UNIVERSITY OF NAIROBI

NEGATIVE BINOMIAL MIXTURES

CONSTRUCTION OF NEGATIVE BINOMIAL MIXTURES AND THEIR PROPERTIES

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SUPERVISOR

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A Dissertation in partial fulfillment for a Master of Science degree in Mathematical Statistics

Declaration

I, the undersigned, declare that this project is my original work and to the best of my knowledge, has not been presented for award of a degree in any other university.

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Declaration by Supervisor

This project report has been submitted for examination with my approval as supervisor.

Signature: ___________________________

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Date: ______________________________

Statement

This dissertation has been submitted in partial of requirements for a Masters of Science degree at the University of Nairobi and is deposited in the University Library to be made available to borrowers under the rules of the library.

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EXECUTIVE SUMMARY

The objective of this project was to construct Negative Binomial mixtures.We consider a class of mixture distributions generated by randomizing the success parameter $p$and fixing parameter r of a Negative Binomial distribution where we obtained a number of mixtures. We parametized $p\ to\ e^{-\lambda}\ and\ \ p\ to\ 1-e^{-\lambda}$.

The mixing distributions used are Exponential, Gamma, Exponeniated Exponential, Beta Exponential, Variate Gamma, Variate Exponential, Inverse Gaussian, and Lindely.

Some of the results were expressed in the explicit, expectations and recursive form. The explicit involves using $f(x) = \int \binom{r+x-1}{x} p^r (1-p)^x g(p) dp$ where $x = 0,1,2, ...$ and g(p) is the mixing distribution. By using this method the Negative Binomial – Exponential mixture wasobtained as $f(x) = -\frac{\lambda\Gamma(x+r)\Gamma(r+\lambda-1)}{\Gamma(r+\lambda+x-1)\Gamma r}$.Other mixtures could not be obtained using explicit since integration was not possible.

The Expectations method involved using the Laplace or method of moments where$f(x) = \binom{r+x-1}{x} \sum_{k=0}^{x} \binom{x}{k} (-1)^k L_t(r+k)$for $x > 0; r > 0; andk = 0,1,2 ..... x$. The mixtures obtained using this method are NB- Exponential $(x) = \binom{r+x-1}{x} \sum_{k=0}^{x} \binom{x}{k} (-1)^k \frac{\lambda}{r+k+\lambda}$, NB-Lindely $prob(x) = \frac{\theta^2}{(\theta+1)} \binom{r+x-1}{x} \sum_{k=0}^{x} \binom{x}{k} (-1)^k \frac{(\theta+r+k+1)}{(\theta+r+k)^2}$,NB- Inverse Gaussian

$p(X = x) =$
$\binom{r+x-1}{x} \sum_{k=0}^{x} \binom{x}{k} (-1)^k \exp\left(\frac{\phi}{\mu}\right[1-$

1−2r+kμ2ϕ,NB−Exponentiated exponentialfx=r+x−1x∝k=0xxk−1kB2β+r+kβ,∝,

Gamma $f(x) = \binom{r+x-1}{x} \sum_{k=0}^{x} \binom{x}{k} \frac{(-1)^k}{(r+k+1)^{\alpha}}$Beta

Exponential$p(X = x) = \binom{r+x-1}{x} \sum_{k=0}^{x} \binom{x}{k} (-1)^k \frac{B\left(b+\frac{r+k}{c},a\right)}{B(a,b)}$

, Variate Gamma $f(x) = \binom{r+x-1}{x} \frac{1}{ln\left(\frac{b}{a}\right)} \left[\ln\left(\frac{r+k+b}{r+k+a}\right) + \frac{a}{r+k+a} - \frac{b}{r+k+b}\right] \sum_{k=0}^{x} \binom{x}{k} (-1)^k$, Variate Exponential$f(x) = \binom{r+x-1}{x} \frac{ln\left(\frac{b+r+k}{a+r+k}\right)}{ln\left(\frac{b}{a}\right)} \sum_{k=0}^{x} \binom{x}{k} (-1)^k$.

NB - Inverse Gaussian distribution was also obtained using recursive relations as $P_r\,(x) = \frac{r+x-1}{x}\left[P_r(x-1) - \frac{r}{r+x-1}P_{r+1}\,(x-1)\right]$.

Geometric mixtures have beenobtained by putting$r = 1$ in the Negative Binomial mixtures; we came up with Geometric-exponential,Geometric- Gamma, Geometric-Beta Exponential, Geometric-inverse Gaussian, and Geometric-Lindley mixtures.

Cases in which the parameters *p* is fixed and *r* is a random variable where ithas a continuous mixing distribution is considered, the probability generating function used is $G(s) = \sum_{k=0}^{\infty} p_k s^k$ where $p_k$is a Negative Binomialmixture. The results obtained were: NB- Exponential $G(s) = \frac{\lambda}{\lambda + log\left(\frac{1-qs}{p}\right)}$,NB- Gamma $G(s) = \left[1 + \beta log\left(\frac{1-qs}{p}\right)\right]^{-\alpha}$, NB-Beta Exponential $G(s) = \frac{c}{B(a,b)} \frac{\sum_{j=0}^{\infty}\binom{a-1}{j}(-1)^j}{log\left(\frac{1-qs}{p}\right)+c(b+j)}$, NB- Exponentiated exponential $G(s) = \frac{\lambda\alpha\sum_{j=0}^{\infty}\binom{\alpha-1}{j}(-1)^j}{log\left(\frac{1-qs}{p}\right)+\lambda r(1+j)}$, NB- Inverse Gaussian $G(s) = \left(\frac{\alpha\phi + log\frac{1-qs}{p}}{\pi}\right)^{\frac{1}{2}} e^{\phi(2\alpha)^{\frac{1}{2}}}.\,2K_{-\frac{1}{2}}\sqrt{\left(2\phi\left(\alpha\phi + log\frac{1-qs}{p}\right)\right)}$where $K_r(w)$ is a Bessel function of the third kind.

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CHAPTER 1

GENERAL INTRODUCTION

1.1. Background Information

Basic ideas of mixture distributions have been given by Johnson et al. 2005 as follows; An important class of distributions consists of mixing one distribution with another. The notion of mixing has a physical interpretation. For example the random variable concerned may be the result of the actual mixing of a number of different populations, such as the number of car insurance claims varies with category of driver. Alternatively, it may come from one of a number of different unknown sources; a mixturerandom variable is then the outcome of ascribing a probability distribution to the possible sources.

Sometimes, however, "mixing" is just a mechanism for constructing new distributions for which empirical justification must later be sought.

The term "compounding" has often been used in plane of "mixing".

1.2. Categories

Two important categories of mixtures of discrete distribution are

(i) Finite mixtures
(ii) Countable or continuous mixtures

1.2.1. Finite mixtures

Under problems for solution in chapter XII.6 of Feller Vol 1 (1966), let $\{f_i\}$ and $\{g_i\}$ be two probability distributions. If α>0, β>0, α+ β=1, then $\{\alpha f_i + \beta g_i\}$ is also a probability distribution.
Generalizing to more than two distributions we have the following:
A k- component finite-mixture distribution is formed from k different component distributions with cumulative distributions (cdfs) $\mathrm{F}_1(x), \mathrm{F}_2(x), \dots \dots \mathrm{F}_k(x)$, with mixing weights $w_{1,} w_{2,} \dots . . w_{k,}$ where $w_{j,} > 0$ and $\sum_{j=1}^{k} w_{j,} = 1$,
by taking the weighted average

$$F(x) = \sum_{j=1}^{k} w_j F_j(x)$$

as the cdf of a new (mixture) distribution. It corresponds to the actual mixing of a number of different distributions and is sometimes called a superposition.
In the theory of insurance $w_j, j = 1,2, \dots . \mathit{is\ called\ the\ risk\ function}$
The mixed probability mass function is given by

$$Prb(X = x) = \sum_{j=1}^{k} w_j P_j(x)$$

where

$$P_j(x) = F_j(x) - F_j(x-1)$$

The support of the outcome for this type of mixture is the union of the supports for the individual components of the mixture.

1.2.2. Varying Parameters

A mixture distribution also arises when the cdf of a random variable depends on the parameters $\theta_{1,}\theta_{2,\dots\dots\dots,}\theta_m$; i.e., it has the form $F\left(x\backslash\theta_{1,}\theta_{2,\dots\dots\dots,}\theta_m\right)$ and some or all those perameters vary. The new distribution then has the cdf

$$E\left[F\left(x/\theta_{1,}\theta_{2,\dots\dots\dots,}\theta_m\right)\right],$$

where the expectation is with respect to the joint distribution of the k parameters that vary. This includes situations where the source of a random variable is unknown.

Suppose now that only one parameter varies. It is convenient to denote a mixture distribution of this type by the symbolic form

$$F_A \wedge F_B$$

where $F_A$represents the original distribution and F_B the mixing distribution, i.e., the distribution θ.

When θ has a discrete distribution, we will call the outcome a countable mixture, the pmf is

$$\sum_{\theta} F(x/\theta)\, g(\theta)$$

$$P_k = Prob(X = x) = \sum_{\theta} F(x/\theta)\, g(\theta)$$

When the points of increase of the mixing distribution are continuous, we will call the outcome a continuous mixture.

The pdf is

$$f(x) = \int F(x\backslash\theta) g(\theta)\, dH(\theta) = \int F(x\backslash\theta) g(\theta)\, h(\theta) d(\theta)$$

1.3. Bayesian Interpretation

There is an interesting interpretation of mixtures via Baye's Theorem.

From

$$P_x = \sum_{\theta} f(x/\theta)\, g(\theta)$$

we have

$$1 = \sum_{\theta} \frac{f(x/\theta)g(\theta)}{f(x)}$$

∴

$$\frac{f(x/\theta)g(\theta)}{f(x)}\ is\ a\ pmf\ for\ a\ discrete\ prior\ distribution.$$

Then $\frac{f(x/\theta)g(\theta)}{f(x)}$ is regarded as a pmf for posterior distribution.

For continuous mixing distribution we have

from

$$f(x) = \int f(x\backslash\theta)\, h(\theta) d(\theta)$$

$$1 = \int \frac{f(x\backslash\theta)h(\theta)d(\theta)}{f(x)}$$

∴

$$\frac{f(x\backslash\theta)h(\theta)}{f(x)}$$

is the posterior pdf for the prior pdf $h(\theta)$.

1.4.Negative Binomial Distribution

Many different models give rise to the negative binomial distribution, and consequently there is a variety of definitions in the literature.

The two main dichotomies are:

(a) Between parameterizations

(b) Between points of support

Formally, the negative binomial distribution can be defined in terms of the expansion of

$$(Q-P)^{-N} = \sum_{x=0}^{\infty} \binom{-N}{x} Q^{-N-x}(-P)^x$$

$$= \sum_{x=0}^{\infty} (-1)^x \binom{-N}{x} P^x Q^{-N-x}$$

$$= \sum_{x=0}^{\infty} \binom{N+x-1}{x} P^x Q^{-N-x}$$

$$= \sum_{x=0}^{\infty} \binom{N+x-1}{N-1} \left(\frac{P}{Q}\right)^x \left(\frac{1}{Q}\right)^N$$

∴

$$1 = \sum_{x=0}^{\infty} \binom{N+x-1}{N-1} \left(\frac{P}{Q}\right)^x \left(\frac{1}{Q}\right)^N \frac{1}{(Q-P)^{-N}}$$

$$= \sum_{x=0}^{\infty} \binom{N+x-1}{N-1} \left(\frac{P}{Q}\right)^x \left(\frac{Q-P}{Q}\right)^N$$

$$= \sum_{x=0}^{\infty} \binom{N+x-1}{N-1} \left(\frac{P}{Q}\right)^x \left(1-\frac{P}{Q}\right)^N$$

$$Prob(X=x) = \binom{N+x-1}{N-1} \left(\frac{P}{Q}\right)^x \left(1-\frac{P}{Q}\right)^N \quad for\ x = 0,1,2,\ldots$$

and $Q - P = 1 \Rightarrow Q = 1 + P$.

$\Rightarrow \mu = \mathrm{N}P$ and $\mu_2 = \mathrm{N}P(1+p)$, $G(s) = (1 + \mathrm{p} - \mathrm{ps})^{-N}$

<u>Parameterizations</u>

<u>Fisher (1941)</u> used $\mu = \mathrm{N}P$ and $\mu_2 = \mathrm{N}P(1+p)$as the parameterization.

<u>Jeffreys (1941)</u> had $\mathrm{b} = \frac{P}{1+p}$ and $\rho = \mathrm{N}P$

$\therefore \mu = \rho, \mu_2 = \frac{\rho}{1-b}$ and $G(x) = \frac{1-bs}{(1-b)^{\rho-\frac{\rho}{b}}}$

<u>Ariscombe (1950</u> used $\alpha = \mathrm{N}, \mathrm{N} = \mathrm{N}P$ giving $\mu = \mathrm{m}(1+\mathrm{a}), G(s) = (1 + \mathrm{a} - \mathrm{as})^{-\frac{m}{a}}$

<u>Another parameterization</u>

$\mathrm{p} = \frac{1}{1+p}$ and $\mathrm{q} = \frac{P}{1+p}$

giving

$\mu = \frac{\text{Nq}}{1-q}$ and $\mu_2 = \frac{\text{Nq}}{(1-q)^2}$

sometimes $\lambda = \frac{\text{p}}{1+p}$ is used to avoid confusion with the binomial parameter q.

1.5. Literature Review

The various works that have been doneon Negative Binomial mixtures are given as follows:

Bowman et al (1992) derived a large number of new Binomial mixtures distributions by assuming that the probability parameter *p* varied according to some laws, mostly derived from frullani integrals. They used the transformation $p = e^{-t}$ and considered various densities for the transformed variables. They also gave graphical representations for some of the more significant distributions.

Alanko and Duffy (1996) developed a class of Binomial mixtures arising from transformations of the Binomial parameter *p* as $1 - e^{-\lambda}$ where λ was treated as a random variable. They showed that this formulation provided closed forms for the marginal probabilities in the compound distribution if the Laplace transform of the mixing distribution could be written in a closed form. They gave examples of the derived compound Binomial distributions; simple properties, and parameter estimates from moments and maximum likelihood estimation. They further illustrate the use of these models by examples from consumption processes.

G'omez (2006) proposed a new compound negative binomial distribution by mixing the p negative binomial parameter with inverse Gaussian distribution. Basic properties of the new distribution were given, three estimation of parameters method were given using method of moments, maximum likelihood method and zero proportion method. Finally, examples of application for both univariate and bivariate cases were given.

Zamani and Ismail (2010) came up with negative binomial – Lindley distribution which provides a better fit compared to the Poisson and the negative binomial for count data where the probability at zero has a large value. They gave simple properties, and parameter estimates from method of moments and maximum likelihood estimation. They also illustrated the use of model by examples from insurance count data.

Bodhisuwan and Zeephongsekul (2012) introduced a new distribution and a more flexible alternative to Poisson distribution when count data are over-dispersed in the form of a Negative Binomial – Beta Exponential (NB – BE) distribution. They gave properties and parameters estimation using maximum likelihood method.

1.6. Statement of the problem and objectives of the study

The negative Binomial distribution can be expressed in three forms;

i. From the direct use of p

$$\binom{r+x-1}{x} p^r q^x, x = 0,1,2,3, \dots \dots$$

ii. From transformation of p to e^{-t}

$$\binom{r+x-1}{x} e^{-tr}(1-e^{-t})^x, x = 0,1,2,3, \dots \dots$$

iii. From transformation of p to $1-e^{-t}$

$$\binom{r+x-1}{x} (1-e^{-t})^r e^{-tx}, x = 0,1,2,3, \dots \dots$$

The corresponding mixed Negative Binomial distributions are;

i. $f(x) = \binom{r+x-1}{x} \int_0^1 p^r (1-p)^x g(p) dp$

ii. $f(x) = \binom{r+x-1}{x} \int_0^\infty e^{-tr}(1-e^{-t})^x g(t) dt$

iii. $f(x) = \binom{r+x-1}{x} \int_0^\infty (1-e^{-t})^r e^{-tx} g(t) dt$

Most of the studies on Binomial and Negative Binomial mixtures used p in $[0,1]$ domain hence there is a need to investigate on cases where p is in $[0,\infty]$domain. This calls for the parameterization of $p = e^{-\lambda} and\ p = 1-e^{-\lambda}$.

Zamani and Ismail (2010)came up with NB – Lindely distribution and G'omez (2006) proposed a new compound negative binomial distribution by mixing the p negative binomial parameter with inverse Gaussian distributionbut the procedure of obtaining these distributions was not provided hence there is a need to explore on the procedures.

Gamma, Exponeniated Exponential, Variate Gamma and Variate Exponential aresome of mixing distributions that have not beenused to construct Negative Binomial mixtures. Therefore there is need to generate information on these new mixtures. This study will take into consideration the introduction of these new mixtures and their properties.

Research Questions:

1. How do we generate mixtures for the Negative Binomial in the $[0,\infty]$ domain?
2. What are the other forms of expressing the Negative Binomial mixtures?

Objectives

1. To generate more mixtures and to apply distributions in the $[0, \infty]$ domain as mixtures for the Negative Binomial, when the success parameter is transformed, such that $p = e^{-t}\, or\, p = 1 - e^{-t}$.
2. To examine alternative forms of the Negative Binomial mixed distributions.

1.7. Probability Distributions

Probability distribution is a major area of statistics.

If X is a continuous random variable then it has a probability density function $f(x)$ and therefore its probability of falling into a given interval, say [*a,b*]is given by the integral

$$Pr[a \leq x \leq b] = \int_a^b f(x)dx$$

If X is a discrete random variable, then it has a probability mass function $f(x)$ and its probability is given by:

$$\sum_u pr(X = u) = 1$$

There, are various methods for constructingNegative Binomial distribution.These are:

· Power series based distributions

· Transformation based distributions

· Distributions based on mixtures

· Distributions based on recursive relations in probabilities

· Distributions based on hazard functions of survival analysis

· Distributions emerging from stochastic processes

· Sum of independent random variables

NEGATIVE BINOMIAL DISTRIBUTION

The Negative Binomial (NB) distribution is another distribution for count data. The NB distribution is often employed in case where a distribution is over-dispersed, i.e., its variance is greater than the mean which relaxes the equality of mean and variance property of the Poisson distribution. If X denotes a random variable distributed under a NB distribution with parameter *r* and *p*, then its probability mass function (pmf) is given by:

$$f(x) = \binom{x+r-1}{x} p^r (1-p)^x, \;\; x = 0,1,2, \dots . \, and \; 0 < p < 1.$$

Case 1:

Let $X =$ the total number of failures before the r^{th}success

$\therefore X + r - 1$ =the total number of trials before the r^{th} success.

$\therefore$

$$\mathrm{p_k} = \; prob(X = k)$$

= [Probability of having $(r-1)$ successes out of $(X + r - 1)$trials]×[probability of achieving the r^{th}success]

$$= \left[\binom{x+r-1}{r-1} p^{r-1} q^x\right] p$$

$$= \binom{x+r-1}{r-1} q^x p^r$$

$$= \binom{x+r-1}{k} q^k p^r, \;\; k = 0,1,2, \dots ..$$

Case 2:

Let Y be the total number of trails required to achieve r successes. If $Y = k$, then $k - 1 =$ the number of trials required to obtain the first $(r-1)$ successes

$\therefore$

$$\mathrm{p}_k = \; prob(Y = k)$$

= [Probability of obtaining (r−1) successes out of (k−1)trials]×[probability of obtaining the r^{th}success]

$$= \left[\binom{k-1}{r-1} p^{r-1} q^{k-r}\right] p$$

$$= \binom{k-1}{r-1} q^{k-r} p^r$$

$$= \binom{k-1}{k-r} q^{k-r} p^r, \;\; k = r, r+1, r+2, \dots ..$$

1.8. Constructing Negative Binomial Distribution

1.8.1 Distribution based on recursive relations in probabilities

a. Consider the following recursive relation in probabilities

$$\frac{f(x+1)}{f(x)} = \frac{P(x)}{Q(x)}$$

where $p(x)$ and $Q(x)$ are polynomials in x.

$f(x)$is a probability mass function in particular.

Let

$$\frac{p(x)}{Q(x)} = \frac{\alpha + \beta x}{x+1}$$

∴

$$\frac{f(x+1)}{f(x)} = \frac{\alpha + \beta x}{x+1} \;, x = 0,1,2, \dots \dots$$

∴

$$f(x+1) = \frac{\alpha + \beta x}{x+1} f(x) \;, x = 0,1,2, \dots \dots$$

Let $\alpha \neq 0$ and $\beta \neq 0$

Then

$$f(x+1) = \frac{\alpha + \beta x}{x+1} f(x) \;, x = 0,1,2, \dots \dots$$

When $x = 0$

$$f(1) = \alpha f(0)$$

When x = 1,

$$f(2) = \frac{\alpha + \beta}{2} f(1) = \left(\frac{\alpha + \beta}{2}\right) \alpha f(0)$$

Whenx = 2,

$$f(3) = \frac{\alpha + 2\beta}{3} f(2) = \left(\frac{\alpha + 2\beta}{3}\right)\left(\frac{\alpha + \beta}{2}\right) \alpha f(0)$$

When x = 3

$$f(4) = \frac{\alpha + 3\beta}{4} f(3) = \left(\frac{\alpha + 3\beta}{4}\right)\left(\frac{\alpha + 2\beta}{3}\right)\left(\frac{\alpha + \beta}{2}\right) \alpha f(0)$$

When $x = k - 1$

$$f(k) = \left(\frac{\alpha + (k-1)\beta}{k}\right)\left(\frac{\alpha + (k-2)\beta}{k-1}\right) \dots\dots\dots.. \left(\frac{\alpha + 3\beta}{4}\right)\left(\frac{\alpha + 2\beta}{3}\right)\left(\frac{\alpha + \beta}{2}\right) \alpha f(0)$$

$$= \frac{\left[\beta\left(\frac{\alpha}{\beta} + k - 1\right)\right]\left[\beta\left(\frac{\alpha}{\beta} + k - 2\right)\right] \dots \left[\beta\left(\frac{\alpha}{\beta} + 2\right)\right]\left[\beta\left(\frac{\alpha}{\beta} + 1\right)\right]\left[\beta\left(\frac{\alpha}{\beta}\right)\right]}{k!} f(0)$$

$$= \frac{\beta^k}{k!}\left(\frac{\alpha}{\beta} + k - 1\right) f(0)$$

∴

$$f(k) = \beta^k \binom{\frac{\alpha}{\beta} + k - 1}{k} f(0)\ , k = 1,2, \dots..$$

$$f(0) + \sum_{k=1}^{\infty} f(k) = 1$$

$$f(0) + \sum_{k=1}^{\infty} \beta^k \binom{\frac{\alpha}{\beta} + k - 1}{k} f(0) = 1$$

$$f(0) \sum_{k=0}^{\infty} \binom{\frac{\alpha}{\beta} + k - 1}{k} \beta^k = 1$$

∴

$$f(0) = \frac{1}{\sum_{k=0}^{\infty} \binom{\frac{\alpha}{\beta} + k - 1}{k} \beta^k}$$

∴

$$f(k)=\frac{\beta^k\binom{\frac{\alpha}{\beta}+k-1}{k}}{\sum_{k=0}^{\infty}\beta^k\binom{\frac{\alpha}{\beta}+k-1}{k}},k=0,1,2,\dots\dots$$

If $r=\frac{\alpha}{\beta}$ *is a positive integer*

$$\binom{\frac{\alpha}{\beta}+k-1}{k}=\binom{r+k-1}{k}=(-1)^k\binom{-r}{k}$$

∴

$$\sum_{k=0}^{\infty}\beta^k\binom{\frac{\alpha}{\beta}+k-1}{k}=\sum_{k=0}^{\infty}(-\beta)^k\binom{-r}{k}$$

$$=\sum_{k=0}^{\infty}\binom{-r}{k}(-\beta)^k$$

$$=(1-\beta)^{-r},\ \ 0<\beta<1$$

Conclusion

$$f(x+1)=\frac{\alpha+\beta x}{x+1}f(x)\,,x=0,1,2,\dots\dots$$

For

1. $\frac{\alpha}{\beta}=r$ *is positive integer*
2. $0<\beta<1$
3. $\alpha>0$

∴

$$f(k)=\frac{\beta^k\binom{\frac{\alpha}{\beta}+k-1}{k}}{(1-\beta)^{-r}}$$

$$=\beta^k\binom{\frac{\alpha}{\beta}+k-1}{k}(1-\beta)^r$$

$$=\binom{\frac{\alpha}{\beta}+k-1}{k}\beta^k(1-\beta)^r,k=0,1,2,\dots.$$

$$f(k) = \binom{r+k-1}{k} \beta^k (1-\beta)^r, k = 0,1,2, \dots.$$

which is a negative binomial distribution.

b. Using the pgf technique

We have

$$f(x+1) = \frac{\alpha + \beta x}{x+1} f(x)\,, x = 0,1,2, \dots \dots$$

when $\alpha \neq 0$ and $\beta \neq 0$

$$\sum_{x=0}^{\infty} (x+1) f(x+1) s^x = \sum_{x=0}^{\infty} (\alpha + \beta x) f(x)\, s^x$$

Define

$$G(s) = \sum_{x=0}^{\infty} f(x)\, s^x$$

∴

$$G'(s) = \sum_{x=0}^{\infty} x f(x)\, s^{x-1}$$

$$= \sum_{x=0}^{\infty} (1+x) f(x+1)\, s^x$$

∴

$$G'^{(s)} = \alpha \sum_{x=0}^{\infty} f(x)\, s^x + \beta \sum_{x=0}^{\infty} f(x)\, s^x$$

$$= \alpha\, G(s) + \beta s \sum_{x=0}^{\infty} x f(x)\, s^{x-1}$$

∴

$$G'^{(s)} = \alpha\, G(s) + \beta s G'(s)$$

$$(1 - \beta s) G'^{(s)} = \alpha\, G(s)$$

$$(1-\beta s)\frac{dG}{ds} = \alpha\, G(s)$$

$$\int \frac{dG}{ds} = \int \frac{\alpha}{(1-\beta s)}\, ds$$

$$lnG(s) = \frac{\alpha}{-\beta} ln(1-\beta s) + lnC$$

$$lnG(s) = ln(1-\beta s)^{\frac{-\alpha}{\beta}} + lnC$$

∴

$$lnG(s) = lnC(1-\beta s)^{\frac{-\alpha}{\beta}}$$

To determine C put $s = 1$

$$lnG(1) = lnC(1-\beta)^{\frac{-\alpha}{\beta}}$$

$$1 = C(1-\beta)^{\frac{-\alpha}{\beta}}$$

∴

$$C = (1-\beta)^{\frac{\alpha}{\beta}}$$

∴

$$G(s) = \left(\frac{1-\beta}{1-\beta s}\right)^{\frac{\alpha}{\beta}}$$

$$G(s) = \left(\frac{1-\beta s}{1-\beta}\right)^{\frac{-\alpha}{\beta}}$$

Let $\frac{\alpha}{\beta} = r$, be a positive interger

∴

$$G(s) = \left(\frac{1-\beta s}{1-\beta}\right)^{-r} = \left(\frac{1-\beta}{1-\beta s}\right)^{r}$$

$$= \left(\frac{p}{1-(1-p)s}\right)^{r}$$

Which is a negative binomial distribution with $p = 1-\beta, 0 < \beta < 1, 0 < p < 1,$

1.8.2. Negative Binomial Based On Mixtures

Poisson – Gamma mixture

Suppose that λ has a Gamma distribution with scale parameter α and shape parameter β. Then the pdf of λ is given by

$$g(\lambda) = \frac{\alpha^{\beta}}{\Gamma\beta}\lambda^{\beta-1}e^{-\alpha\lambda}, \qquad \lambda > 0$$

The pdf of a Poisson distribution is given by

$$P(X = x/\lambda = \lambda)g(\lambda) = \frac{e^{-\lambda}\lambda^{x}}{x!}, x = 0,1,2, \dots \dots$$

Then the joint density of X and λ is

$$P(X = n/\lambda = \lambda)g(\lambda) = \frac{e^{-\lambda}\lambda^{x}}{x!}\frac{\alpha^{\beta}}{\Gamma\beta}\lambda^{\beta-1}e^{-\alpha\lambda} \dots \dots (i)$$

The unconditional distribution of X is obtained by summing out λ in (i)

$$P(X = x) = \int_0^\infty P(X = n/\lambda = \lambda)g(\lambda)d\lambda$$

$$= \int_0^\infty \frac{e^{-\lambda}\lambda^{x}}{x!}\frac{\alpha^{\beta}}{\Gamma\beta}\lambda^{\beta-1}e^{-\alpha\lambda}d\lambda$$

$$= \int_0^\infty \frac{\alpha^{\beta}}{x!\,\Gamma\beta}\lambda^{x+\beta-1}e^{-(\alpha+1)\lambda}d\lambda$$

$$= \frac{\alpha^{\beta}}{x!\,\Gamma\beta}\frac{\Gamma(x+\beta)}{(a+1)^{x+\beta}}\int_0^\infty \frac{(a+1)^{x+\beta}}{\Gamma(x+\beta)}\lambda^{x+\beta-1}e^{-(\alpha+1)\lambda}d\lambda$$

$$= \frac{\alpha^{\beta}}{x!\,\Gamma\beta}\frac{\Gamma(x+\beta)}{(a+1)^{x+\beta}}$$

$$= \frac{\Gamma(x+\beta)}{(a+1)^{x+\beta}\Gamma\beta}\left(\frac{\alpha}{\alpha+1}\right)^{\beta}\left(\frac{1}{\alpha+1}\right)^{x}$$

$$= \binom{x+\beta-1}{x}\left(\frac{\alpha}{\alpha+1}\right)^{\beta}\left(\frac{1}{\alpha+1}\right)^{x}, \qquad x = 0,1,2,..$$

which is a negative binomial distribution with parameter $r = \beta \; and \; p = \frac{\alpha}{\alpha+1}$

1.8.3. Binomial Expansion / Power Series

(a) Consider $(a+b)^n$

Where n is a negative integer

Let $n = -r$ where r is a positive integer

$$(a+b)^{-r} = \binom{-r}{0} a^{-r} b^0 + \binom{-r}{1} a^{-r-1} b^1 + \binom{-r}{2} a^{-r-2} b^2 + \cdots + \binom{-r}{k} a^{-r-k} b^k + \cdots$$

$$= \sum_{k=0}^{\infty} \binom{-r}{k} a^{-r-k} b^k$$

Put $a = 1 \; and \; b = -s$

$\therefore$

$$(1-s)^{-r} = \sum_{k=0}^{\infty} \binom{-r}{k} (-s)^k$$

$$= \sum_{k=0}^{\infty} (-1) \binom{-r}{k} (s)^k$$

$\therefore (1-s)^{-r}$ is the generating function for the sequence $\left\{(-1)\binom{-r}{k}: k = 0,1,2,\ldots\right\}$

If n is a positive integer

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

In general

$$\binom{n}{k} = \frac{n(n-1)(n-2)\ldots\ldots(n-(k-1))}{1.2.3\ldots\ldots..k}$$

But

$$\binom{-r}{k} = \frac{-r(-r-1)(-r-2)\ldots\ldots(-r-(k-1))}{1.2.3\ldots\ldots..k}$$

$$= \frac{(-1)^k r(r+1)(r+2)\dots\dots(r+(k-1))}{k!}$$

$$= \frac{(-1)^k \big(r+(k-1)\big)\big(r+(k-2)\big)\dots\dots\dots..(r+(k-k))}{k!}$$

$$= (-1)^k \binom{r+k-1}{k}$$

∴

$$(-1)^k \binom{-r}{k} = \binom{r+k-1}{k}$$

∴$(1-s)^{-r}$ is the generating function for

$$\left\{\left((-1)^k \binom{-r}{k}\right) : k = 0,1,2,\dots\right\}$$

From

$$(1-s)^{-r} = \sum_{k=0}^{\infty} (-1) \binom{-r}{k} (s)^k$$

$$= \sum_{k=0}^{\infty} \binom{r+k-1}{k} (s)^k$$

(b)

Let

$$1 = p^r p^{-r} = p^r (1-q)^{-r}$$

$$= p^r \sum_{k=0}^{\infty} \binom{-r}{k} (-q)^x$$

The n^{th} term in the series above is

$$\binom{-r}{k} p^r (-q)^x = (-1) \binom{-r}{k} p^r (q)^x$$

$$= \binom{r+k-1}{k} p^r (1-p)^x$$

Which is the probability that $\mathrm{X} = x$ where X ~negative binomial with parameter *r* and *p*

1.8.4. Distribution Based On Sum of Independent Random Variables

Let $S_N = X_1 + X_2 + \ldots\ldots + X_N$, where the $X_i's$ are independent identical distribution random variables

Let

$$G(s) = E(S^x), the\ pgf\ of\ X$$

$$F(s) = E(S^N), the\ pgf\ of\ N$$

$$H(s) = E(S^{SN}), the\ pgf\ of\ SN$$

If X $\sim NB(r,p)$

$$H(s) = E(S^{SN}) = EE(S^{SN}\backslash N)$$

$$= E(S^{x_1+x_2+\cdots+x_N})$$

$$= E(S^{x_1})E(S^{x_2})E(S^{x_3}) \ldots . E(S^{x_N})$$

Since $X_i's$ are iid then

$$H(s) = [E(S^x)]^N$$

But

$$G(s) = E(S^x) = \left(\frac{p}{1-qs}\right)^r$$

∴

$$H(s) = [G(s)]^N = \left(\frac{p}{1-qs}\right)^{rN}$$

$$= \prod_{i=1}^{N}\left(\frac{p_i}{1-q_i s}\right)^{r_i}$$

where $\left(\frac{p}{1-qs}\right)^r$ is the pgf of a negative binomial distribution with parameter *p* and *r*.

1.9. Mixtures

From Feller (1957) we can develop a class of probability distributions in the following manner.

Let F_x be a distribution function depending on the parameter θ, and let F be another distribution function. Then

$$F_y(y) = \int_{-\infty}^{\infty} F_x(y \backslash \theta) dF\,(\theta)$$

is also a distribution function.

Fellar calls distributions generated in this manner, mixtures.

Mixtures can thus be generated by randomizing a parameter(s) in a parent distribution.

1.9.1. Negative Binomial Mixtures

The Negative Binomial distribution has two parameters *r* and *p* where either may be randomized, to give a Negative Binomial mixture.

This project discusses cases in which the parameter*p*is continuous mixing distribution with probability density $g(p)$ so that

$$f(x) = \int \binom{r+x-1}{x} p^r (1-p)^x g(p) dp$$

Where $f(x)$ is a Negative Binomial mixture.

1.10. Application

- In automobile insurance, the negative binomial is preferred to the Poisson because is over-dispersed and present experience shows that this is certainly observed in the field of automobile insurance. Because the NBIG distribution is over-dispersed other than the traditional Poisson-gamma and Poisson-inverse Gaussian, the NBIG is chosen for computing automobile insurance premiums. (See Gomez, (2006))
- It is a very useful tool for analyzing crash data characterized with a large amount of zeros. (see (Lord and Geedipally, (2011))

CHAPTER 2

NEGATIVE BINOMIAL MIXTURES BASED ON TRANSFORMATION OF THE PARAMETER p

2.1 Introduction

The mixing distributions are within the interval [0,∞]. The parameter p is transformed to be equal to e^{-t} for $t > 0$

The following methods are used when mixing or compounding negative binomial distribution with other distributions

1. Direct substitution and subsequent integration
2. Recursive method
3. Moments methods or Laplace transform

The mixing distributions considered in this chapter are:

- Exponential
- Gamma
- Exponeniated Exponential
- Beta Exponential
- Variate Gamma
- Variate Exponential
- Inverse Gaussian
- Lindely

1. Method of moments

$$f(x) = \binom{r+x-1}{x} \int_0^1 p^r (1-p)^x g(p) dp$$

$$f(x) = \binom{r+x-1}{x} \sum_{k=0}^{x} \binom{x}{k} (-1)^k \int_0^1 p^{(r+k)} g(p) dp$$

$$f(x) = \binom{r+x-1}{x} \sum_{j=r}^{x} \binom{x}{j-r} (-1)^{j-r} \int_0^1 p^j g(p) dp$$

$$f(x) = \binom{r+x-1}{x} \sum_{j=r}^{x} \binom{x}{j-r} E(p^j)$$

$$f(x) = \sum_{j=r}^{x} \frac{(r+x-1)!}{(r-1)!\,x!} \frac{x!\,(-1)^{j-r}}{(x-j+r)!\,(j-r)!} E(p^j)$$

$$f(x) = \sum_{j=r}^{x} \frac{(r+x-1)!\,(-1)^{j-r}}{(r-1)!\,(x-j+r)!\,(j-r)!} E(p^j)$$

$$for\ j \geq r\ and\ 0\ when\ j < r$$

$$and\ E(p^j) is\ the\ moment\ of\ order\ j\ about\ the\ origin\ of\ the\ mixing\ distribution$$

$$(see\ Sivagenesan\ and\ Berger (1993))$$

When $p = e^{-t}$ *the negative binomial mixture is expressed as*

$$f(x) = \binom{r+x-1}{x} \int_0^\infty p^r (1-p)^x g(t) dt$$

$$= \binom{r+x-1}{x} \int_0^\infty e^{-tr} (1-e^{-t})^x g(t) dt$$

$$= \binom{r+x-1}{x} \int_0^\infty e^{-tr} \sum_{k=0}^{x} \binom{x}{k} (-e^{-t})^k g(t) dt$$

$$= \binom{r+x-1}{x} \sum_{k=0}^{x} \binom{x}{k} (-1)^k \int_0^\infty e^{-t(r+k)} g(t) dt$$

Where $\int_0^\infty e^{-t(r+k)} g(t) dt$ is the laplace transform i.e.

$E\left(e^{-t(r+k)}\right) of\ g(t)$denoted by $L_t(r+k)$

and hence

$$f(x) = \binom{r+x-1}{x} \sum_{k=0}^{x} \binom{x}{k} (-1)^k L_t(r+k)$$

For $x > 0; r > 0; andk = 0,1,2 \ldots \ldots x$

This is the formula for mixing negative binomial with other distributions using the Laplace transform with the value of p transformed into e^{-t}

Obtaining the pgf of the mixed distribution:

$$f(x) = \binom{r+x-1}{x} \sum_{k=0}^{x} \binom{x}{k} (-1)^k L_t(r+k)$$

$$G_X(s) = \sum_{k=0}^{x} f(x) s^x$$

$$= \sum_{x=0}^{\infty} \left\{ \binom{r+x-1}{x} \sum_{k=0}^{x} \binom{x}{k} (-1)^k L_t(r+k) s^k \right\}$$

$$= \sum_{x=0}^{\infty} \sum_{k=0}^{x} (-1)^k \binom{r+x-1}{x} \binom{x}{k} s^k L_t(r+k)$$

$$= \sum_{x=0}^{\infty} \sum_{k=0}^{\infty} (-1)^k \binom{r+x-1}{x} \binom{x}{k} s^k L_t(r+k)$$

$$= \sum_{k=0}^{\infty} \left\{ \sum_{x=0}^{\infty} \binom{r+x-1}{x} \binom{x}{k} s^k \right\} (-1)^k L_t(r+k)$$

Properties

Consider mean of $f(x)$

$$E(X^j) = E_t[E(X^j/T)]$$

E_t denotes the expectation with respect to the distribution of T

hence

$$E(X) = E\big(E(x/p)\big) = E\left(\frac{r(1-p)}{p}\right) = rE\left(\frac{1-p}{p}\right)$$

but $p = e^{-t}$

$$E(X) = rE\left(\frac{1-e^{-t}}{e^{-t}}\right)$$

$$= rE(e^{t}-1)$$

$$= r[E(e^{t}) - E(1)]$$

$$= rE(e^{t}) - r$$

$$= rL_t(-1) - r$$

Variance

$$var(x) = E(X^2) - [E(X)]^2$$

$$E\big(X^j\big) = E_t[E\big(X^j/T\big)]$$

$$E(X^2) = E_t[E(X^2/p = e^{-t})]$$

$$E(X^2/p) = \frac{\mathrm{r}(1-\mathrm{p})(1+\mathrm{r}(1-\mathrm{p}))}{\mathrm{p}^2}$$

$$E(X^2/p = e^{-t}) = \frac{\mathrm{r}(1-e^{-t})(1+\mathrm{r}(1-e^{-t}))}{e^{-2t}}$$

$$E(X^2) = E_t\left[\frac{\mathrm{r}(1-e^{-t})\big(1+\mathrm{r}(1-e^{-t})\big)}{e^{-2t}}\right]$$

$$= \mathrm{r}E_t\left[\frac{(1-e^{-t})\big(1+\mathrm{r}(1-e^{-t})\big)}{e^{-2t}}\right]$$

$$= \mathrm{r}E_t\left[\frac{1+\mathrm{r}-\mathrm{r}e^{-t}-e^{-t}-re^{-t}+re^{-2t}}{e^{-2t}}\right]$$

$$= \mathrm{r}E_t[e^{2t}+\mathrm{r}e^{2t}-\mathrm{r}e^{t}-e^{t}-re^{t}+r]$$

$$= \mathrm{r}E_t[(1+r)e^{2t}-(2\mathrm{r}+1)e^{t}+r]$$

$$= r[(1+r)E_t(e^{2t}) - (2\text{r}+1)E_t(e^t) + r]$$

$$= r[(1+r)L_t(-2) - (2\text{r}+1)L_t(-1) + r]$$

$$var(X) = E(X^2) - [E(X)]^2$$

$$var(X) = r[(1+r)L_t(-2) - (2\text{r}+1)L_t(-1) + r] - r^2[L_t(-1) - 1]^2$$

2.2. Negative Binomial –Exponential distribution

2.2.1. Exponential distribution

Construction

Exponential distribution is one of the distributions that are based on power series

The pdf of an exponential distribution is given by

$$g(t) = \lambda e^{-t\lambda} \qquad ; t > 0;\ \lambda > 0$$

2.2.2. Properties of exponential distribution

The laplace transform of $g(t)$is

$$L_t(s) = E(e^{-ts}) = \int_0^\infty \lambda e^{-t(s+\lambda)} dt$$

$$= \lambda \int_0^\infty e^{-t(s+\lambda)} dt$$

$$= \frac{\lambda}{s+\lambda} \int_0^\infty (s+\lambda) e^{-t(s+\lambda)} dt$$

But

$$= \int_0^\infty (s+\lambda) e^{-t(s+\lambda)} dt = 1$$

Therefore

$$L_t(s) = \frac{\lambda}{s+\lambda}$$

Moment generating function for Exponential distribution

$$M_t(s) = E(e^{ts}) = \int_0^\infty \lambda e^{-t(\lambda-s)} dt$$

$$= \frac{\lambda}{\lambda - s} \int_0^\infty (\lambda - s) e^{-t(\lambda-s)} dt$$

But

$$\int_0^\infty (\lambda - s) e^{-t(\lambda-s)} dt = 1$$

Therefore

$$M_t(s) = \frac{\lambda}{\lambda - s}$$

2.2.2. Negative Binomial – Exponential distribution mixing

(a). using Laplace transform

$$f(x) = \binom{r+x-1}{x} \sum_{k=0}^{x} \binom{x}{k} (-1)^k L_t(r+k)$$

$$f(x) = \binom{r+x-1}{x} \sum_{k=0}^{x} \binom{x}{k} (-1)^k \frac{\lambda}{r+k+\lambda}$$

$$f(x) = \binom{r+x-1}{x} \lambda \sum_{k=0}^{x} \binom{x}{k} \frac{(-1)^k}{r+k+\lambda}$$

$$r > 0;\ \lambda > 0, x > 0$$

Properties of negative binomial exponential distribution mixture

$$E(X) = r(L_t(-1) - 1)$$

$$= r\left(\frac{\lambda}{\lambda - 1} - 1\right)$$

$$= r\left(\frac{1}{\lambda - 1}\right)$$

$$= \frac{r}{\lambda - 1}$$

Variance

(b). Mixing using the moment Generating Function technique

$$f(x) = \binom{r + x - 1}{x} \sum_{k=0}^{x} \binom{x}{k} (-1)^k M_t(-(r + k))$$

$$= \binom{r + x - 1}{x} \sum_{k=0}^{x} \binom{x}{k} \frac{(-1)^k \lambda}{\lambda + r + k}$$

Note: similar result compared to the mixture using Laplace transform

c. Explicit mixing

$$f(x) = \binom{r + x - 1}{x} \int_0^{\infty} e^{-tr} (1 - e^{-t})^x g(t) dt$$

Where

$$g(t) = \lambda e^{-t\lambda} \qquad ; t > 0;\ \lambda > 0$$

$$f(x) = \binom{r + x - 1}{x} \lambda \int_0^{\infty} e^{-tr} (1 - e^{-t})^x e^{-t\lambda} dt$$

$$f(x) = \binom{r + x - 1}{x} \lambda \int_0^{\infty} e^{-t(r+\lambda)} (1 - e^{-t})^x dt$$

$$\boldsymbol{let\ p} = e^{-t}$$

$$\ln e^{-t} = \ln p$$

$$t = -lnp$$

$$dt = -\frac{dp}{p}$$

$$f(x) = \binom{r+x-1}{x}\lambda\int_0^\infty p^{(r+\lambda)}(1-p)^x\left(-\frac{dp}{p}\right)$$

$$f(x) = -\lambda\binom{r+x-1}{x}\int_0^\infty p^{(r+\lambda-1)}(1-p)^x dp$$

$$= -\frac{\lambda\binom{r+x-1}{x}}{\binom{r+\lambda+x-2}{x}}\int_0^\infty \binom{r+\lambda+x-2}{x}p^{(r+\lambda-1)}(1-p)^x dp$$

But

$$\int_0^\infty \binom{r+\lambda+x-2}{x}p^{(r+\lambda-1)}(1-p)^x dp = 1$$

This is a negative binomial pdf

Therefore

$$f(x) = \frac{\lambda\binom{r+x-1}{x}}{\binom{r+\lambda+x-2}{x}}$$

$$f(x) = -\frac{\lambda\Gamma(x+r)\Gamma(r+\lambda-1)}{\Gamma(r+\lambda+x-1)\Gamma r}$$

2.3. Negative binomial distribution-Beta Exponential –

2. 3.1 beta exponential distribution

$$g(t) = \begin{cases} \frac{c}{B(a,b)}e^{-bct}(1-e^{-ct})^{a-1} & x > 0; for\ a, b\ and\ c > 0 \\ 0 & elsewhere \end{cases}$$

$$\boldsymbol{p = e^{-cx}}$$

2.3.2 Properties of beta exponential distribution

(a). moment generating function

$$\boldsymbol{M_t(s)} = E(e^{ts})$$

$$\boldsymbol{M_t(s)} = \frac{c}{B(a,b)} \int_0^\infty e^{-ct\left(b-\frac{s}{c}\right)} (1-e^{-ct})^{a-1} dt$$

But

$$\frac{B(a,b)}{c} = \int_0^\infty e^{-ct(b)} (1-e^{-ct})^{a-1} dt$$

Therefore

$$\boldsymbol{M_t(s)} = \frac{c}{B(a,b)} \int_0^\infty e^{-ct\left(b-\frac{s}{c}\right)} (1-e^{-ct})^{a-1} dt$$

$$\boldsymbol{M_t(s)} = \frac{c}{B(a,b)} \frac{B\left(b-\frac{s}{c}, a\right)}{c}$$

$$\boldsymbol{M_t(s)} = \frac{B\left(b-\frac{s}{c}, a\right)}{B(a,b)}$$

(b). Laplace Transpose

$$\boldsymbol{L_t(s)} = E(e^{-ts})$$

$$\boldsymbol{L_t(s)} = \frac{c}{B(a,b)} \int_0^\infty e^{-ct\left(b+\frac{s}{c}\right)} (1-e^{-ct})^{a-1} dt$$

But

$$\frac{B(a,b)}{c} = \int_0^\infty e^{-ct(b)} (1-e^{-ct})^{a-1} dt$$

Therefore

$$\boldsymbol{L_t(s)} = \frac{c}{B(a,b)} \int_0^\infty e^{-ct\left(b+\frac{s}{c}\right)} (1-e^{-ct})^{a-1} dt$$

$$\boldsymbol{L_t(s)} = \frac{c}{B(a,b)} \frac{B\left(b+\frac{s}{c}, a\right)}{c}$$

$$\boldsymbol{L_t(s)} = \frac{B\left(b+\frac{s}{c}, a\right)}{B(a,b)}$$

2.3.3 Negative Binomial - Beta Exponential Mixing

(a). using Laplace transpose

$$p(X=x) = \binom{r+x-1}{x} \sum_{k=0}^{x} \binom{x}{k} (-1)^k L_t(r+k)$$

$$p(X=x) = \binom{r+x-1}{x} \sum_{k=0}^{x} \binom{x}{k} (-1)^k \frac{B\left(b+\frac{r+k}{c}, a\right)}{B(a,b)}$$

(b). using MGF technique

$$p(X=x)=\binom{r+x-1}{x}\sum_{k=0}^{x}\binom{x}{k}(-1)^k M_t(-(r+k))$$

$$p(X=x)=\binom{r+x-1}{x}\sum_{k=0}^{x}\binom{x}{k}(-1)^k \frac{B\left(b+\frac{r+k}{c},a\right)}{B(a,b)}$$

©.Using explicit format

$$f(x)=\binom{r+x-1}{x}\int_0^{\infty} e^{-tr}(1-e^{-t})^x g(t)dt$$

Where

$$g(t)=\lambda e^{-t\lambda} \quad ;t>0;\ \lambda>0$$

$$f(x)=\binom{r+x-1}{x}\lambda\int_0^{\infty} e^{-tr}(1-e^{-t})^x e^{-t\lambda}dt$$

$$f(x)=\binom{r+x-1}{x}\lambda\int_0^{\infty} e^{-t(r+\lambda)}(1-e^{-t})^x dt$$

$$let\ p=e^{-t}$$

$$\ln e^{-t}=\ln p$$

$$t=-lnp$$

$$dt=-\frac{dp}{p}$$

$$f(x)=\binom{r+x-1}{x}\lambda\int_0^{\infty} p^{(r+\lambda)}(1-p)^x\left(-\frac{dp}{p}\right)$$

$$f(x) = -\lambda \binom{r+x-1}{x} \int_0^\infty p^{(r+\lambda-1)}(1-p)^x dp$$

$$f(x) = -\frac{\lambda \binom{r+x-1}{x}}{\binom{r+\lambda+x-2}{x}} \int_0^\infty \binom{r+\lambda+x-2}{x} p^{(r+\lambda-1)}(1-p)^x dp$$

But

$$\int_0^\infty \binom{r+\lambda+x-2}{x} p^{(r+\lambda-1)}(1-p)^x dp = 1$$

This is a negative binomial pdf

Therefore

$$f(x) = -\frac{\lambda \binom{r+x-1}{x}}{\binom{r+\lambda+x-2}{x}}$$

$$f(x) = -\frac{\lambda \Gamma(x+r)\Gamma(r+\lambda-1)}{\Gamma(r+\lambda+x-1)\Gamma r}$$

2.4.Negative Binomial Distribution–Gamma (I)

2.4.1. Gamma distribution with 1 parameter distribution

(a). Construction

A gamma function can be expressed as follows

$$\Gamma\alpha = e^{-t}t^{\alpha-1}$$

When you divide both sides by $\Gamma\alpha$ we will have

$$1 = \frac{e^{-t}t^{\alpha-1}}{\Gamma\alpha}$$

Since the right hand side is equivalent to 1 the above function forms a pdf and is referred to as a gamma distribution with one parameter α and is expressed as

$$f(t)=\begin{cases}\dfrac{e^{-t}t^{\alpha-1}}{\Gamma\alpha} & t>0;\ \alpha>0\\ 0 & elsewhere\end{cases}$$

(b). Properties of gamma distribution with one parameter

- **Laplace transform for Gamma(I) distribution**

$$L_t(s)=E(e^{-ts})=\int_0^\infty \frac{e^{-t(1+s)}t^{\alpha-1}}{\Gamma\alpha}dt$$

$$let\ t(s+1)=u$$

$$t=\frac{u}{s+1}$$

$$dt=\frac{du}{s+1}$$

Hence

$$L_t(s)=E(e^{-ts})=\frac{1}{(s+1)^\alpha}\int_0^\infty \frac{e^{-u}u^{\alpha-1}}{\Gamma\alpha}du$$

$$L_t(s)=\frac{1}{(s+1)^\alpha}$$

- **Moment generating function for Gamma(I) distribution**

$$M_t(s)=E(e^{ts})=\int_0^\infty \frac{e^{-t(s-1)}t^{\alpha-1}}{\Gamma\alpha}dt$$

$$let\ t(s-1)=u$$

$$t = \frac{u}{s-1}$$

$$dt = \frac{du}{s-1}$$

Hence

$$M_t(s) = E(e^{ts}) = \frac{1}{(s-1)^{\alpha}} \int_0^{\infty} \frac{e^{-u}u^{\alpha-1}}{\Gamma\alpha} du$$

$$M_t(s) = \frac{1}{(s-1)^{\alpha}}$$

2.4.4 Negative Binomial –Gamma (I) distribution mixing

a) Using Laplace transform

$$f(x) = \binom{r+x-1}{x} \sum_{k=0}^{x} \binom{x}{k} (-1)^k L_t(r+k)$$

$$= \binom{r+x-1}{x} \sum_{k=0}^{x} \binom{x}{k} \frac{(-1)^k}{(r+k+1)^{\alpha}}$$

b) Using MGF

$$f(x) = \binom{r+x-1}{x} \sum_{k=0}^{x} \binom{x}{k} (-1)^k M_t(-(r+k))$$

$$f(x) = \binom{r+x-1}{x} \sum_{k=0}^{x} \binom{x}{k} \frac{(-1)^k}{(r+k+1)^\alpha}$$

Note:

Both mixtures based on laplace transpose and moment generating function results to the same solution.

Properties of the mixture Gamma (I) – Negative Binomial Mixture

$$E(X) = r(L_t(-1) - 1)$$

$$= r\left(\frac{1}{(1-1)^\propto} - 1\right)$$

This is undefined and therefore it is not possible to evaluate the expectation of this mixture.

Since the mean is undefined, and we use the mean in finding the variance, it definitely means the variance is equally undefined.

2.5. Negative Binomial – Gamma with two parameters distribution

2.5.1. Gamma with two parameters distribution

(a). Construction of Gamma with two parameters distribution

Consider a Gamma distribution with one parameter

$$g(x) = \begin{cases} \frac{e^{-x}x^{\alpha-1}}{\Gamma\alpha} & x > 0; \alpha > 0 \\ 0 & elsewhere \end{cases}$$

Let $x = t\beta$

$$g(t) = \frac{e^{-t\beta}(t\beta)^{\alpha-1}}{\Gamma\alpha}|J|$$

$$t > 0;\ \alpha, \beta > 0$$

$$|J| = \left|\frac{\delta x}{\delta t}\right|$$

$$= \beta$$

$$g(t) = \begin{cases} \dfrac{e^{-t\beta} t^{\alpha-1} \beta^{\propto}}{\Gamma\alpha} & t > 0;\ \alpha, \beta > 0 \\ 0 & elsewhere \end{cases}$$

This is the pdf of a Gamma (II) distribution with parameters $\alpha\ and\ \beta$

(b). Properties of Gamma with two parameters distribution

- **Laplace transpose**

$$L_t(s) = E(e^{-ts}) = \beta^{\alpha} \int_0^{\infty} \frac{t^{\alpha-1} e^{-t(\beta+s)}}{\Gamma\alpha} dt$$

Let

$$t(\beta + s) = m$$

$$t = \frac{m}{\beta + s}$$

$$dt = \frac{dm}{\beta + s}$$

$$L_t(s) = \frac{\beta^{\alpha}}{(\beta + s)^{\alpha}} \int_0^{\infty} \frac{m^{\alpha-1} e^{-m}}{\Gamma\alpha} dm$$

$$L_t(s) = \left[\frac{\beta}{\beta + s}\right]^{\alpha}$$

- **Moment generating function**

$$M_t(s) = E(e^{ts}) = \beta^{\alpha} \int_0^{\infty} \frac{t^{\alpha-1} e^{-t(\beta-s)}}{\Gamma\alpha} dt$$

Let

$$t(\beta - s) = m$$

$$t = \frac{m}{\beta - s}$$

$$dt = \frac{dm}{\beta - s}$$

$$M_t(s) = \frac{\beta^{\alpha}}{(\beta - s)^{\alpha}} \int_0^{\infty} \frac{m^{\alpha-1} e^{-m}}{\Gamma\alpha} dm$$

$$M_t(s) = \left[\frac{\beta}{\beta - s}\right]^{\alpha}$$

2.5.2. Negative binomial distribution– Gamma (II) mixing

a. **Mixing using Laplace**

$$f(x) = \binom{r+x-1}{x} \sum_{k=0}^{x} \binom{x}{k} (-1)^k L_t(r+k)$$

$$f(x) = \binom{r+x-1}{x} \sum_{k=0}^{x} \binom{x}{k} (-1)^k \left[\frac{\beta}{\beta + r + k}\right]^{\alpha}$$

b. Mixing using Moment Generating function

$$f(x) = \binom{r+x-1}{x} \sum_{k=0}^{x} \binom{x}{k} (-1)^k M_t(-(r+k))$$

$$f(x) = \binom{r+x-1}{x} \sum_{k=0}^{x} \binom{x}{k} (-1)^k \left[\frac{\beta}{\beta+r+k}\right]^{\alpha}$$

2.5.3. Properties of Gamma (II) – Negative binomial distribution

$$E(X) = r(L_t(-1) - 1)$$

$$= r\left[\left[\frac{\beta}{\beta-1}\right]^{\alpha} - 1\right]$$

$$\boldsymbol{var(X) = r[(1+r)L_t(-2) - (2\mathrm{r}+1)L_t(-1) + r] - r^2[L_t(-1) - 1]^2}$$

$$var(x) = \boldsymbol{r}\left[(\boldsymbol{1}+\boldsymbol{r})\left[\frac{\beta}{\beta-2}\right]^{\alpha} - \left[\frac{\beta}{\beta-1}\right]^{\alpha} (\boldsymbol{2\mathrm{r}+1})\boldsymbol{L_t} + \boldsymbol{r}\right] - \boldsymbol{r^2}\left[\left[\frac{\beta}{\beta-1}\right]^{\alpha} - \boldsymbol{1}\right]^{\boldsymbol{2}}$$

2.6.Negative Binomial –Exponentiated exponential with one parameter Distribution

2.6.1. Exponentiated exponential Distribution with one parameter

a. Construction

$$F(x) = [G(x)]^{\alpha}$$

$$g(x) = \lambda e^{-\lambda t}$$

$$f(t) = \lambda\alpha(1 - e^{-\lambda t})^{\alpha-1}e^{-\lambda t}$$

When $\lambda = 1$

$$f(x) = \alpha(1 - e^{-t})^{\alpha-1}e^{-t}, t > 0;\ \alpha > 0$$

b. Properties of Exponentiated exponential Distribution with one parameter

- **Laplace transform**

$$L_t(s) = E(e^{-ts})$$

$$=\propto \int_0^\infty (1 - e^{-t})^{\propto-1}e^{-t(s+1)}dt$$

Remember

$$B(\alpha, \beta) = \int_0^\infty e^{-t(\alpha-1)}(1 - e^{-t})^{\beta-1}$$

Where $B(\alpha, \beta)$ is a Beta function with parameters$\alpha\ and \beta$

Thus

$$L_t(s) =\propto B(s + 2, \propto)$$

2.6.2. Moment Generating Function of Exponentiated exponential with one parameter

$$M_t(s) = E(e^{ts})$$

$$=\propto \int_0^\infty (1 - e^{-t})^{\propto-1}e^{-t(1-s)}dt$$

Remember

$$B(\alpha, \beta) = \int_0^\infty e^{-t(\alpha-1)}(1 - e^{-t})^{\beta-1}$$

Where $B(\alpha,\beta)$ is a Beta function with parameters $\alpha\ and\beta$

Thus

$$M_t(s) = \propto B(2-s,\propto)$$

2.6.2. Negative Binomial –Exponentiated exponential with one parameter Distribution mixing

a. Mixing using Laplace transform

$$f(x) = \binom{r+x-1}{x}\sum_{k=0}^{x}\binom{x}{k}(-1)^k L_t(r+k)$$

$$f(x) = \binom{r+x-1}{x}\alpha\sum_{k=0}^{x}\binom{x}{k}(-1)^k B(2+r+k,\alpha)$$

b. Mixing using MGF

$$f(x) = \binom{r+x-1}{x}\sum_{k=0}^{x}\binom{x}{k}(-1)^k M_t(-(r+k))$$

$$f(x) = \binom{r+x-1}{x}\propto\sum_{k=0}^{x}\binom{x}{k}(-1)^k B(2+r+k,\propto)$$

2.6.4. Properties of Exponentiated exponential with one parameter – Negative Binomial Distribution mixture

a). Mean

$$E(X) = r(L_t(-1) - 1)$$

$$= r(\alpha B(1,\alpha) - 1)$$

b). Variance

$$\boldsymbol{var(X) = r[(1+r)L_t(-2) - (2r+1)L_t(-1) + r] - r^2[L_t(-1) - 1]^2}$$

$$\boldsymbol{var(X) = r[(1+r)\alpha B(0,\alpha) - (2r+1)\alpha B(1,\alpha) + r] - r^2[\alpha B(1,\alpha) - 1]^2}$$

2.7. Negative Binomial –Exponentiated exponential with 2 parameter Distribution

2.7.1 Exponentiated exponential Distribution with 2 parameter

a. Construction

Consider a **Exponentiated exponential with 1 parameter**

$$g(\theta) = \begin{cases} \propto \left(1 - e^{-\theta}\right)^{\propto -1} e^{-\theta} & \theta > 0;\ \propto > 0 \\ 0 & elsewhere \end{cases}$$

$$let\ \theta = t\beta t > 0$$

Such that

$$g_1(t) = \propto \left(1 - e^{-t\beta}\right)^{\propto -1} e^{-t\beta} |J|$$

$$|J| = \left|\frac{\delta\theta}{\delta t}\right| = \beta$$

$$g(t) = \begin{cases} \propto \beta \left(1 - e^{-t\beta}\right)^{\propto -1} e^{-t\beta} & t > 0;\ \propto > 0 \\ 0 & elsewhere \end{cases}$$

This is the pdf of Exponentiated exponential distribution with 2 parameter $\alpha\ and\beta$ which is denoted as Exponentiated exponential (II) distribution

b. Properties of Exponentiated exponential Distribution with two parameter

- **Laplace transform**

$$L_t(s) = E(e^{-ts})$$

$$=\propto \beta \int_0^{\infty} (1 - e^{-t\beta})^{\propto -1} e^{-t(s+\beta)} dt$$

Let $\mu = t\beta$

Then

$$t = \frac{\mu}{\beta}$$

$$dt = \frac{d\mu}{\beta}$$

Thus

$$L_t(s) =\propto \beta \int_0^{\infty} (1 - e^{-\mu})^{\propto -1} e^{-\mu\left(\frac{s+\beta}{\beta}\right)} \frac{d\mu}{\beta}$$

$$L_t(s) =\propto \int_0^{\infty} (1 - e^{-\mu})^{\propto -1} e^{-\mu\left(\frac{s+\beta}{\beta}\right)} d\mu$$

Remember

$$B(\alpha, \beta) = \int_0^{\infty} e^{-t(\alpha-1)} (1 - e^{-t})^{\beta - 1} dt$$

Where $B(\alpha, \beta)$ is a Beta function with parameters $\alpha\ and\beta$

Thus

$$L_t(s) = \propto B\left(\frac{2\beta + s}{\beta}, \propto\right)$$

2.7.2. Moment Generating Function of Exponentiated exponential with one parameter

$$M_t(s) = E(e^{ts})$$

$$= \propto \beta \int_0^\infty (1 - e^{-\beta t})^{\propto - 1} e^{-t(\beta - s)} dt$$

Let $\mu = t\beta$

Then

$$t = \frac{\mu}{\beta}$$

$$dt = \frac{d\mu}{\beta}$$

Thus

$$M_t(s) = \propto \beta \int_0^\infty (1 - e^{-\mu})^{\propto - 1} e^{-\mu\left(\frac{\beta - s}{\beta}\right)} \frac{d\mu}{\beta}$$

$$M_t(s) = \propto \int_0^\infty (1 - e^{-\mu})^{\propto - 1} e^{-\mu\left(\frac{\beta - s}{\beta}\right)} d\mu$$

Remember

$$B(\alpha, \beta) = \int_0^\infty e^{-t(\alpha - 1)} (1 - e^{-t})^{\beta - 1} dt$$

Where $B(\alpha, \beta)$ is a Beta function with parameters $\alpha\ and \beta$

Thus

$$M_t(s) = \propto B\left(\frac{2\beta - s}{\beta}, \propto\right)$$

2.7.3. Negative Binomial –Exponentiated exponential with 2 parameter Distribution mixing

a. Mixing using Laplace transform

$$f(x) = \binom{r+x-1}{x} \sum_{k=0}^{x} \binom{x}{k} (-1)^k L_t(r+k)$$

$$f(x) = \binom{r+x-1}{x} \propto \sum_{k=0}^{x} \binom{x}{k} (-1)^k B\left(\frac{2\beta+r+k}{\beta}, \propto\right)$$

b. Mixing using MGF

$$f(x) = \binom{r+x-1}{x} \sum_{k=0}^{x} \binom{x}{k} (-1)^k M_t(-(r+k))$$

$$f(x) = \binom{r+x-1}{x} \propto \sum_{k=0}^{x} \binom{x}{k} (-1)^k B\left(\frac{2\beta+r+k}{\beta}, \propto\right)$$

2.7.4. Properties of Negative Binomial –Exponentiated exponential with two parameter Distribution mixture

a). Mean

$$E(X) = r(L_t(-1) - 1)$$

$$= r\left(\alpha B\left(\frac{2\beta-1}{\beta}, \alpha\right) - 1\right)$$

b). Variance

$$\boldsymbol{var(X) = r[(1+r)L_t(-2) - (2\mathrm{r}+1)L_t(-1) + r] - r^2[L_t(-1) - 1]^2}$$

$$\boldsymbol{var(X) = r\left[(1+r)\alpha B(\frac{2(\beta-1)}{\beta}, \alpha) - (2\mathrm{r}+1)\alpha B\left(\frac{2\beta-1}{\beta}, \alpha\right) + r\right] - r^2[\alpha B\left(\frac{2\beta-1}{\beta}, \alpha\right) - 1]^2}$$

2.8. Negative Binomial –Variate exponential Distribution

2.8.1 Variate exponential Distribution

a. Construction

Consider a **Variate exponential Distribution**

$$g(t) = \begin{cases} \lambda e^{-\lambda t} & \theta > 0;\ \lambda > 0 \\ 0 & elsewhere \end{cases}$$

Where λ is randomized and takes on the distribution $\frac{1}{\lambda\left(ln\left(\frac{b}{a}\right)\right)}$ with $0 < a \leq \lambda < b$

The verified exponential distribution $g_1(t)$ becomes

$$g_1(t) = \frac{1}{ln(\frac{b}{a})}\int_0^\infty e^{-\lambda t}\, d\lambda$$

$$= \frac{1}{ln\left(\frac{b}{a}\right)}\left[\frac{e^{-\lambda t}}{-t}\right]_a^b$$

$$g_1(t) = \frac{e^{-at} - e^{-bt}}{tln\left(\frac{b}{a}\right)} \qquad t > 0; 0 < a < b$$

This is the pdf of variate exponential

b. Properties of Variate exponential Distribution with two parameter

- **Laplace transform**

$$L_t(s) = E(e^{-ts})$$

$$= \frac{1}{ln\left(\frac{b}{a}\right)} \int_0^\infty e^{-ts} \frac{e^{-at} - e^{-bt}}{t} dt$$

$$= \frac{1}{ln\left(\frac{b}{a}\right)} \int_0^\infty \frac{e^{-t(a+s)} - e^{-t(b+s)}}{t} dt$$

But according to $Frullani\ integral$

$$\int_0^\infty \frac{f(at) - f(bt)}{t} dt = [f(0) - f(\infty)] \ln\left(\frac{b}{a}\right)$$

And hence

$$L_t(s) = \frac{ln\left(\frac{s+b}{s+a}\right)}{ln\left(\frac{b}{a}\right)}$$

2.8.2. Moment Generating Function of Variate exponential with 2 parameter

$$M_t(s) = E(e^{ts})$$

$$= \frac{1}{ln\left(\frac{b}{a}\right)} \int_0^\infty e^{ts} \frac{e^{-at} - e^{-bt}}{t} dt$$

$$= \frac{1}{ln\left(\frac{b}{a}\right)} \int_0^\infty \frac{e^{-t(a-s)} - e^{-t(b-s)}}{t} dt$$

But according to $Frullani\ integral$

$$\int_0^\infty \frac{f(at) - f(bt)}{t} dt = [f(0) - f(\infty)] \ln\left(\frac{b}{a}\right)$$

And hence

$$M_t(s) = \frac{ln\left(\frac{b-s}{a-s}\right)}{ln\left(\frac{b}{a}\right)}$$

2.8.3. Negative Binomial –Variate exponential distribution with 2 parameter Distribution mixing

a. Mixing using Laplace transform

$$f(x) = \binom{r+x-1}{x} \sum_{k=0}^{x} \binom{x}{k} (-1)^k L_t(r+k)$$

$$f(x) = \binom{r+x-1}{x} \frac{ln\left(\frac{b+r+k}{a+r+k}\right)}{ln\left(\frac{b}{a}\right)} \sum_{k=0}^{x} \binom{x}{k} (-1)^k$$

b. Mixing using MGF

$$f(x) = \binom{r+x-1}{x} \sum_{k=0}^{x} \binom{x}{k} (-1)^k M_t(-(r+k))$$

$$f(x) = \binom{r+x-1}{x} \frac{ln\left(\frac{b+r+k}{a+r+k}\right)}{ln\left(\frac{b}{a}\right)} \sum_{k=0}^{x} \binom{x}{k} (-1)^k$$

2.8.4. Properties of Negative Binomial –Variate Exponential with 2 parameter Distribution mixture

a). Mean

$$E(X) = r(L_t(-1) - 1)$$

$$= r\left(\frac{ln\left(\frac{b-1}{a-1}\right)}{ln\left(\frac{b}{a}\right)} - 1\right)$$

b). Variance

$$\boldsymbol{var(X) = r[(1+r)L_t(-2) - (2r+1)L_t(-1) + r] - r^2[L_t(-1) - 1]^2}$$

$$\boldsymbol{var(X) = r\left[(1+r)\frac{ln\left(\frac{b-1}{a-1}\right)}{ln\left(\frac{b}{a}\right)} - (2r+1)\frac{ln\left(\frac{b-1}{a-1}\right)}{ln\left(\frac{b}{a}\right)} + r\right] - r^2\left[\frac{ln\left(\frac{b-1}{a-1}\right)}{ln\left(\frac{b}{a}\right)} - 1\right]^2}$$

2.9. Negative Binomial –Variate Gamma$(2, \alpha)$ Distribution

2.9.1 Variate Gamma$(2, \alpha)$ Distribution

a. Construction

Consider a **Distribution**

$$g(t) = \begin{cases} \lambda^2 t e^{-\lambda t} & \theta > 0;\ \lambda > 0 \\ 0 & elsewhere \end{cases}$$

Where λ is randomized and takes on the distribution $\frac{1}{\lambda\left(ln\left(\frac{b}{a}\right)\right)}$ with $0 < a \leq \lambda < b$

The variate **Gamma(2, $\boldsymbol{\alpha}$)**distribution $g_1(t)$ becomes

$$g_1(t) = \frac{t}{ln\left(\frac{b}{a}\right)} \int_0^{\infty} \lambda e^{-\lambda t}\, d\lambda$$

Using integration by parts

$$\int udv = uv - \int vdu$$

Let $u = \lambda$ $\qquad du = d\lambda$

$$dv = e^{-\lambda t} \qquad\qquad v = -\frac{e^{-\lambda t}}{t}$$

Therefore

$$g_1(t) = \frac{t}{ln\left(\frac{b}{a}\right)} \left[\left[-\frac{\lambda e^{-\lambda t}}{t}\right]_a^b + \frac{1}{t}\int_0^{\infty} e^{-\lambda t}\, d\lambda\right]$$

$$= \frac{t}{ln\left(\frac{b}{a}\right)} \left[-\frac{\lambda e^{-\lambda t}}{t} - \frac{e^{-\lambda t}}{t^2}\right]_a^b$$

$$= \frac{t}{ln\left(\frac{b}{a}\right)} \left[\frac{ae^{-at} - be^{-bt}}{t} + \frac{e^{-at} - e^{-bt}}{t^2}\right]_a^b$$

$$= \frac{t}{ln\left(\frac{b}{a}\right)} \left[\frac{ate^{-at} - bte^{-bt} + e^{-at} - e^{-bt}}{t^2}\right]_a^b$$

$$= \frac{1}{ln\left(\frac{b}{a}\right)}\left[\frac{ate^{-at} - bte^{-bt} + e^{-at} - e^{-bt}}{t}\right]$$

$$= \frac{1}{tln\left(\frac{b}{a}\right)}[(at+1)e^{-at} - (bt+1)e^{-bt}]$$

With $t > 0; 0 < a < b$

b. Properties of Variate Gamma (2,α) Distribution

- **Laplace transform**

$$L_t(s) = E(e^{-ts})$$

$$= \frac{1}{ln\left(\frac{b}{a}\right)}\int_0^{\infty}\frac{e^{-ts}}{t}[(at+1)e^{-at} - (bt+1)e^{-bt}]dt$$

$$= \frac{1}{ln\left(\frac{b}{a}\right)}\int_0^{\infty}\left[\frac{(at+1)e^{-(s+a)t} - (bt+1)e^{-(s+b)t}}{t}\right]dt$$

$$L_t(s) = \frac{1}{ln\left(\frac{b}{a}\right)}\left[\int_0^{\infty}\frac{e^{-(s+a)t} - e^{-(s+b)t}}{t}dt + a\int_0^{\infty}e^{-(s+a)t}\,dt - b\int_0^{\infty}e^{-(s+b)t}\,dt\right]$$

$$L_t(s) = \frac{1}{ln\left(\frac{b}{a}\right)}\left[\ln\left(\frac{s+b}{s+a}\right) + \frac{a}{s+a} - \frac{b}{s+b}\right]$$

- **Moment Generating Function of Variate Gamma (2,α)**

$$M_t(s) = E(e^{ts})$$

$$= \frac{1}{ln\left(\frac{b}{a}\right)} \int_0^\infty \frac{e^{ts}}{t} [(at+1)e^{-at} - (bt+1)e^{-bt}]dt$$

$$= \frac{1}{ln\left(\frac{b}{a}\right)} \int_0^\infty \left[\frac{(at+1)e^{-(a-s)t} - (bt+1)e^{-(b-s)t}}{t}\right] dt$$

$$M_t(s) = \frac{1}{ln\left(\frac{b}{a}\right)} \left[\int_0^\infty \frac{e^{-(a-s)t} - e^{-(b-s)t}}{t} dt + a\int_0^\infty e^{-(a-s)t}\, dt - b\int_0^\infty e^{-(b-s)t}\, dt\right]$$

$$M_t(s) = \frac{1}{ln\left(\frac{b}{a}\right)} \left[\ln\left(\frac{b-s}{a-s}\right) + \frac{a}{a-s} - \frac{b}{b-s}\right]$$

2.9.2. Negative Binomial –Variate Gamma$(2, \propto)$ distribution Distribution mixing

a. Mixing using Laplace transform

$$f(x) = \binom{r+x-1}{x} \sum_{k=0}^{x} \binom{x}{k} (-1)^k L_t(r+k)$$

$$f(x) = \binom{r+x-1}{x} \frac{1}{ln\left(\frac{b}{a}\right)} \left[\ln\left(\frac{r+k+b}{r+k+a}\right) + \frac{a}{r+k+a} - \frac{b}{r+k+b}\right] \sum_{k=0}^{x} \binom{x}{k} (-1)^k$$

$$\boldsymbol{a}, \boldsymbol{b} > 0; r > 0$$

b. Mixing using MGF

$$f(x) = \binom{r+x-1}{x} \sum_{k=0}^{x} \binom{x}{k} (-1)^k M_t(-(r+k))$$

$$f(x) = \binom{r+x-1}{x} \frac{1}{ln\left(\frac{b}{a}\right)} \left[\ln\left(\frac{r+k+b}{r+k+a}\right) + \frac{a}{r+k+a} - \frac{b}{r+k+b}\right] \sum_{k=0}^{x} \binom{x}{k} (-1)^k$$

$$\boldsymbol{a}, \boldsymbol{b} > 0; r > 0$$

Reference: $\boldsymbol{Bowman\ et\ al\ (1992)}$

2.9.3. Properties of Variate gamma(2,α) with 2 parameter – Negative Binomial Distribution mixture

a). Mean

$$E(X) = r(L_t(-1) - 1)$$

$$= r\left(\frac{1}{ln\left(\frac{b}{a}\right)}\left[\ln\left(\frac{b-1}{a-1}\right) + \frac{a}{a-1} - \frac{b}{b-1}\right] - 1\right)$$

b). Variance

$$\boldsymbol{var(X) = r[(1+r)L_t(-2) - (2r+1)L_t(-1) + r] - r^2[L_t(-1) - 1]^2}$$

$$\boldsymbol{var(X)} = \boldsymbol{r}\left[(\mathbf{1}+\boldsymbol{r})\frac{1}{ln\left(\frac{b}{a}\right)}\left[\ln\left(\frac{b-2}{a-2}\right)+\frac{a}{a-1}-\frac{b}{b-1}\right]\right.$$
$$\left. -(\mathbf{2r}+\mathbf{1})\frac{1}{ln\left(\frac{b}{a}\right)}\left[\ln\left(\frac{b-1}{a-1}\right)+\frac{a}{a-1}-\frac{b}{b-1}\right]+\boldsymbol{r}\right]$$
$$-\boldsymbol{r}^{\mathbf{2}}\left[\frac{1}{ln\left(\frac{b}{a}\right)}\left[\ln\left(\frac{b-1}{a-1}\right)+\frac{a}{a-1}-\frac{b}{b-1}\right]-\mathbf{1}\right]^{\mathbf{2}}$$

2.10. Negative Binomial –Inverse Gaussian Distribution

2.10.1 Inverse Gaussian Distribution

a. Construction

Consider the pdf of Inverse Gaussian distribution given below

$$g(\lambda) = \left(\frac{\phi}{2\pi\lambda^3}\right)^{\frac{1}{2}} \exp\left(-\frac{\phi(\lambda-\mu)^2}{2\mu^2\lambda}\right)$$

Put $\mu = (2\alpha)^{-1/2}$ $\quad$ *and* $\quad$ $\mu^2 = (2\alpha)^{-1}$

Then

$$g(\lambda) = \left(\frac{\phi}{2\pi\lambda^3}\right)^{\frac{1}{2}} \exp\left(-\frac{\phi(\lambda-(2\alpha)^{-1/2})^2}{2\lambda(2\alpha)^{-1}}\right)$$

$$g(\lambda) = \left(\frac{\phi}{2\pi\lambda^3}\right)^{\frac{1}{2}} \exp\left(-\frac{\phi(\lambda-(2\alpha)^{-1/2})^2(2\alpha)}{2\lambda}\right)$$

$$= \left(\frac{\phi}{2\pi\lambda^3}\right)^{\frac{1}{2}} \exp\left(-\frac{\phi 2\alpha\left(\lambda^2 - 2\lambda(2\alpha)^{-\frac{1}{2}} + (2\alpha)^{-1}\right)}{2\lambda}\right)$$

$$= \left(\frac{\phi}{2\pi\lambda^3}\right)^{\frac{1}{2}} \exp\left(-\lambda\alpha\phi + \phi(2\alpha)^{\frac{1}{2}} - \frac{\phi}{2\lambda}\right)$$

This is the Inverse Gaussian distribution we are going to use in this mixture

b. properties of Inverse Gaussian distribution

- **Laplace transform**

$$L_t(s) = \int_0^\infty e^{-ts} g(t) dt$$

$$= \int_0^\infty \left(\frac{\phi}{2\pi t^3}\right)^{\frac{1}{2}} \exp\left(-st - t\alpha\phi + \phi(2\alpha)^{\frac{1}{2}} - \frac{\phi}{2t}\right) dt$$

$$= \exp\left(\phi(2\alpha)^{\frac{1}{2}}\right) \int_0^\infty \left(\frac{\phi}{2\pi t}\right)^{\frac{1}{2}} \exp\left(-st - t\alpha\phi - \frac{\phi}{2t}\right) dt$$

Factorize the like terms

$$L_t(s) = \exp\left(\phi(2\alpha)^{\frac{1}{2}}\right) \int_0^\infty \left(\frac{\phi}{2\pi t^3}\right)^{\frac{1}{2}} \exp\left(-t(s + \alpha\phi) - \frac{\phi}{2t}\right) dt$$

$$= \exp\left(\phi(2\alpha)^{\frac{1}{2}}\right)\int_0^{\infty}\left(\frac{\phi}{2\pi t^3}\right)^{\frac{1}{2}}\exp\left(-t\left(\frac{s\phi}{\phi}+\alpha\phi\right)-\frac{\phi}{2t}\right)dt$$

$$= \exp\left(\phi(2\alpha)^{\frac{1}{2}}\right)\int_0^{\infty}\left(\frac{\phi}{2\pi t^3}\right)^{\frac{1}{2}}\exp\left(-t\phi\left(\frac{s}{\phi}+\alpha\right)-\frac{\phi}{2t}\right)dt \;\ldots\ldots..eq3$$

Consider the exponential

$$\phi(2\alpha)^{\frac{1}{2}} - t\alpha\phi - \frac{\phi}{2t}$$

Suppose we substitute $\alpha \; by \; \alpha + \frac{s}{\phi}$

$$\phi\left(2\left(\alpha+\frac{s}{\phi}\right)\right)^{\frac{1}{2}} - t\phi\left(2\alpha+\frac{s}{\phi}\right) - \frac{\phi}{2t}$$

This is almost similar to equation 3 but without the middle term

$$L_t(s) = \exp\left(\phi(2\alpha)^{\frac{1}{2}}\right)\exp\left(-\phi\left(2\left(\alpha+\frac{s}{\phi}\right)\right)^{\frac{1}{2}}\right)\int_0^{\infty}\left(\frac{\phi}{2\pi t^3}\right)^{\frac{1}{2}}\exp\left(-t\phi\left(\frac{s}{\phi}+\alpha\right) - \phi\left(2\left(\alpha+\frac{s}{\phi}\right)\right)^{\frac{1}{2}} - \frac{\phi}{2t}\right)dt$$

$$L_t(s) = \exp\left(\phi(2\alpha)^{\frac{1}{2}}\right)\exp\left(-\phi\left(2\left(\alpha+\frac{s}{\phi}\right)\right)^{\frac{1}{2}}\right)X1$$

$$L_t(s) = \exp\left(\phi(2\alpha)^{\frac{1}{2}} - \phi\left(2\left(\alpha + \frac{s}{\phi}\right)\right)^{\frac{1}{2}}\right)$$

Remember initially we had let

$$\mu = (2\alpha)^{-1/2} \qquad and \qquad \mu^2 = (2\alpha)^{-1}$$

Replacing these on the equation we have

$$L_t(s) = \exp\left(\phi(2\alpha)^{\frac{1}{2}} - \phi\left(2\left(\alpha + \frac{s}{\phi}\right)\right)^{\frac{1}{2}}\right)$$

$$L_t(s) = \exp\left(\frac{\phi}{\mu} - \phi\left(2\left(\frac{1}{2\mu^2} + \frac{s}{\phi}\right)\right)^{\frac{1}{2}}\right)$$

$$= \exp\left(\frac{\phi}{\mu} - \phi\left(\frac{\phi + 2s\mu^2}{\phi\mu^2}\right)^{\frac{1}{2}}\right)$$

$$= \exp\left(\frac{\phi}{\mu}\left[1 - \left(\frac{\phi + 2s\mu^2}{\phi}\right)^{\frac{1}{2}}\right]\right)$$

As obtained by Tweedie (1957)

If we let

$$\propto = 0$$

$$g(t) = \left(\frac{\phi}{2\pi t^3}\right)^{\frac{1}{2}} \exp\left(-\frac{\phi}{2t}\right)$$

$$\mathrm{L_t(s)} = \exp\left(-\phi\left(2\left[\frac{s}{\phi}\right]\right)^{\frac{1}{2}}\right)$$

$$\mathrm{L_t(s)} = \exp\left(-(2[s\phi])^{\frac{1}{2}}\right)$$

Putting $\phi = \frac{k^2}{2}$ we have

$$g(t) = \left(\frac{k^2}{4\pi t^3}\right)^{\frac{1}{2}} \exp\left(-\frac{k^2}{4t}\right)$$

$$\mathrm{L_t(s)} = \exp\left(-\left(2\left[s\frac{k^2}{2}\right]\right)^{\frac{1}{2}}\right)$$

$$\mathrm{L_t(s)} = \exp\left(-(k(s)^{\frac{1}{2}})\right)$$

As obtained by Bowman et al (1992)

2.10.2.Negative Binomial – Inverse Gaussian Distribution Mixture

a. mixing using Laplace transform

$$p(X = x) = \binom{r + x - 1}{x} \sum_{k=0}^{x} \binom{x}{k} (-1)^k L_t(r + k)$$

But

$$L_t(r+k) = \exp\left(\frac{\phi}{\mu}\left[1-\sqrt{1-\frac{2(r+k)\mu^2}{\phi}}\right]\right)$$

$$p(X=x) = \binom{r+x-1}{x}\sum_{k=0}^{x}\binom{x}{k}(-1)^k \exp\left(\frac{\phi}{\mu}\left[1-\sqrt{1-\frac{2(r+k)\mu^2}{\phi}}\right]\right)$$

b.Mixing using moment generating function technique

$$p(X=x) = \binom{r+x-1}{x}\sum_{k=0}^{x}\binom{x}{k}(-1)^k M_t\big(-(r+k)\big)$$

But

$$M_t(-(r+k)) = \exp\left(\frac{\phi}{\mu}\left[1-\sqrt{1-\frac{2(r+k)\mu^2}{\phi}}\right]\right)$$

$$p(X=x) = \binom{r+x-1}{x}\sum_{k=0}^{x}\binom{x}{k}(-1)^k \exp\left(\frac{\phi}{\mu}\left[1-\sqrt{1-\frac{2(r+k)\mu^2}{\phi}}\right]\right)$$

Properties

$$L_t(s) = \exp\left(\frac{\phi}{\mu}\left[1-\left(\frac{\phi+2s\mu^2}{\phi}\right)^{\frac{1}{2}}\right]\right)$$

$$E(X) = r(L_t(-1) - 1)$$

$$r\left(\exp\left(\frac{\phi}{\mu}\left[1-\left(\frac{\phi-2\mu^2}{\phi}\right)^{\frac{1}{2}}\right]\right)-1\right)$$

$$\boldsymbol{var(X) = r[(1+r)L_t(-2) - (2\mathbf{r}+1)L_t(-1) + r] - r^2[L_t(-1) - 1]^2}$$

$$\boldsymbol{= (r + r^2)L_t(-2) - \mathbf{r}L_t(-1) - r^2L_t^2(-1)}$$

$$= (\boldsymbol{r} + \boldsymbol{r^2}) \exp\left(\frac{\phi}{\mu}\left[1-\left(\frac{\phi-4\mu^2}{\phi}\right)^{\frac{1}{2}}\right]\right)_{\boldsymbol{t}} - \mathbf{r}\exp\left(\frac{\phi}{\mu}\left[1-\left(\frac{\phi-2\mu^2}{\phi}\right)^{\frac{1}{2}}\right]\right)_{\boldsymbol{t}} - \boldsymbol{r^2}\left\{\exp\left(\frac{\phi}{\mu}\left[1-\left(\frac{\phi-2\mu^2}{\phi}\right)^{\frac{1}{2}}\right]\right)\right\}^2$$

c. Mixing using recursive relation

We know when $p = e^{-t}$ the negative binomial distribution will assume the formula below

$$p(x/t) = \binom{r+x-1}{x} e^{-rt}(1-e^{-t})^x \qquad x = 0,1,2 \ldots\ldots$$

Suppose we want to get $p((x-1)/t)$

$$p((x-1)/t) = \binom{r+x-2}{x-1} e^{-rt}(1-e^{-t})^{x-1}$$

$$\frac{p(x/t)}{p((x-1)/t)} = \frac{(r+x-1)(r+x-2)!}{x(x-1)!\,(r-1)!}\frac{(x-1)!\,(r-1)!}{(r+x-2)!}\frac{e^{-rt}(1-e^{-t})^{x}}{e^{-rt}(1-e^{-t})^{x-1}}$$

$$\frac{p(x/t)}{p((x-1)/t)} = \frac{r+x-1}{x}(1-e^{-t})$$

$$p(x/t) = \frac{r+x-1}{x}(1-e^{-t})p((x-1)/t) \dots\dots\dots..eqn\ 4.2$$

Using eqn 4.2 and the Negative binomial distribution

$$P_r\,(x) = \int_0^\infty p(x/t)g(t)dt$$

$$= \int_0^\infty \frac{r+x-1}{x}(1-e^{-t})p((x-1)/t)g(t)dt$$

$$= \frac{r+x-1}{x}\left[\int_0^\infty p\left(\frac{x-1}{t}\right)g(t)dt - \int_0^\infty e^{-t}p\left(\frac{x-1}{t}\right)g(t)dt\right]$$

$$P_r\,(x) = \frac{r+x-1}{x}\left[\int_0^\infty P_r(x-1) - \int_0^\infty e^{-t}p((x-1)/t)g(t)dt\right] \dots\dots..eqn\ 4.00$$

Now consider

$$\int_0^\infty e^{-t} p((x-1)/t) g(t) dt$$

This can be expressed as follows

$$= \int_0^\infty \binom{r+x-2}{x-1} e^{-t(r+1)} (1-e^{-t})^{x-1} g(t) dt \ldots \ldots \ldots . eqn4.21$$

Consider the combination

$$\binom{r+x-2}{x-1} = \frac{(r+x-2)!}{(r-1)!\,(x-1)!}$$

$$= \frac{r}{r+x-1} \binom{r+x-1}{x-1}$$

Or

$$= \frac{r}{r+x-1} \binom{r+x-1}{r}$$

eqn 4.21 therefore becomes

$$= \frac{r}{r+x-1} \int_0^\infty \binom{r+1+x-2}{x-1} e^{-t(r+1)} (1-e^{-t})^{x-1} g(t) dt$$

$$= \frac{r}{r+x-1} P_{r+1}(x-1)$$

eqn 4.00 Therefore becomes

$$P_r\,(x) = \frac{r+x-1}{x}\left[P_r(x-1) - \frac{r}{r+x-1}P_{r+1}\,(x-1)\right]$$

Which is the mixture of negative binomial distribution and inverse Gaussian distribution using recursive relation as obtained by $Gomez\ and\ Deniz\ et\ al$ (2006)

Note this is similar to the mixture of beta and negative binomial using recursive relation.

2.11. Negative binomial – Lindley distribution

Lindley negative binomial mixture. Ref Hussein Zamani and Noriszura Ismail 2010: negative binomial – Lindley distribution and its application.

2.11.1. Properties of Lindley distribution

a. Moment generating function for Lindley distribution

$$M_t(s) = E(e^{ts}) = \int_0^{\infty} \frac{\theta^2(1+t)e^{-\theta t}e^{ts}}{\theta+1}dt$$

$$M_t(s) = \frac{\theta^2}{\theta+1}\int_0^{\infty}(1+t)e^{-t(\theta-s)}dt$$

$$M_t(s) = \frac{\theta^2}{(\theta+1)}\left[\int_0^{\infty} e^{-t(\theta-s)}dt + \int_0^{\infty} te^{-t(\theta-s)}dt\right]$$

Consider

$$\int_0^{\infty} e^{-t(\theta-s)}dt$$

This can be expressed as

$$\frac{1}{(\theta-s)}\int_0^{\infty}(\theta-s)e^{-t(\theta-s)}dt$$

But

$$\int_0^\infty (\theta - s)e^{-t(\theta - s)}dt = 1$$

i.e. exponential pdf

and therefore

$$\int_0^\infty e^{-t(\theta - s)}dt = \frac{1}{(\theta - s)}$$

Next consider

$$\int_0^\infty te^{-t(\theta - s)}dt$$

$$\int \mathrm{udv} = \mathrm{uv} - \int \mathrm{vdu}$$

Now let

$$u = t$$

$$dv = e^{-t(\theta - s)}dt$$

$$du = dt$$

$$v = -\frac{e^{-t(\theta - s)}}{(\theta - s)}$$

$$\int_0^\infty te^{-t(\theta - s)}dt = \left[-\frac{te^{-t(\theta - s)}}{(\theta - s)}\right]_0^\infty + \frac{1}{(\theta - s)}\int_0^\infty e^{-t(\theta - s)}\,dt$$

$$\int_0^\infty te^{-t(\theta - s)}dt = \frac{1}{(\theta - s)^2}$$

The MGF therefore translates to

$$M_t(s) = \frac{\theta^2}{(\theta+1)}\left[\frac{1}{\theta - s} + \frac{1}{(\theta - s)^2}\right]$$

$$M_t(s) = \frac{\theta^2(1 - s + \theta)}{(\theta+1)(\theta - s)^2}$$

b. Laplace transform for Lindley distribution

$$L_t(s) = E(e^{-ts}) = \int_0^\infty \frac{\theta^2(1+t)e^{-\theta t}e^{-ts}}{\theta + 1}dt$$

$$L_t(s) = \frac{\theta^2}{\theta + 1}\int_0^\infty (1+t)e^{-t(\theta+s)}dt$$

$$L_t(s) = \frac{\theta^2}{(\theta+1)}\left[\int_0^\infty e^{-t(\theta+s)}dt + \int_0^\infty te^{-t(\theta+s)}dt\right]$$

Consider

$$\int_0^\infty e^{-t(\theta+s)}dt$$

This can be expressed as

$$\frac{1}{(\theta+s)}\int_0^\infty (\theta+s)e^{-t(\theta+s)}dt$$

But

$$\int_0^\infty (\theta+s)e^{-t(\theta+s)}dt = 1$$

i.e. exponential pdf

and therefore

$$\int_0^{\infty} e^{-t(\theta+s)} dt = \frac{1}{(\theta+s)}$$

Next consider

$$\int_0^{\infty} te^{-t(\theta+s)} dt$$

$$\int \mathrm{udv} = \mathrm{uv} - \int \mathrm{vdu}$$

Now let

$$u = t$$

$$dv = e^{-t(\theta+s)} dt$$

$$du = dt$$

$$v = -\frac{e^{-t(\theta+s)}}{(\theta+s)}$$

$$\int_0^{\infty} te^{-t(\theta+s)} dt = \left[-\frac{te^{-t(\theta+s)}}{(\theta+s)}\right]_0^{\infty} + \frac{1}{(\theta+s)} \int_0^{\infty} e^{-t(\theta+s)}\, dt$$

$$\int_0^{\infty} te^{-t(\theta+s)} dt = \frac{1}{(\theta+s)^2}$$

The Laplace therefore translates to

$$L_t(s) = \frac{\theta^2}{(\theta+1)} \left[\frac{1}{\theta+s} + \frac{1}{(\theta+s)^2}\right]$$

$$L_t(s) = \frac{\theta^2(1+s+\theta)}{(\theta+1)(\theta+s)^2}$$

2.11.2. Negative binomial- lindley distribution mixing

Mixture using MGF

Negative binomial distribution

$$f(x/\lambda) = \binom{r+x-1}{x} e^{-\lambda r}(1-e^{-\lambda})^{x} \qquad for\ p = e^{-\lambda}$$

This can as well be written as

$$f(x/\lambda) = \binom{r+x-1}{x} \sum_{k=0}^{x} \binom{x}{k} (-1)^{k}\, e^{-\lambda(r+j)}$$

The mixture

$$prob(x) = \int_0^{\infty} prob(x/\lambda) g(\lambda,\theta) d\lambda$$

$$= \int_0^{\infty} \binom{r+x-1}{x} \sum_{j=0}^{x} \binom{x}{j} (-1)^{j}\, e^{-\lambda(r+j)} \frac{\theta^2(1+\lambda)e^{-\theta\lambda}}{\theta+1} d\lambda$$

$$= \binom{r+x-1}{x} \sum_{k=0}^{x} \binom{x}{k} (-1)^{k} \int_0^{\infty} e^{-\lambda(r+k)} \frac{\theta^2(1+\lambda)e^{-\theta\lambda}}{\theta+1} d\lambda$$

But

$$\int_0^{\infty} e^{-\lambda(r+k)} \frac{\theta^2(1+\lambda)e^{-\theta\lambda}}{\theta+1} d\lambda = M_{\lambda}(-(r+k))$$

Where $M_{\lambda}(-(r+k))$ is the MGF

From previous proof we have the MGF of Lindley distribution as

$$M_t(s) = \frac{\theta^2(\theta - s + 1)}{(\theta + 1)(\theta - s)^2}$$

In our case $s = -(r + j)$

And hence

$$prob(x) = \binom{r + x - 1}{x} \sum_{k=0}^{x} \binom{x}{k} (-1)^k \, M_\lambda\big(-(r + k)\big)$$

$$= \binom{r + x - 1}{x} \sum_{k=0}^{x} \binom{x}{k} (-1)^k \frac{\theta^2(\theta + r + k + 1)}{(\theta + 1)(\theta + r + k)^2}$$

$$prob(x) = \frac{\theta^2}{(\theta + 1)} \binom{r + x - 1}{x} \sum_{k=0}^{x} \binom{x}{k} (-1)^k \frac{(\theta + r + k + 1)}{(\theta + r + k)^2}$$

2.11.3. Properties of negative binomial – Lindley distribution

Mean

$$E(X) = r\left[\frac{\theta^3}{(\theta + 1)(\theta - 1)^2} - 1\right]$$

$$E(X^2) = (r + r^2)\left[\frac{\theta^2(\theta - 1)}{(\theta + 1)(\theta - 2)^2}\right] - (r + 2r^2)\left[\frac{\theta^3}{(\theta + 1)(\theta - 1)^2}\right] + r^2$$

Variance

$$var(x) = E(X^2) - [E(X)]^2$$

$$[E(X)]^2 = r^2\left[\frac{\theta^3}{(\theta+1)(\theta-1)^2} - 1\right]^2$$

$$= r^2\left[\frac{\theta^6}{(\theta+1)^2(\theta-1)^4} - \frac{2\theta^3}{(\theta+1)(\theta-1)^2} + 1\right]$$

$$Var(X) = \left\{(r+r^2)\left[\frac{\theta^2(\theta-1)}{(\theta+1)(\theta-2)^2}\right] - (r+2r^2)\left[\frac{\theta^3}{(\theta+1)(\theta-1)^2}\right] + r^2\right\}$$
$$- r^2\left[\frac{\theta^6}{(\theta+1)^2(\theta-1)^4} - \frac{2\theta^3}{(\theta+1)(\theta-1)^2} + 1\right]$$

CHAPTER 3

3.1. Introduction

The mixing distribution are within the interval [0,∞]. The parameter p is transformed to be equal to $p = 1 - e^{-t}$ for $t > 0$

In this chapter the following methods are used when mixing or compounding negative binomial distribution with other distributions

1. Direct substitution and subsequent integration
2. Recursive method
3. Moments methods or Laplace transform

The mixing distributions considered in this chapter are:

- Exponential
- Gamma
- Exponeniated Exponential
- Beta Exponential
- Variate Gamma
- Variate Exponential
- Inverse Gaussian
- Lindely

From chapter 2 we have

$$E(X) = r(L_t(-1) - 1)$$

$$\boldsymbol{var(X) = r[(1+r)L_t(-2) - (2\mathbf{r}+1)L_t(-1) + r] - r^2[L_t(-1) - 1]^2}$$

When $p = 1 - e^{-t}$

$$f(x) = \binom{r+x-1}{x} \int_0^\infty p^r (1-p)^x g(t) dt$$

$$= \binom{r+x-1}{x} \int_0^\infty (1-e^{-t})^r e^{-tx} g(t) dt$$

$$= \binom{r+x-1}{x} \sum_{k=0}^{r} \binom{r}{k} (-e^{-t})^k \int_0^\infty e^{-tx} g(t)dt$$

$$= \binom{r+x-1}{x} \sum_{k=0}^{r} \binom{r}{k} (-1)^k \int_0^\infty e^{-t(x+k)} g(t)dt$$

But

$$\int_0^\infty e^{-t(x+k)} g(t)dt = L_t(x+k)$$

Therefore

$$f(x) = \binom{r+x-1}{x} \sum_{k=0}^{r} \binom{r}{k} (-1)^k L_t(x+k)$$

Or

$$f(x) = \binom{r+x-1}{x} \sum_{k=0}^{r} \binom{r}{k} (-1)^k \int_0^\infty e^{-(x+k)t} g(t)dt$$

But

$$\int_0^\infty e^{-(x+k)t} g(t)dt = M_t(-(x+k))$$

And hence

$$f(x) = \binom{r+x-1}{x} \sum_{k=0}^{r} \binom{r}{k} (-1)^k M_t(-(x+k))$$

This is the formula for mixing negative binomial with other distributions using the MGF technique with the value of p transformed into $1 - e^{-t}$

Properties

These properties are similar to those of the transformation of the parameter p to be e^{-t}

3.2. Negative Binomial –Exponential distribution

3.2.1 Exponential distribution

The pdf of an exponential distribution is given by

$$g(t) = \lambda e^{-t\lambda} \qquad ; t > 0;\ \lambda > 0$$

Properties of exponential distribution

The Laplace transform of $g(t)$is

$$L_t(s) = E(e^{-ts}) = \int_0^\infty \lambda e^{-t(s+\lambda)} dt$$

$$= \lambda \int_0^\infty e^{-t(s+\lambda)} dt$$

$$= \frac{\lambda}{s+\lambda} \int_0^\infty (s+\lambda) e^{-t(s+\lambda)} dt$$

But

$$= \int_0^\infty (s+\lambda) e^{-t(s+\lambda)} dt = 1$$

Therefore

$$L_t(s) = \frac{\lambda}{s+\lambda}$$

Moment generating function for Exponential distribution

$$M_t(s) = E(e^{ts}) = \int_0^\infty \lambda e^{-t(\lambda-s)} dt$$

$$= \frac{\lambda}{\lambda - s} \int_0^\infty (\lambda - s) e^{-t(\lambda-s)} dt$$

But

$$\int_0^\infty (\lambda - s) e^{-t(\lambda-s)} dt = 1$$

Therefore

$$M_t(s) = \frac{\lambda}{\lambda - s}$$

3.2.2. Negative Binomial – Exponential distribution mixing

a. using Laplace transform

$$f(x) = \binom{r+x-1}{x} \sum_{k=0}^{r} \binom{r}{k} (-1)^k L_t(k+x)$$

$$f(x) = \binom{r+x-1}{x} \sum_{k=0}^{r} \binom{r}{k} (-1)^k \frac{\lambda}{k+x+\lambda}$$

$$f(x) = \binom{r+x-1}{x} \lambda \sum_{k=0}^{r} \binom{r}{k} \frac{(-1)^k}{k+x+\lambda}$$

$$r > 0;\ \lambda > 0, x > 0$$

Properties of negative binomial exponential distribution mixture

b. Mixing using the moment Generating Function technique

$$f(x) = \binom{r+x-1}{x} \sum_{k=0}^{r} \binom{r}{k} (-1)^k M_t(-(k+x))$$

$$f(x) = \binom{r+x-1}{x} \sum_{k=0}^{r} \binom{r}{k} \frac{(-1)^k \lambda}{\lambda+x+k}$$

Note: similar result compared to the mixture using Laplace transform

c. Explicit mixing

$$f(x) = \binom{r+x-1}{x} \int_0^\infty e^{-tr}(1-e^{-t})^x g(t)dt$$

Where

$$g(t) = \lambda e^{-t\lambda} \quad ; t > 0;\ \lambda > 0$$

$$f(x) = \binom{r+x-1}{x} \lambda \int_0^\infty (1-e^{-t})^r e^{-tx} e^{-t\lambda} dt$$

$$f(x) = \binom{r+x-1}{x} \lambda \int_0^\infty e^{-t(x+\lambda)} (1-e^{-t})^r dt$$

$$let\ u = e^{-t}$$

$$\ln e^{-t} = \ln u$$

$$t = -\ln u$$

$$dt = -\frac{du}{u}$$

$$f(x) = \binom{r+x-1}{x} \lambda \int_0^\infty u^{(x+\lambda)} (1-u)^r \left(-\frac{du}{u}\right)$$

$$f(x) = -\lambda \binom{r+x-1}{x} \int_0^\infty u^{(x+\lambda-1)} (1-u)^r du$$

$$= -\frac{\lambda \binom{r+x-1}{x}}{\binom{x+\lambda-1+r-1}{r}} \int_0^\infty \binom{x+\lambda-1+r-1}{r} u^{(x+\lambda-1)} (1-u)^r du$$

But

$$\int_0^\infty \binom{x+\lambda-1+r-1}{r} u^{(x+\lambda-1)} (1-u)^r du = 1$$

This is the pdf of negative binomial

Therefore

$$f(x) = \frac{\lambda \binom{r+x-1}{x}}{\binom{r+\lambda+x-2}{r}}$$

3.3. Negative Binomial –Gamma (I) Distribution

3.3.1. Gamma distribution with 1 parameter distribution

a. Construction

A gamma function can be expressed as follows

$$\Gamma\alpha = e^{-t}t^{\alpha-1}$$

When you divide both sides by $\Gamma\alpha$ we will have Type equation here.

$$1 = \frac{e^{-t}t^{\alpha-1}}{\Gamma\alpha}$$

Since the right hand side is equivalent to 1 the above function forms a pdf and is referred to as a gamma distribution with one parameter α and is expressed as

$$f(t) = \begin{cases} \frac{e^{-t}t^{\alpha-1}}{\Gamma\alpha} & t > 0;\ \alpha > 0 \\ 0 & elsewhere \end{cases}$$

b. Properties of gamma distribution with one parameter

- **Laplace transform for Gamma(I) distribution**

$$L_t(s) = E(e^{-ts}) = \int_0^\infty \frac{e^{-t(1+s)}t^{\alpha-1}}{\Gamma\alpha}dt$$

$$let\ t(s+1) = u$$

$$t = \frac{u}{s+1}$$

$$dt = \frac{du}{s+1}$$

Hence

$$L_t(s) = E(e^{-ts}) = \frac{1}{(s+1)^\alpha}\int_0^\infty \frac{e^{-u}u^{\alpha-1}}{\Gamma\alpha}du$$

$$L_t(s) = \frac{1}{(s+1)^{\alpha}}$$

- **Moment generating function for Gamma(I) distribution**

$$M_t(s) = E(e^{ts}) = \int_0^{\infty} \frac{e^{-t(s-1)}t^{\alpha-1}}{\Gamma\alpha} dt$$

$$let\ t(s-1) = u$$

$$t = \frac{u}{s-1}$$

$$dt = \frac{du}{s-1}$$

Hence

$$M_t(s) = E(e^{ts}) = \frac{1}{(s-1)^{\alpha}} \int_0^{\infty} \frac{e^{-u}u^{\alpha-1}}{\Gamma\alpha} du$$

$$M_t(s) = \frac{1}{(s-1)^{\alpha}}$$

3.3.2.Negative Binomial –Gamma (I) distribution mixing

a. **Using Laplace transform**

$$f(x) = \binom{r+x-1}{x} \sum_{k=0}^{r} \binom{r}{k} (-1)^k L_t(x+k)$$

$$f(x) = \binom{r+x-1}{x} \sum_{k=0}^{r} \binom{r}{k} \frac{(-1)^k}{(x+k+1)^{\alpha}}$$

b. Using MGF

$$f(x) = \binom{r+x-1}{x} \sum_{k=0}^{r} \binom{r}{k} (-1)^k M_t(-(x+k))$$

$$f(x) = \binom{r+x-1}{x} \sum_{k=0}^{r} \binom{r}{k} \frac{(-1)^k}{(x+k+1)^{\alpha}}$$

Note:

Both mixtures based on Laplace transpose and moment generating function results to the same solution.

Properties of the mixture Negative Binomial –Gamma (I) Mixture

Mean

This is undefined and therefore it is not possible to evaluate the expectation of this mixture.

Since the mean is undefined, and we use the mean in finding the variance, it definitely means the variance is equally undefined.

3.4. Negative Binomial – Gamma with two parameters distribution

3.4.1. Gamma with two parameters distribution

a. Construction of Gamma with two parameters distribution

Consider a Gamma distribution with one parameter

$$g(x) = \begin{cases} \frac{e^{-x}x^{\alpha-1}}{\Gamma\alpha} & x > 0; \alpha > 0 \\ 0 & elsewhere \end{cases}$$

Let $x = t\beta$

$$g(t) = \frac{e^{-t\beta}(t\beta)^{\alpha-1}}{\Gamma\alpha}|J|$$

$$t > 0;\ \alpha, \beta > 0$$

$$|J| = \left|\frac{\delta x}{\delta t}\right|$$

$$= \beta$$

$$g(t) = \begin{cases} \frac{e^{-t\beta}t^{\alpha-1}\beta^{\propto}}{\Gamma\alpha} & t > 0;\ \alpha, \beta > 0 \\ 0 & elsewhere \end{cases}$$

This is the pdf of a Gamma (II) distribution with parameters $\alpha\ and\ \beta$

b. Properties of Gamma with two parameters distribution

- **Laplace transpose**

$$L_t(s) = E(e^{-ts}) = \beta^{\alpha}\int_0^{\infty}\frac{t^{\alpha-1}e^{-t(\beta+s)}}{\Gamma\alpha}dt$$

Let

$$t(\beta + s) = m$$

$$t = \frac{m}{\beta + s}$$

$$dt = \frac{dm}{\beta + s}$$

$$L_t(s) = \frac{\beta^\alpha}{(\beta + s)^\alpha} \int_0^\infty \frac{m^{\alpha-1} e^{-m}}{\Gamma\alpha} dm$$

$$L_t(s) = \left[\frac{\beta}{\beta + s}\right]^\alpha$$

- **Moment generating function**

$$M_t(s) = E(e^{ts}) = \beta^\alpha \int_0^\infty \frac{t^{\alpha-1} e^{-t(\beta-s)}}{\Gamma\alpha} dt$$

Let

$$t(\beta - s) = m$$

$$t = \frac{m}{\beta - s}$$

$$dt = \frac{dm}{\beta - s}$$

$$M_t(s) = \frac{\beta^\alpha}{(\beta - s)^\alpha} \int_0^\infty \frac{m^{\alpha-1} e^{-m}}{\Gamma\alpha} dm$$

$$M_t(s) = \left[\frac{\beta}{\beta - s}\right]^\alpha$$

3.4.2.Negative binomial –Gamma (II) distribution mixing

a. Mixing using Laplace transpose

$$f(x) = \binom{r + x - 1}{x} \sum_{k=0}^{r} \binom{r}{k} (-1)^k L_t(x + k)$$

$$f(x) = \binom{r+x-1}{x} \sum_{k=0}^{r} \binom{r}{k} (-1)^k \left[\frac{\beta}{\beta + x + k}\right]^{\alpha}$$

b. **Mixing using Moment Generating function**

$$f(x) = \binom{r+x-1}{x} \sum_{k=0}^{r} \binom{r}{k} (-1)^k M_t(-(x+k))$$

$$f(x) = \binom{r+x-1}{x} \sum_{k=0}^{r} \binom{r}{k} (-1)^k \left[\frac{\beta}{\beta + x + k}\right]^{\alpha}$$

3.4.3. Properties of Negative binomial – Gamma (II) distribution

3.5. Negative Binomial –Exponentiated exponential with one parameter Distribution

3.5.1. Exponentiated exponential Distribution with one parameter

a. Construction

$$F(x) = [G(x)]^{\alpha}$$

$$g(x) = \lambda e^{-\lambda t}$$

$$f(t) = \lambda\alpha(1 - e^{-\lambda t})^{\alpha-1} e^{-\lambda t}$$

This distribution can be expressed as follows

$$g(t) = \begin{cases} \alpha(1 - e^{-t})^{\alpha-1} e^{-t} & t > 0;\ \propto > 0 \\ 0 & elsewhere \end{cases}$$

b. Properties of Exponentiated exponential Distribution with one parameter

- **Laplace transform**

$$L_t(s) = E(e^{-ts})$$

$$= \alpha \int_0^\infty (1 - e^{-t})^{\alpha-1} e^{-t(s+1)} dt$$

Remember

$$B(\alpha, \beta) = \int_0^\infty e^{-t(\alpha-1)} (1 - e^{-t})^{\beta-1}$$

Where $B(\alpha, \beta)$ is a Beta function with parameters $\alpha\ and \beta$

Thus

$$L_t(s) = \alpha B(s + 2, \alpha)$$

- **Moment Generating Function of Exponentiated exponential with one parameter**

$$M_t(s) = E(e^{ts})$$

$$= \alpha \int_0^\infty (1 - e^{-t})^{\propto-1} e^{-t(1-s)} dt$$

Remember

$$B(\alpha, \beta) = \int_0^\infty e^{-t(\alpha-1)} (1 - e^{-t})^{\beta-1}$$

Where $B(\alpha, \beta)$ is a Beta function with parameters $\alpha\ and \beta$

Thus

$$M_t(s) = \propto B(2 - s, \propto)$$

3.5.2. Negative Binomial –Exponentiated exponential with one parameter Distribution mixing

a. Mixing using Laplace transform

$$f(x) = \binom{r+x-1}{x} \sum_{k=0}^{r} \binom{r}{k} (-1)^k L_t(x+k)$$

$$f(x) = \binom{r+x-1}{x} \propto \sum_{k=0}^{r} \binom{r}{k} (-1)^k B(2+x+k, \propto)$$

b. Mixing using MGF

$$f(x) = \binom{r+x-1}{x} \sum_{k=0}^{r} \binom{r}{k} (-1)^k M_t(-(x+k))$$

$$f(x) = \binom{r+x-1}{x} \propto \sum_{k=0}^{r} \binom{r}{k} (-1)^k B(2+x+k, \propto)$$

3.5.3. Properties of Negative Binomial –Exponentiated exponential with one parameter Distribution mixture

3.6. Negative Binomial – Exponentiated exponential with 2 parameter Distribution

3.6.1.Exponentiated exponential Distribution with 2 parameter

a. Construction

Consider a **Exponentiated exponential with 1 parameter**

$$g(\theta) = \begin{cases} \propto \left(1 - e^{-\theta}\right)^{\propto - 1} e^{-\theta} & \theta > 0;\ \propto > 0 \\ 0 & elsewhere \end{cases}$$

$$let\ \theta = t\beta t > 0$$

Such that

$$g_1(t) = \propto \left(1 - e^{-t\beta}\right)^{\propto - 1} e^{-t\beta} |J|$$

$$|J| = \left|\frac{\delta\theta}{\delta t}\right| = \beta$$

$$g(t) = \begin{cases} \propto \beta\left(1 - e^{-t\beta}\right)^{\propto - 1} e^{-t\beta} & t > 0;\ \propto > 0 \\ 0 & elsewhere \end{cases}$$

This is the pdf of **Exponentiated** exponential distribution with 2 parameter $\alpha\ and \beta$ which is denoted as **Exponentiated** exponential (II) distribution

3.6.2. Properties of Exponentiated exponential Distribution with two parameter

a. Laplace transform

$$L_t(s) = E(e^{-ts})$$

$$= \propto \beta \int_0^\infty (1 - e^{-t\beta})^{\propto - 1} e^{-t(s+\beta)} dt$$

Let $\mu = t\beta$

Then

$$t = \frac{\mu}{\beta}$$

$$dt = \frac{d\mu}{\beta}$$

Thus

$$L_t(s) = \propto \beta \int_0^{\infty} (1 - e^{-\mu})^{\propto - 1} e^{-\mu\left(\frac{s+\beta}{\beta}\right)} \frac{d\mu}{\beta}$$

$$L_t(s) = \propto \int_0^{\infty} (1 - e^{-\mu})^{\propto - 1} e^{-\mu\left(\frac{s+\beta}{\beta}\right)} d\mu$$

Remember

$$B(\alpha, \beta) = \int_0^{\infty} e^{-t(\alpha-1)} (1 - e^{-t})^{\beta - 1} dt$$

Where $B(\alpha, \beta)$ is a Beta function with parameters $\alpha \, and \beta$

Thus

$$L_t(s) = \propto B\left(\frac{2\beta + s}{\beta}, \propto\right)$$

c. Moment Generating Function of Generalized exponential with one parameter

$$M_t(s) = E(e^{ts})$$

$$= \propto \beta \int_0^{\infty} (1 - e^{-\beta t})^{\propto - 1} e^{-t(\beta - s)} dt$$

Let $\mu = t\beta$

Then

$$t = \frac{\mu}{\beta}$$

$$dt = \frac{d\mu}{\beta}$$

Thus

$$M_t(s) = \propto \beta \int_0^\infty (1 - e^{-\mu})^{\propto -1} e^{-\mu\left(\frac{\beta - s}{\beta}\right)} \frac{d\mu}{\beta}$$

$$M_t(s) = \propto \int_0^\infty (1 - e^{-\mu})^{\propto -1} e^{-\mu\left(\frac{\beta - s}{\beta}\right)} d\mu$$

Remember

$$B(\alpha, \beta) = \int_0^\infty e^{-t(\alpha - 1)} (1 - e^{-t})^{\beta - 1} dt$$

Where $B(\alpha, \beta)$ is a Beta function with parameters $\alpha \ and \beta$

Thus

$$M_t(s) = \propto B\left(\frac{2\beta - s}{\beta}, \propto\right)$$

3.6.3. Negative Binomial –Exponentiated exponential with 2 parameter Distribution mixing

a. Mixing using Laplace transform

$$f(x) = \binom{r + x - 1}{x} \sum_{k=0}^{r} \binom{r}{k} (-1)^k L_t(x + k)$$

$$f(x) = \binom{r+x-1}{x} \propto \sum_{k=0}^{r} \binom{r}{k} (-1)^k B\left(\frac{2\beta + x + k}{\beta}, \propto\right)$$

b. Mixing using MGF

$$f(x) = \binom{r+x-1}{x} \sum_{k=0}^{r} \binom{r}{k} (-1)^k M_t(-(x+k))$$

$$f(x) = \binom{r+x-1}{x} \propto \sum_{k=0}^{r} \binom{r}{k} (-1)^k B\left(\frac{2\beta + x + k}{\beta}, \propto\right)$$

3.6.4. Properties ofNegative Binomial –exponentiaed exponential with two parameter Distribution mixture

3.7. Negative Binomial – Variate exponential Distribution

3.7.1. Variate exponential Distribution

a. Construction

Consider a **Variate exponential Distribution**

$$g(t) = \begin{cases} \lambda e^{-\lambda t} & \theta > 0;\ \lambda > 0 \\ 0 & elsewhere \end{cases}$$

Where λ is randomized and takes on the distribution $\frac{1}{\lambda\left(ln\left(\frac{b}{a}\right)\right)}$ with $0 < a \leq \lambda < b$

The verified exponential distribution $g_1(t)$ becomes

$$g_1(t) = \frac{1}{ln(\frac{b}{a})} \int_0^{\infty} e^{-\lambda t}\, d\lambda$$

$$= \frac{1}{ln\left(\frac{b}{a}\right)} \left[\frac{e^{-\lambda t}}{-t}\right]_a^b$$

$$g_1(t) = \frac{e^{-at} - e^{-bt}}{tln\left(\frac{b}{a}\right)} \qquad t > 0; 0 < a < b$$

This is the pdf of variate exponential

b. Properties of Variate exponential Distribution

- **Laplace transform**

$$L_t(s) = E(e^{-ts})$$

$$= \frac{1}{ln\left(\frac{b}{a}\right)} \int_0^{\infty} e^{-ts} \frac{e^{-at} - e^{-bt}}{t} dt$$

$$= \frac{1}{ln\left(\frac{b}{a}\right)} \int_0^{\infty} \frac{e^{-t(a+s)} - e^{-t(b+s)}}{t} dt$$

But according to $Frullani\ integral$

$$\int_0^{\infty} \frac{f(at) - f(bt)}{t} dt = [f(0) - f(\infty)] \ln\left(\frac{b}{a}\right)$$

And hence

$$L_t(s) = \frac{ln\left(\frac{s+b}{s+a}\right)}{ln\left(\frac{b}{a}\right)}$$

- **Moment Generating Function of Variate exponential**

$$M_t(s) = E(e^{ts})$$

$$= \frac{1}{ln\left(\frac{b}{a}\right)} \int_0^\infty e^{ts} \frac{e^{-at} - e^{-bt}}{t} dt$$

$$= \frac{1}{ln\left(\frac{b}{a}\right)} \int_0^\infty \frac{e^{-t(a-s)} - e^{-t(b-s)}}{t} dt$$

But according to $Frullani\ integral$

$$\int_0^\infty \frac{f(at) - f(bt)}{t} dt = [f(0) - f(\infty)] \ln\left(\frac{b}{a}\right)$$

And hence

$$M_t(s) = \frac{ln\left(\frac{b-s}{a-s}\right)}{ln\left(\frac{b}{a}\right)}$$

3.7.2. Negative Binomial –Variate exponential distribution Distribution mixing

a. Mixing using Laplace transform

$$f(x) = \binom{r+x-1}{x} \sum_{k=0}^{r} \binom{r}{k} (-1)^k L_t(x+k)$$

$$f(x) = \binom{r+x-1}{x} \frac{ln\left(\frac{b+x+k}{a+x+k}\right)}{ln\left(\frac{b}{a}\right)} \sum_{k=0}^{r} \binom{r}{k} (-1)^k$$

b. Mixing using MGF

$$f(x) = \binom{r+x-1}{x} \sum_{k=0}^{r} \binom{r}{k} (-1)^k M_t(-(x+k))$$

$$f(x) = \binom{r+x-1}{x} \frac{ln\left(\frac{b+x+k}{a+x+k}\right)}{ln\left(\frac{b}{a}\right)} \sum_{k=0}^{r} \binom{r}{k} (-1)^k$$

3.7.3. Properties of Negative Binomial –Variate Exponential with 2 parameter Distribution mixture

a). Mean

3.8. Negative Binomial –Variate Gamma$(2, \alpha)$ Distribution

3.8.1. Variate Gamma$(2, \alpha)$ Distribution

a. Construction

Consider a **Distribution**

$$g(t) = \begin{cases} \lambda^2 t e^{-\lambda t} & \theta > 0;\ \lambda > 0 \\ 0 & elsewhere \end{cases}$$

Where λ is randomized and takes on the distribution $\frac{1}{\lambda\left(ln\left(\frac{b}{a}\right)\right)}$ with $0 < a \leq \lambda < b$

The variate **Gamma(2, $\boldsymbol{\alpha}$)**distribution $g_1(t)$ becomes

$$g_1(t) = \frac{t}{ln\left(\frac{b}{a}\right)} \int_0^\infty \lambda e^{-\lambda t}\, d\lambda$$

Using integration by parts

$$\int udv = uv - \int vdu$$

Let $u = \lambda$ $\qquad du = d\lambda$

$$dv = e^{-\lambda t} \qquad\qquad v = -\frac{e^{-\lambda t}}{t}$$

Therefore

$$g_1(t) = \frac{t}{ln\left(\frac{b}{a}\right)}\left[\left[-\frac{\lambda e^{-\lambda t}}{t}\right]_a^b + \frac{1}{t}\int_0^\infty e^{-\lambda t}\, d\lambda\right]$$

$$= \frac{t}{ln\left(\frac{b}{a}\right)}\left[-\frac{\lambda e^{-\lambda t}}{t} - \frac{e^{-\lambda t}}{t^2}\right]_a^b$$

$$= \frac{t}{ln\left(\frac{b}{a}\right)}\left[\frac{ae^{-at} - be^{-bt}}{t} + \frac{e^{-at} - e^{-bt}}{t^2}\right]_a^b$$

$$= \frac{t}{ln\left(\frac{b}{a}\right)}\left[\frac{ate^{-at} - bte^{-bt} + e^{-at} - e^{-bt}}{t^2}\right]_a^b$$

$$= \frac{1}{ln\left(\frac{b}{a}\right)}\left[\frac{ate^{-at} - bte^{-bt} + e^{-at} - e^{-bt}}{t}\right]$$

$$= \frac{1}{tln\left(\frac{b}{a}\right)}[(at+1)e^{-at} - (bt+1)e^{-bt}]$$

With $t > 0; 0 < a < b$

b. Properties of Variate Gamma (2,α) Distribution

- **Laplace transform**

$$L_t(s) = E(e^{-ts})$$

$$= \frac{1}{ln\left(\frac{b}{a}\right)}\int_0^\infty \frac{e^{-ts}}{t}[(at+1)e^{-at} - (bt+1)e^{-bt}]dt$$

$$= \frac{1}{ln\left(\frac{b}{a}\right)}\int_0^\infty \left[\frac{(at+1)e^{-(s+a)t} - (bt+1)e^{-(s+b)t}}{t}\right]dt$$

$$L_t(s) = \frac{1}{ln\left(\frac{b}{a}\right)}\left[\int_0^\infty \frac{e^{-(s+a)t} - e^{-(s+b)t}}{t}dt + a\int_0^\infty e^{-(s+a)t}\,dt - b\int_0^\infty e^{-(s+b)t}\,dt\right]$$

$$L_t(s) = \frac{1}{ln\left(\frac{b}{a}\right)}\left[\ln\left(\frac{s+b}{s+a}\right) + \frac{a}{s+a} - \frac{b}{s+b}\right]$$

- **Moment Generating Function of Variate Gamma (2,α)**

$$M_t(s) = E(e^{ts})$$

$$= \frac{1}{ln\left(\frac{b}{a}\right)}\int_0^\infty \frac{e^{ts}}{t}[(at+1)e^{-at} - (bt+1)e^{-bt}]dt$$

$$= \frac{1}{ln\left(\frac{b}{a}\right)}\int_0^\infty \left[\frac{(at+1)e^{-(a-s)t} - (bt+1)e^{-(b-s)t}}{t}\right]dt$$

$$M_t(s) = \frac{1}{ln\left(\frac{b}{a}\right)}\left[\int_0^\infty \frac{e^{-(a-s)t} - e^{-(b-s)t}}{t}dt + a\int_0^\infty e^{-(a-s)t}\,dt - b\int_0^\infty e^{-(b-s)t}\,dt\right]$$

$$M_t(s) = \frac{1}{ln\left(\frac{b}{a}\right)}\left[\ln\left(\frac{b-s}{a-s}\right) + \frac{a}{a-s} - \frac{b}{b-s}\right]$$

3.8.2. Negative Binomial –Variate Gamma(2, ∝) distribution mixing

a. Mixing using Laplace transform

$$f(x) = \binom{r+x-1}{x}\sum_{k=0}^{r}\binom{r}{k}(-1)^k L_t(x+k)$$

$$f(x) = \binom{r+x-1}{x} \frac{1}{ln\left(\frac{b}{a}\right)} \left[\ln\left(\frac{x+k+b}{x+k+a}\right) + \frac{a}{x+k+a} - \frac{b}{x+k+b}\right] \sum_{k=0}^{r} \binom{r}{k} (-1)^k$$

$$\boldsymbol{a}, \boldsymbol{b} > 0; r > 0$$

b. Mixing using MGF

$$f(x) = \binom{r+x-1}{x} \sum_{k=0}^{r} \binom{r}{k} (-1)^k M_t(-(x+k))$$

$$f(x) = \binom{r+x-1}{x} \frac{1}{ln\left(\frac{b}{a}\right)} \left[\ln\left(\frac{x+k+b}{x+k+a}\right) + \frac{a}{x+k+a} - \frac{b}{x+k+b}\right] \sum_{k=0}^{r} \binom{r}{k} (-1)^k$$

$$\boldsymbol{a}, \boldsymbol{b} > 0; r > 0$$

Reference: $\boldsymbol{Bowman\ et\ al\ (1992)}$

3.8.3. Properties of Negative Binomial– Variate gamma(2,α) with 2 parameter Distribution mixture

a). Mean

3.9. Negative Binomial– Inverse Gaussian Distribution

3.9.1. Inverse Gaussian Distribution

a. Construction

Consider the pdf of Inverse Gaussian distribution given below

$$g(\lambda) = \left(\frac{\phi}{2\pi\lambda^3}\right)^{\frac{1}{2}} \exp\left(-\frac{\phi(\lambda-\mu)^2}{2\mu^2\lambda}\right)$$

Put $\mu = (2\alpha)^{-1/2}$ $\quad$ *and* $\quad$ $\mu^2 = (2\alpha)^{-1}$

Then

$$g(\lambda) = \left(\frac{\phi}{2\pi\lambda^3}\right)^{\frac{1}{2}} \exp\left(-\frac{\phi(\lambda-(2\alpha)^{-1/2})^2}{2\lambda(2\alpha)^{-1}}\right)$$

$$g(\lambda) = \left(\frac{\phi}{2\pi\lambda^3}\right)^{\frac{1}{2}} \exp\left(-\frac{\phi(\lambda-(2\alpha)^{-1/2})^2(2\alpha)}{2\lambda}\right)$$

$$= \left(\frac{\phi}{2\pi\lambda^3}\right)^{\frac{1}{2}} \exp\left(-\frac{\phi 2\alpha\left(\lambda^2 - 2\lambda(2\alpha)^{-\frac{1}{2}} + (2\alpha)^{-1}\right)}{2\lambda}\right)$$

$$= \left(\frac{\phi}{2\pi\lambda^3}\right)^{\frac{1}{2}} \exp\left(-\lambda\alpha\phi + \phi(2\alpha)^{\frac{1}{2}} - \frac{\phi}{2\lambda}\right)$$

This is the Inverse Gaussian distribution we are going to use in this mixture

b. properties of Inverse Gaussian distribution

- **Laplace transform**

$$L_t(s) = \int_0^\infty e^{-ts} g(t) dt$$

$$= \int_0^\infty \left(\frac{\phi}{2\pi t^3}\right)^{\frac{1}{2}} \exp\left(-st - t\alpha\phi + \phi(2\alpha)^{\frac{1}{2}} - \frac{\phi}{2t}\right) dt$$

$$= \exp\left(\phi(2\alpha)^{\frac{1}{2}}\right) \int_0^\infty \left(\frac{\phi}{2\pi t}\right)^{\frac{1}{2}} \exp\left(-st - t\alpha\phi - \frac{\phi}{2t}\right) dt$$

Factorize the like terms

$$L_t(s) = \exp\left(\phi(2\alpha)^{\frac{1}{2}}\right)\int_0^{\infty}\left(\frac{\phi}{2\pi t^3}\right)^{\frac{1}{2}}\exp\left(-t(s+\alpha\phi)-\frac{\phi}{2t}\right)dt$$

$$= \exp\left(\phi(2\alpha)^{\frac{1}{2}}\right)\int_0^{\infty}\left(\frac{\phi}{2\pi t^3}\right)^{\frac{1}{2}}\exp\left(-t\left(\frac{s\phi}{\phi}+\alpha\phi\right)-\frac{\phi}{2t}\right)dt$$

$$= \exp\left(\phi(2\alpha)^{\frac{1}{2}}\right)\int_0^{\infty}\left(\frac{\phi}{2\pi t^3}\right)^{\frac{1}{2}}\exp\left(-t\phi\left(\frac{s}{\phi}+\alpha\right)-\frac{\phi}{2t}\right)dt \ldots\ldots..eq3$$

Consider the exponential

$$\phi(2\alpha)^{\frac{1}{2}} - t\alpha\phi - \frac{\phi}{2t}$$

Suppose we substitute $\alpha\ by\ \alpha + \frac{s}{\phi}$

$$\phi\left(2\left(\alpha+\frac{s}{\phi}\right)\right)^{\frac{1}{2}} - t\phi\left(2\alpha+\frac{s}{\phi}\right)-\frac{\phi}{2t}$$

This is almost similar to equation 3 but without the middle term

$$L_t(s) = \exp\left(\phi(2\alpha)^{\frac{1}{2}}\right)\exp\left(-\phi\left(2\left(\alpha+\frac{s}{\phi}\right)\right)^{\frac{1}{2}}\right)\int_0^{\infty}\left(\frac{\phi}{2\pi t^3}\right)^{\frac{1}{2}}\exp\left(-t\phi\left(\frac{s}{\phi}+\alpha\right)\right.$$
$$\left. -\phi\left(2\left(\alpha+\frac{s}{\phi}\right)\right)^{\frac{1}{2}}-\frac{\phi}{2t}\right)dt$$

$$L_t(s) = \exp\left(\phi(2\alpha)^{\frac{1}{2}}\right)\exp\left(-\phi\left(2\left(\alpha+\frac{s}{\phi}\right)\right)^{\frac{1}{2}}\right)X1$$

$$L_t(s) = \exp\left(\phi(2\alpha)^{\frac{1}{2}} - \phi\left(2\left(\alpha+\frac{s}{\phi}\right)\right)^{\frac{1}{2}}\right)$$

Remember initially we had let

$$\mu = (2\alpha)^{-1/2} \qquad and \qquad \mu^2 = (2\alpha)^{-1}$$

Replacing these on the equation we have

$$L_t(s) = \exp\left(\phi(2\alpha)^{\frac{1}{2}} - \phi\left(2\left(\alpha+\frac{s}{\phi}\right)\right)^{\frac{1}{2}}\right)$$

$$L_t(s) = \exp\left(\frac{\phi}{\mu} - \phi\left(2\left(\frac{1}{2\mu^2}+\frac{s}{\phi}\right)\right)^{\frac{1}{2}}\right)$$

$$= \exp\left(\frac{\phi}{\mu} - \phi\left(\frac{\phi+2s\mu^2}{\phi\mu^2}\right)^{\frac{1}{2}}\right)$$

$$= \exp\left(\frac{\phi}{\mu}\left[1-\left(\frac{\phi+2s\mu^2}{\phi}\right)^{\frac{1}{2}}\right]\right)$$

As obtained by Tweedie (1957)

If we let

$$\propto = 0$$

$$g(t) = \left(\frac{\phi}{2\pi t^3}\right)^{\frac{1}{2}} \exp\left(-\frac{\phi}{2t}\right)$$

$$\mathrm{L_t(s)} = \exp\left(-\phi\left(2\left[\frac{s}{\phi}\right]\right)^{\frac{1}{2}}\right)$$

$$\mathrm{L_t(s)} = \exp\left(-(2[s\phi])^{\frac{1}{2}}\right)$$

Putting $\phi = \frac{k^2}{2}$ we have

$$g(t) = \left(\frac{k^2}{4\pi t^3}\right)^{\frac{1}{2}} \exp\left(-\frac{k^2}{4t}\right)$$

$$\mathrm{L_t(s)} = \exp\left(-\left(2\left[s\frac{k^2}{2}\right]\right)^{\frac{1}{2}}\right)$$

$$\mathrm{L_t(s)} = \exp\left(-(k(s)^{\frac{1}{2}})\right)$$

As obtained by Bowman et al (1992)

- **Moment generating function for inverse Gaussian distribution**

3.9.2.Negative Binomial –Inverse Gaussian Distribution Mixture

a. mixing using Laplace transform

$$p(X=x)=\binom{r+x-1}{x}\sum_{k=0}^{r}\binom{r}{k}(-1)^k L_t(x+k)$$

But

$$L_t(x+k)=\exp\left(\frac{\phi}{\mu}\left[1-\sqrt{1-\frac{2(x+k)\mu^2}{\phi}}\right]\right)$$

$$p(X=x)=\binom{r+x-1}{x}\sum_{k=0}^{r}\binom{r}{k}(-1)^k \exp\left(\frac{\phi}{\mu}\left[1-\sqrt{1-\frac{2(x+k)\mu^2}{\phi}}\right]\right)$$

b. Mixing using moment generating function technique

$$p(X=x)=\binom{r+x-1}{x}\sum_{k=0}^{r}\binom{r}{k}(-1)^k M_t(-(x+k))$$

But

$$M_t(-(x+k))=\exp\left(\frac{\phi}{\mu}\left[1-\sqrt{1-\frac{2(x+k)\mu^2}{\phi}}\right]\right)$$

$$p(X=x)=\binom{r+x-1}{x}\sum_{k=0}^{r}\binom{r}{k}(-1)^k \exp\left(\frac{\phi}{\mu}\left[1-\sqrt{1-\frac{2(x+k)\mu^2}{\phi}}\right]\right)$$

c. **Mixing using recursive relation**

We know when $p = 1 - e^{-t}$ the negative binomial distribution will assume the formula below

$$p(x/t) = \binom{r+x-1}{x}(1-e^{-t})^r e^{-xt} \qquad x = 0,1,2 \ldots\ldots$$

Suppose we want to get $p((x-1)/t)$

$$p((x-1)/t) = \binom{r+x-2}{x-1}(1-e^{-t})^r e^{-t(x-1)}$$

$$\frac{p(x/t)}{p((x-1)/t)} = \frac{(r+x-1)(r+x-2)!}{x(x-1)!\,(r-1)!}\frac{(x-1)!\,(r-1)!}{(r+x-2)!}\frac{e^{-t(x)}(1-e^{-t})^r}{e^{-t(x-1)}(1-e^{-t})^r}$$

$$\frac{p(x/t)}{p((x-1)/t)} = \frac{r+x-1}{x}e^{-t}$$

$$p(x/t) = \frac{r+x-1}{x}e^{-t}p((x-1)/t) \ldots\ldots\ldots\ldots eqn\ 4.3$$

Using eqn 4.3 and the Negative binomial distribution

$$P_r(x) = \int_0^\infty p(x/t)g(t)dt$$

$$P_r(x) = \int_0^\infty \frac{r+x-1}{x}e^{-t}p((x-1)/t)g(t)dt$$

$$P_r\,(x) = \frac{r + x - 1}{x} \int_0^\infty e^{-t} p((x-1)/t) g(t) dt$$

Now consider

$$\int_0^\infty e^{-t} p((x-1)/t) g(t) dt$$

This can be expressed as follows

$$\int_0^\infty e^{-t} p((x-1)/t) g(t) dt = \binom{r + x - 2}{x - 1} \int_0^\infty (1 - e^{-t})^r e^{-tx} g(t) dt \ldots eqn 4.31$$

… … …

Consider the combination

$$\binom{r + x - 2}{x - 1} = \frac{(r + x - 2)!}{(r - 1)!\,(x - 1)!}$$

$$= \frac{r}{r + x - 1} \binom{r + x - 1}{x - 1}$$

Or

$$= \frac{r}{r + x - 1} \binom{r + x - 1}{r}$$

eqn 4.31 therefore becomes

$$= \frac{r}{r+x-1} \int_0^\infty \binom{r+x-1}{r} (1-e^{-t})^r e^{-tx} g(t) dt$$

$$= \frac{r}{r+x-1} P_x(r)$$

eqn 4.00 therefore becomes

$$P_r(x) = \frac{r+x-1}{x} X \frac{r}{r+x-1} P_x(r)$$

$$P_r(x) = \frac{r}{x} P_x(r)$$

Which is the mixture of negative binomial and inverse Gaussian distribution using recursive relation as obtained by $Gomez\ and\ Deniz\ et\ al$ (2006)

3.9.3. Properties of IGNB distribution

$$E(X) = L_t(-1) - 1$$

$$\exp\left(\frac{\phi}{\mu}\left[1-\left(\frac{\phi-2\mu^2}{\phi}\right)^{\frac{1}{2}}\right]\right) - 1$$

$$\boldsymbol{var}(\boldsymbol{X}) = [\mathbf{2}\boldsymbol{L_t}(-\mathbf{2}) - \mathbf{3}\boldsymbol{L_t}(-\mathbf{1}) + \mathbf{1}] - [\boldsymbol{L_t}(-\mathbf{1}) - \mathbf{1}]^{\mathbf{2}}$$
$$= \mathbf{2}\boldsymbol{L_t}(-\mathbf{2}) - \boldsymbol{L_t}(-\mathbf{1}) - \boldsymbol{L_t^2}(-\mathbf{1})$$

$$= \mathbf{2}\exp\left(\frac{\phi}{\mu}\left[1-\left(\frac{\phi-4\mu^2}{\phi}\right)^{\frac{1}{2}}\right]\right)_{\boldsymbol{t}} - \exp\left(\frac{\phi}{\mu}\left[1-\left(\frac{\phi-2\mu^2}{\phi}\right)^{\frac{1}{2}}\right]\right)_{\boldsymbol{t}}$$
$$-\left\{\exp\left(\frac{\phi}{\mu}\left[1-\left(\frac{\phi-2\mu^2}{\phi}\right)^{\frac{1}{2}}\right]\right)\right\}^2$$

3.10. Negative binomial – Lindley distribution

Alternative Lindley negative binomial mixture. Ref Hussein Zamani and Noriszura Ismail 2010: negative binomial – Lindley distribution and its application.

$$g(t) = \frac{\theta^2(1+t)e^{-\theta t}}{\theta+1}$$

Moment generating function for Lindley distribution

$$E(e^{ts}) = \int_0^\infty \frac{\theta^2(1+t)e^{-\theta t}e^{ts}}{\theta+1}dt$$

$$= \frac{\theta^2(1+t)}{\theta+1}\int_0^\infty e^{-t(\theta-s)}dt$$

$$= \frac{\theta^2(1+t)}{(\theta+1)(\theta-s)}\int_0^\infty (\theta-s)e^{-t(\theta-s)}dt$$

In the notes

3.10.1. Mixture using MGF

Negative binomial distribution

$$f(x/\lambda) = \binom{r+x-1}{x} e^{-\lambda x}(1-e^{-\lambda})^r \qquad for\ p = e^{-\lambda}$$

This can as well be written as

$$f(x/\lambda) = \binom{r+x-1}{x} \sum_{k=0}^{r} \binom{r}{k} (-1)^k\, e^{-\lambda(x+j)}$$

The mixture

$$prob(x) = \int_0^\infty prob(x/\lambda) g(\lambda, \theta) d\lambda$$

$$= \int_0^\infty \binom{r+x-1}{x} \sum_{j=0}^{r} \binom{r}{j} (-1)^j \, e^{-\lambda(x+j)} \frac{\theta^2(1+\lambda)e^{-\theta\lambda}}{\theta+1} d\lambda$$

$$= \binom{r+x-1}{x} \sum_{k=0}^{x} \binom{x}{k} (-1)^k \int_0^\infty e^{-\lambda(x+k)} \frac{\theta^2(1+\lambda)e^{-\theta\lambda}}{\theta+1} d\lambda$$

But

$$\int_0^\infty e^{-\lambda(x+k)} \frac{\theta^2(1+\lambda)e^{-\theta\lambda}}{\theta+1} d\lambda = M_\lambda(-(x+k))$$

Where $M_\lambda(-(r+k))$ is the MGF

From previous proof we have the MGF of Lindley distribution as

$$M_x(z) = \frac{\theta^2(\theta - z + 1)}{(\theta+1)(\theta - z)^2}$$

In our case $z = -(r+j)$

And hence

$$prob(x) = \binom{r+x-1}{x} \sum_{k=0}^{r} \binom{r}{k} (-1)^k \, M_\lambda\big(-(x+k)\big)$$

$$= \binom{r+x-1}{x} \sum_{k=0}^{r} \binom{r}{k} (-1)^k \frac{\theta^2(\theta + x + k + 1)}{(\theta+1)(\theta + x + k)^2}$$

$$prob(x) = \frac{\theta^2}{(\theta+1)} \binom{r+x-1}{x} \sum_{k=0}^{r} \binom{r}{k} (-1)^k \frac{(\theta + x + k + 1)}{(\theta + x + k)^2}$$

3.10.2. Properties of negative binomial – Lindley distribution

Mean

$$E(X) = r \left[\frac{\theta^3}{(\theta+1)(\theta-1)^2} - 1 \right]$$

$$E(X^2) = (r + r^2) \left[\frac{\theta^2(\theta-1)}{(\theta+1)(\theta-2)^2} \right] - (r + 2r^2) \left[\frac{\theta^3}{(\theta+1)(\theta-1)^2} \right] + r^2$$

Variance

$$var(x) = E(X^2) - [E(X)]^2$$

$$[E(X)]^2 = r^2 \left[\frac{\theta^3}{(\theta+1)(\theta-1)^2} - 1 \right]^2$$

$$= r^2 \left[\frac{\theta^6}{(\theta+1)^2(\theta-1)^4} - \frac{2\theta^3}{(\theta+1)(\theta-1)^2} + 1 \right]$$

$$Var(X) = \left\{ (r + r^2) \left[\frac{\theta^2(\theta-1)}{(\theta+1)(\theta-2)^2} \right] - (r + 2r^2) \left[\frac{\theta^3}{(\theta+1)(\theta-1)^2} \right] + r^2 \right\} - r^2 \left[\frac{\theta^6}{(\theta+1)^2(\theta-1)^4} - \frac{2\theta^3}{(\theta+1)(\theta-1)^2} + 1 \right]$$

CHAPTER 4

GEOMETRIC MIXTURES FROM NEGATIVE BINOMIAL MIXTURES BASED ON TRANSFORMATION OF THE PARAMETER $p= e^{-t}$

4.1 Introduction

The mixing distributions are within the interval [0,∞]. The parameter$p$ is transformed toe^{-t}. In this chapter we have considered when $r = 1$ in theNegative Binomial distribution, the distribution becomes a Geometric distribution hence Geometric mixtures.

4.2. Geometric – Exponential distribution

4.2.1.Exponential distribution

The pdf of an exponential distribution is given by

$$g(t) = \lambda e^{-t\lambda} \qquad ; t > 0;\ \lambda > 0$$

From the previous chapters the Negative Binomial – Exponential distributionis expressed in the following formats after mixing

a. Negative Binomial – Exponential distribution from Laplace transform

$$f(x) = \binom{r+x-1}{x} \lambda \sum_{k=0}^{x} \binom{x}{k} \frac{(-1)^k}{r+k+\lambda}$$

$$r > 0;\ \lambda > 0, x > 0$$

Properties of negative binomial exponential distribution mixture

$$E(X) = \frac{r}{\lambda - 1}$$

Variance

$$var(X) = (r + r^2)M_t(2) - rM_t(1) + r^2 M_t^2(1)$$

$$var(X) = (r + r^2)\frac{\lambda}{\lambda - 2} - r\frac{\lambda}{\lambda - 1} + r^2\left[\frac{\lambda}{\lambda - 1}\right]^2$$

When $r = 1$, we obtain Geometric – Exponential mixture from Laplace transformas follows:

$$f(x) = \lambda \sum_{k=0}^{x} \binom{x}{k} \frac{(-1)^k}{1+k+\lambda}$$

$$\lambda > 0, x > 0$$

Properties of geometric - exponential distribution mixture

$E(X) = \frac{1}{\lambda-1}$, λ>1

Variance

$var(X) = \frac{2\lambda}{\lambda-2} - \frac{\lambda}{\lambda-1} + \left[\frac{\lambda}{\lambda-1}\right]^2$, λ>2

b. Negative binomial exponential Mixture from the moment Generating Function technique

$$f(x) = \binom{r+x-1}{x} \sum_{k=0}^{x} \binom{x}{k} \frac{(-1)^k \lambda}{\lambda+r+k}$$

When $r = 1$, we obtain Geometric – Exponential mixture from the moment Generating Function technique as follows:

$$f(x) = \sum_{k=0}^{x} \binom{x}{k} \frac{(-1)^k \lambda}{\lambda+1+k}$$

c. Negative binomial exponential distribution from Explicit mixing

$$f(x) = \frac{\lambda \binom{r+x-1}{x}}{\binom{r+\lambda+x-2}{x}}$$

or

$$f(x) = -\frac{\lambda \Gamma(x+r)\Gamma(r+\lambda-1)}{\Gamma(r+\lambda+x-1)\Gamma r}$$

When $r = 1$ the above mixture assumes the geometric – exponential mixture as outlined below

$$f(x) = \frac{\lambda}{\binom{\lambda+x-1}{x}}$$

4.3. Geometric - Beta exponential distribution

4.3.1. Beta exponential distribution

$$g(t) = \begin{cases} \frac{c}{B(a,b)} e^{-bct}(1-e^{-ct})^{a-1} & x > 0; for\ a, b\ and\ c > 0 \\ 0 & elsewhere \end{cases}$$

$$\boldsymbol{p = e^{-cx}}$$

a. Beta exponential – negative binomial mixture using Laplace transform

$$p(X = x) = \binom{r+x-1}{x} \sum_{k=0}^{x} \binom{x}{k} (-1)^k \frac{B\left(b + \frac{r+k}{c}, a\right)}{B(a,b)}$$

When $r = 1$ the above mixture assumes the Geometric - Beta exponential mixture as outlined below

$$p(X = x) = \sum_{k=0}^{x} \binom{x}{k} (-1)^k \frac{B\left(b + \frac{1+k}{c}, a\right)}{B(a,b)}$$

b. Beta exponential – negative binomial mixture using MGF technique

$$p(X = x) = \binom{r + x - 1}{x} \sum_{k=0}^{x} \binom{x}{k} (-1)^k \frac{B\left(b + \frac{r+k}{c}, a\right)}{B(a, b)}$$

When $r = 1$ the above mixture assumes the Geometric - Beta exponentialmixture as outlined below

$$p(X = x) = \sum_{k=0}^{x} \binom{x}{k} (-1)^k \frac{B\left(b + \frac{1+k}{c}, a\right)}{B(a, b)}$$

c. Beta exponential – negative binomial mixture using explicit format

When $r = 1$the above mixture assumes the Geometric - Beta exponentialmixture as outlined below

$$p(X = x) = \sum_{k=0}^{x} \binom{x}{k} (-1)^k \frac{B\left(b + \frac{1+k}{c}, a\right)}{B(a, b)}$$

4.4. Geometric – Gamma (I) distribution

4.4.1. Gamma (I)

The pdf of **Gamma (I)** distribution is given by

$$f(t) = \begin{cases} \frac{e^{-t} t^{\alpha-1}}{\Gamma\alpha} & t > 0;\ \alpha > 0 \\ 0 & elsewhere \end{cases}$$

a.Geometric – Gamma (I) distribution from Laplace transform

- **Gamma (I) – Negative Binomial distribution mixing using Laplace transform**

$$f(x) = \binom{r + x - 1}{x} \sum_{k=0}^{x} \binom{x}{k} \frac{(-1)^k}{(r + k + 1)^\alpha}$$

When $r = 1$the above mixture assumes the Geometric – Gamma (I)mixture as outlined below

$$f(x) = \sum_{k=0}^{x} \binom{x}{k} \frac{(-1)^k}{(k+2)^\alpha}$$

b. Geometric – Gamma (I) distribution using MGF

- **Gamma (I) – Negative Binomial Mixture using MGF**

$$f(x) = \binom{r+x-1}{x} \sum_{k=0}^{x} \binom{x}{k} \frac{(-1)^k}{(r+k+1)^\alpha}$$

Properties of the mixture Gamma (I) – Negative Binomial Mixture

$$E(X) = r(L_t(-1) - 1)$$

$$= r\left(\frac{1}{(1-1)^\alpha} - 1\right)$$

This is undefined and therefore it is not possible to evaluate the expectation of this mixture.

Since the mean is undefined, and we use the mean in finding the variance, it definitely means the variance is equally undefined.

When $r = 1$the above mixture assumes the Geometric – Gamma (I)mixture as outlined below

$$f(x) = \sum_{k=0}^{x} \binom{x}{k} \frac{(-1)^k}{(k+2)^\alpha}$$

- **Properties of Geometric- Gamma (I) Mixture**

$$E(X) = L_t(-1) - 1$$

$$= \frac{1}{(1-1)^{\propto}} - 1$$

This is undefined and therefore it is not possible to evaluate the expectation of this mixture.

Since the mean is undefined, and we use the mean in finding the variance, it definitely means the variance is equally undefined.

4.5. Geometric -Gamma (II) mixture

4.5.1. Gamma with two parameters distribution

The pdf of Gamma (II) distribution with parameters α and β is given

$$g(t) = \begin{cases} \frac{e^{-t\beta} t^{\alpha-1} \beta^{\propto}}{\Gamma\alpha} & t > 0;\ \alpha, \beta > 0 \\ 0 & elsewhere \end{cases}$$

a. **Gamma (II) – Negative binomial mixture using Laplace transform**

$$f(x) = \binom{r+x-1}{x} \sum_{k=0}^{x} \binom{x}{k} (-1)^k \left[\frac{\beta}{\beta+r+k}\right]^{\alpha}$$

- **Properties of Gamma (II) – Negative binomial distribution**

$$E(X) = r(L_t(-1) - 1)$$

$$= r\left[\left[\frac{\beta}{\beta-1}\right]^{\alpha} - 1\right]$$

$$\boldsymbol{var(X) = r[(1+r)L_t(-2) - (2r+1)L_t(-1) + r] - r^2[L_t(-1) - 1]^2}$$

$$var(x) = \boldsymbol{r}\left[(\mathbf{1}+\boldsymbol{r})\left[\frac{\beta}{\beta-2}\right]^{\alpha} - \left[\frac{\beta}{\beta-1}\right]^{\alpha}(\mathbf{2r}+\mathbf{1})\boldsymbol{L_t}+\boldsymbol{r}\right] - \boldsymbol{r}^{\mathbf{2}}\left[\left[\frac{\beta}{\beta-1}\right]^{\alpha} - \mathbf{1}\right]^{\mathbf{2}}$$

When $r = 1$the above mixture assumes the Geometric – Gamma (II)mixture as outlined below

$$f(x) = \sum_{k=0}^{x}\binom{x}{k}(-1)^k\left[\frac{\beta}{\beta+1+k}\right]^{\alpha}$$

- **Properties of Geometric - Gamma (II) distribution**

$$E(X) = L_t(-1) - 1$$

$$= \left[\frac{\beta}{\beta-1}\right]^{\alpha} - 1$$

$$\boldsymbol{var}(\boldsymbol{X}) = [\mathbf{2}\boldsymbol{L_t}(-\mathbf{2}) - \mathbf{3}\boldsymbol{L_t}(-\mathbf{1}) + \mathbf{1}] - [\boldsymbol{L_t}(-\mathbf{1}) - \mathbf{1}]^{\mathbf{2}}$$

$$var(x) = \left[\mathbf{2}\left[\frac{\beta}{\beta-2}\right]^{\alpha} - \left[\frac{\beta}{\beta-1}\right]^{\alpha}\mathbf{3}\boldsymbol{L_t}+\mathbf{1}\right] - \left[\left[\frac{\beta}{\beta-1}\right]^{\alpha} - \mathbf{1}\right]^{\mathbf{2}}$$

b. Gamma (II) – Negative binomial Mixture using Moment Generating function

$$f(x) = \binom{r+x-1}{x}\sum_{k=0}^{x}\binom{x}{k}(-1)^k\left[\frac{\beta}{\beta+r+k}\right]^{\alpha}$$

When $r = 1$ the above mixture assumes the Geometric – Gamma (II)mixture as outlined below

$$f(x) = \sum_{k=0}^{x} \binom{x}{k} (-1)^k \left[\frac{\beta}{\beta + 1 + k}\right]^{\alpha}$$

4.6. Geometric – Exponentiated Exponential with one parameter

4.6.1. Exponentiated exponential with one parameter

The pdf of Exponentiated exponential with one parameter

$$g(t) = \begin{cases} \propto (1 - e^{-t})^{\propto - 1} e^{-t} & t > 0;\ \propto > 0 \\ 0 & elsewhere \end{cases}$$

a. Exponentiated exponential with one parameter – Negative Binomial Distribution mixing using Laplace transform

$$f(x) = \binom{r + x - 1}{x} \propto \sum_{k=0}^{x} \binom{x}{k} (-1)^k B(2 + r + k, \propto)$$

When $r = 1$the above mixture assumes the Geometric – Exponentiated exponential with one parameter mixture as outlined below

$$f(x) = \propto \sum_{k=0}^{x} \binom{x}{k} (-1)^k B(3 + k, \propto)$$

b.Exponentiated exponential with one parameter – Negative Binomial Distribution mixing using MGF

$$f(x) = \binom{r + x - 1}{x} \propto \sum_{k=0}^{x} \binom{x}{k} (-1)^k B(2 + r + k, \propto)$$

4.6.2 Properties of Exponentiated exponential with one parameter – Negative Binomial Distribution mixture

a). Mean

$$E(X) = r(L_t(-1) - 1)$$

$$= r(\alpha B(1,\alpha) - 1)$$

b). Variance

$$\boldsymbol{var(X) = r[(1+r)L_t(-2) - (2r+1)L_t(-1) + r] - r^2[L_t(-1) - 1]^2}$$

$$\boldsymbol{var(X) = r[(1+r)\alpha B(0,\alpha) - (2r+1)\alpha B(1,\alpha) + r] - r^2[\alpha B(1,\alpha) - 1]^2}$$

When $r = 1$the above mixture assumes the Geometric - Exponentiatedexponential with one parametermixture as outlined below

$$f(x) = \propto \sum_{k=0}^{x} \binom{x}{k} (-1)^k B(3+k,\propto)$$

4.6.3 Properties of Geometric -Exponentiated exponential with one parameter mixture

a). Mean

$$E(X) = L_t(-1) - 1$$

$$= \alpha B(1,\alpha) - 1$$

b). Variance

$$\boldsymbol{var(X) = [2L_t(-2) - 3L_t(-1) + r] - [L_t(-1) - 1]^2}$$

$$\boldsymbol{var(X) = [2\alpha B(0,\alpha) - 3\alpha B(1,\alpha) + 1] - [\alpha B(1,\alpha) - 1]^2}$$

4.7. Geometric - Exponentiated exponential with 2 parameter

4.7.1. Exponentiated exponential with 2 parameter

The Pdf of Exponentiatedexponential with 2 parameter is given as

$$g(\theta) = \begin{cases} \propto \left(1 - e^{-\theta}\right)^{\propto -1} e^{-\theta} \theta > 0;\ \propto > 0 \\ 0 \quad\quad\quad\quad\quad\quad\quad elsewhere \end{cases}$$

$$let\ \theta = t\beta t > 0$$

a. Exponentiated exponential with 2 parameter – Negative Binomial mixture using Laplace transform

$$f(x) = \binom{r + x - 1}{x} \propto \sum_{k=0}^{x} \binom{x}{k} (-1)^k B\left(\frac{2\beta + r + k}{\beta}, \propto\right)$$

When $r = 1$ the above mixture assumes the Geometric -Exponentiated exponential with 2 parametermixture as outlined below

$$f(x) = \propto \sum_{k=0}^{x} \binom{x}{k} (-1)^k B\left(\frac{2\beta + 1 + k}{\beta}, \propto\right)$$

b. Exponentiated exponential with 2 parameter – Negative Binomial mixture using MGF

$$f(x) = \binom{r + x - 1}{x} \propto \sum_{k=0}^{x} \binom{x}{k} (-1)^k B\left(\frac{2\beta + r + k}{\beta}, \propto\right)$$

When $r = 1$the above mixture assumes the Geometric – Exponentiated exponential with 2 parameter mixture as outlined below

$$f(x) =\propto \sum_{k=0}^{x} \binom{x}{k} (-1)^k B\left(\frac{2\beta+1+k}{\beta}, \propto\right)$$

4.7.2. Properties of Exponentiated exponential with two parameter – Negative Binomial Distribution mixture

a). Mean

$$E(X) = r(L_t(-1) - 1)$$

$$= r\left(\alpha B\left(\frac{2\beta-1}{\beta}, \alpha\right) - 1\right)$$

b). Variance

$$\boldsymbol{var(X) = r[(1+r)L_t(-2) - (2\mathbf{r}+1)L_t(-1) + r] - r^2[L_t(-1) - 1]^2}$$

$$\boldsymbol{var(X) = r\left[(1+r)\alpha B(\frac{2(\beta-1)}{\beta}, \alpha) - (2\mathbf{r}+1)\alpha B\left(\frac{2\beta-1}{\beta}, \alpha\right) + r\right] - r^2[\alpha B\left(\frac{2\beta-1}{\beta}, \alpha\right) - 1]^2}$$

4.7.3. Properties of Geometric - Exponentiated exponential with two parameter – Negative Binomial mixture

a). Mean

$$E(X) = L_t(-1) - 1$$

$$= \alpha B\left(\frac{2\beta-1}{\beta}, \alpha\right) - 1$$

b). Variance

$$\boldsymbol{var(X) = [2L_t(-2) - 3L_t(-1) + 1] - [L_t(-1) - 1]^2}$$

$$\boldsymbol{var(X)} = \left[\boldsymbol{2\alpha B}(\frac{2(\beta-1)}{\beta}, \boldsymbol{\alpha}) - \boldsymbol{3\alpha B}\left(\frac{2\beta-1}{\beta}, \boldsymbol{\alpha}\right) + \boldsymbol{1}\right] - [\boldsymbol{\alpha B}\left(\frac{2\beta-1}{\beta}, \boldsymbol{\alpha}\right) - \boldsymbol{1}]^{\boldsymbol{2}}$$

4.8. Geometric - Variate exponential mixture

4.8.1. Variate exponential

The pdf of variate exponential is given as

$$g_1(t) = \frac{e^{-at} - e^{-bt}}{tln\left(\frac{b}{a}\right)} \qquad t > 0; 0 < a < b$$

a. Variate exponential distribution with 2 parameter – Negative Binomial mixture using Laplace transform

$$f(x) = \binom{r+x-1}{x} \frac{ln\left(\frac{b+r+k}{a+r+k}\right)}{ln\left(\frac{b}{a}\right)} \sum_{k=0}^{x} \binom{x}{k} (-1)^k$$

When $r = 1$ the above mixture assumes the Geometric – Variate exponential with 2 parameter mixture as outlined below

$$f(x) = \frac{ln\left(\frac{b+1+k}{a+1+k}\right)}{ln\left(\frac{b}{a}\right)} \sum_{k=0}^{x} \binom{x}{k} (-1)^k$$

b. Variate exponential distribution with 2 parameter – Negative Binomial mixture using MGF

$$f(x) = \binom{r+x-1}{x} \frac{ln\left(\frac{b+r+k}{a+r+k}\right)}{ln\left(\frac{b}{a}\right)} \sum_{k=0}^{x} \binom{x}{k} (-1)^k$$

When $r = 1$ the above mixture assumes the Geometric – Variate exponential with 2 parameter mixture as outlined below

$$f(x) = \frac{ln\left(\frac{b+1+k}{a+1+k}\right)}{ln\left(\frac{b}{a}\right)} \sum_{k=0}^{x} \binom{x}{k} (-1)^k$$

4.8.2. Properties of Variate Exponential with 2 parameter – Negative Binomial Distribution mixture

a). Mean

$$E(X) = r(L_t(-1) - 1)$$

$$= r\left(\frac{ln\left(\frac{b-1}{a-1}\right)}{ln\left(\frac{b}{a}\right)} - 1\right)$$

b). Variance

$$\boldsymbol{var(X) = r[(1+r)L_t(-2) - (2r+1)L_t(-1) + r] - r^2[L_t(-1) - 1]^2}$$

$$\boldsymbol{var(X) = r\left[(1+r)\frac{ln\left(\frac{b-1}{a-1}\right)}{ln\left(\frac{b}{a}\right)} - (2r+1)\frac{ln\left(\frac{b-1}{a-1}\right)}{ln\left(\frac{b}{a}\right)} + r\right] - r^2\left[\frac{ln\left(\frac{b-1}{a-1}\right)}{ln\left(\frac{b}{a}\right)} - 1\right]^2}$$

4.8.3. Properties of Geometric -Variate Exponential with 2 parameters mixture

a). Mean

$$E(X) = L_t(-1) - 1$$

$$E(X) = \frac{ln\left(\frac{b-1}{a-1}\right)}{ln\left(\frac{b}{a}\right)} - 1$$

b). Variance

$$\boldsymbol{var(X) = [2L_t(-2) - 3L_t(-1) + 1] - [L_t(-1) - 1]^2}$$

$$\boldsymbol{var(X)} = \left[\mathbf{2}\frac{ln\left(\frac{b-1}{a-1}\right)}{ln\left(\frac{b}{a}\right)} - \mathbf{3}\frac{ln\left(\frac{b-1}{a-1}\right)}{ln\left(\frac{b}{a}\right)} + \mathbf{1}\right] - \left[\frac{ln\left(\frac{b-1}{a-1}\right)}{ln\left(\frac{b}{a}\right)} - \mathbf{1}\right]^{\mathbf{2}}$$

4.9. Geometric - Variate Gamma $(2, \alpha)$ Distribution

4.9.1. Variate Gamma $(2, \alpha)$ Distribution

The Pdf of Variate Gamma $(2, \alpha)$ Distribution is given as

$$g_1(t) = \frac{1}{tln\left(\frac{b}{a}\right)}\left[(at+1)e^{-at} - (bt+1)e^{-bt}\right]$$

With $t > 0; 0 < a < b$

a. Variate Gamma $(2, \propto)$ distribution – Negative Binomial Distribution mixture using Laplace transform

$$f(x) = \binom{r+x-1}{x}\frac{1}{ln\left(\frac{b}{a}\right)}\left[\ln\left(\frac{r+k+b}{r+k+a}\right) + \frac{a}{r+k+a} - \frac{b}{r+k+b}\right]\sum_{k=0}^{x}\binom{x}{k}(-1)^k$$

$$\boldsymbol{a}, \boldsymbol{b} > 0; r > 0$$

When $r = 1$ the above mixture assumes the Geometric – Variate Gamma $(2, \alpha)$ mixture as outlined below

$$f(x) = \frac{1}{ln\left(\frac{b}{a}\right)}\left[\ln\left(\frac{1+k+b}{1+k+a}\right) + \frac{a}{1+k+a} - \frac{b}{1+k+b}\right]\sum_{k=0}^{x}\binom{x}{k}(-1)^k$$

$$\boldsymbol{a}, \boldsymbol{b} > 0$$

b. Variate Gamma($2, \propto$) distribution – Negative Binomial Distribution mixture using MGF

$$f(x) = \binom{r+x-1}{x}\frac{1}{ln\left(\frac{b}{a}\right)}\left[\ln\left(\frac{r+k+b}{r+k+a}\right) + \frac{a}{r+k+a} - \frac{b}{r+k+b}\right]\sum_{k=0}^{x}\binom{x}{k}(-1)^k$$

$$\boldsymbol{a}, \boldsymbol{b} > 0; r > 0$$

Reference: $\boldsymbol{Bowman\ et\ al\ (1992)}$

When $r = 1$ the above mixture assumes the Geometric - Variate Gamma $(2, \alpha)$ mixture as outlined below

$$f(x) = \frac{1}{ln\left(\frac{b}{a}\right)}\left[\ln\left(\frac{1+k+b}{1+k+a}\right) + \frac{a}{1+k+a} - \frac{b}{1+k+b}\right]\sum_{k=0}^{x}\binom{x}{k}(-1)^k$$

$$\boldsymbol{a}, \boldsymbol{b} > 0$$

4.9.2. Properties of Variate gamma (2,α) – Negative Binomial Distribution mixture

a). Mean

$$E(X) = r(L_t(-1) - 1)$$

$$= r\left(\frac{1}{ln\left(\frac{b}{a}\right)}\left[\ln\left(\frac{b-1}{a-1}\right) + \frac{a}{a-1} - \frac{b}{b-1}\right] - 1\right)$$

b). Variance

$$\boldsymbol{var}(\boldsymbol{X}) = \boldsymbol{r}[(\mathbf{1}+\boldsymbol{r})\boldsymbol{L_t}(-\mathbf{2}) - (\mathbf{2r}+\mathbf{1})\boldsymbol{L_t}(-\mathbf{1}) + \boldsymbol{r}] - \boldsymbol{r}^2[\boldsymbol{L_t}(-\mathbf{1}) - \mathbf{1}]^2$$

$$\begin{aligned}\boldsymbol{var}(\boldsymbol{X}) = \boldsymbol{r}\Bigg[&(\mathbf{1}+\boldsymbol{r})\frac{1}{ln\left(\frac{b}{a}\right)}\left[\ln\left(\frac{b-2}{a-2}\right) + \frac{a}{a-1} - \frac{b}{b-1}\right] \\ &- (\mathbf{2r}+\mathbf{1})\frac{1}{ln\left(\frac{b}{a}\right)}\left[\ln\left(\frac{b-1}{a-1}\right) + \frac{a}{a-1} - \frac{b}{b-1}\right] + \boldsymbol{r}\Bigg] \\ &- \boldsymbol{r}^2\left[\frac{1}{ln\left(\frac{b}{a}\right)}\left[\ln\left(\frac{b-1}{a-1}\right) + \frac{a}{a-1} - \frac{b}{b-1}\right] - \mathbf{1}\right]^2\end{aligned}$$

4.9.3. Properties of Geometric -Variate gamma (2,α) mixture

a). Mean

$$E(X) = L_t(-1) - 1$$

$$= \frac{1}{ln\left(\frac{b}{a}\right)}\left[\ln\left(\frac{b-1}{a-1}\right) + \frac{a}{a-1} - \frac{b}{b-1}\right] - 1$$

b). Variance

$$\boldsymbol{var}(\boldsymbol{X}) = [\mathbf{2}\boldsymbol{L_t}(-\mathbf{2}) - \mathbf{3}\boldsymbol{L_t}(-\mathbf{1}) + \mathbf{1}] - [\boldsymbol{L_t}(-\mathbf{1}) - \mathbf{1}]^2$$

$$\boldsymbol{var(X)} = \left[\mathbf{2}\frac{1}{ln\left(\frac{b}{a}\right)}\left[\ln\left(\frac{b-2}{a-2}\right)+\frac{a}{a-1}-\frac{b}{b-1}\right]-\mathbf{3}\frac{1}{ln\left(\frac{b}{a}\right)}\left[\ln\left(\frac{b-1}{a-1}\right)+\frac{a}{a-1}-\frac{b}{b-1}\right] + \mathbf{1}\right] - \left[\frac{1}{ln\left(\frac{b}{a}\right)}\left[\ln\left(\frac{b-1}{a-1}\right)+\frac{a}{a-1}-\frac{b}{b-1}\right]-\mathbf{1}\right]^{\mathbf{2}}$$

4.10. Geometric - Inverse Gaussian Distribution

4.10.1. Inverse Gaussian Distribution

The pdf of Inverse Gaussian distribution is given as

$$g(\lambda) = \left(\frac{\phi}{2\pi\lambda^3}\right)^{\frac{1}{2}} \exp\left(-\lambda\alpha\phi + \phi(2\alpha)^{\frac{1}{2}} - \frac{\phi}{2\lambda}\right)$$

a. Inverse Gaussian – Negative Binomial Distribution Mixture using Laplace transform

$$p(X=x) = \binom{r+x-1}{x}\sum_{k=0}^{x}\binom{x}{k}(-1)^k \exp\left(\frac{\phi}{\mu}\left[1-\sqrt{1-\frac{2(r+k)\mu^2}{\phi}}\right]\right)$$

When $r = 1$ the above mixture assumes the Geometric - Inverse Gaussian mixture as outlined below

$$p(X=x) = \sum_{k=0}^{x}\binom{x}{k}(-1)^k \exp\left(\frac{\phi}{\mu}\left[1-\sqrt{1-\frac{2(1+k)\mu^2}{\phi}}\right]\right)$$

b. Inverse Gaussian – Negative Binomial Distribution Mixture using moment generating function technique

$$p(X=x) = \binom{r+x-1}{x}\sum_{k=0}^{x}\binom{x}{k}(-1)^k \exp\left(\frac{\phi}{\mu}\left[1-\sqrt{1-\frac{2(r+k)\mu^2}{\phi}}\right]\right)$$

When $r = 1$the above mixture assumes the Geometric - Inverse Gaussian mixture as outlined below

$$p(X=x)=\sum_{k=0}^{x}\binom{x}{k}(-1)^k\exp\left(\frac{\phi}{\mu}\left[1-\sqrt{1-\frac{2(1+k)\mu^2}{\phi}}\right]\right)$$

4.10.2. Properties Inverse Gaussian – Negative Binomial Distribution Mixture

$$E(X)=r(L_t(-1)-1)$$

$$r\left(\exp\left(\frac{\phi}{\mu}\left[1-\left(\frac{\phi-2\mu^2}{\phi}\right)^{\frac{1}{2}}\right]\right)-1\right)$$

$$\boldsymbol{var(X)=r[(1+r)L_t(-2)-(2\mathrm{r}+1)L_t(-1)+r]-r^2[L_t(-1)-1]^2}$$

$$\boldsymbol{=(r+r^2)L_t(-2)-\mathrm{r}L_t(-1)-r^2L_t^2(-1)}$$

$$=(\boldsymbol{r}+\boldsymbol{r^2})\exp\left(\frac{\phi}{\mu}\left[1-\left(\frac{\phi-4\mu^2}{\phi}\right)^{\frac{1}{2}}\right]\right)_{\boldsymbol{t}}-\mathbf{r}\exp\left(\frac{\phi}{\mu}\left[1-\left(\frac{\phi-2\mu^2}{\phi}\right)^{\frac{1}{2}}\right]\right)_{\boldsymbol{t}}$$
$$-\boldsymbol{r^2}\left\{\exp\left(\frac{\phi}{\mu}\left[1-\left(\frac{\phi-2\mu^2}{\phi}\right)^{\frac{1}{2}}\right]\right)\right\}^2$$

4.10.3. Properties Geometric - Inverse Gaussian Mixture

$$E(X)=L_t(-1)-1$$

$$\exp\left(\frac{\phi}{\mu}\left[1-\left(\frac{\phi-2\mu^2}{\phi}\right)^{\frac{1}{2}}\right]\right)-1$$

$$\boldsymbol{var(X) = [2L_t(-2) - 3L_t(-1) + 1] - [L_t(-1) - 1]^2}$$

$$= \boldsymbol{2L_t(-2) - L_t(-1) - L_t^2(-1)}$$

$$= \mathbf{2}\exp\left(\frac{\phi}{\mu}\left[1-\left(\frac{\phi-4\mu^2}{\phi}\right)^{\frac{1}{2}}\right]\right)_{\boldsymbol{t}} - \exp\left(\frac{\phi}{\mu}\left[1-\left(\frac{\phi-2\mu^2}{\phi}\right)^{\frac{1}{2}}\right]\right)_{\boldsymbol{t}} - \left\{\exp\left(\frac{\phi}{\mu}\left[1-\left(\frac{\phi-2\mu^2}{\phi}\right)^{\frac{1}{2}}\right]\right)\right\}^2$$

c. Inverse Gaussian – Negative Binomial Distribution Mixture using recursive relation

$$P_r(x) = \frac{r+x-1}{x}\left[P_r(x-1) - \frac{r}{r+x-1}P_{r+1}(x-1)\right]$$

$$Gomez\ and\ Deniz\ et\ al\ (2006)$$

When $r = 1$the above mixture assumes the Geometric - Inverse Gaussian mixture as outlined below

$$P_1(x) = \left[P_1(x-1) - \frac{1}{x}P_2(x-1)\right]$$

4.11. Geometric - Lindley Mixture

4.11.1. Lindley distribution

The pdf of **Lindley** distribution is given by

$$f(x) = \frac{\theta^2(1+t)e^{-\theta t}}{\theta + 1}$$

Lindley negative binomial mixtures.

a. Geometric - Lindley Mixture using MGF

Negative Binomial - Lindley Mixture using MGF

$$prob(x) = \frac{\theta^2}{(\theta+1)}\binom{r+x-1}{x}\sum_{k=0}^{x}\binom{x}{k}(-1)^k\frac{(\theta+r+k+1)}{(\theta+r+k)^2}$$

Properties of negative binomial – Lindley distribution

Mean

$$E(X) = r\left[\frac{\theta^3}{(\theta+1)(\theta-1)^2} - 1\right]$$

$$E(X^2) = (r+r^2)\left[\frac{\theta^2(\theta-1)}{(\theta+1)(\theta-2)^2}\right] - (r+2r^2)\left[\frac{\theta^3}{(\theta+1)(\theta-1)^2}\right] + r^2$$

Variance

$$var(x) = E(X^2) - [E(X)]^2$$

$$Var(X) = \left\{(r+r^2)\left[\frac{\theta^2(\theta-1)}{(\theta+1)(\theta-2)^2}\right] - (r+2r^2)\left[\frac{\theta^3}{(\theta+1)(\theta-1)^2}\right] + r^2\right\} - r^2\left[\frac{\theta^6}{(\theta+1)^2(\theta-1)^4} - \frac{2\theta^3}{(\theta+1)(\theta-1)^2} + 1\right]$$

When $r = 1$the above mixture assumes the Geometric – Lindley mixture as outlined below

$$prob(x) = \frac{\theta^2}{(\theta+1)}\sum_{k=0}^{x}\binom{x}{k}(-1)^k\frac{(\theta+k+2)}{(\theta+1+k)^2}$$

Properties of Geometric – Lindley distribution

Mean

$$E(X) = \frac{\theta^3}{(\theta+1)(\theta-1)^2} - 1$$

$$E(X^2) = 2\left[\frac{\theta^2(\theta-1)}{(\theta+1)(\theta-2)^2}\right] - 3\left[\frac{\theta^3}{(\theta+1)(\theta-1)^2}\right] + 1$$

Variance

$$var(x) = E(X^2) - [E(X)]^2$$

$$Var(X) = \left\{2\left[\frac{\theta^2(\theta-1)}{(\theta+1)(\theta-2)^2}\right] - 3\left[\frac{\theta^3}{(\theta+1)(\theta-1)^2}\right] + 1\right\} - \left[\frac{\theta^6}{(\theta+1)^2(\theta-1)^4} - \frac{2\theta^3}{(\theta+1)(\theta-1)^2} + 1\right]$$

CHAPTER 5

MIXTURES OF NEGATIVE BINOMIAL DISTRIBUTIONS VARYING THE PARAMETER r

5.1. Introduction

In this chapter we will consider cases in which the parameters *p* is fixed and *r* is a random variable.*r*has a continuous mixing distribution.

Let X be a random variable with probability mass function $P_k = prob(X = k)$.

If $prob(X = k/R = r) \quad = \binom{r+k-1}{k} p^k q^r, where\ k = 0,1,2,\dots, q = 1 - p\ and\ r > o.$

$$p_k = \int \binom{r+k-1}{k} p^k q^r Prob(R = r)dr$$

where p_k is a Negative Binomial mixture.

The pgf of X is given by

$$G(s) = \sum_{k=0}^{\infty} p_k s^k$$

5.2. Negative Binomial – Exponential Distribution

Let R be a random variable with pfd with

$$g(r) = \lambda e^{-\lambda r} \quad , r > 0$$

Then

$$P_k = \int_0^{\infty} prob(X = k/R = r) prob(R = r)dr$$

$$= \int_0^{\infty} \binom{r+k-1}{k} p^r q^k \lambda e^{-\lambda r} dr$$

∴

$$G(s) = \sum_{k=0}^{\infty} p_k s^k = \lambda \int_0^{\infty} p^r e^{-\lambda r} \sum_{k=0}^{\infty} \binom{r+k-1}{k} (qs)^k \, dr$$

$$= \lambda \int_0^{\infty} p^r e^{-\lambda r} \sum_{k=0}^{\infty} (-1)^k \binom{-r}{k} (qs)^k \, dr$$

$$= \lambda \int_0^{\infty} p^r e^{-\lambda r} \sum_{k=0}^{\infty} \binom{-r}{k} (-qs)^k \, dr$$

$$= \lambda \int_0^{\infty} p^r e^{-\lambda r} (1 - qs)^{-r} dr$$

$$= \lambda \int_0^{\infty} e^{-\lambda r} \left(\frac{p}{1 - qs} \right)^r dr$$

$$= \lambda \int_0^{\infty} e^{-\lambda r} \left(\frac{1 - qs}{p} \right)^{-r} dr$$

$$= \lambda \int_0^{\infty} e^{-\lambda r} e^{log\left(\frac{1-qs}{p}\right)^{-r}} dr$$

$$= \lambda \int_0^{\infty} e^{-\lambda r - r log\left(\frac{1-qs}{p}\right)} dr$$

$$= \lambda \int_0^{\infty} e^{-\left[\lambda + log\left(\frac{1-ps}{q}\right)\right] r} dr$$

Let

$$\mathrm{y} = \left[\lambda + log\left(\frac{1 - qs}{p} \right) \right] r$$

∴

$$\mathrm{r} = \frac{\mathrm{y}}{\lambda + log\left(\frac{1-qs}{p}\right)} \; and \; dr = \frac{dy}{\lambda + log\left(\frac{1-qs}{p}\right)}$$

∴

$$G(s) = \lambda \int_0^{\infty} \frac{e^{-y}}{\lambda + log\left(\frac{1-qs}{p}\right)} dy$$

$$G(s) = \frac{\lambda}{\lambda + log\left(\frac{1-qs}{p}\right)} \int_0^{\infty} e^{-y} \, dy$$

but $\int_0^{\infty} e^{-y} \, dy = 1$

∴

$$G(s) = \frac{\lambda}{\lambda + log\left(\frac{1-qs}{p}\right)}$$

5.3. Negative Binomial – Gamma Distribution

Let R be a random variable with pfd with

$$g(r) = \frac{1}{\beta^{\alpha}\Gamma\alpha} e^{-\frac{r}{\beta}} r^{\alpha-1}, r > 0$$

Then

$$P_k = \int_0^{\infty} prob(X = k/R = r) prob(R = r) dr$$

$$= \int_0^{\infty} \binom{r+k-1}{k} p^k q^r \frac{1}{\beta^{\alpha}\Gamma\alpha} e^{-\frac{r}{\beta}} r^{\alpha-1} dr$$

$$= \frac{1}{\beta^{\alpha}\Gamma\alpha} \int_0^{\infty} \binom{r+k-1}{k} p^r q^k e^{-\frac{r}{\beta}} r^{\alpha-1} dr$$

The pgf of X is given by

$$G(s) = \sum_{k=0}^{\infty} p_k s^k$$

i.e.

$$G(s) = \frac{1}{\beta^{\alpha}\Gamma\alpha} \int_0^{\infty} p^r e^{-\frac{r}{\beta}} r^{\alpha-1} \sum_{k=0}^{\infty} \binom{r+k-1}{k} (qs)^k dr$$

$$= \frac{1}{\beta^{\alpha}\Gamma\alpha} \int_0^{\infty} p^r e^{-\frac{r}{\beta}} r^{\alpha-1} \sum_{k=0}^{\infty} (-1)^k \binom{-r}{k} (qs)^k dr$$

$$= \frac{1}{\beta^{\alpha}\Gamma\alpha} \int_0^{\infty} p^r e^{-\frac{r}{\beta}} r^{\alpha-1} \sum_{k=0}^{\infty} \binom{-r}{k} (-qs)^k dr$$

$$= \frac{1}{\beta^{\alpha}\Gamma\alpha}\int_{0}^{\infty} p^{r} e^{-\frac{r}{\beta}} r^{\alpha-1} (1-qs)^{-r} dr$$

$$= \frac{1}{\beta^{\alpha}\Gamma\alpha}\int_{0}^{\infty} e^{-\frac{r}{\beta}} \left(\frac{p}{1-qs}\right)^{r} r^{\alpha-1} dr$$

$$= \frac{1}{\beta^{\alpha}\Gamma\alpha}\int_{0}^{\infty} e^{-\frac{r}{\beta}} \left(\frac{1-qs}{p}\right)^{-r} r^{\alpha-1} dr$$

$$= \frac{1}{\beta^{\alpha}\Gamma\alpha}\int_{0}^{\infty} e^{-\frac{r}{\beta}} e^{log\left(\frac{1-qs}{p}\right)^{-r}} r^{\alpha-1} dr$$

$$= \frac{1}{\beta^{\alpha}\Gamma\alpha}\int_{0}^{\infty} e^{-\frac{r}{\beta}-rlog\frac{1-qs}{p}} r^{\alpha-1} dr$$

$$= \frac{1}{\beta^{\alpha}\Gamma\alpha}\int_{0}^{\infty} e^{-\left(\frac{1}{\beta}-log\left(\frac{1-qs}{p}\right)\right)r} r^{\alpha-1} dr$$

∴

$$G(s) = \frac{1}{\beta^{\alpha}\Gamma\alpha}\int_{0}^{\infty} e^{-\left(1-\beta log\left(\frac{1-qs}{p}\right)\right)\frac{r}{\beta}} r^{\alpha-1} dr$$

Let

$$y = \left(1-\beta log\left(\frac{1-qs}{p}\right)\right)\frac{r}{\beta}$$

∴

$$r = \frac{\beta y}{1+\beta log\left(\frac{1-qs}{p}\right)} \; and \; dr = \frac{\beta dy}{1+\beta log\left(\frac{1-qs}{p}\right)}$$

$$G(s) = \frac{1}{\beta^{\alpha}\Gamma\alpha}\int_{0}^{\infty} e^{y} \left[\frac{\beta y}{1+\beta log\left(\frac{1-qs}{p}\right)}\right]^{\alpha-1} \frac{\beta}{1+\beta log\left(\frac{1-qs}{p}\right)} dy$$

$$= \frac{\beta^{\alpha}}{\beta^{\alpha}\Gamma\alpha\left[1+\beta log\left(\frac{1-qs}{p}\right)\right]^{\alpha}}\int_{0}^{\infty} e^{-y} y^{\alpha-1} dy$$

$$= \frac{\beta^{\alpha}\Gamma\alpha}{\beta^{\alpha}\Gamma\alpha\left[1+\beta log\left(\frac{1-qs}{p}\right)\right]^{\alpha}}$$

$$= \frac{1}{\left[1+\beta log\left(\frac{1-qs}{p}\right)\right]^{\alpha}}$$

$$G(s) = \left[1+\beta log\left(\frac{1-qs}{p}\right)\right]^{-\alpha}$$

Mean

$$E(x) = G'(s)$$

$$G(s) = \left[1+\beta log\left(\frac{1-qs}{p}\right)\right]^{-\alpha}$$

Let

$$u = 1+\beta log\left(\frac{1-qs}{p}\right)$$

$$G'(s) = \frac{dG(s)}{du}.\frac{du}{ds}$$

$$u = 1+\beta log(1-qs)-\beta logp$$

$$\frac{du}{ds} = \beta\frac{-q}{(1-qs)}$$

$$\frac{du}{ds} = \frac{-\beta q}{(1-qs)}$$

and

$$G(s) = u^{-\alpha}$$

$$\frac{dG(s)}{du} = -\alpha u^{-\alpha-1} = -\alpha u^{-(\alpha+1)}$$

∴

$$G'(s) = \frac{-\alpha}{\left[1+\beta log\left(\frac{1-qs}{p}\right)\right]^{\alpha+1}} \cdot \frac{-\beta q}{(1-qs)}$$

$$G'(s) = \frac{\alpha\beta q}{(1-qs)\left[1+\beta log\left(\frac{1-qs}{p}\right)\right]^{\alpha+1}}$$

put $s = 1$

$$G'(1) = \frac{\alpha\beta q}{(1-q)[1+\beta log1]^{\alpha+1}}$$

$$= \frac{\alpha\beta q}{(1-q)}$$

∴

$$E(x) = \frac{\alpha\beta q}{(1-q)}$$

Variance

$$var(x) = G''(1) + G'(1) - \left(G'(1)\right)^2$$

$$G'(s) = \frac{\alpha\beta q}{(1-qs)\left[1+\beta log\left(\frac{1-qs}{p}\right)\right]^{\alpha+1}}$$

$$G''(s) = \frac{u'v - v'u}{v^2}$$

Let $u = \alpha\beta q, u' = 0$

$$v = (1-qs)\left[1+\beta log\left(\frac{1-qs}{p}\right)\right]^{\alpha+1}$$

$$v' = a'b + b'a$$

Let $a = 1 - qs, a' = -p$

$$b = \left[1+\beta log\left(\frac{1-qs}{p}\right)\right]^{\alpha+1}$$

Let

$$y = 1 + \beta log\left(\frac{1-qs}{p}\right)$$

$$\frac{db}{ds} = \frac{dy}{ds}.\frac{db}{dy}$$

$$\frac{dy}{ds} = \frac{-\beta q}{(1-qs)}$$

$$b = y^{\alpha+1}$$

$$\Rightarrow \frac{db}{dy} = (\alpha+1)y^{\alpha}$$

$$= (\alpha+1)\left[1 + \beta log\left(\frac{1-qs}{p}\right)\right]^{\alpha}$$

∴

$$\frac{db}{ds} = \frac{-\beta q}{(1-qs)}\left\{(\alpha+1)\left[1 + \beta log\left(\frac{1-qs}{p}\right)\right]^{\alpha}\right\}$$

$$v' = -q\left[1 + \beta log\left(\frac{1-qs}{p}\right)\right]^{\alpha+1} + \frac{-\beta q}{(1-qs)}\left\{(\alpha+1)\left[1 + \beta log\left(\frac{1-qs}{p}\right)\right]^{\alpha}\right\}(1-qs)$$

$$G''(s) = \frac{\alpha\beta q\left\{q\left[1 + \beta log\left(\frac{1-qs}{p}\right)\right]^{\alpha+1} + \beta q\left\{(\alpha+1)\left[1 + \beta log\left(\frac{1-qs}{p}\right)\right]^{\alpha}\right\}\right\}}{\left\{(1-qs)\left[1 + \beta log\left(\frac{1-qs}{p}\right)\right]^{\alpha+1}\right\}^2}$$

put $s = 1$

$$G''(1) = \frac{\alpha\beta q[q + \beta q(\alpha+1)]}{(1-q)^2}$$

$$G''(1) = \frac{\alpha\beta q^2[1 + \beta(\alpha+1)]}{(1-q)^2}$$

$$= \frac{\alpha\beta q^2[1 + \beta\alpha + \beta]}{(1-q)^2}$$

∴

$$var(x) = G''(1) + G'(1) - \left(G'(1)\right)^2$$

$$= \frac{\alpha\beta q^2[1+\beta\alpha+\beta]}{(1-q)^2} + \frac{\alpha\beta q}{(1-q)} - \left(\frac{\alpha\beta q}{(1-q)}\right)^2$$

$$= \frac{\alpha\beta q^2 + \alpha^2\beta^2 q^2 + \alpha\beta^2 q^2 + \alpha^2\beta^2 q^2 - \alpha^2\beta^2 q^2}{(1-q)^2}$$

$$var(x) = \frac{\alpha\beta q^2(1+\alpha\beta+\beta)}{(1-q)^2}$$

5.4. Negative Binomial –Beta Exponential

Let R be a random variable with pfd with

$$g(r) = \frac{c}{B(a,b)} e^{-bcr}(1-e^{-cr})^{a-1} \quad , x > 0; for\ a, b\ and\ c > 0$$

Then

$$P_k = \int_0^\infty prob(X = k/R = r) prob(R = r) dr$$

$$= \int_0^\infty \binom{r+k-1}{k} q^k p^r \frac{c}{B(a,b)} e^{-bcr}(1-e^{-cr})^{a-1} dr$$

∴

$$G(s) = \sum_{k=0}^{\infty} p_k s^k = \frac{c}{B(a,b)} \int_0^\infty p^r e^{-bcr}(1-e^{-cr})^{a-1} \sum_{k=0}^{\infty} \binom{r+k-1}{k} (qs)^k\, dr$$

$$= \frac{c}{B(a,b)} \int_0^\infty p^r e^{-bcr}(1-e^{-cr})^{a-1}(1-qs)^{-r} dr$$

$$= \frac{c}{B(a,b)} \int_0^\infty e^{-bcr} \sum_{j=0}^{\infty} \binom{a-1}{j} (-1)^j e^{-crj} \left(\frac{1-qs}{p}\right)^{-r} dr$$

$$= \frac{c}{B(a,b)} \sum_{j=0}^{\infty} \binom{a-1}{j} (-1)^j \int_0^\infty e^{-bcr} e^{-crj} e^{-r\log\left(\frac{1-qs}{p}\right)} dr$$

$$= \frac{c}{B(a,b)} \sum_{j=0}^{\infty} \binom{a-1}{j} (-1)^j \int_0^\infty e^{-r\left(c(j+b)+\log\left(\frac{1-qs}{p}\right)\right)} dr$$

Let $y = r\left(log\left(\frac{1-qs}{p}\right) + c(b+j)\right)$

$$r = \frac{y}{\left(log\left(\frac{1-qs}{p}\right) + c(b+j)\right)}$$

$$dr = \frac{dy}{\left(log\left(\frac{1-qs}{p}\right) + c(b+j)\right)}$$

∴

$$G(s) = \frac{c}{B(a,b)}\sum_{j=0}^{\infty}\binom{a-1}{j}(-1)^j\int_0^{\infty} e^{-y}\frac{dy}{log\left(\frac{1-qs}{p}\right) + c(b+j)}$$

$$= \frac{c}{B(a,b)}\frac{\sum_{j=0}^{\infty}\binom{a-1}{j}(-1)^j}{log\left(\frac{1-qs}{p}\right) + c(b+j)}\int_0^{\infty} e^{-ydy}$$

$$G(s) = \frac{c}{B(a,b)}\frac{\sum_{j=0}^{\infty}\binom{a-1}{j}(-1)^j}{log\left(\frac{1-qs}{p}\right) + c(b+j)}$$

Mean

$$E(x) = G'(s)$$

$$G(s) = \frac{c}{B(a,b)}\frac{\sum_{j=0}^{\infty}\binom{a-1}{j}(-1)^j}{log\left(\frac{1-qs}{p}\right) + c(b+j)}$$

$$G'(s) = \frac{c}{B(a,b)}\sum_{j=0}^{\infty}\binom{a-1}{j}(-1)^j\frac{d}{ds}\left[\frac{1}{log\left(\frac{1-qs}{p}\right) + c(b+j)}\right]$$

Using

$$\frac{u'v - v'u}{v^2}$$

$$u = 1, u' = 0 \; and$$

$$v = log\left(\frac{1-qs}{p}\right) + c(b+j), v' = \frac{p}{1-qs}.\frac{-q}{p} = \frac{-q}{1-qs}$$

∴

$$G'(s) = \frac{c}{B(a,b)}\sum_{j=0}^{\infty}\binom{a-1}{j}(-1)^j\left[\frac{\frac{q}{1-qs}}{\left[log\left(\frac{1-qs}{p}\right)+c(b+j)\right]^2}\right]$$

$$= \frac{cq}{B(a,b)(1-qs)}\sum_{j=0}^{\infty}\binom{\alpha-1}{j}(-1)^j\left[log\left(\frac{1-qs}{p}\right)+c(b+j)\right]^{-2}$$

∴

$$E(x) = G'(1) = \frac{cq}{B(a,b)(1-q)}\sum_{j=0}^{\infty}\binom{\alpha-1}{j}(-1)^j[c(b+j)]^{-2}$$

$$= \frac{q}{B(a,b)c(1-q)}\sum_{j=0}^{\infty}\binom{a-1}{j}(-1)^j(b+j)^{-2}$$

$$G''(s) = \frac{cq}{B(a,b)}\sum_{j=0}^{\infty}\binom{a-1}{j}(-1)^j\frac{d}{ds}\left\{\left[log\left(\frac{1-qs}{p}\right)+c(b+j)\right]^{-2}\frac{1}{1-qs}\right\}$$

$$\sum_{j=0}^{\infty}\binom{a-1}{j}(-1)^j\frac{d}{ds}\left\{\left[log\left(\frac{1-qs}{p}\right)+\right]^{-2}\frac{1}{1-qs}\right\}$$

$$= \frac{cq}{B(a,b)}\sum_{j=0}^{\infty}\binom{a-1}{j}(-1)^j\frac{d}{ds}\left[\frac{1}{\left[log\left(\frac{1-qs}{p}\right)+c(b+j)\right]^2(1-qs)}\right]$$

$$v' = 2(1-qs)\left(\frac{-q}{1-qs}\right)\left[log\left(\frac{1-qs}{p}\right)+c(b+j)\right] - q\left[log\left(\frac{1-qs}{p}\right)+c(b+j)\right]^2$$

$$G''(s) = \frac{cq}{B(a,b)}\sum_{j=0}^{\infty}\binom{a-1}{j}(-1)^j\frac{q\left[log\left(\frac{1-qs}{p}\right)+c(b+j)\right]^2 - 2(1-qs)\left(\frac{q}{1-qs}\right)\left[log\left(\frac{1-qs}{p}\right)+c(b+j)\right]}{(1-qs)^2\left[log\left(\frac{1-qs}{p}\right)+c(b+j)\right]^4}$$

$$G''(1) = \frac{cq}{B(a,b)} \sum_{j=0}^{\infty} \binom{a-1}{j} (-1)^j \frac{q[c(b+j)]^2 - 2(1-q)\left(\frac{q}{1-q}\right)[c(b+j)]}{(1-q)^2[c(b+j)]^4}$$

$$G''(1) = \frac{cq^2}{B(a,b)} \sum_{j=0}^{\infty} \binom{a-1}{j} (-1)^j \frac{[c(b+j)]^2 + 2[c(b+j)]}{p^2[c(b+j)]^4}$$

$$G''(1) = \frac{q^2}{B(a,b)} \sum_{j=0}^{\infty} \binom{\alpha-1}{j} (-1)^j \frac{c(b+j)+2}{(pc)^2[(b+j)]^3}$$

$$var(x) = G''(1) + G'(1) - \left(G'(1)\right)^2$$

$$var(x) = \frac{q^2}{B(a,b)} \sum_{j=0}^{\infty} \binom{\alpha-1}{j} (-1)^j \frac{c(b+j)+2}{(pc)^2[(b+j)]^3} + \frac{cq}{B(a,b)(1-q)} \sum_{j=0}^{\infty} \binom{\alpha-1}{j} (-1)^j [c(b+j)]^{-2} - \left(\frac{cq}{B(a,b)(1-q)} \sum_{j=0}^{\infty} \binom{\alpha-1}{j} (-1)^j [c(b+j)]^{-2}\right)^2$$

$$var(x) = \frac{q\sum_{j=0}^{\infty}\binom{a-1}{j}(-1)^j}{B(a,b)} \left\{ q\frac{c(b+j)+2}{(pc)^2[(b+j)]^3} + \frac{c}{(1-q)}[c(b+j)]^{-2} - \left(\frac{c}{(1-q)}[c(b+j)]^{-2}\right)^2 \right\}$$

5.5. Negative Binomial –Exponentiated Exponential

Let r be a random variable with pfd with

$$g(r) = \lambda\alpha(1-e^{-\lambda r})^{\alpha-1}e^{-\lambda r} \quad ,x > 0$$

Then

$$P_k = \int_0^{\infty} prob(X = k/R = r)prob(R = r)dr$$

$$= \int_0^\infty \binom{r+k-1}{k} q^k p^r \lambda\alpha (1-e^{-\lambda r})^{\alpha-1} e^{-\lambda r} dr$$

∴

$$G(s) = \sum_{k=0}^{\infty} p_k s^k = \lambda\alpha \int_0^\infty p^r e^{-\lambda r} (1-e^{-\lambda r})^{\alpha-1} \sum_{k=0}^{\infty} \binom{r+k-1}{k} (qs)^k \, dr$$

$$= \lambda\alpha \int_0^\infty p^r e^{-\lambda r} (1-e^{-\lambda r})^{\alpha-1} (1-qs)^{-r} dr$$

$$= \lambda\alpha \int_0^\infty \left(\frac{p}{1-qs}\right)^r e^{-\lambda r} (1-e^{-\lambda r})^{\alpha-1} dr$$

$$= \lambda\alpha \int_0^\infty \left(\frac{p}{1-qs}\right)^r \sum_{j=0}^{\infty} \binom{\alpha-1}{j} (-1)^j e^{-\lambda r j} e^{-\lambda r} dr$$

$$= \lambda\alpha \sum_{j=0}^{\infty} \binom{\alpha-1}{j} (-1)^j \int_0^\infty \left(\frac{p}{1-qs}\right)^r e^{-\lambda r(1+j)} dr$$

$$= \lambda\alpha \sum_{j=0}^{\infty} \binom{\alpha-1}{j} (-1)^j \int_0^\infty e^{-r log\left(\frac{1-qs}{p}\right)} e^{-\lambda r(1+j)} dr$$

$$= \lambda\alpha \sum_{j=0}^{\infty} \binom{\alpha-1}{j} (-1)^j \int_0^\infty e^{-r\left(log\left(\frac{1-qs}{p}\right)+\lambda r(1+j)\right)} dr$$

Let $r\left(log\left(\frac{1-qs}{p}\right) + \lambda r(1+j)\right) = y$

$$r = \frac{y}{log\left(\frac{1-qs}{p}\right) + \lambda r(1+j)}$$

$$dr = \frac{dy}{log\left(\frac{1-qs}{p}\right) + \lambda r(1+j)}$$

∴

$$G(s) = \lambda\alpha \sum_{j=0}^{\infty} \binom{\alpha-1}{j} (-1)^j \int_0^\infty e^{-y} \frac{dy}{log\left(\frac{1-qs}{p}\right) + \lambda r(1+j)}$$

$$G(s) = \frac{\lambda\alpha \sum_{j=0}^{\infty} \binom{\alpha-1}{j} (-1)^j}{log\left(\frac{1-qs}{p}\right) + \lambda r(1+j)} \int_0^{\infty} e^{-y}\, dy$$

$$G(s) = \frac{\lambda\alpha \sum_{j=0}^{\infty} \binom{\alpha-1}{j} (-1)^j}{log\left(\frac{1-qs}{p}\right) + \lambda r(1+j)}$$

Mean

$$E(x) = G'(s)$$

$$G(s) = \lambda\alpha \sum_{j=0}^{\infty} \binom{\alpha-1}{j} (-1)^j \frac{1}{log\left(\frac{1-qs}{p}\right) + \lambda r(1+j)}$$

$$G'(s) = \lambda\alpha \sum_{j=0}^{\infty} \binom{\alpha-1}{j} (-1)^j \frac{d}{ds}\left[\frac{1}{log\left(\frac{1-qs}{p}\right) + \lambda r(1+j)}\right]$$

Using

$$\frac{u'v - v'u}{v^2}$$

$$= \lambda\alpha \sum_{j=0}^{\infty} \binom{\alpha-1}{j} (-1)^j \left[\frac{\frac{q}{1-qs}}{\left[log\left(\frac{1-qs}{p}\right) + \lambda(1+j)\right]^2}\right]$$

$$= \frac{\lambda\alpha q}{1-qs} \sum_{j=0}^{\infty} \binom{\alpha-1}{j} (-1)^j \left[log\left(\frac{1-qs}{p}\right) + \lambda(1+j)\right]^{-2}$$

∴

$$E(x) = G'(1) = \frac{\lambda\alpha q}{1-q} \sum_{j=0}^{\infty} \binom{\alpha-1}{j} (-1)^j [\lambda(1+j)]^{-2}$$

$$= \frac{\alpha q}{\lambda(1-q)} \sum_{j=0}^{\infty} \binom{\alpha-1}{j} (-1)^j (1+j)^{-2}$$

$$G''(s) = \lambda\alpha q \sum_{j=0}^{\infty} \binom{\alpha-1}{j} (-1)^j \frac{d}{ds}\left\{\left[log\left(\frac{1-qs}{p}\right) + \lambda(1+j)\right]^{-2} \frac{1}{1-qs}\right\}$$

$$= \lambda\alpha q \sum_{j=0}^{\infty} \binom{\alpha-1}{j} (-1)^j \frac{d}{ds} \left[\frac{1}{\left[log\left(\frac{1-qs}{p}\right) + \lambda(1+j) \right]^2 (1-qs)} \right]$$

$$v' = 2(1-qs)\left(\frac{-q}{1-qs}\right)\left[log\left(\frac{1-qs}{p}\right) + \lambda(1+j)\right] - q\left[log\left(\frac{1-qs}{p}\right) + \lambda(1+j)\right]^2$$

$G''(s)$

$$= \lambda\alpha q \sum_{j=0}^{\infty} \binom{\alpha-1}{j} (-1)^j \frac{q\left[log\left(\frac{1-qs}{p}\right) + \lambda(1+j)\right]^2 - 2(1-qs)\left(\frac{q}{1-qs}\right)\left[log\left(\frac{1-qs}{p}\right) + \lambda(1+j)\right]}{(1-qs)^2\left[log\left(\frac{1-qs}{p}\right) + \lambda(1+j)\right]^4}$$

$$G''(1) = \lambda\alpha q \sum_{j=0}^{\infty} \binom{\alpha-1}{j} (-1)^j \frac{q[\lambda(1+j)]^2 - 2(1-q)\left(\frac{q}{1-q}\right)[\lambda(1+j)]}{(1-q)^2[\lambda(1+j)]^4}$$

$$G''(1) = \lambda\alpha q^2 \sum_{j=0}^{\infty} \binom{\alpha-1}{j} (-1)^j \frac{[\lambda(1+j)]^2 + 2[\lambda(1+j)]}{p^2[\lambda(1+j)]^4}$$

$$G''(1) = \alpha q^2 \sum_{j=0}^{\infty} \binom{\alpha-1}{j} (-1)^j \frac{\lambda(1+j) + 2}{(p\lambda)^2[(1+j)]^3}$$

$$var(x) = G''(1) + G'(1) - \left(G'(1)\right)^2$$

$$= \alpha q^2 \sum_{j=0}^{\infty} \binom{\alpha-1}{j} (-1)^j \frac{\lambda(1+j) + 2}{(p\lambda)^2[(1+j)]^3} + \frac{\lambda\alpha q}{1-q} \sum_{j=0}^{\infty} \binom{\alpha-1}{j} (-1)^j [\lambda(1+j)]^{-2}$$
$$- \left(\frac{\lambda\alpha q}{1-q} \sum_{j=0}^{\infty} \binom{\alpha-1}{j} (-1)^j [\lambda(1+j)]^{-2} \right)^2$$

$$var(x) = \alpha q \left\{ q \sum_{j=0}^{\infty} \binom{\alpha-1}{j} (-1)^j \frac{\lambda(1+j) + 2}{(p\lambda)^2[(1+j)]^3} + \frac{\lambda}{1-q} \sum_{j=0}^{\infty} \binom{\alpha-1}{j} (-1)^j [\lambda(1+j)]^{-2} \right.$$
$$\left. - \frac{\lambda^2 \alpha q}{(1-q)^2} \left(\sum_{j=0}^{\infty} \binom{\alpha-1}{j} (-1)^j [\lambda(1+j)]^{-2} \right)^2 \right\}$$

5.6.Negative Binomial –Inverse Gaussian Distribution

Let r be a random variable with pfd with

$$g(r) = \left(\frac{\phi}{2\pi r^3}\right)^{\frac{1}{2}} \exp\left(-r\alpha\phi + \phi(2\alpha)^{\frac{1}{2}} - \frac{\phi}{2r}\right)$$

Then

$$P_k = \int_0^\infty prob(X = k/R = r)prob(R = r)dr$$

$$= \int_0^\infty \binom{r+k-1}{k} p^r q^k \left(\frac{\phi}{2\pi r^3}\right)^{\frac{1}{2}} \exp\left(-r\alpha\phi + \phi(2\alpha)^{\frac{1}{2}} - \frac{\phi}{2r}\right) dr$$

∴

$$G(s) = \sum_{k=0}^{\infty} p_k s^k = \left(\frac{\phi}{2\pi}\right)^{\frac{1}{2}} \int_0^\infty p^r r^{\frac{3}{2}} e^{-r\alpha\phi+\phi(2\alpha)^{\frac{1}{2}}-\frac{\phi}{2r}} \sum_{k=0}^{\infty} \binom{r+k-1}{k} (qs)^k\, dr$$

$$= \left(\frac{\phi}{2\pi}\right)^{\frac{1}{2}} \int_0^\infty p^r r^{\frac{3}{2}} e^{-r\alpha\phi+\phi(2\alpha)^{\frac{1}{2}}-\frac{\phi}{2r}} (1-qs)^{-r}\, dr$$

$$= \left(\frac{\phi}{2\pi}\right)^{\frac{1}{2}} \int_0^\infty r^{\frac{-3}{2}} e^{-r\alpha\phi+\phi(2\alpha)^{\frac{1}{2}}-\frac{\phi}{2r}} \left(\frac{1-qs}{p}\right)^{-r} dr$$

$$= \left(\frac{\phi}{2\pi}\right)^{\frac{1}{2}} \int_0^\infty r^{\frac{-3}{2}} e^{-r\alpha\phi+\phi(2\alpha)^{\frac{1}{2}}-\frac{\phi}{2r}} e^{-rlog\frac{1-qs}{p}} dr$$

$$= \left(\frac{\phi}{2\pi}\right)^{\frac{1}{2}} \int_0^\infty r^{\frac{-3}{2}} e^{-\left(r\alpha\phi-\phi(2\alpha)^{\frac{1}{2}}-\frac{\phi}{2r}+rlog\frac{1-qs}{p}\right)} dr$$

$$= \left(\frac{\phi}{2\pi}\right)^{\frac{1}{2}} e^{\phi(2\alpha)^{\frac{1}{2}}} \int_0^\infty r^{\frac{-3}{2}} e^{-\left(r\alpha\phi+\frac{\phi}{2r}+rlog\frac{1-qs}{p}\right)} dr$$

$$G(s) = \left(\frac{\phi}{2\pi}\right)^{\frac{1}{2}} e^{-\phi(2\alpha)^{\frac{1}{2}}} \int_0^\infty r^{\frac{-1}{2}-1} e^{-\left[r\left(\alpha\phi+log\frac{1-qs}{p}\right)+\frac{\phi}{2r}\right]} dr$$

$$= \left(\frac{\phi}{2\pi}\right)^{\frac{1}{2}} e^{-\phi(2\alpha)^{\frac{1}{2}}} \int_0^\infty r^{\frac{-1}{2}-1} e^{-\left(\alpha\phi+log\frac{1-qs}{p}\right)\left[r+\frac{\phi \qquad 1}{2(\alpha\phi+log\frac{1-qs}{p}r}\right]} dr$$

Let

$$r = \sqrt{\left(\frac{\phi}{2(\alpha\phi + log\frac{1-qs}{p}}\right)x}$$

$$\therefore dr = \sqrt{\left(\frac{\phi}{2\left(\alpha\phi + log\frac{1-qs}{p}\right)}\right)}dx$$

$$\therefore G(s) = \left(\frac{\phi}{2\pi}\right)^{\frac{1}{2}} e^{\phi(2\alpha)^{\frac{1}{2}}} \int_0^\infty \left(\frac{\phi}{2(\alpha\phi+log\frac{1-qs}{p}}\right)^{-\frac{1}{2}-1} .exp\left\{-\left(\alpha\phi + log\frac{1-qs}{p}\right)\left(\frac{\phi}{2\left(\alpha\phi+log\frac{1-qs}{p}\right)}\right)^{\frac{1}{2}} x +\right.$$

ϕ2αϕ+log1−qsp121x.ϕ2αϕ+log1−qsp−12dx

$$\therefore G(s) = \left(\frac{\phi}{2\pi}\right)^{\frac{1}{2}} e^{\phi(2\alpha)^{\frac{1}{2}}} \left(\frac{\phi}{2(\alpha\phi + log\frac{1-qs}{p})}\right)^{-\frac{1}{2}} \int_0^\infty x^{-\frac{1}{2}-1} exp\left\{\sqrt{\left(\frac{\phi}{2}\left(\alpha\phi + log\frac{1-qs}{p}\right)\right)}\left(x+\frac{1}{x}\right)\right\}dx$$

$$G(s) = \left(\frac{\alpha\phi + log\frac{1-qs}{p}}{\pi}\right)^{\frac{1}{2}} e^{\phi(2\alpha)^{\frac{1}{2}}} \int_0^\infty x^{-\frac{1}{2}-1} exp\left\{\sqrt{\left(2\phi\left(\alpha\phi + log\frac{1-qs}{p}\right)\right)}\frac{1}{2}\left(x+\frac{1}{x}\right)\right\}dx$$

$$= \left(\frac{\alpha\phi + log\frac{1-qs}{p}}{\pi}\right)^{\frac{1}{2}} e^{\phi(2\alpha)^{\frac{1}{2}}}.2K_{-\frac{1}{2}}\sqrt{\left(2\phi\left(\alpha\phi + log\frac{1-qs}{p}\right)\right)}$$

where $K_r(w)$ is a Bessel function of the third kind.

CHAPTER 6

SUMMARY AND CONCLUSIONS

6.1. Summary

This workis on the construction of Negative Binomial mixture distributions based on continuous mixing distributions for the success parameter p, where$p$is transformed to e^{-t} and $1 - e^{-t}$, and is in the range of$[0, \infty]$.

In chapter one, we have the introduction of Negative Binomial distribution which is used to come up with the mixtures.

In chapter two and three, we focused on Negative Binomial mixtures based on transformation of p.Bowman et al (1992) and Alanko & Duffy, (1996)used the transformations$p = e^{-t}$ and $p = 1 - e^{-t}$ where p is the interval $[0, \infty]$ through the random variable t for $t > 0$.

In chapter four, we have dealt with geometric mixtures from negative binomial mixtures based on transformation of the parameter $p= e^{-t}$and $p = 1 - e^{-t}$.

In chapter five we considered cases in which the parameters p is fixed and r is a random variable. r has a continuous mixing distribution.

The mixing distributions used were Exponential, Gamma, Exponeniated Exponential, Beta Exponential, Variate Gamma, Variate Exponential, Inverse Gaussian, and Lindely.

Mixtures were obtained using explicit method, recursive relations, Expectations (Laplace transform or moment generating function). The explicit involved using$f(x) = \int \binom{r+x-1}{x} p^r (1-p)^x g(p) dp$ where $x = 0,1,2, \ldots.$ and g(p) is the mixing distribution. By using this method the Negative Binomial – Exponential mixture was obtained as $f(x) = -\frac{\lambda\Gamma(x+r)\Gamma(r+\lambda-1)}{\Gamma(r+\lambda+x-1)\Gamma r}$. Other mixtures could not be obtained using explicit since integration was not possible.

NB - Inverse Gaussian distribution was obtained using recursive relations as $P_r(x) = \frac{r+x-1}{x}\left[P_r(x-1) - \frac{r}{r+x-1} P_{r+1}(x-1)\right]$.

A major limitation to our work was using all the methods was not possible in obtaining some of the mixtures. The table below gives summary of these results.

Summary of the mixtures of Negative Binomial

Distributions based on the transformation of the parameter P		
$\boldsymbol{p = e^{-t}}$		
Mixing Distribution	**Methods of mixing**	**Mixture**
Exponential distribution	Laplace transform	$f(x) = \binom{r+x-1}{x} \sum_{k=0}^{x} \binom{x}{k} (-1)^k \frac{\lambda}{r+k+\lambda}$
$g(t) = \lambda e^{-t\lambda} \quad ; t > 0;\ \lambda > 0$	Explicit	$f(x) = -\frac{\lambda\Gamma(x+r)\Gamma(r+\lambda-1)}{\Gamma(r+\lambda+x-1)\Gamma r}$

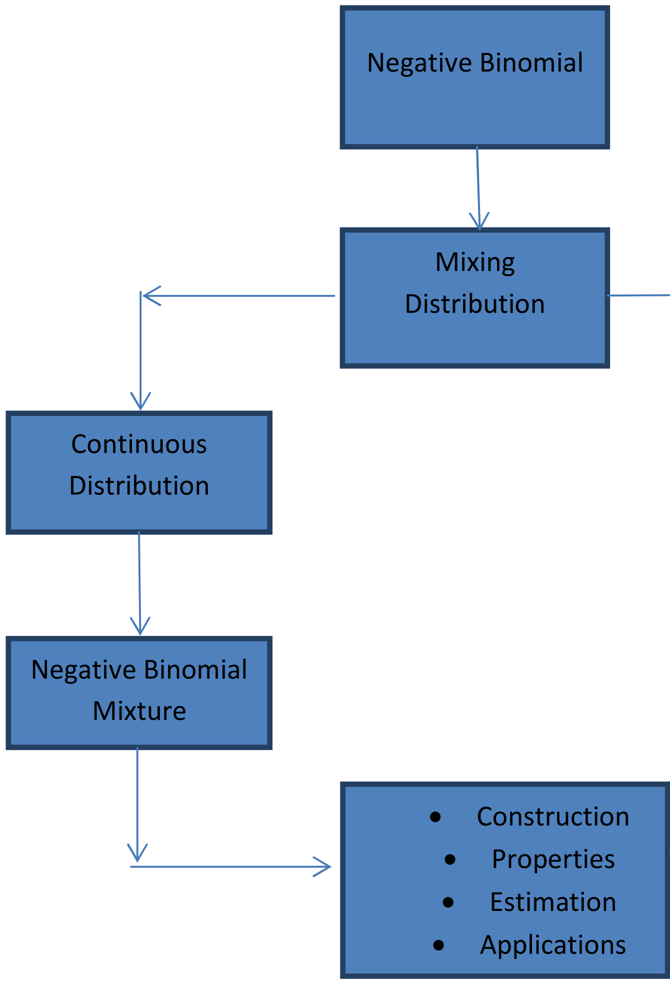
Lindley (I) distribution $$g(t)=\int_0^\infty \frac{\theta^2(1+t)e^{-\theta t}}{\theta+1}dt$$	Laplace transform	$$prob(x)=\frac{\theta^2}{(\theta+1)}\binom{r+x-1}{x}\sum_{k=0}^{x}\binom{x}{k}(-1)^k\frac{(\theta+r+k+1}{(\theta+r+k)^2}$$
Inverse Gaussian $$g(\lambda)=\left(\frac{\phi}{2\pi\lambda^3}\right)^{\frac{1}{2}}\exp\left(-\lambda\alpha\phi+\phi(2\alpha)^{\frac{1}{2}}-\frac{\phi}{2\lambda}\right)$$	Laplace transform	$$p(X=x)=\binom{r+x-1}{x}\sum_{k=0}^{x}\binom{x}{k}(-1)^k\exp\left(\frac{\phi}{\mu}\left[1-\sqrt{1-\frac{2(r+k)\mu^2}{\phi}}\right]\right)$$
	Recursive relation	$$P_r(x)=\frac{r+x-1}{x}\left[P_r(x-1)-\frac{r}{r+x-1}P_{r+1}(x-1)\right]$$
Exponentiated exponential with two parameter $$g(t)=\begin{cases}\alpha\beta\left(1-e^{-t\beta}\right)^{\alpha-1}e^{-t\beta} & ;t>0;\alpha\beta\\ 0 & elsewhere\end{cases}$$	Moment generating function	$$f(x)=\binom{r+x-1}{x}\alpha\sum_{k=0}^{x}\binom{x}{k}(-1)^kB\left(\frac{2\beta+r+k}{\beta},\alpha\right)$$
Exponentiated exponential with one parameter $$g(t)=\begin{cases}\alpha(1-e^{-t})^{\alpha-1}e^{-t} & t>0;\alpha>\\ 0 & elsewhere\end{cases}$$	Laplace transform	$$f(x)=\binom{r+x-1}{x}\alpha\sum_{k=0}^{x}\binom{x}{k}(-1)^kB(2+r+k,\alpha)$$
Gamma (I) distribution $$f(t)=\begin{cases}\frac{e^{-t}t^{\alpha-1}}{\Gamma\alpha} & t>0;\ \alpha>0\\ 0 & elsewhere\end{cases}$$	Laplace transform	$$f(x)=\binom{r+x-1}{x}\sum_{k=0}^{x}\binom{x}{k}\frac{(-1)^k}{(r+k+1)^\alpha}$$
Gamma (II) distribution $$g(t)=\begin{cases}\frac{e^{-t\beta}t^{\alpha-1}\beta^\alpha}{\Gamma\alpha} & t>0;\ \alpha,\beta>0\\ 0 & elsewhere\end{cases}$$	Laplace transform	$$f(x)=\binom{r+x-1}{x}\sum_{k=0}^{x}\binom{x}{k}(-1)^k\left[\frac{\beta}{\beta+r+k}\right]^\alpha$$
Variate Exponential with 2 parameter	Laplace transform	$$f(x)=\binom{r+x-1}{x}\frac{ln\left(\frac{b+r+k}{a+r+k}\right)}{ln\left(\frac{b}{a}\right)}\sum_{k=0}^{x}\binom{x}{k}(-1)^k$$

$g_1(t) = \frac{e^{-at} - e^{-bt}}{tln\left(\frac{b}{a}\right)} \quad t > 0; 0 < a < b$		
Variate Gamma(2, ∝) distribution $g(t) = \frac{1}{tln\left(\frac{b}{a}\right)}[(at+1)e^{-at} - (bt+1)e^{-bt}]$ With $t > 0; 0 < a < b$	Laplace transform	$f(x) = \binom{r+x-1}{x} \frac{1}{ln\left(\frac{b}{a}\right)} \left[\ln\left(\frac{r+k+b}{r+k+a}\right) + \frac{a}{r+k+a} - \frac{b}{r+k+b}\right] \sum_{k=0}^{x} \binom{x}{k} (-1)^k$ $\boldsymbol{a}, \boldsymbol{b} > 0; r > 0$
Beta exponential distribution $g(t) = \begin{cases} \frac{c}{B(a,b)} e^{-bct}(1-e^{-ct})^{a-1} & x > \\ 0 & else \end{cases}$	Laplace transform	$p(X = x) = \binom{r+x-1}{x} \sum_{k=0}^{x} \binom{x}{k} (-1)^k \frac{B\left(b + \frac{r+k}{c}, a\right)}{B(a,b)}$

$\boldsymbol{p = 1 - e^{-t}}$		
Mixing Distribution	**Methods of mixing**	**Mixture**
Exponential distribution $g(t) = \lambda e^{-t\lambda} \quad ; t > 0;\ \lambda > 0$	Laplace transform	$f(x) = \binom{r+x-1}{x} \sum_{k=0}^{r} \binom{r}{k} \frac{(-1)^k \lambda}{\lambda + x + k}$
	Explicit	$f(x) = \frac{\lambda \binom{r+x-1}{x}}{\binom{r+\lambda+x-2}{r}}$
Inverse Gaussian $g(\lambda) = \left(\frac{\phi}{2\pi\lambda^3}\right)^{\frac{1}{2}} \exp\left(-\lambda\alpha\phi + \phi(2\alpha)^{\frac{1}{2}} - \frac{\phi}{2\lambda}\right)$	Laplace transform	$p(X = x) = \binom{r+x-1}{x} \sum_{k=0}^{r} \binom{r}{k} (-1)^k \exp\left(\frac{\phi}{\mu}\left[1 - \sqrt{1 - \frac{2(x+k)\mu^2}{\phi}}\right]\right)$
	Recursive relation	$P_r(x) = \frac{r}{x} P_x(r)$

The work in this project can be summarized using the following frame work as a guideline:

A FRAMEWORK FOR NEGATIVE BINOMIAL MIXTURES



6.2. Conclusions

This work can be extended to study other properties of Negative Binomial mixturesother than mean and variance, the estimationof parameters and applications of the Negative Binomial mixtures.

Negative Binomial – Exponential and Negative Binomial – Inverse Gaussian distributions were expressed in two forms, therefore itwould be interesting to focus on proofing these identities

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