + \frac{p_n(0)}{s+\lambda} \qquad \textbf{(3.18)}$$

When $n = n_0$, equation **(3.18)** becomes

$$L[p_{n_0}(t)] = \frac{\lambda}{\lambda + s} L[p_{n_0-1}(t)] + \frac{p_{n_0}(0)}{s+\lambda}$$

From the initial conditions, $p_{n_0}(0) = 1$ and $p_{n_0-1}(t) = 0$. Thus,

$$L[p_{n_0}(t)] = \frac{1}{s+\lambda} \qquad \textbf{(3.19)}$$

When $n = n_0 + 1$, equation **(3.18)** becomes

$$L[p_{n_0+1}(t)] = \frac{\lambda}{s+\lambda} L[p_{n_0}(t)] + \frac{p_{n_0+1}(0)}{s+\lambda}$$

From the initial conditions, $p_{n_0+1}(0) = 0$. Also $L(p_{n_0}(t))$ is as derived in **(3.19).** Thus

$$L[p_{n_0+1}(t)] = \frac{\lambda}{\lambda+s} \times \frac{1}{\lambda+s}$$

Thus,

$$L\left[p_{n_0+1}(t)\right] = \frac{\lambda}{(\lambda+s)^2} \tag{3.20}$$

When $n = n_0 + 2$ equation **(3.18)** becomes

$$L\left[p_{n_0+2}(t)\right] = \frac{\lambda}{s+\lambda} L\left[p_{n_0+1}(t)\right] + \frac{p_{n_0+2}(0)}{s+\lambda}$$

From the initial conditions, $P_{n_0+2}(0) = 0$. Also**,** from **(3.20),** $L\left[p_{n_0+1}(t)\right] = \frac{\lambda}{(\lambda+s)^2}$

Therefore.

$$L\left[p_{n_0+2}(t)\right] = \frac{\lambda}{\lambda+s} \times \frac{\lambda}{(\lambda+s)^2}$$

Thus,

$$L\left[p_{n_0+2}(t)\right] = \frac{\lambda^2}{(\lambda+s)^3} \tag{3.21}$$

By induction, assume that for $n = n_0 + k - 1$, then $L\left[p_{n_0+k-1}(t)\right] = \frac{\lambda^{k-1}}{(\lambda+s)^k}$

When $n = n_0 + k$, equation **(2.13)** becomes

$$L\left[[p_{n_0+k}(t)\right] = \frac{\lambda}{s+\lambda} L\left[p_{n_0+k-1}(t)\right] + \frac{p_{n_0+k}(0)}{s+\lambda}$$

From initial conditions: When t = 0, $p_{n_0+k}(0) = 0$. Thus,

$$L\left[p_{n_0+k}(t)\right] = \frac{\lambda}{s+\lambda} L\left[p_{n_0+k-1}(t)\right]$$

$$= \frac{\lambda}{s+\lambda} \times \frac{\lambda^{k-1}}{(\lambda+s)^k}$$

$$= \frac{\lambda^k}{(\lambda+s)^{k+1}}$$

Thus

$$L\left[p_n(t)\right] = \frac{\lambda^k}{(\lambda+s)^{k+1}} \tag{3.22}$$

Where $n = n_0 + k$ and $k = 0, 1, 2, 3, ...$

Taking $L^{-1}$ of both sides of equation **(3.22)**, we have

$$L^{-1}\left\{L\left[p_n(t)\right]\right\} = L^{-1}\left\{\frac{\lambda^n}{(\lambda + s)^{n+1}}\right\}$$ or equivalently,

$$p_n(t) = \lambda^n \ L^{-1}\left[\frac{1}{(\lambda + s)}\right]^{n+1}$$

$-\lambda$ is a pole of order $n + 1$.

The residue of $f(s) = \left[\frac{1}{(\lambda + s)}\right]^{n+1}$ around $s = -\lambda$ is given by

$$\text{Res}(f, -\lambda) = \frac{1}{n!} \lim_{s \to -\lambda} \frac{d^n}{ds^n} e^{st}\left[(s + \lambda)^{n+1} \times \left[\frac{1}{(\lambda + s)}\right]^{n+1}\right]$$

$$= \frac{1}{n!} \lim_{s \to -\lambda} \frac{d^n}{ds^n}\left[e^{st}\right]$$

$$= \frac{1}{n!} \lim_{s \to -\lambda} \left[t^n \times e^{st}\right]$$

$$= \frac{1}{n!} \times t^n \times e^{-\lambda t}$$

Therefore,

$$p_n(t) = \lambda^n \times \frac{1}{n!} \times t^n \times e^{-\lambda t} = \frac{(\lambda t)^n}{n!} e^{-\lambda t} \quad \text{for } n = \{n_0 + k;\ k = 0, 1, 2, 3, ..\}$$

Thus

$$\mathbf{p_n(t) = \frac{(\lambda t)^n e^{-\lambda t}}{n!}} \quad n = n_0 + k;\ k = 0, 1, 2, 3, .. \qquad \textbf{(3.23)}$$

This is a Poisson distribution.

### 3.4 Probability Generating Function Method

### 3.4.1 Determining $p_n(t)$ by PGF Method

***Initial conditions: When t = 0, X(0) = 0.***

Equation **(3.5)** is $p_n'(t) = -\lambda p_n(t) + \lambda p_{n-1}(t), \quad n \geq 1$

Multiplying both sides of the equation by $s^n$ and summing the results over n

$$\sum_{n=1}^{\infty} p_n'(t)s^n = -\lambda\sum_{n=1}^{\infty} p_n(t)s^n + \lambda\sum_{n=1}^{\infty} p_{n-1}(t)s^n \quad \textbf{(3.24)}$$

Define

$$\left.\begin{aligned} G(s,t) &= \sum_{n=0}^{\infty} p_n(t)s^n = p_0(t) + \sum_{n=1}^{\infty} p_n(t)s^n \\ \frac{d(G(s,t))}{dt} &= \sum_{n=0}^{\infty} p_n'(t)s^n = p_0'(t) + \sum_{n=1}^{\infty} p_n'(t)s^n \\ \frac{d(G(s,t))}{ds} &= \sum_{n=0}^{\infty} n\,p_n(t)s^{n-1} = \sum_{n=1}^{\infty} n\,p_n(t)s^{n-1} \end{aligned}\right\} \quad \textbf{(3.25)}$$

Equation (**3.24)** now becomes

$$\frac{d}{dt}[G(s,t)] - p_0'(t) = -\lambda\left[G(s,t) - p_0(t)\right] + \lambda s\sum_{n=1}^{\infty} p_{n-1}(t)s^{n-1}$$

$$= -\lambda G(s,t) + \lambda p_o(t) + \lambda s G(s,t)$$

Rearranging the above equation,

$$\frac{d(G(s,t))}{dt} - p_0'(t) - \lambda p_0(t) = -\lambda G(s,t) + \lambda s G(s,t)$$

$$= -\lambda(1-s)G(s,t)$$

But from the equation **(3.4)**, $p_0'(t) = -\lambda p_0(t)$. Therefore

$$\frac{d(G(s,t))}{dt} - p_0'(t) + p_0'(t) = -\lambda(1-s)G(s,t)$$

$$\frac{d(G(s,t))}{dt} = -\lambda(1-s)G(s,t)$$

$$\frac{\frac{d}{dt}(G(s,t))}{G(s,t)} = -\lambda(1-s)dt$$

$$\frac{d}{dt}[\ln(G(s,t)] = -\lambda(1-s)$$

Integrating both sides with respect to *t*,

$$\int d\left[\ln(G(s,t))\right]dt = \int -\lambda(1-s)dt$$

$$\ln(G(s,t)) = -\lambda(1-s)t + c$$

Taking the exponential of both sides, we have

$$G(s,t) = e^{c-\lambda(1-s)t} = ke^{-\lambda(1-s)t}$$

Initial Conditions: When t = 0, we have $G(s,0) = ke^{-\lambda(1-s)\times 0} = k.$

But using the definition, $G(s,t) = \sum_{n=0}^{\infty} p_n(t)s^n = p_0(t) + \sum_{n=1}^{\infty} p_n(t)s^n$. Putting $t = 0$ we have $G(s,0) = p_0(0) + \sum_{n=1}^{\infty} p_n(0)s^n$. But from the initial conditions, at t = 0, $X(0) = 0$.

This implies that $p_0(0) = 1$ and $p_n(t) = 0$ for $n \neq 0$.

$$\therefore \quad G(s,0) = 1$$

$$\Rightarrow \quad k = 1$$

Therefore

$$G(s,t) = e^{-\lambda t(1+s)} \qquad \textbf{(3.26)}$$

Now $p_n(t)$ is the coefficient of $s^n$ on $G(s,t)$.

$$G(s,t) = e^{-\lambda t(1-s)} = e^{-\lambda t}e^{\lambda ts}$$

$$= e^{-\lambda t}\left\{\sum_{n=0}^{\infty}\frac{(\lambda ts)^n}{n!}\right\}$$

$$= e^{-\lambda t}\sum_{n=0}^{\infty}\frac{(\lambda t)^n}{n!}s^n$$

$$= \sum_{n=0}^{\infty}\frac{e^{-\lambda t}(\lambda t)^n}{n!}s^n$$

$$\mathbf{p_k(t) = \frac{e^{-\lambda t}(\lambda t)^k}{k!}} \qquad k = 1, 2, 3, \ldots \qquad \textbf{(3.27)}$$

### 3.4.2 Mean and Variance

(i) **Mean**

Recall that

$$G(s,t) = E\left(s^{x}\right) \Rightarrow \frac{d}{ds}G(s,t) = E\left(x\,s^{x-1}\right) = E\left(x\right).\ \text{When } s = 1,\ G'(1,t) = E\left(x\right).$$

$$G(s,t) = e^{-\lambda t(1-s)}$$

$$E\left[X\left(t\right) = k\right] = \frac{d}{ds}G\left(s,t\right)\Big|_{s=1}$$

$$= \lambda t\, e^{-\lambda t(1-1)}$$

Thus,

$$E\left[X\left(t\right) = k\right] = \lambda t \qquad \textbf{(3.28)}$$

(ii) **Variance**

Now,

$$\text{var } x = E\left(x^2\right) - \left[E\left(x\right)\right]^2$$

$$= E\left[x\left(x-1\right) + x\right] - \left[E\left(x\right)\right]^2$$

$$= E\left[x\left(x-1\right)\right] + E\left(x\right) - \left[E\left(x\right)\right]^2$$

$$= G''\left(1\right) + G'\left(1\right) - \left[G'\left(1\right)\right]^2$$

$$\frac{d^2}{ds^2}\left(G\left(s,t\right)\right) = \frac{d}{ds}\lambda t\ e^{-\lambda t\,(1-s)}$$

$$= \left(\lambda t\right)^2\ e^{-\lambda t(1-s)}$$

Put s = 1

$$\frac{d^2}{ds^2}\left(G\left(s,t\right)\right)\Big|_{s=1} = \left(\lambda t\right)^2\ e^{-\lambda t(1-1)} = \left(\lambda t\right)^2$$

Thus,

$$\text{var } x = \left(\lambda t\right)^2 + \lambda t - \left(\lambda t\right)^2$$

$$= \lambda t \qquad \textbf{(3.29)}$$

## 3.5 Method of Moments to Determine Mean and Variance

### 3.5.1 Mean

Define

$$M_1(t) = \sum_{n=1}^{\infty} n\, p_n(t) \quad \Rightarrow \quad M_1'(t) = \sum_{n=1}^{\infty} n\, p_n(t) \qquad \textbf{(3.30)}$$

Initial conditions are that at $t = 0$, $X(0) = 0$. Thus $M_1(0) = 0$ and $M_2(0) = 0$

Now, multiply equation **(3.5)** by n and sum the results over n

$$\sum_{n=1}^{\infty} n\, p_n(t) = -\lambda \sum_{n=1}^{\infty} n\, p_n(t) + \lambda \sum_{n=1}^{\infty} n\, p_{n-1}(t)$$

$$= -\lambda \sum_{n=1}^{\infty} n\, p_n(t) + \lambda \sum_{n=1}^{\infty} (n - 1 + 1) p_{n-1}(t)$$

$$\sum_{n=1}^{\infty} n\, p_n(t) = -\lambda \sum_{n=1}^{\infty} n\, p_n(t) + \lambda \sum_{n=1}^{\infty} (n - 1) p_{n-1}(t) + \lambda \sum_{n=1}^{\infty} p_{n-1}(t)$$

Therefore,

$$M_1'(t) = -\lambda M_1(t) + \lambda M_1(t) + \lambda$$

$$= \lambda$$

This implies that

$$\frac{d}{dt} M_1(t) = \lambda$$

$$\Rightarrow$$

$$M_1(t) = \lambda t + c$$

The initial conditions are; When $t = 0, M_1(0) = 0.$

Therefore,

$$c = 0$$

Therefore

$$M_1(t) = \lambda t \quad \Rightarrow \quad E[X(t)] = \lambda t \qquad \textbf{(3.31)}$$

### 3.5.2 Variance

Define

$$M_2(t) = \sum_{n=}^{\infty} n^2\, p_n(t) \quad \Rightarrow \quad M_2'(t) = \sum_{n=}^{\infty} n^2\, p_n'(t)$$

Now, multiply equation **(2.1b)** by $n^2$ and sum the results over n

$$\sum_{n=1}^{\infty} n^2 p_n(t) = -\lambda \sum_{n=1}^{\infty} n^2 p_n(t) + \lambda \sum_{n=1}^{\infty} n^2 p_{n-1}(t)$$

$$= -\lambda \sum_{n=1}^{\infty} n^2 p_n(t) + \lambda \sum_{n=1}^{\infty} (n - 1 + 1)^2 p_{n-1}(t)$$

$$= -\lambda \sum_{n=1}^{\infty} n^2 p_n(t) + \lambda \sum_{n=1}^{\infty} \left\{(n - 1)^2 + 2(n - 1) + 1\right\} p_{n-1}(t)$$

$$= -\lambda \sum_{n=1}^{\infty} n^2 p_n(t) + \lambda \sum_{n=1}^{\infty} (n - 1)^2 p_{n-1}(t) + 2\lambda \sum_{n=1}^{\infty} (n - 1) p_{n-1}(t) + \lambda \sum_{n=1}^{\infty} p_{n-1}(t)$$

Using definitions,

$$\sum_{n=1}^{\infty} n^2 p_n(t) = -\lambda M_2(t) + \lambda M_2(t) + 2\lambda M_1(t) + \lambda$$

$$= 2\lambda M_1(t) + \lambda$$

$$= 2\lambda \times \lambda t + \lambda$$

$$= 2\lambda^2 t + \lambda$$

Therefore,

$$M_2'(t) = 2\lambda^2 t + \lambda$$

$$\frac{d}{dt} M_2(t) = 2\lambda^2 t + \lambda$$

Integrating both sides with respect to t, we have

$$M_2(t) = 2\lambda^2 \int t\,dt + \int \lambda\,dt$$

$$= (\lambda t)^2 + \lambda t + c$$

When $t = 0$, $M_2(t) = 0$. Therefore $c = 0$.

$$M_2(t) = (\lambda t)^2 + \lambda t \qquad \textbf{(3.32)}$$

Now,

$$\text{Var}\left[X(t)\right] = M_2(t) - \left[M_1(t)\right]^2$$

$$= (\lambda t)^2 + \lambda t - [\lambda t]^2$$

$$= \lambda t$$

Therefore,

$$\text{Var}\left[X(t)\right] = \lambda t \qquad \textbf{(3.33)}$$

# CHAPTER FOUR
# SIMPLE BIRTH PROCESS

## 4.1 Introduction

Consider a population whose members can (by splitting or otherwise) give birth to new members but cannot die. Assume that the probability is approximately $\lambda\Delta t$ that in a short interval of length $\Delta t$, a member will create a new Member. More precisely, assume that if $X(t)$ is the size of the population at time t, then $\{X(t),\ t \geq 0\}$ is a pure birth process with $\lambda_n = n\lambda$ for n = 0, 1, 2,…

The simple birth process with a linear birth rate is also called yule process or the Furry – Yule process. The objective of this Chapter is to solve the basic difference differential equations derived in Chapter 2 (refer to equations **(2.1)** and **(2.2)**) when $\lambda_n = n\lambda$. We shall specifically look at four methods namely iterative method, the Laplace transform and the Lagranges Method. We shall look at each of these cases when the initial conditions are (i) X(0) = 1 and (ii) $X(0) = n_0$.

When $\lambda_n = n\lambda$, the difference differential equations become

$$p_0'(t) = 0 \qquad \textbf{(4.1a)}$$

$$p_n'(t) = -n\lambda\, p_n(t) + (n-1)\lambda\, p_{n-1}(t), \quad n \geq 1 \qquad \textbf{(4.1b)}$$

## 4.2 Iteration Method

### 4.2.1 Determining $p_n(t)$ using iteration Method

***Initial Condition: When t = 0 is $X(0) = 1$***

When $n = 1$

Substituting $n = 1$ in equation **(4.1b)**, we have

$$p_1'(t) = -1\times\lambda\times p_1(t) + (1-1)\lambda\, p_{1-1}(t)$$

$$= -\lambda\, p_1(t)$$

Dividing both sides of the equation by $p_1(t)$.

$$\frac{p_1'(t)}{p_1(t)} = -\lambda$$

Equivalently

$$\frac{d}{dt}(\ln p_1(t)) = -\lambda$$

Integrating both sides with respect to t

$$\int d(\ln p_1(t))\,dt = -\int \lambda\,dt$$

$$\ln p_1(t) = -\lambda t + c$$

Taking the exponential of both sides, we have

$$p_1(t) = e^{-\lambda t + c} = e^{c} \times e^{-\lambda t} = k\,e^{-\lambda t}$$

Initial conditions that when $t = 0,\ X(0) = 1$. This implies that $p_1(0) = 1,\ \ p_n(0) = 0\ \forall\ n \neq 1$.

Substituting $t = 0$ in the equation above, we have

$$p_1(0) = k\,e^{-\lambda \times 0}$$

$$1 = k \times 1$$

$$k = 1$$

Therefore

$$p_1(t) = e^{-\lambda t} \qquad \textbf{(4.1)}$$

When $n = 2$

Equation **(4.1b)** becomes

$$p_2'(t) = -2\lambda p_2(t) + \lambda p_1(t)$$

$$p_2'(t) = -2\lambda p_2(t) + \lambda \times e^{-\lambda t}$$

Rearranging

$$p_2'(t) + 2\lambda p_2(t) = \lambda e^{-\lambda t}$$

Next, we integrate the above equation by use of integrating factor method.

Integrating factor $= e^{\int 2\lambda\,dt} = e^{2\lambda t}$. Multiplying both sides of the equation above by the integrating factor

$$p_2'(t) \times e^{2\lambda t} + 2\lambda p_2(t) \times e^{2\lambda t} = \lambda \times e^{-\lambda t} \times e^{2\lambda t} = \lambda e^{\lambda t}$$

$$\frac{d}{dt}\left[e^{2\lambda t} p_2(t)\right] = \lambda e^{\lambda t}$$

Integrating both sides with respect to *t*

$$\int d\left[e^{2\lambda t}p_2(t)\right]dt = \int \lambda e^{\lambda t}dt$$

$$e^{2\,t}p_2(t) = \lambda \times \frac{e^{\lambda t}}{\lambda} + c = e^{\lambda t} + c$$

Equivalently,

$$p_2(t) = \left(e^{\lambda t} + c\right)e^{-2\lambda t}$$

Initial conditions: When $t = 0,\ \ p_2(0) = 0.$ Therefore,

$$p_2(0) = \left(e^{\lambda\times 0} + c\right)e^{-2\times\lambda\times 0}$$

Therefore,

$$0 = (1 + c)$$

$$c = -1$$

Therefore

$$p_2(t) = \left(e^{\lambda t} - 1\right)e^{-2\lambda t}$$

$$= e^{-\lambda t} - e^{-2\lambda t}$$

$$= e^{-\lambda t}\left(1 - e^{-\lambda t}\right)$$

Therefore

$$p_2(t) = e^{-\lambda t}\left(1 - e^{-\lambda t}\right) \qquad \textbf{(4.2)}$$

When $n = 3$

Equation **(4.1b)** becomes

$$p_3'(t) = -3\lambda p_3(t) + 2\lambda p_2(t)$$

$$= -3\lambda p_3(t) + 2\lambda\times e^{-\lambda t}\left(1 - e^{-\lambda t}\right)$$

Rearranging

$$p_3'(t) + 3\lambda p_3(t) = 2\lambda e^{-\lambda t}\left(1 - e^{-\lambda t}\right)$$

Next, we integrate the above equation by use of integrating factor method.

Integrating factor $= e^{\int 3\lambda dt} = e^{3\lambda t}$

Multiplying both sides of the equation above by the integrating factor

$$e^{3\lambda t} \times p_3'(t) + 3\lambda \times e^{3\lambda t} \times p_3(t) = 2\lambda \times e^{2\lambda t}\left(1 - e^{-\lambda t}\right)$$

$$\frac{d}{dt}\left(e^{3\lambda t} p_3(t)\right) = 2\lambda e^{2\lambda t} - 2\lambda e^{\lambda t}$$

Integrating both sides with respect to t, we get

$$\int d\left(e^{3\lambda t} p_3(t)\right)dt = \int 2\lambda e^{2\lambda t}dt - \int 2\lambda e^{\lambda t}dt$$

$$e^{3\lambda t} p_3(t) = e^{2\lambda t} - 2e^{\lambda t} + c$$

Equivalently,

$$p_3(t) = \left(e^{2\lambda t} - 2e^{\lambda t} + c\right)e^{-3\lambda t}$$

Using the initial conditions, at $t = 0$, $p_3(0) = 0$

$$p_3(0) = \left(e^{2\lambda \times 0} - 2e^{\lambda \times 0} + c\right)e^{3\lambda \times 0}$$

$$0 = (1 - 2 + c)$$

$$c = 1$$

Therefore,

$$p_3(t) = \left(e^{2\lambda t} - 2e^{\lambda t} + 1\right)e^{-3\lambda t}$$

$$= e^{-\lambda t} - 2e^{-2\lambda t} + e^{-3\lambda t}$$

$$= e^{-\lambda t}\left(1 - 2e^{-\lambda t} + e^{-2\lambda t}\right)$$

$$= e^{-\lambda t}\left(1 - e^{-\lambda t}\right)^2$$

$$p_3(t) = e^{-\lambda t}\left(1 - e^{-\lambda t}\right)^2 \qquad \textbf{(4.3)}$$

Generalizing,

$$p_1(t) = e^{-\lambda t}$$

$$p_2(t) = e^{-\lambda t}\left(1 - e^{-\lambda t}\right)$$

$$p_3(t) = e^{-\lambda t}\left(1 - e^{-\lambda t}\right)^2$$

By induction, assume that $p_{n-1}(t) = e^{-\lambda t}\left(1 - e^{-\lambda t}\right)^{n-2}$

When n = n, equation **(4.1b)** then becomes

$$p_n'(t) = -n\lambda\, p_n(t) + (n-1)\lambda\, p_{n-1}(t)$$

$$= -n\lambda\, p_n(t) + (n-1)\lambda e^{-\lambda t}\left(1 - e^{-\lambda t}\right)^{n-2}$$

Re arranging,

$$p_n'(t) + n\lambda\, p_n(t) = (n-1)\lambda e^{-\lambda t}\left(1 - e^{-\lambda t}\right)^{n-2}$$

Next, we integrate the above equation by use of integrating factor method.

Integrating Factor $= e^{\int n\lambda\, dt} = e^{n\lambda t}$

Multiplying the equation above by the integrating factor, we have

$$e^{n\lambda t} p_n'(t) + e^{n\lambda t} n\lambda\, p_n(t) = e^{n\lambda t}(n-1)\lambda e^{-\lambda t}\left(1 - e^{-\lambda t}\right)^{n-2}$$

Therefore

$$\frac{d}{dt}\left[e^{n\lambda t} p_n(t)\right] = e^{n\lambda t}(n-1)\lambda e^{-\lambda t}\left(1 - e^{-\lambda t}\right)^{n-2}$$

$$\frac{d}{dt}\left[e^{n\lambda t} p_n(t)\right] = e^{n\lambda t}(n-1)\lambda e^{-\lambda t}\left(\frac{e^{\lambda t} - 1}{e^{\lambda t}}\right)^{n-2}$$

$$= (n-1)\lambda e^{-\lambda t} e^{n\lambda t} e^{-(n-2)\lambda t}\left(e^{\lambda t} - 1\right)^{n-2}$$

$$= (n-1)\lambda e^{-\lambda t + n\lambda t - n\lambda t + 2\lambda t}\left(e^{\lambda t} - 1\right)^{n-2}$$

$$= (n-1)\lambda e^{\lambda t}\left(e^{\lambda t} - 1\right)^{n-2}$$

Integrating both sides, we have

$$e^{n\lambda t} p_n(t) = (n-1)\lambda \int e^{\lambda t}\left(e^{\lambda t} - 1\right)^{n-2} dt$$

Let $e^{\lambda t} - 1 = u \quad \Rightarrow \quad \frac{du}{dt} = \lambda e^{\lambda t}$ or $dt = \frac{du}{\lambda e^{\lambda t}}$.

Therefore

$$e^{n\lambda t} p_n(t) = (n-1)\lambda \int e^{\lambda t} u^{n-2} \frac{du}{\lambda e^{\lambda t}}$$

$$= (n-1)\int u^{n-2}\, du$$

$$= (n-1)\frac{u^{n-1}}{(n-1)} + c$$

$$= \left(e^{\lambda t} - 1\right)^{n-1} + c$$

Equivalently

$$p_n(t) = \left(e^{\lambda t} - 1\right)^{n-1} e^{-n\lambda t} + c\, e^{-n\lambda t}$$

From the initial conditions: When t = 0, $X(0) = 0$ and $p_n(0) = 0 \ \forall \ n \neq 0$.

Substituting t = 0, we have

$$p_n(0) = (1 - 1)^{n-1} \times 1 + c \times 1$$

$$0 = 0 + c$$

$$c = 0$$

Therefore

$$p_n(t) = e^{-n\lambda t}\left(e^{\lambda t} - 1\right)^{n-1}$$

$$= e^{-n\lambda t}\left[e^{\lambda t}\left(1 - e^{-\lambda t}\right)^{n-1}\right]$$

$$p_n(t) = e^{-n\lambda t} e^{\lambda t(n-1)}\left(1 - e^{-\lambda t}\right)^{n-1}$$

$$= e^{-\lambda t}\left(1 - e^{-\lambda t}\right)^{n-1}$$

Therefore,

$$\mathbf{p_n(t) = e^{-\lambda t}\left(1 - e^{-\lambda t}\right)^{n-1}} \qquad \textbf{(4.4)}$$

This is geometric distribution of the form $p_j(t) = p q^{j-1}$. j = 1,2,3,… and $p = e^{-\lambda t}$ and $q = 1 - e^{-\lambda t}$

**General case (Population at time t=0 is $n_0$, no ≥1)**

Initial conditions;

When $t = 0$, $X(0) = n_0 \Rightarrow p_{n_0}(0) = 1$, $p_n(0) = 0 \ \forall \ n \neq n_0$ and $p_n(t) = 0 \ \forall \ n < n_0$.

When $n = n_0$ then equation **(4.1b)** becomes

$$p'_{n_0}(t) = -n_0 \lambda p_{n_0}(t) + (n_0 - 1)\lambda p_{n_0 - 1}(t)$$

This can be rewritten as

$$p'_{n_0}(t) + n_0 \lambda p_{n_0}(t) = (n_0 - 1)\lambda p_{n_0 - 1}(t)$$

Next, we integrate the above equation by use of integrating factor method.

The integrating factor $= e^{\int \lambda n_0 dt} = e^{n_0 \lambda t}$

Multiplying both sides of the equation above by the integrating factor, we get

$$e^{n_0 \lambda t} \times p'_{n_0}(t) + e^{n_0 \lambda t} \times n_0 \lambda p_{n_0}(t) = e^{n_0 \lambda t} \times (n_0 - 1) \lambda p_{n_0 - 1}(t)$$

$$\frac{d}{dt}\left(e^{n_0 \lambda t} \times p_{n_0}(t)\right) = e^{n_0 \lambda t} \times (n_0 - 1) \lambda \times 0 \text{ as } p_{n_0 - 1}(t) = 0$$

Integrating both sides with respect to t, we have

$$\int d\left(e^{n_0 \lambda t} \times p_{n_0}(t)\right) dt = \int 0 \, dt$$

$$e^{n_0 \lambda t} \times p_{n_0}(t) = c$$

But from the initial conditions, when t = 0, $p_{n_0}(0) = 1$ and $p_{n_i}(0) = 0 \ \forall \ i \neq 0$.

Thus, substituting t = 0 in the above equation, we have

$$e^{n_0 \lambda \times 0} \times p_{n_0}(0) = c$$

$$1 \times 1 = c$$

$$c = 1$$

Substituting the value of c, we now have

$$e^{n_0 \lambda t} \times p_{n_0}(t) = 1$$

Thus

$$p_{n_0}(t) = e^{-n_0 \lambda t} \qquad \textbf{(4.5)}$$

When $n = n_0 + 1$, equation **(3.1b)** becomes

$$p'_{n_0 + 1}(t) = -(n_0 + 1) \lambda p_{n_0 + 1}(t) + (n_0 + 1 - 1) \lambda p_{n_0 + 1 - 1}(t)$$

$$= -(n_0 + 1) \lambda p_{n_0 + 1}(t) + n_0 \lambda p_{n_0}(t)$$

Rewriting the above equation

$$p'_{n_0 + 1}(t) + (n_0 + 1) \lambda p_{n_0 + 1}(t) = n_0 \lambda p_{n_0}(t)$$

Next, we integrate the above equation by use of integrating factor method.

Integrating factor $= e^{\int (n_0 + 1) \lambda dt} = e^{(n_0 + 1) \lambda t}$

Multiplying both sides of the equation above by the integrating factor, we have

$$e^{(n_0 + 1) \lambda t} \times p'_{n_0 + 1}(t) + e^{(n_0 + 1) \lambda t} \times (n_0 + 1) \lambda p_{n_0 + 1}(t) = e^{(n_0 + 1) \lambda t} \times n_0 \lambda p_{n_0}(t)$$

Equivalently

$$\frac{d}{dt}\left[e^{(n_0+1)\lambda t}p_{n_0+1}(t)\right] = e^{(n_0+1)\lambda t}\times n_0\lambda\times e^{-n_0\lambda t}$$

$$= e^{\lambda t}n_0\lambda$$

Integrating both sides with respect to t, we have

$$\int d\left(e^{(n_0+1)\lambda t}p_{n_0+1}(t)\right)dt = \int e^{\lambda t}n_0\lambda dt = n_0\lambda\int e^{\lambda t}dt$$

$$e^{(n_0+1)\lambda t}p_{n_0+1}(t) = \frac{n_0\lambda e^{\lambda t}}{\lambda} + c$$

$$= n_0 e^{\lambda t} + c$$

The initial conditions are. When $t=0,\ X(0) = n_0$ where $n_0 \geq 0$. This implies that $p_{n_0+1}(0) = 0$.

Substituting the initial conditions in the equation above, we have

$$e^{(n_0+1)\lambda t}p_{n_0+1}(0) = n_0 e^{\lambda\times 0} + c$$

$$0 = n_0 + c$$

$$c = -n_0$$

Substituting,

$$e^{(n_0+1)\lambda t}p_{n_0+1}(t) = n_0 e^{\lambda t} - n_0$$

$$p_{n_0+1}(t) = n_0 e^{-(n_0+1)\lambda t}\left(e^{\lambda t} - 1\right) = n_0\left(e^{-n_0\lambda t} - e^{-n_0\lambda t}\times e^{-\lambda t}\right) = n_0 e^{-n_0\lambda t}\left(1 - e^{-\lambda t}\right)$$

Thus

$$p_{n_0+1}(t) = n_0 e^{-n_0\lambda t}\left(1 - e^{-\lambda t}\right) \qquad \textbf{(4.6)}$$

When $n = n_0 + 2$, equation **(4.2b)** becomes

$$p'_{n_0+2}(t) = -(n_0+2)\lambda p_{n_0+2}(t) + (n_0 - 1 + 2)\lambda p_{n_0-1+2}(t)$$

$$= -(n_0+2)\lambda p_{n_0+2}(t) + (n_0+1)\lambda p_{n_0+1}(t)$$

Rearranging,

$$p'_{n_0+2}(t) + (n_0+2)\lambda p_{n_0+2}(t) = (n_0+1)\lambda p_{n_0+1}(t)$$

Next, we integrate the above equation by use of integrating factor method.

Integrating factor $= e^{\int (n_0+2)\lambda dt} = e^{(n_0+2)\lambda t}$

Multiplying both sides of the equation above by the integrating factor, we have

$$e^{(n_0+2)\lambda t}\times p'_{n_0+2}(t) + e^{(n_0+2)\lambda t}\times(n_0+2)\lambda\, p_{n_0+2}(t) = e^{(n_0+2)\lambda t}\times(n_0+1)\lambda\, p_{n_0+1}(t)$$

Equivalently,

$$\frac{d}{dt}\left(e^{(n_0+2)\lambda t}\times p_{n_0+2}(t) + e^{(n_0+2)\lambda t}\right) = e^{(n_0+2)\lambda t}\times(n_0+1)\lambda\times n_0\, e^{-n_0\lambda t}\left(1-e^{-\lambda t}\right)$$

$$= n_0(n_0+1)\lambda e^{2\lambda t}\left(1-e^{-\lambda t}\right)$$

Integrating both sides with respect to t

$$\int d\left(e^{(n_0+2)\lambda t}\times P_{n_0+2}(t) + e^{(n_0+2)\lambda t}\right)dt = \int n_0(n_0+1)\lambda e^{2\lambda t}\left(1-e^{-\lambda t}\right)dt$$

Therefore,

$$e^{(n_0+2)\lambda t}\times p_{n_0+2}(t) = n_0(n_0+1)\lambda\left(\int e^{2\lambda t}\,dt - \int e^{\lambda t}dt\right)$$

$$= n_0(n_0+1)\lambda\left[\frac{e^{2\lambda t}}{2\lambda} - \frac{e^{\lambda t}}{\lambda}\right] + c$$

$$= n_0(n_0+1)\left[\frac{e^{2\lambda t}}{2} - e^{\lambda t}\right] + c$$

The initial conditions are: When t = 0, $X(0)=n_0$. This implies that $p_{n_0+2}(0) = 0$. Substituting the initial conditions in the equation above, we have

$$e^{(n_0+2)\lambda\times 0}\times p_{n_0+2}(0) = n_0(n_0+1)\left[\frac{e^{2\lambda\times 0}}{2} - e^{\lambda\times 0}\right] + c$$

$$0 = n_0(n_0+1)\left[\frac{1}{2} - 1\right] + c$$

$$c = \frac{n_0(n_0+1)}{2}$$

Substituting the value of c, our equation becomes

$$e^{(n_0+2)\lambda t}\times p_{n_0+2}(t) = n_0(n_0+1)\left[\frac{e^{2\lambda t}}{2} - e^{\lambda t}\right] + \frac{n_0(n_0+1)}{2}$$

$$= \frac{n_0(n_0+1)}{2}\left[e^{2\lambda t} - 2e^{\lambda t} + 1\right]$$

$$= \frac{n_0(n_0+1)}{2}\left(e^{\lambda t} - 1\right)^2$$

Equivalently,

$$p_{n_0+2}(t) = \frac{n_0(n_0+1)}{2}\left(e^{\lambda t} - 1\right)^2 \times e^{-(n_0+2)\lambda t}$$

Simplifying and rearranging,

$$p_{n_0+2}(t) = \frac{n_0(n_0+1)}{2}\, e^{-(n_0+2)\lambda t}\left(e^{\lambda t} - 1\right)^2 = \frac{n_0(n_0+1)}{2}\, e^{-(n_0+2)\lambda t}\left(e^{\lambda t}\left(1 - e^{-\lambda t}\right)\right)^2$$

$$= \frac{n_0(n_0+1)}{2}\, e^{-(n_0+2)\lambda t} \times e^{2\lambda t}\left(1 - e^{-\lambda t}\right)^2$$

$$= \frac{n_0(n_0+1)}{2}\, e^{-n_0\lambda t}\left(1 - e^{-\lambda t}\right)^2$$

Thus

$$p_{n_0+2}(t) = \frac{n_0(n_0+1)}{2}\, e^{-n_0\lambda t}\left(1 - e^{-\lambda t}\right)^2 \qquad \textbf{(4.7)}$$

When $n = n_0 + 3$, equation **(4.1b)** becomes

$$p'_{n_0+3}(t) = -\left(n_0 + 3\right)\lambda\, p_{n_0+3}(t) + \left(n_0 + 3 - 1\right)\lambda\, p_{n_0+3-1}(t)$$

$$= -\left(n_0 + 3\right)\lambda\, p_{n_0+3}(t) + \left(n_0 + 2\right)\lambda\, p_{n_0+2}(t)$$

Rearranging

$$p'_{n_0+3}(t) + \left(n_0 + 3\right)\lambda\, p_{n_0+3}(t) = \left(n_0 + 2\right)\lambda\, p_{n_0+2}(t)$$

Next, we integrate the above equation by use of integrating factor method.

Integrating factor $= e^{\int (n_0+3)\lambda dt} = e^{(n_0+3)\lambda t}$

Multiplying both sides of the equation above by the integrating factor, we get

$$e^{(n_0+3)\lambda t} \times p'_{n_0+3}(t) + e^{(n_0+3)\lambda t} \times n_0\lambda\, p_{n_0+3}(t) = e^{(n_0+3)\lambda t} \times \left(n_0 + 2\right)\lambda\, p_{n_0+2}(t)$$

$\Rightarrow$

$$\frac{d}{dt}\left(e^{(n_0+3)\lambda t} \times p_{n_0+3}(t)\right) = e^{(n_0+3)\lambda t} \times \left(n_0 + 2\right) \times \lambda \times \frac{n_0(n_0+1)}{2}\, e^{-n_0\lambda t}\left(1 - e^{-\lambda t}\right)^2$$

$$= \frac{n_0(n_0+1)(n_0+2)}{2} \times \lambda \times e^{3\lambda t} \times \left(1 - e^{-\lambda t}\right)^2$$

$$= \frac{n_0(n_0+1)(n_0+2)}{2} \times \lambda \times e^{3\lambda t} \times \left(1 - 2e^{-\lambda t} + e^{-2\lambda t}\right)$$

Equivalently,

$$\frac{d}{dt}\left(e^{(n_0+3)\lambda t}\times p_{n_0+3}(t)\right)=\frac{n_0(n_0+1)(n_0+2)}{2}\times\lambda\times\left(e^{3\lambda t}-2e^{2\lambda t}+e^{\lambda t}\right)$$

Integrating both sides with respect to t, we have

$$\int d\left(e^{(n_0+3)\lambda t}\times p_{n_0+3}(t)\right)=\int\frac{n_0(n_0+1)(n_0+2)}{2}\times\lambda\times\left(e^{3\lambda t}-2e^{2\lambda t}+e^{\lambda t}\right)dt$$

Equivalently

$$e^{(n_0+3)\lambda t}\times p_{n_0+3}(t)=\frac{n_0(n_0+1)(n_0+2)}{2}\times\lambda\times\int\left(e^{3\lambda t}-2e^{2\lambda t}+e^{\lambda t}\right)dt$$

$$=\frac{n_0(n_0+1)(n_0+2)}{2}\times\lambda\times\left[\frac{e^{3\lambda t}}{3\lambda}-\frac{2e^{2\lambda t}}{2\lambda}+\frac{e^{\lambda t}}{\lambda}\right]+c$$

$$=\frac{n_0(n_0+1)(n_0+2)}{2}\times\left[\frac{e^{3\lambda t}}{3}-e^{2\lambda t}+e^{\lambda t}\right]+c$$

But from the initial conditions, When $t=0, X(0)=n_0$. This implies that $p_{n_0+3}(0)=0$.

Substituting the initial conditions in the equation above, we have

$$e^{(n_0+3)\lambda\times 0}\times p_{n_0+3}(0)=\frac{n_0(n_0+1)(n_0+2)}{2}\times\left[\frac{e^{3\lambda\times 0}}{3}-e^{2\lambda\times 0}+e^{\lambda\times 0}\right]+c$$

$$0=\frac{n_0(n_0+1)(n_0+2)}{2}\times\left[\frac{1}{3}-1+1\right]+c$$

$$c=-\frac{n_0(n_0+1)(n_0+2)}{2\times 3}$$

Substituting,

$$e^{(n_0+3)\lambda t}\times p_{n_0+3}(t)=\frac{n_0(n_0+1)(n_0+2)}{2}\times\left[\frac{e^{3\lambda t}}{3}-e^{2\lambda t}+e^{\lambda t}\right]-\frac{n_0(n_0+1)(n_0+2)}{2\times 3}$$

$$=\frac{n_0(n_0+1)(n_0+2)}{2}\times\left[\frac{e^{3\lambda t}}{3}-e^{2\lambda t}+e^{\lambda t}\right]-\frac{n_0(n_0+1)(n_0+2)}{2\times 3}$$

$$=\frac{n_0(n_0+1)(n_0+2)}{2\times 3}\times\left[e^{3\lambda t}-3e^{2\lambda t}+3e^{\lambda t}\right]-\frac{n_0(n_0+1)(n_0+2)}{2\times 3}$$

$$=\frac{n_0(n_0+1)(n_0+2)}{2\times 3}\left\{\left(e^{3\lambda t}-3e^{2\lambda t}+3e^{\lambda t}\right)-1\right\}$$

Therefore,

$$e^{(n_0+3)\lambda t} \times p_{n_0+3}(t) = \frac{n_0(n_0+1)(n_0+2)}{2\times3} \times \left(e^{3\lambda t} - 3e^{2\lambda t} + 3e^{\lambda t} - 1\right)$$

$$= \frac{n_0(n_0+1)(n_0+2)}{2\times3} \times \left(e^{\lambda t} - 1\right)^3$$

$$= \frac{n_0(n_0+1)(n_0+2)}{2\times3} \times \left(e^{\lambda t}\left(1 - e^{-\lambda t}\right)\right)^3$$

$$= \frac{n_0(n_0+1)(n_0+2)}{2\times3} \times e^{3\lambda t} \times \left(1 - e^{-\lambda t}\right)^3$$

Therefore,

$$p_{n_0+3}(t) = \left\{\frac{n_0(n_0+1)(n_0+2)}{2\times3} \times e^{3\lambda t} \times \left(1 - e^{-\lambda t}\right)^3\right\} \times e^{-(n_0+3)\lambda t}$$

$$= \frac{n_0(n_0+1)(n_0+2)}{2\times3} \times e^{-n_0\lambda t} \times \left(1 - e^{-\lambda t}\right)^3$$

Thus

$$p_{n_0+3}(t) = \frac{n_0(n_0+1)(n_0+2)}{2\times3} \times e^{-n_0\lambda t} \times \left(1 - e^{-\lambda t}\right)^3 \qquad \textbf{(4.8)}$$

When $n = n_0 + 4$, equation **(4.1b)** becomes

$$p'_{n_0+4}(t) = -(n_0+4)\lambda\, p_{n_0+4}(t) + (n_0 + 4 - 1)\lambda\, p_{n_0+4-1}(t)$$

$$= -(n_0+4)\lambda\, p_{n_0+4}(t) + (n_0+3)\lambda\, p_{n_0+3}(t)$$

Re arranging, we have

$$p'_{n_0+4}(t) + (n_0+4)\lambda\, p_{n_0+4}(t) = (n_0+3)\lambda\, p_{n_0+3}(t)$$

Next, we integrate the above equation by use of integrating factor method.

Integrating factor $= e^{\int (n_0+4)\lambda dt} = e^{(n_0+4)\lambda t}$

Multiplying both sides of the equation above by the integrating factor, we get

$$e^{(n_0+4)\lambda t} \times p'_{n_0+4}(t) + e^{(n_0+4)\lambda t} \times (n_0+4)\lambda\, p_{n_0+4}(t) = e^{(n_0+4)\lambda t} \times (n_0+3)\lambda\, p_{n_0+3}(t)$$

Equivalently

$$\frac{d}{dt}\left(e^{(n_0+4)\lambda t}\times p_{n_0+4}(t)\right) = e^{(n_0+4)\lambda t}\times(n_0+3)\times\lambda\times\frac{n_0(n_0+1)(n_0+2)}{2\times3}\times e^{-n_0\lambda t}\times\left(1-e^{-\lambda t}\right)^3$$

$$= \frac{n_0(n_0+1)(n_0+2)(n_0+3)}{2\times3}\times\lambda\times e^{4\lambda t}\times\left(1-e^{-\lambda t}\right)^3$$

Integrating both sides with respect to t, we have

$$\int d\left(e^{(n_0+4)\lambda t}\times p_{n_0+4}(t)\right)dt = \frac{n_0(n_0+1)(n_0+2)(n_0+3)}{2\times3}\times\lambda\times\int e^{4\lambda t}\times\left(1-3e^{-\lambda t}+3e^{-2\lambda t}-e^{-3\lambda t}\right)^3 dt$$

Simplifying

$$e^{(n_0+4)\lambda t}\times p_{n_0+4}(t) = \frac{n_0(n_0+1)(n_0+2)(n_0+3)}{2\times3}\times\lambda\times\int\left(e^{4\lambda t}-3e^{3\lambda t}+3e^{2\lambda t}-e^{\lambda t}\right)dt$$

$$= \frac{n_0(n_0+1)(n_0+2)(n_0+3)}{2\times3}\times\lambda\times\left[\frac{e^{4\lambda t}}{4\lambda}-\frac{3e^{3\lambda t}}{3\lambda}+\frac{3e^{2\lambda t}}{2\lambda}-\frac{e^{\lambda t}}{\lambda}\right]+c$$

$$= \frac{n_0(n_0+1)(n_0+2)(n_0+3)}{2\times3}\times\left[\frac{e^{4\lambda t}}{4}-e^{3\lambda t}+\frac{3e^{2\lambda t}}{2}-e^{\lambda t}\right]+c$$

But from the initial conditions, when $t=0, X(0)=n_0$. This implies that $p_{n_0+4}(0)=0$.

Substituting the initial conditions in the equation above, we have

$$e^{(n_0+4)\lambda\times0}\times p_{n_0+4}(0) = \frac{n_0(n_0+1)(n_0+2)(n_0+3)}{2\times3}\times\left[\frac{e^{4\lambda\times0}}{4}-e^{3\lambda\times0}+\frac{3e^{2\lambda\times0}}{2}-e^{\lambda\times0}\right]+c$$

$$0 = \frac{n_0(n_0+1)(n_0+2)(n_0+3)}{2\times3}\times\left[\frac{1}{4}-1+\frac{3}{2}-1\right]+c$$

$$c = \frac{n_0(n_0+1)(n_0+2)(n_0+3)}{2\times3}\times\frac{1}{4} = \frac{n_0(n_0+1)(n_0+2)(n_0+3)}{2\times3\times4}$$

Therefore, substituting the value of c in the equation above, we have

$$e^{(n_0+4)\lambda t}\times p_{n_0+4}(t) = \frac{n_0(n_0+1)(n_0+2)(n_0+3)}{2\times3}\times\left[\frac{e^{4\lambda t}}{4}-e^{3\lambda t}+\frac{3e^{2\lambda t}}{2}-e^{\lambda t}\right]+\frac{n_0(n_0+1)(n_0+2)(n_0+3)}{2\times3\times4}$$

$$= \frac{n_0(n_0+1)(n_0+2)(n_0+3)}{2\times3}\times\left[\frac{e^{4\lambda t}-4e^{3\lambda t}+6e^{2\lambda t}-4e^{\lambda t}}{4}\right]+\frac{n_0(n_0+1)(n_0+2)(n_0+3)}{2\times3\times4}$$

$$e^{(n_0+4)\lambda t} \times p_{n_0+4}(t) = \frac{n_0(n_0+1)(n_0+2)(n_0+3)}{2\times3\times4}\times\left\{\left(e^{4\lambda t} - 4e^{3\lambda t} + 6e^{2\lambda t} - 4e^{\lambda t}\right) + 1\right\}$$

$$= \frac{n_0(n_0+1)(n_0+2)(n_0+3)}{2\times3\times4}\times\left\{\left(e^{\lambda t} - 1\right)^4\right\}$$

$$= \frac{n_0(n_0+1)(n_0+2)(n_0+3)}{2\times3\times4}\times\left\{e^{\lambda t}\left(1 - e^{-\lambda t}\right)\right\}^4$$

$$= \frac{n_0(n_0+1)(n_0+2)(n_0+3)}{2\times3\times4}\times e^{4\lambda t}\times\left(1 - e^{-\lambda t}\right)^4$$

Therefore

$$p_{n_0+4}(t) = \frac{n_0(n_0+1)(n_0+2)(n_0+3)}{2\times3\times4}\times e^{4\lambda t}\times\left(1 - e^{-\lambda t}\right)^4\times e^{-(n_0+4)\lambda t}$$

$$= \frac{n_0(n_0+1)(n_0+2)(n_0+3)}{2\times3\times4}\times\left(1 - e^{-\lambda t}\right)^4\times e^{-n_0\lambda t}$$

Thus

$$p_{n_0+4}(t) = \frac{n_0(n_0+1)(n_0+2)(n_0+3)}{2\times3\times4}\times e^{-n_0\lambda t}\times\left(1 - e^{-\lambda t}\right)^4 \qquad \textbf{(4.9)}$$

Generalizing,

$$p_{n_0}(t) = e^{-n_0\lambda t}$$

$$p_{n_0+1}(t) = n_0\, e^{-n_0\lambda t}\left(1 - e^{-\lambda t}\right)$$

$$p_{n_0+2}(t) = \binom{n_0+1}{2} e^{-n_0\lambda t}\left(1 - e^{-\lambda t}\right)^2$$

$$p_{n_0+3}(t) = \binom{n_0+2}{3}\times e^{-n_0\lambda t}\times\left(1 - e^{-\lambda t}\right)^3$$

$$p_{n_0+4}(t) = \binom{n_0+3}{4}\times e^{-n_0\lambda t}\times\left(1 - e^{-\lambda t}\right)^4$$

By induction, assume that for $n = n_0 + k - 1$

$$p_{n_0+k-1}(t) = \binom{n_0+k-2}{k-1}\left(e^{-\lambda t}\right)^{n_0}\left(1 - e^{-\lambda t}\right)^{k-1}$$

When $n = n_0 + k$, equation **(4.1b)** then becomes

$$p'_{n_0+k}(t) = -(n_0+k)\lambda p_{n_0+k}(t) + (n_0 + k - 1)\lambda p_{n_0+k-1}(t)$$

Rearranging,

$$p'_{n_0+k}(t) + (n_0+k)\lambda p_{n_0+k}(t) = (n_0 + k - 1)\lambda p_{n_0+k-1}(t)$$

$$= (n_0 + k - 1)\lambda \binom{n_0 + k - 2}{k - 1} \left(e^{-\lambda t}\right)^{n_0} \left(1 - e^{-\lambda t}\right)^{k-1}$$

Next, we integrate the above equation by use of integrating factor method.

Integrating factor $= e^{\int (n_0 + k)\lambda dt} = e^{(n_0 + k)\lambda t}$

Multiplying equation (3.2) with the integrating factor, we have

$$e^{(n_0 + k)\lambda t} p'_{n_0+k}(t) + e^{(n_0 + k)\lambda t} (n_0+k)\lambda p_{n_0+k}(t) = e^{(n_0 + k)\lambda t} (n_0 + k - 1)\lambda \binom{n_0 + k - 2}{k - 1} \left(e^{-\lambda t}\right)^{n_0} \left(1 - e^{-\lambda t}\right)^{k-1}$$

Therefore,

$$\frac{d}{dt}\left[e^{(n_0 + k)\lambda t} p_{n_0+k}(t)\right] = (n_0 + k - 1)\lambda \binom{n_0 + k - 2}{k - 1} e^{-n_0\lambda t + k\lambda t + n_0\lambda t} \left(1 - e^{-\lambda t}\right)^{k-1}$$

$$= (n_0 + k - 1)\lambda \binom{n_0 + k - 2}{k - 1} e^{k\lambda t} \left(1 - e^{-\lambda t}\right)^{k-1}$$

$$= (n_0 + k - 1)\lambda \binom{n_0 + k - 2}{k - 1} e^{k\lambda t} \left(1 - \frac{1}{e^{\lambda t}}\right)^{k-1}$$

$$= (n_0 + k - 1)\lambda \binom{n_0 + k - 2}{k - 1} e^{k\lambda t} \left(\frac{e^{\lambda t} - 1}{e^{\lambda t}}\right)^{k-1}$$

$$= (n_0 + k - 1)\lambda \binom{n_0 + k - 2}{k - 1} e^{k\lambda t} e^{-(k - 1)\lambda t} \left(e^{\lambda t} - 1\right)^{k-1}$$

$$= (n_0 + k - 1)\lambda \binom{n_0 + k - 2}{k - 1} e^{\lambda t} \left(e^{\lambda t} - 1\right)^{k-1}$$

Integrating both sides with respect to t, we have

$$e^{(n_0 + k)\lambda t} p_{n_0+k}(t) = (n_0 + k - 1)\lambda \binom{n_0 + k - 2}{k - 1} \int e^{\lambda t} \left(e^{\lambda t} - 1\right)^{k-1} dt$$

Let $u = e^{\lambda t} - 1$. Therefore, $\frac{du}{dt} = \lambda e^{\lambda t}$ or $dt = \frac{du}{\lambda e^{\lambda t}}$.

Therefore

$$e^{(n_0 + k)\lambda t} p_{n_0+k}(t) = (n_0 + k - 1)\lambda \binom{n_0 + k - 2}{k - 1} \int e^{\lambda t} u^{k-1} \frac{du}{\lambda e^{\lambda t}}$$

$$= (n_0 + k - 1)\binom{n_0 + k - 2}{k - 1} \frac{u^k}{k} + c$$

$$= (n_0 + k - 1)\binom{n_0 + k - 2}{k - 1} \frac{(e^{\lambda t} - 1)^k}{k} + c$$

Therefore

$$p_{n_0+k}(t) = \frac{(n_0 + k - 1)}{k} \times \frac{(n_0 + k - 2)!}{(k - 1)!(n_0 - 1)!}(e^{\lambda t} - 1)^k e^{-(n_0 + k)\lambda t} + c e^{-(n_0 + k)\lambda t}$$

$$= \frac{(n_0 + k - 1)!}{k!(n_0 - 1)}(e^{\lambda t} - 1)^k e^{-(n_0 + k)\lambda t} + c e^{-(n_0 + k)\lambda t}$$

$$= \binom{n_0 + k - 1}{k}(e^{\lambda t} - 1)^k e^{-(n_0 + k)\lambda t} + c e^{-(n_0 + k)\lambda t}$$

From the initial conditions: When t = 0, $X(0) = n_0 \Rightarrow p_{n_0}(0) = 1$ and $p_n(0) = 0 \forall n \neq n_0$.

Substituting t = 0 in the equation above, we have

$$p_{n_0+k}(0) = \binom{n_0 + k - 1}{k}(1 - 1)^k \times 1 + c \times 1$$

$$c = 0.$$

Therefore,

$$p_{n_0+k}(t) = \binom{n_0 + k - 1}{k}(e^{\lambda t} - 1)^k e^{-(n_0 + k)\lambda t}$$

$$= \binom{n_0 + k - 1}{k} \times \left[e^{\lambda t}(1 - e^{-\lambda t})\right]^k e^{-n_0 \lambda t} e^{-k\lambda t}$$

Therefore

$$\mathbf{p_{n_0 + k}(t) = \binom{n_0 + k - 1}{k}(e^{-\lambda t})^{n_0}(1 - e^{-\lambda t})^k} \qquad \textbf{(4.10)}$$

Where $n = n_0 + k$, and k = 0, 1, 2, 3,…,

This is a negative binomial distribution of the form $p_k = \binom{k + r - 1}{k} \times q^k \times p^r$ where $q = 1 - p$.

In our case above, $p = e^{-\lambda t}$ and $q = 1 - e^{-\lambda t}$.

### 4.2.2 Mean and Variance by Definition

$$p_{n_0+k}(t) = \binom{n_0 + k - 1}{k}\left(e^{-\lambda t}\right)^{n_0}\left(1 - e^{-\lambda t}\right)^{k}$$

i.e.

$$\text{Prob}\left[X(t) = n_0 + k\right] = \binom{n_0 + k - 1}{k} p^{n_0} q^{k} \text{ where } p = e^{-\lambda t} \text{ and } q = 1 - e^{-\lambda t}.$$

**Mean**

$$E\left[X(t) = n_0 + k\right] = \sum_{k=0}^{\infty}(n_0 + k)\binom{n_0 + k - 1}{k} p^{n_0} q^{k}$$

$$= \sum_{k=0}^{\infty} n_0 \binom{n_0 + k - 1}{k} p^{n_0} q^{k} + \sum_{k=0}^{\infty} k \binom{n_0 + k - 1}{k} p^{n_0} q^{k}$$

$$= n_0 \sum_{k=0}^{\infty} \binom{n_0 + k - 1}{k} p^{n_0} q^{k} + \sum_{k=0}^{\infty} k \times \frac{(n_0 + k - 1)!}{k!(n_0 - 1)!} p^{n_0} q^{k}$$

$$= n_0 \times 1 + \frac{1}{(n_0 - 1)!} \sum_{k=0}^{\infty} \frac{(n_0 + k - 1)!}{k!} \times \frac{n_0 !}{n_0 !} p^{n_0} q^{k}$$

$$= n_0 \times 1 + \frac{n_0 !}{(n_0 - 1)!} \sum_{k=0}^{\infty} \frac{(n_0 + k - 1)!}{k!\, n_0 !} p^{n_0} q^{k}$$

$$= n_0 + n_0 \sum_{k=0}^{\infty} \binom{n_0 + k - 1}{k} \frac{p^{n_0 + 1}}{p} q^{k-1} \times q$$

$$= n_0 + \frac{n_0 q}{p} \sum_{k=0}^{\infty} \binom{n_0 + k - 1}{k} p^{n_0 + 1} q^{k-1}$$

$$= n_0 + \frac{n_0 q}{p}$$

$$= \frac{n_0 p + n_0 (1 - p)}{p}$$

$$= \frac{n_0}{p}$$

$$= n_0 e^{\lambda t}$$

Therefore

$$E\left[X(t) = n_0 + k\right] = n_0 e^{\lambda t} \qquad \textbf{(4.11)}$$

**Variance**

$$\mathrm{Var}\left[X(t)=n_0+k\right]=E\left[\left(X(t)=n_0+k\right)^2\right]-\left\{E\left[X(t)=n_0+k\right]\right\}^2$$

Now

$$E\left[\left(X(t)=n_0+k\right)^2\right]=\sum_{k=0}^{n}\left(n_0+k\right)^2\binom{n_0+k-1}{k}q^k p^{n_0}$$

$$=\sum_{k=0}^{n}\left\{\left(n_0+k\right)\left(n_0+k-1\right)+\left(n_0+k\right)\right\}\binom{n_0+k-1}{k}q^k p^{n_0}$$

$$=\sum_{k=0}^{\infty}\left(n_0+k\right)\left(n_0+k-1\right)\binom{n_0+k-1}{k}q^k p^{n_0}+\sum_{k=0}^{\infty}\left(n_0+k\right)\binom{n_0+k-1}{k}q^k p^{n_0}$$

$$=\sum_{k=0}^{\infty}\left(n_0+k\right)\left(n_0+k-1\right)\frac{\left(n_0+k-1\right)!}{k!\left(n_0-1\right)!}q^k p^{n_0}+\frac{n_0}{p}$$

$$=\sum_{k=0}^{\infty}\frac{\left(n_0+k\right)!\,n_0}{k!\left(n_0-1\right)!n_0}q^k p^{n_0}\left(n_0+k-1\right)+\frac{n_0}{p}$$

$$=\sum_{k=0}^{\infty}\frac{\left(n_0+k\right)!}{k!\;n_0!}q^k p^{n_0}\left(n_0^2+n_0k-n_0\right)+\frac{n_0}{p}$$

Therefore,

$$E\left[\left(X(t)=n_0+k\right)^2\right]=\sum_{k=0}^{\infty}\binom{n_0+k}{k}q^k p^{n_0}\left(n_0^2+n_0k-n_0\right)+\frac{n_0}{p}$$

Expanding, we have

$$E\left[\left(X(t)=n_0+k\right)^2\right]=n_0^2\sum_{k=0}^{\infty}\binom{n_0+k}{k}q^k p^{n_0}+\sum_{k=0}^{\infty}n_0k\binom{n_0+k}{k}q^k p^{n_0}-n_0\sum_{k=0}^{\infty}\binom{n_0+k}{k}q^k p^{n_0}+\frac{n_0}{p}$$

Working out each term separately, we have

First term

$$n_0^2\sum_{k=0}^{\infty}\binom{n_0+k}{k}q^k p^{n_0}=n_0^2\sum_{k=0}^{\infty}\binom{n_0+k}{k}q^k\frac{p^{n_0+1}}{p}$$

$$=\frac{n_0^2}{p}\sum_{k=0}^{\infty}\binom{n_0+k}{k}q^k p^{n_0+1}$$

$$=\frac{n_0^2}{p}$$

Second term

$$\sum_{k=0}^{\infty} n_0 k \binom{n_0+k}{k} q^k p^{n_0} = \sum_{k=0}^{\infty} n_0 k \frac{(n_0+k)!}{k!\, n_0!} q^k p^{n_0}$$

$$= \sum_{k=0}^{\infty} \frac{(n_0+k)!}{(k-1)!\,(n_0-1)!} q^k p^{n_0}$$

$$= \sum_{k=0}^{\infty} \frac{(n_0+k)!}{(k-1)!\,(n_0-1)!} \times \frac{n_0(n_0+1)}{n_0(n_0+1)} q^k p^{n_0}$$

$$= n_0(n_0+1) \sum_{k=0}^{\infty} \frac{(n_0+k)!}{(k-1)!\,(n_0+1)!} q^k p^{n_0}$$

$$= n_0(n_0+1) \sum_{k=0}^{\infty} \binom{n_0+k}{k-1} q^{k-1}.q.\frac{p^{n_0+2}}{p^2}$$

$$= \frac{n_0(n_0+1)q}{p^2} \sum_{k=0}^{\infty} \binom{n_0+k}{k-1} q^{k-1} p^{n_0+2}$$

$$= \frac{n_0(n_0+1)q}{p^2}$$

Third term

$$n_0 \sum_{k=0}^{\infty} \binom{n_0+k}{k} q^k p^{n_0} = n_0 \sum_{k=0}^{\infty} \binom{n_0+k}{k} q^k \frac{p^{n_0+1}}{p}$$

$$= \frac{n_0}{p} \sum_{k=0}^{\infty} \binom{n_0+k}{k} q^k p^{n_0+1}$$

$$= \frac{n_0}{p}$$

Therefore

$$E\left[\left(X(t)=n_0+k\right)^2\right] = \frac{n_0^2}{p} + \frac{n_0(n_0+1)q}{p^2} - \frac{n_0}{p} + \frac{n_0}{p}$$

$$= \frac{n_0^2}{p} + \frac{n_0(n_0+1)q}{p^2}$$

Therefore

$$\text{Variance} = \frac{n_0^2}{p} + \frac{n_0\left(n_0 + 1\right)q}{p^2} - \left(\frac{n_0}{p}\right)^2$$

$$= \frac{n_0^2}{p} + \frac{n_0\left(n_0 + 1\right)q}{p^2} - \frac{n_0^2}{p^2}$$

$$= \frac{n_0^2 p + n_0\left(n_0^2 + n_0\right)\left(1 - p\right) - n_0^2}{p^2}$$

$$= \frac{n_0^2 p + n_0^2 - n_0^2 p + n_0 - n_0 p - n_0^2}{p^2}$$

$$= \frac{n_0 q}{p^2}$$

$$= n_0 e^{2\lambda t}\left(1 - e^{-\lambda t}\right)$$

Therefore

$$\text{Variance} = n_0 e^{2\lambda t}\left(1 - e^{-\lambda t}\right) \qquad \textbf{(4.12)}$$

### 4.3 Determining $p_n(t)$ by Laplace Method

Recall the from **(2.9)** that

$$L[p_n'(t)] = sL[p_n(t)] - p_n(0)$$

Recall that the differential equation **(4.1b)** in the simple birth process is

$$p_n'(t) = -n\lambda p_n(t) + (n-1)\lambda p_{n-1}(t), \quad n \geq 1$$

Taking the Laplace transform of equation **(4.1b)**, we have

$$L(p_n'(t)) = L(-n\lambda p_n(t)) + L((n-1)\lambda p_{n-1}(t)), \quad n \geq 1$$

$$= -n\lambda L(p_n(t)) + (n-1)\lambda L(p_{n-1}(t)), \quad n \geq 1$$

Taking advantage of the identity above, we have

$$sL(p_n(t)) - p_n(0) = -n\lambda L(p_n(t)) + (n-1)\lambda L(p_{n-1}(t)), \quad n \geq 1$$

Re arranging

$$sL(p_n(t)) + n\lambda L(p_n(t)) = p_n(0) + (n-1)\lambda L(p_{n-1}(t))$$

Grouping the like terms in the equation above together and factorising, we have

$$L(p_n(t))(s+n\lambda) = p_n(0) + (n-1)\lambda L(p_{n-1}(t))$$

Therefore,

$$L(p_n(t)) = \frac{p_n(0)}{(s+n\lambda)} + \frac{(n-1)\lambda L(p_{n-1}(t))}{(s+n\lambda)} \qquad \textbf{(4.13)}$$

The initial conditions are; When $t = 0, X(0) = n_0 \Rightarrow p_{n_0}(0) = 1$ and $p_n(0) = 0 \; \forall n \neq n_0$

and $p_n(t) = 0 \; \forall n < n_0$

When $n = n_0$ in equation **(4.13)** becomes

$$L(p_{n_0}(t)) = \frac{p_{n_0}(0)}{(s+n_0\lambda)} + \frac{(n_0-1)\lambda L(p_{n_0-1}(t))}{(s+n_0\lambda)}$$

Thus

$$L(p_{n_0}(t)) = \frac{1}{(s+n_0\lambda)} \qquad \textbf{(4.14)}$$

When $n = n_0 + 1$, Equation **(4.13)** becomes

$$L\left(p_{n_0+1}(t)\right) = \frac{\left(n_0 + 1 - 1\right)\lambda}{\left(s + \left(n_0 + 1\right)\lambda\right)} L\left(p_{n_0+1-1}(t)\right)$$

$$= \frac{n_0 \lambda}{\left(s + \left(n_0 + 1\right)\lambda\right)} L\left(p_{n_0}(t)\right)$$

$$= \frac{n_0 \lambda}{\left(s + \left(n_0 + 1\right)\lambda\right)} \times \frac{1}{\left(s + n_0 \lambda\right)}$$

$$= \frac{n_0 \lambda}{\left(s + n_0 \lambda\right)\left(s + \left(n_0 + 1\right)\lambda\right)}$$

Thus

$$L\left(p_{n_0+1}(t)\right) = \frac{n_0 \lambda}{\left(s + n_0 \lambda\right)\left(s + \left(n_0 + 1\right)\lambda\right)} \quad \textbf{(4.15)}$$

When $n = n_0 + 2$, Equation **(4.13)** becomes

$$L\left(p_{n_0+2}(t)\right) = \frac{\left(n_0 + 2 - 1\right)\lambda}{\left(s + \left(n_0 + 2\right)\lambda\right)} L\left(p_{n_0+2-1}(t)\right)$$

$$= \frac{\left(n_0 + 1\right)\lambda}{\left(s + \left(n_0 + 2\right)\lambda\right)} L\left(p_{n_0+1}(t)\right)$$

$$= \frac{\left(n_0 + 1\right)\lambda}{\left(s + \left(n_0 + 2\right)\lambda\right)} \times \frac{n_0 \lambda}{\left(s + n_0 \lambda\right)\left(s + \left(n_0 + 1\right)\lambda\right)}$$

Thus

$$L\left(p_{n_0+2}(t)\right) = \frac{n_0\left(n_0 + 1\right)\lambda^2}{\left(s + n_0 \lambda\right)\left(s + \left(n_0 + 1\right)\lambda\right)\left(s + \left(n_0 + 2\right)\lambda\right)} \quad \textbf{(4.16)}$$

When $n = n_0 + 3$, Equation **(4.13)** becomes

$$L\left(p_{n_0+3}(t)\right) = \frac{\left(n_0 + 3 - 1\right)\lambda}{\left(s + \left(n_0 + 3\right)\lambda\right)} L\left(p_{n_0+3-1}(t)\right)$$

$$= \frac{\left(n_0 + 2\right)\lambda}{\left(s + \left(n_0 + 3\right)\lambda\right)} L\left(p_{n_0+2}(t)\right)$$

$$= \frac{\left(n_0 + 2\right)\lambda}{\left(s + \left(n_0 + 3\right)\lambda\right)} \times \frac{n_0\left(n_0 + 1\right)\lambda^2}{\left(s + n_0 \lambda\right)\left(s + \left(n_0 + 1\right)\lambda\right)\left(s + \left(n_0 + 2\right)\lambda\right)}$$

Thus

$$L\left(p_{n_0+3}(t)\right) = \frac{n_0\left(n_0+1\right)\left(n_0+2\right)\lambda^3}{\left(s+n_0\lambda\right)\left(s+\left(n_0+1\right)\lambda\right)\left(s+\left(n_0+2\right)\lambda\right)\left(s+\left(n_0+3\right)\lambda\right)} \qquad \textbf{(4.17)}$$

Generalizing,

$$p_n\left(t\right)=p_{n_0+k}\left(t\right), k=0,1,2,3..... \text{ and } n=n_0, n_0+1,\ n_0+2,\ ....$$

$$L\left(p_n\left(t\right)\right) = L\left(p_{n_0+k}\left(t\right)\right) = \frac{\prod_{i=0}^{k-1}\left(n_0+i\right)\lambda^k}{\prod_{i=0}^{k}\left(s+\left(n_0+i\right)\lambda\right)},\ \ i = 0, 1, 2, 3,..... \qquad \textbf{(4.18)}$$

Taking the laplace inverse of equation **(3.18)** above, we have

$$p_n\left(t\right) = p_{n_0+k}\left(t\right) = L^{-1}\left[p_{n_0+k}\left(t\right)\right]=L^{-1}\left[\frac{\prod_{i=0}^{k-1}\left(n_0+i\right)\lambda^k}{\prod_{i=0}^{k}\left(s+\left(n_0+i\right)\lambda\right)}\right]=\prod_{i=0}^{k-1}\left(n_0+i\right)\lambda^k\ L^{-1}\left[\frac{1}{\prod_{i=0}^{k}\left(s+\left(n_0+i\right)\lambda\right)}\right]$$

These are $k+1$simple poles (Singularities)

$$L^{-1}\left[\frac{1}{\prod_{i=0}^{k}\left(s+\left(n_0+i\right)\lambda\right)}\right] = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{e^{st}}{\prod_{i=0}^{k}\left(s+\left(n_0+i\right)\lambda\right)}$$

$$= \sum_{j=0}^{k}\frac{e^{-\left(n_0+j\right)\lambda t}}{\prod_{i=0}^{k}\left(-\left(n_0+j\right)\lambda+\left(n_0+i\right)\lambda\right)},\quad j \neq i$$

$$= \sum_{j=0}^{k}\frac{e^{-\left(n_0+j\right)\lambda t}}{\lambda^k\prod_{i=0}^{k}\left(-n_0-j+n_0+i\right)}$$

$$= \sum_{j=0}^{k}\frac{e^{-\left(n_0+j\right)\lambda t}}{\lambda^k\prod_{i=0}^{k}\left(i-j\right)}$$

Thus,

$$p_n\left(t\right) = \prod_{i=0}^{k-1}\left(n_0+i\right)\lambda^k \times \sum_{j=0}^{k}\frac{e^{-\left(n_0+j\right)\lambda t}}{\lambda^k\prod_{i=0}^{k}\left(i-j\right)}$$

Equivalently,

$$p_n(t) = \prod_{i=0}^{k-1}(n_0 + i) \times \sum_{j=0}^{k} \frac{e^{-(n_0 + j)\lambda t}}{\prod_{i=0}^{k}(i - j)} \qquad \textbf{(4.19)}$$

But

$$\prod_{i=0}^{k-1}(n_0 + i) = n_0(n_0 + 1)(n_0 + 2).......(n_0 + (k-1)) = \frac{(n_0 + k - 1)!}{(n_0 - 1)!}$$

Further,

$$\prod_{i=0}^{k}(i - j) = \sum_{i=0}^{j-1}(i - j) \times \sum_{i=j+1}^{k}(i - j), \quad i \neq j$$

But

$$\sum_{i=0}^{j-1}(i - j) = (-j)(1 - j)(2 - j).......(-3)(-2)(-1)$$

$$= (-1)^j \; j!$$

Similarly,

$$\sum_{i=j+1}^{k}(i - j) = 1 \,.\, 2 \,.\, 3 \,.\, ......(k - 1 - j)(k - j) = (k - j)!$$

Thus

$$\prod_{i=0}^{k}(i - j) = \sum_{i=0}^{j-1}(i - j) \times \sum_{i=j+1}^{k}(i - j)$$

$$= (-1)^j \; j! \times (k - j)!$$

$$= (-1)^j \; j!(k - j)!$$

Equation **(4.19)** now becomes

$$p_n(t) = \frac{(n_0 + k - 1)!}{(n_0 - 1)!} \times \sum_{j=0}^{k} \frac{e^{-(n_0 + j)\lambda t}}{(-1)^j \; j!(k - j)!}$$

$$= \frac{(n_0 + k - 1)!}{(n_0 - 1)!} \times \sum_{j=0}^{k} \frac{(-1)^j e^{-n_0 \lambda t} e^{j\lambda t}}{j!(k - j)!}$$

$$= \frac{(n_0 + k - 1)!}{(n_0 - 1)!} e^{-n_0 \lambda t} \sum_{j=0}^{k} \frac{(-1)^j e^{j\lambda t}}{j!(k - j)!} \times \frac{k!}{k!}$$

Equivalently

$$p_n(t) = \frac{(n_0 + k - 1)!}{(n_0 - 1)!k} e^{-n_0 \lambda t} \sum_{j=0}^{k} \frac{(-e^{\lambda t})^j k!}{j!(k-j)!}$$

$$= \binom{n_0 + k - 1}{n_0 - 1} (e^{-\lambda t})^{n_0} \sum_{j=0}^{k} \binom{k}{j} (-e^{\lambda t})^j$$

But $\sum_{j=0}^{k} \binom{k}{j} (-e^{\lambda t})^j = (1 - e^{\lambda t})^k$. Thus

$$p_n(t) = \binom{n_0 + k - 1}{n_0 - 1} (e^{-\lambda t})^{n_0} \times (1 - e^{\lambda t})^k$$

But $n = n_0 + k \Rightarrow k = n - n_0, \ k = 0, 1, 2, ...$

Substituting $k$ with $n - n_0$ in equation the above, we have

$$p_n(t) = \binom{n_0 + n - n_0 - 1}{n_0 - 1} (e^{-\lambda t})^{n_0} \times (1 - e^{\lambda t})^{n - n_0}$$

$\Rightarrow$

$$\mathbf{p_n(t) = \binom{n - 1}{n_0 - 1} (e^{-\lambda t})^{n_0} \times (1 - e^{\lambda t})^{n - n_0}} \quad n = n_0 + k, \ k = 0,1,2,... \qquad \textbf{(4.20)}$$

Putting $p = e^{-\lambda t}$ and $q = 1 - e^{-\lambda t}$, we have $p_n(t) = \binom{n-1}{n_0 - 1} p^{n_0} q^{n - n_0}$ which is a negative binomial distribution.

***Special case***

If $n_0 = 1$, then

$$p_n(t) = \binom{n-1}{0} e^{-\lambda t} (1 - e^{-\lambda t})^n. \text{ But } \binom{n-1}{0} = 1. \text{ Thus}$$

$$p_n(t) = e^{-\lambda t} (1 - e^{-\lambda t})^n \qquad \textbf{(4.21)}$$

which is of the form $p_n(t) = p\,q^n$ which is a pmf geometric distribution.

## 4.4 Probability Generation Method

### 4.4.1 Determining $p_n(t)$ by PGF Method

***The initial conditions are: When t = 0, $X(0) = n_0$***

$\Rightarrow p_{n_0}(0) = 1,\ p_n(0) = 0\ \forall\ n \neq n_0$ and $p_n(t) = 0\ \ \forall\ n < n_0$.

If $\lambda_n = n\lambda$ for $n = 1, 2, 3, \ldots$ the difference differential equation **(4.2b)** remain as

$$p_n'(t) = -n\lambda\, p_n(t) + (n-1)\lambda\, p_{n-1}(t), \quad n \geq 1$$

***Definitions***

Define

$$\left.\begin{aligned} G(s,t) &= \sum_{n=0}^{\infty} p_n(t) s^n \\ \frac{d}{dt}G(s,t) &= \sum_{n=0}^{\infty} p_n'(t)\, s^n = p_0'(t) + \sum_{n=1}^{\infty} p_n'(t)\, s^n \\ \frac{d}{ds}G(s,t) &= \sum_{n=0}^{\infty} n\, p_n(t)\, s^{n-1} = \frac{1}{s}\sum_{n=1}^{\infty} n\, p_n(t)\, s^n \end{aligned}\right\} \qquad \textbf{(4.22)}$$

From the definition of $G(s,t)$ and the initial conditions, we can make the following deductions

$$G(s,0) = \sum_{n=0}^{\infty} p_n(0)\, s^n = p_{n_0}(0)\, s^{n_0} = s^{n_0}$$

$$\sum_{n=1}^{\infty} p_n'(t)\, s^n = \frac{d}{dt}G(s,t) - p_0'(t) \quad \text{and} \quad \sum_{n=1}^{\infty} n\, p_n(t)\, s^n = s\left[\frac{d}{ds}G(s,t)\right]$$

Multiplying equation **(3.2b)** by $s^n$ and summing over n

$$\sum_{n=1}^{\infty} p_n'(t)s^n = -\lambda\sum_{n=1}^{\infty} n\, p_n(t)s^n + \lambda\sum_{n=1}^{\infty}(n-1)\, p_{n-1}(t)s^n \qquad \textbf{(4.23)}$$

Equivalently,

$$\sum_{n=1}^{\infty} p_n'(t)s^n = -\lambda\sum_{n=1}^{\infty} n\, p_n(t)s^n + \lambda s\sum_{n=1}^{\infty}(n-1)\, p_{n-1}(t)s^{n-1}$$

From the definitions **(4.22)**, initial conditions, and the deductions, equation **(4.23)** becomes

$$\frac{d}{dt}G(s,t) - p_0'(t) = -s\lambda\frac{d}{ds}G(s,t) + \lambda s^2\frac{d}{ds}G(s,t)$$

Factoring out $\frac{d}{ds}G(s,t)$, we have

$$\frac{d}{dt}G(s,t) - p_0'(t) = \frac{d}{ds}G(s,t)\left(-s\lambda + \lambda s^2\right)$$

But from **(4.1a),** $p_0'(t) = 0$. Thus, the linear differential equation is

$$\frac{d}{dt}G(s,t) - \lambda s(s-1)\frac{d}{ds}G(s,t) = 0 \qquad \textbf{(4.24)}$$

Applying lagranges differential equation, the auxiliary equations are

$$\frac{dt}{1} = \frac{ds}{-\lambda s(s-1)} = \frac{dG(s,t)}{0} \qquad \textbf{(4.25)}$$

Taking $\frac{dt}{1} = \frac{ds}{-\lambda s(s-1)}$ and integrating, we have

$$\int \frac{-\lambda dt}{1} = \int \frac{ds}{s(s-1)} \qquad \text{(i)}$$

$$\frac{1}{s(s-1)} = \frac{A}{s} + \frac{B}{(s-1)}$$

$\Rightarrow$

$$\frac{1}{s(s-1)} = \frac{A(s-1)+Bs}{s(s-1)} = \frac{-A+s(A+B)}{s(s-1)}$$

$\Rightarrow$

$A = -1$ and $B = 1$

Thus, equation (i) above becomes

$$\int -\lambda dt = -\int \frac{1}{s} ds + \int \frac{1}{s-1} ds$$

$\Rightarrow$

$$-\lambda t = -\ln s + \ln(s-1) + c_1$$

Thus,

$$-\lambda t + \ln s - \ln(s-1) = c_1 \qquad \text{(ii)}$$

Taking $\frac{dt}{1} = \frac{dG(s,t)}{0}$ and integrating, we have

$$\int 0\, dt = \int dG(s,t)$$

$$c_2 = G(s,t) \qquad \text{(iii)}$$

Therefore, from (ii) and (iii),

$$G(s,t) = F(-\lambda t + \ln s - \ln(s-1)) = F\left(-\lambda t + \ln\left(\frac{s}{s-1}\right)\right) \qquad \textbf{(4.26)}$$

Let $F\left(-\lambda t + \ln\left(\frac{s}{s-1}\right)\right) = w$ and $w = e^{-\lambda t + \ln\left(\frac{s}{s-1}\right)}$. Therefore

$$w = e^{-\lambda t}\left(\frac{s}{s-1}\right)$$

But $G(s,0) = s^{n_0}$

$$w = \frac{s}{s-1} \Rightarrow s = \frac{w}{w-1}$$

$$G(s,0) = \left(\frac{w}{w-1}\right)^{n_0}$$

$$G(s,t) = \left[\frac{e^{-\lambda t}\dfrac{s}{s-1}}{e^{-\lambda t}\dfrac{s}{s-1} - 1}\right]^{n_0}$$

$$= \left[\frac{\dfrac{se^{-\lambda t}}{s-1}}{\dfrac{se^{-\lambda t} - s + 1}{s-1}}\right]^{n_0} = \left[\frac{se^{-\lambda t}}{se^{-\lambda t} - s + 1}\right]^{n_0}$$

But $se^{-\lambda t} - s + 1 = 1 - s\left(1 - e^{-\lambda t}\right)$

Thus

$$G(s,t) = \left[\frac{se^{-\lambda t}}{se^{-\lambda t} - s + 1}\right]^{n_0} = \left[\frac{se^{-\lambda t}}{1 - s\left(1 - e^{-\lambda t}\right)}\right]^{n_0} \qquad \textbf{(4.27)}$$

Let $p = e^{-\lambda t}$ and $q = 1 - e^{-\lambda t}$

Then

$$G(s,t) = \left[\frac{ps}{1-qs}\right]^{n_0}$$

This is the p.m.f of a negative binomial distribution.

$p_n(t)$ is the coefficient of $s^n$ in $G(s,t)$.

$$G(s,t) = \left[\frac{ps}{1-qs}\right]^{n_0} = (ps)^{n_0}(1-qs)^{n_0}$$

$$= (ps)^{n_0} \sum_{k=0}^{\infty} \binom{n_0+k-1}{k} (qs)^k$$

$$= p^{n_0} \sum_{k=0}^{\infty} \binom{n_0+k-1}{k} q^k s^{n_0+k}$$

$$= \sum_{k=0}^{\infty} p^{n_0} \binom{n_0+k-1}{k} q^k s^{n_0+k}$$

Thus,

$$p_n(t) = p_{n_0+k}(t) = p^{n_0} \binom{n_0+k-1}{k} q^k$$

$$= \binom{n_0+k-1}{k} p^{n_0} q^k$$

But $p = e^{-\lambda t}$ and $q = 1 - e^{-\lambda t}$.

Thus

$$\mathbf{P_n(t) = \binom{n_0+k-1}{k} \left(e^{-\lambda t}\right)^{n_0} \left(1 - e^{-\lambda t}\right)^k} \quad k = 0, 1, 2, 3, \ldots \quad \text{and } n_0 \geq 1. \qquad \mathbf{(4.28)}$$

This is a negative binomial distribution with parameters $p = e^{-\lambda t}$ and $q = 1 - e^{-\lambda t}$.

### 4.4.2 Mean and Variance by use of Generating Function

**(i) Mean**

Recall that $G(s,t) = \left[\frac{ps}{1-qs}\right]^{n_0}$

Now $E(X) = \frac{d}{ds} G(s,t)\Big|_{s=1}$

$$\frac{d}{ds}\left[\frac{ps}{1-qs}\right]^{n_0} = \frac{d}{ds}\left\{(ps)^{n_0}(1-qs)^{-n_0}\right\}$$

$$= n_0 p^{n_0} s^{n_0-1}(1-qs)^{-n_0} + (ps)^{n_0} n_0 q (1-qs)^{-n_0-1}$$

Putting $s = 1$.

$$\frac{d}{ds}G(s,t)\bigg|_{s=1} = n_0 p^{n_0}(1-q)^{-n_0} + (p)^{n_0} n_0 q (1-q)^{-n_0-1}$$

$$= n_0 + n_0 q p^{-1} = n_0\left(1+\frac{q}{p}\right)$$

Thus,

$$E(X) = n_0\left(1+\frac{q}{p}\right) \qquad \textbf{(4.29)}$$

**(ii) Variance**

Recall

$$\frac{d}{ds}G(s,t) = n_0 p^{n_0} s^{n_0-1}(1-qs)^{-n_0} + (ps)^{n_0} n_0 q (1-qs)^{-n_0-1}$$

$$= n_0 p^{n_0}\left\{s^{n_0-1}(1-qs)^{-n_0} + s^{n_0} q (1-qs)^{-n_0-1}\right\}$$

Therefore

$$\frac{d^2}{ds^2}G(s,t) = n_0 p^{n_0}\Big\{\Big[(n_0-1)s^{n_0-2}(1-qs)^{-n_0} + s^{n_0-1}(-n_0)(1-qs)^{-n_0-1}(-q)\Big] +$$

$$\Big[n_0 q s^{n_0-1}(1-qs)^{-(n_0+1)} + q s^{n_0}\Big(-(n_0+1)(1-qs)^{-(n_0+2)}(-q)\Big)\Big]\Big\}$$

$$\frac{d^2}{ds^2}G(s,t) = n_0 p^{n_0}\Big\{\Big[(n_0-1)s^{n_0-2}(1-qs)^{-n_0} + n_0 q s^{n_0-1}(1-qs)^{-n_0-1}\Big] +$$

$$\Big[n_0 q s^{n_0-1}(1-qs)^{-(n_0+1)} + (n_0+1)q^2 s^{n_0}(1-qs)^{-(n_0+2)}\Big]\Big\}$$

Now put s = 1

$$\frac{d^2}{ds^2}G(s,t)\bigg|_{s=1} = n_0 p^{n_0}\Big\{(n_0-1)(1-q)^{-n_0} + n_0 q (1-q)^{-n_0-1} +$$

$$n_0 q (1-q)^{-(n_0+1)} + (n_0+1)q^2(1-q)^{-(n_0+2)}\Big\}$$

$$= n_0 p^{n_0}\Big\{(n_0-1)p^{-n_0} + n_0 q p^{-(n_0+1)} + n_0 q p^{-(n_0+1)} + (n_0+1)q^2 p^{-(n_0+2)}\Big\}$$

$$= \frac{n_0 p^{n_0}}{p^{n_0}}\left\{(n_0-1) + 2n_0\frac{q}{p} + \frac{(n_0+1)q^2}{p^2}\right\}$$

Therefore

$$\frac{d^2}{ds^2}G(s,t)\Big|_{s=1} = n_0\left\{(n_0 - 1) + \frac{2n_0 q}{p} + \frac{(n_0 + 1)q^2}{p^2}\right\}$$

Now, since $\operatorname{Var} X = G''(1,t) + G'(1,t) - \left[G'(1,t)\right]^2$

$$\operatorname{Var} X = n_0\left\{(n_0 - 1) + \frac{2n_0 q}{p} + \frac{(n_0 + 1)q^2}{p^2}\right\} + n_0\left(1 + \frac{q}{p}\right) - \left[n_0\left(1 + \frac{q}{p}\right)\right]^2$$

$$= n_0(n_0 - 1) + \frac{2n_0^2 q}{p} + \frac{(n_0 + 1)n_0 q^2}{p^2} + n_0 + \frac{n_0 q}{p} - n_0^2\left(1 + \frac{2q}{p} + \frac{q^2}{p^2}\right)$$

$$= n_0^2 - n_0 + \frac{2n_0^2 q}{p} + \frac{n_0^2 q^2}{p^2} + \frac{n_0 q^2}{p^2} + n_0 + \frac{n_0 q}{p} - n_0^2 - \frac{2n_0^2 q}{p} - \frac{n_0^2 q^2}{p^2}$$

$$= \frac{n_0 q^2}{p^2} + \frac{n_0 q}{p} = \frac{n_0 q}{p}\left(\frac{q}{p} + 1\right)$$

$$= \frac{n_0 q}{p}\left(\frac{q + p}{p}\right)$$

Therefore

$$\operatorname{Var} X = \frac{n_0 q}{p^2}(1)$$

$$= \frac{n_0 q}{p^2}$$

$$= \frac{n_0\left(1 - e^{-\lambda t}\right)}{\left(e^{-\lambda t}\right)^2}$$

$$= n_0 e^{2\lambda t}\left(1 - e^{-\lambda t}\right)$$

Therefore,

$$\operatorname{Var} X = n_0 e^{2\lambda t}\left(1 - e^{-\lambda t}\right) \qquad \textbf{(4.30)}$$

## 4.5 Method of Moments to Determine Mean and Variance

### 4.5.1 Mean

Multiply the basic difference differential equations by n and then sum over n

$$\sum_{n=1}^{\infty} n\, p_n'(t) = -\lambda \sum_{n=1}^{\infty} n^2 p_n(t) + \lambda \sum_{n=1}^{\infty} n(n-1) p_{n-1}(t) \qquad \textbf{(4.31)}$$

Define

$$M_1(t) = \sum_{n=1}^{\infty} n\, p_n(t) \text{ and } M_2(t) = \sum_{n=1}^{\infty} n^2 p_n(t) \qquad \textbf{(4.32)}$$

Therefore,

$$M_1'(t) = \sum_{n=1}^{\infty} n\, p_n'(t) \text{ and } M_2'(t) = \sum_{n=1}^{\infty} n^2 p_n'(t) \qquad \textbf{(4.33)}$$

Substituting the definitions above in equation **(4.31)**, we have

$$M_1'(t) = -\lambda M_2(t) + \lambda \sum_{n=1}^{\infty} (n-1+1)(n-1) p_{n-1}(t)$$

$$= -\lambda M_2(t) + \lambda \left\{ \sum_{n=1}^{\infty} (n-1)^2 p_{n-1}(t) + \sum_{n=1}^{\infty} (n-1) p_{n-1}(t) \right\}$$

$$= -\lambda M_2(t) + \lambda \left\{ M_2(t) + M_1(t) \right\}$$

Therefore,

$$M_1'(t) = \lambda M_1(t)$$

$$\Rightarrow$$

$$\frac{M_1'(t)}{M_1(t)} = \lambda$$

$$\Rightarrow$$

$$\frac{d}{dt}\left(\ln M_1(t)\right) = \lambda$$

Integrating both sides with respect to t, we have

$$\int \frac{d}{dt}\left(\ln M_1(t)\right) dt = \int \lambda \, dt$$

$$\Rightarrow$$

$$\ln\left(M_1(t)\right) = \lambda t + c$$

Taking the exponential of both sides

$$M_1(t) = e^{\lambda t + c} = k e^{\lambda t}$$

Thus,

$$M_1(t) = k e^{\lambda t} \qquad \textbf{(4.34)}$$

When $t = 0$,

$$M_1(0) = k e^{\lambda \times 0} = k \qquad \textbf{(4.35)}$$

But from the definition of the first moment

$$M_1(0) = \sum_{n=1}^{\infty} n\, p_n(0) = p_1(0) + 2p_2(0) + ... + n_0\, p_{n_0}(0) + ... = n_0$$

(Recall that at time $t = 0$, $X(0) = n_0 \quad \Rightarrow \quad p_{n_0}(0) = 1$ and $P_n(0) = 0 \ \forall \ n \neq n_0$)

Thus

$k = n_0$ (From equation **(4.35)**)

Equation **(4.34)** now becomes

$$M_1(t) = n_0 e^{\lambda t} \quad \Rightarrow \quad E\left[X(t)\right] = n_0 e^{\lambda t} \qquad \textbf{(4.36)}$$

### 4.5.2 Variance

Multiply the basic difference differential equations by n and then sum over n

$$\sum_{n=1}^{\infty} n^2 p_n'(t) = -\lambda \sum_{n=1}^{\infty} n^3 p_n(t) + \lambda \sum_{n=1}^{\infty} n^2 (n-1) p_{n-1}(t) \qquad \textbf{(4.37)}$$

Substituting the definitions **(4.32)** and **(4.33)** above in equation **(4.37)**, we have

$$M_2'(t) = -\lambda M_3(t) + \lambda \sum_{n=1}^{\infty} (n-1+1)^2 (n-1) p_{n-1}(t)$$

$$= -\lambda M_2(t) + \lambda \sum_{n=1}^{\infty} \left[(n-1)^2 + 2(n-1) + 1\right](n-1) p_{n-1}(t)$$

$$= -\lambda M_2(t) + \lambda \sum_{n=1}^{\infty} \left[(n-1)^3 P_{n-1}(t) + 2(n-1)^2 p_{n-1}(t) + (n-1) p_{n-1}(t)\right]$$

$$= -\lambda M_2(t) + \lambda \left\{M_3(t) + 2M_2(t) + M_1(t)\right\}$$

Therefore,

$$M_2'(t) = 2\lambda M_2(t) + \lambda M_1(t)$$

Equivalently,

$$M_2'(t) - 2\lambda M_2(t) = \lambda M_1(t) \qquad \textbf{(4.38)}$$

Integrating factor $= e^{\int -2\lambda\, dt} = e^{-2\lambda t}$

Multiplying equation **(4.38)** by the integrating factor, we have

$$e^{-2\lambda t} M_2'(t) - 2\lambda e^{-2\lambda t} M_2(t) = \lambda e^{-2\lambda t} M_1(t)$$

But from **(4.36),** $M_1(t) = n_0 e^{\lambda t}$. Thus

$$e^{-2\lambda t} M_2'(t) - 2\lambda e^{-2\lambda t} M_2(t) = \lambda e^{-2\lambda t} \times n_0 e^{\lambda t} = n_0 \lambda e^{-\lambda t}$$

or

$$e^{-2\lambda t} M_2'(t) - 2\lambda e^{-2\lambda t} M_2(t) = n_0 \lambda e^{-\lambda t}$$

Equivalently

$$\frac{d}{dt}\left[e^{-2\lambda t} M_2(t)\right] = n_0 \lambda e^{-\lambda t}$$

Therefore

$$e^{-2\lambda t} M_2(t) = n_0 \lambda \int e^{-\lambda t} dt$$

$$= -\frac{n_0 \lambda}{\lambda} e^{-\lambda t} + c$$

$$= -n_0 e^{-\lambda t} + c \qquad \textbf{(4.39)}$$

When $t = 0$,

$$M_2(0) = -n_0 + c$$

Therefore

$$\sum_{n=1}^{\infty} n^2 p_n(0) = -n_0 + c$$

Thus

$$n_0^2 = -n_0 + c$$

$$\Rightarrow$$

$$c = n_0^2 + n_0$$

Recall that from the definition of the second moment

$$M_2(0) = \sum_{n=1}^{\infty} n^2 p_n(0) = p_1(0) + 2^2 p_2(0) + ... + n_0^2 p_{n_0}(0) + ... = n_0^2$$

Thus, equation **(4.39)** becomes

$$e^{-2\lambda t} M_2(t) = -n_0 e^{-\lambda t} + n_0^2 + n_0$$

$$\Rightarrow$$

$$M_2(t) = -n_0 e^{\lambda t} + n_0^2 e^{2\lambda t} + n_0 e^{2\lambda t} \qquad \textbf{(4.40)}$$

Therefore,

$$\begin{aligned} \operatorname{Var}[X(t)] &= M_2(t) - [M_1(t)]^2 \\ &= -n_0 e^{\lambda t} + n_0^2 e^{2\lambda t} + n_0 e^{2\lambda t} - n_0^2 e^{2\lambda t} \\ &= -n_0 e^{\lambda t} + n_0 e^{2\lambda t} \\ &= n_0 e^{2\lambda t} - n_0 e^{\lambda t} \\ &= n_0 e^{\lambda t}\left(e^{\lambda t} - 1\right) \end{aligned}$$

Therefore,

$$\operatorname{Var}[X(t)] = n_0 e^{\lambda t}\left(e^{\lambda t} - 1\right) \qquad \textbf{(4.41)}$$

# CHAPTER FIVE

# SIMPLE BIRTH PROCESS WITH IMMIGRATION

## 5.1 Introduction

Consider a pure birth process $\{X(t), t \geq 0\}$ with intensity function $\lambda_n(t) = v(t) + n\lambda(t)$, $n = 0, 1, 2, \ldots$ One can regard $X(t)$ as the size at time t of a population into which individuals immigrate in accord with a Poisson process with intensity function $v(t)$ and then give rise to offspring in accord with a pure birth process with linear birth rate.

The objective in this chapter is to solve the basic difference differential equations given in Chapter 2 (refer to equations **(2.1)** and **(2.2)**) when $\lambda_n = n\lambda + v.$ $n = 0, 1, 2, 3, \ldots$ We shall specifically look at three methods namely the iterative method, the Laplace transform and the Lagrages method. In all these cases, we will work with the initial conditions being (i) X(0) = 0 and (ii) $X(0) = n_0$

When $\lambda_n = n\,\lambda + v$ for $n = 0, 1, 2, 3, \ldots$. the basic difference differential equations become

$$p_0'(t) = -(0\times\lambda + v)\,p_0(t) \;\Rightarrow$$

$$p_0'(t) = -v\,p_0(t) \qquad \textbf{(5.1a)}$$

$$p_n'(t) = -(n\lambda + v)\,p_n(t) + ((n-1)\lambda + v)\,p_{n-1}(t), \quad n \geq 1 \qquad \textbf{(5.1b)}$$

## 5.2 Iteration Method

### 5.2.1 Determining $p_n(t)$ using iteration Method

***Initial Conditions: When t = 0, $X(0) = n_0$***

When $n = n_0$, equation **(5.1b)** becomes

$$p_{n_0}'(t) = -(n_0\lambda + v)\,p_{n_0}(t) + ((n-1)\lambda + v)\,p_{n_0-1}(t)$$

But $p_{n_0-1}(t) = 0.$ Therefore

$$p_{n_0}'(t) = -(n_0\lambda + v)\,P_{n_0}(t)$$

$$\frac{p_{n_0}'(t)}{p_{n_0}(t)} = -(n_0\lambda + v)$$

$$\frac{d}{dt}\log p_{n_0}(t) = -(n_0\lambda + v)$$

Integrating both sides, we have

$$\int d\log p_{n_0}(t) = -\int \left(n_0\lambda + v\right)dt$$

$$\log p_{n_0}(t) = -\left(n_0\lambda + v\right)t + c$$

Taking exponential both sides, we have

$$p_{n_0}(t) = e^{-\left(n_0\lambda + v\right)t + c} = e^{-\left(n_0\lambda + v\right)t}\, e^{c} = k\, e^{-\left(n_0\lambda + v\right)t}$$

Using the initial condition, $X(0) = n_0$, $p_{n_0}(0) = 1$.

$$1 = k$$

Thus,

$$p_{n_0}(t) = e^{-\left(n_0\lambda + v\right)t} \qquad \textbf{(5.1)}$$

When $n = n_0 + 1$, equation **(5.1b)** becomes

$$p'_{n_0+1}(t) = -\left(\left(n_0 + 1\right)\lambda + v\right) p_{n_0+1}(t) + \left(n_0\lambda + v\right) p_{n_0}(t)$$

Re arranging, we have

$$p'_{n_0+1}(t) + \left(\left(n_0 + 1\right)\lambda + v\right) p_{n_0+1}(t) = \left(n_0\lambda + v\right) e^{-\left(n\lambda + v\right)t}$$

Next, we solve the equation above using the integrating factor method

Integrating factor $= e^{\int \left(\left(n_0 + 1\right)\lambda + v\right)dt} = e^{\left(\left(n_0 + 1\right)\lambda + v\right)t}$

Multiplying equation above by the integrating factor, we have

$$e^{\left(\left(n_0 + 1\right)\lambda + v\right)t} p'_{n_0+1}(t) + \left(\left(n_0 + 1\right)\lambda + v\right) e^{\left(\left(n_0 + 1\right)\lambda + v\right)t} p_{n_0+1}(t) = \left(n_0\lambda + v\right) e^{\left(\left(n_0 + 1\right)\lambda + v\right)t} e^{-\left(n_0\lambda + v\right)t}$$

$$= \left(n_0\lambda + v\right)e^{\lambda t}$$

$$\frac{d}{dt}\left[e^{\left(\left(n_0 + 1\right)\lambda + v\right)t} p_{n_0+1}(t)\right] = \left(n_0\lambda + v\right)e^{\lambda t}$$

Integrating

$$e^{\left(\left(n_0 + 1\right)\lambda + v\right)t} p_{n_0+1}(t) = \left(n_0\lambda + v\right)\int e^{\lambda t}dt$$

$$= \left(n_0\lambda + v\right)\frac{e^{\lambda t}}{\lambda} + c$$

Therefore

$$p_{n_0+1}(t) = \left(\left(n_0\lambda + v\right)\frac{e^{\lambda t}}{\lambda} + c\right) e^{-\left(\left(n_0 + 1\right)\lambda + v\right)t}$$

When t = 0,

$$p_{n_0+1}(0) = \frac{(n_0\lambda + v)e^0}{\lambda} + c$$

$$0 = \left(n_0 + \frac{v}{\lambda}\right) + c$$

$$c = -\left(n_0 + \frac{v}{\lambda}\right)$$

Substituting for c, we get

$$p_{n_0+1}(t) = \left(\left(n_0 + \frac{v}{\lambda}\right)e^{\lambda t} - \left(n_0 + \frac{v}{\lambda}\right)\right)e^{-((n_0+1)\lambda + v)t}$$

$$= \left(n_0 + \frac{v}{\lambda}\right)\left(e^{\lambda t} - 1\right)e^{-((n_0+1)\lambda + v)t}$$

$$= \left(n_0 + \frac{v}{\lambda}\right)e^{-(n_0\lambda + v)t}\left(1 - e^{-\lambda t}\right)$$

Therefore, $p_{n_0+1}(t) = \left(n_0 + \frac{v}{\lambda}\right)e^{-(n_0\lambda + v)t}\left(1 - e^{-\lambda t}\right)$ **(5.2)**

When $n = n_0 + 2,$ equation **(5.2b)** becomes

$$p'_{n_0+2}(t) = -((n_0+2)\lambda + v)\,p_{n_0+2}(t) + ((n_0+1)\lambda + v)\,p_{n_0+1}(t)$$

Re arranging, we have

$$p'_{n_0+2}(t) + ((n_0+2)\lambda + v)\,p_{n_0+2}(t) = ((n_0+1)\lambda + v)\left(n_0 + \frac{v}{\lambda}\right)e^{-(n_0\lambda + v)t}\left(1 - e^{-\lambda t}\right)$$

$$= \left(n_0 + \frac{v}{\lambda}\right)\left(n_0 + \frac{v}{\lambda} + 1\right)e^{-(n_0\lambda + v)t}\left(1 - e^{-\lambda t}\right)$$

Next, we solve the above equation using the integrating factor Method

Integrating factor $= e^{\int((n_0+2)\lambda + v)dt} = e^{((n_0+2)\lambda + v)t}$

Multiplying equation above by the integrating factor, we get

$$e^{((n_0+2)\lambda+v)t}p_{n_0+2}(t) + e^{((n_0+2)\lambda+v)t}((n_0+2)\lambda + v)\,p_{n_0+2}(t) = e^{((n_0+2)\lambda+v)t}((n_0+1)\lambda+v)\left(n_0+\tfrac{v}{\lambda}\right)e^{-(n_0\lambda+v)t}\left(1-e^{-\lambda t}\right)$$

$$\frac{d}{dt}\left[e^{((n_0+2)\lambda+v)t}p_{n_0+2}(t)\right] = \left(n_0+\frac{v}{\lambda}\right)\left(n_0+\frac{v}{\lambda}+1\right)\lambda e^{2\lambda t}\left(1-e^{-\lambda t}\right)$$

$$= \left(n_0+\frac{v}{\lambda}\right)\left(n_0+\frac{v}{\lambda}+1\right)\lambda\left(e^{2\lambda t}-e^{\lambda t}\right)$$

Integrating both sides with respect to t, we have

$$e^{((n_0+2)\lambda+v)t}p_{n_0+2}(t) = \left(n_0+\frac{v}{\lambda}\right)\left(n_0+\frac{v}{\lambda}+1\right)\lambda\int\left(e^{2\lambda t}-e^{\lambda t}\right)dt$$

$$= \left(n_0+\frac{v}{\lambda}\right)\left(n_0+\frac{v}{\lambda}+1\right)\lambda\left(\frac{e^{2\lambda t}}{2\lambda}-\frac{e^{\lambda t}}{\lambda}\right)$$

$$e^{((n_0+2)\lambda+v)t}p_{n_0+2}(t) = \left(n_0+\frac{v}{\lambda}\right)\left(n_0+\frac{v}{\lambda}+1\right)\left(\frac{e^{2\lambda t}-2e^{\lambda t}}{2}\right)+c$$

Initial Condition t = 0, $p_{n_0+2}(t) = 0$

$$0 = \left(n_0+\frac{v}{\lambda}\right)\left(n_0+\frac{v}{\lambda}+1\right)\left(\frac{1-2}{2}\right)+c$$

$$c = \left(n_0+\frac{v}{\lambda}\right)\left(n_0+\frac{v}{\lambda}+1\right)\frac{1}{2}$$

$$e^{((n_0+2)\lambda+v)t}p_{n_0+2}(t) = \left(n_0+\frac{v}{\lambda}\right)\left(n_0+\frac{v}{\lambda}+1\right)\lambda\left(\frac{e^{2\lambda t}-2e^{\lambda t}}{2}\right)+\left(n_0+\frac{v}{\lambda}\right)\left(n_0+\frac{v}{\lambda}+1\right).\frac{1}{2}$$

$$= \frac{\left(n_0+\frac{v}{\lambda}\right)\left(n_0+\frac{v}{\lambda}+1\right)}{2}\left\{\left(e^{2\lambda t}-2e^{\lambda t}\right)+1\right\}$$

$$= \frac{\left(n_0+\frac{v}{\lambda}\right)\left(n_0+\frac{v}{\lambda}+1\right)}{2}\left\{e^{2\lambda t}-2e^{\lambda t}+1\right\}$$

Equivalently,

$$p_{n_0+2}(t) = \frac{\left(n_0+\frac{v}{\lambda}\right)\left(n_0+\frac{v}{\lambda}+1\right)}{2}e^{-((n_0+2)\lambda+v)t}\left\{e^{2\lambda t}-2e^{\lambda t}+1\right\}$$

$$= \frac{\left(n_0+\frac{v}{\lambda}\right)\left(n_0+\frac{v}{\lambda}+1\right)}{2}e^{-\left(n_0+\frac{v}{\lambda}\right)\lambda t}\left\{1-2e^{-\lambda t}+e^{-2\lambda t}\right\}$$

$$= \frac{\left(n_0+\frac{v}{\lambda}\right)\left(n_0+\frac{v}{\lambda}+1\right)}{2}e^{-\left(n_0+\frac{v}{\lambda}\right)\lambda t}\left(1-e^{-\lambda t}\right)^2$$

$$= \frac{\left(n_0+\frac{v}{\lambda}+1\right)!}{\left(n_0+\frac{v}{\lambda}-1\right)!\,2!}e^{-\lambda t\left(n_0+\frac{v}{\lambda}\right)}\left(1-e^{-\lambda t}\right)^2$$

$$= \binom{n_0+\frac{v}{\lambda}+1}{2}e^{-\lambda t\left(n_0+\frac{v}{\lambda}\right)}\left(1-e^{-\lambda t}\right)^2$$

Thus

$$p_{n_0+2}(t) = \binom{n_0+\frac{v}{\lambda}+1}{2}e^{-\lambda t\left(n_0+\frac{v}{\lambda}\right)}\left(1-e^{-\lambda t}\right)^2 \qquad \textbf{(5.3)}$$

When $n = n_0 + 3$, equation **(5.2b)** becomes

$$p'_{n_0+3}(t) = -\left((n_0+3)\lambda + v\right) p_{n_0+3}(t) + \left((n_0+2)\lambda + v\right) p_{n_0+2}(t)$$

Re arranging, we have

$$p'_{n_0+3}(t) + \left((n_0+3)\lambda + v\right) p_{n_0+3}(t) = \left((n_0+2)\lambda + v\right) \frac{\left(n_0 + \frac{v}{\lambda} + 1\right)!}{\left(n_0 + \frac{v}{\lambda} - 1\right)!\, 2!} e^{-\lambda t\left(n_0 + \frac{v}{\lambda}\right)} \left(1 - e^{-\lambda t}\right)^2$$

$$= \left((n_0+2)\lambda + v\right) \frac{\left(n_0 + \frac{v}{\lambda} + 1\right)!}{\left(n_0 + \frac{v}{\lambda} - 1\right)!\, 2!} e^{-\lambda t\left(n_0 + \frac{v}{\lambda}\right)} \left(1 - e^{-\lambda t}\right)^2$$

Therefore,

$$p'_{n_0+3}(t) + \left((n_0+3)\lambda + v\right) p_{n_0+3}(t) = \frac{\left(n_0 + \frac{v}{\lambda} + 2\right)\left(n_0 + \frac{v}{\lambda} + 1\right)!}{\left(n_0 + \frac{v}{\lambda} - 1\right)!\, 2!} \lambda e^{-\lambda t\left(n_0 + \frac{v}{\lambda}\right)} \left(1 - e^{-\lambda t}\right)^2$$

Next, we integrate the equation above using the integrating factor method

Integrating factor $= e^{\int\left((n_0+3)\lambda + v\right)dt} = e^{\left((n_0+3)\lambda + v\right)t}$

Multiplying equation the equation above by the integrating factor, we have

$$e^{\left((n_0+3)\lambda+v\right)t} p'_{n_0+3}(t) + e^{\left((n_0+3)\lambda+v\right)t} \left((n_0+3)\lambda+v\right) p_{n_0+3}(t) = e^{\left((n_0+3)\lambda+v\right)t} \frac{\left(n_0 + \frac{v}{\lambda} + 2\right)!}{\left(n_0 + \frac{v}{\lambda} - 1\right)!\, 2!} \lambda e^{-\lambda t\left(n_0 + \frac{v}{\lambda}\right)} \left(1 - e^{-\lambda t}\right)^2$$

Equivalently,

$$\frac{d}{dt}\left[e^{\left((n_0+3)\lambda+v\right)t} p_{n_0+3}(t)\right] = \frac{\left(n_0 + \frac{v}{\lambda} + 2\right)!}{\left(n_0 + \frac{v}{\lambda} - 1\right)!\, 2!} \lambda e^{\left((n_0+3)\lambda+v\right)t} e^{-\lambda t\left(n_0 + \frac{v}{\lambda}\right)} \left(1 - e^{-\lambda t}\right)^2$$

$$= \frac{\left(n_0 + \frac{v}{\lambda} + 2\right)!}{\left(n_0 + \frac{v}{\lambda} - 1\right)!\, 2!} \lambda e^{3\lambda t} \left(1 - e^{-\lambda t}\right)^2$$

$$= \frac{\left(n_0 + \frac{v}{\lambda} + 2\right)!}{\left(n_0 + \frac{v}{\lambda} - 1\right)!\, 2!} \lambda e^{3\lambda t} \left(1 - 2e^{-\lambda t} + e^{-2\lambda t}\right)$$

$$= \frac{\left(n_0 + \frac{v}{\lambda} + 2\right)!\lambda}{\left(n_0 + \frac{v}{\lambda} - 1\right)!\, 2!} \left(e^{3\lambda t} - 2e^{2\lambda t} + e^{\lambda t}\right)$$

Integrating both sides with respect to t, we have

$$e^{\left((n_0+3)\lambda+v\right)t} p_{n_0+3}(t) = \frac{\left(n_0 + \frac{v}{\lambda} + 2\right)!\lambda}{\left(n_0 + \frac{v}{\lambda} - 1\right)!\, 2!} \left(\frac{e^{3\lambda t}}{3\lambda} - \frac{2e^{2\lambda t}}{2\lambda} + \frac{e^{\lambda t}}{\lambda}\right) + c$$

$$= \frac{\left(n_0 + \frac{v}{\lambda} + 2\right)!\lambda}{\left(n_0 + \frac{v}{\lambda} - 1\right)!\, 2!} \left(\frac{e^{3\lambda t}}{3} - e^{2\lambda t} + e^{\lambda t}\right) + c$$

Initial Condition t = 0, $X(0) = n_0, P_{n_0+2}(0) = 0$

$$0 = \frac{\left(n_0+\frac{v}{\lambda}+2\right)!\lambda}{\left(n_0+\frac{v}{\lambda}-1\right)!\,2!}\left(\frac{1}{3}-1+1\right)+c$$

$$c = -\frac{\left(n_0+\frac{v}{\lambda}+2\right)!\lambda}{\left(n_0+\frac{v}{\lambda}-1\right)!\,3!}$$

$$e^{((n_0+3)\lambda+v)t}p_{n_0+3}(t) = \frac{\left(n_0+\frac{v}{\lambda}+2\right)!\lambda}{\left(n_0+\frac{v}{\lambda}-1\right)!\,2!}\left(\frac{e^{3\lambda t}}{3}-e^{2\lambda t}+e^{\lambda t}\right)-\frac{\left(n_0+\frac{v}{\lambda}+2\right)!\lambda}{\left(n_0+\frac{v}{\lambda}-1\right)!\,3!}$$

$$= \frac{\left(n_0+\frac{v}{\lambda}+2\right)!\lambda}{\left(n_0+\frac{v}{\lambda}-1\right)!\,3!}\left(e^{3\lambda t}-3e^{2\lambda t}+3e^{\lambda t}\right)-\frac{\left(n_0+\frac{v}{\lambda}+2\right)!\lambda}{\left(n_0+\frac{v}{\lambda}-1\right)!\,3!}$$

$$= \frac{\left(n_0+\frac{v}{\lambda}+2\right)!\lambda}{\left(n_0+\frac{v}{\lambda}-1\right)!\,3!}\left\{e^{3\lambda t}-3e^{2\lambda t}+3e^{\lambda t}-1\right\}$$

Equivalently,

$$p_{n_0+3}(t) = \frac{\left(n_0+\frac{v}{\lambda}+2\right)!\lambda}{\left(n_0+\frac{v}{\lambda}-1\right)!\,3!}e^{-((n_0+3)\lambda+v)t}\left\{e^{3\lambda t}-3e^{2\lambda t}+3e^{\lambda t}-1\right\}$$

$$= \binom{n_0+\frac{v}{\lambda}+2}{3}e^{-\lambda t\left(n_0+\frac{v}{\lambda}\right)}\left\{1-3e^{-\lambda t}+3e^{-2\lambda t}-e^{-3\lambda t}\right\}$$

$$= \binom{n_0+\frac{v}{\lambda}+2}{3}e^{-\lambda t\left(n_0+\frac{v}{\lambda}\right)}\left(1-e^{-\lambda t}\right)^3$$

Thus

$$p_{n_0+3}(t) = \binom{n_0+\frac{v}{\lambda}+2}{3}e^{-\lambda t\left(n_0+\frac{v}{\lambda}\right)}\left(1-e^{-\lambda t}\right)^3 \qquad \textbf{(5.4)}$$

By induction, assume that when $n = n_0 + j - 1$

$$p_{n_0+j-1}(t)=\binom{n_0+\frac{v}{\lambda}+j-2}{j-1}e^{-\lambda t\left(n_0+\frac{v}{\lambda}\right)}\left(1-e^{-\lambda t}\right)^{j-1}$$

When $n = n_0 + j$, equation **(5.1b)** becomes

$$p'_{n_0+j}(t) = -\left((n_0+j)\lambda+v\right)p_{n_0+j}(t) + \left((n_0+j-1)\lambda+v\right)p_{n_0+j-1}(t)$$

$$p'_{n_0+j}(t) + \left((n_0+j)\lambda+v\right)p_{n_0+j}(t) = \left((n_0+j-1)\lambda+v\right)\binom{n_0+\frac{v}{\lambda}+j-2}{j-1}e^{-\lambda t\left(n_0+\frac{v}{\lambda}\right)}\left(1-e^{-\lambda t}\right)^{j-1}$$

$$= \left(n_0+\frac{v}{\lambda}+j-1\right)\lambda\frac{\left(n_0+\frac{v}{\lambda}+j-2\right)!}{(j-1)!\left(n_0+\frac{v}{\lambda}-1\right)!}e^{-\lambda t\left(n_0+\frac{v}{\lambda}\right)}\left(1-e^{-\lambda t}\right)^{j-1}$$

Re arranging the above equation, we have

$$p'_{n_0+j}(t) + \left(\left(n_0 + j\right)\lambda + v\right) p_{n_0+j}(t) = \frac{\left(n_0 + \frac{v}{\lambda} + j\text{-}1\right)!\, \lambda}{\left(j\text{-}1\right)!\left(n_0 + \frac{v}{\lambda}\text{-}1\right)!} e^{-\lambda t\left(n_0 + \frac{v}{\lambda}\right)} \left(1\text{-}e^{-\lambda t}\right)^{j-1}$$

Next we integrate the above equation using integrating factor method

Integrating factor $= e^{\int\left[(n_0+j)\lambda + v\right]dt} = e^{\left[(n_0+j)\lambda + v\right]t}$

Multiplying the equation above with the integrating factor, we get

$$e^{\left[(n_0+j)\lambda+v\right]t} p'_{n_0+j}(t) + e^{\left[(n_0+j)\lambda+v\right]t} \left(\left(n_0 + j\right)\lambda + v\right) p_{n_0+j}(t) = e^{\left[(n_0+j)\lambda+v\right]t} \frac{\left(n_0 + \frac{v}{\lambda} + j\text{-}1\right)!\, \lambda}{\left(j\text{-}1\right)!\left(n_0 + \frac{v}{\lambda}\text{-}1\right)!} e^{-\lambda t\left(n_0 + \frac{v}{\lambda}\right)} \left(1\text{-}e^{-\lambda t}\right)^{j-1}$$

Equivalently,

$$\frac{d}{dt}\left[e^{\left[(n_0+j)\lambda+v\right]t} p_{n_0+j}(t)\right] = e^{\left[(n_0+j)\lambda+v\right]t} \frac{\left(n_0 + \frac{v}{\lambda} + j\text{-}1\right)!\, \lambda}{\left(j\text{-}1\right)!\left(n_0 + \frac{v}{\lambda}\text{-}1\right)!} e^{-\lambda t\left(n_0 + \frac{v}{\lambda}\right)} \left(1\text{-}e^{-\lambda t}\right)^{j-1}$$

$$= \frac{\left(n_0 + \frac{v}{\lambda} + j\text{-}1\right)!}{\left(n_0 + \frac{v}{\lambda}\text{-}1\right)!\left(j\text{-}1\right)!} e^{j\lambda t} \left(1\text{-}e^{-\lambda t}\right)^{j-1} \lambda$$

$$= \frac{\left(n_0 + \frac{v}{\lambda} + j\text{-}1\right)!}{\left(n_0 + \frac{v}{\lambda}\text{-}1\right)!\left(j\text{-}1\right)!} e^{j\lambda t} \left(\frac{e^{\lambda t} - 1}{e^{\lambda t}}\right)^{j-1} \lambda$$

$$= \frac{\left(n_0 + \frac{v}{\lambda} + j\text{-}1\right)!}{\left(n_0 + \frac{v}{\lambda}\text{-}1\right)!\left(j\text{-}1\right)!} e^{\lambda t} \left(e^{\lambda t} - 1\right)^{j-1} \lambda$$

Integrating both sides with respect to t, we have

$$e^{\left[(n_0+j)\lambda+v\right]t} p_{n_0+j}(t) = \frac{\left(n_0 + \frac{v}{\lambda} + j\text{-}1\right)!}{\left(n_0 + \frac{v}{\lambda}\text{-}1\right)!\left(j\text{-}1\right)!} \lambda \int e^{\lambda t} \left(e^{\lambda t} - 1\right)^{j-1} dt$$

We wish to integrate $\int e^{\lambda t}\left(e^{\lambda t} - 1\right)^{j-1} dt$

Let $u = e^{\lambda t} - 1 \;\Rightarrow\; \frac{du}{dt} = \lambda e^{\lambda t}$ or $dt = \frac{e^{-\lambda t}\, du}{\lambda}$

Therefore,

$$e^{\left[(n_0+j)\lambda+v\right]t} p_{n_0+j}(t) = \frac{\left(n_0 + \frac{v}{\lambda} + j\text{-}1\right)!}{\left(n_0 + \frac{v}{\lambda}\text{-}1\right)!\left(j\text{-}1\right)!} \lambda \int e^{\lambda t} u^{j-1} e^{-\lambda t} \frac{du}{\lambda}$$

$$= \frac{\left(n_0 + \frac{v}{\lambda} + j\text{-}1\right)!}{\left(n_0 + \frac{v}{\lambda}\text{-}1\right)!\left(j\text{-}1\right)!} \int u^{j-1} du$$

$$= \frac{\left(n_0 + \frac{v}{\lambda} + j\text{-}1\right)!}{\left(n_0 + \frac{v}{\lambda}\text{-}1\right)!\left(j\text{-}1\right)!} \frac{u^j}{j} + c$$

Equivalently,

$$e^{[(n_0+j)\lambda+v]t}\, p_{n_0+j}(t) = \frac{(n_0+\frac{v}{\lambda}+j-1)!}{(n_0+\frac{v}{\lambda}-1)!\ j!}\left(e^{\lambda t}-1\right)^j + c$$

$$= \binom{n_0+\frac{v}{\lambda}+j-1}{j}\left(e^{\lambda t}-1\right)^j + c$$

$$= \binom{n_0+\frac{v}{\lambda}+j-1}{j}\left(e^{\lambda t}-1\right)^j + c$$

Using initial conditions, $X(0) = n_0 \quad \Rightarrow \quad p_{n_0}(0) = 1 \text{ and } p_n(t) = 0\ \forall\, n \neq n_0$. Substituting t = 0 in the equation above, we have

$$0 = \binom{n_0+\frac{v}{\lambda}+j-1}{j}(1-1)^j + c$$

$$c = 0.$$

Therefore,

$$e^{[(n_0+j)\lambda+v]t}\, p_{n_0+j}(t) = \binom{n_0+\frac{v}{\lambda}+j-1}{j}\left(e^{\lambda t}-1\right)^j$$

Equivalently,

$$p_{n_0+j}(t) = \binom{n_0+\frac{v}{\lambda}+j-1}{j} e^{-[(n_0+j)\lambda+v]t}\left(e^{\lambda t}-1\right)^j$$

$$= \binom{n_0+\frac{v}{\lambda}+j-1}{j} e^{-[(n_0+j)\lambda+v]t}\left[e^{\lambda t}\left(1-e^{-\lambda t}\right)\right]^j$$

$$= \binom{n_0+\frac{v}{\lambda}+j-1}{j} e^{-n_0\lambda - \lambda jt - vt}\left[e^{\lambda t}\left(1-e^{-\lambda t}\right)\right]^j$$

$$= \binom{n_0+\frac{v}{\lambda}+j-1}{j} e^{-n_0\lambda - \lambda jt - vt + \lambda jt}\left(1-e^{-\lambda t}\right)^j$$

$$= \binom{n_0+\frac{v}{\lambda}+j-1}{j} e^{-n_0\lambda - vt}\left(1-e^{-\lambda t}\right)^j$$

Therefore

$$\mathbf{p_{n_0+j}(t) = \binom{n_0+\frac{v}{\lambda}+j-1}{j} e^{-\lambda t\left(n_0+\frac{v}{\lambda}\right)}\left(1-e^{-\lambda t}\right)^j} \qquad \textbf{(5.5)}$$

This is a negative binomial distribution.

### 5.2.2 Mean and Variance by Definition

Now, let $p = e^{-\lambda t}$, $q = \left(1 - e^{-\lambda t}\right)$ and j = k. Equation **(5.5)** now becomes

$$\text{Prob}\left[X(t) = n_0 + k\right] = \binom{n_0 + \frac{v}{\lambda} + k - 1}{k} p^{\left(n_0 + \frac{v}{\lambda}\right)} q^k$$

**Mean**

$$E\left[X(t) = n_0 + k\right] = \sum_{k=0}^{\infty}(n_0 + k)\binom{n_0 + \frac{v}{\lambda} + k - 1}{k} p^{\left(n_0 + \frac{v}{\lambda}\right)} q^k$$

$$E\left[X(t) = n_0 + k\right] = n_0 \sum_{k=0}^{\infty}\binom{n_0 + \frac{v}{\lambda} + k - 1}{k} p^{\left(n_0 + \frac{v}{\lambda}\right)} q^k + \sum_{k=0} k\binom{n_0 + \frac{v}{\lambda} + k - 1}{k} p^{\left(n_0 + \frac{v}{\lambda}\right)} q^k$$

$$= n_0 \times 1 + \sum_{k=0}^{\infty} k \times \frac{\left(n_0 + \frac{v}{\lambda} + k - 1\right)!}{k!\left(n_0 + \frac{v}{\lambda} - 1\right)!} p^{\left(n_0 + \frac{v}{\lambda}\right)} q^k$$

$$= n_0 + \sum_{k=0}^{\infty} \frac{\left(n_0 + \frac{v}{\lambda} + k - 1\right)!}{(k-1)!\left(n_0 + \frac{v}{\lambda} - 1\right)!} p^{\left(n_0 + \frac{v}{\lambda}\right)} q^k$$

$$= n_0 + \frac{1}{\left(n_0 + \frac{v}{\lambda} - 1\right)!} \sum_{k=0}^{\infty} \frac{\left(n_0 + \frac{v}{\lambda} + k - 1\right)!}{(k-1)!} \times \frac{\left(n_0 + \frac{v}{\lambda}\right)!}{\left(n_0 + \frac{v}{\lambda}\right)!} \times \frac{p^{\left(n_0 + \frac{v}{\lambda}\right)+1}}{p} q^{k-1} . q$$

$$= n_0 + \frac{\left(n_0 + \frac{v}{\lambda}\right)!}{\left(n_0 + \frac{v}{\lambda} - 1\right)!} \times \frac{q}{p} \sum_{k=0}^{\infty} \frac{\left(n_0 + \frac{v}{\lambda} + k - 1\right)!}{(k-1)!\left(n_0 + \frac{v}{\lambda}\right)!} \times p^{\left(n_0 + \frac{v}{\lambda}\right)+1} \times q^{k-1}$$

$$= n_0 + \left(n_0 + \frac{v}{\lambda}\right) \times \frac{q}{p} \sum_{k=0}^{\infty} \binom{n_0 + \frac{v}{\lambda} + k - 1}{k - 1} \times p^{\left(n_0 + \frac{v}{\lambda}\right)+1} \times q^{k-1}$$

$$= n_0 + \left(n_0 + \frac{v}{\lambda}\right) \times \frac{q}{p}$$

$$= \frac{n_0 p + \left(n_0 + \frac{v}{\lambda}\right) q}{p}$$

$$= \frac{n_0 p + \left(n_0 + \frac{v}{\lambda}\right)(1 - p)}{p}$$

$$= \frac{n_0 p + n_0 - n_0 p + \frac{v}{\lambda} - \frac{v}{\lambda} p}{p}$$

$$= \frac{n_0 + \frac{v}{\lambda} - \frac{v}{\lambda} p}{p}$$

$$= \frac{n_0 + \frac{v}{\lambda}(1 - p)}{p}$$

Re substituting back the values of p and q, we have

$$E\left[X(t) = n_0 + k\right] = \frac{n_0 + \frac{v}{\lambda}\left(1 - e^{-\lambda t}\right)}{e^{-\lambda t}} \qquad \textbf{(5.6)}$$

**Variance**

$$E\left[\left(X(t) = n_0 + k\right)^2\right] = \sum_{k=0}^{\infty}\left(n_0 + k\right)^2 \binom{n_0 + \frac{v}{\lambda} + k - 1}{k} p^{\left(n_0 + \frac{v}{\lambda}\right)} q^k$$

$$= \sum_{k=0}^{\infty}\left(n_0^2 + 2n_0 k + k^2\right) \binom{n_0 + \frac{v}{\lambda} + k - 1}{k} p^{\left(n_0 + \frac{v}{\lambda}\right)} q^k$$

$$E\left[\left(X(t) = n_0 + k\right)^2\right] = n_0^2 \sum_{k=0} \binom{n_0 + \frac{v}{\lambda} + k - 1}{k} p^{\left(n_0 + \frac{v}{\lambda}\right)} q^k + 2n_0 \sum_{k=0} k \binom{n_0 + \frac{v}{\lambda} + k - 1}{k} p^{\left(n_0 + \frac{v}{\lambda}\right)} q^k + \sum_{k=0} k^2 \binom{n_0 + \frac{v}{\lambda} + k - 1}{k} p^{\left(n_0 + \frac{v}{\lambda}\right)} q^k$$

Working out each term separately, we have

1st term

$$n_0^2 \sum_{k=0}^{\infty} \binom{n_0 + \frac{v}{\lambda} + k - 1}{k} p^{\left(n_0 + \frac{v}{\lambda}\right)} q^k = n_0^2 \times 1 = n_0^2$$

2nd term

$$2n_0 \sum_{k=0}^{\infty} k \binom{n_0 + \frac{v}{\lambda} + k - 1}{k} p^{\left(n_0 + \frac{v}{\lambda}\right)} q^k = 2n_0 \sum_{k=0}^{\infty} k \times \frac{\left(n_0 + \frac{v}{\lambda} + k - 1\right)!}{k!\left(n_0 + \frac{v}{\lambda} - 1\right)!} p^{\left(n_0 + \frac{v}{\lambda}\right)} q^k$$

$$= \frac{2n_0}{\left(n_0 + \frac{v}{\lambda} - 1\right)!} \sum_{k=0}^{\infty} \frac{\left(n_0 + \frac{v}{\lambda} + k - 1\right)!}{(k-1)!} p^{\left(n_0 + \frac{v}{\lambda}\right)} q^k$$

$$= \frac{2n_0}{\left(n_0 + \frac{v}{\lambda} - 1\right)!} \sum_{k=0}^{\infty} \frac{\left(n_0 + \frac{v}{\lambda} + k - 1\right)!}{(k-1)!} \times \frac{\left(n_0 + \frac{v}{\lambda}\right)!}{\left(n_0 + \frac{v}{\lambda}\right)!} \times p^{\left(n_0 + \frac{v}{\lambda}\right)} q^k$$

$$= \frac{2n_0\left(n_0 + \frac{v}{\lambda}\right)!}{\left(n_0 + \frac{v}{\lambda} - 1\right)!} \sum_{k=0}^{\infty} \frac{\left(n_0 + \frac{v}{\lambda} + k - 1\right)!}{(k-1)!\left(n_0 + \frac{v}{\lambda}\right)!} \times \frac{p^{n_0 + \frac{v}{\lambda} + 1}}{p} \times q^{k-1}.\, q$$

$$= 2n_0\left(n_0 + \frac{v}{\lambda}\right) \times \frac{q}{p} \sum_{k=0}^{\infty} \binom{n_0 + \frac{v}{\lambda} + k - 1}{k - 1} \times p^{n_0 + \frac{v}{\lambda} + 1} q^{k-1}$$

$$= 2n_0\left(n_0 + \frac{v}{\lambda}\right) \times \frac{q}{p}$$

3rd term

$$\sum_{k=0}^{\infty} k^2 \binom{n_0+\frac{v}{\lambda}+k-1}{k} p^{\left(n_0+\frac{v}{\lambda}\right)} q^k = \sum_{k=0}^{\infty}\left[k(k-1)+k\right]\binom{n_0+\frac{v}{\lambda}+k-1}{k} p^{\left(n_0+\frac{v}{\lambda}\right)} q^k$$

$$= \sum_{k=0}^{\infty} k(k-1)\binom{n_0+\frac{v}{\lambda}+k-1}{k} p^{\left(n_0+\frac{v}{\lambda}\right)} q^k + \sum_{k=0}^{\infty} k\binom{n_0+\frac{v}{\lambda}+k-1}{k} p^{\left(n_0+\frac{v}{\lambda}\right)} q^k$$

$$\sum_{k=0}^{\infty} k^2 \binom{n_0+\frac{v}{\lambda}+k-1}{k} p^{\left(n_0+\frac{v}{\lambda}\right)} q^k = \sum_{k=0}^{\infty} k(k-1)\times\frac{\left(n_0+\frac{v}{\lambda}+k-1\right)!}{k!\left(n_0+\frac{v}{\lambda}-1\right)!} p^{\left(n_0+\frac{v}{\lambda}\right)} q^k + \left(n_0+\frac{v}{\lambda}\right)\times\frac{q}{p}$$

$$= \frac{1}{\left(n_0+\frac{v}{\lambda}-1\right)!}\sum_{k=0}^{\infty}\frac{\left(n_0+\frac{v}{\lambda}+k-1\right)!}{(k-2)!} p^{\left(n_0+\frac{v}{\lambda}\right)} q^k + \left(n_0+\frac{v}{\lambda}\right)\times\frac{q}{p}$$

$$= \frac{1}{\left(n_0+\frac{v}{\lambda}-1\right)!}\sum_{k=0}^{\infty}\frac{\left(n_0+\frac{v}{\lambda}+k-1\right)!}{(k-2)!}\times\frac{\left(n_0+\frac{v}{\lambda}+1\right)!}{\left(n_0+\frac{v}{\lambda}+1\right)!}\times p^{\left(n_0+\frac{v}{\lambda}\right)} q^k + \left(n_0+\frac{v}{\lambda}\right)\times\frac{q}{p}$$

$$= \frac{\left(n_0+\frac{v}{\lambda}+1\right)!}{\left(n_0+\frac{v}{\lambda}-1\right)!}\sum_{k=0}^{\infty}\frac{\left(n_0+\frac{v}{\lambda}+k-1\right)!}{(k-2)!\left(n_0+\frac{v}{\lambda}+1\right)!}\times p^{\left(n_0+\frac{v}{\lambda}\right)} q^k + \left(n_0+\frac{v}{\lambda}\right)\times\frac{q}{p}$$

$$= \left(n_0+\frac{v}{\lambda}+1\right)\left(n_0+\frac{v}{\lambda}\right)\sum_{k=0}^{\infty}\frac{\left(n_0+\frac{v}{\lambda}+k-1\right)!}{(k-2)!\left(n_0+\frac{v}{\lambda}+1\right)!}\times\frac{p^{n_0+\frac{v}{\lambda}+2}}{p^2} q^{k-2} q^2 + \left(n_0+\frac{v}{\lambda}\right)\frac{q}{p}$$

$$= \left(n_0+\frac{v}{\lambda}+1\right)\left(n_0+\frac{v}{\lambda}\right)\times\left(\frac{q}{p}\right)^2 \sum_{k=0}^{\infty}\binom{n_0+\frac{v}{\lambda}+k-1}{k-2}\times p^{n_0+\frac{v}{\lambda}+2} q^{k-2} + \left(n_0+\frac{v}{\lambda}\right)\frac{q}{p}$$

$$= \left(n_0+\frac{v}{\lambda}+1\right)\left(n_0+\frac{v}{\lambda}\right)\times\left(\frac{q}{p}\right)^2 \times 1 + \left(n_0+\frac{v}{\lambda}\right)\times\frac{q}{p}$$

$$= \left(n_0+\frac{v}{\lambda}+1\right)\left(n_0+\frac{v}{\lambda}\right)\times\left(\frac{q}{p}\right)^2 + \left(n_0+\frac{v}{\lambda}\right)\times\frac{q}{p}$$

Therefore

$$E\left[\left(X(t)=n_0+k\right)^2\right] = n_0^2 + 2n_0\left(n_0+\frac{v}{\lambda}\right)\times\frac{q}{p} + \left(n_0+\frac{v}{\lambda}+1\right)\left(n_0+\frac{v}{\lambda}\right)\times\left(\frac{q}{p}\right)^2 + \left(n_0+\frac{v}{\lambda}\right)\times\frac{q}{p}$$

Therefore,

$$\text{Var}\left(X(t)=n_0+k\right)=$$

$$= n_0^2 + 2n_0\left(n_0+\tfrac{v}{\lambda}\right)\times\frac{q}{p} + \left(n_0+\tfrac{v}{\lambda}+1\right)\left(n_0+\tfrac{v}{\lambda}\right)\times\left(\frac{q}{p}\right)^2 + \left(n_0+\tfrac{v}{\lambda}\right)\times\frac{q}{p} - \left[n_0+\left(n_0+\tfrac{v}{\lambda}\right)\frac{q}{p}\right]^2$$

$$= n_0^2 + 2n_0\left(n_0+\tfrac{v}{\lambda}\right)\frac{q}{p} + \left(n_0+\tfrac{v}{\lambda}+1\right)\left(n_0+\tfrac{v}{\lambda}\right)\left(\frac{q}{p}\right)^2 + \left(n_0+\tfrac{v}{\lambda}\right)\frac{q}{p} - \left[n_0^2+2n_0\left(n_0+\tfrac{v}{\lambda}\right)\frac{q}{p}+\left(n_0+\tfrac{v}{\lambda}\right)^2\left(\frac{q}{p}\right)^2\right]$$

$$= n_0^2 + 2n_0\left(n_0+\tfrac{v}{\lambda}\right)\frac{q}{p} + \left(n_0+\tfrac{v}{\lambda}+1\right)\left(n_0+\tfrac{v}{\lambda}\right)\left(\frac{q}{p}\right)^2 + \left(n_0+\tfrac{v}{\lambda}\right)\frac{q}{p} - n_0^2 - 2n_0\left(n_0+\tfrac{v}{\lambda}\right)\frac{q}{p} - \left(n_0+\tfrac{v}{\lambda}\right)^2\left(\frac{q}{p}\right)^2$$

$$= \left(n_0+\tfrac{v}{\lambda}+1\right)\left(n_0+\tfrac{v}{\lambda}\right)\left(\frac{q}{p}\right)^2 + \left(n_0+\tfrac{v}{\lambda}\right)\frac{q}{p} - \left(n_0+\tfrac{v}{\lambda}\right)^2\left(\frac{q}{p}\right)^2$$

Therefore,

$$\text{Var}\left(X(t)=n_0+k\right) = \left(n_0+\tfrac{v}{\lambda}\right)\frac{q}{p} + \left(n_0+\tfrac{v}{\lambda}\right)\left(\frac{q}{p}\right)^2\left\{n_0+\tfrac{v}{\lambda}+1-n_0-\tfrac{v}{\lambda}\right\}$$

$$= \left(n_0+\tfrac{v}{\lambda}\right)\frac{q}{p} + \left(n_0+\tfrac{v}{\lambda}\right)\left(\frac{q}{p}\right)^2$$

$$= \left(n_0+\tfrac{v}{\lambda}\right)\frac{q}{p}\left\{1+\frac{q}{p}\right\}$$

$$= \left(n_0+\tfrac{v}{\lambda}\right)\frac{q}{p}\left\{\frac{p+q}{p}\right\}$$

Equivalently,

$$\text{Var}\left(X(t)=n_0+k\right) = \left(n_0+\tfrac{v}{\lambda}\right)\frac{q}{p^2}$$

Substituting back the values of p and q, we have

$$\text{Var}\left[X(t)=n_0+k\right] = \left(n_0+\tfrac{v}{\lambda}\right)\frac{\left(1-e^{-\lambda t}\right)}{e^{-2\lambda t}}$$

$$= \left(n_0+\tfrac{v}{\lambda}\right)e^{2\lambda t}\left(1-e^{-\lambda t}\right)$$

$$= \left(n_0+\tfrac{v}{\lambda}\right)\left(e^{2\lambda t}-e^{\lambda t}\right)$$

$$= \left(n_0+\tfrac{v}{\lambda}\right)e^{\lambda t}\left(e^{\lambda t}-1\right)$$

Thus

$$\text{Var}\left[X(t)=n_0+k\right] = \left(n_0+\tfrac{v}{\lambda}\right)e^{\lambda t}\left(e^{\lambda t}-1\right) \qquad \textbf{(5.7)}$$

### 5.3 Determining $p_n(t)$ by Laplace Method

***Initial conditions: When t = 0, $X(0) = n_0$***

Recall that equation **(5.1a)** and **(5.1b)** are

$$p_0'(t) = -v\,p_0(t)$$

$$p_n'(t) = -(n\lambda + v)\,p_n(t) + ((n-1)\lambda + v)\,p_{n-1}(t), \quad n \geq 1$$

Taking the Laplace transform of both sides of **(5.1b)**

$$L(p_n'(t)) = L(-(n\lambda + v)\,p_n(t)) + L(((n-1)\lambda + v)\,p_{n-1}(t))$$

$$= -(n\lambda + v)\,L(p_n(t)) + ((n-1)\lambda + v)L(p_{n-1}(t))$$

But, $L[p_n'(t)] = sL[p_n(t)] - p_n(0)$ (from Chapter 2, section 2.5 formulae **(2.9)**).

Substituting in the equation above, we have

$$sL[p_n(t)] - p_n(0) = -(n\lambda + v)\,L(p_n(t)) + ((n-1)\lambda + v)L(p_{n-1}(t))$$

Grouping the like terms together and factorizing, we have

$$(s + (n\lambda + v))L[p_n(t)] = p_n(0) + ((n-1)\lambda + v)L(p_{n-1}(t))$$

$$\Rightarrow$$

$$L[p_n(t)] = \frac{p_n(0)}{(s + (n\lambda + v))} + \frac{((n-1)\lambda + v)}{(s + (n\lambda + v))}L(p_{n-1}(t)) \qquad \textbf{(5.8)}$$

Initial condition: At time $t = 0$, the population is $n_0$. Thus, $p_{n_0}(0) = 1$, $p_n(0) = 0 \ \forall\ n \neq n_0$ and $p_n(t) = 0 \ \forall\ n < n_0$.

When $n = n_0$, equation **(5.8)** becomes

$$L[p_{n_0}(t)] = \frac{p_{n_0}(0)}{(s + (n_0\lambda + v))} + \frac{((n_0 - 1)\lambda + v)}{(s + (n_0\lambda + v))}L(p_{n_0-1}(t))$$

But $L(p_{n_0-1}(t)) = 0$. This above equation is thus equivalent to

$$L[p_{n_0}(t)] = \frac{1}{(s + (n_0\lambda + v))} \qquad \textbf{(5.9)}$$

When $n = n_0 + 1$, equation **(5.8)** becomes

$$L\left[p_{n_0+1}(t)\right] = \frac{p_{n_0+1}(0)}{\left(s + \left((n_0+1)\lambda + v\right)\right)} + \frac{\left((n_0 + 1 - 1)\lambda + v\right)}{\left(s + \left((n_0+1)\lambda + v\right)\right)} L\left( p_{n_0+1-1}(t)\right)$$

But $p_{n_0+1}(0) = 0$. Thus

$$L\left[p_{n_0+1}(t)\right] = \frac{(n_0\lambda + v)}{\left(s + \left((n_0+1)\lambda + v\right)\right)} L\left( p_{n_0}(t)\right)$$

But $L\left[p_{n_0}(t)\right] = \dfrac{1}{\left(s + (n_0\lambda + v)\right)}$. Therefore

$$L\left[p_{n_0+1}(t)\right] = \frac{(n_0\lambda + v)}{\left(s + \left((n_0+1)\lambda + v\right)\right)} \times \frac{1}{\left(s + (n_0\lambda + v)\right)} \qquad \textbf{(5.10)}$$

When $n = n_0 + 2$, equation **(5.8)** becomes

$$L\left[p_{n_0+2}(t)\right] = \frac{p_{n_0+2}(0)}{\left(s + \left((n_0+2)\lambda + v\right)\right)} + \frac{\left((n_0 + 2 - 1)\lambda + v\right)}{\left(s + \left((n_0+2)\lambda + v\right)\right)} L\left( p_{n_0+2-1}(t)\right)$$

Equivalently,

$$L\left[p_{n_0+2}(t)\right] = \frac{\left((n_0+1)\lambda + v\right)}{\left(s + \left((n_0+2)\lambda + v\right)\right)} L\left( p_{n_0+1}(t)\right) \text{ as } p_{n_0+2}(0) = 0.$$

But $L\left[p_{n_0+1}(t)\right] = \dfrac{(n_0\lambda + v)}{\left(s + \left((n_0+1)\lambda + v\right)\right)} \times \dfrac{1}{\left(s + (n_0\lambda + v)\right)}$

Therefore,

$$L\left[p_{n_0+2}(t)\right] = \frac{\left((n_0+1)\lambda + v\right)}{\left(s + \left((n_0+2)\lambda + v\right)\right)} \times \frac{(n_0\lambda + v)}{\left(s + \left((n_0+1)\lambda + v\right)\right)} \times \frac{1}{\left(s + (n_0\lambda + v)\right)} \qquad \textbf{(5.11)}$$

When $n = n_0 + 3$, equation **(5.8)** becomes

$$L\left[P_{n_0+3}(t)\right] = \frac{p_{n_0+3}(0)}{\left(s + \left((n_0+3)\lambda + v\right)\right)} + \frac{\left((n_0 + 3 - 1)\lambda + v\right)}{\left(s + \left((n_0+3)\lambda + v\right)\right)} L\left( p_{n_0+3-1}(t)\right)$$

$$= \frac{\left((n_0+2)\lambda + v\right)}{\left(s + \left((n_0+3)\lambda + v\right)\right)} L\left( p_{n_0+2}(t)\right)$$

Therefore

$$L\left[P_{n_0+3}(t)\right]=\frac{\left((n_0+2)\lambda+v\right)}{\left(s+\left((n_0+3)\lambda+v\right)\right)}\times\frac{\left((n_0+1)\lambda+v\right)}{\left(s+\left((n_0+2)\lambda+v\right)\right)}\times\frac{\left(n_0\lambda+v\right)}{\left(s+\left((n_0+1)\lambda+v\right)\right)}\times\frac{1}{\left(s+\left(n_0\lambda+v\right)\right)} \quad \textbf{(5.12)}$$

Generalizing by letting $n=n_0,\ n_0+1,\ n_0+2,\ n_0+3,\ ....$

Then $L\left(p_n(t)\right)=L\left(p_{n_0+k}(t)\right),\ \ k=0,1,2,3,....$

Thus,

$$L\left(p_n(t)\right)=\frac{\prod_{i=0}^{k-1}\left((n_0+i)\lambda+v\right)}{\prod_{i=0}^{k}\left(s+v+(n_0+i)\lambda\right)}=\frac{\prod_{i=0}^{k-1}\left(n_0\lambda+v+i\lambda\right)}{\prod_{i=0}^{k}\left(s+v+n_0\lambda+i\lambda\right)} \quad \textbf{(5.13)}$$

The numerator in equation **(5.13)** can further be simplified as shown below

$$\prod_{i=0}^{k-1}\left(n_0\lambda+v+i\lambda\right)=\prod_{i=0}^{k-1}\left(n_0\lambda+v+i\lambda\right)\times\frac{\lambda}{\lambda}=\prod_{i=0}^{k-1}\left(\left(n_0+\frac{v}{\lambda}\right)+i\right)\lambda=\lambda^{k-1}\prod_{i=0}^{k-1}\left(\left(n_0+\frac{v}{\lambda}\right)+i\right)$$

Let $m=n_0+\dfrac{v}{\lambda}$

Then,

$$\lambda^{k-1}\prod_{i=0}^{k-1}\left(\left(n_0+\frac{v}{\lambda}\right)+i\right)=\lambda^{k-1}\prod_{i=0}^{k-1}(m+i)=\lambda^{k-1}\left\{m(m+1)(m+2)(m+3)...(m+k-2)(m+k-1)\right\}$$

Thus

$$\lambda^{k-1}\prod_{i=0}^{k-1}\left(\left(n_0+\frac{v}{\lambda}\right)+i\right)=\lambda^{k-1}\frac{(m+k-1)!}{(m-1)!}$$

Equation **(5.13)** can now be written as

$$L\left(p_n(t)\right)=\frac{\lambda^{k-1}\dfrac{(m+k-1)!}{(m-1)!}}{\prod_{i=0}^{k}\left(s+v+n_0\lambda+i\lambda\right)}=\lambda^{k-1}\times\frac{(m+k-1)!}{(m-1)!}\times\frac{1}{\prod_{i=0}^{k}\left(s+v+(n_0+i)\lambda\right)}$$

Equivalently,

$$L\left(p_n(t)\right)=\lambda^{k-1}\times\frac{(m+k-1)!}{(m-1)!}\times\frac{1}{\prod_{i=0}^{k}\left(s+v+(n_0+i)\lambda\right)} \quad \textbf{(5.14)}$$

Taking the inverse of the Laplace transform, equation **(5.14)** becomes

$$p_n(t) = L^{-1}\left(L\left(p_n(t)\right)\right) = L^{-1}\left[\lambda^{k-1} \times \frac{(m+k-1)!}{(m-1)!} \times \frac{1}{\prod_{i=0}^{k}\left(s+v+(n_0+i)\lambda\right)}\right]$$

$$= \lambda^{k-1} \times \frac{(m+k-1)!}{(m-1)!} \sum_{i=0}^{k} \frac{e^{-(v+(n_0+i)\lambda)t}}{\prod_{j=0}^{k}\left(-(v+(n_0+i)\lambda)+v+(n_0+j)\lambda\right)} \quad i \neq j$$

$$= \lambda^{k-1} \times \frac{(m+k-1)!}{(m-1)!} \sum_{i=0}^{k} \frac{e^{-(v+n_0\lambda+i\lambda)t}}{\prod_{j=0}^{k}\left(-v-n_0\lambda-i\lambda+v+n_0\lambda+j\lambda\right)}$$

$$= \lambda^{k-1} \times \frac{(m+k-1)!}{(m-1)!} \sum_{i=0}^{k} \frac{e^{-(v+n_0\lambda+i\lambda)t}}{\prod_{j=0}^{k}(-i+j)\lambda} \quad i \neq j$$

$$= \lambda^{k-1} \times \frac{(m+k-1)!}{(m-1)!} \sum_{i=0}^{k} \frac{e^{-(v+n_0\lambda)t} \times e^{-i\lambda t}}{\lambda^{k-1}\prod_{j=0}^{k}(j-i)}$$

$$= \frac{\lambda^{k-1}}{\lambda^{k-1}} \times \frac{(m+k-1)!}{(m-1)!} \times e^{-(v+n_0\lambda)t} \sum_{i=0}^{k} \frac{e^{-i\lambda t}}{\prod_{j=0}^{k}(j-i)} \quad i \neq j$$

$$= \frac{(m+k-1)!}{(m-1)!} \times e^{-(v+n_0\lambda)t} \sum_{i=0}^{k} \frac{e^{-i\lambda t}}{\prod_{j=0}^{i-1}(j-i)\prod_{j=i+1}^{k}(j-i)}$$

Equivalently,

$$p_n(t) = \frac{(m+k-1)!}{(m-1)!} \times e^{-(v+n_0\lambda)t} \sum_{i=0}^{k} \frac{e^{-i\lambda t}}{\prod_{j=0}^{i-1}(j-i)\prod_{j=i+1}^{k}(j-i)} \qquad \textbf{(5.15)}$$

Simplifying the denominator inside the summation

$$\prod_{j=0}^{i-1}(j-i) = (-i)(1-i)(2-i)(3-i).....(-2)(-1) = (-1)^i\, i!$$

$$\prod_{j=i+1}^{k}(j-i) = 1\times 2\times 3\times ................\times(k-i-2)\times(k-i-1)\times(k-i) = (k-i)!$$

Now equation **(5.15)** becomes

$$p_n(t) = \frac{(m+k-1)!}{(m-1)!} \times e^{-(v+n_0\lambda)t} \sum_{i=0}^{k} \frac{e^{-i\lambda t}}{(-1)^i \, i! \times (k-i)!}$$

This can also be re written as

$$p_n(t) = \frac{(m+k-1)!}{(m-1)!} \times e^{-(v+n_0\lambda)t} \sum_{i=0}^{k} \frac{(-1)^i e^{-i\lambda t} \times k!}{i! \times (k-i)! \times k!}$$

$$= \frac{(m+k-1)!}{(m-1)! \times k!} \times e^{-(v+n_0\lambda)t} \sum_{i=0}^{k} \frac{(-e)^{-i\lambda t} \times k!}{i! \times (k-i)!}$$

$$= \frac{(m+k-1)!}{(m-1)! \times k!} \times e^{-(v+n_0\lambda)t} \sum_{i=0}^{k} \frac{k!}{i!(k-i)!}\left(-e^{-\lambda t}\right)^i$$

$$= \binom{m+k-1}{m-1} \times e^{-(v+n_0\lambda)t} \sum_{i=0}^{k} \binom{k}{i}\left(-e^{-\lambda t}\right)^i$$

$$= \binom{m+k-1}{m-1} \times e^{-(v+n_0\lambda)t} \left(1-e^{-\lambda t}\right)^k$$

In short

$$p_n(t) = \binom{m+k-1}{m-1} \times e^{-(v+n_0\lambda)t} \left(1-e^{-\lambda t}\right)^k \qquad \textbf{(5.15)}$$

But $m = n_0 + \frac{v}{\lambda} \;\Rightarrow\; m\lambda = n_0\lambda + v$

Equation **(5.15)** becomes

$$p_n(t) = \binom{m+k-1}{m-1} \times e^{-m\lambda t} \left(1-e^{-\lambda t}\right)^k$$

$$= \binom{k+m-1}{m-1} \times \left(e^{-\lambda t}\right)^m \left(1-e^{-\lambda t}\right)^k$$

Thus

$$\mathbf{p_n(t) = \binom{k+m-1}{k}\left(e^{-\lambda t}\right)^m \left(1-e^{-\lambda t}\right)^k} \quad k = 0, 1, 2, \ldots \text{ and } m = n_0 + \frac{v}{\lambda}. \qquad \textbf{(5.16)}$$

Equivalently, If we take $p = e^{-\lambda t}$ and $q = 1 - e^{-\lambda t}$, then

$$p_n(t) = \binom{k+m-1}{k} p^m q^k$$

This is a negative binomial distribution.

## 5.4 Probability Generating Function Method

### 5.4.1 Determining $p_n(t)$ by PGF Method

**The initial condition: When $t = 0$, $X(0) = 1$**

Recall that equation **(5.1a)** and **(5.1b)** are

$p_0'(t) = -v\,p_0(t)$ and

$p_n'(t) = -(n\lambda + v)\,p_n(t) + ((n-1)\lambda + v)\,p_{n-1}(t)$, $n \geq 1$ respectively.

From the initial conditions, when $t = 0$, $X(0) = 1$, then $p_1(0) = 1$.

Further $p_n(0) = 0 \; \forall \; n \neq 1$ and $p_0(t) = 0$.

**Definitions**

Let

$$\left.\begin{aligned} G(s,t) &= \sum_{n=0}^{\infty} p_n(t)s^n = p_0(t) + \sum_{n=1}^{\infty} p_n(t)s^n \\ \frac{d}{dt}\left[G(s,t)\right] &= \sum_{n=0}^{\infty} p_n'(t)s^n = p_0'(t) + \sum_{n=1}^{\infty} p_n'(t)s^n \\ \frac{d}{ds}\left[G(s,t)\right] &= \sum_{n=0}^{\infty} n\,p_n(t)s^{n-1} = \frac{1}{s}\sum_{n=1}^{\infty} n\,p_n(t)s^n \end{aligned}\right\} \tag{5.17}$$

Further $G(s,0) = s$.

Multiplying both sides of **(4.2)** by $s^n$ and summing over n.

$$\sum_{n=1}^{\infty} p_n(t)s^n = -\sum_{n=1}^{\infty}(n\lambda + v)\,p_n(t)s^n + \sum_{n=1}^{\infty}((n-1)\lambda + v)\,p_{n-1}(t)s^n$$

$$= -\lambda\sum_{n=1}^{\infty} n\,p_n(t)s^n - v\sum_{n=1}^{\infty} p_n(t)s^n + \lambda\sum_{n=1}^{\infty}(n-1)p_{n-1}(t)s^n + v\sum_{n=1}^{\infty} p_{n-1}(t)s^n$$

Thus

$$\sum_{n=1}^{\infty} p_n(t)s^n = -\lambda\sum_{n=1}^{\infty} n\,p_n(t)s^n - v\sum_{n=1}^{\infty} p_n(t)\,s^n + \lambda s\sum_{n=1}^{\infty}(n-1)p_{n-1}(t)s^{n-1} + vs\sum_{n=1}^{\infty} p_{n-1}(t)s^{n-1} \tag{5.18}$$

Taking advantage of the definitions **(5.17)** above, equation **(5.18)** can now be written as

$$\frac{d}{dt}G(s,t) = -\lambda s\frac{d}{ds}G(s,t) - vG(s,t) + \lambda s^2\frac{d}{ds}G(s,t) + vsG(s,t)$$

Equivalently, by grouping like terms together, the above equation becomes

$$\frac{d}{dt}G(s,t) = -\lambda s(1-s)\frac{d}{ds}G(s,t) - v(1-s)G(s,t)$$

Re arranging, the above equation can be rewritten in the form below

$$\frac{d}{dt}G(s,t) + \lambda s(1 - s)\frac{d}{ds}G(s,t) = - v(1 - s)G(s,t) \quad \textbf{(5.19)}$$

Applying Langrages method of solving this kind of differential equations, the auxiliary equations are as below

$$\frac{dt}{1} = \frac{ds}{\lambda s(1 - s)} = \frac{d\left[G(s,t)\right]}{v(1 - s)G(s,t)} \quad \textbf{(5.20)}$$

Taking $\frac{dt}{1} = \frac{ds}{\lambda s(1 - s)}$ and integrating, we have

$$\int \lambda\, dt = \int \frac{1}{s(1 - s)}\, ds.$$

But $\frac{1}{s(1 - s)} = \frac{1}{s} + \frac{1}{1 - s}$.

Therefore,

$$- \int \lambda\, dt = \int \left[\frac{1}{s} + \frac{1}{1 - s}\right] ds$$

$\Rightarrow$

$$- \lambda t = \ln s - \ln(1 - s) + c \quad \text{or } c_1^* = - \lambda t + \ln s - \ln(1 - s)$$

Equivalently

$$c_1^* = - \lambda t + \ln \frac{s}{(1 - s)}$$

Taking exponential of both sides, we have

$$k = e^{-\lambda t}\left(\frac{s}{1 - s}\right) \quad \text{(i)}$$

Taking $\frac{ds}{\lambda s(1 - s)} = - \frac{d\left[G(s,t)\right]}{v(1 - s)G(s,t)}$ and integrating, we have

$$-\frac{v}{\lambda}\int \frac{1}{s}\, ds = \int d \ln\left[G(s,t)\right]$$

$\Rightarrow$

$$-\frac{v}{\lambda} \ln s = \ln\left[G(s,t)\right] + c_2$$

Thus

$$\ln\left[G(s,t)\right] + \frac{v}{\lambda}\ln s = c_2 \text{ or } c_2 = \ln\left(s^{\frac{v}{\lambda}}G(s,t)\right)$$

Taking the exponential of both sides,

$$k^* = s^{\frac{v}{\lambda}} G(s,t)$$

From (i) and (ii)

$$s^{\frac{v}{\lambda}} G(s,t) = \psi\left(e^{-\lambda t}\left(\frac{s}{1-s}\right)\right) \quad \textbf{(5.21)}$$

When $t = 0,\ G(s,0) = s$

$$s^{\frac{v}{\lambda}} G(s,0) = \psi\left(\frac{s}{1-s}\right)$$

$$s^{\frac{v}{\lambda}} \times s = \psi\left(\frac{s}{1-s}\right)$$

Therefore

$$s^{1+\frac{v}{\lambda}} = \psi\left(\frac{s}{1-s}\right)$$

Let $w = \dfrac{s}{1-s}$

$\Rightarrow w - sw = s$ or $s + sw = w$ or equivalently $s = \dfrac{w}{1+w}$

$$\left(\frac{w}{1+w}\right)^{1+\frac{v}{\lambda}} = \psi(w)$$

Substituting these changes in equation **(5.21)**, we have

$$s^{\frac{v}{\lambda}}G(s,t) = \psi\left(e^{-\lambda t}w\right)$$

$$G(s,t) = s^{-\frac{v}{\lambda}}\psi\left(e^{-\lambda t}w\right)$$

$$= s^{-\frac{v}{\lambda}}\left(\frac{e^{-\lambda t}w}{1+e^{-\lambda t}w}\right)^{1+\frac{v}{\lambda}}$$

Equivalently

$$G(s,t) = s^{-\frac{v}{\lambda}} \times \left( \frac{e^{-\lambda t}\left(\frac{s}{1-s}\right)}{1 + e^{-\lambda t}\left(\frac{s}{1-s}\right)} \right)^{1+\frac{v}{\lambda}}$$

$$= s^{-\frac{v}{\lambda}} \times \left( \frac{e^{-\lambda t} s}{1 - s + e^{-\lambda t} s} \right)^{1+\frac{v}{\lambda}}$$

Let $p = e^{-\lambda t}$ and $q = 1 - p = 1 - e^{-\lambda t}$

$$G(s,t) = s^{-\frac{v}{\lambda}} \times \left( \frac{e^{-\lambda t} s}{1 - s\left(1 - e^{-\lambda t}\right)} \right)^{1+\frac{v}{\lambda}}$$

$$G(s,t) = s \times \left( \frac{e^{-\lambda t}}{1 - \left(1 - e^{-\lambda t}\right)s} \right)^{1+\frac{v}{\lambda}}$$

$$= s\left[ \frac{p}{1 - qs} \right]^{1+\frac{v}{\lambda}}$$

Therefore

$$G(s,t) = s\left[ \frac{p}{1 - qs} \right]^{1+\frac{v}{\lambda}} \qquad \textbf{(5.22)}$$

$p_n(t) = p_{1+k}(t)$ is the coefficient of $s^n$. $k = 0, 1, 2, 3, \ldots$..

$$G(s,t) = s\left[ \frac{p}{1 - qs} \right]^{1+\frac{v}{\lambda}} = s\,p^{1+\frac{v}{\lambda}}\left(1 - qs\right)^{-\left(1+\frac{v}{\lambda}\right)}$$

$\Rightarrow$

$$G(s,t) = \sum_{k=0}^{\infty} \binom{-\left(1 + \frac{v}{\lambda}\right)}{k} s\,p^{\left(1+\frac{v}{\lambda}\right)} \left(-qs\right)^k$$

$$= \sum_{k=0}^{\infty} (-1)^k (-1)^k \binom{\left(1 + \frac{v}{\lambda}\right) + k - 1}{k} p^{\left(1+\frac{v}{\lambda}\right)} q^k s^{k+1}$$

Therefore,

$$G(s,t) = \sum_{k=0}^{\infty} \binom{\left(1+\frac{v}{\lambda}\right)+k-1}{k} p^{\left(1+\frac{v}{\lambda}\right)} q^k s^{k+1}$$

Thus

$$\mathbf{p_n(t) = \binom{\frac{v}{\lambda}+k}{k} \left(e^{-\lambda t}\right)^{\left(1+\frac{v}{\lambda}\right)} \left(1 - e^{-\lambda t}\right)^{k}} \quad \text{where } n = 1 + k \text{ and } k = 0, 1, 2, \ldots \qquad \textbf{(5.23)}$$

**Initial conditions: when t = 0, $\mathbf{X(0) = n_0}$.**

This implies that $p_{n_0}(t) = 1$, $p_n(t) = 0 \;\forall\; n \neq n_0$ and $p_n(t) = 0 \;\forall\; n < n_0$.

$$p_0'(t) = -v\,p_0(t) \text{ as } n = 0 \qquad \textbf{(5.1a)}$$

$$p_n'(t) = -(n\lambda + v)\,p_n(t) + \left((n-1)\lambda + v\right)p_{n-1}(t), \quad n \geq 1 \qquad \textbf{(5.1b)}$$

**Definitions**

Let

$$\left.\begin{aligned} G(s,t) &= \sum_{n=0}^{\infty} p_n(t)s^n = p_0(t) + \sum_{n=1}^{\infty} p_n(t)s^n \\ \frac{d}{dt}\left[G(s,t)\right] &= \sum_{n=0}^{\infty} p_n'(t)s^n = p_0'(t) + \sum_{n=1}^{\infty} p_n'(t)s^n \\ \frac{d}{ds}\left[G(s,t)\right] &= \sum_{n=0}^{\infty} n\,p_n(t)s^{n-1} = \frac{1}{s}\sum_{n=1}^{\infty} n\,p_n(t)s^n \end{aligned}\right\} \qquad \textbf{(5.25)}$$

Further $G(1, t) = 1$, $G(0,t) = p_0(t)$ and $G(s,0) = s^{n_0}$.

Multiplying both sides of **(5.1b)** by $s^n$ and summing over n.

$$\sum_{n=1}^{\infty} p_n'(t)s^n = -\sum_{n=1}^{\infty}(n\lambda + v)\,p_n(t)s^n + \sum_{n=1}^{\infty}\left((n-1)\lambda + v\right)p_{n-1}(t)s^n$$

$$\sum_{n=1}^{\infty} p_n'(t)s^n = -\lambda\sum_{n=1}^{\infty} n\,p_n(t)s^n - v\sum_{n=1}^{\infty} p_n(t)s^n + \lambda\sum_{n=1}^{\infty}(n-1)p_{n-1}(t)s^n + v\sum_{n=1}^{\infty} p_{n-1}(t)s^n$$

$$= -\lambda\sum_{n=1}^{\infty} n\,p_n(t)s^n - v\sum_{n=1}^{\infty} p_n(t)s^n + \lambda s\sum_{n=1}^{\infty}(n-1)p_{n-1}(t)s^{n-1} + vs\sum_{n=1}^{\infty} p_{n-1}(t)s^{n-1}$$

Thus`

$$\sum_{n=1}^{\infty} p_n'(t) s^n = -\lambda \sum_{n=1}^{\infty} n\, p_n(t) s^n - v\sum_{n=1}^{\infty} p_n(t) s^n + \lambda s \sum_{n=1}^{\infty} (n-1) p_{n-1}(t) s^{n-1} + vs\sum_{n=1}^{\infty} p_{n-1}(t) s^{n-1} \qquad \textbf{(5.26)}$$

Taking advantage of the definitions **(5.25)** above, equation **(5.26)** can now be written as

$$\frac{d}{dt}G(s,t) = -\lambda s \frac{d}{ds}G(s,t) - vG(s,t) + \lambda s^2 \frac{d}{ds}G(s,t) + vsG(s,t)$$

Equivalently, by grouping like terms together, we have

$$\frac{d}{dt}G(s,t) = -\lambda s(1-s)\frac{d}{ds}G(s,t) + v(1-s)G(s,t)$$

The above equation can further be rewritten in the form

$$\frac{d}{dt}G(s,t) + \lambda s(1-s)\frac{d}{ds}G(s,t) = -v(1-s)G(s,t) \qquad \textbf{(5.27)}$$

Applying Lagrages method of solving this kind of differential equations, the auxiliary equations are as below

$$\frac{dt}{1} = \frac{ds}{-\lambda s(1-s)} = \frac{d\left[G(s,t)\right]}{v(1-s)G(s,t)} \qquad \textbf{(5.28)}$$

Taking $\frac{dt}{1} = \frac{ds}{-\lambda s(1-s)}$ and integrating, we have

$$-\int \lambda\, dt = \int \frac{1}{s(1-s)}\, ds \,.$$

But $\frac{1}{s(1-s)} = \frac{1}{s} + \frac{1}{(1-s)}$ .

Therefore,

$$-\int \lambda\, dt = \int \left[ -\frac{1}{s} + \frac{1}{1-s}\right] ds$$

$\Rightarrow$

$$\lambda t = \ln s - \ln(1-s) + c \quad \text{or } \lambda t = \ln \frac{s}{(1-s)} + c$$

Therefore,

$$c^* = -\lambda t + \ln\left(\frac{s}{1-s}\right)$$

Taking the exponential of both sides

$$k = e^{-\lambda t}\left(\frac{s}{1-s}\right) \qquad \text{(i)}$$

Taking $\frac{ds}{\lambda s(1-s)} = -\frac{d\left[G(s,t)\right]}{v(1-s)G(s,t)}$ and integrating, we have

$$-\frac{v}{\lambda}\int \frac{1}{s}\,ds = \int d\ln\left[G(s,t)\right]$$

$\Rightarrow$

$$-\frac{v}{\lambda}\ln s = \ln\left[G(s,t)\right] + c.$$

Equivalently

$$c_2^* = \ln G(s,t) + \frac{v}{\lambda}\ln s = \ln\left[s^{\frac{v}{\lambda}}\, G(s,t)\right]$$

Taking the exponential of both sides,

$$k^* = s^{\frac{v}{\lambda}} G(s,t) \qquad \text{(ii)}$$

From (i) and (ii)

$$s^{\frac{v}{\lambda}} G(s,t) = \psi\left(e^{-\lambda t}\frac{s}{1-s}\right) \qquad \textbf{(5.29)}$$

When $t = 0$, $G(s,0) = s^{n_0}$. Put $t = 0$ in the equation above.

$$s^{\frac{v}{\lambda}} G(s,0) = \psi\left(\frac{s}{1-s}\right)$$

$\Rightarrow$

$$s^{\frac{v}{\lambda}+n_0} = \psi\left(\frac{s}{1-s}\right)$$

Let $w = \frac{s}{1-s} \Rightarrow w - sw = s$ or $w = s + sw$ or equivalently $s = \frac{w}{1+w}$. Thus,

$$\psi(w) = \left(\frac{w}{1+w}\right)^{n_0+\frac{v}{\lambda}}$$

From equation **(5.29)** above,

$$s^{\frac{v}{\lambda}} G(s,t) = \psi\left(e^{-\lambda t} \frac{s}{1-s}\right) = \psi\left(e^{-\lambda t}\ w\right) = \left[\frac{e^{-\lambda t} w}{1+e^{-\lambda t} w}\right]^{n_0 + \frac{v}{\lambda}}$$

Therefore,

$$G(s,t) = s^{-\frac{v}{\lambda}} \left[\frac{e^{-\lambda t} w}{1+e^{-\lambda t} w}\right]^{n_0 + \frac{v}{\lambda}}$$

$$= s^{-\frac{v}{\lambda}} \left[\frac{e^{-\lambda t} \frac{s}{1-s}}{1+e^{-\lambda t} \frac{s}{1-s}}\right]^{n_0 + \frac{v}{\lambda}}$$

$$= s^{-\frac{v}{\lambda}} \left[\frac{\frac{e^{-\lambda t} s}{1-s}}{\frac{1-s+e^{-\lambda t} s}{1-s}}\right]^{n_0 + \frac{v}{\lambda}}$$

$$= s^{-\frac{v}{\lambda}} \left[\frac{e^{-\lambda t} s}{1-s+e^{-\lambda t} s}\right]^{n_0 + \frac{v}{\lambda}}$$

$$= s^{n_0} \left[\frac{e^{-\lambda t}}{1-s+e^{-\lambda t} s}\right]^{n_0 + \frac{v}{\lambda}}$$

If we take $p = e^{-\lambda t}$ and $q = 1 - e^{-\lambda t}$ we have

$$G(s,t) = s^{n_0} \left[\frac{p}{1-qs}\right]^{n_0 + \frac{v}{\lambda}} \qquad \textbf{(5.30)}$$

$p_n(t)$ is the coefficient of $s^n$, $n = n_0 + k$, $k = 0, 1, 2, 3, \ldots$.

$$G(s,t) = s^{n_0} p^{n_0 + \frac{v}{\lambda}} (1-qs)^{-\left(n_0 + \frac{v}{\lambda}\right)}$$

Let $n_0 + \dfrac{v}{\lambda} = r$. Then,

$$G(s,t) = s^{n_0} p^r (1-qs)^{-r}$$

$$= s^{n_0} p^r \sum_{k=0}^{\infty} \binom{-r}{k} (-qs)^k$$

Equivalently,

$$G(s,t) = s^{n_0} p^r \sum_{k=0}^{\infty} (-1)^k (-1)^k \binom{r+k-1}{k} (qs)^k$$

$$= \sum_{k=0}^{\infty} \binom{r+k-1}{k} p^r q^k s^{n_0+k}$$

Therefore

$$p_{n_0+k}(t) = p_n(t) = \binom{r+k-1}{k} p^r q^k$$

$$= \binom{\left(n_0 + \frac{v}{\lambda}\right) + k - 1}{k} p^{n_0 + \frac{v}{\lambda}} q^k$$

$$= \binom{\left(n_0 + \frac{v}{\lambda}\right) + k - 1}{k} \left(e^{-\lambda t}\right)^{n_0 + \frac{v}{\lambda}} \left(1 - e^{-\lambda t}\right)^k$$

$$\mathbf{p_n(t) = \binom{n_0 + \frac{v}{\lambda} + k - 1}{k} \left(e^{-\lambda t}\right)^{n_0 + \frac{v}{\lambda}} \left(1 - e^{-\lambda t}\right)^k} \qquad k = 0, 1, 2, 3, \ldots\ldots \qquad \textbf{(5.31)}$$

### 5.4.2 Mean and Variance by use of the Generating Function

**(i) Mean**

$$E(X) = \frac{d}{ds} G(s,t)\Big|_{s=1}$$

$$G(s,t) = s^{n_0} p^r (1-qs)^{-r}$$

$$= p^r s^{n_0} (1-qs)^{-r}$$

$$\frac{d}{ds} G(s,t) = p^r \left[ n_0 s^{n_0-1} (1-qs)^{-r} + s^{n_0} (-r)(1-qs)^{-r-1}(-q) \right]$$

$$= p^r \left[ n_0 s^{n_0-1} (1-qs)^{-r} + s^{n_0} r q (1-qs)^{-(r+1)} \right]$$

$$= p^r \left[ n_0 s^{n_0-1} (1-qs)^{-r} + r q s^{n_0} (1-qs)^{-(r+1)} \right]$$

$$= p^r \left[ s^{n_0} (1-qs)^{-r} \right] \left[ n_0 s^{-1} + r q (1-qs)^{-1} \right]$$

Put $s = 1$

$$\frac{d}{ds}G(s,t)\Big|_{s=1} = p^r\left[(1-q)^{-r}\right]\left[n_0 + rq(1-q)^{-1}\right]$$

$$= p^r p^{-r}\left[n_0 + rq(1-q)^{-1}\right]$$

$$= n_0 + \frac{rq}{p}$$

$$= n_0 + \frac{\left(n_0 + \frac{v}{\lambda}\right)\left(1 - e^{-\lambda t}\right)}{e^{-\lambda t}}$$

$$= \frac{n_0 e^{-\lambda t} + \left(n_0 + \frac{v}{\lambda}\right)\left(1 - e^{-\lambda t}\right)}{e^{-\lambda t}}$$

$$= \frac{n_0 + \frac{v}{\lambda}\left(1 - e^{-\lambda t}\right)}{e^{-\lambda t}}$$

Therefore

$$E(X(t)) = \frac{n_0 + \frac{v}{\lambda}\left(1 - e^{-\lambda t}\right)}{e^{-\lambda t}} \qquad \textbf{(5.32)}$$

**(ii) Variance**

$$\operatorname{Var} X = G''(1,t) + G'(1,t) - \left[G'(1,t)\right]^2$$

Therefore

$$\frac{d}{ds}G(s,t) = p^r\left[n_0 s^{n_0-1}(1-qs)^{-r} + rqs^{n_0}(1-qs)^{-(r+1)}\right]$$

$$\frac{d^2}{ds^2}G(s,t) = \frac{d}{ds}\left\{p^r\left[n_0 s^{n_0-1}(1-qs)^{-r} + rqs^{n_0}(1-qs)^{-(r+1)}\right]\right\}$$

$$= \frac{d}{ds}n_0 p^r s^{n_0-1}(1-qs)^{-r} + \frac{d}{ds}rqp^r s^{n_0}(1-qs)^{-(r+1)}$$

$$= n_0 p^r\left[(n_0-1)s^{n_0-2}(1-qs)^{-r} + s^{n_0-1}(-r)(1-qs)^{-(r+1)}(-q)\right]$$

$$+ rqp^r\left[n_0 s^{n_0-1}(1-qs)^{-(r+1)} + s^{n_0}(-(r+1))(1-qs)^{-(r+2)}(-q)\right]$$

$$\frac{d^2}{ds^2}G(s,t) = n_0 p^r \left[ (n_0 - 1)s^{n_0 - 2}(1-qs)^{-r} + rqs^{n_0-1}(1-qs)^{-(r+1)} \right]$$

$$+ rqp^r \left[ n_0 s^{n_0-1}(1-qs)^{-(r+1)} + (r+1)qs^{n_0}(1-qs)^{-(r+2)} \right]$$

Put $s = 1$

$$\frac{d^2}{ds^2}G(s,t)\Big|_{s=1} = n_0 p^r (1-q)^{-r}\left[ (n_0 - 1) + rq(1-q)^{-1} \right] + rqp^r(1-q)^{-(r+1)}\left[ n_0 + (r+1)q(1-q)^{-1} \right]$$

$$= n_0 p^r p^{-r}\left[ (n_0 - 1) + \frac{rq}{p} \right] + rqp^r p^{-(r+1)}\left[ n_0 + (r+1)\frac{q}{p} \right]$$

$$= n_0\left[ (n_0 - 1) + \frac{rq}{p} \right] + \frac{rq}{p}\left[ n_0 + (1+r)\frac{q}{p} \right]$$

$$= n_0(n_0 - 1) + \frac{n_0 rq}{p} + \frac{n_0 rq}{p} + \frac{r(1+r)q^2}{p^2}$$

$$= n_0(n_0 - 1) + \frac{2n_0 rq}{p} + \frac{r(1+r)q^2}{p^2}$$

Therefore,

$$\operatorname{Var} X = n_0(n_0 - 1) + \frac{2n_0 rq}{p} + \frac{r(1+r)q^2}{p^2} + n_0 + \frac{rq}{p} - \left( n_0 + \frac{rq}{p} \right)^2$$

$$= n_0^{\,2} - n_0 + \frac{2n_0 rq}{p} + \frac{r(1+r)q^2}{p^2} + n_0 + \frac{rq}{p} - \left( n_0^2 + \frac{2n_0 rq}{p} + \frac{r^2q^2}{p^2} \right)$$

$$= n_0^{\,2} - n_0 + \frac{2n_0 rq}{p} + \frac{r^2q^2}{p^2} + \frac{rq^2}{p^2} + n_0 + \frac{rq}{p} - n_0^2 - \frac{2n_0 rq}{p} - \frac{r^2q^2}{p^2}$$

$$= \frac{rq^2}{p^2} + \frac{rq}{p}$$

$$= \frac{rq}{p}\left( \frac{p+q}{p} \right)$$

$$= \frac{rq}{p^2}$$

$$= \frac{\left( n_0 + \frac{v}{\lambda} \right)\left( 1 - e^{-\lambda t} \right)}{\left( e^{-\lambda t} \right)^2}$$

Therefore

$$\text{Var}\,X = \left(n_0 + \frac{v}{\lambda}\right)\left(1 - e^{-\lambda t}\right)e^{2\lambda t}$$

Equivalently,

$$\text{Var}\;X = \left(n_0 + \frac{v}{\lambda}\right)e^{\lambda t}\left(e^{\lambda t} - 1\right) \qquad \textbf{(5.33)}$$

**5.1 Method of Moments to Determine Mean and Variance**

**5.5.1 Mean**

**Initial conditions; When $t = 0$, $X(0) = 0$.**

Multiply the basic difference – differential equations by n and then sum the results over n.

$$\sum_{n=0}^{\infty} n\,p_n'(t) = \sum_{n=0}^{\infty} n\left[(n-1)\lambda + v\right]p_{n-1}(t) - \sum_{n=0}^{\infty} n\left[n\lambda + v\right]p_n(t) \qquad \textbf{(5.34)}$$

Define

$$M_1(t) = \sum_{n=0}^{\infty} n\,p_n(t) \text{ and } M_2(t) = \sum_{n=0}^{\infty} n^2\,p_n(t)$$

Therefore,

$$M_1'(t) = \sum_{n=0}^{\infty} n\,p_n'(t) \text{ and } M_2'(t) = \sum_{n=0}^{\infty} n^2\,p_n'(t)$$

Therefore, equation **(5.34)** now becomes

$$M_1'(t) = \sum_{n=0}^{\infty}(n - 1 + 1)\left[(n-1)\lambda + v\right]p_{n-1}(t) - \sum_{n=0}^{\infty}\left[n^2\lambda + nv\right]p_n(t)$$

$$= \lambda\sum_{n=0}^{\infty}(n-1)^2\,p_{n-1}(t) + v\sum_{n=1}^{\infty}(n-1)p_{n-1}(t) + \lambda\sum_{n=1}^{\infty}(n-1)p_{n-1}(t) + v\sum_{n=1}^{\infty}p_{n-1}(t) - \lambda\sum_{n=1}^{\infty}n^2p_n(t)$$

$$- v\sum_{n=1}^{\infty} n\,p_n(t)$$

$$= \lambda M_2(t) + vM_1(t) + \lambda M_1(t) + v - \lambda M_2(t) - vM_1(t)$$

$$= \lambda M_1(t) + v$$

Therefore,

$$M'(t)_1 - \lambda M_1(t) = v \qquad \textbf{(5.35)}$$

Let the integrating factor $I = e^{\int -\lambda\,dt} = e^{-\lambda t}$

Multiplying equation **(5.35)** by the integrating factor, we have

$$e^{-\lambda t}M_1'(t) - \lambda e^{-\lambda t}M_1(t) = ve^{-\lambda t}$$

Equivalently, the equation above can also be written in the form

$$\frac{d}{dt}\left[e^{-\lambda t}M_1(t)\right] = ve^{-\lambda t}$$

Integrating both sides with respect to t, we have

$$e^{-\lambda t}M_1(t) = \int ve^{-\lambda t}\,dt$$

$$= \frac{ve^{-\lambda t}}{-\lambda} + c$$

Therefore

$$M_1(t) = ce^{\lambda t} - \frac{v}{\lambda} \qquad \textbf{(5.36)}$$

When $t = 0$,

$$M_1(0) = c - \frac{v}{\lambda} \qquad \textbf{(5.37)}$$

From definition, $M_1(0) = \sum_{n=1}^{\infty} n\,p_n(0)$

The initial condition is $p_0(0) = 1$ and $p_n(0) = 0$ for $n \neq 0$. Therefore, $M_1(0) = 0$.

Equation **(5.37)** now becomes

$$0 = c - \frac{v}{\lambda}$$

$$\Rightarrow$$

$$c = \frac{v}{\lambda}$$

Equation **(5.36)** now becomes

$$M_1(t) = \frac{v}{\lambda}\left(e^{\lambda t} - 1\right) \Rightarrow E\left(X(t) = n\right) = \frac{v}{\lambda}\left(e^{\lambda t} - 1\right) \qquad \textbf{(5.38)}$$

### 5.5.2 Variance

Next, multiply the basic difference- differential equations by $n^2$ and then sum the results over n.

$$\sum_{n=0}^{\infty} n^2 p_n'(t) = -\sum_{n=0}^{\infty} n^2\left[n\lambda + v\right]p_n(t) + \sum_{n=0}^{\infty} n^2\left[(n-1)\lambda + v\right]p_{n-1}(t) \qquad \textbf{(5.39)}$$

Define

$$M_1(t) = \sum_{n=1}^{\infty} n\,p_n(t) \quad \Rightarrow \quad M_1'(t) = \sum_{n=1}^{\infty} n\,p_n'(t)$$

$$M_2(t) = \sum_{n=1}^{\infty} n^2\,p_n(t) \quad \Rightarrow \quad M_2'(t) = \sum_{n=1}^{\infty} n^2\,p_n'(t)$$

$$M_3(t) = \sum_{n=1}^{\infty} n^3\,p_n(t) \quad \Rightarrow \quad M_3'(t) = \sum_{n=1}^{\infty} n^3\,p_n'(t)$$

Equation **(5.39)** now becomes

$$M_2'(t) = -\lambda M_3(t) - vM_2(t) + \sum_{n=0}^{\infty}(n-1+1)^2\left[(n-1)\lambda + v\right]p_{n-1}(t)$$

$$= -\lambda M_3(t) - vM_2(t) + \sum_{n=0}^{\infty}\left[(n-1)^2 + 2(n-1) + 1\right]\left[(n-1)\lambda + v\right]p_{n-1}(t)$$

$$= -\lambda M_3(t) - vM_2(t) + \lambda M_3(t) + 2\lambda M_2(t) + \lambda M_1(t) + vM_3(t) + 2vM_1(t) + v$$

$$M_2'(t) = 2\lambda M_2(t) + \lambda M_1(t) + 2vM_1(t) + v$$

Therefore,

$$M_2'(t) - 2\lambda M_2(t) = (\lambda + 2v)M_1(t) + v \qquad \textbf{(5.40)}$$

But from equation **(5.38)**, $M_1(t) = \frac{v}{\lambda}\left(e^{\lambda t} - 1\right)$

Equation **(5.40)** now becomes

$$M_2'(t) - 2\lambda M_2(t) = (\lambda + 2v)\times\frac{v}{\lambda}\left(e^{\lambda t} - 1\right) + v \qquad \textbf{(5.41)}$$

Let the integrating factor $I = e^{-2\lambda\int dt} = e^{-2\lambda t}$

Multiplying equation **(5.41)** by the integrating factor, we have

$$e^{-2\lambda t}M_2'(t) - 2\lambda e^{-2\lambda t}M_2(t) = (\lambda + 2v)e^{-2\lambda t}\times\frac{v}{\lambda}\left(e^{\lambda t} - 1\right) + ve^{-2\lambda t}$$

The above equation can be written in the form

$$\frac{d}{dt}\left[e^{-2\lambda t}M_2(t)\right] = (\lambda + 2v)e^{-2\lambda t}\times\frac{v}{\lambda}\left(e^{\lambda t} - 1\right) + ve^{-2\lambda t}$$

$$= \left(v + \frac{2v^2}{\lambda}\right)e^{-2\lambda t}\times\left(e^{\lambda t} - 1\right) + ve^{-2\lambda t}$$

$$= \left(ve^{-2\lambda t} + \frac{2v^2}{\lambda}e^{-2\lambda t}\right)\times\left(e^{\lambda t} - 1\right) + ve^{-2\lambda t}$$

$$\frac{d}{dt}\left[e^{-2\lambda t}M_2(t)\right] = ve^{-\lambda t} - ve^{-2\lambda t} + \frac{2v^2}{\lambda}e^{-\lambda t} - \frac{2v^2}{\lambda}e^{-2\lambda t} + ve^{-2\lambda t}$$

$$= ve^{-\lambda t} + \frac{2v^2}{\lambda}e^{-\lambda t} - \frac{2v^2}{\lambda}e^{-2\lambda t}$$

$$= \left(v + \frac{2v^2}{\lambda}\right)e^{-\lambda t} - \frac{2v^2}{\lambda}e^{-2\lambda t}$$

Therefore

$$\frac{d}{dt}\left[e^{-2\lambda t}M_2(t)\right] = \left(v + \frac{2v^2}{\lambda}\right)e^{-\lambda t} - \frac{2v^2}{\lambda}e^{-2\lambda t} \quad \textbf{(5.42)}$$

Integrating both sides of equation **(5.42)** with respect to t, we have

$$e^{-2\lambda t}M_2(t) = \int\left(v + \frac{2v^2}{\lambda}\right)e^{-\lambda t}dt - \frac{2v^2}{\lambda}\int e^{-2\lambda t}dt + c$$

$$= \left(v + \frac{2v^2}{\lambda}\right)\frac{e^{-\lambda t}}{-\lambda} - \frac{2v^2}{\lambda}\times\frac{e^{-2\lambda t}}{-2\lambda} + c$$

$$= \left(-\frac{v}{\lambda} - \frac{2v^2}{\lambda^2}\right)e^{-\lambda t} + \frac{v^2}{\lambda^2}e^{-2\lambda t} + c$$

Therefore,

$$M_2(t) = -\left(\frac{v}{\lambda} + \frac{2v^2}{\lambda^2}\right)e^{\lambda t} + \frac{v^2}{\lambda^2} + ce^{2\lambda t} \quad \textbf{(5.43)}$$

When $t = 0$, equation **(5.43)** becomes

$$M_2(0) = -\left(\frac{v}{\lambda} + \frac{2v^2}{\lambda^2}\right) + \frac{v^2}{\lambda^2} + c$$

$$= -\frac{v}{\lambda} - \frac{v^2}{\lambda^2} + c$$

From definition, $M_2(0) = \sum_{n=0}^{\infty} n^2 p_n(0) = 0$

Therefore

$$0 = -\frac{v}{\lambda} - \frac{v^2}{\lambda^2} + c \quad \Rightarrow \quad c = \frac{v}{\lambda} + \frac{v^2}{\lambda^2}$$

Equation **(5.43)** now becomes,

$$M_2(t) = -\left(\frac{v}{\lambda} + \frac{2v^2}{\lambda^2}\right)e^{\lambda t} + \frac{v^2}{\lambda^2} + \left(\frac{v}{\lambda} + \frac{v^2}{\lambda^2}\right)e^{2\lambda t}$$

$$= -\left[\frac{v}{\lambda} + \left(\frac{v}{\lambda}\right)^2 + \left(\frac{v}{\lambda}\right)^2\right]e^{\lambda t} + \left(\frac{v}{\lambda}\right)^2 + \left[\frac{v}{\lambda} + \left(\frac{v}{\lambda}\right)^2\right]e^{2\lambda t}$$

$$= -\left[\frac{v}{\lambda} + \left(\frac{v}{\lambda}\right)^2\right]e^{\lambda t} - \left(\frac{v}{\lambda}\right)^2 e^{\lambda t} + \left(\frac{v}{\lambda}\right)^2 + \left[\frac{v}{\lambda} + \left(\frac{v}{\lambda}\right)^2\right]e^{2\lambda t}$$

$$= \left[\frac{v}{\lambda} + \left(\frac{v}{\lambda}\right)^2\right]\left[e^{2\lambda t} - e^{\lambda t}\right] - \left(\frac{v}{\lambda}\right)^2\left(e^{\lambda t} - 1\right)$$

$$= \frac{v}{\lambda}e^{\lambda t}\left(1 + \frac{v}{\lambda}\right)\left(e^{\lambda t} - 1\right) - \left(\frac{v}{\lambda}\right)^2\left(e^{\lambda t} - 1\right)$$

$$= \frac{v}{\lambda}\left(e^{\lambda t} - 1\right)\left[\left(1 + \frac{v}{\lambda}\right)e^{\lambda t} - \frac{v}{\lambda}\right]$$

$$= \frac{v}{\lambda}\left(e^{\lambda t} - 1\right)\left[e^{\lambda t} + \frac{v}{\lambda}e^{\lambda t} - \frac{v}{\lambda}\right]$$

$$= \frac{v}{\lambda}\left(e^{\lambda t} - 1\right)\left[e^{\lambda t} + \frac{v}{\lambda}\left(e^{\lambda t} - 1\right)\right]$$

$$= \frac{v}{\lambda}e^{\lambda t}\left(e^{\lambda t} - 1\right) - \left[\frac{v}{\lambda}\left(e^{\lambda t} - 1\right)\right]^2$$

Therefore,

$$M_2(t) = \frac{v}{\lambda}e^{\lambda t}\left(e^{\lambda t} - 1\right) - \left[\frac{v}{\lambda}\left(e^{\lambda t} - 1\right)\right]^2 \qquad \textbf{(5.44)}$$

$$\text{Variance} = M_2(t) - \left[M_1(t)\right]^2$$

$$= \frac{v}{\lambda}e^{\lambda t}\left(e^{\lambda t} - 1\right) - \left[\frac{v}{\lambda}\left(e^{\lambda t} - 1\right)\right]^2 - \left[\frac{v}{\lambda}\left(e^{\lambda t} - 1\right)\right]^2$$

$$= \frac{v}{\lambda}e^{\lambda t}\left(e^{\lambda t} - 1\right)$$

Therefore,

$$\text{Variance} = \frac{v}{\lambda}e^{\lambda t}\left(e^{\lambda t} - 1\right) \qquad \textbf{(5.45)}$$

# CHAPTER SIX

# POLYA PROCESS

## 6.1 Introduction

The objective in this topic is to solve the basic difference differential equations derived in Chapter 2 (refer to equations (**2.1)** and **(2.2)** of Chapter 1) when $\lambda_n = \lambda\left(\frac{1 + an}{1 + \lambda at}\right)$. We shall specifically look at three methods namely the iterative method, the Laplace transform and the Lagranges method. In each of these cases, we shall work with the initial conditions being (i) X(0) = 0 and (ii) $X(0) = n_0$

When $\lambda_n = \lambda\left(\frac{1 + an}{1 + \lambda at}\right)$ for $n = 0, 1, 2, 3, \ldots$. the basic difference differential equations become

$$p_0'(t) = -\left(\frac{\lambda}{1 + \lambda at}\right)p_0(t) \qquad \textbf{(6.1a)}$$

$$p_n'(t) = -\lambda\left(\frac{1 + an}{1 + \lambda at}\right)p_n(t) + \lambda\left(\frac{1 + a(n-1)}{1 + \lambda at}\right)p_{n-1}(t), \quad n \geq 1 \qquad \textbf{(6.1b)}$$

## 6.2 Iteration Method

### 6.2.1 Deriving $p_n(t)$ using Iteration Method

**Initial Condition: X(0) = 0**

When $n = 0$, we use the equation **(5.1a)**

$$p_0'(t) = -\lambda\left(\frac{1}{1 + \lambda at}\right)p_0(t)$$

Dividing both sides by $p_0(t)$

$$\frac{p_0'(t)}{p_0(t)} = \frac{-\lambda}{1 + \lambda at}$$

$$\frac{d}{dt}\left(\ln p_0(t)\right) = \frac{-\lambda}{1 + \lambda at}$$

Integrating both sides of the equation with respect to t, we have

$$\int d\left(\ln p_0(t)\right)dt = \int \frac{-\lambda}{1 + \lambda at}dt$$

$$\ln p_0(t) = \frac{-\lambda}{\lambda a} \ln(1 + \lambda at) + \ln k$$

$$= -\frac{1}{a} \ln(1 + \lambda at) + \ln k$$

$$= \ln(1 + \lambda at)^{-\frac{1}{a}} + \ln k$$

$$= \ln\left[k(1 + \lambda a t)^{-\frac{1}{a}}\right]$$

Therefore

$$p_0(t) = k(1 + \lambda a t)^{-\frac{1}{a}}$$

Using the initial condition: When $t = 0,\ X(0) = 0$ which implies that $p_0(0) = 1$ and substituting,

$$p_0(0) = k(1 + \lambda \times a \times 0)^{-\frac{1}{a}}$$

$$k = 1$$

Therefore

$$p_0(t) = (1 + \lambda a t)^{-\frac{1}{a}} \text{ or } \quad p_0(t) = \left(\frac{1}{1 + \lambda a t}\right)^{\frac{1}{a}} \qquad \textbf{(6.2)}$$

We can use the second difference differential equation to obtain recursive relation when $n \geq 1$. When $n = 1$.

$$p_1'(t) = -\lambda\left(\frac{1 + a}{1 + \lambda a t}\right) p_1(t) + \lambda\left(\frac{1}{1 + \lambda a t}\right) p_0(t)$$

$$= -\lambda\left(\frac{1 + a}{1 + \lambda a t}\right) p_1(t) + \lambda\left(\frac{1}{1 + \lambda a t}\right) \times \left(\frac{1}{1 + \lambda a t}\right)^{\frac{1}{a}}$$

$$= -\lambda\left(\frac{1 + a}{1 + \lambda a t}\right) p_1(t) + \lambda\left(\frac{1}{1 + \lambda a t}\right)^{1 + \frac{1}{a}}$$

Re arranging

$$p_1'(t) + \lambda\left(\frac{1 + a}{1 + \lambda a t}\right) p_1(t) = \lambda\left(\frac{1}{1 + \lambda a t}\right)^{1 + \frac{1}{a}}$$

Next, we integrate the above equation by integrating factor Method

Integrating factor $= e^{\int \frac{1 + a}{1 + \lambda at} dt} = (1 + \lambda a t)^{1 + \frac{1}{a}}$

Multiplying both sides by the integrating factor

$$\left(1+\lambda a t\right)^{1+\frac{1}{a}}\times p_1'\left(t\right)+\left(1+\lambda a t\right)^{1+\frac{1}{a}}\times\lambda\left(\frac{1+a}{1+\lambda a t}\right)\ p_1\left(t\right)=\left(1+\lambda a t\right)^{1+\frac{1}{a}}\times\lambda\left(\frac{1}{1+\lambda a t}\right)^{1+\frac{1}{a}}$$

This simplifies to

$$\frac{d}{dt}\left[\left(1+\lambda a t\right)^{1+\frac{1}{a}}p_1\left(t\right)\right]=\lambda$$

Integrating both sides with respect to t

$$\int d\left[\left(1+\lambda a t\right)^{1+\frac{1}{a}}p_1\left(t\right)\right]dt=\int\lambda\,dt$$

$$\left(1+\lambda a t\right)^{1+\frac{1}{a}}p_1\left(t\right)=\lambda t+c$$

The initial conditions are: When $t=0,\ X\left(0\right)=0,$ which implies that $p_1\left(t\right)=0.$ Substituting in the above equation, we have

$$\left(1+\lambda a\times 0\right)^{1+\frac{1}{a}}p_1\left(0\right)=\lambda\times 0+c$$

$$c=0$$

Thus,

$$p_1\left(t\right)=\frac{\lambda t}{\left(1+\lambda a t\right)^{1+\frac{1}{a}}}\qquad\text{or equivalently}$$

$$p_1\left(t\right)=\frac{1}{a}\left(\frac{a t}{1+\lambda a t}\right)\left(\frac{1}{1+\lambda a t}\right)^{\frac{1}{a}}\tag{6.3}$$

When $n=2.$

$$p_2'(t)=-\lambda\left(\frac{1+2a}{1+\lambda a t}\right)p_2(t)+\lambda\left(\frac{1+a}{1+\lambda a t}\right)p_1(t)$$

$$=-\lambda\left(\frac{1+2a}{1+\lambda a t}\right)p_2(t)+\lambda\left(\frac{1+a}{1+\lambda a t}\right)\times p_1(t)$$

Rewriting the equation above

$$p_2'(t)+\lambda\left(\frac{1+2a}{1+\lambda a t}\right)p_2(t)=\lambda\left(\frac{1+a}{1+\lambda a t}\right)\times p_1(t)$$

Next, we integrate the above equation by integrating factor Method.

Integrating factor $=e^{\lambda\int\frac{1+2a}{1+\lambda at}dt}=\left(1+\lambda a t\right)^{2+\frac{1}{a}}$

Multiplying both sides of the equation by the integrating factor

$$\left(1+\lambda a t\right)^{2+\frac{1}{a}}\times p_2'(t)+\left(1+\lambda a t\right)^{2+\frac{1}{a}}\times\lambda\left(\frac{1+2a}{1+\lambda a t}\right)p_2(t)=\left(1+\lambda a t\right)^{2+\frac{1}{a}}\times\lambda\left(\frac{1+a}{1+\lambda a t}\right)p_1(t)$$

Equivalently,

$$\frac{d}{dt}\left[\left(1+\lambda a t\right)^{2+\frac{1}{a}}p_2\left(t\right)\right]=\left(1+\lambda a t\right)^{2+\frac{1}{a}}\times\lambda\left(\frac{1+a}{1+\lambda a t}\right)\times\frac{1}{a}\left(\frac{a t}{1+\lambda a t}\right)\left(\frac{1}{1+\lambda a t}\right)^{\frac{1}{a}}$$

$$=\lambda^2 t\left(1+a\right)$$

Integrating both sides with respect to t

$$\int d\left(\left(1+\lambda a t\right)^{2+\frac{1}{a}}p_2\left(t\right)\right)dt=\int\lambda^2 t\left(1+a\right)dt$$

$$\Rightarrow\ \left(1+\lambda a t\right)^{2+\frac{1}{a}}p_2\left(t\right)=\frac{\lambda^2 t^2}{2}\left(1+a\right)+c$$

The initial conditions are: When $t=0,\ X\left(t\right)=0.$ This implies that $p_2\left(0\right)=0.$ Substituting in the equation above

$$\left(1+\lambda a\times 0\right)^{2+\frac{1}{a}}p_2\left(0\right)=\frac{\lambda^2\times 0^2}{2}\left(1+a\right)+c$$

$$\Rightarrow\ c=0$$

Thus,

$$\left(1+\lambda a t\right)^{2+\frac{1}{a}}p_2\left(t\right)=\frac{\lambda^2 t^2}{2}\left(1+a\right)$$

Therefore,

$$p_2\left(t\right)=\frac{\lambda^2 t^2}{2}\left(1+a\right)\left(1+\lambda a t\right)^{-(2+\frac{1}{a})}$$

$$=\frac{\lambda^2 a^2 t^2}{2a^2}\times\frac{1}{\left(1+\lambda at\right)^{(2+\frac{1}{a})}}\times\left(1+a\right)$$

$$=\frac{\left(1+a\right)}{2a^2}\times\left(\frac{\lambda at}{1+\lambda a t}\right)^2\times\left(\frac{1}{1+\lambda at}\right)^{\frac{1}{a}}$$

$$=\frac{\left(\frac{1}{a}+1\right)\frac{1}{a}}{2}\times\left(\frac{\lambda at}{1+\lambda at}\right)^2\times\left(\frac{1}{1+\lambda at}\right)^{\frac{1}{a}}$$

$$=\frac{\left(\frac{1}{a}+1\right)!}{\left(\frac{1}{a}-1\right)!\,2!}\times\left(\frac{\lambda at}{1+\lambda at}\right)^2\times\left(\frac{1}{1+\lambda at}\right)^{\frac{1}{a}}$$

Thus,

$$p_2(t) = \begin{pmatrix} \frac{1}{a} + 1 \\ \frac{1}{a} - 1 \end{pmatrix} \left( \frac{1}{1 + \lambda a t} \right)^{\frac{1}{a}} \left( \frac{\lambda a t}{1 + \lambda a t} \right)^2 \qquad \textbf{(6.4)}$$

When n = 3,

$$p_3'(t) = -\lambda \left( \frac{1 + a \times 3}{1 + \lambda at} \right) p_3(t) + \lambda \left( \frac{1 + a \times 2}{1 + \lambda at} \right) p_2(t)$$

$$= -\lambda \left( \frac{1 + a \times 3}{1 + \lambda at} \right) p_3(t) + \lambda \left( \frac{1 + a \times 2}{1 + \lambda at} \right) \times \begin{pmatrix} \frac{1}{a} + 1 \\ \frac{1}{a} - 1 \end{pmatrix} \left( \frac{1}{1 + \lambda a t} \right)^{\frac{1}{a}} \left( \frac{\lambda a t}{1 + \lambda a t} \right)^2$$

Re writing, we have

$$p_3'(t) + \lambda \left( \frac{1 + a \times 3}{1 + \lambda at} \right) p_3(t) = \lambda \left( \frac{1 + 2a}{1 + \lambda at} \right) \times \left( \frac{1 + a}{2a^2} \right) \left( \frac{1}{1 + \lambda a t} \right)^{\frac{1}{a}} \left( \frac{\lambda a t}{1 + \lambda a t} \right)^2$$

$$= \frac{\lambda(1 + a)(1 + 2a)}{2a^2} \times \left( \frac{1}{1 + \lambda a t} \right)^{\frac{1}{a} + 1} \left( \frac{\lambda a t}{1 + \lambda a t} \right)^2$$

$$= \frac{\lambda(1 + \frac{1}{a})(2 + \frac{1}{a})}{2} \times \left( \frac{1}{1 + \lambda a t} \right)^{\frac{1}{a} + 1} \left( \frac{\lambda a t}{1 + \lambda a t} \right)^2$$

Next, we integrate the above equation by integrating factor Method.

$$\text{Integrating factor} = e^{\int \lambda \left( \frac{1 + 3a}{1 + \lambda at} \right) dt} = e^{\lambda(1 + 3a) \int \left( \frac{1}{1 + \lambda at} \right) dt}$$

$$\text{Let } 1 + \lambda at = u \quad \Rightarrow \quad \frac{du}{dt} = \lambda a. \text{ Therefore } dt = \frac{du}{\lambda a}.$$

$$\text{Integrating factor} = e^{\lambda(1 + 3a) \int \frac{1}{u} \cdot \left( \frac{du}{\lambda a} \right)} = e^{\left( \frac{1 + 3a}{a} \right) \ln u} = (1 + \lambda at)^{\frac{1}{a} + 3}$$

Multiplying the above equation by the integrating factor, we have

$$(1+\lambda at)^{\frac{1}{a}+3} p_3'(t) + \lambda (1+\lambda at)^{\frac{1}{a}+3} \left( \frac{1+3a}{1+\lambda at} \right) p_3(t) = \frac{\lambda(1+\frac{1}{a})(2+\frac{1}{a})}{2} (1+\lambda at)^{\frac{1}{a}+3} \left( \frac{1}{1+\lambda a t} \right)^{\frac{1}{a}+1} \left( \frac{\lambda a t}{1+\lambda a t} \right)^2$$

Equivalently,

$$\frac{d}{dt} \left\{ (1 + \lambda at)^{\frac{1}{a} + 3} p_3(t) \right\} = \frac{\lambda(1 + \frac{1}{a})(2 + \frac{1}{a})}{2} (1 + \lambda at)^{\frac{1}{a} + 3} \left( \frac{1}{1 + \lambda a t} \right)^{\frac{1}{a} + 1} \left( \frac{\lambda a t}{1 + \lambda a t} \right)^2$$

$$\frac{d}{dt}\left\{(1+\lambda at)^{\frac{1}{a}+3} p_3(t)\right\} = \frac{\lambda\left(\frac{1}{a}+2\right)!}{2!\,\frac{1}{a}!}(1+\lambda at)^2\left(\frac{\lambda a t}{1+\lambda a t}\right)^2$$

$$= \lambda\binom{\frac{1}{a}+2}{2}(\lambda at)^2$$

Integrating both sides,

$$(1+\lambda at)^{\frac{1}{a}+3} p_3(t) = a^2\lambda^3\binom{\frac{1}{a}+2}{2}\int t^2\, dt$$

$$(1+\lambda at)^{\frac{1}{a}+3} p_3(t) = a^2\lambda^3\binom{\frac{1}{a}+2}{2}\times\frac{t^3}{3}+c$$

When t = 0, $p_3(0) = 0$. Thus c = 0.

Therefore

$$p_3(t) = a^2\lambda^3\binom{\frac{1}{a}+2}{2}\times\frac{t^3}{3}\times(1+\lambda at)^{-\left(\frac{1}{a}+3\right)}$$

$$= a^2\lambda^3\binom{\frac{1}{a}+2}{2}\times\frac{t^3}{3}\times\left(\frac{1}{1+\lambda at}\right)^{\frac{1}{a}}\times\left(\frac{1}{1+\lambda at}\right)^3$$

$$= \frac{1}{3a}\binom{\frac{1}{a}+2}{2}\left(\frac{1}{1+\lambda at}\right)^{\frac{1}{a}}\left(\frac{a\lambda t}{1+\lambda at}\right)^3$$

$$= \frac{\frac{1}{a}}{3}\times\frac{\left(\frac{1}{a}+2\right)\left(\frac{1}{a}+1\right)}{2}\times\left(\frac{1}{1+\lambda at}\right)^{\frac{1}{a}}\left(\frac{a\lambda t}{1+\lambda at}\right)^3$$

$$= \frac{\left(\frac{1}{a}+2\right)\left(\frac{1}{a}+1\right)\left(\frac{1}{a}\right)}{3\times 2}\times\left(\frac{1}{1+\lambda at}\right)^{\frac{1}{a}}\left(\frac{a\lambda t}{1+\lambda at}\right)^3$$

$$= \frac{\left(\frac{1}{a}+2\right)!}{\left(\frac{1}{a}-1\right)!\,3!}\times\left(\frac{1}{1+\lambda at}\right)^{\frac{1}{a}}\left(\frac{a\lambda t}{1+\lambda at}\right)^3$$

$$= \binom{\frac{1}{a}+2}{3}\left(\frac{1}{1+\lambda at}\right)^{\frac{1}{a}}\left(\frac{a\lambda t}{1+\lambda at}\right)^3$$

Thus

$$p_3(t) = \binom{\frac{1}{a}+2}{3}\left(\frac{1}{1+\lambda at}\right)^{\frac{1}{a}}\left(\frac{a\lambda t}{1+\lambda at}\right)^3 \qquad \textbf{(6.5)}$$

By induction, assume that

$$p_{k-1}(t) = \binom{\frac{1}{a}+k-2}{k-2}\left(\frac{1}{1+\lambda at}\right)^{\frac{1}{a}}\left(\frac{a\lambda t}{1+\lambda at}\right)^{k-1}$$

For n = k, equation **(6.1b)** becomes

$$p_k'(t) = -\lambda\left(\frac{1+ak}{1+\lambda at}\right)p_k(t) + \lambda\left(\frac{1+a(k-1)}{1+\lambda at}\right)p_{k-1}(t)$$

$$p_k'(t) = -\lambda\left(\frac{1+ak}{1+\lambda at}\right)p_k(t) + \lambda\left(\frac{1+a(k-1)}{1+\lambda at}\right)\times\binom{\frac{1}{a}+k-2}{k-2}\left(\frac{1}{1+\lambda at}\right)^{\frac{1}{a}}\left(\frac{a\lambda t}{1+\lambda at}\right)^{k-1}$$

$$= -\lambda\left(\frac{1+ak}{1+\lambda at}\right)p_k(t) + \lambda(1+ak-a)\times\binom{\frac{1}{a}+k-2}{k-2}\left(\frac{1}{1+\lambda at}\right)^{\frac{1}{a}+1}\left(\frac{a\lambda t}{1+\lambda at}\right)^{k-1}$$

Re writing

$$p_k'(t) + \lambda\left(\frac{1+ak}{1+\lambda at}\right)p_k(t) = \lambda(1+ak-a)\times\binom{\frac{1}{a}+k-2}{k-2}\left(\frac{1}{1+\lambda at}\right)^{\frac{1}{a}+1}\left(\frac{a\lambda t}{1+\lambda at}\right)^{k-1}$$

Next, we integrate the above equation by integrating factor method.

$$\text{Integrating factor } = e^{\int\lambda\left(\frac{1+ak}{1+\lambda at}\right)dt} = e^{\lambda(1+ak)\int\frac{1}{1+\lambda at}dt}$$

$$\text{Let } 1+\lambda at = u \quad\Rightarrow\quad \frac{du}{dt} = a\lambda. \text{ Therefore } dt = \frac{du}{a\lambda}.$$

$$\text{Integrating factor } = e^{\lambda(1+ak)\int\frac{1}{u}\times\frac{du}{a\lambda}} = e^{\frac{1+ak}{a}\ln u} = (1+\lambda at)^{\frac{1}{a}+k}$$

Multiplying the above equation by the integrating factor, we have

$$\frac{d}{dt}\left\{p_k(t)\times(1+\lambda at)^{\frac{1}{a}+k}\right\} = \lambda(1+ak-a)\times(1+\lambda at)^{\frac{1}{a}+k}\binom{\frac{1}{a}+k-2}{k-2}\left(\frac{1}{1+\lambda at}\right)^{\frac{1}{a}+1}\left(\frac{a\lambda t}{1+\lambda at}\right)^{k-1}$$

$$= \lambda(1+ak-a)\binom{\frac{1}{a}+k-2}{k-2}(1+\lambda at)^{k-1}\left(\frac{a\lambda t}{1+\lambda at}\right)^{k-1}$$

$$= \lambda(1+ak-a)\binom{\frac{1}{a}+k-2}{k-2}(\lambda at)^{k-1}$$

Integrating both sides, we have

$$p_k(t)\times(1+\lambda at)^{\frac{1}{a}+k} = \lambda(1+ak-a)\binom{\frac{1}{a}+k-2}{k-2}\lambda^{k-1}a^{k-1}\int t^{k-1}\,dt$$

Equivalently,

$$p_k(t)\times(1+\lambda at)^{\frac{1}{a}+k} = \lambda(1+ak-a)\binom{\frac{1}{a}+k-2}{k-2}\lambda^{k-1}a^{k-1}\frac{t^k}{k}+c$$

When t = 0, $p_k(t) = 0$. Thus c = 0. Therefore

$$p_k(t) = \lambda(1+ak-a)\binom{\frac{1}{a}+k-2}{k-1}\times\frac{(a\lambda t)^k}{a\lambda k}\times(1+\lambda at)^{-(\frac{1}{a}+k)}$$

$$= \frac{(1+ak-a)}{ak}\times\frac{(\frac{1}{a}+k-2)!}{(k-1)!(\frac{1}{a}-1)!}\times(a\lambda t)^k\times(1+\lambda at)^{-\frac{1}{a}}\times(1+\lambda at)^{-k}$$

$$= \frac{(\frac{1}{a}+k-1)}{k}\times\frac{(\frac{1}{a}+k-2)!}{(k-1)!(\frac{1}{a}-1)!}\times(a\lambda t)^k\times(1+\lambda at)^{-\frac{1}{a}}\times(1+\lambda at)^{-k}$$

$$= \frac{(\frac{1}{a}+k-1)!}{k!\,(\frac{1}{a}-1)!}\times\left(\frac{1}{1+\lambda at}\right)^{\frac{1}{a}}\times\left(\frac{a\lambda t}{1+\lambda at}\right)^k$$

$$= \binom{\frac{1}{a}+k-1}{k}\times\left(\frac{1}{1+\lambda at}\right)^{\frac{1}{a}}\times\left(\frac{a\lambda t}{1+\lambda at}\right)^k$$

Thus

$$\mathbf{p_n(t) = \binom{\frac{1}{a}+n-1}{n}\times\left(\frac{1}{1+\lambda at}\right)^{\frac{1}{a}}\times\left(\frac{a\lambda t}{1+\lambda at}\right)^n} \qquad \textbf{(6.6)}$$

Equivalently,

$$p_j(t) = \binom{j+\frac{1}{a}-1}{j}q^j\times p^{\frac{1}{a}} \quad \text{where } p = \left(\frac{1}{1+\lambda at}\right) \text{ and } q = 1-p = \left(\frac{\lambda at}{1+\lambda at}\right)$$

This is a negative binomial with parameters p and q defined as above.

**Initial conditions: $\mathbf{X(0) = n_0}$ with $\mathbf{n_0 \geq 1}$**

Therefore, $p_{n_0}(0) = 1$, $p_n(0) = 0\ \forall\ n \neq n_0$ and $p_n(t)\ \forall\ n < n_0$.

When $n = n_0$, equation **(6.1b)** becomes

$$p'_{n_0}(t) = -\lambda\left(\frac{1+a\,n_0}{1+\lambda at}\right)p_{n_0}(t) + \lambda\left(\frac{1+a(n_0-1)}{1+\lambda at}\right)p_{n_0-1}(t), \quad n_0 \geq 1$$

$$= -\lambda\left(\frac{1+a\,n_0}{1+\lambda at}\right)p_{n_0}(t)$$

Therefore, dividing both sides by $p_{n_0}(t)$, we have

$$\frac{p'_{n_0}(t)}{p_{n_0}(t)} = -\lambda\left(\frac{1 + an_0}{1 + \lambda at}\right)$$

Equivalently,

$$\frac{d}{dt}\left\{\ln\left(p_{n_0}(t)\right)\right\} = -\lambda\left(\frac{1 + an_0}{1 + \lambda at}\right)$$

Integrating both sides with respect to t, we have

$$\ln\left(p_{n_0}(t)\right) = -\lambda\left(1 + an_0\right)\int\frac{1}{\left(1 + \lambda at\right)}\,dt$$

From the previous section, we know that $\int\frac{1}{\left(1 + \lambda at\right)}\,dt = \frac{1}{\lambda a}\ln\left(1 + \lambda at\right)$.

Therefore,

$$\ln\left(p_{n_0}(t)\right) = -\lambda\left(1 + an_0\right)\times\frac{1}{\lambda a}\ln\left(1 + \lambda at\right) + c$$

$$= -\left(\frac{1}{a} + n_0\right)\ln\left(1 + \lambda at\right) + c$$

Therefore,

$$p_{n_0}(t) = e^{-\left(\frac{1}{a} + n_0\right)\ln\left(1 + \lambda at\right) + c}$$

$$= e^{\ln\left(1 + \lambda at\right)^{-\left(\frac{1}{a} + n_0\right)}}\times k$$

$$= \left(1 + \lambda at\right)^{-\left(\frac{1}{a} + n_0\right)}\times k$$

Putting in the initial conditions, we have

$$p_{n_0}(0) = \left(1 + \lambda a\times 0\right)^{-\left(\frac{1}{a} + n_0\right)}\times k$$

$$\Rightarrow$$

$$k = 1$$

Therefore,

$$p_{n_0}(t) = \left(1 + \lambda at\right)^{-\left(\frac{1}{a} + n_0\right)}$$

$$= \frac{1}{\left(1 + \lambda at\right)^{\left(\frac{1}{a} + n_0\right)}} \qquad \textbf{(6.7)}$$

When $n = n_0 + 1$, equation **(6.1b)** becomes

$$p'_{n_0+1}(t) = -\lambda\left(\frac{1 + a(n_0+1)}{1 + \lambda at}\right)p_{n_0+1}(t) + \lambda\left(\frac{1 + a(n_0 + 1 - 1)}{1 + \lambda at}\right)p_{n_0+1-1}(t)$$

$$= -\lambda\left(\frac{1 + an_0 + a}{1 + \lambda at}\right)p_{n_0+1}(t) + \lambda\left(\frac{1 + an_0}{1 + \lambda at}\right)p_{n_0}(t)$$

$$= -\lambda\left(\frac{1 + an_0 + a}{1 + \lambda at}\right)p_{n_0+1}(t) + \lambda\left(\frac{1 + an_0}{1 + \lambda at}\right)\times\frac{1}{(1 + \lambda at)^{\left(\frac{1}{a}+n_0\right)}}$$

$$= -\lambda\left(\frac{1 + an_0 + a}{1 + \lambda at}\right)p_{n_0+1}(t) + \frac{\lambda(1 + an_0)}{(1 + \lambda at)^{\left(\frac{1}{a}+n_0+1\right)}}$$

Re arranging,

$$p'_{n_0+1}(t) + \lambda\left(\frac{1 + an_0 + a}{1 + \lambda at}\right)p_{n_0+1}(t) = \frac{\lambda(1 + an_0)}{(1 + \lambda at)^{\left(\frac{1}{a}+n_0+1\right)}}$$

We will now solve the above equation using the integrating factor method.

$$\text{Integrating factor } = e^{\lambda\left(\frac{1 + an_0 + a}{1 + \lambda at}\right)} = e^{\lambda(1 + an_0 + a)\int\frac{1}{1 + \lambda at}dt} = e^{\lambda(1 + an_0 + a)\frac{1}{\lambda a}\ln(1 + \lambda at)} = e^{a\lambda\left(\frac{1}{a} + n_0 + 1\right)\frac{1}{\lambda a}\ln(1 + \lambda at)}$$

$$= e^{\ln(1 + \lambda at)^{\left(\frac{1}{a} + n_0 + 1\right)}} = (1 + \lambda at)^{\left(\frac{1}{a} + n_0 + 1\right)}$$

Multiplying the above equation by the integrating factor, we have

$$\frac{d}{dt}\left\{(1 + \lambda at)^{\left(\frac{1}{a} + n_0 + 1\right)} p_{n_0+1}(t)\right\} = \frac{\lambda(1 + an_0)}{(1 + \lambda at)^{\left(\frac{1}{a}+n_0+1\right)}}\times(1 + \lambda at)^{\left(\frac{1}{a} + n_0 + 1\right)}$$

$$= \lambda(1 + an_0)$$

Integrating both sides with respect to t, we have

$$(1 + \lambda at)^{\left(\frac{1}{a} + n_0 + 1\right)} p_{n_0+1}(t) = \lambda(1 + an_0)\int dt$$

$$= \lambda(1 + an_0)t + c$$

Putting in the initial conditions, we have

$$(1 + \lambda at)^{\left(\frac{1}{a} + n_0 + 1\right)} p_{n_0+1}(0) = \lambda(1 + an_0)\times 0 + c$$

$\Rightarrow$

$c = 0$

Therefore,

$$\left(1+\lambda at\right)^{\left(\frac{1}{a}+n_0+1\right)} p_{n_0+1}(t) = \lambda\left(1+an_0\right)t$$

Equivalently,

$$p_{n_0+1}(t) = \frac{\lambda t\left(1+an_0\right)}{\left(1+\lambda at\right)^{\left(\frac{1}{a}+n_0+1\right)}} \tag{6.8}$$

When $n = n_0 + 2$, equation **(6.1b)** becomes

$$p'_{n_0+2}(t) = -\lambda\left(\frac{1+a\left(n_0+2\right)}{1+\lambda at}\right)p_{n_0+2}(t) + \lambda\left(\frac{1+a\left(n_0+2-1\right)}{1+\lambda at}\right)p_{n_0+2-1}(t)$$

$$= -\lambda\left(\frac{1+an_0+2a}{1+\lambda at}\right)p_{n_0+2}(t) + \lambda\left(\frac{1+a\left(n_0+1\right)}{1+\lambda at}\right)p_{n_0+1}(t)$$

$$= -\lambda\left(\frac{1+an_0+2a}{1+\lambda at}\right)p_{n_0+2}(t) + \lambda\left(\frac{1+a\left(n_0+1\right)}{1+\lambda at}\right)\times\frac{\lambda t\left(1+an_0\right)}{\left(1+\lambda at\right)^{\left(\frac{1}{a}+n_0+1\right)}}$$

$$= -\lambda\left(\frac{1+an_0+2a}{1+\lambda at}\right)p_{n_0+2}(t) + \frac{\lambda^2 t\left(1+an_0\right)\left(1+an_0+a\right)}{\left(1+\lambda at\right)^{\left(\frac{1}{a}+n_0+2\right)}}$$

Re arranging,

$$p'_{n_0+2}(t) + \lambda\left(\frac{1+an_0+2a}{1+\lambda at}\right)p_{n_0+2}(t) = \frac{\lambda^2 t\left(1+an_0\right)\left(1+an_0+a\right)}{\left(1+\lambda at\right)^{\left(\frac{1}{a}+n_0+2\right)}}$$

We will now solve the above equation using the integrating factor method.

$$\text{Integrating factor } = e^{\lambda\left(\frac{1+an_0+2a}{1+\lambda at}\right)} = e^{\lambda\left(1+an_0+2a\right)\int\frac{1}{1+\lambda at}dt} = e^{\lambda\left(1+an_0+2a\right)\frac{1}{\lambda a}\ln\left(1+\lambda at\right)}$$

$$= e^{a\lambda\left(\frac{1}{a}+n_0+2\right)\frac{1}{\lambda a}\ln\left(1+\lambda at\right)} = e^{\ln\left(1+\lambda at\right)^{\left(\frac{1}{a}+n_0+2\right)}} = \left(1+\lambda at\right)^{\left(\frac{1}{a}+n_0+2\right)}$$

Multiplying the above equation by the integrating factor, we have

$$\frac{d}{dt}\left\{\left(1+\lambda at\right)^{\left(\frac{1}{a}+n_0+2\right)}p_{n_0+2}(t)\right\} = \frac{\lambda^2 t\left(1+an_0\right)\left(1+an_0+a\right)}{\left(1+\lambda at\right)^{\left(\frac{1}{a}+n_0+2\right)}}\times\left(1+\lambda at\right)^{\left(\frac{1}{a}+n_0+2\right)}$$

$$= \lambda^2 t\left(1+an_0\right)\left(1+an_0+a\right)$$

Integrating both sides with respect to t, we have

$$\left(1+\lambda at\right)^{\left(\frac{1}{a}+n_0+2\right)} p_{n_0+2}(t) = \lambda^2\left(1+an_0\right)\left(1+an_0+a\right)\int t\,dt$$

$$= \lambda^2\left(1+an_0\right)\left(1+an_0+a\right)\frac{t^2}{2}+c$$

Putting in the initial conditions, we have

$$\left(1+\lambda a\times 0\right)^{\left(\frac{1}{a}+n_0+2\right)} p_{n_0+2}(0) = \lambda^2\left(1+an_0\right)\left(1+an_0+a\right)\frac{0^2}{2}+c$$

$$\Rightarrow$$

$$c = 0$$

Therefore,

$$\left(1+\lambda at\right)^{\left(\frac{1}{a}+n_0+2\right)} p_{n_0+2}(t) = \lambda^2\left(1+an_0\right)\left(1+an_0+a\right)\frac{t^2}{2}$$

Equivalently,

$$p_{n_0+2}(t) = \frac{(a\lambda t)^2\left(\frac{1}{a}+n_0\right)\left(\frac{1}{a}+n_0+1\right)}{2\left(1+\lambda at\right)^{\left(\frac{1}{a}+n_0+2\right)}} \qquad \textbf{(6.9)}$$

When $n = n_0+3$, equation **(6.1b)** becomes

$$p'_{n_0+3}(t) = -\lambda\left(\frac{1+a\left(n_0+3\right)}{1+\lambda at}\right)p_{n_0+3}(t) + \lambda\left(\frac{1+a\left(n_0+3-1\right)}{1+\lambda at}\right)p_{n_0+3-1}(t)$$

$$= -\lambda\left(\frac{1+an_0+3a}{1+\lambda at}\right)p_{n_0+3}(t) + \lambda\left(\frac{1+a\left(n_0+2\right)}{1+\lambda at}\right)p_{n_0+2}(t)$$

$$= -\lambda\left(\frac{1+an_0+3a}{1+\lambda at}\right)p_{n_0+3}(t) + \lambda\left(\frac{1+a\left(n_0+2\right)}{1+\lambda at}\right)\times\frac{(a\lambda t)^2\left(\frac{1}{a}+n_0\right)\left(\frac{1}{a}+n_0+1\right)}{2\left(1+\lambda at\right)^{\left(\frac{1}{a}+n_0+2\right)}}$$

$$= -\lambda\left(\frac{1+an_0+3a}{1+\lambda at}\right)p_{n_0+3}(t) + \frac{a^2\lambda^3t^2\left(\frac{1}{a}+n_0\right)\left(\frac{1}{a}+n_0+1\right)\left(1+an_0+2a\right)}{2\left(1+\lambda at\right)^{\left(\frac{1}{a}+n_0+3\right)}}$$

Re arranging,

$$p'_{n_0+3}(t) + \lambda\left(\frac{1+an_0+3a}{1+\lambda at}\right)p_{n_0+3}(t) = \frac{a^2\lambda^3t^2\left(\frac{1}{a}+n_0\right)\left(\frac{1}{a}+n_0+1\right)\left(1+an_0+2a\right)}{2\left(1+\lambda at\right)^{\left(\frac{1}{a}+n_0+3\right)}}$$

We will now solve the above equation using the integrating factor method.

Integrating factor $= e^{\lambda\left(\frac{1 + an_0 + 3a}{1 + \lambda at}\right)} = e^{\lambda(1 + an_0 + 3a)\int\frac{1}{1 + \lambda at}dt} = e^{\lambda(1 + an_0 + 3a)\frac{1}{\lambda a}\ln(1 + \lambda at)}$

$$= e^{a\lambda\left(\frac{1}{a} + n_0 + 3\right)\frac{1}{\lambda a}\ln(1 + \lambda at)} = e^{\ln(1 + \lambda at)^{\left(\frac{1}{a} + n_0 + 3\right)}} = (1 + \lambda at)^{\left(\frac{1}{a} + n_0 + 3\right)}$$

Multiplying the above equation by the integrating factor, we have

$$\frac{d}{dt}\left\{(1 + \lambda at)^{\left(\frac{1}{a} + n_0 + 3\right)} p_{n_0+3}(t)\right\} = \frac{a^2\lambda^3t^2\left(\frac{1}{a} + n_0\right)\left(\frac{1}{a} + n_0 + 1\right)(1 + an_0 + 2a)}{2(1 + \lambda at)^{\left(\frac{1}{a} + n_0 + 3\right)}} \times (1 + \lambda at)^{\left(\frac{1}{a} + n_0 + 3\right)}$$

$$= \frac{a^2\lambda^3t^2}{2}\left(\frac{1}{a} + n_0\right)\left(\frac{1}{a} + n_0 + 1\right)(1 + an_0 + 2a)$$

Integrating both sides with respect to t, we have

$$(1 + \lambda at)^{\left(\frac{1}{a} + n_0 + 3\right)} p_{n_0+3}(t) = \frac{a^2\lambda^3}{2}\left(\frac{1}{a} + n_0\right)\left(\frac{1}{a} + n_0 + 1\right)(1 + an_0 + 2a)\int t^2\,dt$$

$$= \frac{a^2\lambda^3t^3}{2.3}\left(\frac{1}{a} + n_0\right)\left(\frac{1}{a} + n_0 + 1\right)(1 + an_0 + 2a) + c$$

Putting in the initial conditions, we have

$$(1 + \lambda a \times 0)^{\left(\frac{1}{a} + n_0 + 3\right)} p_{n_0+3}(0) = \frac{a^2\lambda^3 \times 0^3}{2.3}\left(\frac{1}{a} + n_0\right)\left(\frac{1}{a} + n_0 + 1\right)(1 + an_0 + 2a) + c$$

$\Rightarrow$

$c = 0$

Therefore,

$$(1 + \lambda at)^{\left(\frac{1}{a} + n_0 + 3\right)} p_{n_0+3}(t) = \frac{a^2\lambda^3t^3}{2.3}\left(\frac{1}{a} + n_0\right)\left(\frac{1}{a} + n_0 + 1\right)(1 + an_0 + 2a)$$

Equivalently,

$$p_{n_0+3}(t) = \frac{(a\lambda t)^3\left(\frac{1}{a} + n_0\right)\left(\frac{1}{a} + n_0 + 1\right)\left(\frac{1}{a} + n_0 + 2\right)}{2.3(1 + \lambda at)^{\left(\frac{1}{a} + n_0 + 3\right)}}$$

$$= \frac{\left(\frac{1}{a} + n_0 + 2\right)!}{\left(\frac{1}{a} + n_0 - 1\right)!\,3!}\frac{(a\lambda t)^3}{(1 + \lambda at)^{\left(\frac{1}{a} + n_0 + 3\right)}}$$

$$= \frac{\left(\frac{1}{a} + n_0 + 2\right)!}{\left(\frac{1}{a} + n_0 - 1\right)!\,3!} \times \left(\frac{a\lambda t}{1 + \lambda at}\right)^3 \times \frac{1}{(1 + \lambda at)^{\left(\frac{1}{a} + n_0\right)}}$$

In a simplified form,

$$p_{n_0+3}(t) = \binom{\frac{1}{a} + n_0 + 2}{3} \times \left(\frac{a\lambda t}{1+\lambda at}\right)^3 \times \left(\frac{1}{1+\lambda at}\right)^{\left(\frac{1}{a}+n_0\right)} \qquad \textbf{(6.10)}$$

Assume that for $n = n_0 + k - 1$

$$p_{n_0+k-1}(t) = \binom{\frac{1}{a} + n_0 + k - 2}{k - 1} \times \left(\frac{a\lambda t}{1+\lambda at}\right)^{k-1} \times \left(\frac{1}{1+\lambda at}\right)^{\left(\frac{1}{a}+n_0\right)}$$

For $n = n_0 + k$, equation **(6.1b)** becomes

$$p'_{n_0+k}(t) = -\lambda\left(\frac{1+a(n_0+k)}{1+\lambda at}\right)p_{n_0+k}(t) + \lambda\left(\frac{1+a(n_0+k-1)}{1+\lambda a t}\right)p_{n_0+k-1}(t)$$

But $p_{n_0+k-1}(t)$ is as assumed above. Therefore

$$p'_{n_0+k}(t) = -\lambda\left(\frac{1+a(n_0+k)}{1+\lambda at}\right)p_{n_0+k}(t) + \lambda\left(\frac{1+a(n_0+k-1)}{1+\lambda at}\right)\times\binom{\frac{1}{a}+n_0+k-2}{k-1}\times\left(\frac{a\lambda t}{1+\lambda at}\right)^{k-1}\times\left(\frac{1}{1+\lambda at}\right)^{\left(\frac{1}{a}+n_0\right)}$$

$$= -\lambda\left(\frac{1+a(n_0+k)}{1+\lambda at}\right)p_{n_0+k}(t) + \lambda\left(\frac{1+a(n_0+k-1)}{1+\lambda at}\right)\times\frac{\left(\frac{1}{a}+n_0+k-2\right)!}{\left(\frac{1}{a}+n_0-1\right)!(k-1)!}\times\left(\frac{a\lambda t}{1+\lambda at}\right)^{k-1}\times\left(\frac{1}{1+\lambda at}\right)^{\left(\frac{1}{a}+n_0\right)}$$

$$= -\lambda\left(\frac{1+a(n_0+k)}{1+\lambda at}\right)p_{n_0+k}(t) + \lambda^k a^k t^{k-1}\times\frac{\left(\frac{1}{a}+n_0+k-1\right)!}{\left(\frac{1}{a}+n_0-1\right)!(k-1)!}\times\left(\frac{1}{1+\lambda at}\right)^{\left(\frac{1}{a}+n_0+k\right)}$$

Re writing

$$p'_{n_0+k}(t) + \lambda\left(\frac{1+a(n_0+k)}{1+\lambda at}\right)p_{n_0+k}(t) = \lambda^k a^k t^{k-1}\times\frac{\left(\frac{1}{a}+n_0+k-1\right)!}{\left(\frac{1}{a}+n_0-1\right)!(k-1)!}\times\left(\frac{1}{1+\lambda at}\right)^{\left(\frac{1}{a}+n_0+k\right)}$$

Next, we integrate the above equation by integrating factor method.

Integrating factor $= e^{\int\lambda\left(\frac{1+a(n_0+k)}{1+\lambda at}\right)dt} = e^{\lambda(1+an_0+ak)\int\frac{1}{1+\lambda at}dt} = e^{\lambda(1+an_0+ak)\frac{1}{a\lambda}\ln(1+\lambda at)}$

$$= e^{a\lambda\left(\frac{1}{a}+n_0+k\right)\frac{1}{a\lambda}\ln(1+\lambda at)} = e^{\ln(1+\lambda at)\left(\frac{1}{a}+n_0+k\right)} = (1+\lambda at)^{\left(\frac{1}{a}+n_0+k\right)}$$

Multiplying the above equation by the integrating factor, we have

$$\frac{d}{d}\left\{p_{n_0+k}(t)\times(1+\lambda at)^{\left(\frac{1}{a}+n_0+k\right)}\right\} = \lambda^k a^k t^{k-1}\times\frac{\left(\frac{1}{a}+n_0+k-1\right)!}{\left(\frac{1}{a}+n_0-1\right)!(k-1)!}\times\left(\frac{1}{1+\lambda at}\right)^{\left(\frac{1}{a}+n_0+k\right)}\times(1+\lambda at)^{\left(\frac{1}{a}+n_0+k\right)}$$

$$= \lambda^k a^k t^{k-1}\times\frac{\left(\frac{1}{a}+n_0+k-1\right)!}{\left(\frac{1}{a}+n_0-1\right)!(k-1)!}$$

Integrating both sides, we have

$$p_{n_0+k}(t)\times(1+\lambda at)^{\left(\frac{1}{a}+n_0+k\right)} = \lambda^k a^k \times \frac{\left(\frac{1}{a}+n_0+k-1\right)!}{\left(\frac{1}{a}+n_0-1\right)!(k-1)!}\int t^{k-1}\,dt$$

$$= \lambda^k a^k \times \frac{\left(\frac{1}{a}+n_0+k-1\right)!}{\left(\frac{1}{a}+n_0-1\right)!(k-1)!}\times\frac{t^k}{k} + c$$

Substituting the initial conditions,

$$p_{n_0+k}(0)\times(1+\lambda a\times 0)^{\left(\frac{1}{a}+n_0+k\right)} = \lambda^k a^k \times \frac{\left(\frac{1}{a}+n_0+k-1\right)!}{\left(\frac{1}{a}+n_0-1\right)!(k-1)!}\times\frac{0^k}{k} + c$$

$\Rightarrow$

$c = 0$

Therefore,

$$p_{n_0+k}(t)\times(1+\lambda at)^{\left(\frac{1}{a}+n_0+k\right)} = \lambda^k a^k \times \frac{\left(\frac{1}{a}+n_0+k-1\right)!}{\left(\frac{1}{a}+n_0-1\right)!(k-1)!}\times\frac{t^k}{k}$$

$$= (\lambda at)^k \frac{\left(\frac{1}{a}+n_0+k-1\right)!}{\left(\frac{1}{a}+n_0-1\right)!k!}$$

$$= \binom{\frac{1}{a}+n_0+k-1}{k}(\lambda at)^k$$

Equivalently,

$$p_{n_0+k}(t) = \binom{\frac{1}{a}+n_0+k-1}{k}\frac{(\lambda at)^k}{(1+\lambda at)^{\left(\frac{1}{a}+n_0+k\right)}}$$

Thus

$$\mathbf{p_{n_0+k}(t) = \binom{\frac{1}{a}+n_0+k-1}{k}\left(\frac{\lambda at}{1+\lambda at}\right)^k\left(\frac{1}{1+\lambda at}\right)^{\left(\frac{1}{a}+n_0\right)}} \qquad \mathbf{(6.11)}$$

Equivalently,

This is a negative binomial with parameters p and q defined as above.

### 6.2.2 Mean and Variance by Definition (for $X(0) = 0$)

From **(6.6)** above, let $p_j(t) = \binom{j+\frac{1}{a}-1}{j}q^j \times p^{\frac{1}{a}}$ where $p = \left(\frac{1}{1+\lambda at}\right)$, $q = 1 - p = \left(\frac{\lambda at}{1+\lambda at}\right)$

and j = k. Then $\text{Prob}\left[X(t) = n\right] = \binom{n+\frac{1}{a}-1}{n}q^n \times p^{\frac{1}{a}}$

**Mean**

$$E\left[X(t)=n\right]=\sum_{n=0}^{\infty} n\binom{\frac{1}{a}+n-1}{n} q^n \times p^{\frac{1}{a}}$$

$$=\sum_{n=0}^{\infty} n\times\frac{\left(\frac{1}{a}+n-1\right)!}{n!\left(\frac{1}{a}-1\right)!}\, q^n \times p^{\frac{1}{a}}$$

$$=\frac{1}{\left(\frac{1}{a}-1\right)!}\sum_{n=0}\frac{\left(\frac{1}{a}+n-1\right)!}{(n-1)!}\times\frac{\left(\frac{1}{a}\right)!}{\left(\frac{1}{a}\right)!}\, q^{n-1}.\, q\times\frac{p^{\frac{1}{a}+1}}{p}$$

$$=\frac{\left(\frac{1}{a}\right)!\times q}{\left(\frac{1}{a}-1\right)!\times p}\sum_{n=0}\frac{\left(\frac{1}{a}+n-1\right)!}{(n-1)!\,\left(\frac{1}{a}\right)!}\; q^{n-1} p^{\frac{1}{a}+1}$$

$$=\frac{1}{a}\times\frac{q}{p}\sum_{n=0}\binom{\frac{1}{a}+n-1}{n-1} q^{n-1} p^{\frac{1}{a}+1}$$

$$=\frac{1}{a}\times\frac{q}{p}$$

$$=\frac{1}{a}\times\left(\frac{\lambda at}{1+\lambda at}\right)\div\left(\frac{1}{1+\lambda at}\right)$$

$$=\frac{1}{a}\times\left(\frac{\lambda at}{1+\lambda at}\times\frac{1+\lambda at}{1}\right)$$

$$=\lambda t$$

Therefore

$$E\left[X(t)=n\right]=\lambda t \qquad \textbf{(6.12)}$$

**Variance**

$$E\left(X(t)=n\right)^2=\sum_{n=0}^{\infty} n^2\binom{n+\frac{1}{a}-1}{n} q^n \times p^{\frac{1}{a}}$$

$$=\sum_{n=0}^{\infty}\left[n(n-1)+n\right]\binom{n+\frac{1}{a}-1}{n} q^n \times p^{\frac{1}{a}}$$

$$=\sum_{n=0}^{\infty} n(n-1)\binom{n+\frac{1}{a}-1}{n} q^n \times p^{\frac{1}{a}}+\sum_{n=0}^{\infty} n\binom{n+\frac{1}{a}-1}{n} q^n \times p^{\frac{1}{a}}$$

$$E\left(X(t)=n\right)^2=\sum_{n=0}^{\infty} n(n-1)\frac{\left(n+\frac{1}{a}-1\right)!}{n!\left(\frac{1}{a}-1\right)!}q^n\times p^{\frac{1}{a}}+\lambda t$$

$$=\frac{1}{\left(\frac{1}{a}-1\right)!}\sum_{n=0}^{\infty}\frac{\left(n+\frac{1}{a}-1\right)\left(\frac{1}{a}+1\right)!}{(n-2)!\left(\frac{1}{a}+1\right)!}q^{n-2}.q^2\times\frac{p^{\frac{1}{a}+2}}{p^2}+\lambda t$$

$$=\frac{\left(\frac{1}{a}+1\right)!\,q^2}{\left(\frac{1}{a}-1\right)!\,p^2}\sum_{n=0}^{\infty}\binom{n+\frac{1}{a}-1}{n-2}q^{n-2}p^{\frac{1}{a}+2}+\lambda t$$

$$=\tfrac{1}{a}\left(\tfrac{1}{a}+1\right)\frac{q^2}{p^2}\times 1+\lambda t$$

Therefore

$$\mathrm{Var}\left[X(t)=n\right]=\left\{\tfrac{1}{a}\left(\tfrac{1}{a}+1\right)\frac{q^2}{p^2}\times 1+\lambda t\right\}-(\lambda t)^2$$

$$=\left\{\tfrac{1}{a}\left(\tfrac{1}{a}+1\right)\left(\frac{q}{p}\right)^2+\lambda t\right\}-(\lambda t)^2$$

Now

$$\frac{q}{p}=\frac{\lambda at}{1+\lambda at}\times\frac{1+\lambda at}{1}=\lambda at.$$

Therefore

$$\mathrm{Var}\left[X(t)=n\right]=\left\{\tfrac{1}{a}\left(\tfrac{1+a}{a}\right)(\lambda at)^2+\lambda t\right\}-(\lambda t)^2$$

$$=\left\{\left(\tfrac{1+a}{a^2}\right)\times\lambda^2a^2t^2+\lambda t\right\}-(\lambda t)^2$$

$$=\left\{(1+a)\lambda^2t^2+\lambda t\right\}-(\lambda t)^2$$

$$=\lambda^2t^2+a\lambda^2t^2+\lambda t-(\lambda t)^2$$

$$=\lambda t(a\lambda t+1)$$

Therefore

$$\mathrm{Var}\left[X(t)=n\right]=\lambda t(a\lambda t+1) \qquad \textbf{(6.13)}$$

## 6.3 Determining $p_n(t)$ by Laplace Method

## 6.4 Probability Generating Factor Method

### 6.4.1 Deriving $p_n(t)$ by PGF Method

**The initial conditions: When $X(0) = 0$.**

This implies that $p_0(0)=1, p_n(0)=0 \ \forall \ n \neq 0$ and $p_0(t)=0$.

The difference differential equations to be solved are **(6.1a)** and **(6.1b)**.

$$p_0'(t) = -\left(\frac{\lambda}{1+\lambda a t}\right) p_0(t)$$

$$p_n'(t) = -\lambda\left(\frac{1+an}{1+\lambda a t}\right) p_n(t) + \lambda\left(\frac{1+a(n-1)}{1+\lambda a t}\right) p_{n-1}(t), \quad n \geq 1$$

**Definitions**

Let

$$\left.\begin{aligned} G(s,t) &= \sum_{n=0}^{\infty} p_n(t)s^n = p_0(t) + \sum_{n=1}^{\infty} p_n(t)s^n \\ \frac{d}{dt}\left[G(s,t)\right] &= \sum_{n=0}^{\infty} p_n'(t)s^n = p_0'(t) + \sum_{n=1}^{\infty} p_n'(t)s^n \\ \frac{d}{ds}\left[G(s,t)\right] &= \sum_{n=0}^{\infty} n\,p_n(t)s^{n-1} = \frac{1}{s}\sum_{n=1}^{\infty} n\,p_n(t)s^n \end{aligned}\right\} \quad \textbf{(6.14)}$$

NB: Notice that $G(1, t)=1, G(0,t)=P_0(t)$ and $G(s,0) = 1$.

Take the equation **(6.1b)** multiply by $s^n$ and sum over n

$$\sum_{n=1}^{\infty} p_n'(t)\, s^n = -\frac{\lambda}{1+\lambda a t}\left[\sum_{n=1}^{\infty}(1+an)\, p_n(t)\, s^n - \sum_{n=1}^{\infty}\left(1+a(n-1)\right) p_{n-1}(t)\, s^n\right]$$

$$= -\frac{\lambda}{1+\lambda a t}\left[\sum_{n=1}^{\infty} p_n(t)\, s^n + a\sum_{n=1}^{\infty} n\, p_n(t)\, s^n - \sum_{n=1}^{\infty} p_{n-1}(t)\, s^n - a\sum_{n=1}^{\infty}(n-1)\, p_{n-1}(t)\, s^n\right]$$

$$= -\frac{\lambda}{1+\lambda a t}\left[\sum_{n=1}^{\infty} p_n(t)\, s^n + a\sum_{n=1}^{\infty} n\, p_n(t)\, s^n - s\sum_{n=1}^{\infty} p_{n-1}(t)\, s^{n-1} - as\sum_{n=1}^{\infty}(n-1)\, p_{n-1}(t)\, s^{n-1}\right]$$

Equivalently

$$\sum_{n=1}^{\infty} p_n'(t)\, s^n = -\frac{\lambda}{1+\lambda a t}\left[\sum_{n=1}^{\infty} p_n(t)\, s^n + as\sum_{n=1}^{\infty} np_n(t)\, s^{n-1} - s\sum_{n=1}^{\infty} p_{n-1}(t)\, s^{n-1} - as^2\sum_{n=2}^{\infty}(n-1)p_{n-1}(t)\, s^{n-2}\right] \quad \textbf{(6.15)}$$

Taking advantage of the definitions in **(6.14)** above, equation **(6.15)** above can now be written as

$$\frac{d}{dt}G(s,t) - p_0'(t) = -\frac{\lambda}{1+\lambda a t}\left[G(s,t) - p_0(t) + as\times\frac{d}{ds}G(s,t) - sG(s,t) - as^2\times\frac{d}{ds}G(s,t)\right]$$

$$= -\frac{\lambda}{1+\lambda a t}\left[(1-s)G(s,t) + as(1-s)\frac{d}{ds}G(s,t)\right] + \frac{\lambda p_0(t)}{1+\lambda a t}$$

$$\frac{d}{dt}G(s,t) = -\frac{\lambda}{1+\lambda a t}\left[(1-s)G(s,t) + as(1-s)\frac{d}{ds}G(s,t)\right] + \frac{\lambda p_0(t)}{1+\lambda a t} + p_0'(t)$$

From equation **(6.1a)**, $p_0'(t) = -\left(\frac{\lambda}{1+\lambda a t}\right)p_0(t)$ Therefore,

$$\frac{d}{dt}G(s,t) = -\frac{\lambda}{1+\lambda a t}\left[(1-s)G(s,t) + as(1-s)\frac{d}{ds}G(s,t)\right]$$

The above equation can also be written in the form

$$\frac{d}{dt}G(s,t) + \frac{as(1-s)\lambda}{1+\lambda a t}\frac{d}{ds}G(s,t) = -\frac{\lambda(1-s)}{1+\lambda a t}G(s,t) \qquad \textbf{(6.16)}$$

Solving this equation using Lagrange's method, the auxiliary equation is

$$\frac{dt}{1} = \frac{(1+\lambda a t)ds}{as(1-s)\lambda} = -\frac{(1+\lambda a t)}{\lambda(1-s)}\frac{dG(s,t)}{G(s,t)} \qquad \textbf{(6.17)}$$

Taking $\frac{dt}{1} = \frac{1+\lambda a t}{as(1-s)\lambda}ds \quad \Rightarrow \quad \frac{a\lambda\, dt}{1+\lambda a t} = \frac{ds}{s(1-s)}$ and integrating, we have

$$a\lambda\int\frac{1}{(1+\lambda a t)}dt = \int\frac{ds}{s(1-s)}$$

But from

$$\frac{1}{s(1-s)} = \frac{A}{s} + \frac{B}{1-s}$$

$$1 = A(1-s) + Bs$$

$$\Rightarrow$$

$$A = 1,\ B = 1$$

Thus

$$\frac{1}{s(1-s)} = \frac{1}{s} + \frac{1}{1-s}$$

Thus

$$a\lambda\int\frac{1}{(1+\lambda a t)}\,dt=\int\frac{ds}{s(1-s)}=\int\frac{1}{s}\,ds+\int\frac{1}{1-s}ds$$

$$\lambda a\times\frac{1}{\lambda a}\ \ln\left(1+\lambda a t\right)=\ln s-\ln\left(1-s\right)+c$$

$$\Rightarrow$$

$$\ln\left(1+\lambda a t\right)=\ln s-\ln\left(1-s\right)+c$$

$$\Rightarrow$$

$$\ln\left(1+\lambda a t\right)=\ln\frac{s}{1-s}+c$$

Taking the exponential of both sides,

$$1+\lambda a t=\frac{s}{1-s}\,.\,k$$

$$k=\frac{1-s}{s}\left(1+\lambda a t\right) \qquad \text{(i)}$$

Taking $\frac{(1+\lambda a t)ds}{as(1-s)\lambda}=-\frac{(1+\lambda a t)}{\lambda(1-s)}\ \frac{dG(s,t)}{G(s,t)}\qquad\Rightarrow\frac{ds}{as}=-\frac{d\,G(s,t)}{G(s,t)}$ and integrating, we have

$$\int\frac{ds}{as}=\int-\frac{d\,G(s,t)}{G(s,t)}$$

$$\frac{1}{a}\ln s=-\int d\left(\ln G(s,t)\right)$$

$$\frac{1}{a}\ln s+c=-\ln G(s,t)\ \text{ or }\ \ln G(s,t)=-\frac{1}{a}\ln s-c$$

$$\Rightarrow$$

$$G(s,t)=s^{-\frac{1}{a}}c^*$$

$$c^*=G(s,t)\,s^{\frac{1}{a}} \qquad \text{(ii)}$$

Integrating both sides

From equations (i) and (ii), we have

$$G(s,t)\times s^{\frac{1}{a}}=\psi\left(\frac{1-s}{s}\left(1+\lambda at\right)\right) \qquad \textbf{(6.18)}$$

Recall that $G(s,0) = P_0(0) = 1$ (from definitions)

$$G(s,0)\times s^{\frac{1}{a}} = \psi\left(\frac{1 - s}{s}\right)$$

$$\Rightarrow$$

$$s^{\frac{1}{a}} = \psi\left(\frac{1 - s}{s}\right)$$

Let $w = \dfrac{1 - s}{s} \Rightarrow sw = 1 - s \Rightarrow sw + s = 1 \Rightarrow s = \dfrac{1}{1 + w}$

$$\left(\frac{1}{1 + w}\right)^{\frac{1}{a}} = \psi(w)$$

From **(6.18)**

$$G(s,t)\times s^{\frac{1}{a}} = \psi\left(w\,(1 + \lambda at)\right)$$

$$\Rightarrow$$

$$G(s,t) = s^{-\frac{1}{a}}\left(\frac{1}{1 + w\,(1 + a\lambda t)}\right)^{\frac{1}{a}}$$

$$\Rightarrow$$

$$G(s,t) = s^{-\frac{1}{a}}\left(\frac{1}{1 + \left(\frac{1 - s}{s}\right)(1 + a\lambda t)}\right)^{\frac{1}{a}}$$

$$= s^{-\frac{1}{a}}\left(\frac{s}{s + (1 - s)(1 + a\lambda t)}\right)^{\frac{1}{a}}$$

$$= \left(\frac{1}{s + 1 + a\lambda t - s - a\lambda ts}\right)^{\frac{1}{a}}$$

$$= \left(\frac{1}{1 + a\lambda t - a\lambda ts}\right)^{\frac{1}{a}}$$

Therefore,

$$G(s,t) = \left( \frac{\frac{1}{1+a\lambda t}}{1 - \frac{a\lambda ts}{1+a\lambda t}} \right)^{\frac{1}{a}}$$

Let $p = \frac{1}{1+a\lambda t}$, $q = 1 - p = 1 - \frac{1}{1+a\lambda t} = \frac{1+a\lambda t - 1}{1+a\lambda t} = \frac{a\lambda t}{1+a\lambda t}$. Then

$$G(s,t) = \left[ \frac{p}{1-qs} \right]^{\frac{1}{a}} \text{ where } p = \frac{1}{1+a\lambda t} \text{ and } q = \frac{a\lambda t}{1+a\lambda t}. \quad \textbf{(6.19)}$$

$p_n(t) =$ Coefficient of $s^n$ in $G(s,t)$.

$$G(s,t) = \left[ \frac{p}{1-qs} \right]^{\frac{1}{a}} = p^{\frac{1}{a}} (1-qs)^{-\frac{1}{a}}$$

$$= p^{\frac{1}{a}} \sum_{n=0}^{\infty} \binom{-\frac{1}{a}}{n} (-qs)^n$$

$$= p^{\frac{1}{a}} \sum_{n=0}^{\infty} (-1)^n \binom{-\frac{1}{a}}{n} q^n s^n$$

$$= \sum_{n=0}^{\infty} \binom{\frac{1}{a} + n - 1}{n} p^{\frac{1}{a}} q^n s^n$$

Thus

$$\mathbf{p_n(t)} = \binom{\frac{1}{a} + n - 1}{n} \left( \frac{1}{1+a\lambda t} \right)^{\frac{1}{a}} \left( \frac{a\lambda t}{1+a\lambda t} \right)^n \qquad n = 0, 1, 2, 3, \ldots \text{ and} \quad \textbf{(6.20)}$$

Equivalently

$$p_n(t) = \binom{\frac{1}{a} + n - 1}{n} p^{\frac{1}{a}} q^n \quad n = 0, 1, 2, 3, \ldots \text{ where } p = \frac{1}{1+a\lambda t} \text{ and } q = \frac{a\lambda t}{1+a\lambda t}.$$

***Initial conditions: When $t = 0$, $X(0) = n_0$ where $n_0 \geq 1$.***

Take the equation **(6.1b)** multiply by $s^n$ and sum over n

$$\sum_{n=1}^{\infty} p_n'(t)\, s^n = -\frac{\lambda}{1+at}\left[\sum_{n=1}^{\infty}(1+an)\, p_n(t)\, s^n - \sum_{n=1}^{\infty}\left(1+a(n-1)\right) p_{n-1}(t)\, s^n\right]$$

$$= -\frac{\lambda}{1+at}\left[\sum_{n=1}^{\infty} p_n(t)\, s^n + a\sum_{n=1}^{\infty} n\, p_n(t)\, s^n - \sum_{n=1}^{\infty} p_{n-1}(t)\, s^n - a\sum_{n=1}^{\infty}(n-1)\, p_{n-1}(t)\, s^n\right]$$

$$= -\frac{\lambda}{1+at}\left[\sum_{n=1}^{\infty} p_n(t)\, s^n + a\sum_{n=1}^{\infty} n\, p_n(t)\, s^n - s\sum_{n=1}^{\infty} p_{n-1}(t)\, s^{n-1} - as\sum_{n=1}^{\infty}(n-1)\, p_{n-1}(t)\, s^{n-1}\right]$$

Thus,

$$\sum_{n=1}^{\infty} p_n'(t)\, s^n = -\frac{\lambda}{1+at}\left[\sum_{n=1}^{\infty} p_n(t) s^n + a\sum_{n=1}^{\infty} n\, p_n(t) s^n - s\sum_{n=1}^{\infty} p_{n-1}(t)\, s^{n-1} - as\sum_{n=1}^{\infty}(n-1)\, p_{n-1}(t)\, s^{n-1}\right] \quad \textbf{(6.21)}$$

**Definitions**

Let

$$\left.\begin{aligned}
G(s,t) &= \sum_{n=0}^{\infty} p_n(t) s^n = p_0(t) + \sum_{n=1}^{\infty} p_n(t) s^n \\
\frac{d}{dt}\left[G(s,t)\right] &= \sum_{n=0}^{\infty} p_n'(t) s^n = p_0'(t) + \sum_{n=1}^{\infty} p_n'(t) s^n \\
\frac{d}{ds}\left[G(s,t)\right] &= \sum_{n=0}^{\infty} n\, p_n(t) s^{n-1} = \frac{1}{s}\sum_{n=1}^{\infty} n\, p_n(t) s^n
\end{aligned}\right\} \quad \textbf{(6.22)}$$

Notice that $G(1, t) = 1$, $G(0,t) = p_0(t)$ and $p_{n_0}(t) = 1$, $p_n(t) = 0 \;\forall\; n \neq n_0$ and $p_0(t) = 0$.

Taking advantage of the definitions above, equation **(6.21)** can now be written as

$$\frac{d}{dt}G(s,t) = -\frac{\lambda}{1+at}\left[G(s,t) + as \times \frac{d}{ds}G(s,t) - s\, G(s,t) - as^2 \times \frac{d}{ds}G(s,t)\right]$$

$$= \frac{\lambda}{1+at}\left[(s-1)G(s,t) + as(s-1)\frac{d}{ds}G(s,t)\right]$$

The above equation can also be written in the form

$$\frac{d}{dt}G(s,t) - \frac{\lambda \times as(s-1)}{1+at}\frac{d}{ds}G(s,t) = \frac{\lambda(s-1)}{1+at}\, G(s,t) \quad \textbf{(6.23)}$$

Solving this equation using langranges formula, the auxiliary equation is

$$\frac{dt}{1} = \frac{(1+at)ds}{-\lambda as(s-1)} = \frac{(1+at)\;\; d\,G(s,t)}{\lambda(s-1)\, G(s,t)} \quad \textbf{(6.24)}$$

Taking $\frac{dt}{1} = \frac{(1 + a\,t)ds}{-\lambda a s(s - 1)}$ and integrating, we have

$$-\lambda a \int \frac{1}{(1 + a\,t)}\, dt = \int \frac{ds}{s(s - 1)}$$

$$\Rightarrow$$

$$-\lambda a \times \frac{1}{a} \ln\,(1 + a\,t) = \int \left( \frac{1}{s - 1} - \frac{1}{s} \right) ds = \ln\,(s - 1) - \ln\, s + c$$

$$\Rightarrow$$

$$-\lambda \ln\,(1 + a\,t) = \ln\,(s - 1) - \ln\, s + c$$

$$\Rightarrow$$

$$\ln\,(1 + a\,t)^{-\lambda} + \ln\, s - \ln\,(s - 1) = c$$

$$\Rightarrow$$

$$\ln \left[ (1 + at)^{-\lambda} \times \frac{s}{s - 1} \right] = c$$

Taking the exponential of both sides,

$$(1 + at)^{-\lambda} \times \frac{s}{s - 1} = c_1^* \qquad \text{(i)}$$

Simplifying the second pair, we have

$$\frac{(1 + a\,t)ds}{-\lambda a s(s - 1)} = \frac{(1 + a\,t) \quad d\,G(s,t)}{\lambda(s - 1)\,G(s,t)}$$

$$\Rightarrow$$

$$-\frac{ds}{a s} = \frac{d\,G(s,t)}{G(s,t)} = d\,\ln\,G(s,t)$$

Integrating both sides

$$-\frac{1}{a}\int \frac{1}{s} ds = \int d \ln\, G(s,t)$$

$$\Rightarrow$$

$$-\frac{1}{a} \ln\, s = \ln\, G(s,t) + c$$

$\Rightarrow$

$$\ln G(s,t) + \frac{1}{a}\ln s = c \text{ or}$$

$$\ln\left[G(s,t)\times s^{\frac{1}{a}}\right] = c$$

Taking the exponential of both sides,

$$G(s,t)\times s^{\frac{1}{a}} = c_2^* \qquad \text{(ii)}$$

From (i) and (ii)

$$G(s,t)\times s^{\frac{1}{a}} = f\left((1+at)^{-\lambda}\times\frac{s}{s-1}\right)$$

$\Rightarrow$

$$G(s,t) = s^{-\frac{1}{a}}\times f\left((1+at)^{-\lambda}\times\frac{s}{s-1}\right)$$

Let $w = (1+a\,t)^{-\lambda}\times\left(\frac{s}{s-1}\right)$

At $t = 0,\ w = \frac{s}{s-1}$

$$\therefore\ ws - w = s \text{ or } ws - s = w$$

$$\Rightarrow s = \frac{w}{w-1}$$

$$G(s,0) = s^{-\frac{1}{a}} f(w) = s^{n_0}$$

$\Rightarrow$

$$f(w) = s^{\frac{1}{a}}\times s^{n_0} = s^{n_0+\frac{1}{a}} = \left(\frac{w}{w-1}\right)^{n_0+\frac{1}{a}}$$

This implies that $G(s,t)$ is of the form $G(s,t) = s^{-\frac{1}{a}}\times\left(\frac{w}{w-1}\right)^{n_0+\frac{1}{a}}$ where $w = (1+a\,t)^{-\lambda}\left(\frac{s}{s-1}\right)$

$\Rightarrow$

$$G(s,t) = s^{-\frac{1}{a}} \times \left[ \frac{\frac{(1+at)^{-\lambda} s}{s-1}}{\frac{(1+at)^{-\lambda} s}{s-1} - 1} \right]^{n_0 + \frac{1}{a}}$$

$$= s^{-\frac{1}{a}} \times \left[ \frac{(1+at)^{-\lambda} s}{(1+at)^{-\lambda} \times s - s + 1} \right]^{n_0 + \frac{1}{a}}$$

$$= s^{-\frac{1}{a}} \times \left[ \frac{(1+at)^{-\lambda} s}{1 - s\left[1 - (1+at)^{-\lambda}\right]} \right]^{n_0 + \frac{1}{a}}$$

This can further be written as

$$G(s,t) = s^{-\frac{1}{a}} \times \left[ \frac{(1+at)^{-\lambda} s}{1 - s\left[1 - (1+at)^{-\lambda}\right]} \right]^{\frac{1}{a}} \times \left[ \frac{(1+at)^{-\lambda} s}{1 - s\left[1 - (1+at)^{-\lambda}\right]} \right]^{n_0}$$

$$= \left[ \frac{(1+at)^{-\lambda}}{1 - s\left[1 - (1+at)^{-\lambda}\right]} \right]^{\frac{1}{a}} \times \left[ \frac{(1+at)^{-\lambda} s}{1 - s\left[1 - (1+at)^{-\lambda}\right]} \right]^{n_0}$$

If we take $p = (1+at)^{-\lambda}$ and $q = 1 - (1+at)^{-\lambda}$, we have

$$G(s,t) = \left[ \frac{p}{1-qs} \right]^{\frac{1}{a}} \times \left[ \frac{ps}{1-qs} \right]^{n_0}$$

Equivalently,

$$G(s,t) = s^{-\frac{1}{a}} \left[ \frac{ps}{1-qs} \right]^{n_0 + \frac{1}{a}} \qquad \textbf{(6.25)}$$

Now, $p_n(t)$ is the coefficient of $s^n$ in the expansion of $G(s,t)$. Therefore,

$$G(s,t) = s^{-\frac{1}{a}} p^{n_0 + \frac{1}{a}} s^{n_0 + \frac{1}{a}} (1-qs)^{-(n_0 + \frac{1}{a})}$$

$$= p^{n_0 + \frac{1}{a}} s^{n_0} (1-qs)^{-(n_0 + \frac{1}{a})}$$

$$= p^{n_0 + \frac{1}{a}} s^{n_0} \sum_{k=0}^{\infty} \binom{-(n_0 + \frac{1}{a})}{k} (-qs)^k$$

$$G(s,t) = p^{n_0+\frac{1}{a}} s^{n_0} \sum_{k=0}^{\infty} \binom{(n_0 + \frac{1}{a}) + k - 1}{k} (qs)^k$$

$$= \sum_{k=0}^{\infty} p^{n_0+\frac{1}{a}} \binom{(n_0 + \frac{1}{a}) + k - 1}{k} q^k\ s^{n_0+k}$$

But $n = n_0 + k$ where k = 0, 1, 2, …. Then

$$p_n(t) = p_{n_0+k}(t) = p^{n_0+\frac{1}{a}} \binom{(n_0 + \frac{1}{a}) + k - 1}{k} q^k$$

$$= \binom{(n_0 + \frac{1}{a}) + k - 1}{k} p^{n_0+\frac{1}{a}} q^k$$

Substituting back the values of p and q, we have

$$\mathbf{p_{n_0+k}(t) = \binom{(n_0 + \frac{1}{a}) + k - 1}{k} \left(\frac{1}{1 + a\lambda t}\right)^{n_0+\frac{1}{a}} \left(1 - \frac{1}{1 + a\lambda t}\right)^{k}} \qquad \textbf{(6.26)}$$

This is a negative binomial distribution.

### 6.4.2 Mean and Variance Using Generating Function Method

**(i) Mean**

**Initial Conditions: When t = 0, X(0) = 0**

From **(6.19)**, $G(s,t) = \left[\frac{p}{1 - qs}\right]^{\frac{1}{a}}$ where $p = \frac{1}{1 + a\lambda t}$ and $q = \frac{a\lambda t}{1 + a\lambda t}$. Therefore,

$$E(X(t)) = \frac{d}{ds} G(s,t)\Big|_{s=1} = \frac{d}{ds}\left( p^{\frac{1}{a}} (1 - qs)^{-\frac{1}{a}} \right)\Big|_{s=1}$$

$$\frac{d}{ds}\left( p^{\frac{1}{a}} (1 - qs)^{-\frac{1}{a}} \right)\Big|_{s=1} = p^{\frac{1}{a}}\left[ -\frac{1}{a}(1 - qs)^{-\frac{1}{a}-1} . (-q) \right]$$

$$= p^{\frac{1}{a}}\ \frac{q}{a}\ (1 - qs)^{-\left(\frac{1}{a}+1\right)}$$

At s = 1

$$E\left[X(t)\right] = p^{\frac{1}{a}}\ \frac{q}{a}\ (1 - q)^{-\left(\frac{1}{a}+1\right)}$$

$$= p^{\frac{1}{a}}\ \frac{q}{a}\ p^{-\left(\frac{1}{a}+1\right)}$$

$$= \frac{1}{a} \cdot \frac{q}{p}$$

$$= \frac{1}{a} \cdot \left\{ \frac{a\lambda t}{1 + a\lambda t} \Big/ \frac{1}{1 + a\lambda t} \right\}$$

Therefore,

$$E\left[X(t)\right] = \frac{1}{a} \cdot \left\{ \frac{a\lambda t}{1 + a\lambda t} \cdot \frac{1 + a\lambda t}{1} \right\}$$

$$= \frac{1}{a} \cdot a\lambda t$$

$$= \lambda t \qquad \textbf{(6.27)}$$

**(ii) Variance**

$$\text{Var}\,X = G''(1,t) + G'(1,t) - \left[G'(1,t)\right]^2$$

Recall

$$\frac{d}{ds}\left( p^{\frac{1}{a}} (1 - qs)^{-\frac{1}{a}} \right)\Bigg|_{s=1} = p^{\frac{1}{a}}\left[ -\frac{1}{a}(1 - qs)^{-\frac{1}{a}-1} . (-q) \right]$$

$$\Rightarrow$$

$$\frac{d^2}{ds^2}G(s,t) = \frac{d}{ds}\left[ p^{\frac{1}{a}}\ \frac{q}{a}\ (1 - qs)^{-\left(\frac{1}{a}+1\right)} \right]$$

$$= \frac{p^{\frac{1}{a}}q}{a}\left( -\left(1 + \frac{1}{a}\right)(1 - qs)^{-1-\frac{1}{a}-1}(-q) \right)$$

$$= \frac{p^{\frac{1}{a}}q^2}{a}\left(1 + \frac{1}{a}\right)(1 - qs)^{-\left(2+\frac{1}{a}\right)}$$

At s = 1

$$\frac{d^2}{ds^2}G(s,t)\bigg|_{s=1} = \frac{p^{\frac{1}{a}}q^2}{a}\left(1+\frac{1}{a}\right)(1-q)^{-\left(2+\frac{1}{a}\right)}$$

$$= \frac{p^{\frac{1}{a}}q^2}{a}\left(1+\frac{1}{a}\right)(1-q)^{-\left(2+\frac{1}{a}\right)}$$

$$= \frac{p^{\frac{1}{a}}q^2}{a}\left(\frac{a+1}{a}\right)p^{-\left(2+\frac{1}{a}\right)}$$

$$= \left(\frac{a+1}{a^2}\right)\frac{q^2}{p^2}$$

$$= (a+1)\,.\,\frac{1}{a^2}\left(\frac{a\lambda t}{1+a\lambda t}\right)^2.\left(\frac{1+a\lambda t}{1}\right)^2$$

$$= (a+1)\,.\,(\lambda t)^2$$

$$\mathrm{Var}(X(t)) = (a+1)\,.\,(\lambda t)^2 + \lambda t - [\lambda t]^2$$

$$= a(\lambda t)^2 + (\lambda t)^2 + \lambda t - [\lambda t]^2$$

$$= a(\lambda t)^2 + \lambda t$$

$$= \lambda t(a\lambda t + 1)$$

$$= \lambda t(1 + a\lambda t) \qquad \textbf{(6.28)}$$

## 6.5 Method of Moments to Determine Mean and Variance

### 6.5.1 Mean

**Initial Condition: when t = 0, X(t) = $n_0$**

Definitions

$$M_1(t) = \sum_{n=1}^{\infty} n\,p_n(t) \;\Rightarrow\; M_1'(t) = \sum_{n=1}^{\infty} n\,p_n'(t) \qquad \textbf{(6.29)}$$

$$M_2(t) = \sum_{n=1}^{\infty} n^2\,p_n(t) \;\Rightarrow\; M_2'(t) = \sum_{n=1}^{\infty} n^2\,p_n'(t) \qquad \textbf{(6.30)}$$

$$M_3(t) = \sum_{n=1}^{\infty} n^3\,p_n(t) \;\Rightarrow\; M_3'(t) = \sum_{n=1}^{\infty} n^3\,p_n'(t) \qquad \textbf{(6.31)}$$

Multiply the Polya basic difference – differential equations by n and then sum the results over n.

$$\sum_{n=1}^{\infty} n\, p_n'(t) = \frac{-\lambda}{1+\lambda a t}\sum_{n=1}^{\infty}(1+a n)\, n\, p_n(t) + \frac{\lambda}{1+\lambda a t}\sum_{n=1}^{\infty}\left[1+a(n-1)\right] n\, p_{n-1}(t) \quad \textbf{(6.32)}$$

Substituting the definitions **(6.29)** and **(6.30)** above in equation **(6.31)**, we have

$$M_1'(t) = \left[\frac{-\lambda}{1+\lambda a t}\right]\left\{M_1(t) + a M_2(t)\right\} + \left[\frac{\lambda}{1+\lambda a t}\right]\left\{\sum_{n=1}^{\infty} n\, p_{n-1}(t) + a\sum_{n=1}^{\infty}(n-1)\, n\, p_{n-1}(t)\right\}$$

$$= \frac{\lambda}{1+\lambda a t}\left\{-M_1(t) - a M_2(t) + \sum_{n=1}^{\infty}(n-1+1)p_{n-1}(t) + a\sum_{n=1}^{\infty}(n-1)(n-1+1)p_{n-1}(t)\right\}$$

$$= \frac{\lambda}{1+\lambda a t}\left\{-M_1(t) - a M_2(t) + \sum_{n=1}^{\infty}(n-1)p_{n-1}(t) + \sum_{n=1}^{\infty}p_{n-1}(t) + a\sum_{n=1}^{\infty}(n-1)^2 p_{n-1}(t)\right.$$

$$\left. + a\sum_{n=1}^{\infty}(n-1)p_{n-1}(t)\right\}$$

$$= \frac{\lambda}{1+\lambda a t}\left\{-M_1(t) - a M_2(t) + M_1(t) + 1 + a M_2(t) + a M_1(t)\right\}$$

$$= \frac{\lambda}{1+\lambda a t}\left\{1 + a M_1(t)\right\}$$

Therefore,

$$M_1'(t) = \frac{\lambda}{1+\lambda a t}\left\{1 + a M_1(t)\right\}$$

Equivalently,

$$M_1'(t) - \frac{a\lambda}{1+\lambda a t} M_1(t) = \frac{\lambda}{1+\lambda a t} \qquad \textbf{(6.33)}$$

$$\text{Integrating factor} = e^{\int -\frac{a\lambda}{1+\lambda at}dt}$$

$$\text{Let } u = 1+\lambda at \;\Rightarrow\; \frac{du}{dt} = a\lambda \;\Rightarrow\; dt = \frac{du}{a\lambda}$$

$$\int -\frac{a\lambda}{1+\lambda at}dt = -\int \frac{a\lambda}{u}\times\frac{du}{a\lambda} = -\int \frac{1}{u}du = -\ln u = \ln(1+\lambda at)^{-1}$$

$$\text{Integrating factor} = e^{\ln(1+\lambda at)^{-1}} = (1+\lambda at)^{-1} = \frac{1}{1+\lambda at}$$

Multiplying equation **(6.33)** by the integrating factor, we have

$$\frac{1}{1+\lambda at}\times M_1'(t)-\frac{1}{1+\lambda at}\times\frac{a\lambda}{1+\lambda a t}\,M_1(t)=\frac{1}{1+\lambda at}\times\frac{\lambda}{1+\lambda at}$$

$\Rightarrow$

$$\frac{1}{1+\lambda at}\times M_1'(t)-\frac{a\lambda}{(1+\lambda at)^2}\,M_1(t)=\frac{\lambda}{(1+\lambda a t)^2}$$

Equivalently,

$$\frac{d}{dt}\left\{\frac{1}{1+\lambda at}\times M_1(t)\right\}=\frac{\lambda}{(1+\lambda at)^2} \qquad \textbf{(6.34)}$$

Integrating both sides of equation **(6.34)** with respect to t, we have

$$\frac{1}{1+\lambda at}\times M_1(t)=\int\frac{\lambda}{(1+\lambda at)^2}dt \qquad \textbf{(6.35)}$$

Let $u=1+\lambda at \quad\Rightarrow\quad \frac{du}{dt}=a\lambda \quad\Rightarrow\quad dt=\frac{du}{a\lambda}$. Therefore

$$\int\frac{\lambda}{(1+\lambda at)^2}dt=\lambda\int\frac{1}{u^2}\times\frac{du}{a\lambda}=\frac{1}{a}\int\frac{1}{u^2}du=\frac{1}{a}\int u^{-2}du=\frac{1}{a}\times\frac{u^{-2+1}}{-2+1}+c=-\frac{1}{a}\times u^{-1}+c$$

Therefore, equation **(6.35)** becomes

$$\frac{M_1(t)}{1+a\lambda t}=-\frac{1}{a(1+a\lambda t)}+c=-\frac{1}{a}(1+a\lambda t)^{-1}+c$$

Equivalently

$$\frac{M_1(t)}{1+a\lambda t}=-\frac{1}{a}\left(\frac{1}{1+a\lambda t}\right)+c \qquad \textbf{(6.36)}$$

Assume that at $t=0,\ X(0)=n_0 \quad\Rightarrow\quad p_{n_0}(0)=1$

When $t=0$, we have

$$M_1(0)=-\frac{1}{a}+c$$

And by definition

$$M_1(0)=\sum_{n=1}^{\infty}n\,p_n(0)=0\times p_0(0)+1\times p_1(0)+2p_2(0)+\ldots+n_0p_{n_0}(0)+\ldots=n_0$$

Therefore,

$$n_0=-\frac{1}{a}+c \quad\Rightarrow\quad c=n_0+\frac{1}{a}$$

Equation **(6.36)** now becomes

$$\frac{M_1(t)}{1 + a\lambda t} = -\frac{1}{a}\left(\frac{1}{1 + a\lambda t}\right) + \left(n_0 + \frac{1}{a}\right)$$

$$\Rightarrow$$

$$M_1(t) = -\frac{1}{a} + \left(n_0 + \frac{1}{a}\right)(1 + a\lambda t)$$

$$= -\frac{1}{a} + \left\{n_0 + n_0\lambda at + \frac{1}{a} + \lambda t\right\}$$

$$= -\frac{1}{a} + n_0 + n_0\lambda at + \frac{1}{a} + \lambda t$$

$$= n_0 + n_0\lambda at + \lambda t$$

$$= n_0(1 + \lambda at) + \lambda t$$

Therefore

$$M_1(t) = n_0(1 + \lambda at) + \lambda t \quad \Rightarrow E\left[X(t)\right] = n_0(1 + \lambda at) + \lambda t \tag{6.37}$$

Special Case

When $n_0 = 0$, $M_1(t) = \lambda t$.

### 6.5.2 Variance

Next, multiply the basic difference differential equations by $n^2$ and sum the results over n.

$$\sum_{n=1}^{\infty} n^2 p_n'(t) = -\frac{\lambda}{1 + \lambda at}\sum_{n=1}^{\infty}(1 + an)n^2 p_n(t) + \frac{\lambda}{1 + \lambda at}\sum_{n=1}^{\infty}\left[1 + a(n - 1)\right]n^2 p_{n-1}(t)$$

Equivalently,

$$M_2'(t) = -\frac{\lambda}{1 + \lambda at}\left\{M_2(t) + aM_3(t)\right\} + \frac{\lambda}{1 + \lambda at}\left\{\sum_{n=1}^{\infty} n^2 p_{n-1}(t) + a\sum_{n=1}^{\infty}(n - 1)n^2 p_{n-1}(t)\right\}$$

$$= -\frac{\lambda}{1 + \lambda at}\left\{M_2(t) + aM_3(t)\right\} + \frac{\lambda}{1 + \lambda at}\left\{\sum_{n=1}^{\infty}(n-1+1)^2 p_{n-1}(t) + a\sum_{n=1}^{\infty}(n-1)(n-1+1)^2 p_{n-1}(t)\right\}$$

$$= -\frac{\lambda}{1 + \lambda at}\left\{M_2(t) + aM_3(t)\right\} + \frac{\lambda}{1 + \lambda at}\left\{M_2(t) + 2M_1(t) + 1 + aM_3(t) + 2aM_2(t) + aM_1(t)\right\}$$

$$= \frac{\lambda}{1 + \lambda at}\left\{(2 + a)M_1(t) + 1 + 2aM_2(t)\right\}$$

Therefore

$$M_2'(t) = \frac{\lambda}{1 + \lambda at}\left\{(2 + a)M_1(t) + 1 + 2a\,M_2(t)\right\}$$

**Case 1:** Restricting ourselves to $t = 0,\ X(0) = 0$ and therefore $M_1(t) = \lambda t.$

Therefore,

$$M_2'(t) - \frac{2a\lambda}{1 + \lambda at}M_2(t) = \frac{\lambda}{1 + \lambda at}\left\{(2 + a)M_1(t) + 1\right\}$$

$$= \frac{\lambda}{1 + \lambda at}\left\{(2 + a)\lambda t + 1\right\}$$

Equivalently,

$$M_2'(t) - \frac{2a\lambda}{1 + \lambda at}M_2(t) = \frac{\lambda}{1 + \lambda at}\left\{(2 + a)\lambda t + 1\right\} \qquad \textbf{(6.38)}$$

Let

$$I = e^{-\int \frac{2a\lambda}{1+\lambda at}dt} = e^{-2\int \frac{d}{dt}\ln(1+\lambda at)dt} = e^{-2\ln(1+\lambda at)} = e^{\ln(1+\lambda at)^{-2}}$$

Therefore,

$$I = (1 + \lambda at)^{-2}$$

Multiplying equation **(6.38)** by the integrating factor, we have

$$(1 + \lambda at)^{-2}M_2'(t) - \frac{2a\lambda}{(1 + \lambda at)^3}M_2(t) = \frac{\lambda}{(1 + \lambda at)^3}\left\{(2 + a)\lambda t + 1\right\}$$

$$\Rightarrow$$

$$\frac{d}{dt}\left[(1 + \lambda at)^{-2}M_2(t)\right] = \frac{\lambda}{(1 + \lambda at)^3}\left\{(2 + a)\lambda t + 1\right\}$$

Integrating both sides with respect to t, we have

$$(1 + \lambda at)^{-2}M_2(t) = (2 + a)\lambda^2\int \frac{t}{(1 + \lambda at)^3}dt + \lambda\int \frac{dt}{(1 + \lambda at)^3}$$

Let $u = (1 + \lambda at) \ \Rightarrow \ du = \lambda a\ dt$ and $u - 1 = \lambda at \ \Rightarrow \ t = \dfrac{u-1}{\lambda a}$

Therefore

$$(1 + \lambda at)^{-2}M_2(t) = (2 + a)\lambda^2\int \frac{u - 1}{\lambda a\,u^3}\times\frac{du}{\lambda a} + \lambda\int \frac{1}{u^3}\times\frac{dt}{\lambda a} + c$$

$$= \frac{(2 + a)}{a^2}\int\left[\frac{1}{u^2} - \frac{1}{u^3}\right]du + \frac{1}{a}\int \frac{1}{u^3} + c$$

$$(1+\lambda at)^{-2} M_2(t) = \frac{(2+a)}{a^2}\left[\frac{u^{-2+1}}{-1} - \frac{u^{-3+1}}{-2}\right] + \frac{1}{a}\left[\frac{u^{-3+1}}{-2}\right] + c$$

$$= \frac{(2+a)}{a^2}\left[-\frac{1}{u} - \frac{1}{2u^2}\right] - \frac{1}{2au^2} + c$$

$$= \left(\frac{2}{a^2} + \frac{1}{a}\right)\left(-\frac{1}{u} - \frac{1}{2u^2}\right) - \frac{1}{2au^2} + c$$

$$= -\frac{2}{a^2u} + \frac{1}{a^2u^2} - \frac{1}{au} + \frac{1}{2au^2} - \frac{1}{2au^2} + c$$

$$= -\frac{2}{a^2u} + \frac{1}{a^2u^2} - \frac{1}{au} + c$$

Therefore,

$$M_2(t) = (1+\lambda at)^2\left[-\frac{2}{a^2u} + \frac{1}{a^2u^2} - \frac{1}{au}\right] + c(1+\lambda at)^2$$

$$= u^2\left[-\frac{2}{a^2u} + \frac{1}{a^2u^2} - \frac{1}{au}\right] + cu^2$$

$$= -\frac{2u}{a^2} + \frac{1}{a^2} - \frac{u}{a} + cu^2$$

$$= \left(\frac{-2u + 1 - ua}{a^2}\right) + cu^2$$

$$= \frac{1 - u(2+a)}{a^2} + cu^2$$

$$= \frac{1 - (1+\lambda at)(2+a)}{a^2} + c(1+\lambda at)^2$$

$$= \frac{1 - \left[2 + a + 2\lambda at + \lambda a^2 t\right]}{a^2} + c(1+\lambda at)^2$$

$$= \frac{1 - 2 - a - 2\lambda at - \lambda a^2 t}{a^2} + c(1+\lambda at)^2$$

$$= -\frac{2u}{a^2} + \frac{1}{a^2} - \frac{u}{a} + cu^2$$

$$= \left(\frac{-2u + 1 - ua}{a^2}\right) + cu^2$$

$$M_2(t) = \frac{1 - u(2 + a)}{a^2} + cu^2$$

$$= \frac{1 - (1 + \lambda at)(2 + a)}{a^2} + c(1 + \lambda at)^2$$

$$= \frac{1 - \left[2 + a + 2\lambda at + \lambda a^2 t\right]}{a^2} + c(1 + \lambda at)^2$$

$$= \frac{1 - 2 - a - 2\lambda at - \lambda a^2 t}{a^2} + c(1 + \lambda at)^2$$

$$= c(1 + \lambda at)^2 - \frac{(1 + a) - \lambda at(2 + a)}{a^2}$$

$$= c(1 + \lambda at)^2 - \frac{(1 + a) - \lambda at(1+1+a)}{a^2}$$

$$= c(1 + \lambda at)^2 - \frac{(1 + a) - \lambda at - \lambda at(1+a)}{a^2}$$

$$= c(1 + \lambda at)^2 - \left[\frac{(1 + a) + \lambda at(1 + a)}{a^2}\right] + \frac{\lambda t}{a}$$

Therefore,

$$M_2(t) = c(1 + \lambda at)^2 - \frac{(1 + a)(1 + \lambda at)}{a^2} + \frac{\lambda t}{a} \qquad \textbf{(6.39)}$$

Substituting $t = 0$ in equation **(6.39)** we have

$$M_2(0) = c - \frac{(1 + a)}{a^2}$$

From definition,

$M_2(0) = \sum_{n=1}^{\infty} n^2 p_n(0) = 0$ since $p_0(0) = 1$ and $p_n(0) = 0$ for $n \neq 0$. Therefore

$$0 = c - \left(\frac{1 + a}{a^2}\right)$$

Therefore

$$c = \frac{1 + a}{a^2}$$

Equation **(6.32)** now becomes

$$M_2(t) = \frac{(1 + a)}{a^2}(1 + \lambda at)^2 - \frac{(1 + a)(1 + \lambda at)}{a^2} + \frac{\lambda t}{a} \qquad \textbf{(6.40)}$$

Simplifying equation **(6.40)**, we have

$$M_2(t) = \frac{(1 + a)(1 + \lambda at)}{a^2}(1 + \lambda at - 1) + \frac{\lambda t}{a}$$

$$= \frac{(1 + a)(1 + \lambda at)(\lambda at)}{a^2} - \frac{\lambda t}{a} *$$

$$= \frac{(1 + a)(1 + \lambda at)(\lambda t)}{a} - \frac{\lambda t}{a} a$$

$$= \frac{\lambda t}{a}\left[(1 + a)(1 + \lambda at) - 1\right]$$

$$= \frac{\lambda t}{a}\left[1 + \lambda at + a + \lambda a^2 t - 1\right]$$

Therefore,

$$M_2(t) = \frac{\lambda t}{a}\left[\lambda at + a + \lambda a^2 t\right]$$

$$= \lambda t\left[\lambda t + 1 + \lambda at\right] \qquad \textbf{(6.41)}$$

Therefore

$$\text{Variance} = M_2(t) - \left[M_1(t)\right]^2$$

$$= \lambda t\left[\lambda t + 1 + \lambda at\right] - (\lambda t)^2$$

$$= \lambda t\left[\lambda t + 1 + \lambda at - \lambda t\right]$$

$$= \lambda t\left[1 + \lambda at\right] \qquad \textbf{(6.42)}$$

# CHAPTER SEVEN

# GENERAL PURE BIRTH PROCESS

## 7.1 Introduction

The objective in this topic is to solve the basic difference differential equations when using a matrix method. We shall solve three processes using this method. i.e. Pure Birth Process, the Pure birth Process with Immigration and the Polya Process.

From Chapter 2, we derived the two basic difference differential equations below

$$p_0'(t) = -\lambda_0 \, p_0(t) \qquad \textbf{(7.1a)}$$

$$p_n'(t) = \lambda_{n-1} \, p_{n-1}(t) - \lambda_n \, p_n(t), \quad n \geq 1 \qquad \textbf{(7.1b)}$$

If we substitute for n (with n = 1, 2, 3, ..) in equation **(7.1b)**, we would have

$$p_0'(t) = -\lambda_0 \, p_0(t)$$

$$p_1'(t) = \lambda_0 \, p_0(t) - \lambda_1 \, p_1(t)$$

$$p_2'(t) = \lambda_1 \, p_1(t) - \lambda_2 \, p_2(t)$$

$$p_3'(t) = \lambda_2 \, p_2(t) - \lambda_3 \, p_3(t)$$

.

.

$$p_n'(t) = \lambda_{n-1} \, p_{n-1}(t) - \lambda_n \, p_n(t)$$

.

Expressing the above equations in matrix form

$$\begin{bmatrix} p_0'(t) \\ p_1'(t) \\ p_2'(t) \\ p_3'(t) \\ . \\ . \\ p_n'(t) \\ . \end{bmatrix} = \begin{bmatrix} -\lambda_0 & 0 & 0 & 0 & .. & 0 & 0 & .. \\ \lambda_0 & -\lambda_1 & 0 & 0 & .. & 0 & 0 & .. \\ 0 & \lambda_1 & -\lambda_2 & 0 & .. & 0 & 0 & .. \\ 0 & 0 & \lambda_2 & -\lambda_3 & .. & 0 & 0 & .. \\ . & . & . & . & .. & . & . & .. \\ . & . & . & . & .. & . & . & .. \\ 0 & 0 & 0 & 0 & .. & \lambda_{n-1} & -\lambda_n & .. \\ . & . & . & . & .. & . & . & .. \end{bmatrix} \times \begin{bmatrix} p_0(t) \\ p_1(t) \\ p_2(t) \\ p_3(t) \\ . \\ . \\ p_n(t) \\ . \end{bmatrix} \qquad \textbf{(7.2)}$$

We shall now solve this Matrix equation.

### 7.2 Generator Matrix Method

We wish to solve the above matrix equation **(7.2)**.

$$p'(t) = \Lambda\ P(t)$$

$$\frac{p'(t)}{p(t)} = \Lambda$$

$$\frac{d}{dt}\left(\ln\ p(t)\right) = \Lambda$$

Integrating both sides

$$\ln\ p(t) = \Lambda t + c$$

$$p(t) = k e^{\Lambda t}$$

The initial condition; When $t = 0,\ p(0) = I$. This implies that $k = 1$. Therefore

$$p(t) = e^{\Lambda t} \qquad \textbf{(7.3)}$$

To solve this kind of equations, we use the formulae

$$p(t) = R\ e^{(\mathrm{diag}\ \Lambda)t}\ R^{-1}\ p(0)$$

Where R is the matrix of eigen vectors, $e^{(\mathrm{diag}\ \Lambda)t}$ is the diagonal matrix and P(0) is the initial value column Matrix of $p_{n+i}(t)$

Using eigen values and eigen vectors of $\Lambda$.

$$\begin{vmatrix} -\lambda_0 - \lambda & 0 & 0 & 0 & .. & 0 & 0 & .. \\ \lambda_0 & -\lambda_1 - \lambda & 0 & 0 & .. & 0 & 0 & .. \\ 0 & \lambda_1 & -\lambda_2 - \lambda & 0 & .. & 0 & 0 & .. \\ 0 & 0 & \lambda_2 & -\lambda_3 - \lambda & .. & 0 & 0 & .. \\ . & . & . & . & .. & . & . & .. \\ . & . & . & . & .. & . & . & .. \\ 0 & 0 & 0 & 0 & .. & \lambda_{n-1} & -\lambda_n - \lambda & .. \\ . & . & . & . & .. & 0 & 0 & .. \end{vmatrix} = 0$$

Restricting ourselves up to *n* only,

$$(-1)^{n+1}(\lambda_0 + \lambda)(\lambda_1 + \lambda)(\lambda_2 + \lambda)......(\lambda_n + \lambda) = 0$$

Dividing both sides by $(-1)^{n+1}$

$$(\lambda_0 + \lambda)(\lambda_1 + \lambda)(\lambda_2 + \lambda)......(\lambda_n + \lambda) = 0$$

Solving for $\lambda$, we get

$$\lambda = -\lambda_0 \text{ or } \lambda = -\lambda_1 \text{ or } \lambda = -\lambda_2 \text{ or .... or } \lambda = -\lambda_n$$

Let

$$z = \begin{bmatrix} z_0 \\ z_1 \\ z_2 \\ . \\ . \\ z_n \end{bmatrix}$$

Eigen vector $, e_0,$ for eigen value $\lambda = -\lambda_0$

$$\begin{bmatrix} -\lambda_0 & 0 & 0 & 0 & .. & 0 & 0 \\ \lambda_0 & -\lambda_1 & 0 & 0 & .. & 0 & 0 \\ 0 & \lambda_1 & -\lambda_2 & 0 & .. & 0 & 0 \\ 0 & 0 & \lambda_2 & -\lambda_3 & .. & 0 & 0 \\ . & . & . & . & .. & . & . \\ . & . & . & . & .. & . & . \\ 0 & 0 & 0 & 0 & .. & \lambda_{n-1} & -\lambda_n \end{bmatrix} \times \begin{bmatrix} z_0 \\ z_1 \\ z_2 \\ z_3 \\ . \\ . \\ z_n \end{bmatrix} = -\lambda_0 \begin{bmatrix} z_0 \\ z_1 \\ z_2 \\ z_3 \\ . \\ . \\ z_n \end{bmatrix}$$

Extracting equations from the matrix equation above, we have

$$-\lambda_0 z_0 = -\lambda_0 z_0 \qquad (\text{trivial})$$

$$\lambda_0 z_0 - \lambda_1 z_1 = -\lambda_0 z_1$$

$$\Rightarrow \qquad \lambda_0 z_0 = (\lambda_1 - \lambda_0) z_1$$

$$\Rightarrow \qquad z_1 = \frac{\lambda_0}{\lambda_1 - \lambda_0} z_0$$

Let $z_0 = 1$ $\qquad (i)$

$$\Rightarrow \qquad z_1 = \frac{\lambda_0}{\lambda_1 - \lambda_0} \qquad (ii)$$

$$\lambda_1 z_1 - \lambda_2 z_2 = -\lambda_0 z_2$$

$$\Rightarrow \qquad z_2 = \frac{\lambda_1}{\lambda_2 - \lambda_0} z_1 = \frac{\lambda_0}{\lambda_1 - \lambda_0} \times \frac{\lambda_1}{\lambda_2 - \lambda_0} \qquad (iii)$$

$$\lambda_2 z_2 - \lambda_3 z_3 = -\lambda_0 z_3$$

$$\Rightarrow \qquad z_3 = \frac{\lambda_2}{\lambda_3 - \lambda_0} z_2 = \frac{\lambda_0}{\lambda_1 - \lambda_0} \times \frac{\lambda_1}{\lambda_2 - \lambda_0} \times \frac{\lambda_2}{\lambda_3 - \lambda_0} \qquad (iv)$$

$$\lambda_{n-1} z_{n-1} - \lambda_n z_n = -\lambda_0 z_n$$

$$\Rightarrow \qquad z_n = \frac{\lambda_{n-1}}{\lambda_n - \lambda_0} z_{n-1} = \frac{\lambda_0}{\lambda_1 - \lambda_0} \times \frac{\lambda_1}{\lambda_2 - \lambda_0} \times \frac{\lambda_2}{\lambda_3 - \lambda_0} \times \ldots \times \frac{\lambda_{n-1}}{\lambda_n - \lambda_0}$$

$$\Rightarrow \qquad z_n = \prod_{k=1}^{n} \left( \frac{\lambda_{k-1}}{\lambda_k - \lambda_0} \right) \qquad (v)$$

Thus

$$e_0 = \begin{bmatrix} 1 \\ \frac{\lambda_0}{\lambda_1 - \lambda_0} \\ \frac{\lambda_0}{\lambda_1 - \lambda_0} . \frac{\lambda_1}{\lambda_2 - \lambda_0} \\ \frac{\lambda_0}{\lambda_1 - \lambda_0} . \frac{\lambda_1}{\lambda_2 - \lambda_0} . \frac{\lambda_2}{\lambda_3 - \lambda_0} \\ . \\ . \\ \prod_{k=1}^{n} \left( \frac{\lambda_{k-1}}{\lambda_k - \lambda_0} \right) \end{bmatrix}$$

Eigen vector $e_1$ for eigen value $\lambda = -\lambda_1$

$$\begin{bmatrix} -\lambda_0 & 0 & 0 & 0 & .. & 0 & 0 \\ \lambda_0 & -\lambda_1 & 0 & 0 & .. & 0 & 0 \\ 0 & \lambda_1 & -\lambda_2 & 0 & .. & 0 & 0 \\ 0 & 0 & \lambda_2 & -\lambda_3 & .. & 0 & 0 \\ . & . & . & . & .. & . & . \\ . & . & . & . & .. & . & . \\ 0 & 0 & 0 & 0 & .. & \lambda_{n-1} & -\lambda_n \end{bmatrix} \times \begin{bmatrix} z_0 \\ z_1 \\ z_2 \\ z_3 \\ . \\ . \\ z_n \end{bmatrix} = -\lambda_1 \begin{bmatrix} z_0 \\ z_1 \\ z_2 \\ z_3 \\ . \\ . \\ z_n \end{bmatrix}$$

$$-\lambda_0 z_0 = -\lambda_1 z_0$$

$$\Rightarrow \quad z_0 = 0$$

$$\lambda_0 z_0 - \lambda_1 z_1 = -\lambda_1 z_1$$

$$\Rightarrow \quad z_0 = 0 \qquad (vi)$$

$$\lambda_1 z_1 - \lambda_2 z_2 = -\lambda_1 z_2$$

$$\Rightarrow \quad z_2 = \frac{\lambda_1}{\lambda_2 - \lambda_1} z_1$$

$$\text{Let } z_1 = 1 \qquad (vii)$$

$$\Rightarrow \quad z_2 = \frac{\lambda_1}{\lambda_2 - \lambda_1} \qquad (viii)$$

$$\lambda_2 z_2 - \lambda_3 z_3 = -\lambda_1 z_3$$

$$\Rightarrow \quad z_3 = \frac{\lambda_2}{\lambda_3 - \lambda_1} z_2 = \frac{\lambda_2}{\lambda_3 - \lambda_1} \times \frac{\lambda_1}{\lambda_2 - \lambda_1}$$

$$\Rightarrow \quad z_3 = \frac{\lambda_1}{\lambda_2 - \lambda_1} \times \frac{\lambda_2}{\lambda_3 - \lambda_1} \qquad (ix)$$

$$\lambda_3 z_3 - \lambda_4 z_4 = -\lambda_1 z_4$$

$$\Rightarrow \quad z_4 = \frac{\lambda_3}{\lambda_4 - \lambda_1} z_3 = \frac{\lambda_3}{\lambda_4 - \lambda_1} \times \frac{\lambda_1}{\lambda_2 - \lambda_1} \times \frac{\lambda_2}{\lambda_3 - \lambda_1}$$

$$\Rightarrow \quad z_4 = \frac{\lambda_1}{\lambda_2 - \lambda_1} \times \frac{\lambda_2}{\lambda_3 - \lambda_1} \times \frac{\lambda_3}{\lambda_4 - \lambda_1} \qquad (x)$$

$$.$$
$$.$$
$$\lambda_{n-1} z_{n-1} - \lambda_n z_n = -\lambda_1 z_n$$

$$\Rightarrow \quad z_n = \frac{\lambda_{n-1}}{\lambda_n - \lambda_1} z_{n-1} = \frac{\lambda_1}{\lambda_2 - \lambda_1} \times \frac{\lambda_2}{\lambda_3 - \lambda_1} \times \frac{\lambda_3}{\lambda_4 - \lambda_1} \times \ldots \times \frac{\lambda_{n-1}}{\lambda_n - \lambda_1}$$

$$\Rightarrow \quad z_n = \prod_{k=2}^{n} \left( \frac{\lambda_{k-1}}{\lambda_k - \lambda_1} \right) \qquad (xi)$$

Thus

$$e_1 = \begin{bmatrix} 0 \\ 1 \\ \frac{\lambda_1}{\lambda_2 - \lambda_1} \\ \frac{\lambda_1}{\lambda_2 - \lambda_1} . \frac{\lambda_2}{\lambda_3 - \lambda_1} \\ \prod_{k=2}^{4} \frac{\lambda_{k-1}}{\lambda_k - \lambda_1} \\ . \\ . \\ \prod_{k=2}^{4} \frac{\lambda_{k-1}}{\lambda_k - \lambda_1} \end{bmatrix}$$

$$R = \left( e_0 \quad e_1 \quad e_2 \quad . \quad . \quad e_n \right)$$

$$
R = \begin{bmatrix}
1 & 0 & 0 & \cdots & 0 \\
\dfrac{\lambda_0}{\lambda_1 - \lambda_0} & 1 & 0 & \cdots & 0 \\
\dfrac{\lambda_0}{\lambda_1 - \lambda_0} \cdot \dfrac{\lambda_1}{\lambda_2 - \lambda_0} & \dfrac{\lambda_1}{\lambda_2 - \lambda_1} & 1 & \cdots & 0 \\
\prod_{k=1}^{3} \dfrac{\lambda_{k-1}}{\lambda_k - \lambda_0} & \prod_{k=2}^{3} \dfrac{\lambda_{k-1}}{\lambda_k - \lambda_1} & \dfrac{\lambda_2}{\lambda_3 - \lambda_2} & \cdots & 0 \\
. & . & . & \cdots & . \\
. & . & . & \cdots & . \\
\prod_{k=1}^{n} \dfrac{\lambda_{k-1}}{\lambda_k - \lambda_0} & \prod_{k=2}^{n} \dfrac{\lambda_{k-1}}{\lambda_k - \lambda_1} & \prod_{k=3}^{n} \dfrac{\lambda_{k-1}}{\lambda_k - \lambda_2} & \cdots & 1
\end{bmatrix} \qquad \textbf{(7.4)}
$$

$$
\begin{array}{c}
i \\ i+1 \\ i+2 \\ i+3 \\ . \\ . \\ . \\ . \\ i+k
\end{array}
\overset{\begin{array}{ccccc} i & i+1 & i+2 & .\ . & i+k \end{array}}{
\begin{bmatrix}
1 & 0 & 0 & \cdots & 0 \\
\dfrac{\lambda_i}{\lambda_{i+1} - \lambda_i} & 1 & 0 & \cdots & 0 \\
\dfrac{\lambda_i \lambda_{i+1}}{(\lambda_{i+1} - \lambda_i)(\lambda_{i+2} - \lambda_i)} & \dfrac{\lambda_{i+1}}{\lambda_{i+2} - \lambda_{i+1}} & 1 & \cdots & 0 \\
\dfrac{\lambda_i \lambda_{i+1} \lambda_{i+2}}{(\lambda_{i+1} - \lambda_i)(\lambda_{i+2} - \lambda_i)(\lambda_{i+3} - \lambda_i)} & \dfrac{\lambda_{i+1} \lambda_{i+2}}{(\lambda_{i+2} - \lambda_{i+1})(\lambda_{i+3} - \lambda_{i+1})} & \dfrac{\lambda_{i+2}}{\lambda_{i+3} - \lambda_{i+2}} & \cdots & 0 \\
. & . & . & \cdots & . \\
. & . & . & \cdots & . \\
\prod_{j=0}^{k-1} \dfrac{\lambda_{i+j}}{\lambda_{i+j+1} - \lambda_i} & \prod_{j=1}^{k-1} \dfrac{\lambda_{i+j}}{\lambda_{i+j+1} - \lambda_{i+1}} & \prod_{j=2}^{k-1} \dfrac{\lambda_{i+j}}{\lambda_{i+j+1} - \lambda_{i+2}} & \cdots & 1
\end{bmatrix}}
$$

Inverse Matrix

$$
\begin{array}{c}
 \\ i \\ i+1 \\ i+2 \\ i+3 \\ . \\ . \\ . \\ . \\ i+k
\end{array}
\begin{array}{c}
\begin{array}{ccccc} i & i+1 & i+2 & . \quad . & i+k \end{array} \\
\begin{bmatrix}
1 & 0 & 0 & . \; . \; . & 0 \\
\frac{-\lambda_i}{\lambda_{i+1}-\lambda_i} & 1 & 0 & . \; . \; . & 0 \\
\frac{\lambda_i\lambda_{i+1}}{(\lambda_{i+2}-\lambda_i)(\lambda_{i+2}-\lambda_{i+1})} & \frac{-\lambda_{i+1}}{\lambda_{i+2}-\lambda_{i+1}} & 1 & . \; . \; . & 0 \\
\frac{-\lambda_i\lambda_{i+1}\lambda_{i+2}}{(\lambda_{i+3}-\lambda_i)(\lambda_{i+3}-\lambda_{i+1})(\lambda_{i+3}-\lambda_{i+2})} & \frac{\lambda_{i+1}\lambda_{i+2}}{(\lambda_{i+2}-\lambda_{i+1})(\lambda_{i+3}-\lambda_{i+1})} & \frac{-\lambda_{i+2}}{(\lambda_{i+3}-\lambda_{i+2})} & . \; . \; . & 0 \\
. & . & . & . \; . \; . & . \\
. & . & . & . \; . \; . & . \\
(-1)^k \prod_{j=0}^{k-1} \frac{\lambda_{i+j}}{(\lambda_{i+k}-\lambda_{i+j})} & (-1)^{k-1} \prod_{j=0}^{k-2} \frac{\lambda_{i+j+1}}{(\lambda_{i+j+2}-\lambda_{i+1})} & (-1)^{k-2} \prod_{j=0}^{k-3} \frac{\lambda_{i+j+2}}{(\lambda_{i+j+3}-\lambda_{i+2})} & . \; . \; . & 1
\end{bmatrix}
\end{array}
\qquad \textbf{(7.5)}
$$

Now

$X(0) = i \qquad p_i(0) = i \qquad p_j(0) = 0 \qquad \forall \qquad j \neq i$

$$
p(0) = \begin{bmatrix} p_i(0) \\ p_{i+1}(0) \\ p_{i+2}(0) \\ . \\ . \\ p_{i+k}(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ . \\ . \\ 0 \end{bmatrix}
$$

Now, we know that $p(t) = R\, e^{(\text{diag } \Lambda)t}\, R^{-1}\, p(0)$

$$R^{-1}\, p(0) = \begin{bmatrix} 1 \\ \dfrac{\lambda_i}{\lambda_{i+1} - \lambda_i} \\ \dfrac{\lambda_1\ \lambda_{i+1}}{(\lambda_{i+k} - \lambda_i)(\lambda_{i+k} - \lambda_i)} \\ \dfrac{-\lambda_i\ \lambda_{i+1}\ \lambda_{i+2}}{(\lambda_{i+k} - \lambda_i)(\lambda_{i+k} - \lambda_i)(\lambda_{i+k} - \lambda_i)} \\ . \\ . \\ (-1)^k \prod_{j=0}^{k-1} \dfrac{\lambda_{i+j}}{(\lambda_{i+k} - \lambda_{i+j})} \end{bmatrix}$$

But

$$e^{(\text{diag}\,\Lambda)\,t} = \begin{bmatrix} e^{-\lambda_i t} & 0 & 0 & . & . & 0 \\ 0 & e^{-\lambda_{i+1} t} & 0 & . & . & 0 \\ 0 & 0 & e^{-\lambda_{i+2} t} & . & . & 0 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & 0 & 0 & . & . & e^{-\lambda_{i+k} t} \end{bmatrix} \qquad \textbf{(7.6)}$$

Therefore,

$$e^{(\text{diag}\,\Lambda)\,t}\, R^{-1}\, p(0) = \begin{bmatrix} e^{-\lambda_i\ t} \\ \dfrac{-\lambda_i}{\lambda_{i+1} - \lambda_i} e^{-\lambda_{i+1} t} \\ \dfrac{\lambda_i\ \lambda_{i+1}}{(\lambda_{i+2} - \lambda_i)(\lambda_{i+2} - \lambda_{i+1})} e^{-\lambda_{i+2} t} \\ \dfrac{-\lambda_i\ \lambda_{i+1}\ \lambda_{i+2}}{(\lambda_{i+3} - \lambda_i)(\lambda_{i+3} - \lambda_{i+1})(\lambda_{i+3} - \lambda_{i+2})} e^{-\lambda_{i+3} t} \\ . \\ . \\ (-1)^k \prod_{j=0}^{k-1} \dfrac{\lambda_{i+j}}{(\lambda_{i+k} - \lambda_{i+j})} e^{-\lambda_{i+k} t} \end{bmatrix}$$

$$p_{i+k}(t) = \left\{\prod_{j=0}^{k-1}\lambda_{i+j}\right\}\left\{\frac{e^{-\lambda_i t}}{\prod_{j=1}^{k-1}\left(\lambda_{i+j} - \lambda_i\right)} + \frac{e^{-\lambda_{i+1} t}}{\prod_{\substack{j=1\\ j\neq 1}}^{k-1}\left(\lambda_{i+j} - \lambda_{i+1}\right)} + \frac{e^{-\lambda_{i+2} t}}{\prod_{\substack{j=0\\ j\neq 2}}^{k-1}\left(\lambda_{i+j} - \lambda_{i+2}\right)} + \ldots + \frac{e^{-\lambda_{i+k} t}}{\prod_{j\neq k}^{k-1}\left(\lambda_{i+j} - \lambda_{i+k}\right)}\right\}$$

$$p_{i+k}(t) = \left\{\prod_{j=0}^{k-1}\lambda_{i+j}\right\}\left\{\sum_{r=0}^{k}\frac{e^{-\lambda_{i+r} t}}{\prod_{\substack{j=0\\ j\neq r}}^{k-1}\left(\lambda_{i+j} - \lambda_{i+r}\right)}\right\} \quad \textbf{(7.8)}$$

### 7.3 Solution To Simple Birth Process

$$\lambda_n = n\,\lambda \qquad \Rightarrow \qquad \lambda_{i+j} = (i+j)\,\lambda$$

Therefore,

$$p_{i+k}(t) = \left\{\prod_{j=0}^{k-1}(i+j)\,\lambda\right\}\left\{\sum_{r=0}^{k}\frac{e^{-(i+r)\lambda t}}{\prod_{\substack{j=0\\ j\neq r}}^{k-1}\big((i+j)\,\lambda - (i+r)\,\lambda\big)}\right\} \qquad \textbf{(7.9)}$$

$$= \lambda^k\left\{\prod_{j=0}^{k-1}(i+j)\right\}\left\{\sum_{r=0}^{k}\frac{e^{-i\lambda t}e^{-r\lambda t}}{\prod_{\substack{j=0\\ j\neq r}}^{k-1}(i+j-i-r)\lambda}\right\}$$

$$= \lambda^k\left\{\prod_{j=0}^{k-1}(i+j)\right\}\left\{\sum_{r=0}^{k}\frac{e^{-(i+j)\lambda t}}{\lambda^k\prod_{\substack{j=0\\ j\neq r}}^{k-1}(j-r)}\right\}$$

$$= e^{-i\lambda t}\left\{\prod_{j=0}^{k-1}(i+j)\right\}\left\{\sum_{r=0}^{k}\frac{e^{-r\lambda t}}{\prod_{\substack{j=0\\ j\neq i+r}}^{k-1}(j-r)}\right\}$$

Therefore

$$p_{i+k}(t) = e^{-i\lambda t}\left\{\prod_{j=0}^{k-1}(i+j)\right\}\left\{\frac{1}{\prod_{j=1}^{k}(j-0)} + \frac{e^{-\lambda t}}{\prod_{\substack{j=0\\ j\neq 1}}^{k}(j-1)} + \frac{e^{-2\lambda t}}{\prod_{\substack{j=0\\ j\neq 2}}^{k}(j-2)} + \frac{e^{-3\lambda t}}{\prod_{\substack{j=0\\ j\neq 3}}^{k}(j-3)} + \frac{e^{-4\lambda t}}{\prod_{\substack{j=0\\ j\neq 4}}^{k}(j-4)} + \ldots + \frac{e^{-k\lambda t}}{\prod_{j=0}^{k}(j-k)}\right\}$$

$$p_{i+k}(t) = e^{-i\lambda t}\,\frac{(i+k-1)\,!}{(i-1)\,!}\left\{\frac{1}{k!} - \frac{e^{-\lambda t}}{(k-1)!} + \frac{e^{-2\lambda t}}{2\,(k-2)!} - \frac{e^{-3\lambda t}}{2.3(k-3)!} + \frac{e^{-4\lambda t}}{1.2.3.4(k-4)\,!} - \ldots + (-1)^k\,\frac{e^{-k\lambda t}}{k!}\right\}$$

$$= e^{-i\lambda t}\,\frac{(i+k-1)\,!}{(i-1)\,!}\left\{\frac{1}{0!\,k!} - \frac{e^{-\lambda t}}{1!\,(k-1)!} + \frac{e^{-2\lambda t}}{2!\,(k-2)!} - \frac{e^{-3\lambda t}}{3!(k-3)!} + \frac{e^{-4\lambda t}}{4!\,(k-4)\,!} - \ldots + (-1)^k\,\frac{e^{-k\lambda t}}{k!\,(k-k)!}\right\}\frac{k\,!}{k!}$$

$$p_{i+k}(t) = e^{-i\lambda t}\,\frac{(i+k-1)!}{k!\,(i-1)!}\left\{\frac{k!}{0!\,k!} - \frac{k!\,e^{-\lambda t}}{1!\,(k-1)!} + \frac{k!\,e^{-2\lambda t}}{2!\,(k-2)!} - \frac{k!\,e^{-3\lambda t}}{3!(k-3)!} + \frac{k!\,e^{-4\lambda t}}{4!\,(k-4)!} - \dots + (-1)^k\,\frac{k!\,e^{-k\lambda t}}{k!\,(k-k)!}\right\}$$

$$= e^{-i\lambda t}\binom{i+k-1}{i-1}\left\{\binom{k}{0}-\binom{k}{1}e^{-\lambda t}+\binom{k}{2}e^{-2\lambda t}-\binom{k}{3}e^{-3\lambda t}+\binom{k}{4}e^{-4\lambda t} - \dots + (-1)^r\binom{k}{r}e^{-r\lambda t} + \dots + (-1)^k\binom{k}{k}e^{-k\lambda t}\right\}$$

Therefore

$$p_{i+k}(t) = \binom{(i+k-1)}{(i-1)}e^{-i\lambda t}\sum_{r=0}^{k}\binom{k}{r}\left(-e^{-\lambda t}\right)^r$$

$$= \binom{(i+k-1)}{(i-1)}e^{-i\lambda t}\left(1-e^{-\lambda t}\right)^k$$

$$= \binom{(i+k-1)}{(i-1)}\left(e^{-\lambda t}\right)^i\left(1-e^{-\lambda t}\right)^k \qquad \textbf{(7.10)}$$

### 7.4 Birth Process with Immigration

In this case, $\lambda_n = n\lambda + v.$ But the general solution is given as

$$p_{i+k}(t) = \left\{\prod_{j=0}^{k-1}\lambda_{i+j}\right\}\left\{\sum_{r=0}^{k}\frac{e^{-\lambda_{i+r}t}}{\prod_{\substack{j=0\\j\neq r}}^{k}\left(\lambda_{i+j}-\lambda_{i+r}\right)}\right\}$$

Now, $\lambda_{i+j} = (i+j)\lambda + v$ and $\lambda_{i+r} = (i+r)\lambda + v.$

Therefore

$$p_{i+k}(t) = \left\{\sum_{r=0}^{k}\frac{e^{-((i+r)\lambda+v)t}}{\prod_{\substack{j=0\\j\neq r}}^{k}\left((i+j)\lambda+v-((i+r)\lambda+v)\right)}\right\}\left\{\prod_{j=0}^{k-1}\left((i+j)\lambda+v\right)\right\} \qquad \textbf{(7.11)}$$

$$= \left\{\prod_{j=0}^{k-1}\left((i+j)\lambda+v\right)\right\}\left\{\sum_{r=0}^{k}\frac{e^{-(i\lambda+v)t}e^{-r\lambda t}}{\prod_{\substack{j=0\\j\neq r}}^{k}\left(i\lambda+j\lambda+v-i\lambda-r\lambda-v\right)}\right\}$$

$$= \left\{\prod_{j=0}^{k-1}\left((i+j)\lambda+v\right)\right\}\left\{\sum_{r=0}^{k}\frac{e^{-(i\lambda+v)t}e^{-r\lambda t}}{\prod_{\substack{j=0\\j\neq r}}^{k}(j-r)\lambda}\right\}$$

$$= \left\{\prod_{j=0}^{k-1}\left((i+j)\lambda+v\right)e^{-(i\lambda+v)t}\right\}\left\{\sum_{r=0}^{k}\frac{e^{-r\lambda t}}{\prod_{\substack{j=0\\j\neq r}}^{k}(j-r)\lambda}\right\}$$

$$= \left\{\prod_{j=0}^{k-1}\left((i+j)\lambda+v\right)\frac{e^{-(i\lambda+v)t}}{\lambda^k}\right\}\left\{\sum_{r=0}^{k}\frac{e^{-r\lambda t}}{\prod_{\substack{j=0\\j\neq r}}^{k}(j-r)}\right\}$$

$$p_{i+k}(t) = \left[\prod_{j=0}^{k-1}\left(\left(i + j\right) + \frac{v}{\lambda}\right).\lambda\right].\ \frac{e^{-(i\lambda + v)t}}{\lambda^{k}}\left\{\sum_{r=0}^{k}\frac{e^{-r\lambda t}}{\prod_{\substack{j=0\\ j\neq r}}^{k}\left(j-r\right)}\right\}$$

$$= \frac{\lambda^{k}}{\lambda^{k}}\left\{\left(\prod_{j=0}^{k-1}\ \left(i + j\right) + \frac{v}{\lambda}\right)\right\}e^{-(i\lambda + v)t}\left\{\sum_{r=0}^{k}\frac{e^{-r\lambda t}}{\prod_{\substack{j=0\\ j\neq r}}^{k}\left(j-r\right)}\right\}$$

$$= \left\{\prod_{j=0}^{k-1}\left(\left(i + j\right) + \frac{v}{\lambda}\right)\right\}e^{-(i\lambda + v)t}\left\{\sum_{r=0}^{k}\frac{e^{-r\lambda t}}{\prod_{\substack{j=0\\ j\neq r}}^{k}\left(j-r\right)}\right\}$$

$$= \left\{\prod_{j=0}^{k-1}\left(i + j + \frac{v}{\lambda}\right)\right\}\ e^{-\left(i + \frac{v}{\lambda}\right)\lambda t}\left\{\sum_{r=0}^{k}\frac{e^{-r\lambda t}}{\prod_{\substack{j=0\\ j\neq r}}^{k}\left(j-r\right)}\right\}$$

$$P_{i+k}(t) = \left\{\prod_{j=0}^{k-1}\left(j + \left(i + \frac{v}{\lambda}\right)\right)\right\} e^{-\left(i + \frac{v}{\lambda}\right)\lambda t}\left\{\frac{e^{-0\lambda t}}{\prod_{j=1}^{k}(j-0)} + \frac{e^{-\lambda t}}{\prod_{\substack{j=0\\ j\neq 1}}^{k}(j-1)} + \frac{e^{-2\lambda t}}{\prod_{\substack{j=0\\ j\neq 2}}^{k}(j-2)} + \frac{e^{-3\lambda t}}{\prod_{\substack{j=0\\ j\neq 3}}^{k}(j-3)} + \frac{e^{-4\lambda t}}{\prod_{\substack{j=0\\ j\neq 4}}^{k}(j-4)} + .... + \frac{e^{-r\lambda t}}{\prod_{\substack{j=0\\ j\neq r}}^{k}(j-r)} + ... + \frac{e^{-k\lambda t}}{\prod_{j=0}^{k}(j-k)}\right\}$$

Therefore

$$P_{i+k}(t) = e^{-\left(i + \frac{v}{\lambda}\right)\lambda t}\left\{\prod_{j=0}^{k-1}\left(j + \left(i + \frac{v}{\lambda}\right)\right)\right\}\ \left\{\frac{e^{-0\lambda t}}{0!\,k!} - \frac{e^{-\lambda t}}{1!(k-1)!} + \frac{e^{-2\lambda t}}{2!(k-2)!} - \frac{e^{-3\lambda t}}{3!(k-3)!} + \frac{e^{-4\lambda t}}{4!(k-4)!} - .... + (-1)^{r}\frac{e^{-r\lambda t}}{r!(k-r)!} + ... + (-1)^{k}\frac{e^{-k\lambda t}}{k!(k-1)!}\right\}$$

$$p_{i+k}(t) = e^{-\left(i + \frac{v}{\lambda}\right)\lambda t}\ \frac{\left(k + \left(i + \frac{v}{\lambda}\right) - 1\right)!}{\left(\left(i + \frac{v}{\lambda}\right) - 1\right)!}\ \sum_{r=0}^{k}\frac{(-1)^{r}e^{-r\lambda t}}{r!(k-r)!}\ .\ \frac{k!}{k!}$$

$$= \frac{\left(k + \left(i + \frac{v}{\lambda}\right) - 1\right)!}{\left(\left(i + \frac{v}{\lambda}\right) - 1\right)!\,k!}e^{-\left(i + \frac{v}{\lambda}\right)\lambda t}\sum_{r=0}^{k}\frac{k!}{r!(k-r)!}\left(e^{-\lambda t}\right)^{r}$$

$$p_{i+k}(t) = \frac{\left(k + \left(i + \frac{v}{\lambda}\right) - 1\right)!}{\left(\left(i + \frac{v}{\lambda}\right) - 1\right)!\, k\,!} e^{-\left(i + \frac{v}{\lambda}\right)\lambda t} \left\{ \sum_{r=0}^{k} \binom{k}{r} \left(e^{-\lambda t}\right)^{r} \right\}$$

$$= \frac{\left(k + \left(i + \frac{v}{\lambda}\right) - 1\right)!}{\left(\left(i + \frac{v}{\lambda}\right) - 1\right)!\, k\,!} e^{-\left(i + \frac{v}{\lambda}\right)\lambda t} \left(1 - e^{-\lambda t}\right)^{k}$$

$$p_{i+k}(t) = \binom{k + \left(i + \frac{v}{\lambda}\right) - 1}{i + \frac{v}{\lambda} - 1} \left(e^{-\lambda t}\right)^{i + \frac{v}{\lambda}} \left(1 - e^{-\lambda t}\right)^{k} \qquad \textbf{(7.12)}$$

### 7.5 Matrix Method for the Polya process

For the Polya Process, $\lambda_n = \lambda\left(\frac{1 + a\,n}{1 + \lambda a\,t}\right)$. But the general solution is given by

$$p_{i+k}(t) = \left\{\prod_{j=0}^{k-1} \lambda_{i+j}\right\} \sum_{r=0}^{k} \left\{ \frac{e^{-\lambda_{i+r}\,t}}{\prod_{\substack{j=0 \\ j\neq r}}^{k} \left(\lambda_{i+j} - \lambda_{i+r}\right)} \right\}$$

Now, $\lambda_{i+j} = \lambda\left(\frac{1 + a(i+j)}{1 + \lambda at}\right)$ and $\lambda_{i+r} = \lambda\left(\frac{1 + a(i+r)}{1 + \lambda at}\right)$.

Therefore

$$p_{i+k}(t) = \left\{\prod_{j=0}^{k-1} \lambda\left(\frac{1 + a(i+j)}{1 + \lambda at}\right)\right\} \sum_{r=0}^{k} \left\{ \frac{e^{-\left[\lambda\left(\frac{1 + a(i+r)}{1 + \lambda at}\right)\right]t}}{\prod_{\substack{j=0 \\ j\neq r}}^{k} \left(\left[\lambda\left(\frac{1 + a(i+j)}{1 + \lambda at}\right)\right] - \left[\lambda\left(\frac{1 + a(i+r)}{1 + \lambda at}\right)\right]\right)} \right\} \qquad \textbf{(7.13)}$$

$$= \left(\frac{\lambda}{1 + \lambda at}\right)^{k} \left\{\prod_{j=0}^{k-1} \left[\left(1 + a(i+j)\right)\right]\right\} \sum_{r=0}^{k} \left\{ \frac{e^{-\left[\lambda\left(\frac{1 + a(i+r)}{1 + \lambda at}\right)\right]t}}{\left(\frac{\lambda}{1 + \lambda at}\right)^{k} \prod_{\substack{j=0 \\ j\neq r}}^{k} \left(\left[1 + a(i+j)\right] - \left[1 + a(i+r)\right]\right)} \right\}$$

$$= \left\{\prod_{j=0}^{k-1} \left[\left(1 + a(i+j)\right)\right]\right\} \sum_{r=0}^{k} \left\{ \frac{e^{-\lambda\left[\frac{1+ai}{1+\lambda at}\right]t}\; e^{-\lambda\left[\frac{ar}{1+\lambda at}\right]t}}{\prod_{\substack{j=0 \\ j\neq r}}^{k} \left(1 + ai + aj - 1 - ai - ar\right)} \right\}$$

$$= \left\{\prod_{j=0}^{k-1} \left[\left(1 + a(i+j)\right)\right]\right\} \sum_{r=0}^{k} \left\{ \frac{e^{-\lambda\left[\frac{1+ai}{1+\lambda at}\right]t}\; e^{-\lambda\left[\frac{ar}{1+\lambda at}\right]t}}{\prod_{\substack{j=0 \\ j\neq r}}^{k} a(j - r)} \right\}$$

$$= a^{k} \left\{\prod_{j=0}^{k-1} \left[i + j + \frac{1}{a}\right]\right\} \sum_{r=0}^{k} \left\{ \frac{e^{-\lambda\left[\frac{1+ai}{1+\lambda at}\right]t}\; e^{-\lambda\left[\frac{ar}{1+\lambda at}\right]t}}{a^{k} \prod_{\substack{j=0 \\ j\neq r}}^{k} (j - r)} \right\}$$

Cont

$$p_{i+k}(t)=\left\{\prod_{j=0}^{k-1}\left[j+i+\frac{1}{a}\right]\right\}e^{-\lambda\left[\frac{1+ai}{1+\lambda at}\right]t}\sum_{r=0}^{k}\left\{\frac{e^{-\lambda\left[\frac{ar}{1+\lambda at}\right]t}}{\prod\limits_{\substack{j=0\\ j\neq r}}^{k}(j-r)}\right\}$$

Expanding, we have

$$p_{i+k}(t)=\left\{\prod_{j=0}^{k-1}\left[j+i+\frac{1}{a}\right]\right\}e^{-\lambda\left[\frac{1+ai}{1+\lambda at}\right]t}\left\{\frac{e^{-\left[\frac{0\lambda t}{1+\lambda at}\right]}}{\prod\limits_{j=1}^{k}(j-0)}+\frac{e^{-\left[\frac{a\lambda t}{1+\lambda at}\right]}}{\prod\limits_{\substack{j=0\\ j\neq 1}}^{k}(j-1)}+\frac{e^{-\left[\frac{2a\lambda t}{1+\lambda at}\right]}}{\prod\limits_{\substack{j=0\\ j\neq 2}}^{k}(j-2)}+\frac{e^{-\left[\frac{3\lambda at}{1+\lambda at}\right]}}{\prod\limits_{\substack{j=0\\ j\neq 3}}^{k}(j-3)}+\frac{e^{-\left[\frac{4a\lambda t}{1+\lambda at}\right]}}{\prod\limits_{\substack{j=0\\ j\neq 4}}^{k}(j-4)}+\ldots.+\frac{e^{-\left[\frac{ka\lambda t}{1+\lambda at}\right]}}{\prod\limits_{\substack{j=0\\ j\neq k}}^{k}(j-k)}\right\}$$

$$=\left\{\prod_{j=0}^{k-1}\left[j+i+\frac{1}{a}\right]\right\}e^{-\lambda\left[\frac{1+ai}{1+\lambda at}\right]t}\left\{\frac{1}{0!k!}-\frac{e^{-\left[\frac{a\lambda t}{1+\lambda at}\right]}}{1!(k-1)!}+\frac{e^{-\left[\frac{2a\lambda t}{1+\lambda at}\right]}}{2!(k-2)!}-\frac{e^{-\left[\frac{3\lambda at}{1+\lambda at}\right]}}{3!(k-3)!}+\frac{e^{-\left[\frac{4a\lambda t}{1+\lambda at}\right]}}{4!(k-4)!}-\ \ldots.+(-1)^{k}\frac{e^{-\left[\frac{ka\lambda t}{1+\lambda at}\right]}}{k!(k-1)!}\right\}$$

Equivalently,

$$p_{i+k}(t)=\left\{\prod_{j=0}^{k-1}\left[j+i+\frac{1}{a}\right]\right\}e^{-\lambda\left[\frac{1+ai}{1+\lambda at}\right]t}\left\{\sum_{r=0}^{k}(-1)^{r}\frac{e^{-\left[\frac{ra\lambda t}{1+\lambda at}\right]}}{r!\,(k-r)!}.\frac{k!}{k!}\right\}$$

$$=\frac{\left(k+\left(i+\frac{1}{a}\right)-1\right)!}{\left(\left(i+\frac{1}{a}\right)-1\right)!\,k!}e^{-\lambda\left[\frac{1+ai}{1+\lambda at}\right]t}\left\{\sum_{r=0}^{k}\frac{k!}{r!\,(k-r)!}\left(-e^{-\left[\frac{ra\lambda t}{1+\lambda at}\right]}\right)\right\}$$

$$=\binom{k+\left(i+\frac{1}{a}\right)-1}{\left(i+\frac{1}{a}\right)-1}e^{-\lambda\left[\frac{1+ai}{1+\lambda at}\right]t}\left\{\sum_{r=0}^{k}\binom{k}{r}\left(-e^{-\left[\frac{a\lambda t}{1+\lambda at}\right]}\right)^{r}\right\}$$

$$=\binom{k+\left(i+\frac{1}{a}\right)-1}{\left(i+\frac{1}{a}\right)-1}e^{-\lambda\left[\frac{1+ai}{1+\lambda at}\right]t}\left(1-e^{-\left[\frac{a\lambda t}{1+\lambda at}\right]}\right)^{k}$$

Thus,

$$p_{i+k}(t)=\binom{k+\left(i+\frac{1}{a}\right)-1}{\left(i+\frac{1}{a}\right)-1}e^{-\lambda\left[\frac{1+ai}{1+\lambda at}\right]t}\left(1-e^{-\left[\frac{a\lambda t}{1+\lambda at}\right]}\right)^{k} \qquad \textbf{(7.14)}$$

# CHAPTER EIGHT

# CONCLUSION

## 8.1 Summary

This literature was dedicated to the construction of probability distributions arising from the solution of the basic difference differential equations derived in Chapter 1. Our intention was to bring together the scattered different Methods that can be used to solve these differential equations and yet give the same results. In the first Chapter, we derived the basic difference differential equations from the first principles and stated the assumptions involved. We also introduced the various necessary pre requisite tools needed before venturing into solving.

In the second, third, fourth and fifth Chapter, we explored various methods of solving the difference differential equations under different values of parameter $\lambda_n$.

In the sixth Chapter, we applied a matrix Method to solve the same equations. Below is a summary of our findings in the cases where our initial condition was $X(0) = n_0$ unless otherwise specified.

### 8.1.1 Basic Difference Differential Equations

$$p_0'(t) = -\lambda_0 p_0(t) \quad \textbf{(2.1)}$$

and

$$p_n'(t) = -\lambda_n p_n(t) + \lambda_{n-1} p_{n-1}(t), \quad n \geq 1 \quad \textbf{(2.2)}$$

where $p_n(t)$ is the Probability that the population at time t is n.

These two equations are called the basic difference differential equations. When solved using any of the known methods, we end up finding the distribution of $p_n(t)$. A summary of the results of solving the equations for some values of $\lambda_n$ and the various properties of the distributions are given below.

### 8.1.2 Poisson Process

(i) Parameters

In this case $\lambda_n = \lambda \ \forall n$ indicating a constant growth rate.

(ii) Basic Difference Differential Equations

When the equations become

$$p_0'(t) = -\lambda p_0(t) \tag{3.4}$$

$$p_n'(t) = -\lambda p_n(t) + \lambda p_{n-1}(t), \quad n \geq 1 \tag{3.5}$$

(iii) Distribution of $p_n(t)$

$$p_n(t) = \frac{e^{-\lambda t}(\lambda t)^k}{k!} \qquad \text{for } n = n_0 + k \text{ and } k = 0, 1, 2, .... \qquad \textbf{(3.15, 3.23, 3.27)}$$

(iv) Probability Generating function

$$G(s,t) = e^{-\lambda t(1-s)} \tag{3.26}$$

(v) Laplace function

$$L[p_n(t)] = \frac{\lambda^k}{(\lambda + s)^{k+1}} \tag{3.22}$$

(vi) Mean

$$E[X(t)] = \lambda t \qquad \textbf{(3.16, 3.28, 3.31)}$$

(vii) Variance

$$Var[X(t)] = \lambda t \qquad \textbf{(3.17, 3.29, 3.33)}$$

(viii) First Moment

$$M_1(t) = \lambda t \tag{3.31}$$

(ix) Second Moment

$$M_2(t) = (\lambda t)^2 + \lambda t \tag{3.32}$$

### 8.1.3 Simple Birth Process

(i) Parameters

In this case $\lambda_n = n\lambda \ \forall \ n.$

(ii) Basic Difference Differential Equations

$$p_0'(t) = 0 \tag{4.1a}$$

$$p_n'(t) = -n\lambda p_n(t) + (n-1)\lambda p_{n-1}(t), \quad n \geq 1 \tag{4.1b}$$

(iii) Distribution of $p_n(t)$

$$p_n(t) = \binom{k + n_0 - 1}{k} \times \left(1 - e^{-\lambda t}\right)^k \times \left(e^{-\lambda t}\right)^{n_0} \qquad \text{for } n = n_0 + k \text{ and } k = 0, 1, 2, \ldots.$$

**(4.10, 4.20, 4.28, 7.10)**

(iv) Probability Generating function

$$G(s,t) = \left[\frac{ps}{1 - qs}\right]^{n_0} \quad \text{where } p = e^{-\lambda t} \text{ and } q = 1 - e^{-\lambda t} \qquad \textbf{(4.27)}$$

(v) Laplace function

$$L\left(p_n(t)\right) = \frac{\prod_{i=0}^{k-1}\left(n_0\lambda + v + i\lambda\right)}{\prod_{i=0}^{k}\left(s + v + n_0\lambda + i\lambda\right)}, \; n = n_0, n_0+1, \ldots. \qquad \textbf{(4.18)}$$

(vi) Mean

$$E(X) = n_0\left(1 + \frac{q}{p}\right) \qquad \textbf{(4.11, 4.29, 4.36)}$$

(vii) Variance

$$Var\left[X(t)\right] = n_0 e^{2\lambda t}\left(1 - e^{-\lambda t}\right) \qquad \textbf{(4.12, 4.30, 4.41)}$$

(viii) First Moment

$$M_1(t) = n_0 e^{\lambda t} \qquad \textbf{(4.36)}$$

(ix) Second Moment

$$M_2(t) = -n_0 e^{\lambda t} + n_0^2 e^{2\lambda t} + n_0 e^{2\lambda t} \qquad \textbf{(4.40)}$$

### 8.1.4 Simple Birth Process with Immigration

(i) Parameters

In this case $\lambda_n = n\lambda + v \;\; \forall n$

(ii) Basic Difference Differential Equations

When the equations become

$$p_0'(t) = -v p_0(t) \qquad \textbf{(5.1a)}$$

$$p_n'(t) = -(n\lambda + v) p_n(t) + \left[(n-1)\lambda + v\right] p_{n-1}(t), \quad n \geq 1 \qquad \textbf{(5.1b)}$$

(iii) Distribution of $p_n(t)$

$$p_n(t) = \binom{n_0 + \frac{v}{\lambda} + k - 1}{k}\left(e^{-\lambda t}\right)^{n_0 + \frac{v}{\lambda}}\left(1 - e^{-\lambda t}\right)^k \quad \textbf{(5.5, 5.16, 5.31, 7.12)}$$

for $n = n_0 + k$ and $k = 0, 1, 2, ....$

(iv) Probability Generating function

$$G(s,t) = s^{n_0}\left[\frac{e^{-\lambda t}}{1 - s + e^{-\lambda t}s}\right]^{n_0 + \frac{v}{\lambda}} \quad \textbf{(5.30)}$$

(v) Laplace function

$$L\left(p_n(t)\right) = \frac{\prod_{i=0}^{k-1}(n_0 + i)\lambda^k}{\prod_{i=0}^{k}\left(s + (n_0 + i)\lambda\right)} \quad n = n_0 + k,\ i = 0, 1, ... \quad \textbf{(5.14)}$$

(vi) Mean

$$E\left[X(t)\right] = \frac{n_0 + \frac{v}{\lambda}\left(1 - e^{-\lambda t}\right)}{e^{-\lambda t}} \quad \textbf{(5.6, 5.32, 5.38)}$$

(vii) Variance (**(5.45)** evaluated for $n_0 = 1$)

$$Var\left[X(t)\right] = \left(n_0 + \frac{v}{\lambda}\right)e^{\lambda t}\left(e^{\lambda t} - 1\right) \quad \textbf{(5.7, 5.33, 5.45)}$$

(viii) First Moment $(n_0 = 1)$

$$M_1(t) = \frac{v}{\lambda}\left(e^{\lambda t} - 1\right) \quad \textbf{(5.38)}$$

(ix) Second Moment $(n_0 = 1)$

$$M_2(t) = \frac{v}{\lambda}e^{\lambda t}\left(e^{\lambda t} - 1\right) - \left[\frac{v}{\lambda}\left(e^{\lambda t} - 1\right)\right]^2 \quad \textbf{(5.44)}$$

### 8.1.5 Polya Process

(i) Parameters

In this case $\lambda_n = \lambda\left(\frac{1 + an}{1 + \lambda at}\right)\ \forall\ n.$

(ii) Basic Difference Differential Equations

When the equations become

$$p_0'(t) = -\left(\frac{\lambda}{1+\lambda a t}\right)p_0(t) \qquad \textbf{(6.1a)}$$

$$p_n'(t) = -\lambda\left(\frac{1+an}{1+\lambda a t}\right)p_n(t) + \lambda\left(\frac{1+a(n-1)}{1+\lambda a t}\right)p_{n-1}(t), \quad n \geq 1 \qquad \textbf{(6.1b)}$$

(iii) Distribution of $p_n(t)$

$$p_{n_0+k}(t) = \binom{\frac{1}{a}+n_0+k-1}{k}\left(\frac{\lambda at}{1+\lambda at}\right)^k\left(\frac{1}{1+\lambda at}\right)^{\left(\frac{1}{a}+n_0\right)} \qquad \textbf{(6.11, 6.26, 7.14)}$$

(iv) Probability Generating function

$$G(s,t) = s^{-\frac{1}{a}} \times \left[\frac{ps}{1-qs}\right]^{n_0+\frac{1}{a}} \qquad \textbf{(6.25)}$$

(v) Laplace Function

The Method did not work.

(vi) Mean

$$E[X(t)] = \lambda t \qquad \textbf{(6.7, 6.26, 6.37)}$$

(vii) Variance

$$Var[X(t)] = \lambda t(1 + a\lambda t) \qquad \textbf{(6.8, 6.28, 6.42)}$$

(viii) First Moment

$$M_1(t) = (1+\lambda at) + \lambda t \qquad \textbf{(6.37)}$$

(ix) Second Moment

$$M_2(t) = \lambda t[\lambda t + 1 + \lambda at] \qquad \textbf{(6.41)}$$

## 8.2 Conclusion

In each case of the pure birth process, all the four Methods resulted in the same distribution. The distributions emerging from difference differential equations of a pure birth process are power series distributions. They are the Geometric distribution, the negative Binomial Distribution and the Poisson distribution.

## 8.3 Recommendation for Further Research

One of the assumptions made in the derivation of the Basic difference differential equations is that in a time interval $\Delta t$ the probability of more than one birth is negligible. It would be interesting to see what kind of distributions would emerge if the probability of two or more births within a time interval $\Delta t$ was not negligible.

If the birth rate was changing over time rather than remain constant (or change at a constant rate over t) , it would introduce a lot of new application areas in real life.

Determine the distribution emerging from pure birth processes when the birth rate is a distribution function.

Determine the distribution emerging from pure birth processes when the birth rate change over certain time intervals

Determine the distribution emerging from pure birth processes when the birth rate is a survival function

## 8.4 Framework

### 8.5 References

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