EMPIRICAL DISTRIBUTION OF RETURNS OF NAIROBI STOCK EXCHANGE 20 SHARE INDEX: 1998-2011

COX LWAKA TAMBA

A Research Project Submitted to the Graduate School in Partial Fulfillment for the Requirements of the Post-Graduate Diploma in Actuarial Science of the University of Nairobi

UNIVERSITY OF NAIROBI

JULY, 2013

DECLARATION AND RECOMMENDATION

Declaration

This research project is my original work and has not been submitted or presented for examination in any other institution.

Signature: ………………………………….. Date: ……………………………..

Cox Lwaka Tamba

I46/80834/2012

Recommendations

This research project has been submitted for examination with our approval as university supervisors.

Signature: …………………………………………. Date: …………………………………..

Prof. J. A. M. Ottieno

School of Mathematics

University of Nairobi

Signature: …………………………………………. Date: ……………………………………

Prof. P. G. O. Weke

School of Mathematics

University of Nairobi

DEDICATION

To

My wife Judy and dad Simon.

ACKNOWLEDGEMENT

First I would like to thank the Lord God Almighty for bringing me this far in my studies by granting me the strength and the grace. Thank you and Glory be to You Lord Jesus!

Secondly, I pass my regards to my supervisor; Prof. J. Otieno for his guidance and tireless support that enabled me to come up with this project work. Thank you for your encouragement and believing in me. My sincere appreciation to Prof. Weke our head of department who relentlessly provided the necessary research materials I needed for this work. To my lecturers in the department, thank you all for your support and the humble and friendly environment you provided. To my classmates, you were an encouragement to me. Thank you so much for your support.

To my beloved family; my wife Judy and baby Victoria, mum Fillister and dad Simon, you've been my encouragement all the way. Thank you for the moral and financial support you have accorded me. You guided me and led me along the right part of life and supported me attain what I want in life. Thank you and God bless you all.

ABSTRSCT

The assumption that daily share index prices are normally distributed has long been disputed by the data. In this project, the normality assumption has been tested using time series data of daily NSE 20-Share Index for the period 1998-2011. It has been confirmed that the share price index does not follow the normal distribution. Other symmetrical distributions have been fit to the data i.e. logistic distribution and t- location scale distribution. With the aid of a programming language; Matlab we have computed the various Maximum Likelihood (ML) estimates from this distributions and tested how well they fit to the data. It has been established that the NSE 20 Share Index returns follows a t-location scale distribution. We recommend that since we have found that the normal inverse gamma mixture best fits the NSE 20 Share Index return, other normal mixtures can be investigated how well they fit this data.

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ABBREVIATIONS

ML	:	Maximum Likelihood
NSE	:	Nairobi Stock Exchange
EM	:	Expectation Maximization
NASI	:	NSE All Share Index

CHAPTER ONE

INTRODUCTION

1.1 Introduction

In this chapter we provide a foundation to our study i.e. the background information and thus leading to the formulation of the problem. The entire chapter is arranged as follows: Section 1.2 provides background information to the present study, the statement of the problem is presented in Section 1.3 while the objectives and the assumptions made in this study are presented in Sections 1.4 and 1.5 respectively. Section 1.6 presents the justification to this study.

1.2 Background Information

A stock market index is a measure of changes in the stocks markets and is usually considered to be reasonably representative of the market as a whole. Indexes are usually tabulated on a daily basis and involve summarizing sample shares price movements (NSE 20 share index) or all the share prices movements (NASI).

Stock prices and returns have been assumed to be normally distributed. The popularity of this assumption may perhaps stem from the fact that normally-distributed stock returns are an implication of the random walk theory of stock prices. However, from a realistic point of view, the normality of stock returns is questionable. Stock prices and returns exhibit a fat tail than expected under the Normal distribution hence a leptokurtic distribution of stock returns should be observed. Empirical evidence against the normality assumption, has been mounting since the pioneering articles by Mandelbrot (1963), Fama (1965). Mandelbrot (1963) argued that price changes can be characterized by a stable Paretian distribution with a characteristic exponent less than 2, thus exhibiting fat tails and an infinite variance. He directly tested the infinite-variance hypothesis by computing the sample variance of a large number of samples containing the returns of cotton prices, and found that the variances did not converge to any limiting value. Rather, they evolved in an erratic fashion, just as would be expected under the infinite-variance hypothesis.

In this work, we test the normality assumption on the NSE 20 Share Index and try to develop a model that adequately fit it. We start by describing the data and testing the normality assumption for NSE 20 Share Index. The statistical distributions to be fitted to the data are discussed in chapter III. In chapter VI we present the results and discussion of our study. The conclusion to our study is presented in Chapter V.

1.3 Statement of the Problem

Many researches have based their findings on the assumption that stock prices and returns are normally distributed. However stock prices and returns exhibit a fat tail than expected under the normal distribution hence a leptokurtic distribution of stock returns should be observed.

It is on this basis that we test this assumption of the normal distribution on the NSE 20 Share Price Index returns. The normality assumption for daily stock prices has been tested and statistical distributions to be fitted to the data have been considered. The logistic and t-location scale distributions have been considered with the aim of establishing the best fit to the data.

1.4 Objectives

1.4.1 General Objective

The overall objective of this study was to establish an empirical distribution for the Nairobi Stock price Index.

1.4.2 Specific Objectives

i) To derive the t- location scale distribution.

ii) To derive Expectation Maximization (EM) algorithm based on the t- location scale distribution.

iii) To fit the normal, logistic, and t- location scale distributions to the Nairobi Stock price index.

iv) To compare the three distributions above.

1.5 Assumption

In the entire study we have assumed that the data is continuous i.e. there are no gaps and missing values due to weekend and holidays.

1.6 Justification

Analysis of the behavior of stock market prices is important in financial economics. Two major questions that are linked have induced a great amount of research. Prediction and distribution of stock market returns and prices constitute the elementary building blocks on which more elaborate financial time series analyses have been advanced. A correct statistical distribution of stock returns is needed first before any proper inferential and predictive analysis can be conducted. The basic empirical fact that the return distributions have bigger tails than normal distribution has been the major research agenda since early 1960s in finance (Mandelbrot, 1963). The concern of this work is to provide a simple and economically justified model of stock market returns and prices. This model captures the heavy tail behavior of stock prices index.

1.7 Definitions

i). Leptokurtosis

A distribution is leptokurtic if it is more peaked in the center and thicker tailed than the normal distribution with the same mean and variance. Occasionally, leptokurtosis is also identified with a moment–based kurtosis measure larger than three.

ii). Return

Let S_t be the price of a financial asset at time *t* e.g the price of stock index. Then the continuous return, r_t, is $r_t = \log(S_t/S_{t-1})$. The discrete return, R_t, is $R_t = S_t/S_{t-1} - 1$. Both are rather similar if $-0.15 < R_t < 0.15$, because $r_t = \log(1 + Rt)$.

iii). Tail

The (upper) tail, denoted by $F(x) = P(X > x)$, characterizes the probability that a random variable X exceeds a certain "large" threshold x. For analytical purposes, "large" is often translated with "as $x \to \infty$". For financial returns, a daily change of 5% is already infinitely large. A Gaussian model essentially excludes such an event.

iv). Tail index

The tail index, or tail exponent, α, characterizes the rate of tail decay if the tail goes to zero, in essence, like a power function, i.e., $F(x) = x^{-\alpha}L(x)$, where L is slowly varying. Moments of order lower (higher) than α are (in) finite.

CHAPTER TWO

LITERATURE REVIEW

2.1 Introduction

Many researches have been carried out to identify the model for stock prices. In this chapter we present the literature review to our study. Section 2.2 provides a review of the NSE 20 Share Index. A review of the models used to model stock prices is presented in Section 2.3.

2.2 NSE 20 Share Index

The NSE currently has two market indices; the NSE 20-Share Index which is price weighted and an all-inclusive NSE All Share Index (NASI) which is market capitalization weighted. Price weighted indices are based on a geometric mean of average prices of the constituent companies which are equally weighted. In line with best practice, the market indices are reviewed periodically to ensure that they reflect an accurate picture of market performance.

The NSE 20 Share index measures the average performance of 20 large cap stocks drawn from different industries. However, experience indicates that most large cap stocks do not record a high performance as compared to low cap stocks. At times small cap counters record growth averaging at 50%, while this is unlikely for large cap stocks. This makes the 20 Share index to be biased towards a large cap counters and thus fails to transmit the right signals on the entire market performance to potential investors.

Below is the list of the current NSE 20-Share Index constituent companies.

i) Agricultural Sector

• Rea Vipingo • Sasini

ii) Commercial and Services Sector

• CMC Holdings • Kenya Airways • Safaricom • Nation Media Group

iii) Finance and Investment Sector

• Barclays Bank of Kenya • Equity Bank • Kenya Commercial Bank • Standard Chartered Bank • Co-operative Bank of Kenya

iv) Industrial and Allied Sector

• Bamburi Cement • British American Tobacco • KenGen • East African Breweries • East African Cables • Kenya Power and Lighting Company • Athi River Mining • Mumias Sugar

v) Alternative Investment Market Segment

• Express Kenya

Earlier February 2008, a new NSE All-Share index (NASI) was introduced to complimentary to the NSE 20 share index. This was part of some of the recommendations by the International Finance Corporation (IFC) and regulators of world stock markets to ensure a comprehensive dissemination of market information to investors. Unlike the 20 Share Index, which measures price movement in selected, relatively stable and best performing 20 listed companies, NASI incorporates all listed companies irrespective of their performance and their time of listing. NASI is calculated based on market capitalization, meaning that it reflects the total value of all listed companies at the NSE.

2.3 Models of Stock Returns

The first complete development of a theory of random walks in security prices is due to Bachelier (1900), whose original work first appeared around the turn of the century. Unfortunately his work did not receive much attention from economists. The Bachelier

(1900) model begins by assuming that price changes from transaction to transaction in an individual security are independent, identically distributed random variables. It further assumes that transactions are fairly uniformly spread across time, and that the distribution of price changes from transaction to transaction has finite variance. If the number of transactions per day, week, or month is very large, then price changes across these differencing intervals will be sums of many independent variables. Under these conditions the central-limit theorem leads us to expect that the daily, weekly, and monthly price changes will each have normal or Gaussian distributions. Moreover, the variances of the distributions will be proportional to the respective time intervals. For example, if σ^2 is the variance of the distribution of the daily changes, then the variance for the distribution of the weekly changes should be approximately $5\sigma^2$. However it has been found out that most of the distributions of price changes are leptokurtic; that is, there are too many values near the mean and too many out in the extreme tails.

Mandelbrot (1962) asserts that, in the past, academic research has too readily neglected the implications of the leptokurtosis usually observed in empirical distributions of price changes. The presence, in general, of leptokurtosis in the empirical distributions seems indisputable. The classic approach to this problem has been to assume that the extreme values are generated by a different mechanism than the majority of the observations. Consequently one tries a posteriori to find 'causal' explanations for the large observations and thus to rationalize their exclusion from any tests carried out on the body of the data. Unlike the statistician, however, the investor cannot ignore the possibility of large price changes before committing his funds, and once he has made his decision to invest, he must consider their effects on his wealth. Mandelbrot (1963) feels that if the outliers are numerous, excluding them takes away much of the significance from any tests carried out on the remainder of the data. This exclusion process is all the more subject to criticism since probability distributions are available which accurately represent the large observations as well as the main body of the data.

The distributions referred to are members of a special class which Mandelbrot (1963) has labelled stable Paretian and it has four parameters; a location parameter, a scale parameter, an index of skewness, and a measure of the height of the extreme tail areas of the distribution which is called the characteristic exponent. When the characteristic exponent is greater than 1, the location parameter is the expectation or mean of the distribution. The scale parameter can be any positive real number, the index of skewness, β and $-1<\beta<1$. When the $\beta=0$, the distribution is symmetric. When $\beta>0$, the distribution is skewed right (i.e., has a long tail to the right). Similarly, when it is less than zero the distribution is skewed left. The characteristic exponent, α of a stable Paretian distribution determines the height of, or total probability contained in, the extreme tails of the distribution, and $0<\alpha<2$. When it is 2, the relevant stable Paretian distribution is the normal or Gaussian distribution. When 0 < α < 2, the extreme tails of the stable Paretian distributions are higher than those of the normal distribution, and the total probability in the extreme tails is larger the smaller the value of α. The most important consequence of this is that the variance exists (i.e., is finite) only in the extreme case α = 2. The mean, however, exists as long as α > 1. Mandelbrot's (1963) hypothesis states that for distributions of price changes in speculative series, α is in the interval 1 < α < 2, so that the distributions have means but their variances are infinite. The Gaussian hypothesis, on the other hand, states that α is exactly equal to 2. Thus both hypotheses assume that the distribution is stable Paretian. The disagreement between them concerns the value of the characteristic exponent α.

Two important properties of stable Paretian distributions are (1) stability or invariance under addition, and (2) the fact that these distributions are the only possible limiting distributions for sums of independent, identically distributed, random variables, Mandelbrot (1963). By definition, a stable Paretian distribution is any distribution that is stable or invariant under addition. That is, the distribution of sums of independent, identically distributed, stable Paretian variables is itself stable Paretian and, except for origin and scale, has the same form as the distribution of the individual summands. Most simply, stability means that the values of the parameters α and β remain constant under addition. The property of stability is responsible for much of the appeal of stable Paretian distributions as descriptions of empirical distributions of price changes. The price change of a stock for any time interval can be regarded as the sum of the changes from transaction to transaction during the interval. If transactions are fairly uniformly spread over time and if the changes between transactions are independent, identically distributed, stable Paretian variables, then daily, weekly, and monthly changes will follow stable Paretian distributions of exactly the same form, except for origin and scale.

Mandelbrot (1963) hypothesis that the distribution of price changes is stable Paretian with characteristic exponent $\alpha < 2$ has far reaching implications. For example, if the variances of distributions of price changes behave as if they are infinite, many common statistical tools which are based on the assumption of a finite variance either will not work or may give very misleading answers.

Fama (1965) discussed first in more detail the theory underlying the random-walk model and then tested the model's empirical validity. The past behaviour of a security's price is rich in information concerning its future behaviour (Fama, 1965). The theory of random walk shows that the successive price changes are independent, identically distributed random variables. Most simply this implies that the series of price changes has no memory, that is, the past cannot be used to predict the future in any meaningful way. The probability distribution of the price changes during time period *t* is independent of the sequence of price changes during the previous time periods i.e,

$$\Pr\left(S_t = s | S_{t-1}, s_{t-1}, \ldots\right) = \Pr\left(S_t = s\right) \quad (1)$$

The actual tests were not performed on the daily prices themselves but on the first differences of their natural logarithms. The variable of interest was

$$u_{t+1} = \log_e S_{t+1} - \log_e S_t , \quad (2)$$

where S_{t+1} is the price of the security at the end of day *t+1*.

Fama (1965) demonstrated that first differences of stock prices seem to follow stable Paretian distributions with characteristic exponent $\alpha < 2$.

According to Officer, (1972), the distribution of stock returns has some characteristics of a non-normal generating process i.e. the results indicate the distribution is "fat- tailed" relative to a normal distribution. However, characteristics were also observed which are inconsistent with a stable non-normal generating process. Evidence is presented illustrating a tendency for longitudinal sums of daily stock returns to become "thinner-tailed" for larger sums, but not to the extent that a normal distribution approximates the distribution. This confirms that the normal distribution is not a good fit for the distribution of stock and index prices.

Praetz (1972) presented both theoretical and empirical evidence about a probability distribution which describes the behaviour of share price changes. Osborne's Brownian motion theory of share price changes was modified to account for the changing variance of the share market. This produced a scaled t-distribution which is an excellent fit to series of share price indices. This distribution was the only known simple distribution to fit changes in share prices. It provided a far better fit to the data than the stable Paretian, compound process, and normal distributions (Praetz, 1972).

The Student and symmetric-stable distributions, as models for daily rates of return on common stocks, have been discussed and empirically evaluated (Blattberg, 1974). Both models were derived using the framework of subordinated stochastic processes. Some important theoretical and empirical implications of these models were also discussed. The descriptive validity of each model, relative to the other, was assessed by applying each model to actual daily rates of return. Interpretations of empirical results were guided by results from a Monte Carlo investigation of the properties of estimators and model-comparison methods. The major inference of this report was that, for daily rates of return, the Student model has greater descriptive validity than the symmetric-stable model.

Many researches have investigate on the adequacy of the Gaussian distribution on modelling stock returns; Aparicio et al., (1900-95) on Scandinavian securities markets, Richer et al., (2008) on U.S.A stock exchange . Not surprisingly, the distributions of daily stock returns analysed show fat tails and high peaks, as well as skewness in different directions.

Hung et al, (2007) studied the variation of Taiwan stock market using the statistical methods developed by econophysicists. The Taiwan market was found to have a fat tail as found in the markets of other countries, but it did not follow a power law as the others. The cumulative distribution of daily returns in Taiwan stock index could be fitted quite well using the log-normal distribution, and even better by a power law with an exponential cut-off. They believed that the distinct behaviour of Taiwan market was mainly due to the protective measures taken by the government.

Barndorff, (1977) introduced a family of continuous type distributions such that the logarithm of the probability (density) function is a hyperbola (or, in several dimensions, a hyperboloid).

The focus was on the mass-size distribution of aeolian sand deposits however, this distribution has been widely used to model stock returns. Barndorff (1978) discussed the generalised hyperbolic distributions which includes the hyperbolic distributions and some distributions which induce distributions on hyperbolae or hyperboloids analogous to the von Mises-Fisher distributions on spheres. It is, among other things, shown that distributions of this kind are a mixture of normal distributions. Eberlin (1995) based on a data set consisting of the daily prices of the 30 DAX share over three years period, investigated the distributional form of compound returns. A class of hyperbolic distribution was fit to the empirical returns with high accuracy. Hyperbolic distributions fit empirical return adequately, (Kuchler et al. (1999) and Bibby (2003)).

The logistic distribution is a general stochastic measurement model; it has been used in measuring risk incurred in financial assets returns, (Osu, 2010). It has been observed that the initial stage of growth of the worth of a business enterprise is approximately exponential. At a time the growth slows down. Osu, 2010 asserts that this could be due to diversification (investing in more than one stock), since the returns on different stocks do not move exactly in the way all the time. Ultimately the growth of the firm is stable and it may not be affected by risk since diversification reduces risk.

Stock returns turn out to be quite sensitive to the degree to which distributions are thick tailed and asymmetric. Lack of encoding information about asymmetry and leptokurtosis is a well-known drawback of the Normal distribution. This has led to a search for alternative distributions. In this work, we test the normality assumption on the NSE 20 Share Price Index. We also fit the logistic and the t- location scale distribution to find the best fit to this data.

CHAPTER THREE

MATERIALS AND METHODS

3.1 Introduction

In this chapter we present the methods i.e. the distributions used in this study. The normal distribution is fully discussed in Section 3.2. Section 3.3 presents the logistic distribution whereas the t- location scale is presented in Section 3.4. In all this sections, we review the construction, properties and estimation of these distributions. In Section 3.5 we consider the various comparison tests employed in this study. It is important to mention that from this point onwards, the sample herein referred to consists of the NSE 20 Share Index for the year 1998 to 2008. These indices are published daily in the local newspapers. The series analyzed, is the series of returns, where returns are defined as,

$$R_t = 100\left(\log\ S_{t+1} - \log\ S_t\right). \tag{3}$$

3.2 Normal Distribution

3.2.1 Construction

Let us review the construction of a normal random variable, R_t. Let

$$I = \int_{-\infty}^{\infty} e^{-\frac{y^2}{2}}\, dy$$

Therefore

$$I^2 = I.I = \int_{-\infty}^{\infty} e^{-\frac{y^2}{2}}\, dy \int_{-\infty}^{\infty} e^{-\frac{z^2}{2}}\, dz$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(y^2+z^2\right)} dydz$$

Let, $y = r\cos\theta$ and $z = r\sin\theta$

Therefore,

$$y^2 + z^2 = r^2\left[\cos^2\theta + \sin^2\theta\right]$$
$$= r^2$$

Transforming,

$$I^2 = \int_0^{2\pi}\int_0^{\infty} e^{\left(-\frac{1}{2}r^2\right)} |J| dr d\theta$$

Where,

$$|J| = \begin{vmatrix} \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} \end{vmatrix}$$
$$= \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta r & \cos\theta \end{vmatrix}$$
$$= r(\cos^2\theta + \sin^2\theta)$$
$$= r$$

Therefore,

$$I^2 = \int_0^{2\pi}\int_0^{\infty} re^{\left(-\frac{1}{2}r^2\right)} |J| dr d\theta$$

and let

$$x = e^{\left(-\frac{1}{2}r^2\right)} \Rightarrow dx = -re^{\left(-\frac{1}{2}r^2\right)} dr$$

Therefore,

Thus,

$$
\begin{aligned}
I^2 &= \int_0^{2\pi} \left[-\int_1^0 dx \right] d\theta \\
&= \int_0^{2\pi} \left[\int_0^1 dx \right] d\theta \\
&= \int_0^{2\pi} [x]_0^1 d\theta \\
&= \int_0^{2\pi} 1 d\theta \\
&= 2\pi \\
I &= \sqrt{2\pi} \\
\sqrt{2\pi} &= \int_{-\infty}^{\infty} e^{-\frac{y^2}{2}} dy \\
1 &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy
\end{aligned}
$$

Therefore the integrand is a probability distribution function

$$f(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}, -\infty < y < \infty$$

which is a standard normal distribution. Using the transform technique, let $Y = \frac{R_t - u}{\sigma}$ and therefore

$$f(r_t) = f(y)|J|$$

Where,

$$J = \frac{d}{dx}\left(\frac{r_t - u}{\sigma}\right) = \frac{1}{\sigma}$$

Thus,

$$f(r_t) = \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} . \frac{1}{\sigma}$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} \exp - \left(\frac{\left(r_{t_i} - \mu\right)^2}{2\sigma^2} \right) \qquad for \quad -\infty \le r_{t_i} \le \infty, \mu \in \mathbb{R}, \sigma > 0. \tag{4}$$

which is a normal distribution with mean, μ and variance, σ^2 (Mood, 2001).

3.2.2 Properties

The normal distribution $f(r_{t_i})$ with any mean μ and variance σ^2 is symmetric around the point $x = \mu$, which is at the same time the mode, the median and the mean of the distribution. It is unimodal. Its first derivative is positive for $x < \mu$, negative for $x > \mu$, and zero only at $x = \mu$. It has two inflection points (where the second derivative of f is zero and changes sign), located one standard deviation away from the mean, namely at $x = \mu - \sigma$ and $x = \mu + \sigma$. It is log-concave. If a random variable R_{t_i} is a normal random variable then its moment generating function $m_{R_{t_i}}(s)$ is given by,

$$m_{R_{t_i}}(s) = E(e^{sR_{t_i}}) = e^{s\mu} E(e^{s(R_{t_i} - \mu)})$$

$$= e^{s\mu} \int_{-\infty}^{\infty} (e^{s(r_{t_i} - \mu)}) \frac{1}{\sqrt{2\pi\sigma^2}} \exp - \left(\frac{\left(r_{t_i} - \mu\right)^2}{2\sigma^2} \right) dr_{t_i}$$

$$= e^{s\mu} \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} \exp - \left(\frac{\left(r_{t_i} - \mu\right)^2 - 2\sigma^2 s(r_{t_i} - \mu)}{2\sigma^2} \right) dr_{t_i} \tag{5}$$

$$= e^{s\mu} e^{\sigma^2 t^2 / 2} \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} \exp - \left(\frac{\left(r_{t_i} - \mu - \sigma^2 t^2\right)}{2\sigma^2} \right) dr_{t_i}$$

$$= e^{s\mu + \sigma^2 t^2 / 2}$$

From this equation we can differentiate to get the raw moments and hence the central moments of R_{t_i} as

$$E(R_{t_i}) = \mu\,.$$

For r>2 the central moments are

$$\mu_r = 0\text{, r odd; } \mu_r = \frac{r!}{\left(r/2\right)}\frac{\sigma^r}{2^{r/2}}\text{, r even.} \tag{6}$$

The normal distribution is symmetric distribution and hence its skewness is zero. This is clearly shown in the ratio below which is called the coefficient of skewness, which is often used to measure skewness;

$$\begin{aligned}\gamma_1 &= \frac{\mu_3}{\mu_2^{3/2}} \\ &= \frac{0}{\sigma^3} \quad . \\ &= 0\end{aligned} \tag{7}$$

However, a coefficient of skewness equal to zero does not mean that the distribution must be symmetric.

The coefficient of kurtosis is given by the following ratio:

$$\begin{aligned}\gamma_1 &= \frac{\mu_4}{\mu_2^{\,2}} \\ &= \frac{3\sigma^4}{\sigma^4} \\ &= 3\end{aligned} \tag{8}$$

For a normal distribution this ratio is equal to 3. Sometimes the coefficient of excess kurtosis is used as a measure of kurtosis. This is given by

$$\gamma_1 = \frac{\mu_4}{\mu_2^{\,2}} - 3$$

and for a normal distribution this is zero.

3.2.3 Estimation

Using the maximum likelihood estimation method, the likelihood function of a random sample of independent and identically distributed random variables from a normal distribution with mean μ and variance σ^2 is

$$\begin{aligned} L\left(\underline{R_{t_i}} \mid \mu, \sigma^2\right) &= \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} \exp -\left(\frac{\left(r_{t_i} - \mu\right)^2}{2\sigma^2} \right) \\ &= \left(\frac{1}{2\pi\sigma^2} \right)^{n/2} \exp -\left[\frac{1}{2\sigma^2} \sum_{i=1}^{n} \left(r_{t_i} - \mu\right)^2 \right] \end{aligned} \tag{9}$$

The logarithm of the likelihood function is

$$\log L = -n/2 \log 2\pi - n/2 \log \sigma^2 - \frac{1}{2\sigma^2} \sum_{i=1}^{n} \left(r_{t_i} - \mu\right)^2$$

Differentiating with respect to μ and σ^2 we have

$$\begin{aligned} \frac{\partial \log L}{\partial \mu} &= \frac{1}{2} \sum_{i=1}^{n} \left(r_{t_i} - \mu\right) \\ \frac{\partial \log L}{\partial \sigma^2} &= -n/2\sigma^2 + \frac{1}{2\sigma^4} \sum_{i=1}^{n} \left(r_{t_i} - \mu\right)^2 \end{aligned} \tag{10}$$

and putting these derivatives equal to 0 and solving the resulting equation for μ and σ^2, we find the estimates,

$$\mu = \frac{\sum_{i=1}^{n} r_{t_i}}{n} = \overline{r_t}$$
$$\sigma^2 = \frac{\sum_{i=1}^{n} \left(r_{t_i} - \overline{r_t} \right)^2}{n} \qquad (11)$$

Since the σ^2 given above is a biased estimator for σ^2 we use the unbiased estimator,

$$\sigma^2 = \frac{\sum_{i=0}^{n} \left(r_{t_i} - \mu \right)^2}{n-1} . \qquad (12)$$
$$= S^2$$

3.3 The Logistic Distribution

The density function of the logistic distribution is given by

$$f(r_{t_i} \mid \mu, \sigma) = \frac{\exp\left(\frac{\left(r_{t_i} - \mu \right)}{\sigma} \right)}{\sigma \left[1 + \exp\left(\frac{\left(r_{t_i} - \mu \right)}{\sigma} \right) \right]^2} \ , \ for \ -\infty \le r_{t_i} \le \infty \qquad (13)$$

where μ (-∞ < μ < ∞) is a location parameter and σ (σ >0) is a dispersion (or scale) parameter. (Walck, 2007).

3.3.1 Properties

This distribution, which is very similar to the normal in that it is symmetric but has thicker tails, and it has been first suggested as appropriate to model stock return. The characteristic function is given by

$$\phi(s) = E(e^{isr_t}) = \int_{-\infty}^{\infty} e^{isr_t} \frac{\exp\left(\frac{(r_{t_i} - \mu)}{\sigma}\right)}{\sigma\left[1+\exp\left(\frac{(r_{t_i} - \mu)}{\sigma}\right)\right]^2} \; ds$$

$$= e^{is\mu} \int_{-\infty}^{\infty} e^{isr_t} \frac{\mathrm{e}^{isz\sigma}\,\mathrm{e}^{z}}{\sigma\left[1+\mathrm{e}^{z}\right]^2} \sigma \; dz$$

$$= e^{is\mu} \int_{-\infty}^{\infty} \frac{y^{is\sigma}}{[1+y]^2} \; \frac{dy}{y}$$

$$= e^{is\mu} B(1+is\sigma,\, 1-is\sigma)$$

$$= e^{is\mu} \frac{\Gamma(1+is\sigma)\Gamma(1-is\sigma)}{\Gamma 2}$$

$$= e^{is\mu} \frac{is\sigma\pi}{\sin \pi is\sigma} \tag{14}$$

We have used the transformation. $z = \frac{r_t - \mu}{\sigma}$ and $y = e^z$ in simplifying the integral in the end identifying the beta function using the relation of this in terms of gamma function and their properties (Walck, 2007). Using the characteristic function to find the moments might be messy. We use,

$$\ln\phi(s) = is\mu + \ln\Gamma(1+is\sigma) + \ln\Gamma(1-is\sigma) \tag{15}$$

to determine the cumulants of the distribution. In the process we utilize derivatives of $\ln\phi(s)$ which involves poly-gamma functions. We find that all cumulants of odd order except $\mu_1^1 = \mu$ vanish and that for even orders,

$$\mu_{2n}{}^1 = 2\sigma^{2n}\psi^{2n+1}(1) \quad , n = 1, 2, 3, \ldots \tag{16}$$

where, $\psi(.)$ is the digamma function. Using this formula, lower order moments are found to be

$$\mu_1^1 = \mu,$$

$$\mu_2 = \frac{\sigma^2\pi^2}{3},$$

$$\mu_3 = 0,$$

$$\mu_4 = \mu_4^1 + 3\mu_2^2 = \frac{2\sigma^4\pi^4}{15} + \frac{\sigma^4\pi^4}{3} = \frac{7\sigma^4\pi^4}{15} \tag{17}$$

Then the coefficients of skewness and kurtosis are

$$\gamma_1 = 0 \tag{18}$$

and

$$\gamma_2 = 1.2. \tag{19}$$

3.3.2 Estimation

Let $R_1, \ldots, R_n$ be iid with a logistic distribution. The log of the likelihood function simplifies to:

$$\begin{aligned} l(\mu,\sigma) &= \sum_{i=1}^{n} \log f(r_{t_i} \mid \mu,\sigma) \\ &= -n\log\sigma + \frac{1}{\sigma}\sum_{i=1}^{n}\left(r_{t_i} - \mu\right) + 2\sum_{i=1}^{n}\log(1+\exp\{\frac{1}{\sigma}(r_{t_i} - \mu)\}) \end{aligned} \tag{20}$$

Using this, the first derivative is

$$\frac{\partial l}{\partial \mu} = -\frac{1}{\sigma}\sum_{i=1}^{n}\left(r_{t_i} - \mu\right) - \frac{2}{\sigma}\sum_{i=1}^{n}\frac{\exp\{\frac{1}{\sigma}(r_{t_i} - \mu)\}}{(1+\exp\{\frac{1}{\sigma}(r_{t_i} - \mu)\})} \tag{21}$$

And

$$\frac{\partial l}{\partial \sigma} = -\frac{1}{\sigma^2}\sum_{i=1}^{n}\left(r_{t_i}-\mu\right)-\frac{n}{\sigma}-\frac{2}{\sigma^2}\sum_{i=1}^{n}\frac{(r_{t_i}-\mu)\exp\{\frac{1}{\sigma}(r_{t_i}-\mu)\}}{(1+\exp\{\frac{1}{\sigma}(r_{t_i}-\mu)\})}. \tag{22}$$

The maximum likelihood estimators of this distribution are obtained by maximizing the likelihood function i.e

$$\frac{1}{\sigma}\sum_{i=1}^{n}\left(r_{t_i}-\mu\right)+\frac{2}{\sigma}\sum_{i=1}^{n}\frac{\exp\{\frac{1}{\sigma}(r_{t_i}-\mu)\}}{(1+\exp\{\frac{1}{\sigma}(r_{t_i}-\mu)\})}=0 \tag{23}$$

$$\frac{1}{\sigma^2}\sum_{i=1}^{n}\left(r_{t_i}-\mu\right)+\frac{n}{\sigma}+\frac{2}{\sigma^2}\sum_{i=1}^{n}\frac{(r_{t_i}-\mu)\exp\{\frac{1}{\sigma}(r_{t_i}-\mu)\}}{(1+\exp\{\frac{1}{\sigma}(r_{t_i}-\mu)\})}=0 \tag{24}$$

We can only solve this equation numerically though the estimates obtained do not converge. Matlab uses the Expectation Maximization (EM) Criteria to obtain better estimates.

3.4 t-Location Scale Distribution

The t-location-scale distribution is useful for modeling data distributions with heavier tails (more prone to outliers) than the normal distribution. It is a normal inverse gamma mixture. Let us begin by deriving the inverse gamma distribution.

3.4.1 Inverse Gamma Distribution

Let us derive the inverse gamma distribution from a gamma distribution with parameters k and θ. Then

$$f(x,k,\theta)=\frac{1}{\theta^k\Gamma k}x^{k-1}e^{-x/\theta} \tag{25}$$

Defining the transformation $Y = \frac{1}{X}$, then $J = \frac{dx}{dy} = -\frac{1}{y^2}$

Then,

$$\begin{aligned} f(y) &= f(x)\left|J\right| \\ &= \frac{1}{\theta^k \Gamma k}\left(\frac{1}{y}\right)^{k-1} e^{-1/y\theta}\left(\frac{1}{y^2}\right) \\ &= \frac{1}{\theta^k \Gamma k} y^{-k-1} e^{-1/y\theta} \end{aligned} \tag{26}$$

Replacing k with α , θ^{-1} with β, and y with x we have

$$f(x,\alpha,\beta) = \frac{\beta^\alpha}{\Gamma\alpha} x^{-\alpha-1} e^{-\beta/x} \tag{27}$$

which is an inverse gamma with shape parameter α and scale parameter β. This distribution has mean and variance given below,

$$\begin{aligned} E(X) &= \frac{\beta}{\alpha-1} \\ Var(X) &= \frac{\beta^2}{(\alpha-1)^2(\alpha-2)} \end{aligned} \tag{28}$$

and,

$$\begin{aligned} E(\log X) &= \log\beta - \psi(\alpha) \\ \psi(x) &= \frac{d \log \Gamma x}{dx} \end{aligned} \tag{29}$$

as shown by Wenbo (2006).

3.4.2 Normal-Inverse Gamma Distribution

Let us assume that the returns of share price or an index R_t has a distribution whose variance, $\sigma^2\tau_i$ over unit time interval, is not a constant which is the case. Really, in practice this is so, because any share market often has long periods of relative activity, followed by long periods of relative inactivity. The information which affects prices does not come uniformly, but rather in bursts of activity.

Let R_t be normally distributed with the variance $\sigma^2\tau_i$ of returns of share price changing, i.e. it's a random variable with distribution function $g(\tau_i)$. A t-location scale is a mean variance mixture with a reciprocal gamma distribution as a mixing distribution. It is one of the standard non-normal distributions used in financial economics. Let us examine how this distribution if formed:

Let R_t be distributed as follows

$$f(r_{t_i} \mid \mu, \sigma^2\tau_i) = \frac{1}{\sqrt{2\pi\sigma^2\tau_i}} \exp - \left(\frac{\left(r_{t_i} - \mu\right)^2}{2\sigma^2\tau_i} \right) \qquad \mathit{for} \quad -\infty \le r_{t_i} \le \infty. \tag{30}$$

with an inverse gamma distribution as the mixing distribution. Consider a special inverse gamma with shape parameter $\nu/2$ and scale parameter $\nu/2$ as shown below:

$$g\left(\tau_i\right) = \frac{\left(\nu/2\right)^{\nu/2}}{\Gamma\left(\nu/2\right)} \tau_i^{\left(-\nu/2-1\right)} e^{-\nu/2\tau_i} \tag{31}$$

and so the distribution,

$$f\left(r_{t_i},\mu,\sigma^2\right)=\int_0^{\infty} f\left(r_{t_i},\mu,\sigma^2\tau_i\right)g\left(\tau_i\right)d\tau_i \qquad (32)$$

which is derived below:

$$\begin{aligned}
f\left(r_{t_i},\mu,\sigma^2\right) &= \int_0^{\infty}\frac{1}{\sqrt{2\pi\sigma^2\tau_i}}\exp-\left(\frac{\left(r_{t_i}-\mu\right)^2}{2\sigma^2\tau_i}\right)\frac{\left(\nu/2\right)^{\nu/2}}{\Gamma\left(\nu/2\right)}\tau_i^{\left(-\nu/2-1\right)}e^{-\nu/2\tau_i}d\tau_i \\
&= \frac{\left(\nu/2\right)^{\nu/2}}{\Gamma\left(\nu/2\right)}\frac{1}{\sqrt{2\pi\sigma^2}}\int_0^{\infty}\tau_i^{\left(-(\nu+1)/2-1\right)}\exp-\frac{1}{\tau_i}\left(\frac{\left(r_{t_i}-\mu\right)^2}{2\sigma^2}+\frac{\nu}{2}\right)d\tau_i \\
&= \frac{\Gamma\left(\nu+1/2\right)}{\left(\frac{\left(r_{t_i}-\mu\right)^2}{2\sigma^2}+\frac{\nu}{2}\right)^{\frac{\nu+1}{2}}}\frac{1}{\sigma\sqrt{2\pi}}\frac{\left(\nu/2\right)^{\nu/2}}{\Gamma\left(\nu/2\right)}\int_0^{\infty}\left\{\frac{\left(\frac{\left(r_{t_i}-\mu\right)^2}{2\sigma^2}+\frac{\nu}{2}\right)^{\frac{\nu+1}{2}}}{\Gamma\left(\nu+1/2\right)}\tau_i^{\left(-(\nu+1)/2-1\right)}\right. \\
&\qquad \left.\exp-\frac{1}{\tau_i}\left(\frac{\left(r_{t_i}-\mu\right)^2}{2\sigma^2\tau_i}+\frac{\nu}{2}\right)\right\}d\tau_i \\
&= \frac{\Gamma\left(\nu+1/2\right)}{\left(\frac{\left(r_{t_i}-\mu\right)^2}{2\sigma^2}+\frac{\nu}{2}\right)^{\frac{\nu+1}{2}}}\frac{1}{\sigma\sqrt{2\pi}}\frac{\left(\nu/2\right)^{\nu/2}}{\Gamma\left(\nu/2\right)}
\end{aligned} \qquad (33)$$

$$f\left(r_{t_i},\mu,\sigma^2\right)=\frac{\Gamma\left(\nu+1/2\right)}{\sigma}\frac{\left(\frac{\left(r_{t_i}-\mu\right)^2}{\nu\sigma^2}+1\right)^{-\frac{\nu+1}{2}}}{\sqrt{\nu\pi}}\frac{1}{\Gamma\left(\nu/2\right)}\quad,\quad for\ -\infty\le r_{t_i}\le\infty,\mu\in R,\sigma>0$$

This is a t- location scale distribution as shown by (Barndorff, 1988). If a random variable R_t has a t- location with parameters μ,σ and ν then the random variable $\frac{r_t-\mu}{\sigma}$ has a student t-distribution with ν degrees of freedom. The mean, variance and kurtosis for this distribution are μ, $\frac{\nu\sigma^2}{\nu-2}$ and $\frac{3(\nu-2)}{\nu-4}$ respectively (Wenbo, 2006).

3.4.3 Estimation of the Parameters of t-location Scale Distribution

We want to find the maximum likelihood estimate for a set of parameters Θ given a set of observed data X by maximizing Pr (R_t |Θ). We assume that it is hard to solve this problem directly but that it is relatively easy to evaluate Pr (R_t, Z |Θ) where Z is a set of latent variables such that

$$\Pr (R_t |\Theta) = \Pr (R_t, Z |\Theta). \tag{34}$$

The EM method then involves the following steps.

i). Write down the complete data log likelihood; log Pr (R_t, Z |Θ).

ii). Write down the posterior latent distribution, Pr (Z | R_t, Θ).

iii). E step: write down the expectations under the distribution Pr (Z | R_t, Θ_0) for all terms in the complete data log likelihood (step (i)).

iv). Write down the function to maximize,

$Q(\Theta,\Theta_0) = \Pr (Z \mid R_t,\Theta_0) \log \Pr (R_t, Z |\Theta)$,

replacing integrals with the expectations from the E step.

v). M step: solve $\frac{\partial Q}{\partial \Theta} = 0$ to yield the update equations.

Once all of the expectation update equations (from step 3) and maximization update equations (from step 5) are known, we initialize our current estimate of the parameters Θ and update them by iterating the E and M updates until convergence. Note that a subscript 0 is used here and in the rest of the article to denote the old setting of the parameters. With each iteration, the new parameter setting (found in step 5) will replace the old one.

3.4.4 Derivation of the EM Update Equations for the t-Location Scale Distribution

We derive the Expectation Maximization (EM) algorithm for this distribution. We write the likelihood for a single data point;

$$\Pr(r_{t_i} \mid \Theta) = t - location\, scale(r_{t_i} \mid \mu, \sigma, \nu). \tag{35}$$

By looking at this as an infinite mixture normal distribution

$$\Pr(r_{t_i} \mid \Theta) = \int_{\tau_i} Normal(r_{t_i} \mid \mu, \sigma^2 \tau_i)\, Inverse\, Gamma(\tau_i \mid \nu/2, \nu/2). \tag{36}$$

We let $R_t = \{r_{t_i}\}$, $Z = \{\tau_i\}$ and $\Theta = \{\mu, \sigma^2, \nu\}$.

Step 1: The complete likelihood function is

$\Pr(R_t, Z \mid \Theta) = \prod_{i=1}^{N} Normal(r_{t_i} \mid \mu, \sigma^2 \tau_i) Inverse\ \ Gamma(\tau_i \mid \nu/2, \nu/2)$. The complete log likelihood function is given by

$$\begin{aligned} l &= \sum_{i=1}^{n} \log Normal(r_{t_i} \mid \mu, \sigma^2 \tau_i) + \log Gamma\, Inverse\, (\tau_i \mid \nu/2, \nu/2) \\ &= \sum_{i=1}^{n} -\frac{1}{2}\log 2\pi - \frac{1}{2}\log \sigma^2 - \frac{1}{2}\log \pi - \frac{(r_{t_i} - \mu)^2}{2\sigma^2 \tau_i} + \nu/2 \log \nu/2 - \log \left|\nu/2\right. \\ &\quad - \left(\nu/2 + 1\right)\log \tau_i - \nu/2\tau_i . \end{aligned} \tag{37}$$

Step 2: The posterior latent distribution is given,

$$\Pr\left(Z, R_t \mid \Theta\right) \propto \Pr(R_t, Z \mid \Theta)$$

$$\Pr\left(R_t, Z \mid \Theta\right) = \prod_{i=1}^{N} Normal(r_{t_i} \mid \mu, \sigma^2 \tau_i) Inverse\ \ Gamma(\tau_i \mid {}^{\nu}\!/_{2}, {}^{\nu}\!/_{2}) \tag{38}$$

$$\propto \prod_{i=1}^{N} Inverse\,Gamma\,(\tau_i \mid a_i, b_i).$$

Since inverse Gamma is the conjugate prior for the normal distribution with unknown precision. We get a_i and b_i by combining factor from normal and inverse Gamma.

$$\begin{aligned}\Pr\left(X, Z \mid \theta\right) &= \prod_{i=1}^{n} Normal(x_i \mid \mu, \sigma^2 \tau_i) Inverse\ \ Gamma(\tau_i \mid {}^{\nu}\!/_{2}, {}^{\nu}\!/_{2}) \\ &\propto \prod_{i=1}^{n} \left(\frac{1}{\tau_i}\right)^{\left({}^{\nu}\!/_{2} + {}^{3}\!/_{2}\right)} e^{-\frac{1}{\tau_i}\left[{}^{\nu}\!/_{2} + \left(\frac{x-\mu}{\sigma}\right)^2 {}^{1}\!/_{2}\right]} \\ &\propto \prod_{i=1}^{n} Inverse\,Gamma\left[\tau_i \mid \frac{\nu+1}{2}, \frac{\nu}{2} + \left(\frac{x_i - \mu}{\sigma}\right)^2\right]\end{aligned} \tag{39}$$

And hence,

$$\begin{aligned} a_i &= \frac{\nu + 1}{2} \\ b_i &= \frac{\nu}{2} + \frac{1}{2}\left(\frac{x_i - \mu}{\sigma}\right). \end{aligned} \tag{40}$$

Step 3: The Expectation

By looking at step (1) we find that we need to calculate the expectations of 1, τ_i and log τ_i under the posterior latent distribution step (2)

$$
\begin{aligned}
E(1) &= 1 \\
E(\log \tau_i) &= \int_Z \log \tau_i \prod_{j=1}^{N} Inverse\,Gamma(\tau_j, a_j, b_j) \\
&= \int_{\tau_i} \log \tau_i Inverse\,Gamma(\tau_i, a_i, b_i) \\
&= \psi(a_i) - \log b_i \\
&= \log \frac{\nu_0}{2} + \frac{1}{2}\left(\frac{r_{\tau_i} - \mu_0}{\sigma_0}\right)^2 - \psi\left(\frac{\nu_0 + 1}{2}\right) \\
E(\tau_i^{-1}) &= \int_Z \tau_i^{-1} \prod_{j=1}^{N} Inverse\,Gamma(\tau_j, a_j, b_j) \\
&= \int_{\tau_i} \tau_i^{-1} Inverse\,Gamma(\tau_i, a_i, b_i) \\
&= \frac{a_i}{b_i} \\
&= \frac{\nu_0 + 1}{\nu_0 + \left(\frac{r_{\tau_i} - \mu_0}{\sigma_0}\right)^2}
\end{aligned} \tag{41}
$$

Where $\psi(.)$ is called the digamma function.

Step 4: Function to optimize

$$
\begin{aligned}
Q(\Theta, \Theta_o) &= \int_Z \Pr(Z \mid X, \Theta_0) \log \Pr(X, Z \mid \Theta_0) \\
&= -\frac{N}{2}\log 2\pi - \frac{N}{2}\log \sigma^2 - \sum_{i=1}^{N} E(\log \tau_i) - \frac{1}{2\sigma^2}\sum_{i=1}^{N}(r_{t_i} - \mu)^2 E\left(\frac{1}{\tau_i}\right) \\
&\quad - N \log \left\lceil \frac{\nu}{2} + N\frac{\nu}{2}\log\frac{\nu}{2} - \left(\frac{\nu+1}{2}\right)\sum_{i=1}^{N} E(\log \tau_i) - \frac{\nu}{2} E\left(\frac{1}{\tau_i}\right). \right.
\end{aligned} \tag{42}
$$

It is clear that the elements of Θ_0 are now explicit in the expectations.

Step 5: Maximization

$$\frac{\partial \Theta}{\partial \mu} = 0 \Rightarrow \frac{1}{\sigma^2} \sum_{i=1}^{N} (r_{t_i} - \mu) E\left(\frac{1}{\tau_i}\right) = 0$$

$$\Rightarrow \mu = \frac{\sum_{i=1}^{N} r_{t_i} E\left(\frac{1}{\tau_i}\right)}{E\left(\frac{1}{\tau_i}\right)}$$

$$\frac{\partial \Theta}{\partial \sigma^2} = 0 \Rightarrow -\frac{N}{2\sigma^2} + \frac{1}{2\left(\sigma^2\right)^2} \sum_{i=1}^{N} (r_{t_i} - \mu)^2 E\left(\frac{1}{\tau_i}\right) = 0 \quad (43)$$

$$\Rightarrow \frac{N}{2\sigma^2} = \frac{1}{2\left(\sigma^2\right)^2} \sum_{i=1}^{N} (r_{t_i} - \mu)^2 E\left(\frac{1}{\tau_i}\right)$$

$$\Rightarrow \sigma^2 = \frac{1}{N} \sum_{i=1}^{N} (r_{t_i} - \mu)^2 E\left(\frac{1}{\tau_i}\right)$$

Note that we need the updated value of μ and σ^2 to find ν.

$$\frac{\partial \Theta}{\partial \nu} = 0 \Rightarrow -N\psi\left(\frac{\nu}{2}\right) + \frac{N}{2}\log\frac{\nu}{2} + \frac{N}{2} - \frac{1}{2}\sum_{i=1}^{N} E(\log \tau_i) - \frac{1}{2} E\left(\frac{1}{\tau_i}\right) = 0$$

$$\Rightarrow -2\psi\left(\frac{\nu}{2}\right) + \log\frac{\nu}{2} + 1 = \frac{1}{N}\sum_{i=1}^{N} E(\log \tau_i) + E\left(\frac{1}{\tau_i}\right) \quad (44)$$

There is no closed form solution of ν we use a numerical approach to find it. We implement this EM algorithm in Matlab to obtain the ML estimates shown in the Appendix.

3.5 Goodness of Fit Tests

The goodness of fit of a statistical model describes how well it fits a set of observations. Measures of goodness of fit typically summarize the discrepancy between observed values and the values expected under the model in question. In this study we have used the following tests of fit:

i) Ansari-Bradley Test

Ansari-Bradley tests the hypothesis that two independent samples, in the vectors x and y, come from the same distribution, against the alternative that they come from distributions that have the same median and shape but different dispersions (e.g. variances). x and y can also be matrices or *N*-dimensional arrays. For matrices, Ansari-bradley test performs separate tests along each column, and returns a vector of results.

ii) Kolmogorov Smirnov test

A two-sample Kolmogorov-Smirnov test is used to compare the distributions of the values in the two data vectors x and y. The null hypothesis is that x and y are from the same continuous distribution. The alternative hypothesis is that they are from different continuous distributions.

iii) Jarque Bera Test

A Jarque-Bera test is used to test the null hypothesis that the sample in vector x comes from a normal distribution with unknown mean and variance, against the alternative that it does not come from a normal distribution. The Jarque-Bera test is a two-sided goodness-of-fit test suitable when a fully-specified null distribution is unknown and its parameters must be estimated.

CHAPTER FOUR

RESULTS AND DISCUSSION

4.0 Introduction

The behavior of the NSE 20 Share Index considered during this period is shown below in Figure 1. The series analyzed for each market is the series of returns, where returns are defined as, given in Equation (3).

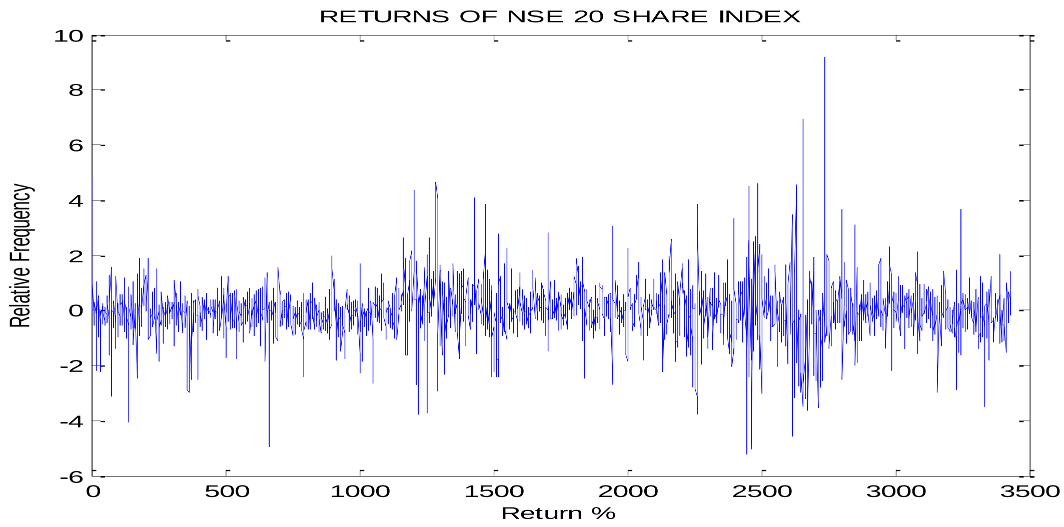

Figure 1: Percentage Returns, R_t of NSE 20 Share Index

where R_t and S_t are the return and the index in day t, respectively. The histogram of this data is shown below;

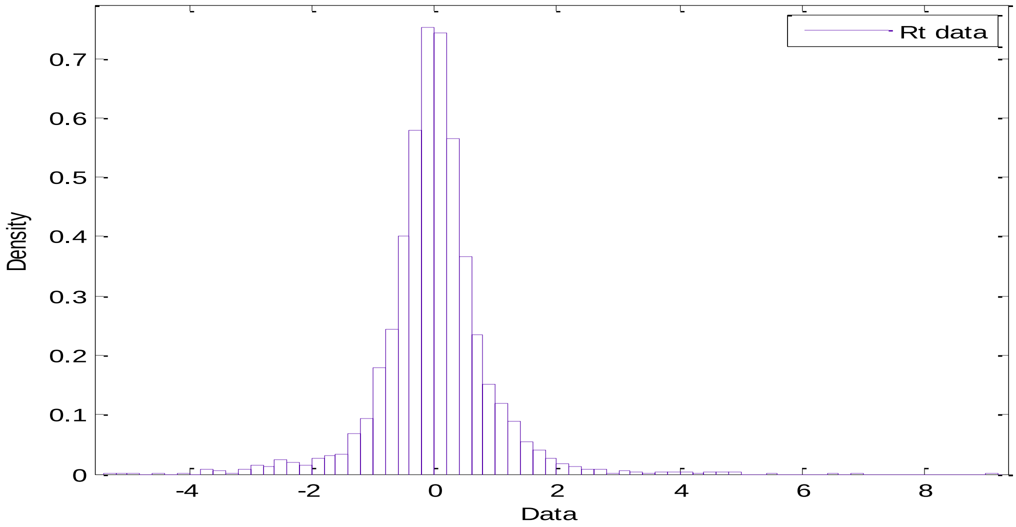

Figure 2: A histogram of Returns Rt of NSE 20 Share Index

Table 1 below summarizes some relevant information about the empirical distributions of stock returns under consideration. The statistics reported are the mean, standard deviation, minimum and maximum return during the sample period, coefficients of skewness and kurtosis.

Mean	Variance	Minimum Value	Maximum Value	Skewness	Kurtosis
7.9454×10^{-4}	0.7881	-5.2339	9.1782	0.5970	13.1608

Table 1: Sample Moments of the Distributions NSE 20 Share Index

The third central moment is often called a measure of asymmetry or skewness in the distribution. From our descriptive results shown above, the distribution of this data is approximately symmetrical though it shows signs of being positively skewed. The coefficient of skewness is almost 0.5970. This is well depicted from the histogram in Figure 2 above.

Kurtosis measures a different type of departure from normality by indicating the extent of the peak (or the degree of flatness near its center) in a distribution. We see that this is the ratio of the fourth central moment divided by the square of the variance. If the distribution is normal, then this ratio is equal to 3. A ratio greater than 3 indicates more values in the neighborhood of the mean (is more peaked than the normal distribution). Our data has a fat tail or excess kurtosis as shown in the coefficient of kurtosis (i.e. kurtosis of 13.1608). The distribution of this data is leptokurtic.

4.1 Normality Assumption Test

Using Equations (11) and (12) we compute the ML estimates of the normal distribution with their corresponding standard errors are shown below in Table 2 below;

Parameter	Estimate	Std error
μ	0.000794536	0.0151517
σ	0.887763	0.0107162

Table 2: ML Estimates of the Normal Distribution

The distribution of financial returns over horizons shorter than a month is not well described by a normal distribution. In particular, the empirical return distributions, while unimodal and approximately symmetric, are typically found to exhibit considerable leptokurtosis. The typical shape of the return distribution, as compared to a fitted normal is presented in Figure 3 below.

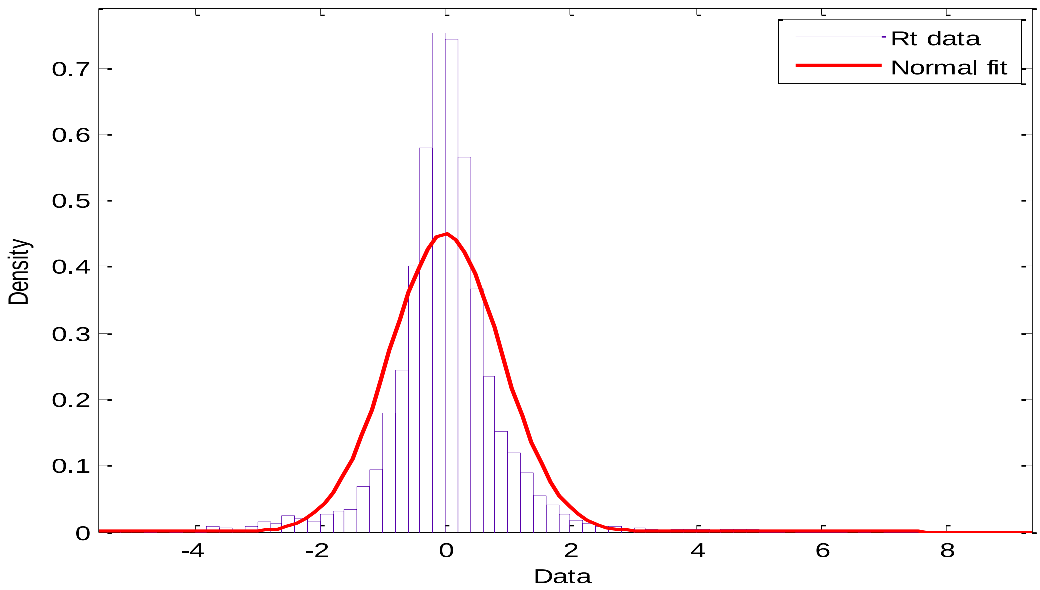

Figure 3: A normal Fit to the Returns R_t Data

The coefficients of standardized skewness and kurtosis provide strong evidence about departures from normality, but more formal conclusions can be reached through the tests of normality reported below in Table 3.

	Ansari Bradley		Kolmogorov-Smirnov		Jarque-Bera	
Market	Statistic	P-value	Statistic	P-value	Statistic	P-value
NSE 20 Share Index	6516176	8.6874×10^{-052}	0.1181	4.0112×10^{-42}	$\mathbf{1.4972}\times10^{4}$	1.0×10^{-003}

Table 3: Goodness of Fit Tests for the Normal Distribution

All these tests are done at the 5% significance level. The results in Table 3 above does not come as no surprise; virtually all studies that use daily data also reject the normality of stock returns. This is due to its failure to capture the fat tail of the returns data. Another reason why this model failed is due to the assumption that the variance σ^2 of price changes over unit time interval is a constant. This is clearly depicted when we plot a probability plot to compare the distribution of our data and the normal distribution as shown in Figure 4.

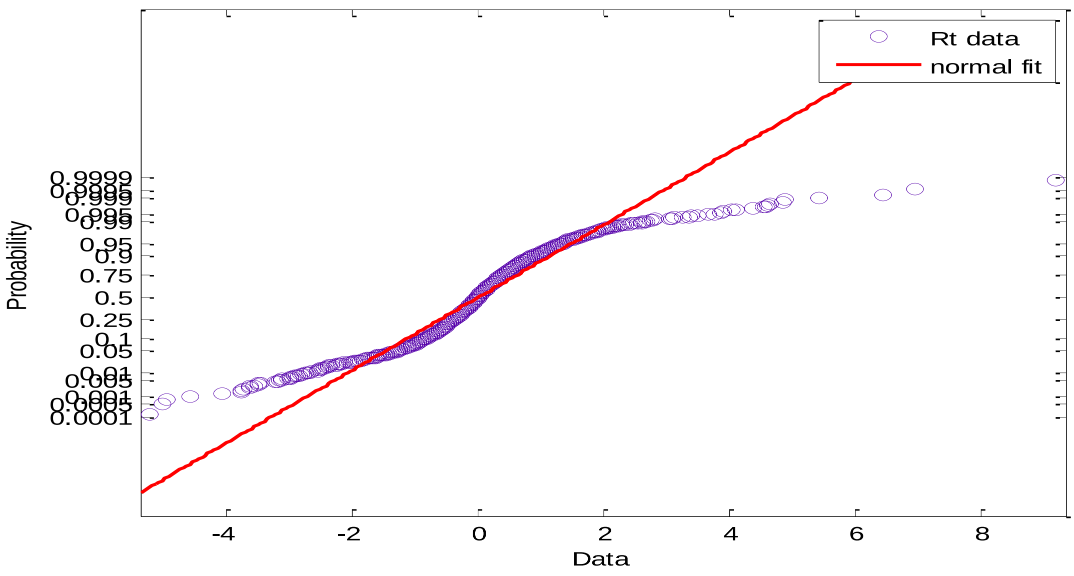

Figure 4: A Probability Plot of the Normal Fit

The goodness of fit coupled with the probability plot give a clear evidence against the normal distribution as a fit of the Returns of NSE 20 Share Price Index. In order to test what specification describes the data better than the Normal distribution, we consider in the next part two alternative distributions that allow for the characteristics of the data discussed above; we then fit such distributions to the data in the following part.

4.2 Fitting a Logistic Distribution

Utilizing Equations (23) and (24) and the use of Matlab codes in the appendix the ML estimates of this distribution are obtained using the Expectation Maximization (EM) algorithm.

Parameter	Estimate	Std error
μ	-0.00461803	0.0121681
σ	0.423535	0.00620854

Table 4: ML Estimates for the Logistic Distribution

The estimates in Table 4 above are used to fit the logistic distribution to the returns NSE 20 Share Index data. The shape of the return distribution, as compared to a fitted logistic distribution is presented in Figure 4 below.

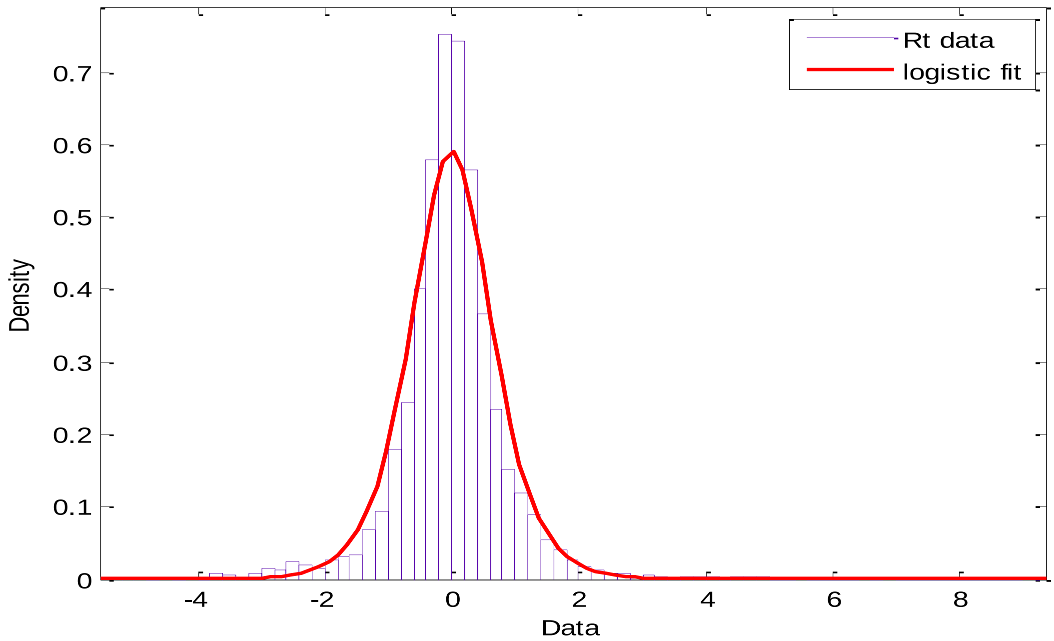

Figure 5: A logistic Distribution Fit to the Returns R_t Data

Tests of goodness of fit results at 5% level of significance are presented in Table 5 below;

	Ansari Bradley		Kolmogorov-Smirnov	
Market	Statistic	P-value	Statistic	P-value

NSE 20 Share Index	6155059	2.2026×10^{-010}	0.0460	0.0013

Table 5: Goodness of Fit Tests for the Logistic Distribution

The probability plot of the distribution of our data and the logistic distribution is represented in Figure 6 below;

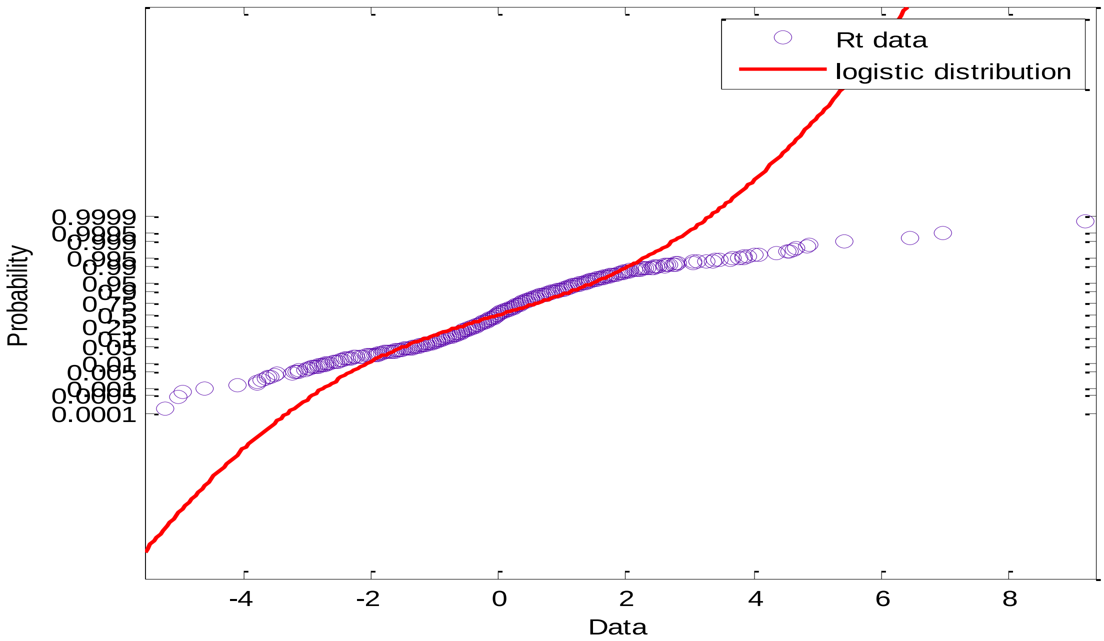

Figure 6: A Probability Plot of the Logistic Distribution Fit

It is also clear that at any level of significance the logistic distribution fails to fit the returns of NSE 20 Share Price Index. The logistic distribution has a fatter tail than normal distribution. It failed to fit the returns of NSE 20 Share Index returns due to the assumption that the variance σ^2 of price changes over unit time interval is a constant.

4.3 Fitting a t-Location Scale Distribution

As we derived the EM algorithms in Equations (42), (43) and (44) of the t- location scale distribution, we implement via Matlab code in the appendix to obtain the ML estimates of the t- location scale distribution.

Parameter	Estimate	Std. Error
μ	-0.00889354	0.0102819
σ	0.480765	0.0115711
ν	2.42046	0.124132

Table 6: ML Estimates for the t-location Scale Distribution

Using these estimates we fit the t-location scale distribution to the NSE 20 Share Index data. The shape of the return distribution, as compared to a fitted t-location scale distribution is presented in Figure 7 below.

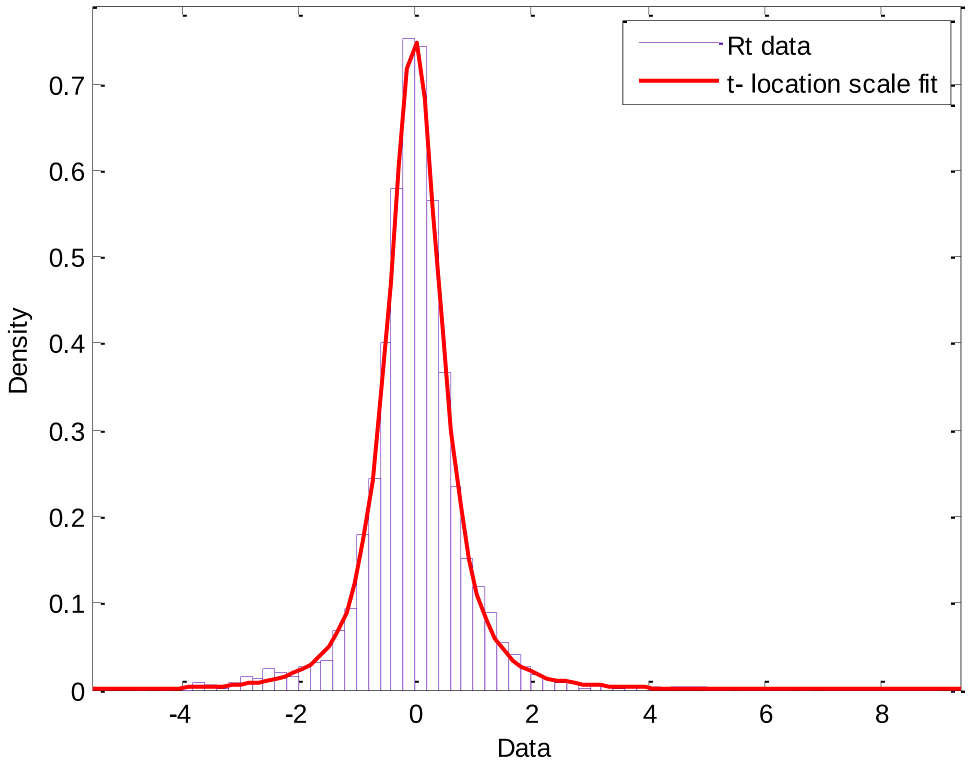

Figure 7: A t- location Scale Distribution fit to the Returns R_t Data

Tests of goodness of fit results at 5% level of significance are presented in Table 7 below;

	Ansari Bradley		Kolmogorov-Smirnov	
Market	Statistic	P-value	Statistic	P-value
NSE 20 Share Index	5922916	0.4883	0.0192	0.5460

Table 7: Goodness of Fit Tests for the t-location Scale Distribution

The probability plot of the return distribution and the t-location scale distribution is given Figure (8) below;

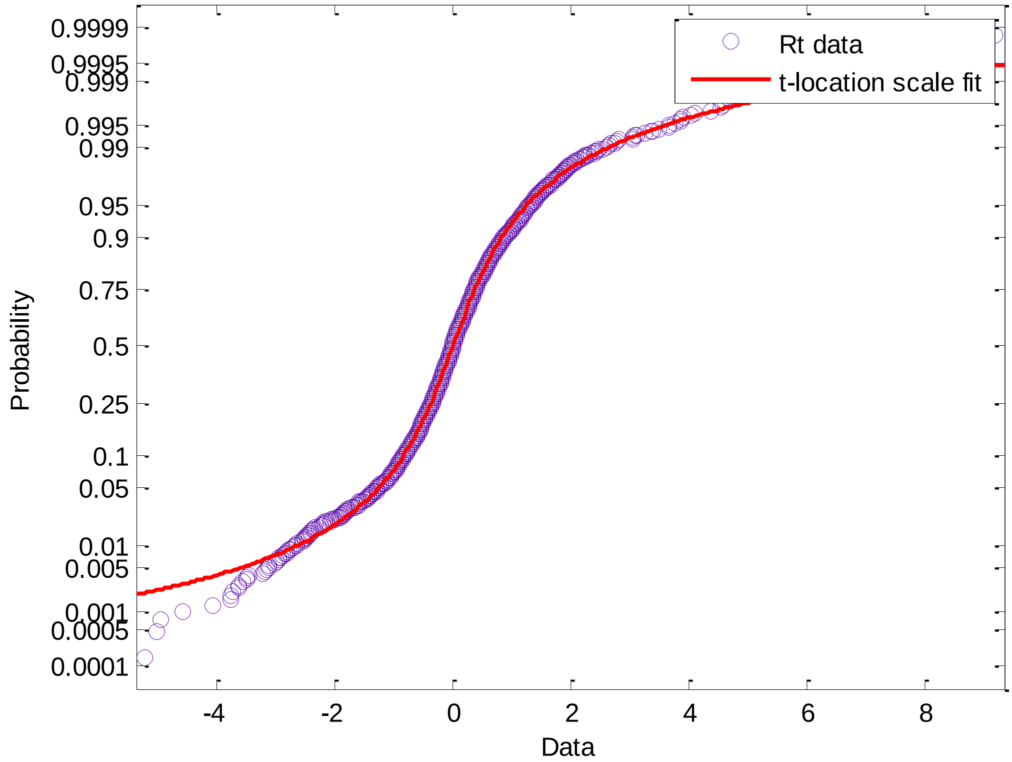

Figure 8: A Probability Plot of the t-location Scale Distribution Fit

We have found strong support for the t-location scale distribution, which cannot be rejected at any reasonable significance level. This is because the t- location scale captures the fat tail exhibited in the NSE 20 Share Index returns. This also provides a clear evidence of the fact that the variance of price changes over unit time interval is a not constant. This is the case in practice because any share market often has long periods of relative activity, followed by long periods of relative inactivity. The information which affects prices does not come uniformly, but rather in bursts of activity. This is a formal evidence of the fact σ^2 varies significantly from year to year, as the degree of activity in the market also varies. Having established that the t-location scale (rather than the Normal) distribution properly describes daily NSE 20 Share Index, we conclude any predictions on the returns of NSE 20 Share Index should be based on the t-location scale distribution and not the normal distribution. Studies in financial economics can be based on the t- location scale distribution.

CHAPTER 5

CONCLUSION

This study was interested in fitting an empirical distribution to the NSE 20 share index. In finding returns we used changes in logarithms prices instead of simple price changes. This is because;

i) The change in log price is the yield, with continuous compounding, from holding the security for that day.

ii) It has been shown that the variability of simple price changes for a given stock is an increasing function of the price level of the stock. Taking logarithms seems to neutralize most of this price level effect.

iii) For changes less than 15 percent, the change in log price is very close to the percentage price change, and for many purposes it is convenient to look at the data in terms of percentage price changes.

After thorough descriptive analysis it was clear that the NSE 20 share price index data is approximately symmetrical though it shows signs of being positively skewed. The coefficient of skewness is almost 0.5970. The data also exhibited a fat tail or excess kurtosis hence leptokurtic (i.e. kurtosis of 13.1608).

In an attempt to fit a normal distribution to the NSE 20 share price index data all tests done at the 5% significance level led to rejection of this distribution. This is due to the fact that the normal distribution fails to capture the fat tail of the returns data. Another reason why this model failed is due to the assumption that the variance σ^2 of price changes over unit time interval is a constant.

Though the logistic distribution has a fatter tail than normal distribution, it was also rejected at 5% level of significance. It failed to fit the returns of NSE 20 Share Index returns due to the assumption that the variance σ^2 of price changes over unit time interval is a constant.

The t-location scale distribution has a fatter tail than the normal and logistic distributions. In the construction of this distribution, the scale parameter (i.e. the variance) is not assumed constant. From our results, we found strong support for the t-location scale distribution, which could not be rejected at any reasonable significance level. This is because it captures the fat tail exhibited in the NSE 20 Share Index returns. This also provides a clear evidence of the fact that the variance of price changes over unit time interval is not a constant. This is a formal evidence of the fact σ^2 varies significantly from year to year, as the degree of activity in the market also varies. From these results, we conclude that any predictions on the returns of NSE 20 Share Index should be based on the t-location scale distribution and not the normal distribution. Studies in financial economics could be based on the t- location scale distribution. We further recommend that since we have found that the normal inverse gamma mixture best fits the NSE 20 Share Index return, other normal mixtures can be investigated how well they fit this data.

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APPENDIX

```
% load and import the nse 20 index data

NSE20Index;

x=log(NSE20Index);

i=[1:3433];

x2=x(i+1)-x(i);

y=100*x2';

Rt=y;

s=skewness(y);

k=kurtosis(y);

st=var(y);

m=mean(y);

dfittool(Rt);

plot(y);

%plot(y)

%plot(NSE20Index)


%RANDOM NUMBER GENERATION FORM NORMAL DISTRIBUTION

dist = ProbDistUnivParam('normal',[7.94536e-006,0.00887763]);

xn=random(dist,1,3433);

NSE20Index;

x=log(NSE20Index);

i=[1:3433];

x2=x(i+1)-x(i);
```

```
y=100*x2';

Rt=y;

alpha=0.05;

%h = chi2gof(y, alpha)

%to test with Ansari-Bradley test

[h,p,stats] = ansaribradley(Rt, xn,alpha);

%to test with a Jarque-Bera test

[h,p,jbstat] = jbtest(Rt, alpha)

%to test with a two-sample Kolmogorov-Smirnov test

[h,p,ksstat] = kstest(Rt,xn,alpha,'unequal' );

%The following figure illustrates the test statistic:

F1 = cdfplot(Rt);

hold on

F2 = cdfplot(xn);

set(F1,'LineWidth',2,'Color','r');

set(F2,'LineWidth',2);

legend([F1 F2],'F1(y)','F2(xn)','Location','NW');

%The test statistic k is the maximum difference between the curves.



%RANDOM NUMBER GENERATION FORM LOGISTIC DISTRIBUTION

dist = ProbDistUnivParam('logistic',[-4.61803e-005,0.00423535]);

xl=random(dist,1,3433);

NSE20Index;

x=log(NSE20Index);
```

```
i=[1:3433];

x2=x(i+1)-x(i);

y=100*x2';

alpha=0.05;

%to test with Ansari-Bradley test

[h,p,stats] = ansaribradley(Rt, xl,alpha);

%to test with a two-sample Kolmogorov-Smirnov test

[h,p,ks2stat] = kstest2(Rt,xl,alpha,'unequal');

%The following figure illustrates the test statistic:

F1 = cdfplot(Rt);

hold on

F2 = cdfplot(xl);

set(F1,'LineWidth',2,'Color','r');

set(F2,'LineWidth',2);

legend([F1 F2],'F1(y)','F2(xl)','Location','NW');

%The test statistic k is the maximum difference between the curves.



%RANDOM NUMBER GENERATION FORM t-LOCATION SCALE DISTRIBUTION

dist = ProbDistUnivParam('tlocationscale',[-0.00889354,0.480765,2.42046]);

xt=random(dist,1,3433);

NSE20Index;

x=log(NSE20Index);

i=[1:3433];

x2=x(i+1)-x(i);
```

```
y=100*x2';

Rt=y;

alpha=0.05;

%to test with Ansari-Bradley test

[h,p,stats] = ansaribradley(Rt, xt,alpha);

%to test with a two-sample Kolmogorov-Smirnov test

[h,p,ks2stat] = kstest2(Rt,xt,alpha,'unequal');

%The following figure illustrates the test statistic:

F1 = cdfplot(Rt);

hold on

F2 = cdfplot(xt);

set(F1,'LineWidth',2,'Color','r');

set(F2,'LineWidth',2);

legend([F1 F2],'F1(y)','F2(xt)','Location','NW');

%The test statistic k is the maximum difference between the curves.

%%%%%also do probability plots.
```