DEPENDENCE MODELLING OF FINANCIAL DATA USING GENENALISED HYPERBOLIC DISTRIBUTION

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Declaration

This is my original work and has never been presented for any academic award in any other learning institution.

Calvin Bitange Maina

Sign: ....

Date: ....

APPROVAL

This is to certify that the project titled "DEPENDENCE MODELLING OF FINANCIAL DATA USING GENENALISED HYPERBOLIC DISTRIBUTION" carried out by above named student has been read and approved for meeting part of the requirements and regulations governing the award of the Master of Science in Actuarial Science degree of University of Nairobi, Kenya.

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Date: ....

Dedication

To my loving wife Edwinah Onyancha and my yet to be born twins.

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Abstract

Generalised Hyberbolic Distributions (GHDs) are very significant in modeling to model returns from financial market variables such as exchange rate, equity prices, and interest rate measured over short time intervals, i.e. daily or weekly. These returns are chacterized by non-normality. The empirical distribution of such returns is more peaked and has fatter tails than the normal distribution, which implies that changes in return occur with a higher frequency than under normality. In addition it is often skewed towards the left tail and has a kurtosis greater than 3. The GHD is a promising distribution for such returns. Its a heavy tailed distribution and thus has kurtosis greater than 3 (leptokurtic). GHD embraces many special cases and limiting distributions. Some examples are the hyperbolic, the Normal Inverse Gaussian(NIG), the (skew) Student's t, Variance Gamma and the Normal itself.

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General Introduction

1.1 Definition

Mixtures models (overdispersion model) provide a general framework for deriving model applicable in situation where standard/simple model fail.

Starting from a distribution family $f(x \mid \theta)$ we may obtain a very rich new family of distribution if we allow the parameter(s) θ to be itself a random variable with distribution function $g(\theta \mid \varphi)$ depending on a vector of parameters φ.

The unconditional distribution of x will be given by:

$$f(x; \varphi) = \int_\theta f(x \mid \theta) \, g(\theta \mid \varphi) \, d\theta \tag{1.1}$$

where θ is continuous, or

$$f(x; \varphi) = \sum_\theta f(x \mid \theta) \, g(\theta \mid \varphi) \tag{1.2}$$

where θ is discrete. $g(\theta \mid \varphi)$ is the mixing distribution. $f(x; \varphi)$ is refered to as mixed distribution.

1.1.1 Particular Case:

Mixed Normal distribution. There are two cases:

- Scale mixture of the Normal distribution which assume that the variance is not fixed for all the members of the population, have been widely used to model heteroscedacity.
- Normal Variance-Mean mixtures assume that variance is not fixed but it is also related to the mean. A rich family of distributions with useful and attractive properties can arise using this scheme.

1.2 The Normal Variance-Mean Mixture Mechanics

The mean and variance of the conditional distribution are in a special (affine-linear) relation.

Suppose Z is a positive r.v, and μ and β are constants,

$$f(x) = \int_0^\infty f_{X|Z}(x \mid z; \mu + \beta z, z)\, f_Z(z)\, dz \tag{1.3}$$

or

$$f(x) = \sum_{\substack{z>0 \\ f_Z(z)\neq 0}} f_{X|Z}(x \mid z; \mu + \beta z, z)\, f_Z(z) \tag{1.4}$$

One of the features of constructing a distribution by mixing is that one can essentially read off the properties of the distribution given the properties of the mixing mixing distribution. Then, we have expected value

$$E(X) = \mu + \beta E(Z) \tag{1.5}$$

Proof: Using the Fubini's theorem:

$$\begin{aligned}
E(X) &= \int_{-\infty}^{\infty} x f_X(x)\, dx \\
&= \int_{0}^{\infty} \int_{-\infty}^{\infty} x f_N(x; \mu + \beta z, z)\, dx f_Z(z)\, dz \\
&= \int_{0}^{\infty} (\mu + \beta z) f_Z(z)\, dz \\
&= \int_{0}^{\infty} \mu f_Z(z)\, dz + \beta \int_{0}^{\infty} z f_Z(z)\, dz = \mu + \beta E(Z)
\end{aligned}$$

Similarly, variance

$$var(X) = E(Z) + \beta^2 var(Z) \tag{1.6}$$

third central moment

$$\mu_3(X) = 3\beta var(Z) + \beta^3 \mu_3(Z) \tag{1.7}$$

The moment generating function

$$M_X(t) = e^{\mu t} M_Z\left(\beta t + t^2/2\right) \tag{1.8}$$

Proof:

$$\begin{aligned}
M_X(t) &= \int_{-\infty}^{\infty} e^{tx} f_X(x)\,dx \\
&= \int_{-\infty}^{\infty} e^{tx} \int_{0}^{\infty} f_N(x;\mu+\beta z, z)\, f_Z(z)\,dz dx \\
&= \int_{0}^{\infty} \int_{-\infty}^{\infty} e^{tx} f_N(x;\mu+\beta z, z)\,dx f_Z(z)\,dz \\
&= \int_{0}^{\infty} \exp\left[(\mu+\beta z)\,t + \frac{z}{2}t^2\right] f_Z(z)\,dz \\
&= e^{\mu t} \int_{0}^{\infty} \exp\left[\left(\beta t + \frac{t^2}{2}\right) z\right] f_Z(z)\,dz \\
&= e^{\mu t} M_Z\left(\beta t + \frac{t^2}{2}\right)
\end{aligned}$$

similarly,and the Characteristic function

$$\varphi_X(\upsilon) = e^{i\mu\upsilon}\varphi_Z\left(\beta\upsilon + i\upsilon^2/2\right) \tag{9}$$

1.3 Problem Statement

In dependence modelling of financial data using copulas, the choice of the marginal distributions is imperative and quite significant. Incorrect choice of marginal distribution can cloud out the effect of the choice of copula in trying to capture the dependence structure of the relevant variables.

The Generalised Hyperbolic Distribution (GHD), which is a Normal variance-Mean Mixture with the Generalised Inverse Gaussian Distribution (GIGD) as the mixing distribution, nests a number of heavy tailed distribution as special cases or limiting cases. We can use the mixing mechanism as a tool to get an overview of the vast number of important subfamilies. That is, every subfamily of a GIGD (Gamma, positive GIG, Inverse Gamma, Inverse Gaussian, exponential, Dirac, Levy) correspond a subfamily of GHD.

The GHD and its subfamilies is quite flexible, but this comes at a price of complexity. The problem therefore is to estimate the parameters of this class of distribution as to make a choice among the subfamilies of the GHD. The aim being to have an easily programmable algorithm for these complex distribution.

1.4 Objectives

Main objective: Determine the Marginal distribution of Bivariate copulas in depence modelling using an EM type Algorithm.

Specific objective:

- To construct the GHDs,
- To determine the subfamilies of the GHDs,
- To study the properties of GHDs
- Fit the marginal distribution identified to copulas.

1.5 Significance of Study

GHDs are largely used in Actuarial application to model returns from financial market variables such as exchange rate, equity prices, and interest rate measured over short time intervals, i.e. daily or weekly. These returns are chacterized by non-normality. The empirical distribution of such returns is more peaked and has fatter tails than the normal distribution, which implies that changes in return occur with a higher frequency than under normality. In addition it is often skewed towards the left tail and has a kurtosis greater than 3.

The GHD is a promising distribution for such returns. Its a heavy tailed distribution and thus has kurtosis greater than 3 (leptokurtic). GHD embraces many special cases and limiting distributions. Some examples are the hyperbolic, the Normal Inverse Gaussian(NIG), the (skew) Student's t, Variance Gamma and the Normal itself.

1.6 Literature Review

The Generalised Hyperbolic Distribution was introduced by Barndorff-Nielsen (1977) in connection modelling grain size distributions of wind blown sands. The original paper focused on the special case of the hyperbolic distribution. The name of the distribution is derived from the fact that its log-density forms a hyperbola, while the log density of a normal distribution forms a parabola. Since then, GHD has been discussed by many authors, particularly in connection with application in finance. Some examples are Barndorff-Nielsen (1979), Barndorff-Nielsen and Blaesild(1981), Eberlein and Keller (1995), Prause (1999), Bibby and Sorensen (2003), Mencia and Sentana (2004) and McNeil et al. (2005). The NIG distribution was introduced by Barndorff-Nielsen (1997). It is able to model symmetric and asymmetric distributions with possibly long tails in both directions. Moreover, the NIG distribution possesses a number of attractive theoretical properties, among others its analytical tractability. For these reasons, it has been used repeatedly for applications in finance, both as the conditional distribution of a GARCH-model (Andersson, 2001; Forsberg and Bollerslev, 2002; Jensen and Lunde, 2001; Venter and de Jongh, 2002) and as the unconditional return distribution (Bølviken and Benth, 2000; Eberlein and Keller, 1995; Lillestøl, 2000; Prause, 1997; Rydberg, 1997). The tail behaviour of NIG is often classified as semi-heavy. That is, the tails are much heavier than in the Gaussian distribution, but it may not be adequate to deal with cases of extremely heavy tails, such as those of Pareto or non-Gaussian stable laws. The skewed Student's t-distribution is a less studied subclass of the GH distribution. It is briefly mentioned by Prause (1999), Barndorff-Nielsen and Shepard (2001), Jones and Faddy (2003), Mencia and Sentana (2004) and Demarta and McNeil (2004). Further attention of it has been studied by Kjersti Aas and Ingrid Haff (2005).

1.7 NORMAL DISTRIBUTION

1.7.1 Construction:

Let

$$I = \int_{-\infty}^{\infty} e^{-\frac{y^2}{2}} dy$$

Therefore

$$\begin{aligned} I^2 &= I.I = \int_{-\infty}^{\infty} e^{-\frac{y^2}{2}} dy \int_{-\infty}^{\infty} e^{-\frac{z^2}{2}} dz \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp\left\{-\frac{1}{2}\left(y^2+z^2\right)\right\} dydz \end{aligned}$$

Let

$$y = r\cos\theta$$

and

$$z = r\sin\theta$$

Therefore,

$$\begin{aligned} y^2+z^2 &= r^2\cos^2\theta + r^2\sin^2\theta \\ &= r^2\cos^2\theta + \sin^2\theta \\ &= r^2 * 1 \\ &= r^2 \end{aligned}$$

transforming,

$$I^2 = \int\limits_0^{2\pi} \int\limits_0^{\infty} \exp\left\{-\frac{1}{2}r^2\right\} |J|\, dr d\theta$$

where,

$$\begin{aligned} |J| &= \begin{vmatrix} \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} \end{vmatrix} \\ &= \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta r & \cos\theta \end{vmatrix} \\ &= r\cos^2\theta + r\sin^2\theta \\ &= r\left(\cos^2\theta + \sin^2\theta\right) \\ &= r * 1 \\ &= r \end{aligned}$$

Therefore,

$$I^2 = \int\limits_0^{2\pi} \int\limits_0^{\infty} r\exp\left\{-\frac{1}{2}r^2\right\} dr d\theta$$

Let

$$x = e^{-\frac{r^2}{2}} \quad \Rightarrow \; dx = -re^{-\frac{r^2}{2}} \quad dr$$

Therefore,

$$\begin{aligned}
I^2 &= \int\limits_0^{2\pi} \left[-\int\limits_1^0 dx \right] d\theta \\
&= \int\limits_0^{2\pi} \left[\int\limits_0^1 dx \right] d\theta \\
&= \int\limits_0^{2\pi} [x]_0^1 \, d\theta \\
&= \int\limits_0^{2\pi} 1 d\theta \\
&= [\theta]_0^{2\pi} \\
&= 2\pi \\
I &= \sqrt{2\pi}
\end{aligned}$$

i.e

$$\begin{aligned}
\sqrt{2\pi} &= \int\limits_{-\infty}^{\infty} e^{-\frac{y^2}{2}} dy \\
1 &= \int\limits_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy
\end{aligned}$$

therefore the integrand is a pdf

$$f(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}, \quad -\infty < y < \infty$$

which is the standard normal distribution.
Using the transformation technique, let

$$Y = \frac{X - \mu}{\sigma}$$

therefore

$$f(x) = f(x)\,|J|$$

where,

$$\begin{aligned} J &= \frac{d}{dx}\left(\frac{x-\mu}{\sigma}\right) \\ &= \frac{1}{\sigma} \end{aligned}$$

thus

$$\begin{aligned} f(x) &= \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} * \frac{1}{\delta} \\ &= \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}, \quad -\infty < x < \infty; -\infty < \mu < \infty; \sigma > 0 \end{aligned}$$

Which is the normal distribution with parameters $\mu \; and \; \sigma^2$.

1.7.2 Properties

$$E(Y) = \mu$$

$$var(Y) = \sigma^2$$

$$kurtosis \; of \; Y = 3$$

The moment generating function is:

$$M_Y\left(t\right) = \exp\left\{\mu t + \frac{1}{2}\sigma^2 t\right\}$$

The characteristic function is:

$$\varphi_Y\left(t\right) = \exp\left\{\mu i t - \frac{1}{2}\sigma^2 t\right\}$$

Chapter 2

The Generalised Inverse Gaussian Distributions (GIGDs) with Different Parameterization

2.1 Introduction

In this chapter, different parameterisation of the GIGDs are considered. Their construction are based on the modified Bessel Function of The Third kind.

2.2 The Bessel Function of the third kind

2.2.1 Definition 1 and its properties

Integral Representation

$$K_v(\omega) = \frac{1}{2}\int_0^{\infty} x^{\lambda-1} e^{-\frac{\omega}{2}\left(x+\frac{1}{x}\right)} dx \tag{2.1}$$

Let

$$x = \frac{\omega}{2t} \Longrightarrow dx = -\frac{\omega}{2t^2}dt$$

Therefore;

$$\begin{aligned}
K_v(\omega) &= \frac{1}{2}\int_{\infty}^{0}\left(\frac{\omega}{2t}\right)^{v-1} e^{-\frac{\omega}{2}\left(\frac{\omega}{2t}+\frac{2t}{\omega}\right)}\left(-\frac{\omega}{2t^2}\right)dt \\
&= \frac{1}{2}\int_{\infty}^{0}\left(\frac{\omega}{2t}\right)^{v-1} e^{-\frac{\omega}{2}\left(\frac{\omega}{2t}+\frac{2t}{\omega}\right)}\frac{\omega}{2t^2}dt \\
&= \frac{1}{2}\left(\frac{\omega}{2}\right)^{v}\int_{\infty}^{0}\frac{1}{t^{v+1}}e^{-\frac{\omega}{2}\left(\frac{\omega}{2t}+\frac{2t}{\omega}\right)}dt \\
&= \frac{1}{2}\left(\frac{\omega}{2}\right)^{v}\int_{\infty}^{0}\frac{1}{t^{v+1}}e^{-\left(\frac{\omega^2}{4t}+t\right)}dt \\
&= \frac{1}{2}\left(\frac{\omega}{2}\right)^{v}\int_{\infty}^{0}t^{-v-1}e^{-t-\frac{\omega^2}{4t}}dt \qquad (2.2)
\end{aligned}$$

2.2.2 Properties

Property 1:

$$K_{\lambda}(\omega) = K_{-\lambda}(\omega) \quad \text{- Symmetry} \qquad (2.3)$$

Proof: Let

$$t = \frac{1}{z}$$

therefore

$$dt = -\frac{dz}{z^2}$$

Substitute these in (1). Then

$$
\begin{aligned}
K_\lambda(\omega) &= \frac{1}{2}\int_\infty^0 \left(\frac{1}{z}\right)^{\lambda-1} e^{-\frac{\omega}{2}\left(\frac{1}{z}+z\right)}\left(-\frac{1}{z^2}\right)dz \\
&= \frac{1}{2}\int_0^\infty z^{-\lambda-1}e^{-\frac{\omega}{2}\left(z+\frac{1}{z}\right)}dz \\
&= K_{-\lambda}(\omega)
\end{aligned}
$$

2.2.3 Derivatives

Derivative 1:

using (1)

$$\frac{\partial}{\partial\omega}K_\lambda(\omega) = -\frac{1}{2}\left[K_{\lambda+1}(\omega)+K_{\lambda-1}(\omega)\right] \tag{2.4}$$

Proof:

From (1)

$$
\begin{aligned}
\frac{\partial}{\partial\omega}K_v(\omega) &= \frac{\partial}{\partial\omega}\frac{1}{2}\int_0^\infty t^{v-1}e^{-\frac{\omega}{2}\left(t+\frac{1}{t}\right)}dt \\
&= \frac{1}{2}\int_0^\infty t^{v-1}\left[-\frac{1}{2}\left(t+\frac{1}{t}\right)\right]e^{-\frac{\omega}{2}\left(t+\frac{1}{t}\right)}dt \\
&= -\frac{1}{2}\left[\frac{1}{2}\int_0^\infty t^{v}e^{-\frac{\omega}{2}\left(t+\frac{1}{t}\right)}dt+\frac{1}{2}\int_0^\infty t^{v-2}e^{-\frac{\omega}{2}\left(t+\frac{1}{t}\right)}dt\right] \\
&= -\frac{1}{2}\left[K_{v+1}(\omega)+K_{v-1}(\omega)\right]
\end{aligned}
$$

Using (2)

$$
\begin{aligned}
\frac{\partial}{\partial\omega}K_v(\omega) &= \frac{\partial}{\partial\omega}\frac{1}{2}\left(\frac{\omega}{2}\right)^v\int_0^\infty t^{-v-1}e^{-t-\frac{\omega^2}{4t}}dt \\
&= \frac{1}{2}\left\{\frac{v}{2}\left(\frac{\omega}{2}\right)^{v-1}\int_0^\infty t^{-v-1}e^{-t-\frac{\omega^2}{4t}}dt \;+\; \left(\frac{\omega}{2}\right)^v\int_0^\infty t^{-v-1}e^{-t-\frac{\omega^2}{4t}}\left(-\frac{2}{4t}\omega\right)dt\right\} \\
&= \frac{1}{2}\left\{\frac{v}{2}\left(\frac{\omega}{2}\right)^{-1}\left(\frac{\omega}{2}\right)^v\int_0^\infty t^{-v-1}e^{-t-\frac{\omega^2}{4t}}dt \;+\; \left(\frac{\omega}{2}\right)^v\int_0^\infty t^{-v-1}e^{-t-\frac{\omega^2}{4t}}\left(-\frac{2}{4t}\omega\right)dt\right\} \\
&= \frac{1}{2}\left\{\frac{v}{\omega}\left(\frac{\omega}{2}\right)^v\int_0^\infty t^{-v-1}e^{-t-\frac{\omega^2}{4t}}dt \;+-\left(\frac{\omega}{2}\right)^v\frac{\omega}{2}\int_0^\infty t^{-v-1-1}e^{-t-\frac{\omega^2}{4t}}dt\right\} \\
&= \frac{v}{\omega}\left[\frac{1}{2}\left(\frac{\omega}{2}\right)^v\int_0^\infty t^{-v-1}e^{-t-\frac{\omega^2}{4t}}dt\right] - \frac{1}{2}\left(\frac{\omega}{2}\right)^{v+1}\int_0^\infty t^{-(v+1)-1}e^{-t-\frac{\omega^2}{4t}}dt \\
&= \frac{v}{\omega}K_v(\omega) - K_{v+1}(\omega)
\end{aligned}
$$

Equating (3) and (4), we get

$$
\begin{aligned}
-\frac{1}{2}K_{v+1}(\omega) - \frac{1}{2}K_{v-1}(\omega) &= \frac{v}{\omega}K_v(\omega) - K_{v+1}(\omega) \\
\frac{1}{2}K_{v+1}(\omega) &= \frac{v}{\omega}K_v(\omega) + \frac{1}{2}K_{v-1}(\omega) \\
K_{v+1}(\omega) &= \frac{2v}{\omega}K_v(\omega) + K_{v-1}(\omega) \qquad (2.5)
\end{aligned}
$$

2.3 Definition 2 and Its Properties

$$
K_\lambda(\omega) = \left(\frac{\omega}{2}\right)^\lambda \frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\lambda+\frac{1}{2}\right)}\int_1^\infty \left(t^2-1\right)^{\lambda-\frac{1}{2}}e^{-\omega t}dt \qquad (2.6)
$$

Property 2:

$$K_{n+\frac{1}{2}}(\omega) = \sqrt{\frac{\pi}{2\omega}} e^{-\omega} \left\{ 1 + \sum_{i=1}^{n} \frac{(n+i)!\,(2\omega)^{-i}}{(n-i)!i!} \right\} \tag{2.7}$$

where $n = 0, 1, 2, ...$

Proof: Using integral representation (2), and letting $\lambda = n + \frac{1}{2}$ for $n = 0, 1, 2, ...,$ we have

$$\begin{aligned} K_{n+\frac{1}{2}}(\omega) &= \left(\frac{\omega}{2}\right)^{n+\frac{1}{2}} \frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma(n+1)} \int_1^\infty \left(t^2-1\right)^n e^{-\omega t} dt \\ &= \frac{\sqrt{\pi}}{2^{n+\frac{1}{2}}} \frac{w^{n+\frac{1}{2}}}{n!} \int_1^\infty \left(t^2-1\right)^n e^{-\omega t} dt \\ &= \frac{\sqrt{\pi}}{2^n 2^{\frac{1}{2}}} \frac{w^{n+\frac{1}{2}}}{n!} \int_1^\infty \left(t^2-1\right)^n e^{-\omega t} dt \end{aligned}$$

Therefore

$$\begin{aligned} K_{n+\frac{1}{2}}(\omega) &= \sqrt{\frac{\omega\pi}{2}} \frac{\omega^n}{2^n} \frac{1}{n!} \int_1^\infty \left(t^2-1\right)^n e^{-\omega t} dt \\ &= \sqrt{\frac{\omega\pi}{2}} \frac{\omega^n}{2^n} \frac{e^{-\omega}}{n!} \int_1^\infty \left(t^2-1\right)^n \frac{e^{-\omega t}}{e^{-\omega}} dt \end{aligned}$$

Therefore

$$\begin{aligned} K_{n+\frac{1}{2}}(\omega) &= \sqrt{\frac{\omega\pi}{2}} \frac{\omega^n}{2^n} \frac{e^{-\omega}}{n!} \int_1^\infty \left[(t-1)(t+1)\right]^n e^{-\omega t+\omega} dt \\ &= \sqrt{\frac{\omega\pi}{2}} \frac{\omega^n}{2^n} \frac{e^{-\omega}}{n!} \int_1^\infty \left[(t-1)(t+1)\right]^n e^{-\omega(t-1)} dt \end{aligned}$$

Let

$$y = \omega(t-1)$$

therefore

$$t-1=\frac{y}{\omega};t+1=2+\frac{y}{\omega}\text{ and }dt=\frac{dy}{\omega}$$

Therefore,

$$\begin{aligned}
K_{n+\frac{1}{2}}\left(\omega\right) &= \sqrt{\frac{\omega\pi}{2}}e^{-\omega}\frac{\omega^n}{2^n n!}\int_0^\infty\left[\frac{y}{\omega}\left(2+\frac{y}{\omega}\right)\right]^n e^{-y}\frac{dy}{\omega} \\
&= \frac{1}{\omega}\sqrt{\frac{\omega\pi}{2}}e^{-\omega}\frac{\omega^n}{2^n n!}\int_0^\infty\left[\frac{2y}{\omega}\left(1+\frac{y}{2\omega}\right)\right]^n e^{-y}dy
\end{aligned}$$

Therefore

$$\begin{aligned}
K_{n+\frac{1}{2}}(\omega) &= \sqrt{\frac{\pi}{2\omega}}\frac{e^{-\omega}}{n!}\frac{\omega^n}{2^n}\int_0^\infty \left(\frac{2y}{\omega}\right)^n \left(1+\frac{y}{2\omega}\right)^n e^{-y}dy \\
&= \sqrt{\frac{\pi}{2\omega}}\frac{e^{-\omega}}{n!}\int_0^\infty y^n \left(1+\frac{y}{2\omega}\right)^n e^{-y}dy \\
&= \sqrt{\frac{\pi}{2\omega}}\frac{e^{-\omega}}{n!}\int_0^\infty y^n \sum_{i=0}^{n}\binom{n}{i}\left(\frac{y}{2\omega}\right)^i e^{-y}dy \\
&= \sqrt{\frac{\pi}{2\omega}}\frac{e^{-\omega}}{n!}\sum_{i=0}^{n}\binom{n}{i}\int_0^\infty \frac{y^{n+i}}{(2\omega)^i}e^{-y}dy \\
&= \sqrt{\frac{\pi}{2\omega}}\frac{e^{-\omega}}{n!}\sum_{i=0}^{n}\binom{n}{i}\frac{1}{(2\omega)^i}\int_0^\infty y^{(n+i+1)-1}e^{-y}dy \\
&= \sqrt{\frac{\pi}{2\omega}}\frac{e^{-\omega}}{n!}\sum_{i=0}^{n}\binom{n}{i}\frac{1}{(2\omega)^i}\Gamma(n+i+1) \\
&= \sqrt{\frac{\pi}{2\omega}}\frac{e^{-\omega}}{n!}\left[\binom{n}{0}\frac{1}{(2\omega)^0}\Gamma(n+1)+\sum_{i=1}^{n}\binom{n}{i}\frac{1}{(2\omega)^i}\Gamma(n+i+1)\right] \\
&= \sqrt{\frac{\pi}{2\omega}}\frac{e^{-\omega}}{n!}\left[\Gamma(n+1)+\sum_{i=1}^{n}\binom{n}{i}(2\omega)^{-i}(n+i)!\right] \\
&= \sqrt{\frac{\pi}{2\omega}}\frac{e^{-\omega}}{n!}\left[n!+\sum_{i=1}^{n}\binom{n}{i}(2\omega)^{-i}(n+i)!\right]
\end{aligned}$$

Therefore

$$\begin{aligned}
K_{n+\frac{1}{2}}(\omega) &= \sqrt{\frac{\pi}{2\omega}}e^{-\omega}\left[1+\sum_{i=1}^{n}\frac{1}{n!}\binom{n}{i}(2\omega)^{-i}(n+i)!\right] \\
&= \sqrt{\frac{\pi}{2\omega}}e^{-\omega}\left[1+\sum_{i=1}^{n}\frac{n!\,(n+i)!}{n!\,(n-i)!i!}(2\omega)^{-i}\right] \\
&= \sqrt{\frac{\pi}{2\omega}}e^{-\omega}\left[1+\sum_{i=1}^{n}\frac{(n+i)!\,(2\omega)^{-i}}{(n-i)!i!}\right]
\end{aligned}$$

Corollary 1 of Property 2:

$$K_{\frac{3}{2}}(\omega) = \sqrt{\frac{\pi}{2\omega}}e^{-\omega}\left(1+\frac{1}{\omega}\right) \tag{2.8}$$

Proof:

Put $n = 1$ in Property 2,

$$\begin{aligned}
K_{\frac{3}{2}}(\omega) &= \sqrt{\frac{\pi}{2\omega}}e^{-\omega}\left[1+\sum_{i=1}^{1}\frac{(n+i)!\,(2\omega)^{-i}}{(n-i)!i!}\right] \\
&= \sqrt{\frac{\pi}{2\omega}}e^{-\omega}\left[1+\frac{(1+1)!\,(2\omega)^{-1}}{(1-1)!1!}\right] \\
&= \sqrt{\frac{\pi}{2\omega}}e^{-\omega}\left[1+\frac{2!\,(2\omega)^{-1}}{0!1!}\right] \\
&= \sqrt{\frac{\pi}{2\omega}}e^{-\omega}\left[1+\frac{2}{2\omega}\right] \\
&= \sqrt{\frac{\pi}{2\omega}}e^{-\omega}\left(1+\frac{1}{\omega}\right)
\end{aligned}$$

Corollary 2 of Property 2

$$K_{\frac{1}{2}}(\omega) = \sqrt{\frac{\pi}{2\omega}} e^{-\omega} \tag{2.9}$$

Proof:

Put $n = 0$ in Property 2. Then

$$\begin{aligned} K_{\frac{1}{2}}(\omega) &= \sqrt{\frac{\pi}{2\omega}} e^{-\omega} [1 + 0] \\ &= \sqrt{\frac{\pi}{2\omega}} e^{-\omega} \end{aligned}$$

Alternatively, using integral representation (2) and letting $\lambda = \frac{1}{2}$, we have

$$\begin{aligned} K_{\frac{1}{2}}(\omega) &= \left(\frac{\omega}{2}\right)^{\frac{1}{2}} \frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma(1)} \int_1^\infty e^{-\omega t} dt \\ &= \left(\frac{\omega}{2}\right)^{\frac{1}{2}} \sqrt{\pi} \left[\frac{e^{-\omega t}}{-\omega}\right]_1^\infty \\ &= \left(\frac{\omega}{2}\right)^{\frac{1}{2}} \sqrt{\pi} \left(-\frac{1}{\omega}\right) \left[e^{-\omega t}\right]_1^\infty \\ &= \left(\frac{\omega}{2}\right)^{\frac{1}{2}} \sqrt{\pi} \left(-\frac{1}{\omega}\right) \left[0 - e^{-\omega}\right] \\ &= \left(\frac{\omega\pi}{2}\right)^{\frac{1}{2}} \frac{1}{\omega} e^{-\omega} \\ &= \left(\frac{\omega\pi}{2\omega^2}\right)^{\frac{1}{2}} e^{-\omega} \\ &= \sqrt{\frac{\pi}{2\omega}} e^{-\omega} \end{aligned}$$

Corollary 3 of Property 2

$$K_{n-\frac{1}{2}}(\omega) = K_{(n-1)+\frac{1}{2}}(\omega) = \sqrt{\frac{\pi}{2\omega}} e^{-\omega} \left\{ 1 + \sum_{i=1}^{n-1} \frac{(n+i-1)!}{i!(n-i-1)!} (2\omega)^{-i} \right\} \tag{2.10}$$

Proof:

Using integral representation (2) and letting $\lambda = n - \frac{1}{2}$ for $n = 0, 1, 2, ...$ we have

$$\begin{aligned}
K_{n-\frac{1}{2}}(\omega) &= \left(\frac{\omega}{2}\right)^{n-\frac{1}{2}} \frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(n-\frac{1}{2}+\frac{1}{2}\right)} \int_1^{\infty} \left(t^2-1\right)^{n-\frac{1}{2}-\frac{1}{2}} e^{-\omega t} dt \\
&= \left(\frac{\omega}{2}\right)^{n-\frac{1}{2}} \frac{\sqrt{\pi}}{\Gamma(n)} \int_1^{\infty} \left(t^2-1\right)^{n-1} e^{-\omega t} dt \\
&= \frac{\sqrt{\pi}}{2^{n-\frac{1}{2}}} \frac{\omega^{n-\frac{1}{2}}}{(n-1)!} \int_1^{\infty} \left(t^2-1\right)^{n-1} e^{-\omega t} dt \\
&= \frac{\sqrt{\pi}}{2^{n-\frac{1}{2}}} \frac{\omega^{n-\frac{1}{2}}}{(n-1)!} e^{-\omega} \int_1^{\infty} \left(t^2-1\right)^{n-1} e^{-\omega t+\omega} dt \\
&= \frac{\sqrt{\pi}}{2^{n-\frac{1}{2}}} \frac{\omega^{n-\frac{1}{2}}}{(n-1)!} e^{-\omega} \int_1^{\infty} \left(t^2-1\right)^{n-1} e^{-\omega(t-1)} dt \\
&= \frac{\sqrt{\pi}}{2^{n-\frac{1}{2}}} \frac{\omega^{n-\frac{1}{2}}}{(n-1)!} e^{-\omega} \int_1^{\infty} (t-1)^{n-1} (t+1)^{n-1} e^{-\omega(t-1)} dt
\end{aligned}$$

Let

$$z = \omega(t-1)$$

therefore

$$dz = \omega dt \Longrightarrow dt = \frac{dz}{\omega}$$

Also

$$t-1 = \frac{z}{\omega} \text{ and } t+1 = 2 + \frac{z}{\omega}$$

Therefore

$$\begin{aligned}
K_{n-\frac{1}{2}}(\omega) &= \frac{\sqrt{\pi}}{2^{n-\frac{1}{2}}}\frac{\omega^{n-\frac{1}{2}}}{(n-1)!}e^{-\omega}\int_0^\infty \left(\frac{z}{\omega}\right)^{n-1}\left(2+\frac{z}{\omega}\right)^{n-1}e^{-z}\frac{dz}{\omega} \\
&= \frac{\sqrt{\pi}}{2^{n-\frac{1}{2}}}\frac{\omega^{n-\frac{1}{2}}}{(n-1)!}e^{-\omega}\int_0^\infty \frac{z^{n-1}}{\omega^{n-1}}2^{n-1}\left(1+\frac{z}{2\omega}\right)^{n-1}\frac{e^{-z}}{\omega}dz
\end{aligned}$$

Therefore

$$\begin{aligned}
K_{n-\frac{1}{2}}(\omega) &= \frac{\sqrt{\pi}}{2^{n-\frac{1}{2}}}\frac{\omega^{n-\frac{1}{2}}}{(n-1)!}e^{-\omega}\frac{2^{n-1}}{\omega^n}\int_0^\infty z^{n-1}\left(1+\frac{z}{2\omega}\right)^{n-1}e^{-z}dz \\
&= \frac{\sqrt{\pi}}{\sqrt{2\omega}}\frac{e^{-\omega}}{(n-1)!}\int_0^\infty z^{n-1}\left(1+\frac{z}{2\omega}\right)^{n-1}e^{-z}dz \\
&= \sqrt{\frac{\pi}{2\omega}}\frac{e^{-\omega}}{(n-1)!}\int_0^\infty z^{n-1}\sum_{i=0}^{n-1}\binom{n-1}{i}\left(\frac{z}{2\omega}\right)^i e^{-z}dz \\
&= \sqrt{\frac{\pi}{2\omega}}\frac{e^{-\omega}}{(n-1)!}\int_0^\infty \sum_{i=0}^{n-1}\binom{n-1}{i}\frac{z^{n+i-1}}{(2\omega)^i}e^{-z}dz \\
&= \sqrt{\frac{\pi}{2\omega}}\frac{e^{-\omega}}{(n-1)!}\sum_{i=0}^{n-1}\binom{n-1}{i}\frac{1}{(2\omega)^i}\int_0^\infty z^{n+i-1}e^{-z}dz \\
&= \sqrt{\frac{\pi}{2\omega}}\frac{e^{-\omega}}{(n-1)!}\sum_{i=0}^{n-1}\binom{n-1}{i}\frac{1}{(2\omega)^i}\Gamma(n+i) \\
&= \sqrt{\frac{\pi}{2\omega}}e^{-\omega}\sum_{i=0}^{n-1}\binom{n-1}{i}\frac{1}{(2\omega)^i}\frac{\Gamma(n+i)}{(n-1)!} \\
&= \sqrt{\frac{\pi}{2\omega}}e^{-\omega}\sum_{i=0}^{n-1}\binom{n-1}{i}\frac{\Gamma(n+i)}{(n-1)!}(2\omega)^{-i}
\end{aligned}$$

Therefore

$$\begin{aligned} K_{n-\frac{1}{2}}(\omega) &= \sqrt{\frac{\pi}{2\omega}}e^{-\omega}\left\{\binom{n-1}{0}\frac{\Gamma(n)}{(n-1)!}(2\omega)^0 + \sum_{i=1}^{n-1}\binom{n-1}{i}\frac{\Gamma(n+i)}{(n-1)!}(2\omega)^{-i}\right\} \\ &= \sqrt{\frac{\pi}{2\omega}}e^{-\omega}\left\{1 + \sum_{i=1}^{n-1}\frac{(n-1)!}{i!(n-i-1)!}\frac{(n+i-1)!}{(n-1)!}(2\omega)^{-i}\right\} \\ &= \sqrt{\frac{\pi}{2\omega}}e^{-\omega}\left\{1 + \sum_{i=1}^{n-1}\frac{(n+i-1)!}{i!(n-i-1)!}(2\omega)^{-i}\right\} \end{aligned}$$

Corollaries:

Thus when $n = 1$, we have

$$\begin{aligned} K_{\frac{1}{2}}(\omega) &= \sqrt{\frac{\pi}{2\omega}}e^{-\omega}\{1+0\} \\ &= \sqrt{\frac{\pi}{2\omega}}e^{-\omega} \end{aligned}$$

When $n = 0$

$$K_{-\frac{1}{2}}(\omega) = \sqrt{\frac{\pi}{2\omega}}e^{-\omega}$$

When $n = 2$

$$\begin{aligned} K_{\frac{3}{2}}(\omega) &= \sqrt{\frac{\pi}{2\omega}}e^{-\omega}\left\{1 + \frac{(2+1-1)!}{1!(2-1-1)!}(2\omega)^{-1}\right\} \\ &= \sqrt{\frac{\pi}{2\omega}}e^{-\omega}\left\{1 + \frac{2!}{1!0!}(2\omega)^{-1}\right\} \\ &= \sqrt{\frac{\pi}{2\omega}}e^{-\omega}\left[1 + \frac{2}{2\omega}\right] \\ &= \sqrt{\frac{\pi}{2\omega}}e^{-\omega}\left(1 + \frac{1}{\omega}\right) \end{aligned}$$

Derivative 3:

$$\frac{\partial}{\partial\omega}\log K_\lambda(\omega)=\frac{\lambda}{\omega}-\frac{K_{\lambda+1}(\omega)}{K_\lambda(\omega)}$$

Proof:

Using (2)

$$\log K_\lambda(\omega)=\lambda\log\frac{\omega}{2}+\log\Gamma\left(\frac{1}{2}\right)-\log\Gamma\left(\lambda+\frac{1}{2}\right)+\log\int_1^\infty\left(t^2-1\right)^{\lambda-\frac{1}{2}}e^{-\omega t}dt$$

$$\begin{aligned}\frac{\partial}{\partial\omega}\log K_\lambda(\omega) &= \frac{\lambda}{\omega}+\frac{\partial}{\partial\omega}\log\int_1^\infty\left(t^2-1\right)^{\lambda-\frac{1}{2}}e^{-\omega t}dt\\ &= \frac{\lambda}{\omega}+\frac{\int_1^\infty -t\left(t^2-1\right)^{\lambda-\frac{1}{2}}e^{-\omega t}dt}{\int_1^\infty\left(t^2-1\right)^{\lambda-\frac{1}{2}}e^{-\omega t}dt}\\ &= \frac{\lambda}{\omega}-\frac{\int_1^\infty t\left(t^2-1\right)^{\lambda-\frac{1}{2}}e^{-\omega t}dt}{\int_1^\infty\left(t^2-1\right)^{\lambda-\frac{1}{2}}e^{-\omega t}dt}\\ &= \frac{\lambda}{\omega}-\frac{\left(\frac{\omega}{2}\right)^\lambda\frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\lambda+\frac{1}{2}\right)}\int_1^\infty t\left(t^2-1\right)^{\lambda-\frac{1}{2}}e^{-\omega t}dt}{K_\lambda(\omega)}\end{aligned}$$

The problem is to show that

$$\left(\frac{\omega}{2}\right)^\lambda\frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\lambda+\frac{1}{2}\right)}\int_1^\infty t\left(t^2-1\right)^{\lambda-\frac{1}{2}}e^{-\omega t}dt=K_{\lambda+1}(\omega)$$

Derivative 5:

$$\left(\log K_\lambda(x)\right)'=\frac{-\lambda}{x}-\frac{K_{\lambda-1}(x)}{K_\lambda(x)} \tag{2.11}$$

2.3.1 Asymptotic Expansions (AE)

AE 1:

$$\lim_{\omega \longrightarrow 0} K_\lambda(\omega) = \frac{1}{2}\left(\frac{\omega}{2}\right)^{-\lambda}\Gamma(\lambda)\,;\lambda > 0 \tag{2.12}$$

AE 2:

$$\lim_{\omega \longrightarrow 0} K_\lambda(\omega) = \frac{1}{2}\left(\frac{\omega}{2}\right)^{\lambda}\Gamma(-\lambda)\,;\lambda > 0 \tag{2.13}$$

AE 3:

$$\lim_{\omega \longrightarrow 0} K_0(\omega) = -\log\omega \tag{2.14}$$

AE 4:

$$K_\lambda(\omega) = \sqrt{\frac{\pi}{2\omega}}e^{-\omega}\left\{\begin{array}{c} 1 + \frac{4\lambda^2-1}{8\omega} + \frac{\left(4\lambda^2-1\right)\left(4\lambda^2-9\right)}{2!(8\omega)^2} + \\ \frac{\left(4\lambda^2-1\right)\left(4\lambda^2-9\right)\left(4\lambda^2-25\right)}{3!(8\omega)^3} + \dots \end{array}\right\} \tag{2.15}$$

Therefore

$$\lim_{\omega \longrightarrow \infty} K_\lambda(\omega) = \sqrt{\frac{\pi}{2\omega}}e^{-\omega}$$

Proof:

Using definition (2)

$$\begin{aligned}
K_\lambda(\omega) &= \left(\frac{\omega}{2}\right)^{\lambda}\frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\lambda+\frac{1}{2}\right)}\int_1^\infty \left(t^2-1\right)^{\lambda-\frac{1}{2}}e^{-\omega t}dt \\
&= \left(\frac{\omega}{2}\right)^{\lambda}\frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\lambda+\frac{1}{2}\right)}e^{-\omega}\int_1^\infty \left(t^2-1\right)^{\lambda-\frac{1}{2}}\frac{e^{-\omega t}}{e^{-\omega}}dt \\
&= \left(\frac{\omega}{2}\right)^{\lambda}\frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\lambda+\frac{1}{2}\right)}e^{-\omega}\int_1^\infty \left(t^2-1\right)^{\lambda-\frac{1}{2}}e^{-\omega t+\omega}dt \\
&= \left(\frac{\omega}{2}\right)^{\lambda}\frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\lambda+\frac{1}{2}\right)}e^{-\omega}\int_1^\infty \left(t^2-1\right)^{\lambda-\frac{1}{2}}e^{-\omega(t-1)}dt
\end{aligned}$$

Therefore

$$K_\lambda(\omega) = \left(\frac{\omega}{2}\right)^\lambda \frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\lambda+\frac{1}{2}\right)} e^{-\omega} \int_1^\infty (t-1)^{\lambda-\frac{1}{2}} (t+1)^{\lambda-\frac{1}{2}} e^{-\omega(t-1)} dt$$

Let

$$y = \omega(t-1) \Longrightarrow t-1 = \frac{y}{\omega}, t+1 = 2+\frac{y}{\omega} \text{ and } dt = \frac{dy}{\omega}$$

Therefore

$$\begin{aligned}
K_\lambda(\omega) &= \left(\frac{\omega}{2}\right)^\lambda \frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\lambda+\frac{1}{2}\right)} e^{-\omega} \int_0^\infty \left(\frac{y}{\omega}\right)^{\lambda-\frac{1}{2}} \left(2+\frac{y}{\omega}\right)^{\lambda-\frac{1}{2}} e^{-y} \frac{dy}{\omega} \\
&= \left(\frac{\omega}{2}\right)^\lambda \frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\lambda+\frac{1}{2}\right)} e^{-\omega} \int_0^\infty \frac{y^{\lambda-\frac{1}{2}}}{\omega^{\lambda-\frac{1}{2}}} \left[2\left(1+\frac{y}{2\omega}\right)\right]^{\lambda-\frac{1}{2}} e^{-y} \frac{dy}{\omega} \\
&= \left(\frac{\omega}{2}\right)^\lambda \frac{\Gamma(\pi)}{\Gamma\left(\lambda+\frac{1}{2}\right)} e^{-\omega} \int_0^\infty \left(\frac{2}{\omega}\right)^{\lambda-\frac{1}{2}} \frac{y^{\lambda-\frac{1}{2}}}{\omega} \left(1+\frac{y}{2\omega}\right)^{\lambda-\frac{1}{2}} e^{-y} dy \\
&= \left(\frac{\omega}{2}\right)^\lambda \left(\frac{2}{\omega}\right)^\lambda \left(\frac{2}{\omega}\right)^{-\frac{1}{2}} \frac{1}{\omega} \frac{\Gamma(\pi)}{\Gamma\left(\lambda+\frac{1}{2}\right)} e^{-\omega} \int_0^\infty y^{\lambda-\frac{1}{2}} \left(1+\frac{y}{2\omega}\right)^{\lambda-\frac{1}{2}} e^{-y} dy \\
&= \left(\frac{\omega}{2}\right)^{\frac{1}{2}} \frac{1}{\omega} \frac{\Gamma(\pi)}{\Gamma\left(\lambda+\frac{1}{2}\right)} e^{-\omega} \int_0^\infty y^{\lambda-\frac{1}{2}} \left(1+\frac{y}{2\omega}\right)^{\lambda-\frac{1}{2}} e^{-y} dy \\
&= \left(\frac{\omega}{2}\frac{\pi}{\omega^2}\right)^{\frac{1}{2}} \frac{e^{-\omega}}{\Gamma\left(\lambda+\frac{1}{2}\right)} \int_0^\infty y^{\lambda-\frac{1}{2}} \left(1+\frac{y}{2\omega}\right)^{\lambda-\frac{1}{2}} e^{-y} dy \\
&= \sqrt{\frac{\pi}{2\omega}} \frac{e^{-\omega}}{\Gamma\left(\lambda+\frac{1}{2}\right)} \int_0^\infty y^{\lambda-\frac{1}{2}} \sum_{i=0}^\infty \binom{\lambda-\frac{1}{2}}{i} \left(\frac{y}{2\omega}\right)^i e^{-y} dy
\end{aligned}$$

Therefore

$$
\begin{aligned}
K_\lambda(\omega) &= \sqrt{\frac{\pi}{2\omega}}\frac{e^{-\omega}}{\Gamma\left(\lambda+\frac{1}{2}\right)}\int_0^\infty \sum_{i=0}^{\infty}\binom{\lambda-\frac{1}{2}}{i}\frac{y^{\lambda+i-\frac{1}{2}}}{(2\omega)^i}e^{-y}dy \\
&= \sqrt{\frac{\pi}{2\omega}}\frac{e^{-\omega}}{\Gamma\left(\lambda+\frac{1}{2}\right)}\sum_{i=0}^{\infty}\left\{\binom{\lambda-\frac{1}{2}}{i}\frac{1}{(2\omega)^i}\int_0^\infty y^{\lambda+i-\frac{1}{2}}e^{-y}dy\right\} \\
&= \sqrt{\frac{\pi}{2\omega}}\frac{e^{-\omega}}{\Gamma\left(\lambda+\frac{1}{2}\right)}\sum_{i=0}^{\infty}\left\{\binom{\lambda-\frac{1}{2}}{i}\frac{1}{(2\omega)^i}\int_0^\infty y^{\left(\lambda+i+\frac{1}{2}\right)-1}e^{-y}dy\right\} \\
&= \sqrt{\frac{\pi}{2\omega}}\frac{e^{-\omega}}{\Gamma\left(\lambda+\frac{1}{2}\right)}\sum_{i=0}^{\infty}\binom{\lambda-\frac{1}{2}}{i}\frac{1}{(2\omega)^i}\Gamma\left(\lambda+i+\frac{1}{2}\right) \\
&= \sqrt{\frac{\pi}{2\omega}}e^{-\omega}\sum_{i=0}^{\infty}\binom{\lambda-\frac{1}{2}}{i}\frac{1}{(2\omega)^i}\frac{\Gamma\left(\lambda+i+\frac{1}{2}\right)}{\Gamma\left(\lambda+\frac{1}{2}\right)} \\
&= \sqrt{\frac{\pi}{2\omega}}e^{-\omega}\left\{1+\sum_{i=1}^{\infty}\binom{\lambda-\frac{1}{2}}{i}\frac{1}{(2\omega)^i}\frac{\Gamma\left(\lambda+i+\frac{1}{2}\right)}{\Gamma\left(\lambda+\frac{1}{2}\right)}\right\} \\
&= \sqrt{\frac{\pi}{2\omega}}e^{-\omega}\left\{\begin{array}{c} 1+\binom{\lambda-\frac{1}{2}}{1}\frac{1}{2\omega}\frac{\Gamma\left(\lambda+1+\frac{1}{2}\right)}{\Gamma\left(\lambda+\frac{1}{2}\right)}+\binom{\lambda-\frac{1}{2}}{2}\frac{1}{(2\omega)^2}\frac{\Gamma\left(\lambda+2+\frac{1}{2}\right)}{\Gamma\left(\lambda+\frac{1}{2}\right)} \\ +\binom{\lambda-\frac{1}{2}}{3}\frac{1}{(2\omega)^3}\frac{\Gamma\left(\lambda+3+\frac{1}{2}\right)}{\Gamma\left(\lambda+\frac{1}{2}\right)}+\ldots \end{array}\right\} \\
&= \sqrt{\frac{\pi}{2\omega}}e^{-\omega}\left\{\begin{array}{c} 1+\frac{\left(\lambda-\frac{1}{2}\right)\left(\lambda+\frac{1}{2}\right)}{1!2\omega}+\frac{\left(\lambda-\frac{1}{2}\right)\left(\lambda-\frac{1}{2}-1\right)\left(\lambda+1+\frac{1}{2}\right)\left(\lambda+\frac{1}{2}\right)}{2!(2\omega)^2} \\ +\frac{\left(\lambda-\frac{1}{2}\right)\left(\lambda-\frac{1}{2}-1\right)\left(\lambda-\frac{1}{2}-2\right)\left(\lambda+2+\frac{1}{2}\right)\left(\lambda+1+\frac{1}{2}\right)\left(\lambda+\frac{1}{2}\right)}{3!(2\omega)^3}+\ldots \end{array}\right\}
\end{aligned}
$$

Therefore

$$K_\lambda(\omega) = \sqrt{\frac{\pi}{2\omega}}e^{-\omega}\left\{\begin{array}{c} 1+\frac{\left(\lambda-\frac{1}{2}\right)\left(\lambda+\frac{1}{2}\right)}{1!2\omega}+\frac{\left[\left(\lambda-\frac{1}{2}\right)\left(\lambda+\frac{1}{2}\right)\right]\left[\left(\lambda-\left(1+\frac{1}{2}\right)\right)\left(\lambda+\left(1+\frac{1}{2}\right)\right)\right]}{2!(2\omega)^2} \\ +\frac{\left(\lambda-\frac{1}{2}\right)\left(\lambda+\frac{1}{2}\right)\left[\left(\lambda-\left(1+\frac{1}{2}\right)\right)\left(\lambda+\left(1+\frac{1}{2}\right)\right)\right]\left[\left(\lambda-\left(2+\frac{1}{2}\right)\right)\left(\lambda+\left(2+\frac{1}{2}\right)\right)\right]}{3!(2\omega)^3}+... \\ +... \\ +\frac{\left(\lambda-\frac{1}{2}\right)\left(\lambda+\frac{1}{2}\right)\left[\left(\lambda-\left(1+\frac{1}{2}\right)\right)\left(\lambda+\left(1+\frac{1}{2}\right)\right)\right]...\left[\left(\lambda-\left(n-1+\frac{1}{2}\right)\right)\left(\lambda+\left(n-1+\frac{1}{2}\right)\right)\right]}{n!(2\omega)^n} \\ +... \end{array}\right\}$$

$$= \sqrt{\frac{\pi}{2\omega}}e^{-\omega}\left\{\begin{array}{c} 1+\frac{\left[\lambda^2-\left(\frac{1}{2}\right)^2\right]}{1!2\omega}+\frac{\left[\lambda^2-\left(\frac{1}{2}\right)^2\right]\left[\lambda^2-\left(1+\frac{1}{2}\right)^2\right]}{2!(2\omega)^2} \\ +\frac{\left[\lambda^2-\left(\frac{1}{2}\right)^2\right]\left[\lambda^2-\left(1+\frac{1}{2}\right)^2\right]\left[\lambda^2-\left(2+\frac{1}{2}\right)^2\right]}{3!(2\omega)^3}+... \\ +... \\ +\frac{\left[\lambda^2-\left(\frac{1}{2}\right)^2\right]\left[\lambda^2-\left(1+\frac{1}{2}\right)^2\right]...\left[\lambda^2-\left(n-1+\frac{1}{2}\right)^2\right]}{n!(2\omega)^n} \\ +... \end{array}\right\}$$

Therefore

$$K_\lambda(\omega) = \sqrt{\frac{\pi}{2\omega}}e^{-\omega}\left\{1+\sum_{n=1}^{\infty}\left[\lambda^2-\left(\frac{1}{2}\right)^2\right]\left[\lambda^2-\left(\frac{3}{2}\right)^2\right]\left[\lambda^2-\left(\frac{5}{2}\right)^2\right]...\left[\lambda^2-\left(\frac{2n-1}{2}\right)^2\right]\frac{\quad}{n!}\right.$$

$$= \sqrt{\frac{\pi}{2\omega}}e^{-\omega}\left\{1+\sum_{n=1}^{\infty}\frac{\left[4\lambda^2-1^2\right]\left[4\lambda^2-3^2\right]\left[4\lambda^2-5^2\right]...\left[4\lambda^2-(2n-1)^2\right]}{n!4^n(2\omega)^n}\right\}$$

$$= \sqrt{\frac{\pi}{2\omega}}e^{-\omega}\left\{1+\sum_{n=1}^{\infty}\frac{\left[4\lambda^2-1^2\right]\left[4\lambda^2-3^2\right]\left[4\lambda^2-5^2\right]...\left[4\lambda^2-(2n-1)^2\right]}{n!(8\omega)^n}\right\}$$

Therefore

$$\lim_{\omega\longrightarrow\infty}K_\lambda(\omega) = \sqrt{\frac{\pi}{2\omega}}e^{-\omega}[1+0]$$

$$= \sqrt{\frac{\pi}{2\omega}}e^{-\omega}$$

2.4 Generalized Inverse Gaussian (GIG) Distributions

2.4.1 Different Parameterizations

From definition (1), that is,

$$K_\lambda(\omega) = \frac{1}{2}\int_0^\infty y^{\lambda-1} e^{-\frac{\omega}{2}\left(y+\frac{1}{y}\right)} dy$$

we shall consider various forms of ω in terms of other parameters. The appropiate transformations of y will be used to obtain GIG distributions.

We shall study the following cases:

Case I:

$$\omega = \frac{\eta}{\xi} \approxeq \frac{\mu}{\beta} \tag{2.16}$$

(Rolski et al; 1999; Willmot, 1993)

Case II:

$$\omega = \sqrt{\delta\gamma} \approxeq \sqrt{x\psi} \tag{2.17}$$

(Sichel, 1975; Jorgensen, 1982; Barndoff - Nielsen, 1977; Gupta and Ong, 2005)

Case III:

$$\omega = \delta\gamma \tag{2.18}$$

(Barndoff - Nielsen, 1978)

Case IV:

$$\omega = 2\delta\theta^{\frac{1}{2}} \tag{2.19}$$

(Allen 1992; Hougaard 1997)

We notice that, for whatever form of ω, the integral part of the Bessel function of the third kind will be

$$
\begin{aligned}
& \int_0^{\infty} x^{\theta-1} \exp\left(-\frac{a}{x} - bx\right) dx \\
= & \int_0^{\infty} x^{\theta-1} \exp\left(-bx - \frac{a}{x}\right) dx \\
= & \int_0^{\infty} x^{\theta-1} \exp\left[-b\left(x + \frac{a}{b} \cdot \frac{1}{x}\right)\right] dx
\end{aligned}
$$

Let

$$x = \sqrt{\frac{a}{b}} z \Longrightarrow dx = \sqrt{\frac{a}{b}} dz$$

Thus

$$
\begin{aligned}
& \int_0^{\infty} x^{\theta-1} \exp\left(-bx - \frac{a}{x}\right) dx \\
= & \int_0^{\infty} \left(\sqrt{\frac{a}{b}} z\right)^{\theta-1} \exp\left[-b\left(\sqrt{\frac{a}{b}} z + \frac{a}{b} \cdot \sqrt{\frac{b}{a}} z^{-1}\right)\right] \sqrt{\frac{a}{b}} dz \\
= & \left(\sqrt{\frac{a}{b}}\right)^{\theta} \int_0^{\infty} z^{\theta-1} \exp\left[-b\sqrt{\frac{a}{b}} \left(z + z^{-1}\right)\right] dz \\
= & \left(\sqrt{\frac{a}{b}}\right)^{\theta} \int_0^{\infty} z^{\theta-1} \exp\left[-\frac{2\sqrt{ab}}{2} \left(z + z^{-1}\right)\right] dz
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\int_0^{\infty} x^{\theta-1} \exp\left(-bx - \frac{a}{x}\right) dx &= z\left(\sqrt{\frac{a}{b}}\right)^{\theta} K_{\theta}\left(2\sqrt{ab}\right) \\
&= 2\left(\frac{a}{b}\right)^{\frac{\theta}{2}} K_{\theta}\left(2\sqrt{ab}\right) \qquad (*)
\end{aligned}
$$

for $a, b > 0$ as given by Rolski et al. (1999 p.371).

This relation will be useful in determining Laplace Transform of a GIGD with different parameterizations. It will also be useful in the construction of Sichel Distributions with their Laplace Transforms.

2.4.2 Parameterization 1

Construction

Let us use notations given by Rolski et al (1999)

$$K_\theta(\omega) = \frac{1}{2}\int_0^\infty x^{\theta-1}\exp\left\{-\frac{\omega}{2}\left(x+x^{-1}\right)\right\}dx$$

Put

$$\omega = \frac{\eta}{\xi}$$

Then

$$K_\theta\left(\frac{\eta}{\xi}\right) = \frac{1}{2}\int_0^\infty x^{\theta-1}\exp\left\{-\frac{\eta}{2\xi}\left(x+x^{-1}\right)\right\}dx$$

Let

$$x = \eta^{-1}z \Longrightarrow dx = \eta^{-1}dz$$

Therefore

$$\begin{aligned} K_\theta\left(\frac{\eta}{\xi}\right) &= \frac{1}{2}\int_0^\infty \left(\eta^{-1}z\right)^{\theta-1}\exp\left\{-\frac{\eta}{2\xi}\left(\eta^{-1}z+\left(\eta^{-1}z\right)^{-1}\right)\right\}\eta^{-1}dz \\ &= \frac{1}{2}\int_0^\infty \eta^{-\theta}z^{\theta-1}\exp\left\{-\frac{(z^2+\eta^2)}{2\xi z}\right\}dz \end{aligned}$$

Therefore

$$1 = \int_0^\infty \frac{\eta^{-\theta}z^{\theta-1}}{2K_\theta\left(\frac{\eta}{\xi}\right)}\exp\left\{-\frac{(z^2+\eta^2)}{2\xi z}\right\}dz$$

Therefore a Generalized Inverse Gaussian pdf is given by

$$f(x) = \frac{\eta^{-\theta} x^{\theta-1}}{2K_\theta\left(\frac{\eta}{\xi}\right)} \exp\left\{-\frac{(x^2+\eta^2)}{2\xi x}\right\}, x > 0; \eta, \xi > 0, \theta \in \mathbb{R} \tag{2.20}$$

The rth Moment

$$\begin{aligned} E(X^r) &= \int_0^\infty x^r \frac{\eta^{-\theta} x^{\theta-1}}{2K_\theta\left(\frac{\eta}{\xi}\right)} \exp\left\{-\frac{(x^2+\eta^2)}{2\xi x}\right\} dx \\ &= \frac{\eta^{-\theta}}{2K_\theta\left(\frac{\eta}{\xi}\right)} \int_0^\infty x^{\theta+r-1} \exp\left\{-\frac{(x^2+\eta^2)}{2\xi x}\right\} dx \\ &= \frac{\eta^{-\theta}}{2K_\theta\left(\frac{\eta}{\xi}\right)} \frac{2K_{\theta+r}\left(\frac{\eta}{\xi}\right)}{\eta^{-(\theta+r)}} \int_0^\infty \frac{\eta^{-(\theta+r)}}{2K_{\theta+r}\left(\frac{\eta}{\xi}\right)} \exp\left\{-\frac{(x^2+\eta^2)}{2\xi x}\right\} dx \\ &= \frac{2\eta^{-\theta} K_{\theta+r}\left(\frac{\eta}{\xi}\right)}{2K_\theta\left(\frac{\eta}{\xi}\right)\eta^{-(\theta+r)}} \cdot 1 \end{aligned}$$

Therefore

$$E(X^r) = \eta^r \frac{K_{\theta+r}\left(\frac{\eta}{\xi}\right)}{K_\theta\left(\frac{\eta}{\xi}\right)} \tag{2.21}$$

Laplace Transform

$$\begin{aligned}
L(s) &= \int_0^\infty e^{-sx} \frac{\eta^{-\theta} x^{\theta-1}}{2K_\theta\left(\frac{\eta}{\xi}\right)} e^{-\frac{\left(x^2+\eta^2\right)}{2\xi x}} dx \\
&= \frac{\eta^{-\theta}}{2K_\theta\left(\frac{\eta}{\xi}\right)} \int_0^\infty x^{\theta-1} e^{-sx-\frac{\left(x^2+\eta^2\right)}{2\xi x}} dx \\
&= \frac{\eta^{-\theta}}{2K_\theta\left(\frac{\eta}{\xi}\right)} \int_0^\infty x^{\theta-1} e^{-sx-\frac{x}{2\xi}-\frac{\eta^2}{2\xi x}} dx \\
&= \frac{\eta^{-\theta}}{2K_\theta\left(\frac{\eta}{\xi}\right)} \int_0^\infty x^{\theta-1} e^{-\left(\frac{1}{2\xi}+s\right)x-\frac{\eta^2}{2\xi}\cdot\frac{1}{x}} dx
\end{aligned}$$

Put

$$a = \frac{\eta^2}{2\xi} \text{ and } b = \frac{1}{2\xi} + s$$

and apply relation $(*)$. Therefore

$$\begin{aligned}
L(s) &= \frac{\eta^{-\theta}}{2K_\theta\left(\frac{\eta}{\xi}\right)} \left\{ 2\left(\frac{\eta^2}{2\xi\left(\frac{1}{2\xi}+s\right)}\right)^{\frac{\theta}{2}} K_\theta\left(2\sqrt{\frac{\eta^2}{2\xi}\left(\frac{1}{2\xi}+s\right)}\right)\right\} \\
&= \frac{1}{K_\theta\left(\frac{\eta}{\xi}\right)} \frac{1}{(1+2\xi s)^{\frac{\theta}{2}}} K_\theta\left(2\sqrt{\frac{\eta^2}{2\xi}\left(\frac{1}{2\xi}+s\right)}\right) \\
&= (1+2\xi s)^{-\frac{\theta}{2}} \frac{K_\theta\left(\frac{\eta}{\xi}\sqrt{1+2\xi s}\right)}{K_\theta\left(\frac{\eta}{\xi}\right)}
\end{aligned}$$

According to Willmot's notations,

$$L(s) = (1+2\beta s)^{-\frac{\alpha}{2}} \frac{K_\alpha\left\{\mu\beta^{-1}(1+2\beta s)^{\frac{1}{2}}\right\}}{K_\alpha\left(\mu\beta^{-1}\right)} \qquad (2.22)$$

Special Cases

Inverse Gaussian Distribution Put

$$\theta = -\frac{1}{2}$$

Then the GIGD becomes

$$f(x) = \frac{\eta^{\frac{1}{2}} x^{-\frac{3}{2}}}{2K_{-\frac{1}{2}}\left(\frac{\eta}{\xi}\right)} \exp\left\{-\left(\frac{x^2+\eta^2}{2\xi x}\right)\right\}$$

By property 1,

$$K_{-\frac{1}{2}}\left(\frac{\eta}{\xi}\right) = K_{\frac{1}{2}}\left(\frac{\eta}{\xi}\right)$$

And by Corollary 2 of Property 2,

$$K_{\frac{1}{2}}\left(\frac{\eta}{\xi}\right) = \sqrt{\frac{\pi\xi}{2\eta}} e^{-\frac{\eta}{\xi}}$$

Therefore

$$\begin{aligned} f(x) &= \frac{\eta^{\frac{1}{2}} x^{-\frac{3}{2}}}{2\sqrt{\frac{\pi\xi}{2\eta}} e^{-\frac{\eta}{\xi}}} \exp\left\{-\frac{x^2+\eta^2}{2\xi x}\right\} \\ &= \eta\left(\frac{1}{2\pi\xi x^3}\right)^{\frac{1}{2}} \exp\left\{-\frac{x^2+\eta^2}{2\xi x} + \frac{\eta}{\xi}\right\} \\ &= \eta\left(\frac{1}{2\pi\xi x^3}\right)^{\frac{1}{2}} \exp\left\{-\frac{(x^2+\eta^2-2\eta x)}{2\xi x}\right\} \\ &= \eta\left(\frac{1}{2\pi\xi x^3}\right)^{\frac{1}{2}} \exp\left\{-\frac{(x-\eta)^2}{2\xi x}\right\} \\ &= \left(\frac{\eta^2}{2\pi\xi x^3}\right)^{\frac{1}{2}} \exp\left\{-\frac{\eta^2 (x-\eta)^2}{2\xi\eta^2 x}\right\} \end{aligned}$$

Therefore

$$f\left(x\right)=\left(\frac{\frac{\eta^2}{\xi}}{2\pi x^3}\right)^{\frac{1}{2}}\exp\left\{-\frac{\eta^2}{\xi}\left(\frac{\left(x-\eta\right)^2}{2\eta^2 x}\right)\right\}$$

$$E\left(X^r\right)=\eta^r\frac{K_{r-\frac{1}{2}}\left(\frac{\eta}{\xi}\right)}{K_{-\frac{1}{2}}\left(\frac{\eta}{\xi}\right)}$$

Using Corollary 3 of Property 2,

$$K_{r-\frac{1}{2}}\left(\frac{\eta}{\xi}\right)=\sqrt{\frac{\pi}{2\frac{\eta}{\xi}}}e^{-\frac{\eta}{\xi}}\left\{1+\sum_{i=1}^{r-1}\frac{(r+i-1)!}{(r-i-1)!}\frac{\left(\frac{2\eta}{\xi}\right)^{-i}}{i!}\right\}$$

Using Corollary 2 of Property 2,

$$K_{-\frac{1}{2}}\left(\frac{\eta}{\xi}\right)=K_{\frac{1}{2}}\left(\frac{\eta}{\xi}\right)=\sqrt{\frac{\pi}{\frac{2\eta}{\xi}}}e^{-\frac{\eta}{\xi}}$$

Therefore

$$E\left(X^r\right)=\eta^r\left\{1+\sum_{i=1}^{r-1}\frac{(r+i-1)!}{(r-i-1)!}\frac{\left(\frac{2\eta}{\xi}\right)^{-i}}{i!}\right\}$$

Therefore

$$E\left(X\right)=\eta$$

$$\begin{aligned}
E\left(X^2\right) &= \eta^2\left\{1+\sum_{i=1}^{r-1}\frac{(2+1-1)!}{(2-1-1)!}\frac{\left(\frac{2\eta}{\xi}\right)^{-1}}{1!}\right\} \\
&= \eta^2\left\{1+\frac{2!}{0!}\frac{\left(\frac{2\eta}{\xi}\right)^{-1}}{1!}\right\} \\
&= \eta^2\left\{1+\frac{2}{\frac{2\eta}{\xi}}\right\} \\
&= \eta^2+\left(1+\frac{\xi}{\eta}\right)
\end{aligned}$$

Therefore

$$VarX = E\left(X^2\right)-\left[E\left(X\right)\right]^2 = \left(1+\frac{\xi}{\eta}\right)$$

$$\begin{aligned}
L\left(s\right) &= \left(\sqrt{(1+2\beta s)}\right)^{\frac{1}{2}}\frac{K_{-\frac{1}{2}}\left\{\mu\beta^{-1}\left(1+2\beta s\right)^{\frac{1}{2}}\right\}}{K_{-\frac{1}{2}}\left\{\mu\beta^{-1}\right\}} \\
&= \left(\sqrt{(1+2\beta s)}\right)^{\frac{1}{2}}\frac{\left(\frac{\pi}{2\mu\beta^{-1}\sqrt{1+2\beta s}}\right)^{\frac{1}{2}}e^{-\mu\beta^{-1}\sqrt{1+2\beta s}}}{\left(\frac{\pi}{2\mu\beta^{-1}}\right)^{\frac{1}{2}}e^{-\mu\beta^{-1}}} \\
&= \left(\frac{\pi\sqrt{(1+2\beta s)}}{2\mu\beta^{-1}\sqrt{1+2\beta s}}\cdot\frac{2\mu\beta^{-1}}{\pi}\right)^{\frac{1}{2}}e^{\mu\beta^{-1}-\mu\beta^{-1}\sqrt{1+2\beta s}} \\
&= \exp\left\{\mu\beta^{-1}\left[1-\sqrt{1+2\beta s}\right]\right\} \\
&= \exp\left\{-\frac{\mu}{\beta}\left[\left(1+2\beta s\right)^{\frac{1}{2}}-1\right]\right\}
\end{aligned}$$

Reciprocal Inverse Gaussian Distribution put;

$$\theta = \frac{1}{2}$$

Then the GIGD becomes

$$\begin{aligned}
f(x) &= \frac{\eta^{-\frac{1}{2}} x^{-\frac{1}{2}}}{2K_{\frac{1}{2}}\left(\frac{\eta}{\xi}\right)} \exp\left\{-\left(\frac{x^2+\eta^2}{2\xi x}\right)\right\} \ , x > 0 \\
&= \frac{1}{2(\eta x)^{\frac{1}{2}}} \frac{1}{\sqrt{\frac{\xi\pi}{2\eta}}} \frac{\exp\left\{-\frac{x^2+\eta^2}{2\xi x}\right\}}{e^{-\frac{\eta}{\xi}}} \\
&= \left(\frac{1}{2\pi\xi x}\right)^{\frac{1}{2}} \exp\left\{-\frac{x^2+\eta^2}{2\xi x} + \frac{\eta}{\xi}\right\} \\
&= \left(\frac{1}{2\pi\xi x}\right)^{\frac{1}{2}} \exp\left\{-\frac{x^2+\eta^2}{2\xi x} + \frac{\eta}{\xi}\right\} \\
&= \left(\frac{1}{2\pi\xi x}\right)^{\frac{1}{2}} \exp\left\{-\frac{(x-\eta)^2}{2\xi x}\right\} \\
&= \left(\frac{\frac{1}{\xi}}{2\pi x}\right)^{\frac{1}{2}} \exp\left\{-\frac{(\eta-x)^2}{2\xi x}\right\} \\
&= \left(\frac{\frac{1}{\xi}}{2\pi x}\right)^{\frac{1}{2}} \exp\left\{-\frac{\eta^2\left(1-\frac{x}{\eta}\right)^2}{2\xi x}\right\} \\
&= \left(\frac{\xi^{-1}}{2\pi x}\right)^{\frac{1}{2}} \exp\left\{-\frac{\eta^2\left(1-\frac{x}{\eta}\right)^2}{2\xi x}\right\} \\
&= \left(\frac{\xi^{-1}}{2\pi x}\right)^{\frac{1}{2}} \exp\left\{\frac{-\eta^2}{2\xi x}\left(1-\frac{x}{\eta}\right)^2\right\}
\end{aligned}$$

according to Willmot's notation,

$$\xi^{-1} = \theta \ and \ \eta = \mu$$

therefore,

$$f\left(x\right) = \left(\frac{\theta}{2\pi x}\right)^{\frac{1}{2}} \exp\left\{\frac{-\theta\mu^2}{2x}\left(1 - \frac{x}{\mu}\right)^2\right\}, \ x > 0 \tag{2.23}$$

$$\begin{aligned} E(X^r) &= \frac{\eta^r K_{r+\frac{1}{2}}\left(\frac{\eta}{\xi}\right)}{K_{\frac{1}{2}}\left(\frac{\eta}{\xi}\right)} \equiv \frac{\mu^r K_{r+\frac{1}{2}}\left(\mu\beta^{-1}\right)}{K_{\frac{1}{2}}\left(\mu\beta^{-1}\right)} \\ &= \frac{\mu^r\sqrt{\frac{\pi}{2\mu\beta^{-1}}}e^{-\mu\beta^{-1}}\left\{1 + \sum_{i=1}^{r}\frac{(r+i)!(2\omega)^{-i}}{(r-i)!i!}\right\}}{\sqrt{\frac{\pi}{2\mu\beta^{-1}}}e^{-\mu\beta^{-1}}} \\ &= \mu^r\left\{1 + \sum_{i=1}^{r}\frac{(r+i)!\left(2\omega\right)^{-i}}{(r-i)!i!}\right\} \end{aligned}$$

Therefore

$$\begin{aligned} E\left(X\right) &= \mu\left\{1 + \sum_{i=1}^{1}\frac{(1+i)!\left(2\omega\right)^{-i}}{(1-i)!i!}\right\} \\ &= \mu\left\{1 + \frac{2!\left(2\omega\right)^{-1}}{0!1!}\right\} \\ &= \mu\left\{1 + \frac{1}{\omega}\right\} \end{aligned}$$

and

$$
\begin{aligned}
E\left(X^2\right) &= \mu^2\left\{1+\sum_{i=1}^{2}\frac{(2+i)!\,(2\omega)^{-i}}{(2-i)!i!}\right\} \\
&= \mu^2\left\{1+\frac{3!\,(2\omega)^{-1}}{1!1!}+\frac{4!\,(2\omega)^{-2}}{0!2!}\right\} \\
&= \mu^2\left\{1+\frac{6}{2\omega}+\frac{24}{2}\frac{1}{4\omega^2}\right\} \\
&= \mu^2\left\{1+\frac{3}{\omega}+\frac{3}{\omega^2}\right\}
\end{aligned}
$$

Therefore;

$$
\begin{aligned}
var\left(X\right) &= \mu^2\left\{1+\frac{3}{\omega}+\frac{3}{\omega^2}\right\}-\left(\mu\left\{1+\frac{1}{\omega}\right\}\right)^2 \\
&= \mu^2\left\{1+\frac{3}{\omega}+\frac{3}{\omega^2}\right\}-\mu^2\left\{1+\frac{1}{\omega}\right\}^2 \\
&= \mu^2\left\{1+\frac{3}{\omega}+\frac{3}{\omega^2}-1-\frac{2}{\omega}-\frac{1}{\omega^2}\right\} \\
&= \mu^2\left\{\frac{1}{\omega}+\frac{2}{\omega^2}\right\} \\
&= \mu^2\left\{\frac{1}{\mu\beta^{-1}}+\frac{2}{\left(\mu\beta^{-1}\right)^2}\right\} \\
&= \mu^2\left\{\frac{1}{\mu\beta^{-1}}+\frac{2}{\mu^2\beta^{-2}}\right\} \\
&= \mu^2\left\{\frac{\beta}{\mu}+\frac{2\beta^2}{\mu^2}\right\} \\
&= \mu\beta+2\beta^2
\end{aligned}
$$

The laplace transform is:

$$\begin{aligned}
\mathcal{L}(s) &= \left(\sqrt{(1+2\beta s)}\right)^{-\frac{1}{2}} \frac{K_{\frac{1}{2}}\left\{\mu\beta^{-1}\sqrt{(1+2\beta s)}\right\}}{K_{\frac{1}{2}}\left\{\mu\beta^{-1}\right\}} \\
&= \left(\sqrt{(1+2\beta s)}\right)^{-\frac{1}{2}} \frac{\left(\frac{\pi}{2\mu\beta^{-1}\sqrt{(1+2\beta s)}}\right)^{\frac{1}{2}} e^{-\mu\beta^{-1}\sqrt{(1+2\beta s)}}}{\left(\frac{\pi}{2\mu\beta^{-1}}\right)^{\frac{1}{2}} e^{-\mu\beta^{-1}}} \\
&= \left(\sqrt{(1+2\beta s)}\right)^{-\frac{1}{2}} \left(\frac{\pi 2\mu\beta^{-1}}{2\mu\beta^{-1}\sqrt{(1+2\beta s)}\pi}\right)^{\frac{1}{2}} \frac{e^{-\mu\beta^{-1}\sqrt{(1+2\beta s)}}}{e^{-\mu\beta^{-1}}} \\
&= \frac{1}{\sqrt{(1+2\beta s)}} e^{-\mu\beta^{-1}\sqrt{(1+2\beta s)}+\mu\beta^{-1}} \\
&= \left(\frac{1}{1+2\beta s}\right)^{\frac{1}{2}} \exp\left\{-\mu\beta^{-1}\left[\sqrt{1+2\beta s}-1\right]\right\} \\
&= \left(\frac{1}{1+\frac{2s}{\theta}}\right)^{\frac{1}{2}} \exp\left\{-\mu\theta\left[\left(1+\frac{2s}{\theta}\right)^{\frac{1}{2}}-1\right]\right\} \\
&= \left(1+\frac{2s}{\theta}\right)^{-\frac{1}{2}} \exp\left\{\mu\theta\left[1-\left(1+\frac{2s}{\theta}\right)^{\frac{1}{2}}\right]\right\}
\end{aligned}$$

Therefore;

$$\mathcal{L}(s) = \left(1+\frac{2s}{\theta}\right)^{-\frac{1}{2}} \exp\left\{\mu\theta\left[1-\left(1+\frac{2s}{\theta}\right)^{\frac{1}{2}}\right]\right\} \tag{2.25}$$

2.4.3 Parameterization 2

construction of GIGD

Let $\omega = \sqrt{\delta\gamma}$ in (1). Then , we have

$$K_\lambda\left(\sqrt{\delta\gamma}\right) = \frac{1}{2}\int_0^\infty y^{\lambda-1} \exp\left\{\frac{-\sqrt{\delta\gamma}}{2}\left(y+\frac{1}{y}\right)\right\} dy$$

If $y = \sqrt{\frac{\gamma}{\delta}} z$, then $dy = \sqrt{\frac{\gamma}{\delta}} dz$
therefore,

$$
\begin{aligned}
K_\lambda\left(\sqrt{\delta\gamma}\right) &= \frac{1}{2}\int_0^\infty \left(\sqrt{\frac{\gamma}{\delta}} z\right)^{\lambda-1} \exp\left\{\frac{-\sqrt{\delta\gamma}}{2}\left(\sqrt{\frac{\gamma}{\delta}} z + \sqrt{\frac{\delta}{\gamma}} z^{-1}\right)\right\} \sqrt{\frac{\gamma}{\delta}} dz \\
&= \frac{1}{2}\left(\sqrt{\frac{\gamma}{\delta}}\right)^\lambda \int_0^\infty z^{\lambda-1} \exp\left\{-\frac{1}{2}\left(\gamma z + \delta z^{-1}\right)\right\} dz
\end{aligned}
$$

normalizing, we get

$$
\begin{aligned}
1 &= \frac{\left(\sqrt{\frac{\gamma}{\delta}}\right)^\lambda}{2K_\lambda\left(\sqrt{\delta\gamma}\right)} \int_0^\infty z^{\lambda-1} \exp\left\{-\frac{1}{2}\left(\gamma z + \delta z^{-1}\right)\right\} dz \\
&= \int_0^\infty \frac{\left(\sqrt{\frac{\gamma}{\delta}}\right)^\lambda}{2K_\lambda\left(\sqrt{\delta\gamma}\right)} z^{\lambda-1} \exp\left\{-\frac{1}{2}\left(\gamma z + \delta z^{-1}\right)\right\} dz
\end{aligned}
$$

Therefore;

$$
f(x) = \frac{\left(\sqrt{\frac{\gamma}{\delta}}\right)^\lambda}{2K_\lambda\left(\sqrt{\delta\gamma}\right)} x^{\lambda-1} \exp\left\{-\frac{1}{2}\left(\gamma x + \frac{\delta}{x}\right)\right\}, \ x > 0 \qquad (2.26)
$$

is a GIG pdf.
Since $\left(\sqrt{\frac{\gamma}{\delta}}\right)^\lambda > 0$, we have the following parameter space:

$$
\begin{aligned}
&\delta \geq 0, \ \gamma > 0, \text{ if } \gamma > 0 \\
&\delta > 0, \ \gamma > 0, \text{ if } \gamma = 0 \\
&\delta > 0, \ \gamma \geq 0, \text{ if } \gamma < 0
\end{aligned}
$$

Introducing the parameters

$$
\omega = \sqrt{\delta\gamma} \text{ and } \eta = \frac{\delta}{\gamma}
$$

then

$$\frac{\omega}{\eta} = \gamma \text{ and } \omega\eta = \delta \ \Rightarrow \ \frac{\gamma}{\delta} = \frac{1}{\eta^2}$$

then (1) becomes

$$\begin{aligned} f(x) &= \frac{\eta^{-\lambda}}{2K_\lambda(\omega)} x^{\lambda-1} \exp\left\{-\frac{1}{2}\left[\frac{\omega}{\eta}x + \frac{\omega\eta}{x}\right]\right\} \\ &= \frac{\eta^{-\lambda}}{2K_\lambda(\omega)} x^{\lambda-1} \exp\left\{-\frac{\omega}{2}\left[\frac{1}{\eta}x + \frac{\eta}{x}\right]\right\}, \ x > 0 \end{aligned} \tag{2.27}$$

for a fixed λ, ω is a concentration parameter and η is a concentration parameter and η is a scale parameter.

Remark: The distribution of the inverse of a GIG variable is again GIG but with a different $-\lambda$.

Proof:

Let Y=$\frac{1}{X}$, where X˜GIG$(\lambda, \delta, \gamma)$.

Then x=$\frac{1}{y}$ $\Rightarrow$ $\frac{dx}{dy}$=-$\frac{1}{y^2}$ $\Rightarrow |J| = \frac{1}{y^2}$

$$\begin{aligned} g(y) &= f(x)|J| \\ &= \frac{\left(\sqrt{\frac{\gamma}{\delta}}\right)^\lambda}{2K_\lambda\left(\sqrt{\delta\gamma}\right)} x^{\lambda-1}|J| \exp\left\{-\frac{1}{2}\left(\gamma x + \frac{\delta}{x}\right)\right\} \\ &= \frac{\left(\sqrt{\frac{\gamma}{\delta}}\right)^\lambda}{2K_\lambda\left(\sqrt{\delta\gamma}\right)} \left(\frac{1}{y}\right)^{\lambda-1}\left(\frac{1}{y^2}\right) \exp\left\{-\frac{1}{2}\left(\frac{\gamma}{y} + \delta y\right)\right\} \\ &= \frac{\left(\sqrt{\frac{\gamma}{\delta}}\right)^\lambda}{2K_\lambda\left(\sqrt{\delta\gamma}\right)} y^{-\lambda-1} \exp\left\{-\frac{1}{2}\left(\gamma y^{-1} + \delta y\right)\right\} \\ &= \frac{\left(\frac{\gamma}{\delta}\right)^{\frac{\lambda}{2}}}{2K_\lambda\left(\sqrt{\delta\gamma}\right)} y^{-\lambda-1} \exp\left\{-\frac{1}{2}\left(\gamma y^{-1} + \delta y\right)\right\} \end{aligned}$$

ie

$$Y = \frac{1}{X}\tilde{\ } GIG(-\lambda, \delta, \gamma).$$

Modality

$$\frac{df}{dx} = 0 \text{ implies}$$

$$\frac{d}{dx}\left(x^{\lambda-1}e^{-\frac{1}{2}\left(\gamma x+\frac{\delta}{x}\right)}\right) = 0$$

ie,

$$(\lambda-1)\,x^{\lambda-2}e^{-\frac{1}{2}\left(\gamma x+\frac{\delta}{x}\right)} + x^{\lambda-1}e^{-\frac{1}{2}\left(\gamma x+\frac{\delta}{x}\right)}\frac{d}{dx}\left[-\frac{1}{2}\left(\gamma x+\frac{\delta}{x}\right)\right] = 0$$

therefore

$$\begin{aligned}
(\lambda-1) + x\left[-\frac{\gamma}{2}+\frac{\delta}{2x^2}\right] &= 0 \\
(\lambda-1) - x\frac{\gamma}{2}+\frac{\delta}{2x} &= 0 \\
2\,(\lambda-1)\,x - \gamma x^2 + \delta &= 0 \\
\gamma x^2 - 2\,(\lambda-1)\,x - \delta &= 0
\end{aligned}$$

Case(i) :

$$\gamma = 0 \;\Rightarrow\; -2\,(\lambda-1)\,x - \delta = 0 \;\Rightarrow\; x = \frac{\delta}{2\,(1-\lambda)}$$

Case (ii):

$$\gamma > 0 \;\Rightarrow\; x = \frac{2\,(\lambda-1) \pm \sqrt{4\,(\lambda-1)^2 + 4\gamma\delta}}{2\gamma}$$

$$x = \frac{(\lambda-1) \pm \sqrt{(\lambda-1)^2 + \gamma\delta}}{\gamma}$$

since $x > 0$,

$$x = \frac{(\lambda-1)+\sqrt{(\lambda-1)^2+\gamma\delta}}{\gamma}$$

Thus the mode is

$$x = \left\{ \begin{array}{c} \frac{(\lambda-1)+\sqrt{(\lambda-1)^2+\gamma\delta}}{\gamma},\ \gamma>0 \\ \frac{\delta}{2(1-\lambda)},\ \gamma=0 \end{array} \right\}$$

The r-th moment

Using (1),

$$\begin{aligned}
E\left(X^r\right) &= \int_0^\infty x^r \frac{\left(\frac{\gamma}{\delta}\right)^{\frac{\lambda}{2}}}{2K_\lambda\left(\sqrt{\delta\gamma}\right)} x^{\lambda-1} \exp\left\{-\frac{1}{2}\left(\gamma x+\frac{\delta}{x}\right)\right\} dx \\
&= \frac{\left(\frac{\gamma}{\delta}\right)^{\frac{\lambda}{2}}}{2K_\lambda\left(\sqrt{\delta\gamma}\right)} \int_0^\infty x^{r+\lambda-1} \exp\left\{-\frac{1}{2}\left(\gamma x+\frac{\delta}{x}\right)\right\} dx \\
&= \frac{\left(\frac{\gamma}{\delta}\right)^{\frac{\lambda}{2}} 2K_{r+\lambda}\left(\sqrt{\delta\gamma}\right)}{2K_\lambda\left(\sqrt{\delta\gamma}\right)\left(\frac{\gamma}{\delta}\right)^{\frac{\lambda+r}{2}}} \int_0^\infty \frac{\left(\frac{\gamma}{\delta}\right)^{\frac{\lambda+r}{2}}}{2K_{r+\lambda}\left(\sqrt{\delta\gamma}\right)} x^{\gamma+\lambda-1} \exp\left\{-\frac{1}{2}\left(\gamma x+\frac{\delta}{x}\right)\right\} dx \\
&= \left(\frac{\gamma}{\delta}\right)^{-\frac{r}{2}} \frac{K_{r+\lambda}\left(\sqrt{\delta\gamma}\right)}{K_\lambda\left(\sqrt{\delta\gamma}\right)} * 1 \\
&= \left(\sqrt{\frac{\delta}{\gamma}}\right)^{\gamma} \frac{K_{r+\lambda}\left(\sqrt{\delta\gamma}\right)}{K_\lambda\left(\sqrt{\delta\gamma}\right)}
\end{aligned}$$

Using(2),

$$
\begin{aligned}
E\left(X^r\right) &= \int_0^\infty x^r \frac{\eta^{-\lambda}}{2K_\lambda(\omega)} x^{\lambda-1} \exp\left\{-\frac{\omega}{2}\left[\frac{1}{\eta}x+\frac{\eta}{x}\right]\right\} dx \\
&= \int_0^\infty x^r \frac{\eta^{-\lambda}}{2K_\lambda(\omega)} x^{\lambda-1} \exp\left\{-\frac{\omega}{2}\left[\eta^{-1}x+\eta x^{-1}\right]\right\} dx \\
&= \frac{\eta^{-\lambda}}{2K_\lambda(\omega)} \int_0^\infty x^{r+\lambda-1} \exp\left\{-\frac{\omega}{2}\left[\eta^{-1}x+\eta x^{-1}\right]\right\} dx \\
&= \frac{\eta^{-\lambda} 2K_{r+\lambda}(\omega)}{2K_\lambda(\omega)\eta^{-(r+\lambda)}} \int_0^\infty \frac{\eta^{-(r+\lambda)}}{2K_{r+\lambda}(\omega)} x^{r+\lambda-1} \exp\left\{-\frac{\omega}{2}\left[\eta^{-1}x+\eta x^{-1}\right]\right\} dx \\
&= \eta^r \frac{K_{r+\lambda}(\omega)}{K_\lambda(\omega)}, \ r \in \mathbb{R}
\end{aligned}
$$

Laplace transform

$$
\begin{aligned}
\mathcal{L}_x(s) &= \int_0^\infty e^{-sx} \frac{\left(\frac{\gamma}{\delta}\right)^{\frac{\lambda}{2}}}{2K_\lambda\left(\sqrt{\delta\gamma}\right)} x^{\lambda-1} \exp\left\{-\frac{1}{2}\left(\gamma x+\frac{\delta}{x}\right)\right\} dx \\
&= \frac{\left(\frac{\gamma}{\delta}\right)^{\frac{\lambda}{2}}}{2K_\lambda\left(\sqrt{\delta\gamma}\right)} \int_0^\infty x^{\lambda-1} e^{-\frac{1}{2}\left(\gamma x+\frac{\delta}{x}\right)-sx} dx \\
&= \frac{\left(\frac{\gamma}{\delta}\right)^{\frac{\lambda}{2}}}{2K_\lambda\left(\sqrt{\delta\gamma}\right)} \int_0^\infty x^{\lambda-1} e^{-\frac{1}{2}\left(\gamma x+\frac{\delta}{x}+2sx\right)} dx \\
&= \frac{\left(\frac{\gamma}{\delta}\right)^{\frac{\lambda}{2}}}{2K_\lambda\left(\sqrt{\delta\gamma}\right)} \int_0^\infty x^{\lambda-1} e^{-\frac{1}{2}(\gamma+2s)x-\frac{\delta}{2}\frac{1}{x}} dx
\end{aligned}
$$

Put $a = \frac{\delta}{2}$ and $b = \frac{1}{2}(\gamma+2s) = \frac{\gamma+2s}{2}$
Using the relation $(*)$,

$$
\begin{aligned}
\mathcal{L}_x(s) &= \frac{\left(\frac{\gamma}{\delta}\right)^{\frac{\lambda}{2}}}{2K_\lambda\left(\sqrt{\delta\gamma}\right)}\left[2\left(\frac{\delta}{2\left(\frac{\gamma+2s}{2}\right)}\right)^{\frac{\lambda}{2}} K_\lambda\left(2\sqrt{\frac{\delta}{2}\left(\frac{\gamma+2s}{2}\right)}\right)\right] \\
&= \frac{\left(\frac{\gamma}{\delta}\right)^{\frac{\lambda}{2}}}{K_\lambda\left(\sqrt{\delta\gamma}\right)}\frac{\delta^{\frac{\lambda}{2}}}{(\gamma+2s)^{\frac{\lambda}{2}}}K_\lambda\left(\sqrt{\delta(\gamma+2s)}\right) \\
&= \left(\frac{\gamma}{\gamma+2s}\right)^{\frac{\lambda}{2}}\frac{K_\lambda\left(\sqrt{\delta(\gamma+2s)}\right)}{K_\lambda\left(\sqrt{\delta\gamma}\right)} \qquad (2.28)
\end{aligned}
$$

Using (2) the transform is:

$$
\begin{aligned}
\mathcal{L}_x(s) &= \int_0^\infty x^{-sx}\frac{\eta^{-\lambda}}{2K_\lambda(\omega)}x^{\lambda-1}\exp\left\{-\frac{\omega}{2}\left[\frac{1}{\eta}x+\frac{\eta}{x}\right]\right\}dx \\
&= \frac{\eta^{-\lambda}}{2K_\lambda(\omega)}\int_0^\infty x^{\lambda-1}e^{-\frac{1}{2}\left[\frac{\omega}{\eta}x+\frac{\omega\eta}{x}\right]-sx}dx \\
&= \frac{\eta^{-\lambda}}{2K_\lambda(\omega)}\int_0^\infty x^{\lambda-1}e^{-\left(\frac{\omega}{2\eta}+s\right)x-\frac{\omega\eta}{2}\frac{1}{x}}dx
\end{aligned}
$$

Put $a=\frac{\omega\eta}{2}$ and $b=\frac{\omega}{2\eta}+s$

$$
\begin{aligned}
\mathcal{L}_x(s) &= \frac{\eta^{-\lambda}}{2K_\lambda(\omega)}\left[2\left(\sqrt{\frac{\omega\eta}{2\left(\frac{\omega}{2\eta}+s\right)}}\right)^{\lambda} K_\lambda\left(2\sqrt{\frac{\omega\eta}{2}\left(\frac{\omega}{2\eta}+s\right)}\right)\right] \\
&= \frac{\eta^{-\lambda}}{K_\lambda(\omega)}\left(\frac{\omega\eta}{2\left(\frac{\omega}{2\eta}+s\right)}\right)^{\frac{\lambda}{2}} K_\lambda\left(2\sqrt{\frac{\omega\eta}{2}\frac{\omega}{2\eta}\left(1+\frac{2\eta}{\omega}s\right)}\right) \\
&= \frac{1}{\eta^{\lambda}K_\lambda(\omega)}\left(\frac{\omega\eta}{2\left(\frac{\omega}{2\eta}+s\right)}\right)^{\frac{\lambda}{2}} K_\lambda\left(\omega\sqrt{\left(1+\frac{2\eta}{\omega}s\right)}\right) \\
&= \frac{1}{\eta^{\lambda}}\frac{1}{K_\lambda(\omega)}\left(\frac{\eta^2}{\left(1+\frac{2\eta s}{\omega}\right)}\right)^{\frac{\lambda}{2}} K_\lambda\left(\omega\sqrt{\left(1+\frac{2\eta}{\omega}s\right)}\right) \\
&= \left(1+\frac{2\eta s}{\omega}\right)^{-\frac{\lambda}{2}} \frac{K_\lambda\left(\omega\sqrt{\left(1+\frac{2\eta}{\omega}s\right)}\right)}{K_\lambda(\omega)} \qquad (2.29)
\end{aligned}
$$

Special cases

2.4.4 Parametarization 3

contruction of GIGD

Put $\omega = \gamma\delta$; therefore,

$$K_\lambda(\gamma\delta) = \frac{1}{2}\int_0^\infty y^{\lambda-1}e^{-\frac{\gamma\delta}{2}\left(y+\frac{1}{y}\right)}dy$$

Let $y = \frac{\gamma}{\delta}z \Rightarrow dy = \frac{\gamma}{\delta}dz$

Therefore;

$$
\begin{aligned}
K_\lambda(\gamma\delta) &= \frac{1}{2}\int_0^\infty \left(\frac{\gamma}{\delta}\right)^{\lambda-1} z^{\lambda-1}\exp\left\{-\frac{\gamma\delta}{2}\left(\frac{\gamma}{\delta}z+\frac{\delta}{\gamma}\frac{1}{z}\right)\right\}\frac{\gamma}{\delta}dz \\
&= \int_0^\infty \frac{\left(\frac{\gamma}{\delta}\right)^\lambda}{2} z^{\lambda-1}\exp\left\{-\frac{1}{2}\left(\gamma^2 z+\frac{\delta^2}{z}\right)\right\}dz \\
1 &= \int_0^\infty \frac{\left(\frac{\gamma}{\delta}\right)^\lambda}{2K_\lambda(\gamma\delta)} z^{\lambda-1}\exp\left\{-\frac{1}{2}\left(\gamma^2 z+\frac{\delta^2}{z}\right)\right\}dz
\end{aligned}
$$

Therefore;

$$
f(x) = \frac{\left(\frac{\gamma}{\delta}\right)^\lambda}{2K_\lambda(\gamma\delta)} x^{\lambda-1}\exp\left\{-\frac{1}{2}\left(\gamma^2 x+\frac{\delta^2}{x}\right)\right\},\ x>0 \tag{2.30}
$$

is a GIG pdf of the third version.
Since $\left(\frac{\gamma}{\delta}\right)^\lambda > 0$, the parameter space is given by

- $\delta \geq 0,\ \gamma > 0$ if $\lambda > 0$
- $\delta > 0,\ \gamma > 0$ if $\lambda = 0$
- $\delta > 0,\ \gamma \geq 0$ if $\lambda < 0$

modality

$$
\frac{df}{dx} = 0 \text{ implies that}
$$

$$
\frac{d}{dx}x^{\lambda-1}\exp\left\{-\frac{1}{2}\left(\gamma^2 x+\frac{\delta^2}{x}\right)\right\} = 0
$$

$$
x^{\lambda-1}\left[\exp\left\{-\frac{1}{2}\left(\gamma^2 x+\frac{\delta^2}{x}\right)\right\}\right]\frac{d}{dx}\left[-\frac{1}{2}\left(\gamma^2 x+\frac{\delta^2}{x}\right)\right]
$$

$$
+(\lambda-1)x^{\lambda-2}\exp\left\{-\frac{1}{2}\left(\gamma^2 x+\frac{\delta^2}{x}\right)\right\} = 0
$$

$$x\frac{d}{dx}\left[-\frac{1}{2}\left(\gamma^2+\frac{\delta^2}{x}\right)\right]+(\lambda-1)=0$$

i.e,

$$x\left(-\frac{\gamma^2}{2}-\frac{\delta^2}{2x^2}\right)+(\lambda-1)=0$$

$$-\frac{\gamma^2}{2}x-\frac{\delta^2}{2x}+(\lambda-1)=0$$

$$-\gamma^2x^2-\delta^2+2(\lambda-1)x=0$$

$$\begin{aligned}\gamma^2x^2+\delta^2-2(\lambda-1)x &= 0\\ \gamma^2x^2-2(\lambda-1)x+\delta^2 &= 0\end{aligned}$$

case (i): when $\gamma^2>0$, then

$$x=\frac{2(\lambda-1)\pm\sqrt{4(\lambda-1)^2-4\gamma^2\delta^2}}{2\gamma^2}$$

since $x>0$, then

$$x=\frac{2(\lambda-1)+\sqrt{4(\lambda-1)^2-4\gamma^2\delta^2}}{2\gamma^2}$$

case (ii): when $\gamma^2=0$, then

$$\begin{aligned}-2(\lambda-1)x+\delta^2 &= 0\\ x &= \frac{\delta^2}{2(\lambda-1)}\end{aligned}$$

Thus the generalized inverse Gaussian distribution are unimodal with mode point given by:

$$\left\{\begin{matrix} \frac{2(\lambda-1)+\sqrt{4(\lambda-1)^2-4\gamma^2\delta^2}}{2\gamma^2} \text{ if } \gamma^2 > 0 \\ \frac{\delta^2}{2(\lambda-1)} \text{ if } \gamma^2 = 0 \end{matrix}\right\}$$

The r^{th} moment

$$\begin{aligned}
E\left(X^r\right) &= \int_0^\infty x^r \frac{\left(\frac{\gamma}{\delta}\right)^\lambda}{2K_\lambda\left(\gamma\delta\right)} x^{\lambda-1} \exp\left\{-\frac{1}{2}\left(\gamma^2 x + \frac{\delta^2}{x}\right)\right\} dx \\
&= \frac{\left(\frac{\gamma}{\delta}\right)^\lambda}{2K_\lambda\left(\gamma\delta\right)} \frac{2K_{\lambda+r}\left(\gamma\delta\right)}{\left(\frac{\gamma}{\delta}\right)^{\lambda+r}} \int_0^\infty \frac{\left(\frac{\gamma}{\delta}\right)^{\lambda+r}}{2K_{\lambda+r}\left(\gamma\delta\right)} x^{\lambda+r-1} \exp\left\{-\frac{1}{2}\left(\gamma^2 x + \frac{\delta^2}{x}\right)\right\} dx \\
&= \frac{\left(\frac{\gamma}{\delta}\right)^\lambda}{2K_\lambda\left(\gamma\delta\right)} \frac{2K_{\lambda+r}\left(\gamma\delta\right)}{\left(\frac{\gamma}{\delta}\right)^{\lambda+r}} * 1 \\
&= \frac{\left(\frac{\gamma}{\delta}\right)^\lambda}{2K_\lambda\left(\gamma\delta\right)} \frac{2K_{\lambda+r}\left(\gamma\delta\right)}{\left(\frac{\gamma}{\delta}\right)^{\lambda+r}} \\
&= \left(\frac{\delta}{\gamma}\right)^r \frac{K_{\lambda+r}\left(\gamma\delta\right)}{K_\lambda\left(\gamma\delta\right)}
\end{aligned}$$

Therefore:

$$E\left(X\right) = \left(\frac{\delta}{\gamma}\right) \frac{K_{\lambda+1}\left(\gamma\delta\right)}{K_\lambda\left(\gamma\delta\right)}$$

and

$$E\left(X^2\right) = \left(\frac{\delta}{\gamma}\right)^2 \frac{K_{\lambda+2}\left(\gamma\delta\right)}{K_\lambda\left(\gamma\delta\right)}$$

Therefore:

$$var(x) = \left(\frac{\delta}{\gamma}\right)^2 \left[\frac{K_{\lambda+2}\left(\gamma\delta\right)}{K_\lambda\left(\gamma\delta\right)} - \frac{K_{\lambda+1}^2\left(\gamma\delta\right)}{K_\lambda^2\left(\gamma\delta\right)}\right]$$

Laplace transform

$$\begin{aligned}
\mathcal{L}_x(s) &= \int_0^\infty e^{-sx} \frac{\left(\frac{\gamma}{\delta}\right)^\lambda}{2K_\lambda(\gamma\delta)} x^{\lambda-1} \exp\left\{-\frac{1}{2}\left(\gamma^2 x + \frac{\delta^2}{x}\right)\right\} dx \\
&= \frac{\left(\frac{\gamma}{\delta}\right)^\lambda}{2K_\lambda(\gamma\delta)} \int_0^\infty e^{-sx} x^{\lambda-1} \exp\left\{-\frac{1}{2}\left(\gamma^2 x + \frac{\delta^2}{x}\right)\right\} dx \\
&= \frac{\left(\frac{\gamma}{\delta}\right)^\lambda}{2K_\lambda(\gamma\delta)} \int_0^\infty x^{\lambda-1} \exp\left\{-\frac{\gamma^2}{2}x - \frac{\delta^2}{2x} - sx\right\} dx \\
&= \frac{\left(\frac{\gamma}{\delta}\right)^\lambda}{2K_\lambda(\gamma\delta)} \int_0^\infty x^{\lambda-1} \exp\left\{-\left(\frac{\gamma^2}{2} + s\right)x - \frac{\delta^2}{2}\frac{1}{x}\right\} dx
\end{aligned}$$

Put:

$$a = \frac{\delta^2}{2} \text{ and } b = \left(\frac{\gamma^2}{2} + s\right) = \frac{\gamma^2 + 2s}{2}$$

Then apply the relation $(*)$,

Therefore:

$$\begin{aligned}
\mathcal{L}_x(s) &= \frac{\left(\frac{\gamma}{\delta}\right)^{\lambda}}{2K_\lambda(\gamma\delta)}\left[2\left(\frac{\delta^2}{2\left(\frac{\gamma^2+2s}{2}\right)}\right)^{\frac{\lambda}{2}} K_\lambda\left(2\sqrt{\frac{\delta^2}{2}\left(\frac{\gamma^2+2s}{2}\right)}\right)\right] \\
&= \frac{\left(\frac{\gamma}{\delta}\right)^{\lambda}}{K_\lambda(\gamma\delta)}\left(\frac{\delta^2}{\gamma^2+2s}\right)^{\frac{\lambda}{2}} K_\lambda\left(\sqrt{\delta^2\left(\gamma^2+2s\right)}\right) \\
&= \left[\left(\frac{\gamma}{\delta}\right)^2\right]^{\frac{\lambda}{2}}\left(\frac{\delta^2}{\gamma^2+2s}\right)^{\frac{\lambda}{2}} \frac{K_\lambda\left(\sqrt{\delta^2\left(\gamma^2+2s\right)}\right)}{K_\lambda(\gamma\delta)} \\
&= \left(\frac{\gamma^2}{\delta^2}\frac{\delta^2}{\gamma^2+2s}\right)^{\frac{\lambda}{2}} \frac{K_\lambda\left(\sqrt{\delta^2\left(\gamma^2+2s\right)}\right)}{K_\lambda(\gamma\delta)} \\
&= \left(\frac{\gamma^2}{\gamma^2+2s}\right)^{\frac{\lambda}{2}} \frac{K_\lambda\left(\sqrt{\delta^2\left(\gamma^2+2s\right)}\right)}{K_\lambda(\gamma\delta)} \qquad (2.31)
\end{aligned}$$

Special cases

2.4.5 Parameterization 4

Construction of GIGD

Let $\omega = 2\delta\theta^{\frac{1}{2}}$ then:

$$K_\lambda\left(2\delta\theta^{\frac{1}{2}}\right) = \frac{1}{2}\int_0^\infty y^{\lambda-1}e^{-\delta\theta^{\frac{1}{2}}\left(y+\frac{1}{y}\right)}dy$$

Let $y = \frac{\theta^{\frac{1}{2}}}{\delta}z \Rightarrow dy = \frac{\theta^{\frac{1}{2}}}{\delta}dz$
Therefore;

$$
\begin{aligned}
K_\lambda\left(2\delta\theta^{\frac{1}{2}}\right) &= \frac{1}{2}\int_0^\infty \left(\frac{\theta^{\frac{1}{2}}}{\delta}\right)^{\lambda-1} z^{\lambda-1} e^{-\delta\theta^{\frac{1}{2}}\left(\frac{\theta^{\frac{1}{2}}}{\delta}z+\frac{\delta}{\theta^{\frac{1}{2}}}\frac{1}{z}\right)} \frac{\theta^{\frac{1}{2}}}{\delta}dz \\
&= \int_0^\infty \frac{1}{2}\left(\frac{\theta^{\frac{1}{2}}}{\delta}\right)^{\lambda} z^{\lambda-1} e^{-\delta\theta^{\frac{1}{2}}\left(\frac{\theta^{\frac{1}{2}}}{\delta}z+\frac{\delta}{\theta^{\frac{1}{2}}}\frac{1}{z}\right)} dz \\
&= \int_0^\infty \frac{1}{2}\left(\frac{\theta^{\frac{1}{2}}}{\delta}\right)^{\lambda} z^{\lambda-1} e^{-\left(\theta z+\frac{\delta^2}{z}\right)} dz \\
1 &= \int_0^\infty \frac{\left(\frac{\theta^{\frac{1}{2}}}{\delta}\right)^{\lambda}}{2K_\lambda\left(2\delta\theta^{\frac{1}{2}}\right)} z^{\lambda-1} e^{-\left(\theta z+\frac{\delta^2}{z}\right)} dz
\end{aligned}
$$

Therefore:

$$
f(x) = \frac{\left(\frac{\theta^{\frac{1}{2}}}{\delta}\right)^{\lambda}}{2K_\lambda\left(2\delta\theta^{\frac{1}{2}}\right)} x^{\lambda-1} e^{-\left(\theta x+\frac{\delta^2}{x}\right)},\ x > 0 \tag{2.32}
$$

Since $\left(\frac{\theta^{\frac{1}{2}}}{\delta}\right)^{\lambda} > 0$, the parameter space is given by:

$$
\begin{aligned}
\theta &> 0,\ \delta \geq 0,\ if\ \lambda > 0 \\
\theta &> 0,\ \delta \geq 0,\ if\ \lambda = 0 \\
\theta &\geq 0,\ \delta > 0,\ if\ \lambda < 0
\end{aligned}
$$

Modality

$$\begin{aligned}
\frac{d}{dx}x^{\lambda-1}e^{-\theta x-\frac{\delta^2}{x}} &= 0\\
\lambda-1x^{\lambda-2}e^{-\theta x-\frac{\delta^2}{x}}+x^{\lambda-1}e^{-\theta x-\frac{\delta^2}{x}}\left[-\theta+\frac{\delta^2}{x^2}\right] &= 0\\
(\lambda-1)+x\left[-\theta+\frac{\delta^2}{x^2}\right] &= 0\\
(\lambda-1)-\theta x+\frac{\delta^2}{x} &= 0\\
(\lambda-1)\,x-\theta x^2+\delta^2 &= 0\\
\theta x^2-(\lambda-1)\,x+\delta^2 &= 0
\end{aligned}$$

$$x=\frac{(\lambda-1)\pm\sqrt{(\lambda-1)^2+4\theta\delta^2}}{2\theta}$$

Since $x>0$,

$$\frac{(\lambda-1)+\sqrt{(\lambda-1)^2+4\theta\delta^2}}{2\theta}$$

The r^{th} moment

$$
\begin{aligned}
E\left(X^r\right) &= \int_0^\infty x^r \frac{\left(\frac{\theta^{\frac{1}{2}}}{\delta}\right)^\lambda}{2K_\lambda\left(2\delta\theta^{\frac{1}{2}}\right)} x^{\lambda-1} e^{-\left(\theta x+\frac{\delta^2}{x}\right)} dx \\
&= \frac{\left(\frac{\theta^{\frac{1}{2}}}{\delta}\right)^\lambda}{2K_\lambda\left(2\delta\theta^{\frac{1}{2}}\right)} \int_0^\infty x^{x+\lambda-1} e^{-\left(\theta x+\frac{\delta^2}{x}\right)} dx \\
&= \frac{\left(\frac{\theta^{\frac{1}{2}}}{\delta}\right)^\lambda}{2K_\lambda\left(2\delta\theta^{\frac{1}{2}}\right)} \frac{2K_{x+\lambda}\left(2\delta\theta^{\frac{1}{2}}\right)}{\left(\frac{\theta^{\frac{1}{2}}}{\delta}\right)^{x+\lambda}} \int_0^\infty \frac{\left(\frac{\theta^{\frac{1}{2}}}{\delta}\right)^{x+\lambda}}{2K_{x+\lambda}\left(2\delta\theta^{\frac{1}{2}}\right)} x^{x+\lambda-1} e^{-\left(\theta x+\frac{\delta^2}{x}\right)} dx \\
&= \left(\frac{\theta^{\frac{1}{2}}}{\delta}\right)^{-r} \frac{K_{x+\lambda}\left(2\delta\theta^{\frac{1}{2}}\right)}{K_\lambda\left(2\delta\theta^{\frac{1}{2}}\right)} \\
&= \left(\frac{\delta}{\theta^{\frac{1}{2}}}\right)^{r} \frac{K_{x+\lambda}\left(2\delta\theta^{\frac{1}{2}}\right)}{K_\lambda\left(2\delta\theta^{\frac{1}{2}}\right)}
\end{aligned}
$$

The Laplace transform

$$
\begin{aligned}
\mathcal{L}_x\left(s\right) &= \int_0^\infty e^{-sx} \frac{\left(\frac{\theta^{\frac{1}{2}}}{\delta}\right)^\lambda}{2K_\lambda\left(2\delta\theta^{\frac{1}{2}}\right)} x^{\lambda-1} e^{-\left(\theta x+\frac{\delta^2}{x}\right)} dx \\
&= \frac{\left(\frac{\theta^{\frac{1}{2}}}{\delta}\right)^\lambda}{2K_\lambda\left(2\delta\theta^{\frac{1}{2}}\right)} \int_0^\infty x^{\lambda-1} e^{-sx-\theta x-\frac{\delta^2}{x}} dx \\
&= \frac{\left(\frac{\theta^{\frac{1}{2}}}{\delta}\right)^\lambda}{2K_\lambda\left(2\delta\theta^{\frac{1}{2}}\right)} \int_0^\infty x^{\lambda-1} e^{-(s+\theta)x-\frac{\delta^2}{x}} dx
\end{aligned}
$$

Put $a = \delta^2$ and $b = (s+\theta)$
Therefore,

$$\begin{aligned}
\mathcal{L}_x(s) &= \frac{\left(\frac{\theta^{\frac{1}{2}}}{\delta}\right)^{\lambda}}{2K_\lambda\left(2\delta\theta^{\frac{1}{2}}\right)} 2\left(\frac{\delta^2}{s+\theta}\right)^{\frac{\lambda}{2}} K_\lambda\left(2\sqrt{\delta^2(s+\theta)}\right) \\
&= \frac{\left(\theta^{\frac{1}{2}}\right)^{\lambda}\delta^{\lambda}}{(s+\theta)^{\frac{\lambda}{2}}} \frac{K_\lambda\left(2\sqrt{\delta^2(s+\theta)}\right)}{K_\lambda\left(2\delta\theta^{\frac{1}{2}}\right)} \\
&= \left(\frac{\theta}{s+\theta}\right)^{\frac{\lambda}{2}} \frac{K_\lambda\left(2\sqrt{\delta^2(s+\theta)}\right)}{K_\lambda\left(2\delta\theta^{\frac{1}{2}}\right)} \qquad (2.33)
\end{aligned}$$

2.5 Special Cases: The Sybfamilies of The GIGD Family

Let;

$$K_\lambda(\chi,\psi) = \int_0^\infty x^{\lambda-1}\exp\left[-\frac{1}{2}\left(\chi x^{-1}+\psi x\right)\right]dx \qquad (2.34)$$

This Integral converges for arbitrary $\lambda \in \mathbb{R}$ and $\chi, \psi > 0$. In this case we have, using notation $\eta = \sqrt{\chi/\psi}$ and $\omega = \sqrt{\chi\psi}$ and the substitution $x = \eta y$,

$$\Longrightarrow \chi = \omega\eta \text{ , } \psi = \omega/\eta \text{ and } dx = \eta dy$$

$$
\begin{aligned}
&\int_0^\infty x^{\lambda-1}\exp\left[-\frac{1}{2}\left(\chi x^{-1}+\psi x\right)\right]dx \\
= &\int_0^\infty x^{\lambda-1}\exp\left[-\frac{1}{2}\left(\omega\eta x^{-1}+\frac{\omega}{\eta}x\right)\right]dx \\
= &\int_0^\infty x^{\lambda-1}\exp\left[-\frac{\omega}{2}\left(\frac{\eta}{x}+\frac{x}{\eta}\right)\right]dx \\
= &\int_0^\infty (\eta y)^{\lambda-1}\exp\left[-\frac{\omega}{2}\left(\frac{\eta}{\eta y}+\frac{\eta y}{\eta}\right)\right]\eta dy \\
= &\ \eta^\lambda\int_0^\infty y^{\lambda-1}\exp\left[-\frac{\omega}{2}\left(\frac{\eta}{\eta y}+\frac{\eta y}{\eta}\right)\right]dy \\
= &\ \eta^\lambda\int_0^\infty y^{\lambda-1}\exp\left[-\frac{\omega}{2}\left(\frac{1}{y}+y\right)\right]dy \\
= &\ 2\eta^\lambda\frac{1}{2}\int_0^\infty y^{\lambda-1}\exp\left[-\frac{\omega}{2}\left(\frac{1}{y}+y\right)\right]dy \\
= &\ 2\eta^\lambda K_\lambda\left(\omega\right) \\
= &\ 2\left(\frac{\chi}{\psi}\right)^{\frac{\lambda}{2}}K_\lambda\left(\sqrt{\chi\psi}\right)
\end{aligned}
$$

$$
K_\lambda\left(\chi,\psi\right)=2\left(\frac{\chi}{\psi}\right)^{\frac{\lambda}{2}}K_\lambda\left(\sqrt{\chi\psi}\right) \tag{2.35}
$$

The integral in (2.34) also converges in two ‘boundary cases’.

Case 1 *If $\chi=0$ and $\psi>0$,it conveges (iff) $\lambda>0$.*

The integral becomes

$$\int_0^\infty x^{\lambda-1} e^{-\frac{1}{2}\psi x} dx$$

Let $y = \frac{1}{2}\psi x \implies x = \frac{2}{\psi} y \implies dx = \frac{2}{\psi} dy$

$$\begin{aligned}
\int_0^\infty x^{\lambda-1} e^{-\frac{1}{2}\psi x} dx &= \int_0^\infty \left(\frac{2}{\psi} y\right)^{\lambda-1} e^{-y} \frac{2}{\psi} dy \\
&= \left(\frac{2}{\psi}\right)^{\lambda} \int_0^\infty y^{\lambda-1} e^{-y} dy \\
&= \left(\frac{2}{\psi}\right)^{\lambda} \Gamma(\lambda)
\end{aligned}$$

$$K_\lambda(0,\psi) = \left(\frac{2}{\psi}\right)^{\lambda} \Gamma(\lambda) \tag{2.36}$$

Case 2 *If* $\chi > 0$ *and* $\psi = 0$*, it converges iff* $\lambda < 0$.

The integral becomes

$$\int_0^\infty x^{\lambda-1} e^{-\frac{1}{2}\chi x^{-1}} dx$$

Let $y = \frac{1}{2}\chi x^{-1} \implies x = \frac{\chi}{2y} \implies dx = -\frac{\chi}{2y^2}dy$

$$\begin{aligned}
\int_0^\infty x^{\lambda-1}e^{-\frac{1}{2}\chi x^{-1}}dx &= \int_0^\infty \left(\frac{\chi}{2y}\right)^{\lambda-1} e^{-y}\frac{\chi}{2y^2}dy \\
&= \int_0^\infty \left(\frac{\chi}{2}\right)^{\lambda-1+1} y^{-\lambda+1-2}e^{-y}dy \\
&= \left(\frac{\chi}{2}\right)^{\lambda} \int_0^\infty y^{-\lambda-1}e^{-y}dy \\
&= \left(\frac{\chi}{2}\right)^{\lambda} \Gamma(-\lambda)
\end{aligned}$$

$$K_\lambda(\chi, 0) = \left(\frac{\chi}{2}\right)^{\lambda} \Gamma(-\lambda) \tag{2.37}$$

Therefore, using (2.34),

$$\begin{aligned}
2\left(\frac{\chi}{\psi}\right)^{\frac{\lambda}{2}} K_\lambda\left(\sqrt{\chi\psi}\right) &= \int_0^\infty x^{\lambda-1}\exp\left[-\frac{1}{2}\left(\chi x^{-1}+\psi x\right)\right]dx \\
1 &= \int_0^\infty \frac{x^{\lambda-1}\exp\left[-\frac{1}{2}\left(\chi x^{-1}+\psi x\right)\right]}{2\left(\frac{\chi}{\psi}\right)^{\frac{\lambda}{2}} K_\lambda\left(\sqrt{\chi\psi}\right)}dx
\end{aligned}$$

Implying that

$$\begin{aligned}
f_{GIG}(x;\lambda,\chi,\psi) &= \frac{x^{\lambda-1}\exp\left[-\frac{1}{2}\left(\chi x^{-1}+\psi x\right)\right]}{2\left(\frac{\chi}{\psi}\right)^{\frac{\lambda}{2}} K_\lambda\left(\sqrt{\chi\psi}\right)} \\
&= \left(\frac{\psi}{\chi}\right)^{\frac{\lambda}{2}} \frac{x^{\lambda-1}\exp\left[-\frac{1}{2}\left(\chi x^{-1}+\psi x\right)\right]}{2K_\lambda\left(\sqrt{\chi\psi}\right)} \qquad (2.38)
\end{aligned}$$

Now,

$$\begin{aligned}
E\left[X^r\right] &= \int_0^\infty x^r \left(\frac{\psi}{\chi}\right)^{\frac{\lambda}{2}} \frac{x^{\lambda-1}\exp\left[-\frac{1}{2}\left(\chi x^{-1}+\psi x\right)\right]}{2K_\lambda\left(\sqrt{\chi\psi}\right)}dx \\
&= \left(\frac{\psi}{\chi}\right)^{\frac{\lambda}{2}} \frac{1}{2K_\lambda\left(\sqrt{\chi\psi}\right)} \int_0^\infty x^r x^{\lambda-1}\exp\left[-\frac{1}{2}\left(\chi x^{-1}+\psi x\right)\right]dx \\
&= \left(\frac{\psi}{\chi}\right)^{\frac{\lambda}{2}} \frac{1}{2K_\lambda\left(\sqrt{\chi\psi}\right)} \int_0^\infty x^{\lambda+r-1}\exp\left[-\frac{1}{2}\left(\chi x^{-1}+\psi x\right)\right]dx \\
&= \left(\frac{\psi}{\chi}\right)^{\frac{\lambda}{2}} \frac{1}{2K_\lambda\left(\sqrt{\chi\psi}\right)} 2\left(\frac{\chi}{\psi}\right)^{\frac{\lambda+r}{2}} K_{\lambda+r}\left(\sqrt{\chi\psi}\right) \\
&= \left(\frac{\chi}{\psi}\right)^{-\frac{\lambda}{2}} \frac{1}{2K_\lambda\left(\sqrt{\chi\psi}\right)} 2\left(\frac{\chi}{\psi}\right)^{\frac{\lambda+r}{2}} K_{\lambda+r}\left(\sqrt{\chi\psi}\right) \\
&= \left(\frac{\chi}{\psi}\right)^{\frac{r}{2}} \frac{K_{\lambda+r}\left(\sqrt{\chi\psi}\right)}{K_\lambda\left(\sqrt{\chi\psi}\right)}
\end{aligned}$$

$$E\left[X\right] = \left(\frac{\chi}{\psi}\right)^{\frac{1}{2}} \frac{K_{\lambda+1}\left(\sqrt{\chi\psi}\right)}{K_\lambda\left(\sqrt{\chi\psi}\right)} \tag{2.39}$$

$$\begin{aligned}
var\left(X\right) &= E\left(X^2\right) - E^2\left[X\right] \\
&= \left(\frac{\chi}{\psi}\right)^{\frac{2}{2}} \frac{K_{\lambda+2}\left(\sqrt{\chi\psi}\right)}{K_\lambda\left(\sqrt{\chi\psi}\right)} - \left(\frac{\chi}{\psi}\right)^{\frac{2}{2}} \frac{K_{\lambda+1}^2\left(\sqrt{\chi\psi}\right)}{K_\lambda^2\left(\sqrt{\chi\psi}\right)} \\
&= \left(\frac{\chi}{\psi}\right)\left[\frac{K_{\lambda+2}\left(\sqrt{\chi\psi}\right)}{K_\lambda\left(\sqrt{\chi\psi}\right)} - \frac{K_{\lambda+1}^2\left(\sqrt{\chi\psi}\right)}{K_\lambda^2\left(\sqrt{\chi\psi}\right)}\right] \\
&= \left(\frac{\chi}{\psi}\right)\left[\frac{K_\lambda\left(\sqrt{\chi\psi}\right)K_{\lambda+2}\left(\sqrt{\chi\psi}\right) - K_{\lambda+1}^2\left(\sqrt{\chi\psi}\right)}{K_\lambda^2\left(\sqrt{\chi\psi}\right)}\right] \qquad (2.40)
\end{aligned}$$

The moment generating function of X can also be determined:

$$
\begin{aligned}
M_{GIG}(t;\lambda,\chi,\psi) &= \int_0^\infty e^{tx}\left(\frac{\psi}{\chi}\right)^{\frac{\lambda}{2}}\frac{x^{\lambda-1}\exp\left[-\frac{1}{2}\left(\chi x^{-1}+\psi x\right)\right]}{2K_\lambda\left(\sqrt{\chi\psi}\right)}dx \\
&= \left(\frac{\psi}{\chi}\right)^{\frac{\lambda}{2}}\frac{1}{2K_\lambda\left(\sqrt{\chi\psi}\right)}\int_0^\infty x^{\lambda-1}\exp\left[-\frac{1}{2}\left(\chi x^{-1}+\psi x-2tx\right)\right]dx \\
&= \left(\frac{\psi}{\chi}\right)^{\frac{\lambda}{2}}\frac{1}{2K_\lambda\left(\sqrt{\chi\psi}\right)}\int_0^\infty x^{\lambda-1}\exp\left[-\frac{1}{2}\left(\chi x^{-1}+\left(\psi-2t\right)x\right)\right]dx \\
&= \left(\frac{\psi}{\chi}\right)^{\frac{\lambda}{2}}\frac{1}{2K_\lambda\left(\sqrt{\chi\psi}\right)}2\left(\frac{\chi}{\psi-2t}\right)^{\frac{\lambda}{2}}K_\lambda\left(\sqrt{\chi\left(\psi-2t\right)}\right) \\
&= \left(\frac{\psi}{\psi-2t}\right)^{\frac{\lambda}{2}}\frac{K_\lambda\left(\sqrt{\chi\left(\psi-2t\right)}\right)}{K_\lambda\left(\sqrt{\chi\psi}\right)} \qquad (2.41)
\end{aligned}
$$

2.5.1 Gamma Distribution.

Using boundary case(I)

$$
\begin{aligned}
\left(\frac{2}{\psi}\right)^{\lambda}\Gamma\left(\lambda\right) &= \int_0^\infty x^{\lambda-1}e^{-\frac{1}{2}\psi x}dx \\
1 &= \int_0^\infty \frac{x^{\lambda-1}e^{-\frac{1}{2}\psi x}}{\left(\frac{2}{\psi}\right)^{\lambda}\Gamma\left(\lambda\right)}dx
\end{aligned}
$$

Implying that

$$
\begin{aligned}
f_{GIG}\left(x;\lambda,0,\psi\right) &= \frac{x^{\lambda-1}e^{-\frac{1}{2}\psi x}}{\left(\frac{2}{\psi}\right)^{\lambda}\Gamma\left(\lambda\right)} \\
&= \frac{\left(\frac{\psi}{2}\right)^{\lambda}}{\Gamma\left(\lambda\right)}e^{-\frac{\psi}{2}x}x^{\lambda-1} \qquad (2.42)
\end{aligned}
$$

Which is a gamma distribution with parameters $(\lambda, \psi/2)$

$$E[X] = \left(\frac{\chi}{\psi}\right)^{\frac{1}{2}} \frac{K_{\lambda+1}\left(\sqrt{\chi\psi}\right)}{K_{\lambda}\left(\sqrt{\chi\psi}\right)} \,|_{\chi=0} \tag{2.43}$$

Recall;

$$K_{\lambda}(0,\psi) = \left(\frac{2}{\psi}\right)^{\lambda} \Gamma(\lambda) \text{ and } K_{\lambda}\left(\sqrt{\chi\psi}\right) = \frac{1}{2}\left(\frac{\chi}{\psi}\right)^{-\frac{\lambda}{2}} K_{\lambda}(\chi,\psi)$$

$$\begin{aligned}
E[X] &= \left(\frac{\chi}{\psi}\right)^{\frac{1}{2}} \frac{\frac{1}{2}\left(\frac{\chi}{\psi}\right)^{-\frac{\lambda+1}{2}} K_{\lambda+1}(0,\psi)}{\frac{1}{2}\left(\frac{\chi}{\psi}\right)^{-\frac{\lambda}{2}} K_{\lambda}(0,\psi)} \\
&= \frac{\frac{1}{2}\left(\frac{\chi}{\psi}\right)^{-\frac{\lambda}{2}} K_{\lambda+1}(0,\psi)}{\frac{1}{2}\left(\frac{\chi}{\psi}\right)^{-\frac{\lambda}{2}} K_{\lambda}(0,\psi)} \\
&= \frac{K_{\lambda+1}(0,\psi)}{K_{\lambda}(0,\psi)} \\
&= \frac{\left(\frac{2}{\psi}\right)^{\lambda+1} \Gamma(\lambda+1)}{\left(\frac{2}{\psi}\right)^{\lambda} \Gamma(\lambda)} \\
&= \frac{2\lambda}{\psi} = \frac{\lambda}{(\psi/2)}
\end{aligned} \tag{2.44}$$

$$
\begin{aligned}
var\left(X\right) &= E\left(X^{2}\right)-E^{2}\left(X\right) \\
&= \frac{\chi}{\psi}\frac{K_{\lambda+2}\left(\sqrt{\chi\psi}\right)}{K_{\lambda}\left(\sqrt{\chi\psi}\right)}-\left(\frac{2\lambda}{\psi}\right)^{2}|_{\chi=0} \\
&= \frac{\chi}{\psi}\frac{\frac{1}{2}\left(\frac{\chi}{\psi}\right)^{-\frac{\lambda+2}{2}}K_{\lambda+2}\left(0,\psi\right)}{\frac{1}{2}\left(\frac{\chi}{\psi}\right)^{-\frac{\lambda}{2}}K_{\lambda}\left(0,\psi\right)}-\left(\frac{2\lambda}{\psi}\right)^{2} \\
&= \frac{\frac{1}{2}\left(\frac{\chi}{\psi}\right)^{-\frac{\lambda}{2}}K_{\lambda+2}\left(0,\psi\right)}{\frac{1}{2}\left(\frac{\chi}{\psi}\right)^{-\frac{\lambda}{2}}K_{\lambda}\left(0,\psi\right)}-\left(\frac{2\lambda}{\psi}\right)^{2} \\
&= \frac{K_{\lambda+2}\left(0,\psi\right)}{K_{\lambda}\left(0,\psi\right)}-\left(\frac{2\lambda}{\psi}\right)^{2} \\
&= \frac{\left(\frac{2}{\psi}\right)^{\lambda+2}\Gamma\left(\lambda+2\right)}{\left(\frac{2}{\psi}\right)^{\lambda}\Gamma\left(\lambda\right)}-\left(\frac{2\lambda}{\psi}\right)^{2} \\
&= \left(\frac{2}{\psi}\right)^{2}\lambda\left(\lambda+1\right)-\left(\frac{2\lambda}{\psi}\right)^{2} \\
&= \left(\frac{2}{\psi}\right)^{2}\lambda^{2}+\left(\frac{2}{\psi}\right)^{2}\lambda-\left(\frac{2\lambda}{\psi}\right)^{2} \\
&= \left(\frac{2}{\psi}\right)^{2}\lambda=\frac{\lambda}{\left(\psi/2\right)^{2}} \qquad (2.45)
\end{aligned}
$$

and

$$
\begin{aligned}
M_{GIG}\left(t;\lambda,0,\psi\right) &= \left(\frac{\psi}{\psi-2t}\right)^{\frac{\lambda}{2}} \frac{K_{\lambda}\left(\sqrt{\chi\left(\psi-2t\right)}\right)}{K_{\lambda}\left(\sqrt{\chi\psi}\right)} \,|_{\chi=0} \\
&= \left(\frac{\psi}{\psi-2t}\right)^{\frac{\lambda}{2}} \frac{\frac{1}{2}\left(\frac{\chi}{\psi-2t}\right)^{-\frac{\lambda}{2}} K_{\lambda}\left(0,\left(\psi-2t\right)\right)}{\frac{1}{2}\left(\frac{\chi}{\psi}\right)^{-\frac{\lambda}{2}} K_{\lambda}\left(0,\psi\right)} \\
&= \left(\frac{\psi}{\psi-2t}\right)^{\frac{\lambda}{2}} \frac{\left(\frac{\chi}{\psi-2t}\right)^{-\frac{\lambda}{2}} \left(\frac{2}{\psi-2t}\right)^{\lambda}\Gamma\left(\lambda\right)}{\left(\frac{\chi}{\psi}\right)^{-\frac{\lambda}{2}} \left(\frac{2}{\psi}\right)^{\lambda}\Gamma\left(\lambda\right)} \\
&= \left(\frac{\psi}{\psi-2t}\right)^{\frac{\lambda}{2}} \left(\frac{\chi}{\psi-2t}\right)^{-\frac{\lambda}{2}} \left(\frac{2}{\psi-2t}\right)^{\lambda} \left(\frac{\psi}{\chi}\right)^{-\frac{\lambda}{2}} \left(\frac{\psi}{2}\right)^{\lambda} \\
&= \left(\frac{\psi}{\psi-2t}\right)^{\frac{\lambda}{2}} \left(\frac{\psi}{\psi-2t}\right)^{-\frac{\lambda}{2}} \left(\frac{\chi}{\chi}\right)^{-\frac{\lambda}{2}} \left(\frac{\psi}{\psi-2t}\right)^{\lambda} \\
&= \left(\frac{\psi}{\psi-2t}\right)^{\lambda} = \left(\frac{\psi-2t}{\psi}\right)^{-\lambda} \\
&= \left(1-\frac{2t}{\psi}\right)^{-\lambda} \qquad (2.46)
\end{aligned}
$$

$$
M_{GIG}\left(t;\lambda,0,\psi\right) = \left(1-\frac{2t}{\psi}\right)^{-\lambda} \quad -\infty < t < \psi/2
$$

2.5.2 The Inverse Gamma Distribution

Using boundary case(II):

$$
\begin{aligned}
\left(\frac{\chi}{2}\right)^{\lambda}\Gamma\left(-\lambda\right) &= \int_{0}^{\infty} x^{\lambda-1}e^{-\frac{1}{2}\chi x^{-1}}dx \\
1 &= \int_{0}^{\infty} \frac{x^{\lambda-1}e^{-\frac{1}{2}\chi x^{-1}}}{\left(\frac{\chi}{2}\right)^{\lambda}\Gamma\left(-\lambda\right)}dx
\end{aligned}
$$

Implying that

$$\begin{aligned} f_{GIG}(x;\lambda,\chi,0) &= \frac{x^{\lambda-1}e^{-\frac{1}{2}\chi x^{-1}}}{\left(\frac{\chi}{2}\right)^{\lambda}\Gamma(-\lambda)} \\ &= \frac{\left(\frac{\chi}{2}\right)^{-\lambda}}{\Gamma(-\lambda)}e^{-\frac{1}{2}\chi x^{-1}}x^{\lambda-1} \end{aligned}$$

which is the inverse gamma distribution IGam($if\ \lambda<0,\chi>0,\psi=0$)

$$f_{IGam}(-\lambda,\chi/2)=\frac{\left(\frac{\chi}{2}\right)^{-\lambda}}{\Gamma(-\lambda)}e^{-\frac{1}{2}\chi x^{-1}}x^{\lambda-1} \tag{2.47}$$

The rth raw moment of $X\sim GIG(\lambda,\chi,0)$ exist iff $r<-\lambda$, and

$$\begin{aligned} E(X^r) &= \int_0^\infty x^r\frac{\left(\frac{\chi}{2}\right)^{-\lambda}}{\Gamma(-\lambda)}e^{-\frac{1}{2}\chi x^{-1}}x^{\lambda-1}dx \\ &= \frac{\Gamma(-\lambda-r)}{\Gamma(-\lambda)}\left(\frac{\chi}{2}\right)^r \end{aligned} \tag{2.48}$$

The mean (and variance) exist if $\lambda<-1$ (and $\lambda<-2$); these are

$$E(X)=-\frac{\chi}{2}\frac{1}{\lambda+1}$$

$$var(X)=-\left(\frac{\chi}{2}\right)^2\frac{1}{(\lambda+1)^2(\lambda+2)}$$

The momement generating function is;

$$M_{GIG}(t;\lambda,\chi,0)=\frac{2K_\lambda\left(\sqrt{-2\chi t}\right)}{\Gamma(-\lambda)(-\chi t/2)^{\lambda/2}},\quad -\infty<t\le 0 \tag{2.49}$$

2.5.3 The exponential distribution Exp

($if\ \lambda=1,\ \chi=0,\ \psi>0$)

$$f_{GIG}(x;1,0,\psi) = \frac{\psi}{2}\exp\left(-\frac{\psi}{2}x\right) = f_{Exp}(\psi/2), \tag{2.50}$$

which is an exponential distribution with parameter $\psi/2$. It is a special case of the gamma distribution mentioned above.

2.5.4 The positive hyperbolic distribution pHyp

$(if\ \lambda = 1,\ \chi > 0,\ \psi > 0)$

$$\begin{aligned} f_{pHyp}(x;\chi,\psi) &= f_{GIG}(x;1,\chi,\psi) \\ &= \frac{1}{2\eta K_1(\omega)}\exp\left[-\frac{1}{2}\left(\chi x^{-1}+\psi x\right)\right] \\ &= \left(\frac{\psi}{\chi}\right)^{\frac{1}{2}}\frac{\exp\left[-\frac{1}{2}\left(\chi x^{-1}+\psi x\right)\right]}{2K_1\left(\sqrt{\chi\psi}\right)} \end{aligned} \tag{2.51}$$

$$E[X] = \left(\frac{\chi}{\psi}\right)^{\frac{1}{2}}\frac{K_2\left(\sqrt{\chi\psi}\right)}{K_1\left(\sqrt{\chi\psi}\right)} \tag{2.52}$$

$$var(X) = \left(\frac{\chi}{\psi}\right)\left[\frac{K_1\left(\sqrt{\chi\psi}\right)K_3\left(\sqrt{\chi\psi}\right) - K_2^2\left(\sqrt{\chi\psi}\right)}{K_1^2\left(\sqrt{\chi\psi}\right)}\right] \tag{2.53}$$

$$M_{GIG}(t;1,\chi,\psi) = \left(\frac{\psi}{\psi-2t}\right)^{\frac{1}{2}}\frac{K_1\left(\sqrt{\chi(\psi-2t)}\right)}{K_1\left(\sqrt{\chi\psi}\right)} \tag{2.54}$$

2.5.5 The Levy distribution

The Levy distribution $\left(if\ \lambda = -\frac{1}{2}, \chi > 0, \psi = 0\right)$

The Levy distribution is a very interesting subfamily of the inverse gamma distribution,

$$Levy(\chi/2) = IGam(-1/2,\chi/2) = GIG(-1/2,\chi,0)$$

$$f_{Levy}\left(x;\chi/2\right)=\left(\frac{\chi}{2\pi}\right)^{1/2}x^{-3/2}\exp\left(-\frac{\chi}{2x}\right) \tag{2.55}$$

The moment generating function of this distribution can be calculated as follows for $t\leq 0$:

$$\begin{aligned}
M_{Levy}\left(t,\chi/2\right) &= \frac{2K_{1/2}\left(\sqrt{-2\chi t}\right)}{\Gamma\left(1/2\right)\left(-\chi t/2\right)^{-1/4}} \\
&= \frac{2\sqrt{\pi/2}\sqrt{-2\chi t}^{-1/2}e^{-\sqrt{-2\chi t}}}{\sqrt{\pi}\left(-\chi t/2\right)^{-1/4}} \\
&= \exp\left(-\sqrt{\chi}\sqrt{-2t}\right) \qquad (2.56)
\end{aligned}$$

Note that for independent r.v.s X and Y,

$$M_{X+Y}\left(t\right)=M_X\left(t\right)M_Y\left(t\right) \tag{2.57}$$

Then it follows that, if $X_i \sim Levy\left(\chi_i/2\right)$, the m.g.f of

$$S=X_1+X_2+\ldots+X_n \tag{2.58}$$

is given by

$$\begin{aligned}
M_S\left(t\right) &= \prod_{i=1}^{n}\exp\left[\sqrt{\chi_i}\sqrt{-2t}\right] \\
&= \exp\left[\sum_{i=1}^{n}\sqrt{\chi_i}\sqrt{-2t}\right] \\
&= \exp\left[\left(\sum_{i=1}^{n}\sqrt{\chi_i}\right)\sqrt{-2t}\right] \\
&= \exp\left[\sqrt{\chi}\sqrt{-2t}\right] \qquad (2.59)
\end{aligned}$$

for all $t\in\left(-\infty,\psi/2\right)$, where $\chi=\left(\chi_1^{1/2}+\chi_2^{1/2}+\ldots+\chi_n^{1/2}\right)^2$. Because the m.g.f of a nonnegative r.v. uniguely determines its distribution,

$$S \sim Levy\left(\chi/2\right). \tag{2.60}$$

2.5.6 The inverse Gaussian distribution IG

The inverse Gaussian distribution IG$\left(if\ \lambda = -\frac{1}{2}, \chi > 0, \psi > 0\right)$ .If $\lambda = \frac{1}{2}$, then the GIG Distribution in the normal case reduces to the inverse Gaussian distribution:

$$IG\left(\chi,\psi\right) := GIG\left(-1/2,\chi,\psi\right),\ \ \chi,\psi > 0$$

$$\begin{aligned} f_{IG}\left(x,\chi,\psi\right) &= f_{GIG}\left(x;-1/2,\chi,\psi\right) \\ &= \frac{1}{K_{1/2}\left(\chi,\psi\right)} x^{-3/2}\exp\left[-\frac{1}{2}\left(\chi x^{-1}+\psi x\right)\right] \end{aligned} \tag{2.61}$$

since;

$$\begin{aligned} K_{1/2}\left(\chi,\psi\right) &= 2\eta^{-1/2}K_{-1/2}\left(\omega\right) \\ &= 2\eta^{-1/2}K_{1/2}\left(\omega\right) \\ &= 2\eta^{-1/2}\sqrt{\frac{\pi}{2\omega}}e^{-\omega} \\ &= 2\sqrt{\frac{\psi}{\chi}}\frac{1}{\sqrt{\chi\psi}}\sqrt{\frac{\pi}{2}}e^{-\omega} \\ &= \sqrt{\frac{2\pi}{\chi}}e^{-\omega} \end{aligned} \tag{2.62}$$

Therefore;

$$\begin{aligned}
f_{IG}(x,\chi,\psi) &= \sqrt{\frac{\chi}{2\pi}} e^{\sqrt{\chi\psi}} x^{-3/2} \exp\left[-\frac{1}{2}\left(\chi x^{-1}+\psi x\right)\right] \\
&= \left(\frac{\chi}{2\pi x}\right)^{-3/2} \exp\left[\sqrt{\chi\psi}-\frac{1}{2}\left(\chi x^{-1}+\psi x\right)\right] \\
&= \left(\frac{\chi}{2\pi x}\right)^{-3/2} \exp\left[-\frac{1}{2}\left(\chi x^{-1}-2\sqrt{\chi\psi}+\psi x\right)\right] \qquad (2.63)
\end{aligned}$$

using the relation

$$\begin{aligned}
\chi x^{-1}-2\sqrt{\chi\psi}+\psi x &= \psi x-2\sqrt{\chi\psi}+\chi x^{-1} \\
&= \psi\frac{x^2-2\eta x+\eta^2}{x} \\
&= \psi\frac{(x-\eta)^2}{x} \\
&= \frac{\chi(x-\eta)^2}{\eta^2 x}
\end{aligned}$$

then

$$\begin{aligned}
f_{IG}(x,\chi,\psi) &= \left(\frac{\chi}{2\pi x}\right)^{-3/2} \exp\left[-\frac{1}{2}\left(\chi x^{-1}-2\sqrt{\chi\psi}+\psi x\right)\right] \\
&= \left(\frac{\chi}{2\pi x}\right)^{-3/2} \exp\left\{-\frac{\chi(x-\eta)^2}{2\eta^2 x}\right\} \qquad (2.64)
\end{aligned}$$

The m.g.f of the IG distrinution can be expressed in the following simple form:

$$
\begin{aligned}
M_{IG}(t;\chi,\psi) &= M_{GIG}(t;-1/2,\chi,\psi) \\
&= \left(\frac{\psi}{\psi-2t}\right)^{-\frac{1}{4}} \frac{K_{1/2}\left(\sqrt{\chi(\psi-2t)}\right)}{K_{1/2}\left(\sqrt{\chi\psi}\right)} \\
&= \left(\frac{\psi}{\psi-2t}\right)^{-\frac{1}{4}} \left(\frac{\pi}{2\sqrt{\chi(\psi-2t)}}\right)^{1/2} e^{-\sqrt{\chi(\psi-2t)}} \left(\frac{\pi}{2\sqrt{\chi\psi}}\right)^{-1/2} e^{\sqrt{\chi\psi}} \\
&= \left(\frac{\psi}{\psi-2t}\right)^{-\frac{1}{4}} \left(\frac{\pi}{2\sqrt{\chi(\psi-2t)}}\right)^{1/2} e^{-\sqrt{\chi(\psi-2t)}} \left(\frac{2\sqrt{\chi\psi}}{\pi}\right)^{1/2} e^{\sqrt{\chi\psi}} \\
&= \left(\frac{\psi}{\psi-2t}\right)^{-\frac{1}{4}} \left(\sqrt{\frac{\psi}{(\psi-2t)}}\right)^{1/2} e^{-\sqrt{\chi(\psi-2t)}+\sqrt{\chi\psi}} \\
&= \left(\frac{\psi}{\psi-2t}\right)^{-\frac{1}{4}} \left(\frac{\psi}{\psi-2t}\right)^{\frac{1}{4}} e^{-\sqrt{\chi(\psi-2t)}+\sqrt{\chi\psi}} \\
&= e^{-\sqrt{\chi(\psi-2t)}+\sqrt{\chi\psi}} \\
&= \exp\left[\sqrt{\chi}\left(\sqrt{\psi}-\sqrt{\psi-2t}\right)\right] \qquad (2.65)
\end{aligned}
$$

$$
M_{IG}(t;\chi,\psi) = \exp\left[\sqrt{\chi}\left(\sqrt{\psi}-\sqrt{\psi-2t}\right)\right], \quad -\infty < t \le \psi/2.
$$

Note that for independent r.v.s X and Y,

$$
M_{X+Y}(t) = M_X(t)\, M_Y(t)
$$

Then it follows that, if $X_i \sim IG(\chi_i, \psi)$, the m.g.f of

$$
S = X_1 + X_2 + \ldots + X_n
$$

is given by

$$\begin{aligned} M_S(t) &= \prod_{i=1}^{n} \exp\left[\sqrt{\chi_i}\left(\sqrt{\psi}-\sqrt{\psi-2t}\right)\right] \\ &= \exp\left[\sum_{i=1}^{n}\sqrt{\chi_i}\left(\sqrt{\psi}-\sqrt{\psi-2t}\right)\right] \\ &= \exp\left[\left(\sum_{i=1}^{n}\sqrt{\chi_i}\right)\left(\sqrt{\psi}-\sqrt{\psi-2t}\right)\right] \\ &= \exp\left[\sqrt{\chi}\left(\sqrt{\psi}-\sqrt{\psi-2t}\right)\right] \end{aligned} \tag{2.66}$$

for all $t \in (-\infty, \psi/2)$, where $\chi = \left(\chi_1^{1/2} + \chi_2^{1/2} + \ldots + \chi_n^{1/2}\right)^2$. Because the m.g.f of a nonnegative r.v. uniguely determines its distribution,

$$S \sim IG(\chi, \psi). \tag{2.67}$$

2.5.7 The degenerate or Dirac distribution as a limiting case

Suppose that X is a constant r.v having some constant value $x \in \mathbb{R}$. Then X has a distribution called the degenerate or Dirac distribution with value x, and is denoted by Δ_x. Let $\lambda \in [0, \infty)$. Then, with $\omega = \sqrt{\chi\psi}$ and $\eta = \sqrt{\chi/\psi}$,

$$GIG(\lambda, \chi, \psi) \longrightarrow \Delta_x \text{ for } \eta \longrightarrow x \text{ and } \omega \longrightarrow \infty \text{ where } \chi, \psi > 0. \tag{2.68}$$

2.5.8 GIG Plots

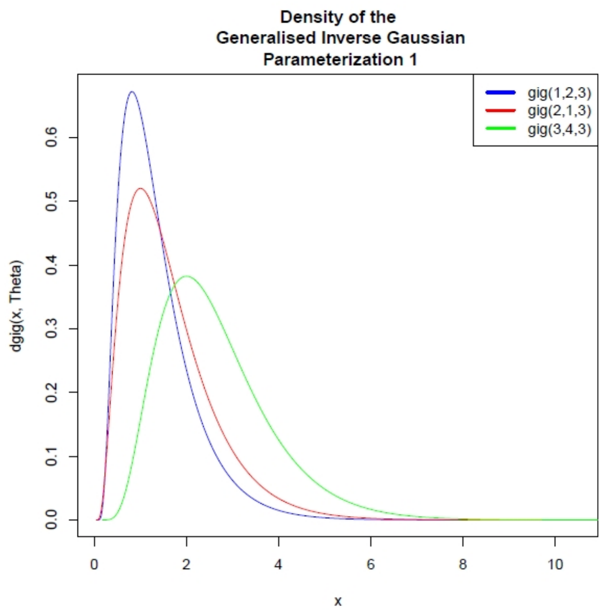


Figure 2.1: Generalized Inverse Gaussian Plot

Chapter 3

The Generalised Hyperbolic Distribution

3.1 Introduction

The generalised hyperbolic distribution is a normal variance-mean mixture where the mixing distribution is generalised inverse Gaussian. Thus if;

$$X|W = w \sim N(\mu + \beta w, w) \tag{3.1}$$

And

$$W \sim GIG(\lambda, \delta, \gamma) \tag{3.2}$$

then the marginal distribution of X will be generalised hyperbolic distribution,*ie*

$$X \sim GH(\lambda, \alpha, \beta, \delta, \mu) \tag{3.3}$$

where

$$\alpha^2 = \beta^2 + \gamma^2 \tag{3.4}$$

3.1.1 Special Cases

- Normal-Inverse Gaussian distibution when the mixing is an inverse Gaussian distribution.
- The variance-gamma mixture where the mixing distribution is a gamma distribution.
- The asymmetric scaled t-distribution is a normal variance-mean mixture with an inverse gamma mixing distribution. As a special case here, we get the well known result that the t-distribution is a normal variance mixture ($\beta = 0$)with an inverse gamma mixing distribution.

The mixing result implies that there is the following simple relationship between the Laplace transform, $\mathcal{L}_x$, of the generalised hyperbolic distribution $H(\lambda, \alpha, \beta, \delta, \mu)$ and that of the $GIG\left(\lambda, \delta, \sqrt{\alpha^2 - \beta^2}\right)$ $-$disribution, L_w;

$$L_x(z) = e^{\mu z} L_w\left(\beta z + \frac{1}{2} z\right) \tag{3.5}$$

Barndorff-Nielsen & Halgreen(1977) Showed that the mixing distribution are Infinitely Divisible.

Using that the GHDs are normal variance-mean mixtures with GIG mixing distributions, they also proved that GHDs are also infinitely divisible.

Halgreen (1979) showed GHDs and GIGDs are even self-decomposable. The properties of infinite divisibility and self-decomposability are important because they allow the construction of certain hyperbolic stochastic process models.

3.1.2 Mixing with parameterization 1

$$f(x|z) = \frac{1}{\sqrt{2\pi z}} \exp\left\{-\frac{1}{2}\left[\frac{(x - (\mu + \beta z))^2}{z}\right]\right\}, \; x \in \mathbb{R}$$

$$f(z) = \frac{\eta^{-\lambda} z^{\lambda-1}}{2K_\lambda(\eta/\xi)} \exp\left\{-\frac{(z^2+\eta^2)}{2\xi z}\right\},\ z > 0$$

Therefore

$$\begin{aligned}
f(x) &= \int_0^\infty \frac{1}{\sqrt{2\pi z}} \exp\left\{-\frac{\left[(x-(\mu+\beta z))^2\right]}{2z}\right\} \\
&\quad \times \frac{\eta^{-\lambda} z^{\lambda-1}}{2K_\lambda(\eta/\xi)} \exp\left\{-\frac{(z^2+\eta^2)}{2\xi z}\right\} dz \\
&= \frac{\eta^{-\lambda}}{\sqrt{2\pi}2K_\lambda(\eta/\xi)} \int_0^\infty z^{(\lambda-1/2)-1} \\
&\quad \times \exp\left\{\frac{-(x-\mu)^2}{2z} + \beta(x-\mu) - \frac{\beta^2 z}{2} - \frac{z}{2\xi} - \frac{\eta^2}{2\xi}\frac{1}{z}\right\} dz \\
&= \frac{\eta^{-\lambda} e^{\beta(x-\mu)}}{\sqrt{2\pi}2K_\lambda(\eta/\xi)} \int_0^\infty z^{(\lambda-1/2)-1} \\
&\quad \times \exp\left\{-\left[\frac{(x-\mu)^2}{2} + \frac{\eta^2}{2\xi}\right]\frac{1}{z} - \left(\frac{\beta^2}{2} + \frac{1}{2\xi}\right) z\right\} dz \\
&= \frac{\eta^{-\lambda} e^{\beta(x-\mu)}}{\sqrt{2\pi}2K_\lambda(\eta/\xi)} \int_0^\infty z^{(\lambda-1/2)-1} \\
&\quad \times \exp\left\{-\left[\frac{\xi(x-\mu)^2+\eta^2}{2\xi}\right]\frac{1}{z} - \left(\frac{\beta^2+1}{2\xi}\right) z\right\} dz \\
&= \frac{\eta^{-\lambda} e^{\beta(x-\mu)}}{\sqrt{2\pi}2K_\lambda(\eta/\xi)} 2 \\
&\quad \times \left(\sqrt{\frac{\xi(x-\mu)^2+\eta^2}{\beta^2+1}}\right)^{\lambda-1/2} K_{\lambda-1/2}\left(\sqrt{\frac{\left(\xi(x-\mu)^2+\eta^2\right)\left(\beta^2+1\right)}{\xi}}\right) \quad (3.6)
\end{aligned}$$

3.1.3 Mixing with parameterization 2

$$f(x|z) = \frac{1}{\sqrt{2\pi z}} \exp\left\{-\frac{1}{2}\left[\frac{(x-(\mu+\beta z))^2}{z}\right]\right\}, \ x \in \mathbb{R}$$

$$f(z) = \frac{\left(\frac{\gamma}{\delta}\right)^{\frac{\lambda}{2}}}{2K_\lambda\left(\sqrt{\gamma\delta}\right)} z^{\lambda-1} \exp\left\{-\frac{1}{2}\left(\delta z^{-1} + \gamma z\right)\right\}, z > 0$$

Therefore

$$\begin{aligned}
f(x) &= \int_0^\infty \frac{1}{\sqrt{2\pi z}} \exp\left\{-\frac{1}{2}\left[\frac{(x-(\mu+\beta z))^2}{z}\right]\right\} \\
&\quad \times \frac{\left(\frac{\gamma}{\delta}\right)^{\frac{\lambda}{2}}}{2K_\lambda\left(\sqrt{\gamma\delta}\right)} z^{\lambda-1} \exp\left\{-\frac{1}{2}\left(\frac{\delta}{z} + \gamma z\right)\right\} dz \\
&= \frac{\left(\frac{\gamma}{\delta}\right)^{\frac{\lambda}{2}}}{\sqrt{2\pi} 2K_\lambda\left(\sqrt{\gamma\delta}\right)} \int_0^\infty z^{(\lambda-1/2)-1} \\
&\quad \times \exp\left\{\frac{-(x-\mu)^2}{2z} + \beta(x-\mu) - \frac{\beta^2 z}{2} - \frac{\gamma z}{2} - \frac{\delta}{2}\frac{1}{z}\right\} dz \\
&= \frac{\left(\frac{\gamma}{\delta}\right)^{\frac{\lambda}{2}} e^{\beta(x-\mu)}}{\sqrt{2\pi} 2K_\lambda\left(\sqrt{\gamma\delta}\right)} \int_0^\infty z^{(\lambda-1/2)-1} \exp\left\{-\frac{\delta + (x-\mu)^2}{2}\frac{1}{z} - \frac{\left(\beta^2+\gamma\right)}{2} z\right\} dz \\
&= \frac{\left(\frac{\gamma}{\delta}\right)^{\frac{\lambda}{2}} e^{\beta(x-\mu)}}{\sqrt{2\pi} 2K_\lambda\left(\sqrt{\gamma\delta}\right)} 2 \times \left(\sqrt{\frac{\delta + (x-\mu)^2}{\left(\beta^2+\gamma\right)}}\right)^{\lambda-1/2} \\
&\quad \times K_{\lambda-1/2}\left(\sqrt{\left(\beta^2+\gamma\right)\left(\delta + (x-\mu)^2\right)}\right) \qquad (3.7)
\end{aligned}$$

3.1.4 Mixing with parameterization 3

$$f(x|z) = \frac{1}{\sqrt{2\pi z}} \exp\left\{-\frac{1}{2}\left[\frac{(x-(\mu+\beta z))^2}{z}\right]\right\}, \ x \in \mathbb{R}$$

$$f(z) = \frac{\left(\frac{\gamma}{\delta}\right)^{\lambda}}{2K_\lambda(\gamma\delta)} z^{\lambda-1} \exp\left\{-\frac{1}{2}\left(\delta^2 z^{-1} + \gamma^2 z\right)\right\}, z > 0$$

Therefore

$$\begin{aligned}
f(x) &= \int_0^\infty \frac{1}{\sqrt{2\pi z}} \exp\left\{-\frac{1}{2}\left[\frac{(x-(\mu+\beta z))^2}{z}\right]\right\} \frac{\left(\frac{\gamma}{\delta}\right)^\lambda}{2K_\lambda(\gamma\delta)} z^{\lambda-1} \\
&\quad \times \exp\left\{-\frac{1}{2}\left(\delta^2 z^{-1} + \gamma^2 z\right)\right\} dz \\
&= \frac{\left(\frac{\gamma}{\delta}\right)^\lambda}{\sqrt{2\pi} 2K_\lambda(\gamma\delta)} \int_0^\infty z^{(\lambda-1/2)-1} \\
&\quad \times \exp\left\{\frac{-(x-\mu)^2}{2z} + \beta(x-\mu) - \frac{\beta^2 z}{2} - \frac{\gamma^2 z}{2} - \frac{\delta^2}{2}\frac{1}{z}\right\} dz \\
&= \frac{\left(\frac{\gamma}{\delta}\right)^\lambda e^{\beta(x-\mu)}}{\sqrt{2\pi} 2K_\lambda(\gamma\delta)} \int_0^\infty z^{(\lambda-1/2)-1} \\
&\quad \times \exp\left\{-\frac{\delta^2 + (x-\mu)^2}{2}\frac{1}{z} - \frac{\left(\beta^2+\gamma^2\right)}{2} z\right\} dz \\
&= \frac{\left(\frac{\gamma}{\delta}\right)^\lambda e^{\beta(x-\mu)}}{\sqrt{2\pi} 2K_\lambda(\gamma\delta)} 2 \times \left(\sqrt{\frac{\delta^2 + (x-\mu)^2}{\left(\beta^2+\gamma^2\right)}}\right)^{\lambda-1/2} \\
&\quad \times K_{\lambda-1/2}\left(\sqrt{\left(\beta^2+\gamma^2\right)\left(\delta^2+(x-\mu)^2\right)}\right) \qquad (3.8)
\end{aligned}$$

3.1.5 Mixing with parameterization 4

$$f(x|z) = \frac{1}{\sqrt{2\pi z}} \exp\left\{-\frac{1}{2}\left[\frac{(x-(\mu+\beta z))^2}{z}\right]\right\}, \ x \in \mathbb{R}$$

$$f(z) = \frac{\left(\frac{\theta^{1/2}}{\delta}\right)^\lambda}{2K_\lambda\left(2\delta\theta^{1/2}\right)} z^{\lambda-1} \exp\left\{-\left(\delta^2 z^{-1} + \theta z\right)\right\}, z > 0$$

Therefore

$$
\begin{aligned}
f(x) &= \int_0^\infty \frac{1}{\sqrt{2\pi z}} \exp\left\{-\frac{1}{2}\left[\frac{(x-(\mu+\beta z))^2}{z}\right]\right\}\left(\frac{\theta^{1/2}}{\delta}\right)^\lambda \\
&\quad \times \frac{1}{2K_\lambda\left(2\delta\theta^{1/2}\right)} z^{\lambda-1}\exp\left\{-\left(\delta^2 z^{-1}+\theta z\right)\right\} dz \\
&= \left(\frac{\theta^{1/2}}{\delta}\right)^\lambda \frac{\left(\sqrt{2\pi}\right)^{-1}}{2K_\lambda\left(2\delta\theta^{1/2}\right)} \int_0^\infty z^{(\lambda-1/2)-1} \\
&\quad \times \exp\left\{\frac{-(x-\mu)^2}{2z}+\beta(x-\mu)-\frac{\beta^2 z}{2}-\theta z-\delta^2\frac{1}{z}\right\} dz \\
&= \left(\frac{\theta^{1/2}}{\delta}\right)^\lambda \frac{\left(\sqrt{2\pi}\right)^{-1} e^{\beta(x-\mu)}}{2K_\lambda\left(2\delta\theta^{1/2}\right)} \int_0^\infty z^{(\lambda-1/2)-1} \\
&\quad \times \exp\left\{-\frac{2\delta^2+(x-\mu)^2}{2}\frac{1}{z}-\frac{\left(\beta^2+2\theta\right)z}{2}\right\} dz \\
&= \left(\frac{\theta^{1/2}}{\delta}\right)^\lambda \frac{\left(\sqrt{2\pi}\right)^{-1}}{2K_\lambda\left(2\delta\theta^{1/2}\right)} 2\times\left(\sqrt{\frac{2\delta^2+(x-\mu)^2}{\beta^2+2\theta}}\right)^{-\lambda-1/2} \\
&\quad \times K_{\lambda-1/2}\left(\sqrt{\left(\beta^2+2\theta\right)\left(2\delta^2+(x-\mu)^2\right)}\right) \qquad (3.9)
\end{aligned}
$$

3.2 Generalization of the Mixing

In this section, we adopt a parameterization of the GIGD ($Barndorff$, 1977), with which the general parameterization in the previous section can be obtained. Further we develop the specific parameterization of the resulting GHD. These specific parameterization will be discussed in details. We refer the approach as Barndorff-Nielsen Approach. See Barndorff-Nielsen, 1977.

Let

$$f\left(x|z\right)=\frac{1}{\sqrt{2\pi z}}\exp\left\{-\frac{1}{2}\left[\frac{\left(x-\left(\mu+\beta z\right)\right)^{2}}{z}\right]\right\},\ x\in\mathbb{R}$$

and

$$f\left(z\right)=\frac{\left(\frac{\psi}{\chi}\right)^{\frac{\lambda}{2}}}{2K_{\lambda}\left(\sqrt{\chi\psi}\right)}z^{\lambda-1}\exp\left\{-\frac{1}{2}\left(\chi z^{-1}+\psi z\right)\right\},z>0$$

Therefore;

$$\begin{aligned}
f(x) &= \int_0^\infty \frac{1}{\sqrt{2\pi z}} \exp\left\{-\frac{1}{2}\left[\frac{(v-(\mu+\beta z))^2}{z}\right]\right\} * \frac{\left(\frac{\psi}{\chi}\right)^{\frac{\lambda}{2}}}{2K_\lambda\left(\sqrt{\chi\psi}\right)}(z)^{\lambda-1} \\
&\quad \times \exp\left\{-\frac{1}{2}\left(\chi(z)^{-1}+\psi z\right)\right\}dz \\
&= \frac{\left(\frac{\psi}{\chi}\right)^{\frac{\lambda}{2}}}{\sqrt{2\pi}2K_\lambda\left(\sqrt{\chi\psi}\right)} \int_0^\infty (z)^{\left(\lambda-\frac{1}{2}\right)-1} \\
&\quad \times \exp\left\{\frac{-\left[(v-\mu)^2 - 2\beta(v-\mu)z + \beta^2(z)^2\right]}{2z} - \frac{1}{2}\left(\chi(z)^{-1}+\psi z\right)\right\}dz \\
&= \frac{\left(\frac{\psi}{\chi}\right)^{\frac{\lambda}{2}}}{\sqrt{2\pi}2K_\lambda\left(\sqrt{\chi\psi}\right)} \int_0^\infty (z)^{\left(\lambda-\frac{1}{2}\right)-1} \\
&\quad \times \exp\left\{-\frac{(v-\mu)^2}{2z} + \beta(v-\mu) - \frac{\beta^2 z}{2} - \frac{\psi}{2}z - \frac{\chi}{2}\frac{1}{z}\right\}dz \\
&= \frac{\left(\frac{\psi}{\chi}\right)^{\frac{\lambda}{2}} e^{\beta(v-\mu)}}{2\sqrt{2\pi}K_\lambda\left(\sqrt{\chi\psi}\right)} \int_0^\infty (z)^{\left(\lambda-\frac{1}{2}\right)-1} \\
&\quad \times \exp\left\{-\left(\frac{\psi+\beta^2}{2}\right)z - \frac{(v-\mu)^2+\chi}{2}\frac{1}{z}\right\}dz \\
&= \frac{\left(\frac{\psi}{\chi}\right)^{\frac{\lambda}{2}} e^{\beta(v-\mu)}}{2\sqrt{2\pi}K_\lambda\left(\sqrt{\chi\psi}\right)} 2\left(\frac{(v-\mu)^2+\chi}{2\left(\frac{\psi+\beta^2}{2}\right)}\right)^{\frac{1}{2}\left(\lambda-\frac{1}{2}\right)} \\
&\quad \times K_{\lambda-\frac{1}{2}}\left(2\sqrt{\frac{\left[(v-\mu)^2+\chi\right]}{2}\frac{\left[\psi+\beta^2\right]}{2}}\right) \\
&= \frac{\left(\frac{\psi}{\chi}\right)^{\frac{\lambda}{2}}}{\sqrt{2\pi}} \frac{e^{\beta(v-\mu)}\left[\sqrt{\left((v-\mu)^2+\chi\right)}\right]^{\left(\lambda-\frac{1}{2}\right)}}{\left[\sqrt{\left(\psi+\beta^2\right)}\right]^{\left(\lambda-\frac{1}{2}\right)}} \\
&\quad \times \frac{K_{\lambda-\frac{1}{2}}\left(\sqrt{\left[(v-\mu)^2+\chi\right]\left[\psi+\beta^2\right]}\right)}{K_\lambda\left(\sqrt{\chi\psi}\right)}
\end{aligned}$$

$$
\begin{aligned}
f(x) &= \frac{\left(\frac{\psi}{\chi}\right)^{\frac{\lambda}{2}}}{\sqrt{2\pi}} \frac{e^{\beta(x-\mu)}\left[\sqrt{\left((x-\mu)^2+\chi\right)}\right]^{\left(\lambda-\frac{1}{2}\right)}}{\left[\sqrt{\left(\psi+\beta^2\right)}\right]^{\left(\lambda-\frac{1}{2}\right)}} \\
&\quad \times \frac{K_{\lambda-\frac{1}{2}}\left(\sqrt{\left[(x-\mu)^2+\chi\right]\left[\psi+\beta^2\right]}\right)}{K_\lambda\left(\sqrt{\chi\psi}\right)}
\end{aligned}
$$

as obtained by Barndoff-Nielsen (1977).

3.2.1 Specific Parameterization

1^{st} Parameterization: (α, β) *parameterization*

- Let $\delta = \sqrt{\chi}$ and $\alpha = \sqrt{\psi + \beta^2}$
- $\Longrightarrow \delta^2 = \chi$ and $\alpha^2 - \beta^2 = \psi$

Therefore;

$$
\begin{aligned}
f_{GH}(x) &= \frac{\left(\frac{\alpha^2-\beta^2}{\delta^2}\right)^{\frac{\lambda}{2}}}{\sqrt{2\pi}\alpha^{\lambda-\frac{1}{2}}} e^{\beta(x-\mu)} \left[\sqrt{(x-\mu)^2+\delta^2}\right]^{\lambda-\frac{1}{2}} \\
&\quad \times \frac{K_{\lambda-\frac{1}{2}}\left(\sqrt{\left[(x-\mu)^2+\delta^2\right]\alpha^2}\right)}{K_\lambda\left(\sqrt{\delta^2\left(\alpha^2-\beta^2\right)}\right)} \\
&= \frac{\left(\alpha^2-\beta^2\right)^{\frac{\lambda}{2}}}{\delta^\lambda\sqrt{2\pi}\alpha^{\lambda-\frac{1}{2}}} e^{\beta(x-\mu)} \left[\sqrt{\delta^2+(x-\mu)^2}\right]^{\lambda-\frac{1}{2}} \\
&\quad \times \frac{K_{\lambda-\frac{1}{2}}\left(\alpha\sqrt{\delta^2+(x-\mu)^2}\right)}{K_\lambda\left(\delta\sqrt{\left(\alpha^2-\beta^2\right)}\right)} \\
&= \frac{\left(\alpha^2-\beta^2\right)^{\frac{\lambda}{2}}}{\delta^\lambda\sqrt{2\pi}\alpha^{\lambda-\frac{1}{2}}} \left[\sqrt{\delta^2+(x-\mu)^2}\right]^{\lambda-\frac{1}{2}} \\
&\quad \times \frac{K_{\lambda-\frac{1}{2}}\left(\alpha\sqrt{\delta^2+(x-\mu)^2}\right)}{K_\lambda\left(\delta\sqrt{\left(\alpha^2-\beta^2\right)}\right)} e^{\beta(x-\mu)} \qquad (3.10)
\end{aligned}
$$

Let;

$$
a(\lambda,\alpha,\beta,\delta) = \frac{\left(\alpha^2-\beta^2\right)^{\frac{\lambda}{2}}}{\delta^\lambda\sqrt{2\pi}\alpha^{\lambda-\frac{1}{2}}K_\lambda\left(\delta\sqrt{\left(\alpha^2-\beta^2\right)}\right)} \qquad (3.11)
$$

Then,

$$
f_{GH}(x) = a(\lambda,\alpha,\beta,\delta)\left[\delta^2+(x-\mu)^2\right]^{\frac{1}{2}\left(\lambda-\frac{1}{2}\right)} K_{\lambda-\frac{1}{2}}\left(\alpha\sqrt{\delta^2+(x-\mu)^2}\right) e^{\beta(x-\mu)} \qquad (3.12)
$$

This density depend on five parameters:

- $\alpha > 0$ as a shape parameter
- β with $0 \leq |\beta| \leq \alpha$ determine the skewness

- $\mu \in \mathbb{R}$ the location parameter
- $\delta > 0$ as a scaling factor
- $\lambda \in \mathbb{R}$ characterizes certain sub-classes and is related to the amount of mass in the tails.

The domain of variations of the parameters are $\mu \in \mathbb{R}$ and:

- $\delta > 0$, $|\beta| \leq \alpha$ if $\lambda < 0$
- $\delta > 0$, $|\beta| < \alpha$ if $\lambda = 0$
- $\delta \geq 0$, $|\beta| < \alpha$ if $\lambda > 0$

2^{nd} Parameterization: (ζ, ρ) *parameterization*
Define

$$\zeta = \delta\sqrt{\alpha^2 - \beta^2} \text{ and } \rho = \beta/\alpha \tag{3.13}$$

3^{rd} Parameterization: (ξ, χ) *parameterization*
4^{th} Parameterization: $(\bar{\alpha}, \bar{\beta})$ *parameterization*
Define

$$\bar{\alpha} = \alpha\delta \text{ and } \bar{\beta} = \beta\delta \tag{3.14}$$

Note: for symmetric distributions $\beta = \bar{\beta} = \rho = \chi = 0$ holds.

Henceforth from here we shall use (α, β) *parameterization* unless otherwise stated.

The m.g.f for Z is as follows:

3.2.2 Effects of the Parameters

In this subsection, the effect of the five parameters of the GHD is considered. This is done by varing the parameter of interest while holding the others constant.

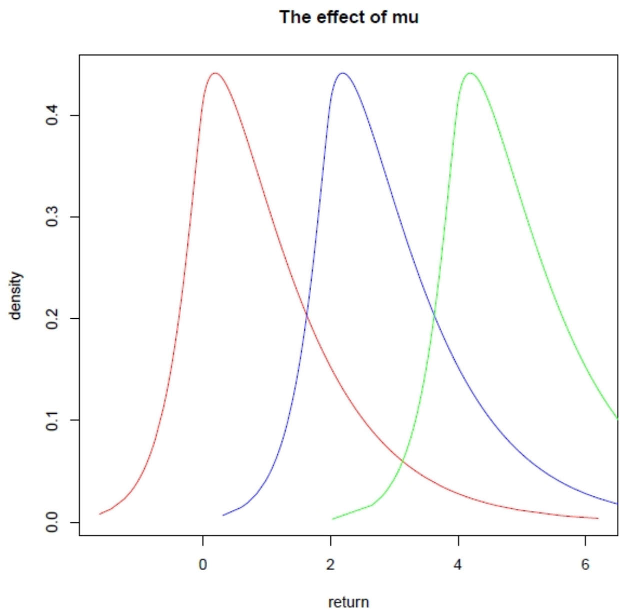


Figure 3.1: pdf curve of three GHDs with $\lambda = 1.5,\ \alpha = 2,\ \beta = 0,\ \delta = 1$ and $\mu = 0$ (red), $\mu = 2$ (blue) and $\mu = 4$ (green).

Effect of the location parameter μ

The location parameter μ is responsible for the horizontal movement of the distribution. An increase in μ moves the pdf curves of the distribution rightward horizontally. Figure 3.1 below illustrate this effect.

Effect of the scale parameter δ

An increase in the value delta flattens the distribution. At the same time, a raise in delta with α remaining constant decreases the kurtosis of the GH distributions. This illustrates that the effect of parameters in GHDs is multifold: one parameter can have an impact on different moments. Figure 3.2 below illustrate this effect.

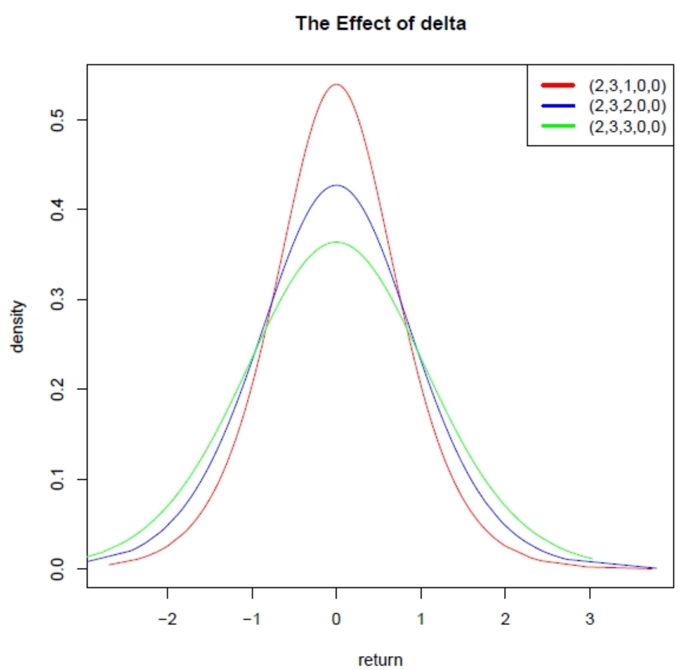


Figure 3.2: pdf curves of three GHDs with $\lambda = 2,\ \alpha = 3,\ \beta = 0,\ \mu = 0$ and $\delta = 1$ (red), $\delta = 2$ (blue) and $\delta = 3$ (green)

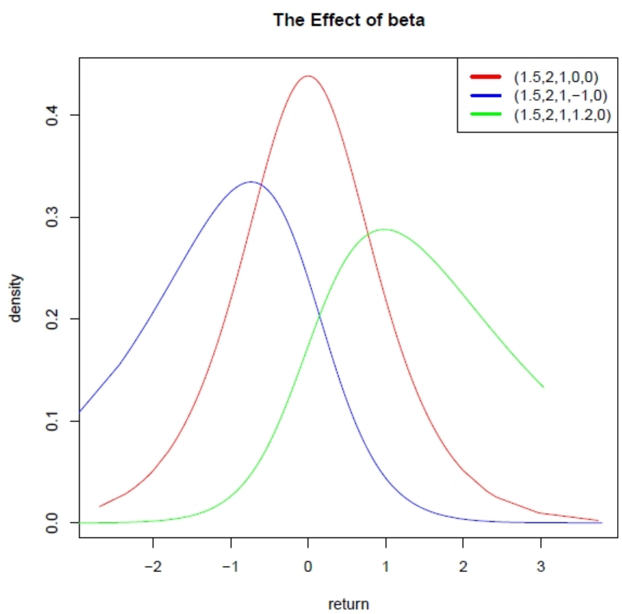


Figure 3.3: pdf curves of three GHs with $\lambda = 1.5,\ \alpha = 2,\ \delta = 1,\ \mu = 0$ and $\beta = 0$ (red), $\beta = -1$ (blue) and $\beta = 1.2$

The effect of the parameter β

The parameter β captures the skewness of the distribution. The pdf curve skews left when $\beta < 0$ and skews right when $\beta > 0$. The pdf is symmetric when $\beta = 0$. With larger values of absolute values of β, the skewness is more obvious. Besides the main effect, β also moves the pdf curve horizontally, which means it changes the mean. Figure 3.4 illustrates The pdf curve moves rightward when β takes a positive value and moves leftward when β takes a negative value. Figure 3.3 below demonstrates the effect of β.

The effect of the parameter α

A decrease in alpha result in an increase in kurtosis, which peaks the pdf curve. At the same time, other parameters remaining constant, a decrease in α also forces the variance to increase, which in contrast flattens the curve,

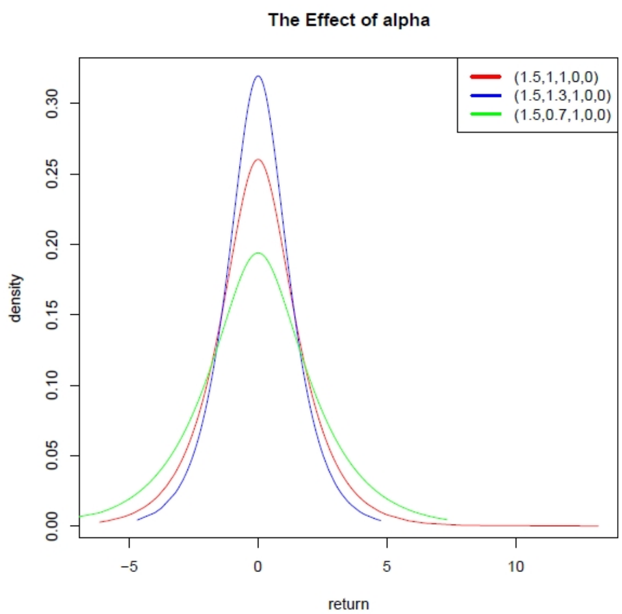


Figure 3.4: pdf curves of three GHDs with $\lambda = 1.5$, $\delta = 1$, $\beta = 0$, $\mu = 0$ and $\alpha = 1$ (red), $\alpha = 1.3$ (blue) and $\alpha = 0.7$ (green)

and vice versa. Figure 3.4 below demonstrates the effect of α.

The combined effect of $\alpha\delta$

An increase in $\alpha\delta$ reflects a decrease in the kurtosis. Figure 3.5 below demonstrates the effect of increasing gradually the value of α from 1 to 2.25, while decreasing gradually the value of δ from 1.2 to 0.8. This has the effect of increasing the product of $\alpha\delta$ from 1.2 to 1.5, and further to 1.8. From the figure, the red pdf curve which has the largest $\alpha\delta$ value, has the fastest decaying speed in tail areas, implying it has the smallest kurtosis while the green curve has the slowest decaying speed.

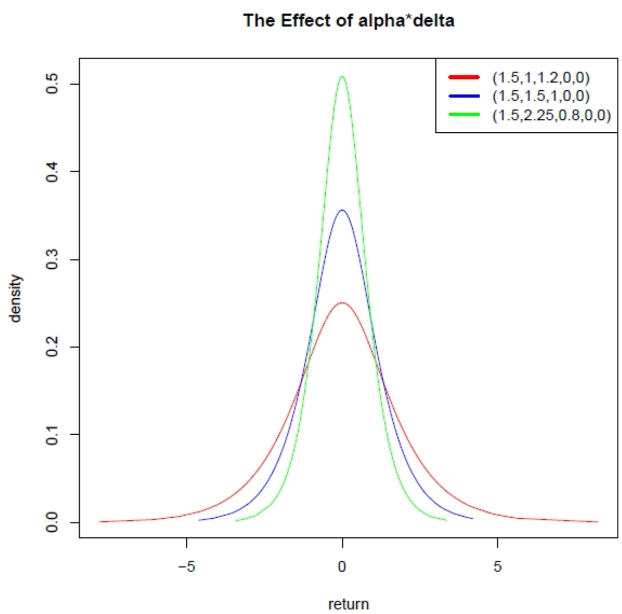


Figure 3.5: pdf curves of three GHDs with $\lambda = 1.5$, $\beta = 0$, $\mu = 0$, and $\alpha = 1$ & $\delta = 1.2$ (red), $\alpha = 1.5$ & $\delta = 1$ (blue) and $\alpha = 2.25$ & $\delta = 0.8$ (green)

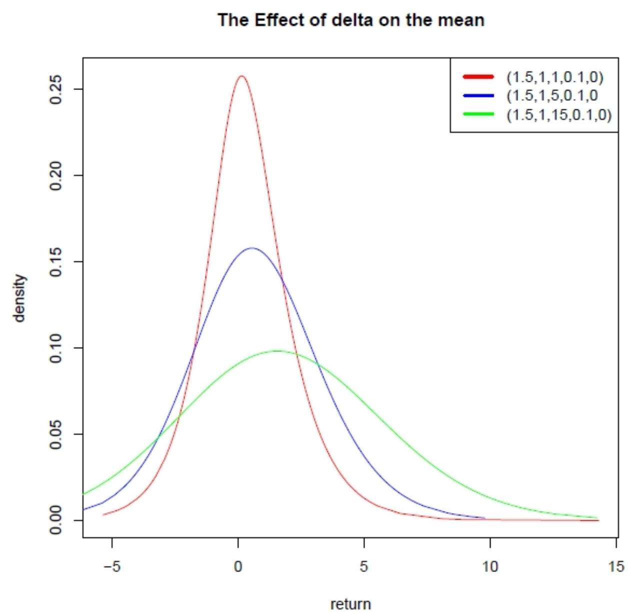


Figure 3.6: pdf curves of three GHDs with $\lambda = 1.5,\ \alpha = 1,\ \beta = 0.1,\ \mu = 0$ and $\delta = 1$ (red), $\delta = 5$ (blue), $\delta = 15$ (green).

The effect of δ on the mean

The mean of GHDs increases in δ under the given parameterisation. In the figure below the curves move rightward with the increase of δ.

The effect of λ on the mean

Similarly, its easy to discern that the mean of the GHDs increases in λ under given parameter settings. The curves move rightward with increase of λ as seen in the figure below.

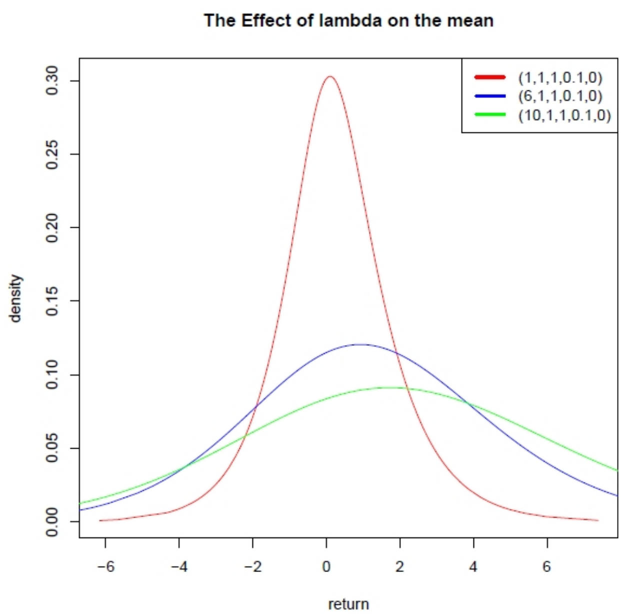


Figure 3.7: pdf curves of three GHDs with $\alpha = 1,\ \beta = 0.1,\ \delta = 1,\ \mu = 0$ and $\lambda = 1$ (red), $\lambda = 6$ (blue) and $\lambda = 10$ (green).

3.2.3 Properties

The Moment Generating Function

$$
\begin{aligned}
M(z) &= E\left[e^{zX}\right] \\
&= \int_0^\infty e^{zX}\left[a(\lambda,\alpha,\beta,\delta)\right]\left[\delta^2+(x-\mu)^2\right]^{\frac{1}{2}\left(\lambda-\frac{1}{2}\right)} \\
&\quad \times e^{\beta(x-\mu)} K_{\lambda-\frac{1}{2}}\left(\alpha\sqrt{\delta^2+(x-\mu)^2}\right)dx \\
&= a(\lambda,\alpha,\beta,\delta)\int_0^\infty \left[\delta^2+(x-\mu)^2\right]^{\frac{1}{2}\left(\lambda-\frac{1}{2}\right)} \\
&\quad \times e^{\beta(x-\mu)+zx} K_{\lambda-\frac{1}{2}}\left(\alpha\sqrt{\delta^2+(x-\mu)^2}\right)dx \\
&= a(\lambda,\alpha,\beta,\delta)\int_0^\infty \left[\delta^2+(x-\mu)^2\right]^{\frac{1}{2}\left(\lambda-\frac{1}{2}\right)} \\
&\quad \times e^{\beta(x-\mu)+z(x-\mu+\mu)} K_{\lambda-\frac{1}{2}}\left(\alpha\sqrt{\delta^2+(x-\mu)^2}\right)dx \\
&= e^{\mu z} a(\lambda,\alpha,\beta,\delta)\int_0^\infty \left[\delta^2+(x-\mu)^2\right]^{\frac{1}{2}\left(\lambda-\frac{1}{2}\right)} \\
&\quad \times e^{(\beta+z)(x-\mu)} K_{\lambda-\frac{1}{2}}\left(\alpha\sqrt{\delta^2+(x-\mu)^2}\right)dx \\
&= e^{\mu z}\frac{a(\lambda,\alpha,\beta,\delta)}{a(\lambda,\alpha,\beta+z,\delta)}\int_0^\infty a(\lambda,\alpha,\beta+z,\delta)\left[\delta^2+(x-\mu)^2\right]^{\frac{1}{2}\left(\lambda-\frac{1}{2}\right)} \\
&\quad \times e^{(\beta+z)(x-\mu)} K_{\lambda-\frac{1}{2}}\left(\alpha\sqrt{\delta^2+(x-\mu)^2}\right)dx \\
&= e^{\mu z}\frac{a(\lambda,\alpha,\beta,\delta)}{a(\lambda,\alpha,\beta+z,\delta)} * 1 \\
&= e^{\mu z}\frac{a(\lambda,\alpha,\beta,\delta)}{a(\lambda,\alpha,\beta+z,\delta)} \\
&= e^{\mu z}\frac{\left(\alpha^2-\beta^2\right)^{\frac{\lambda}{2}}}{\delta^\lambda\sqrt{2\pi}\alpha^{\lambda-\frac{1}{2}}K_\lambda\left(\delta\sqrt{\left(\alpha^2-\beta^2\right)}\right)} \\
&\quad \times\frac{\delta^\lambda\sqrt{2\pi}\alpha^{\lambda-\frac{1}{2}}K_\lambda\left(\delta\sqrt{\left(\alpha^2-(\beta+z)^2\right)}\right)}{\left(\alpha^2-(\beta+z)^2\right)^{\frac{\lambda}{2}}} \\
&\quad e^{\mu z}\left(\alpha^2-\beta^2\right)^{\frac{\lambda}{2}}
\end{aligned}
$$

$$\text{for } (\beta+z)^2 < \alpha^2 \Longrightarrow |\beta+z| < \alpha.$$

$$\text{set } \omega = \delta\sqrt{\alpha^2-\beta^2} = \sqrt{\chi\psi} \text{ then}$$

$$\begin{aligned}
M(z) &= e^{\mu z}\left\{\frac{\alpha^2-\beta^2}{\alpha^2-(\beta+z)^2}\right\}^{\frac{\lambda}{2}} \\
&\quad \times \frac{K_\lambda\left(\delta\sqrt{\left(\alpha^2-(\beta+z)^2\right)}\right)}{K_\lambda\left(\delta\sqrt{\left(\alpha^2-\beta^2\right)}\right)} \\
&= e^{\mu z}\left\{\frac{\left(\frac{\omega}{\delta}\right)^\lambda}{\alpha^2-(\beta+z)^2}\right\}^{\frac{\lambda}{2}} \\
&\quad \times \frac{K_\lambda\left(\delta\sqrt{\alpha^2-(\beta+z)^2}\right)}{K_\lambda(\omega)} \\
&= \frac{\left(\frac{\omega}{\delta}\right)^\lambda}{K_\lambda(\omega)} \frac{e^{\mu z}K_\lambda\left(\delta\sqrt{\alpha^2-(\beta+z)^2}\right)}{\left[\alpha^2-(\beta+z)^2\right]^{\frac{\lambda}{2}}} \\
&= \frac{\left(\alpha^2-\beta^2\right)^{\lambda/2}}{K_\lambda\left(\delta\sqrt{\alpha^2-\beta^2}\right)} \frac{e^{\mu z}K_\lambda\left(\delta\sqrt{\alpha^2-(\beta+z)^2}\right)}{\left[\alpha^2-(\beta+z)^2\right]^{\frac{\lambda}{2}}} \\
&= \exp(\mu z)\left(\frac{\alpha^2-\beta^2}{\alpha^2-(\beta+z)^2}\right)^{\lambda/2} \frac{K_\lambda\left(\delta\sqrt{\alpha^2-(\beta+z)^2}\right)}{K_\lambda\left(\delta\sqrt{\alpha^2-\beta^2}\right)}
\end{aligned}$$

The characteristic function of the generalised hyperbolic distribution is given by:

$$\varphi_{GHD}(z) = \exp(i\mu z)\left(\frac{\alpha^2-\beta^2}{\alpha^2-(\beta+iz)^2}\right)^{\lambda/2}\frac{K_\lambda\left(\delta\sqrt{\alpha^2-(\beta+iz)^2}\right)}{K_\lambda\left(\delta\sqrt{\alpha^2-\beta^2}\right)} \quad (3.16)$$

Therefore;

$$\begin{aligned}
\log M(z) &= \log\left(\frac{\omega}{\delta}\right)^\lambda - \log K_\lambda(\omega) + \mu z + \log K_\lambda\left(\delta\sqrt{\alpha^2-(\beta+z)^2}\right) \\
&\quad - \log\left[\alpha^2-(\beta+z)^2\right]^{\frac{\lambda}{2}} \\
\frac{d}{dz}\log M(z) &= \mu + \frac{K'_\lambda\left(\delta\sqrt{\alpha^2-(\beta+z)^2}\right)\frac{d}{dz}\delta\sqrt{\alpha^2-(\beta+z)^2}}{K_\lambda\left(\delta\sqrt{\alpha^2-(\beta+z)^2}\right)} \\
&\quad - \frac{\lambda\left[\alpha^2-(\beta+z)^2\right]^{\frac{\lambda}{2}-1}\left[-2(\beta+z)\right]}{2\left[\alpha^2-(\beta+z)^2\right]^{\frac{\lambda}{2}}} \\
&= \mu + \frac{K'_\lambda\left(\delta\sqrt{\alpha^2-(\beta+z)^2}\right)\frac{d}{dz}\delta\sqrt{\alpha^2-(\beta+z)^2}}{K_\lambda\left(\delta\sqrt{\alpha^2-(\beta+z)^2}\right)} \\
&\quad + \frac{\lambda(\beta+z)}{\left[\alpha^2-(\beta+z)^2\right]} \qquad (3.17)
\end{aligned}$$

3.2.4 Moments of The Generalized Hyperbolic Distribution

Denote the $k-th$ moment about μ by

$$\bar{M}_k = E(X-\mu)^k \quad (3.18)$$

Then the mixture represenation of GH distribution leads to the following results for $\bar{M}_k$ (Scott et al, 2009)

$$\bar{M}_k = \sum_{l=\lfloor (k+1)/2 \rfloor}^{k} \frac{k!}{(k-l)!\,(2l-k)!2^{k-l}} \beta^{2l-k} E\left(W^l\right) \tag{3.19}$$

where $W \sim gig\left(\lambda, \delta^2, \alpha^2 - \beta^2\right)$.

Proof:

since

$$W \sim gig\left(\lambda, \delta^2, \alpha^2 - \beta^2\right) \text{ and } X \mid W \sim N\left(\mu + \beta W, W\right)$$

$$X \sim gh\left(\lambda, \alpha, \beta, \delta, \mu\right)$$

Suppose $Z \sim N(0,1)$ independently of $W \sim gig\left(\lambda, \delta^2, \alpha^2 - \beta^2\right)$ then it implies that

$$\mu + \beta W + \sqrt{W} Z \sim gh\left(\lambda, \alpha, \beta, \delta, \mu\right)$$

which expression is commonly used to simulate from the GH distribution.

As shown early, if $X \sim gig\left(\lambda, \chi, \psi\right)$ the provided $\chi > 0$ and $\psi > 0$,

$$E\left(X^k\right) = \left(\chi/\psi\right)^{(\lambda+k)/2} K_{\lambda+k}\left(\sqrt{\chi\psi}\right) / K_\lambda\left(\sqrt{\chi\psi}\right)$$

when $\chi = 0$ and $\lambda > 0$, X has a gamma distribution, and the moments are

$$E\left(X^k\right) = \left(2/\psi\right)^k \Gamma\left(\lambda + k\right) / \Gamma\left(\lambda\right)$$

when $\psi = 0$ and $\lambda < 0$, X has an inverse gamma distribution, and the moments are

$$E\left(X^k\right) = \left(\chi/2\right)^k \Gamma\left(-\lambda - k\right) / \Gamma\left(-\lambda\right)$$

provided $-\lambda > k$.

Therefore

$$\begin{aligned}
\bar{M}_k &= E\left[(X-\mu)^k\right] \\
&= E\left[\left(\mu+\beta W+\sqrt{W}Z-\mu\right)^k\right] \\
&= E\left[\left(\beta W+\sqrt{W}Z\right)^k\right] \\
&= E\left[\sum_{i=0}^{k}\binom{k}{i}(\beta W)^{k-i}\left(\sqrt{W}Z\right)^i\right] \\
&= E\left[\sum_{i=0}^{k}\binom{k}{i}\beta^{k-i}W^{k-i}W^{\frac{i}{2}}Z^i\right] \\
&= E\left[\sum_{i=0}^{k}\binom{k}{i}\beta^{k-i}W^{k-\frac{i}{2}}Z^i\right] \\
&= \sum_{i=0}^{k}\binom{k}{i}\beta^{k-i}E\left(W^{k-\frac{i}{2}}\right)E\left(Z^i\right)
\end{aligned}$$

since, for the standard normal distribution,

$$E\left(Z^i\right)=\begin{cases}0 & i\ odd \\ i!\left[2^{i/2}\,(i/2)!\right]^{-1} & i\ even\end{cases}$$

$$\bar{M}_k=\sum_{i=0}^{k}\binom{k}{i}\beta^{k-i}E\left(W^{k-\frac{i}{2}}\right)\frac{i!}{2^{i/2}\,(i/2)!}$$

Let $j = \frac{i}{2} \Longrightarrow i = 2j$

$$\begin{aligned}
\bar{M}_k &= \sum_{j=0}^{\lfloor k/2 \rfloor} \binom{k}{2j} \beta^{k-2j} E\left(W^{k-j}\right) \frac{(2j)!}{2^j j!} \\
&= \sum_{j=0}^{\lfloor k/2 \rfloor} \frac{k!}{(2j)!\,(k-2j)!} \beta^{k-2j} E\left(W^{k-j}\right) \frac{(2j)!}{2^j j!} \\
&= \sum_{j=0}^{\lfloor k/2 \rfloor} \frac{k!}{j!\,(k-2j)!2^j} \beta^{k-2j} E\left(W^{k-j}\right)
\end{aligned}$$

Again let $l = k - j \Longrightarrow j = k - l$

$$\begin{aligned}
\bar{M}_k &= \sum_{j=0}^{\lfloor k/2 \rfloor} \frac{k!}{j!\,(k-2j)!2^j} \beta^{k-2j} E\left(W^{k-j}\right) \\
&= \sum_{\lfloor l=(k+1)/2 \rfloor}^{k} \frac{k!}{(k-l)!\,(2l-k)!2^{k-l}} \beta^{2l-k} E\left(W^{l}\right) \qquad (3.20)
\end{aligned}$$

The moment of the mixing distribution W can be found form the moment of the GIG distribution.

Lemma Suppose $X \sim gh\left(\lambda, \alpha, \beta, \delta, \mu\right)$ .Then for $W \sim gig\left(\lambda, \delta^2, \alpha^2 - \beta^2\right)$

$$E\left(W^k\right) = \left(\delta^2/\zeta\right)^{\lambda+k} K_{\lambda+k}\left(\zeta\right)/K_{\lambda}\left(\zeta\right)$$

provided $\chi > 0$ and $\psi > 0$.

Proof The mixing distribution W is GIG with $\chi = \delta^2$ and $\psi = \alpha^2 - \beta^2$. The results follows form substitution of these values, noting that $\zeta = \delta\sqrt{\alpha^2 - \beta^2}$

This follows that

$$\begin{aligned}\bar{M}_k &= \sum_{\lfloor l=(k+1)/2\rfloor}^{k} \frac{k!}{(k-l)!\,(2l-k)!2^{k-l}}\beta^{2l-k}E\left(W^l\right) \\ &= \sum_{\lfloor l=(k+1)/2\rfloor}^{k} \frac{k!}{(k-l)!\,(2l-k)!2^{k-l}}\beta^{2l-k}\left(\delta^2/\zeta\right)^{\lambda+l}K_{\lambda+l}(\zeta)/K_\lambda(\zeta) \quad (3.21)\end{aligned}$$

If $\beta = 0$ the generalised hyperbolic distribution is symmetric and the moments take a particularly simple form.Note that the mean is just μ in this case and the moment about μ are usually central moments.

Suppose $X \sim gh(\lambda, \alpha, 0, \delta, \mu)$. Then for any integer $k > 0$, $\bar{M}_k$ is the $k-th$ central moment, and $\bar{M}_k = 0$ for k odd, while

$$\begin{aligned}\bar{M}_k &= a_{k,k/2}\left(\delta^2/\zeta\right)^{\lambda+k/2}K_{\lambda+k/2}(\zeta)/K_\lambda(\zeta) \\ &= k!\left[2^{k/2}(k/2)!\right]^{-1}\left(\delta^2/\zeta\right)^{\lambda+k/2}K_{\lambda+k/2}(\zeta)/K_\lambda(\zeta)\end{aligned}$$

for k even.

The expression

$$\bar{M}_k = \sum_{\lfloor l=(k+1)/2\rfloor}^{k} \frac{k!}{(k-l)!\,(2l-k)!2^{k-l}}\beta^{2l-k}\left(\delta^2/\zeta\right)^{\lambda+l}K_{\lambda+l}(\zeta)/K_\lambda(\zeta)$$

gives the moments about μ of any order for the GH distribution. For implementation, it is useful to have a recursive method of obtaining the coefficients in the summation which in (**) and (**). Define the coefficients $a_{k,l}$ for $l = 1, 2, 3, \ldots, k$ and $k = 1, 2, \ldots$ by

$$a_{k,l} = \left\{\begin{matrix} 0 & l < \lfloor(k+1)/2\rfloor \\ k!\left[(k-l)!\,(2l-k)!2^{k-l}\right]^{-1} & \lfloor(k+1)/2\rfloor \ < l < k \end{matrix}\right\}$$

For convinience define $m(k) = \lfloor (k+1)/2 \rfloor$

The coeffient $a_{k,l}$ may be determined recursively as $a_{1,1} = 1$ and

$$a_{k,l} = a_{k-1,l-1} + (2l - k + 1)\, a_{k-1,l}$$

Proof The result is true for $k = 1$.

We show it is true for all k by induction. Assume the result is true for $k - 1$.

For k even and $l = m(k) = k/2$,the recurrence relation above gives

$$\begin{aligned}
a_{k,m(k)} &= a_{k-1,m(k)-1} + (2m(k) - k + 1)\, a_{k-1,m(k)} \\
&= 0 + (2m(k) - k + 1)\, a_{k-1,m(k)} \\
&= (2m(k) - k + 1)\, a_{k-1,m(k)} \\
&= (2(k/2) - k + 1)\, a_{k-1,k/2} \\
&= a_{k-1,k/2} \\
&= \frac{(k-1)!}{(k-1-(k/2))!\,(k-(k-1))!\,(2(k/2)-k)!2^{k-1-(k/2)}} \\
&= \frac{(k-1)!}{(k-k/2-1)!\,(1)!\,(0)!2^{k-k/2-1}} \\
&= \frac{2(k-k/2)(k-1)!}{(k-k/2)!1!2^{k-k/2}} \\
&= \frac{k(k-1)!}{(k-m(k))!0!2^{k-m(k)}} \\
&= \frac{k!}{(k-m(k))!\,(2m(k)-k)!2^{k-m(k)}}
\end{aligned}$$

as required.

For any other values of k and l we have

$$
\begin{aligned}
a_{k,l} &= a_{k-1,l-1} + (2l-k+1)\, a_{k-1,l} \\
&= \frac{(k-1)!}{(k-1-l+1)!\,(2\,(l-1)-(k-1))!2^{k-l}} \\
&\quad + \frac{(2l-k+1)\,(k-1)!}{(k-1-l)!\,(2l-(k-1))!2^{k-1-l}} \\
&= \frac{(k-1)!}{(k-l)!\,(2l-k-1)!2^{k-l}} + \frac{(2l-k+1)\,(k-1)!}{(k-1-l)!\,(2l-k+1)!2^{k-1-l}} \\
&= \frac{(2l-k)\,k\,(k-1)!}{k\,(k-l)!\,(2l-k)!2^{k-l}} + \frac{k\,(k-l)\,2\,(k-1)!}{k\,(k-l)!\,(2l-k)!2^{k-l}} \\
&= \frac{(2l-k)\,k!}{k\,(k-l)!\,(2l-k)!2^{k-l}} + \frac{(k-l)\,2k!}{k\,(k-l)!\,(2l-k)!2^{k-l}} \\
&= \frac{k!}{(k-l)!\,(2l-k)!2^{k-l}} \left[\frac{(2l-k)}{k} + \frac{2\,(k-l)}{k}\right] \\
&= \frac{k!}{(k-l)!\,(2l-k)!2^{k-l}} \left[\frac{2l-k+2k-2l}{k}\right] \\
&= \frac{k!}{(k-l)!\,(2l-k)!2^{k-l}} \left[\frac{k}{k}\right] \\
&= \frac{k!}{(k-l)!\,(2l-k)!2^{k-l}}
\end{aligned}
$$

which is again of the required form, completing the proof.

Though it is possible to compute the moments about μ directly from the expression(**), using the recursion is very stable numerically and avoids the possibility of intermediate expression swell which can occur for the factorial present in equation (**).

The coefficients $a_{k,l}$ for $k = 1, 2, 3, 4$ are

$$\begin{aligned}
a_{1,1} &= 1\\
a_{2,1} &= 1,\ a_{2,2}=1\\
a_{3,1} &= 0,\ a_{3,2}=3,\ a_{3,3}=1\\
a_{4,1} &= 0,\ a_{4,2}=3,\ a_{4,3}=6,\ a_{4,4}=1
\end{aligned}$$

So the first four moments about μ are

$$\bar{M}_1 = \left(\delta^2/\zeta\right)\beta K_{\lambda+1}\left(\zeta\right)/K_{\lambda}\left(\zeta\right) \tag{3.22}$$

$$\bar{M}_2 = \left[\left(\delta^2/\zeta\right)K_{\lambda+1}\left(\zeta\right)+\left(\delta^2/\zeta\right)^2\beta^2 K_{\lambda+2}\left(\zeta\right)\right]/K_{\lambda}\left(\zeta\right) \tag{3.23}$$

$$\bar{M}_3 = \left[3\left(\delta^2/\zeta\right)^2\beta K_{\lambda+2}\left(\zeta\right)+\left(\delta^2/\zeta\right)^3\beta^3 K_{\lambda+3}\left(\zeta\right)\right]/K_{\lambda}\left(\zeta\right) \tag{3.24}$$

$$\bar{M}_4 = \left[3\left(\delta^2/\zeta\right)^2 K_{\lambda+2}\left(\zeta\right)+6\left(\delta^2/\zeta\right)^3\beta^2 K_{\lambda+3}\left(\zeta\right)+\left(\delta^2/\zeta\right)^4\beta^4 K_{\lambda+4}\left(\zeta\right)\right]/K_{\lambda}\left(\zeta\right) \tag{3.25}$$

The calculation of moments about μ (Scott et al, 2009)is mathematically convinient for the GH distribution. Typically though, moments about zero (raw moments) or about the mean (central moments) are required. Changing the point about which the moments are calculated is straight-forward however involving a simple application of the binomial theorem. The computation borrows from the analogy of interchanging between the raw and central moments.

Therefore, for any constant a and b, and any integer k,

$$
\begin{aligned}
E\left[(X-b)^k\right] &= E\left[(X-a-b+a)^k\right] \\
&= E\left[\sum_{i=0}^{k}\binom{k}{i}(a-b)^{k-i}(X-a)^i\right] \\
&= \sum_{i=0}^{k}\binom{k}{i}(a-b)^{k-i}E\left[(X-a)^i\right] \qquad (3.26)
\end{aligned}
$$

Low Order Moments

Define:

$$R_{\lambda,i} = K_{\lambda+i}(\zeta)/K_{\lambda}(\zeta)$$

Denote the central moments by

$$M_k = \left[(X-E(X))^k\right] \text{ for } k = 2,3,...$$

Using the previous section we can obtain expression for the mean, M_2, M_2, and M_4.

$$E(X) = \mu + \frac{\beta\delta}{\sqrt{\alpha^2-\beta^2}}R_{\lambda,1}(\zeta) \qquad (3.27)$$

$$
\begin{aligned}
M_2 &= var(X) = \delta^4\beta^2\zeta^{-1}\left[R_{\lambda,2}(\zeta) - R^2_{\lambda,2}(\zeta)\right] \qquad (3.28) \\
&\quad +\delta^2\zeta^{-1}R_{\lambda,1}(\zeta)
\end{aligned}
$$

$$
\begin{aligned}
M_3 &= \delta^6\beta^3\zeta^{-3}\left[R_{\lambda,3}(\zeta) - 3R_{\lambda,2}(\zeta)R_{\lambda,1}(\zeta) + 2R^3_{\lambda,1}(\zeta)\right] \qquad (3.29) \\
&\quad +3\delta^4\beta\zeta^{-2}\left[R_{\lambda,2}(\zeta) - R^2_{\lambda,1}(\zeta)\right]
\end{aligned}
$$

and

$$\begin{aligned} M_4 &= \delta^8\beta^4\zeta^{-4}\left[\begin{array}{c} R_{\lambda,4}(\zeta)-4R_{\lambda,3}(\zeta)R_{\lambda,1}(\zeta)+ \\ 6R_{\lambda,2}(\zeta)R_{\lambda,1}^2(\zeta)-3R_{\lambda,1}^3(\zeta) \end{array}\right] \\ &\quad +6\delta^6\beta^2\zeta^{-3}\left[R_{\lambda,3}(\zeta)-2R_{\lambda,2}(\zeta)R_{\lambda,1}(\zeta)+R_{\lambda,1}^3(\zeta)\right] \end{aligned} \tag{3.30}$$

Note: One of the features of constructing a distribution by mixing is that one can essentially read off the properties of the distribution given the properties of the chosen weight (Paolella, 2007). The density of X depends on Z only via the p.d.f of Z, hence only through the distribution of Z.

$$E(X)=\mu+\beta E(Z) \tag{3.31}$$

Proof: Using the Fubini's theorem:

$$\begin{aligned} E(X) &= \int_{-\infty}^{\infty} x f_X(x)\,dx \\ &= \int_{-\infty}^{\infty} x \int_{0}^{\infty} f_N(x;\mu+\beta z,z)\,f_Z(z)\,dzdx \\ &= \int_{0}^{\infty}\int_{-\infty}^{\infty} x f_N(x;\mu+\beta z,z)\,dx f_Z(z)\,dz \\ &= \int_{0}^{\infty}(\mu+\beta z)\,f_Z(z)\,dz \\ &= \int_{0}^{\infty}\mu f_Z(z)\,dz+\beta\int_{0}^{\infty} z f_Z(z)\,dz \\ &= \mu+\beta E(Z) \end{aligned}$$

$$var(X)=E(Z)+\beta^2 var(Z) \tag{3.32}$$

Proof: Using the Fubini'theorem:

$$\begin{aligned}
E\left(X^2\right) &= \int_{-\infty}^{\infty} x^2 f_X(x)\,dx \\
&= \int_{-\infty}^{\infty} x^2 \int_{0}^{\infty} f_N(x;\mu+\beta z, z)\, f_Z(z)\,dz dx \\
&= \int_{0}^{\infty} \int_{-\infty}^{\infty} x^2 f_N(x;\mu+\beta z, z)\,dx f_Z(z)\,dz \\
&= \int_{0}^{\infty} \left(z+(\mu+\beta z)^2\right) f_Z(z)\,dz \\
&= \int_{0}^{\infty} \left(z+\mu^2+\beta^2 z^2+2\beta\mu z\right) f_Z(z)\,dz \\
&= E(Z)+\mu^2+\beta^2 E\left(Z^2\right)+2\beta\mu E(Z)
\end{aligned}$$

Therefore

$$\begin{aligned}
var(X) &= E\left(X^2\right)-E^2(X) \\
&= E(Z)+\mu^2+\beta^2 E\left(Z^2\right)+2\beta\mu E(Z)-(\mu+\beta E(Z))^2 \\
&= E(Z)+\mu^2+\beta^2 E\left(Z^2\right)+2\beta\mu E(Z)-\mu^2-2\beta\mu E(Z)-\beta^2 E^2(Z) \\
&= E(Z)+\beta^2 E\left(Z^2\right)-\beta^2 E^2(Z) \\
&= E(Z)+\beta^2\left(E\left(Z^2\right)-E^2(Z)\right) \\
&= E(Z)+\beta^2 var(Z)
\end{aligned}$$

The third central moment

$$\mu_3(X)=3\beta var(Z)+\beta^3\mu_3(Z) \tag{3.33}$$

Proof:

$$
\begin{aligned}
E\left(X^3\right) &= \int\limits_{-\infty}^{\infty} x^3 f_X\left(x\right)dx \\
&= \int\limits_{-\infty}^{\infty} x^3 \int\limits_{0}^{\infty} f_N\left(x;\mu+\beta z, z\right) f_Z\left(z\right)dz dx \\
&= \int\limits_{0}^{\infty} \int\limits_{-\infty}^{\infty} x^3 f_N\left(x;\mu+\beta z, z\right) dx f_Z\left(z\right)dz \\
&= \int\limits_{0}^{\infty} \left(3\mu z + \left(\mu+\beta z\right)^3\right) f_Z\left(z\right)dz \\
&= \int\limits_{0}^{\infty} \left(3\mu z + \mu^3 + 3\mu^2\beta z + 3\mu\beta^2 z^2 + \beta^3 z^3\right) f_Z\left(z\right)dz \\
&= 3\mu E\left(Z\right) + \mu^3 + 3\mu^2\beta E\left(Z\right) + 3\mu\beta^2 E\left(Z^2\right) + \beta^3 E\left(Z^3\right)
\end{aligned}
$$

Now

$$
\begin{aligned}
& E\left[(X - E(X))^3\right] \\
= \;& E\left[X^3 - 3E(X)X^2 + 3E(X)^2 X - E(X)^3\right] \\
= \;& E\left(X^3\right) - 3E(X)E\left[X^2\right] + 3E(X)^2 E[X] - E(X)^3 \\
= \;& 3\mu E(Z) + \mu^3 + 3\mu^2\beta E(Z) + 3\mu\beta^2 E\left(Z^2\right) + \beta^3 E\left(Z^3\right) - \\
& 3E(X)\left(E(Z) + \mu^2 + \beta^2 E\left(Z^2\right) + 2\beta\mu E(Z)\right) \\
& +3E(X)^2(\mu + \beta E(Z)) - E(X)^3 \\
= \;& 3\mu E(Z) + \mu^3 + 3\mu^2\beta E(Z) + 3\mu\beta^2 E\left(Z^2\right) \\
& +\beta^3 E\left(Z^3\right) - 3(\mu + \beta E(Z))E(Z) - \\
& 3\mu^2(\mu + \beta E(Z)) - 3\beta^2 E\left(Z^2\right)(\mu + \beta E(Z)) \\
& -6\beta\mu E(Z)(\mu + \beta E(Z)) + \\
& 3\mu(\mu + \beta E(Z))^2 + 3\beta E(Z)(\mu + \beta E(Z))^2 \\
& -(\mu + \beta E(Z))^3 \\
= \;& 3\beta var(Z) + \beta^3\mu_3(Z)
\end{aligned}
$$

The moment generating function

$$
M_X(t) = e^{\mu t} M_Z\left(\beta t + t^2/2\right) \tag{3.34}
$$

Proof:

$$\begin{aligned} M_X(t) &= \int_{-\infty}^{\infty} e^{tx} f_X(x)\, dx \\ &= \int_{-\infty}^{\infty} e^{tx} \int_{0}^{\infty} f_N(x; \mu + \beta z, z)\, f_Z(z)\, dz dx \\ &= \int_{0}^{\infty} \int_{-\infty}^{\infty} e^{tx} f_N(x; \mu + \beta z, z)\, dx f_Z(z)\, dz \\ &= \int_{0}^{\infty} \exp\left[(\mu + \beta z)\, t + \frac{z}{2} t^2\right] f_Z(z)\, dz \\ &= e^{\mu t} \int_{0}^{\infty} \exp\left[\left(\beta t + \frac{t^2}{2}\right) z\right] f_Z(z)\, dz \\ &= e^{\mu t} M_Z\left(\beta t + \frac{t^2}{2}\right) \end{aligned}$$

and the Characteristic function

$$\varphi_X(\upsilon) = e^{i\mu\upsilon} \varphi_Z\left(\beta\upsilon + i\upsilon^2/2\right) \quad (3.35)$$

Proof:

$$\begin{aligned}
\varphi_X(\upsilon) &= \int_{-\infty}^{\infty} e^{i\upsilon x} f_X(x)\, dx \\
&= \int_{-\infty}^{\infty} e^{i\upsilon x} \int_{0}^{\infty} f_N(x; \mu + \beta z, z)\, f_Z(z)\, dz dx \\
&= \int_{0}^{\infty} \int_{-\infty}^{\infty} e^{i\upsilon x} f_N(x; \mu + \beta z, z)\, dx f_Z(z)\, dz \\
&= \int_{0}^{\infty} \exp\left[(\mu + \beta z)\, i\upsilon + \frac{z}{2}(i\upsilon)^2\right] f_Z(z)\, dz \\
&= e^{\mu i\upsilon} \int_{0}^{\infty} \exp\left[\left(\beta\upsilon + \frac{i\upsilon^2}{2}\right) iz\right] f_Z(z)\, dz \\
&= e^{\mu t} \varphi_X\left(\beta\upsilon + \frac{i\upsilon^2}{2}\right)
\end{aligned}$$

3.3 Special Cases: The Subfamilies of GHD family

Using the notation (Paollela, 2007)

$$\begin{aligned}
K_\lambda(\chi, \psi) &= \int_{0}^{\infty} x^{\lambda-1} \exp\left[-\frac{1}{2}\left(\chi x^{-1} + \psi x\right)\right] dx \\
&= 2\left(\frac{\chi}{\psi}\right)^{\frac{\lambda}{2}} K_\lambda\left(\sqrt{\chi\psi}\right)
\end{aligned}$$

$$
\begin{aligned}
& f_{GHD}\left(x;\lambda,\alpha,\beta,\delta,\mu\right) \\
= & \int_0^\infty f_N\left(x;\mu+\beta z,z\right) f_{GIG}\left(z;\lambda,\delta^2,\alpha^2-\beta^2\right) dz \\
= & \int_0^\infty \frac{1}{\sqrt{2\pi z}} \exp\left\{-\frac{1}{2}\frac{\left(x-\left(\mu+\beta z\right)\right)^2}{z}\right\} \\
& \times \frac{1}{K_\lambda\left(\delta^2,\alpha^2-\beta^2\right)} z^{\lambda-1} \exp\left\{-\frac{1}{2}\left(\delta^2 z^{-1}+\left(\alpha^2-\beta^2\right)z\right)\right\} dz \\
= & \frac{1}{\sqrt{2\pi}K_\lambda\left(\delta^2,\alpha^2-\beta^2\right)} \int_0^\infty z^{\lambda-\frac{1}{2}-1} e^{-\frac{1}{2}\left\{z^{-1}\left[(x-\mu)^2+\delta^2\right]+z\left(\alpha^2-\beta^2+\beta^2\right)-2\beta(x-\mu)\right\}} dz \\
= & \frac{1}{\sqrt{2\pi}K_\lambda\left(\delta^2,\alpha^2-\beta^2\right)} e^{\beta(x-\mu)} \int_0^\infty z^{\lambda-\frac{1}{2}-1} e^{-\frac{1}{2}\left\{z^{-1}\left[(x-\mu)^2+\delta^2\right]+z\left(\alpha^2-\beta^2+\beta^2\right)\right\}} dz \\
= & \frac{1}{\sqrt{2\pi}K_\lambda\left(\delta^2,\alpha^2-\beta^2\right)} e^{\beta(x-\mu)} K_{\lambda-\frac{1}{2}}\left(\left(x-\mu\right)^2+\delta^2,\alpha^2\right)
\end{aligned}
$$

$$
\begin{aligned}
& f_{GHD}\left(x;\lambda,\alpha,\beta,\delta,\mu\right) \\
= & \frac{K_{\lambda-\frac{1}{2}}\left(\left(x-\mu\right)^2+\delta^2,\alpha^2\right)}{\sqrt{2\pi}K_\lambda\left(\delta^2,\alpha^2-\beta^2\right)} e^{\beta(x-\mu)} \\
= & \frac{\left(\alpha^2-\beta^2\right)^{\frac{\lambda}{2}}\left(\sqrt{\delta^2+\left(x-\mu\right)^2}\right)^{\lambda-\frac{1}{2}} K_{\lambda-\frac{1}{2}}\left(\alpha\sqrt{\left(x-\mu\right)^2+\delta^2}\right)}{\sqrt{2\pi}\alpha^{\lambda-\frac{1}{2}}\delta^\lambda K_\lambda\left(\delta\sqrt{\alpha^2-\beta^2}\right)} e^{\beta(x-\mu)}
\end{aligned}
$$

Since

$$
E\left(Z\right) = \frac{K_{\lambda+1}\left(\chi,\psi\right)}{K_\lambda\left(\chi,\psi\right)}
$$

then

$$\begin{aligned} E(X) &= \mu + \beta E(Z) \\ &= \mu + \beta \frac{K_{\lambda+1}(\chi, \psi)}{K_{\lambda}(\chi, \psi)} \end{aligned}$$

Since

$$var(Z) = \frac{K_{\lambda}(\chi, \psi) K_{\lambda+2}(\chi, \psi) - (K_{\lambda}(\chi, \psi))^2}{(K_{\lambda}(\chi, \psi))^2}$$

then

$$\begin{aligned} var(X) &= E(Z) + \beta^2 var(Z) = \\ & \frac{K_{\lambda+1}(\chi, \psi)}{K_{\lambda}(\chi, \psi)} + \beta^2 \frac{K_{\lambda}(\chi, \psi) K_{\lambda+2}(\chi, \psi) - (K_{\lambda}(\chi, \psi))^2}{(K_{\lambda}(\chi, \psi))^2} \end{aligned}$$

Since

$$M_{GIG}(t; \lambda, \chi, \psi) = \frac{K_{\lambda}(\chi, \psi - 2t)}{K_{\lambda}(\chi, \psi)}$$

then

$$\begin{aligned} M_X(t) &= e^{\mu t} M_Z\left(\beta t + t^2/2\right) \\ &= e^{\mu t} \frac{K_{\lambda}(\chi, \psi - 2(\beta t + t^2/2))}{K_{\lambda}(\chi, \psi)} \\ &= e^{\mu t} \frac{K_{\lambda}\left(\delta^2, \alpha^2 - \beta^2 - 2(\beta t + t^2/2)\right)}{K_{\lambda}\left(\delta^2, \alpha^2 - \beta^2\right)} \end{aligned}$$

but

$$\begin{aligned} \alpha^2 - \beta^2 - 2\left(\beta t + t^2/2\right) &= \alpha^2 - \beta^2 - 2\beta t - t^2 \\ &= \alpha^2 - (\beta + t)^2 \end{aligned}$$

so

$$M_X(t) = e^{\mu t}\frac{K_\lambda\left(\delta^2, \alpha^2 - (\beta+t)^2\right)}{K_\lambda\left(\delta^2, \alpha^2 - \beta^2\right)} \quad -\alpha-\beta < t < \alpha-\beta$$

3.3.1 The variance-gamma distribution VG

This is the GHD if $\lambda > 0, \alpha > 0, \beta > 0, \beta \in (-\alpha, \alpha), \delta = 0, \mu \in \mathbb{R}$. Then $\chi = \delta^2 = 0$ and $\psi = \alpha^2 - \beta^2 > 0$, resulting in a gamma mixture of normals.

$$\begin{aligned} & f_{GHD}(x; \lambda, \alpha, \beta, 0, \mu) \\ = & \lim_{\delta \longrightarrow 0} \frac{\left(\alpha^2-\beta^2\right)^{\frac{\lambda}{2}}\left(\sqrt{\delta^2+(x-\mu)^2}\right)^{\lambda-\frac{1}{2}} K_{\lambda-\frac{1}{2}}\left(\alpha\sqrt{(x-\mu)^2+\delta^2}\right)}{\sqrt{2\pi}\alpha^{\lambda-\frac{1}{2}}\delta^\lambda K_\lambda\left(\delta\sqrt{\alpha^2-\beta^2}\right)} e^{\beta(x-\mu)} \end{aligned}$$

$$\begin{aligned} & f_{GHD}(x; \lambda, \alpha, \beta, 0, \mu) \\ = & \frac{\left(\alpha^2-\beta^2\right)^{\frac{\lambda}{2}}\left(\sqrt{(x-\mu)^2}\right)^{\lambda-\frac{1}{2}} K_{\lambda-\frac{1}{2}}\left(\alpha\sqrt{(x-\mu)^2}\right)}{\sqrt{2\pi}\alpha^{\lambda-\frac{1}{2}}\delta^\lambda\left(\frac{1}{2}\left(\frac{\delta\sqrt{\alpha^2-\beta^2}}{2}\right)^{-\lambda}\Gamma(\lambda)\right)} e^{\beta(x-\mu)} \\ = & \frac{\left(\alpha^2-\beta^2\right)^{\frac{\lambda}{2}}(|x-\mu|)^{\lambda-\frac{1}{2}} K_{\lambda-\frac{1}{2}}(\alpha|x-\mu|)}{\sqrt{2\pi}\alpha^{\lambda-\frac{1}{2}}\delta^\lambda\delta^{-\lambda}\left(\frac{1}{2^{-\lambda+1}}\right)\left(\alpha^2-\beta^2\right)^{-\frac{\lambda}{2}}\Gamma(\lambda)} e^{\beta(x-\mu)} \\ = & \frac{\left(\alpha^2-\beta^2\right)^{\lambda}(|x-\mu|)^{\lambda-\frac{1}{2}} K_{\lambda-\frac{1}{2}}(\alpha|x-\mu|)\, 2^{-\lambda+1}}{\sqrt{2\pi}\alpha^{\lambda-\frac{1}{2}}\Gamma(\lambda)} e^{\beta(x-\mu)} \\ = & 2\left(\frac{\alpha^2-\beta^2}{2}\right)^\lambda\left(\frac{|x-\mu|}{\alpha}\right)^{\lambda-\frac{1}{2}} \frac{K_{\lambda-\frac{1}{2}}(\alpha|x-\mu|)\, e^{\beta(x-\mu)}}{\sqrt{2\pi}\Gamma(\lambda)} \end{aligned} \tag{3.36}$$

which is the variance-gamma distribution (VG). It can be obtained directly by using the gamma (α, β) as the mixing distribution in the Normal variance-mean mixture. The distribution was popularised by Madan and

Seneta (1990) in there study of financial returns data.

Here, $Z \sim GIG\left(z;\lambda,0,\psi\right)$ and

$$E\left(Z\right)=\frac{\lambda}{\psi/2},\ var\left(Z\right)=\frac{\lambda}{\left(\psi/2\right)^{2}},\ M_{X}\left(t\right)=e^{\mu t}M_{Z}\left(\beta t+t^{2}/2\right)$$

Therefore

$$\begin{aligned} E\left(X\right) &= \mu+\beta E\left(Z\right) \\ &= \mu+\frac{\lambda}{\psi/2} \end{aligned}$$

$$\begin{aligned} var\left(X\right) &= E\left(Z\right)+\beta^{2}var\left(Z\right) \\ &= \frac{\lambda}{\psi/2}+\beta^{2}\frac{\lambda}{\left(\psi/2\right)^{2}} \\ &= \frac{\lambda}{\psi/2}\left(1+\frac{\beta^{2}}{\psi/2}\right) \\ &= 2\lambda\left(\frac{\psi+2\beta^{2}}{\psi^{2}}\right) \\ &= 2\lambda\left(\frac{\alpha^{2}-\beta^{2}+2\beta^{2}}{\left(\alpha^{2}-\beta^{2}\right)^{2}}\right) \\ &= 2\lambda\left(\frac{\alpha^{2}+\beta^{2}}{\left(\alpha^{2}-\beta^{2}\right)^{2}}\right) \end{aligned}$$

$$\begin{aligned}
M_X(t) &= e^{\mu t} M_Z\left(\beta t + t^2/2\right) \\
&= e^{\mu t}\left(\frac{\psi - 2\left(\beta t + t^2/2\right)}{\psi}\right)^{-\lambda} \\
&= e^{\mu t}\left(\frac{\psi - \left(2\beta t + t^2\right)}{\psi}\right)^{-\lambda} \\
&= e^{\mu t}\left(\frac{\psi}{\psi - \left(2\beta t + t^2\right)}\right)^{\lambda} \\
&= e^{\mu t}\left(\frac{\alpha^2 - \beta^2}{\alpha^2 - \beta^2 - 2\beta t - t^2}\right)^{\lambda} \\
&= e^{\mu t}\left(\frac{\alpha^2 - \beta^2}{\alpha^2 - (\beta + t)^2}\right)^{\lambda}
\end{aligned}$$

3.3.2 The hyperbolic asymmetric $(Student\prime s)$ t-distribution Hat

This is the GHD obtained $if\ \lambda > 0,\ \beta \in \mathbb{R},\ \alpha = |\beta|, \delta > 0, \mu \in \mathbb{R}$

Now, $\chi = \delta^2 > 0$ and $\psi = \alpha^2 - \beta^2 = 0$, and we have an inverse gamma mixture of normals.There are two cases to distinguish, $\alpha = |\beta| > 0$ and $\alpha = \beta = 0$.

For the case $\alpha = |\beta| > 0$

$$f_{GHD}\left(x; \lambda, |\beta|, \beta, \delta, \mu\right) = \lim_{\alpha \longrightarrow |\beta|} \frac{\left(\alpha^2 - \beta^2\right)^{\frac{\lambda}{2}} \left(\sqrt{\delta^2 + (x-\mu)^2}\right)^{\lambda - \frac{1}{2}} K_{\lambda - \frac{1}{2}}\left(\alpha\sqrt{(x-\mu)^2 + \delta^2}\right)}{\sqrt{2\pi}\alpha^{\lambda - \frac{1}{2}}\delta^{\lambda} K_{\lambda}\left(\delta\sqrt{\alpha^2 - \beta^2}\right)} e^{\beta(x-\mu}$$

for $\lambda < 0$;

$$\lim_{\omega \longrightarrow 0} K_{\lambda}(\omega) = \frac{1}{2}\left(\frac{\omega}{2}\right)^{\lambda} \Gamma(-\lambda)$$

therefore

$$
\begin{aligned}
& f_{GHD}\left(x;\lambda,\left|\beta\right|,\beta,\delta,\mu\right) \\
= \; & \frac{\left(\alpha^2-\beta^2\right)^{\frac{\lambda}{2}}\left(\sqrt{\delta^2+\left(x-\mu\right)^2}\right)^{\lambda-\frac{1}{2}} K_{\lambda-\frac{1}{2}}\left(\left|\beta\right|\sqrt{\left(x-\mu\right)^2+\delta^2}\right)}{\sqrt{2\pi}\alpha^{\lambda-\frac{1}{2}}\delta^{\lambda}\left[\frac{1}{2}\left(\frac{\delta\sqrt{\alpha^2-\beta^2}}{2}\right)^{\lambda}\Gamma\left(-\lambda\right)\right]} e^{\beta(x-\mu)} \\
= \; & \frac{\left(\alpha^2-\beta^2\right)^{\frac{\lambda}{2}}\left(\sqrt{\delta^2+\left(x-\mu\right)^2}\right)^{\lambda-\frac{1}{2}} K_{\lambda-\frac{1}{2}}\left(\left|\beta\right|\sqrt{\left(x-\mu\right)^2+\delta^2}\right)}{\sqrt{2\pi}\left|\beta\right|^{\lambda-\frac{1}{2}}\delta^{\lambda}\delta^{\lambda}\frac{1}{2^{\lambda+1}}\left(\alpha^2-\beta^2\right)^{\frac{\lambda}{2}}\Gamma\left(-\lambda\right)} e^{\beta(x-\mu)} \\
= \; & \frac{\left(\sqrt{\delta^2+\left(x-\mu\right)^2}\right)^{\lambda-\frac{1}{2}} K_{\lambda-\frac{1}{2}}\left(\left|\beta\right|\sqrt{\left(x-\mu\right)^2+\delta^2}\right)}{\sqrt{2\pi}\left|\beta\right|^{\lambda-\frac{1}{2}}\delta^{2\lambda}\frac{1}{2^{\lambda+1}}\Gamma\left(-\lambda\right)} e^{\beta(x-\mu)} \\
= \; & \frac{\left(\sqrt{\delta^2+\left(x-\mu\right)^2}\right)^{\lambda-\frac{1}{2}} K_{\lambda-\frac{1}{2}}\left(\left|\beta\right|\sqrt{\left(x-\mu\right)^2+\delta^2}\right)}{\sqrt{2\pi}\left|\beta\right|^{\lambda-\frac{1}{2}}\frac{1}{2}\left(\frac{\delta^2}{2}\right)^{\lambda}\Gamma\left(-\lambda\right)} e^{\beta(x-\mu)} \\
= \; & \frac{2\left(\frac{\delta^2}{2}\right)^{-\lambda}}{\sqrt{2\pi}\Gamma\left(-\lambda\right)}\left(\frac{\sqrt{\delta^2+\left(x-\mu\right)^2}}{\left|\beta\right|}\right)^{\lambda-\frac{1}{2}} K_{\lambda-\frac{1}{2}}\left(\left|\beta\right|\sqrt{\left(x-\mu\right)^2+\delta^2}\right) e^{\beta(x-\mu)} \qquad (3.37)
\end{aligned}
$$

For the case $\alpha = \beta$

$$
\begin{aligned}
& f_{GHD}\left(x;\lambda,0,0,\delta,\mu\right) \\
= & \lim_{\alpha,\beta\longrightarrow 0} \frac{\left(\alpha^2-\beta^2\right)^{\frac{\lambda}{2}}\left(\sqrt{\delta^2+(x-\mu)^2}\right)^{\lambda-\frac{1}{2}} K_{\lambda-\frac{1}{2}}\left(\alpha\sqrt{(x-\mu)^2+\delta^2}\right)}{\sqrt{2\pi}\alpha^{\lambda-\frac{1}{2}}\delta^{\lambda}K_{\lambda}\left(\delta\sqrt{\alpha^2-\beta^2}\right)} e^{\beta(x-\mu)} \\
= & \frac{\left(\alpha^2-\beta^2\right)^{\frac{\lambda}{2}}\left(\sqrt{\delta^2+(x-\mu)^2}\right)^{\lambda-\frac{1}{2}}\left[\frac{1}{2}\left(\frac{\alpha\sqrt{(x-\mu)^2+\delta^2}}{2}\right)^{\lambda-\frac{1}{2}}\Gamma\left(-\lambda+\frac{1}{2}\right)\right]}{\sqrt{2\pi}\alpha^{\lambda-\frac{1}{2}}\delta^{\lambda}\left[\frac{1}{2}\left(\frac{\delta\sqrt{\alpha^2-\beta^2}}{2}\right)^{\lambda}\Gamma\left(-\lambda\right)\right]} \\
= & \frac{\left(\alpha^2-\beta^2\right)^{\frac{\lambda}{2}}\left(\sqrt{\delta^2+(x-\mu)^2}\right)^{\lambda-\frac{1}{2}}\alpha^{\lambda-\frac{1}{2}}\left(\frac{1}{2}\right)^{\lambda+\frac{1}{2}}\left(\sqrt{(x-\mu)^2+\delta^2}\right)^{\lambda-\frac{1}{2}}\Gamma\left(-\lambda+\frac{1}{2}\right)}{\sqrt{\pi}\alpha^{\lambda-\frac{1}{2}}\delta^{\lambda}\delta^{\lambda}\left(\frac{1}{2}\right)^{\lambda+\frac{1}{2}}\left(\alpha^2-\beta^2\right)^{\frac{\lambda}{2}}\Gamma\left(-\lambda\right)} \\
= & \frac{\Gamma\left(-\lambda+\frac{1}{2}\right)}{\Gamma\left(-\lambda\right)}\frac{\left(\delta^2+(x-\mu)^2\right)^{\lambda-\frac{1}{2}}}{\delta^{2\lambda}\sqrt{\pi}} \\
= & \frac{\Gamma\left(\frac{-2\lambda+1}{2}\right)}{\Gamma\left(\frac{-2\lambda}{2}\right)}\frac{\left(\delta^2+(x-\mu)^2\right)^{-\frac{-2\lambda+1}{2}}}{\delta^{2\lambda}\sqrt{\pi}} \\
= & \frac{\Gamma\left(\frac{-2\lambda+1}{2}\right)}{\Gamma\left(\frac{-2\lambda}{2}\right)}\frac{\left(\delta^2+(x-\mu)^2\right)^{-\frac{-2\lambda+1}{2}}}{\delta^{2\lambda-1}\delta\sqrt{\pi}} \\
= & \frac{\Gamma\left(\frac{-2\lambda+1}{2}\right)}{\Gamma\left(\frac{-2\lambda}{2}\right)}\frac{\left(\delta^2+(x-\mu)^2\right)^{-\frac{-2\lambda+1}{2}}}{\left(\delta^2\right)^{\lambda-\frac{1}{2}}\sqrt{\delta^2\pi}} \\
= & \frac{\Gamma\left(\frac{-2\lambda+1}{2}\right)}{\Gamma\left(\frac{-2\lambda}{2}\right)}\frac{1}{\sqrt{\delta^2\pi}}\left(\frac{\delta^2+(x-\mu)^2}{\delta^2}\right)^{-\frac{-2\lambda+1}{2}} \\
= & \frac{\Gamma\left(\frac{-2\lambda+1}{2}\right)}{\Gamma\left(\frac{-2\lambda}{2}\right)}\frac{1}{\sqrt{\delta^2\pi}}\left(1+\frac{(x-\mu)^2}{\delta^2}\right)^{-\frac{-2\lambda+1}{2}} \qquad (3.38)
\end{aligned}
$$

If $\delta^2 = -2\lambda = n$, then this is a student's t density with n degrees of freedom. The parameter δ is the scale parameter of the distribution and μ is a location parameter.

$$f_{GHD}\left(x;\lambda,|\beta|,\beta,\delta,\mu\right) = \frac{2^{\frac{-n+1}{2}}\delta^{n}}{\sqrt{2\pi}\Gamma\left(n/2\right)}\left(\frac{\sqrt{\delta^{2}+\left(x-\mu\right)^{2}}}{|\beta|}\right)^{-\frac{n+1}{2}} \times K_{-\frac{n+1}{2}}\left(|\beta|\sqrt{\left(x-\mu\right)^{2}+\delta^{2}}\right)e^{\beta(x-\mu)} \quad (3.39)$$

Which is the *hyperbolic asymmetric* (*student′s*) t, or HAt, distribution, given by

$$HAt\left(n,\beta,\mu,\delta\right) = GHyp\left(\lambda,|\beta|,\beta,\delta,\mu\right)$$

When $|\beta| = 0$,we have;

$$f_{GHD}\left(x;\lambda,0,0,\delta,\mu\right) = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}\frac{1}{\sqrt{n\pi}}\left(1+\frac{\left(x-\mu\right)^{2}}{n}\right)^{-\frac{n+1}{2}} \quad (3.40)$$

which is the usual t distribution with n degrees of freedom. The t distribution results from a mixture of normal and inverse gamma distributions. we have a student-t distributions as a limit of GH distributions for $\lambda < 0$ and $\alpha = \beta = \mu = 0$. See Barndorff-Nielsen (1978). In the figure below GHD(-2,0,2,0,0) lap over the curve of student-t distribution with degrees of freedom 4 (red), since $\lambda = -n/2,\ \delta = \sqrt{n}$, denoting the degrees of freedom by n. The other curves are for comparison.

Tail comparison

In the final analysis, it is the heavy tail that makes Gh distribution so popular in modelling the time dynamics of financial time series. Generally the GHDs

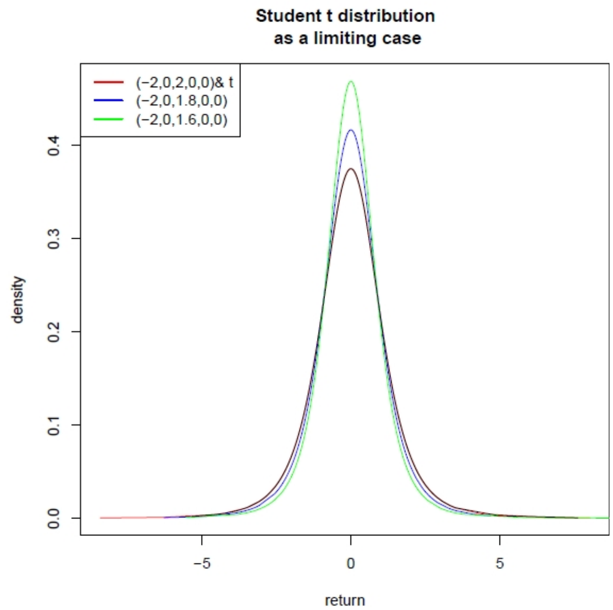


Figure 3.8: Limiting distribution: t distribution

have an exponentially decaying speed

$$f_{GHD}(x;\lambda,\alpha,\beta,\delta,\mu) \sim x^{\lambda-1}\exp\{-(\alpha-\beta)x\} \text{ as } x\to\infty$$

The figure below illustrate the tail behaviour of GH distribution with different values of λ with $\alpha = 1$, $\beta = 0$, $\delta = 1$ and $\mu = 0$. Among the different cases considered, its the GHD(with $\lambda = 1.5$) that has the lowest decaying speed, while the NIG decays fastest.

3.3.3 The Asymmetric Laplace distribution ALap

This is obtained $(if\ \lambda = 1,\ \alpha > 0,\ \beta \in (-\alpha,\alpha),\ \delta = 0,\ \mu \in \mathbb{R})$.Then $\chi = 0,\ \psi > 0$, and $\lambda = 1,\ i.e.,$ we have an exponential mixture of normals. This

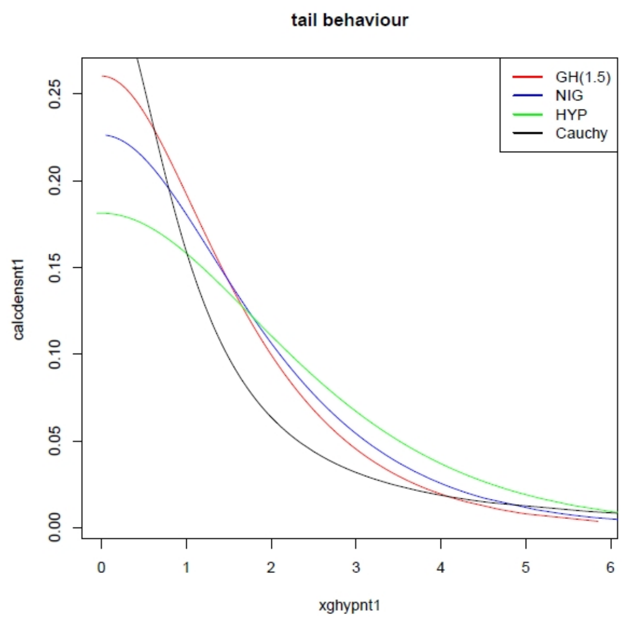


Figure 3.9: Tail comparison between GHDs

is a subfamily of the variance-gamma distribution, the resulting density being

$$f_{ALap}\left(x;\alpha,\beta,\mu\right)=f_{GHD}\left(x;1,\alpha,\beta,0,\mu\right)=\frac{\alpha^2-\beta^2}{2\alpha}e^{-\alpha|x-\mu|+\beta(x-\mu)},\quad(3.41)$$

with distribution notation

$$ALap\left(\alpha,\beta,\mu\right)=GHD\left(1,\alpha,\beta,0,\mu\right)\quad(3.42)$$

Note that, with $\beta=0$,this simplifies to

$$\frac{\alpha}{2}e^{-\alpha|x-\mu|}\quad(3.43)$$

which is the usual location-scale Laplace p.d.f. $i.e$ $GHD\left(1,\alpha,0,0,\mu\right)=Lap\left(\mu,\alpha^{-1}\right).$

3.3.4 The Hyperbolic Distribution

Here, $\lambda = 1, \alpha > 0, \beta \in (-\alpha, \alpha)$, $\delta > 0$, $\mu \in \mathbb{R}$. Then $\chi > 0$, $\psi > 0$ and $\lambda = 1$. The hyperbolic distribution is defined as the distribution

$$
\begin{aligned}
& f_{Hyp}(x; \alpha, \beta, \delta, \mu) \qquad (3.44) \\
= \; & \frac{\sqrt{\alpha^2 - \beta^2}}{2\alpha\beta K_1\left(\delta\sqrt{\alpha^2 - \beta^2}\right)} \exp\left[-\sqrt{\delta^2 + (x-\mu)^2} + \beta(x-\mu)\right]
\end{aligned}
$$

3.3.5 An Asymmetric Cauchy distribution

Here $\lambda = -1/2$, $\beta \in \mathbb{R}$, $\alpha = |\beta|$, $\delta > 0, \mu \in \mathbb{R}$. This is a special case of the hyperbolic asymmetric t distribution HAt, discussed above. If we use the Levy distribution as a mixing weight, then we get a distribution having the following p.d.f.

$$
\begin{aligned}
& f_{HAt}(x; -1/2, \beta, \delta, \mu) \\
= \; & \frac{2\left(\delta^2/2\right)^{1/2}}{\sqrt{2\pi}\Gamma(1/2)} \left(\frac{\sqrt{\delta^2 + (x-\mu)^2}}{|\beta|}\right)^{-1} \\
& \times K_{-1}\left(|\beta|\sqrt{\delta^2 + (x-\mu)^2}\right) e^{\beta(x-\mu)} \\
= \; & \frac{\delta|\beta|}{\pi\sqrt{\delta^2 + (x-\mu)^2}} K_{-1}\left(|\beta|\sqrt{\delta^2 + (x-\mu)^2}\right) e^{\beta(x-\mu)} \qquad (3.45)
\end{aligned}
$$

In the symmetric case, i.e., with $\beta = 0$,we get

$$
f_{HAt}(x; -1/2, 0, \delta, \mu) = \frac{\delta}{\pi\left(\delta^2 + (x-\mu)^2\right)} \qquad (3.46)
$$

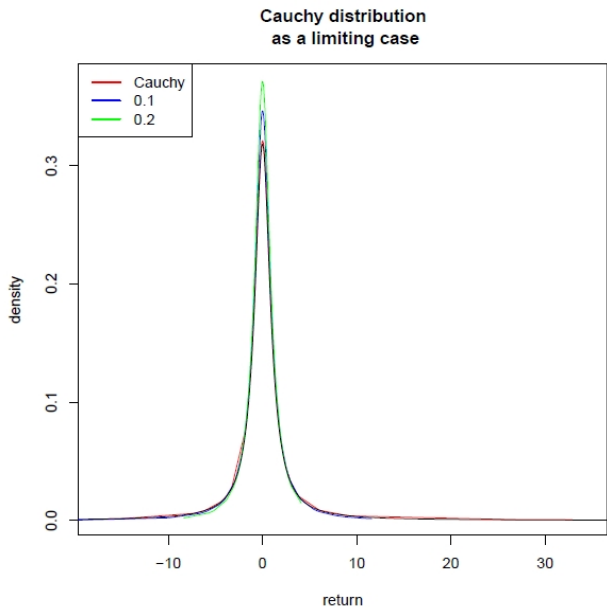


Figure 3.10: Limiting case: Cauchy distribution

$f_{HAt}\left(x;-1/2,\beta,\delta,\mu\right)$ with $\beta\neq 0$ and $-\alpha-\beta\leq t\leq\alpha+\beta$, then

$$M_X\left(t\right)=e^{\mu t}e^{-\delta\sqrt{\alpha^2-(\beta+t)^2}}$$

3.3.6 The Normal Inverse Gaussian Distribution NIG

This is the case when $\lambda=-1/2,\ \alpha>0,\ \beta\in(-\alpha,\alpha),\ \delta>0,\ \mu\in\mathbb{R}$. The inverse Gaussian (NIG) distribution is defined as

$$NIG\left(\alpha,\beta,\delta,\mu\right)=GHD\left(-1/2,\alpha,\beta,\delta,\mu\right)$$

and has the density

$$f_{NIG}(x;\alpha,\beta,\delta,\mu) = e^{\delta\sqrt{\alpha^2-\beta^2}}\frac{\alpha\delta}{\pi\sqrt{\delta^2+(x-\mu)^2}} \times K_1\left(\alpha\sqrt{\delta^2+(x-\mu)^2}\right)e^{\beta(x-\mu)} \tag{3.47}$$

The formulae for the mean, variance and skewness for the NIG distribution take a much simpler form than the general case of GHD. If $X \sim NIG(\alpha,\beta,\delta,\mu)$, then

$$E(X) = \mu + \beta\left(\frac{\chi}{\psi}\right)^{\lambda/2}$$

$$\begin{aligned} var(X) &= \eta + \beta^2\frac{\eta^2}{\omega} \\ &= \left(\frac{\chi}{\psi}\right)^{\lambda/2} + \beta^2\frac{\left(\frac{\chi}{\psi}\right)^{\lambda/2}}{\sqrt{\chi\psi}} \end{aligned}$$

$$\mu_3(X) = 3\beta\frac{\eta^2}{\omega} + 3\beta^2\frac{\eta^3}{\omega^2}$$

$$M_X(t) = e^{\mu t}e^{\delta\left(\sqrt{\alpha^2-\beta^2}-\sqrt{\alpha^2-(\beta+t)^2}\right)} \quad -\alpha-\beta \le t \le \alpha+\beta$$

3.3.7 Normal Distribution

The normal distribution can be derived as a limiting case. Let $\sigma_0^2 > 0$ and $\mu_0, \beta_0 \in \mathbb{R}$. Then, for all $\lambda \in \mathbb{R}$,

$$GHD(\lambda,\alpha,\beta,\delta,\mu) \longrightarrow N\left(\mu_0 + \beta_0\sigma_0^2, \sigma_0^2\right)$$

as $\alpha \longrightarrow \infty$, $\delta \longrightarrow \infty$, $\mu \longrightarrow \mu_0$, and $\beta \longrightarrow \beta_0$ with $\delta/\alpha \longrightarrow \sigma_0^2$.see Barndorff-Nielsen (1978). In the figure below, The GHD with parameters

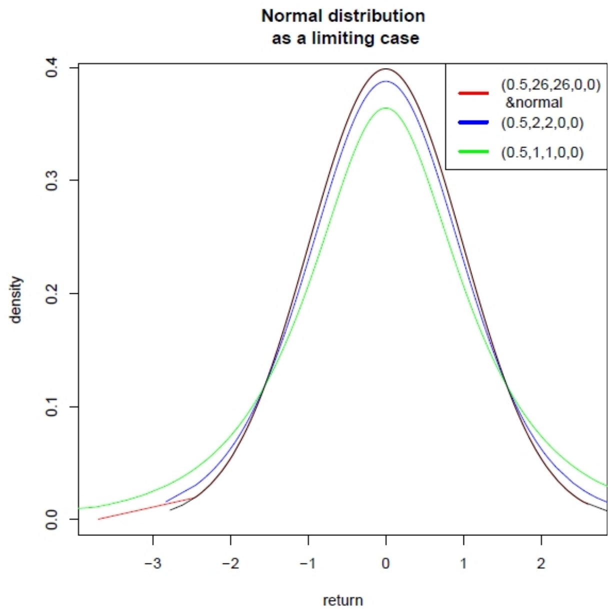


Figure 3.11: Limiting distribution: Normal distribution

(0.5,26,26,0,0) laps over the pdf curve of standard normal distribution (red). The blue and green curves are placed here as comparison.

3.3.8 GIG Distribution

It was shown by Eberlein and Hammerstein (2004) that the GIG Distribution can be obtained as a limiting case. Let $\lambda_0 \in \mathbb{R}$ and $\psi_0, \chi_0 \geq 0$ such that $(\lambda_0, \psi_0, \chi_0)$ is in the parameter space of the GIG distribution. Then

$$GHD\,(\lambda_0, \alpha, \beta, \delta, 0) \longrightarrow GIG\,(\lambda_0, \psi_0, \chi_0)$$

as $\alpha \longrightarrow \infty$, $\beta \longrightarrow \infty$, and $\delta \longrightarrow 0$ such that $\alpha - \beta \longrightarrow \psi_0/2$ and $\alpha\delta^2 \longrightarrow \chi_{0.}$

3.3.9 Reflecting GIG Distribution

Let $\lambda_0 \in \mathbb{R}$ and $\psi_0, \chi_0 \geq 0$ such that $(\lambda_0, \psi_0, \chi_0)$ is in the parameter space of the GIG distribution. Then

$$GHD\,(\lambda_0, \alpha, \beta, \delta, 0) \longrightarrow -GIG\,(\lambda_0, \psi_0, \chi_0)$$

as $\alpha \longrightarrow \infty$, $\beta \longrightarrow -\infty$, and $\delta \longrightarrow 0$ such that $\alpha + \beta \longrightarrow \psi_0/2$ and $\alpha\delta^2 \longrightarrow \chi_{0.}$

Chapter 4

HYPERBOLA DISTRIBUTION

Let

$$\ln y = -\alpha\sqrt{1+v^2} + \beta v \ , \quad -\infty < v < +\infty$$

if

$$\phi = \alpha + \beta$$

and

$$\gamma = \alpha - \beta$$

then

$$\phi + \gamma = 2\alpha$$

and

$$\phi - \gamma = 2\beta$$

which implies that,

$$\alpha = \frac{\phi + \gamma}{2}$$

$$\beta = \frac{\phi - \gamma}{2}$$

Therefore,

$$\begin{aligned}
\ln y &= -\left(\frac{\phi + \gamma}{2}\right)\sqrt{1+v^2} + \left(\frac{\phi - \gamma}{2}\right) v \\
&= -\frac{\phi}{2}\sqrt{1+v^2} - \frac{\gamma}{2}\sqrt{1+v^2} + \frac{\phi}{2}v - \frac{\gamma}{2}v \\
&= -\frac{\phi}{2}\left(\sqrt{1+v^2} - v\right) - \frac{\gamma}{2}\left(\sqrt{1+v^2} + v\right)
\end{aligned}$$

thus

$$y = \exp\left\{-\frac{\phi}{2}\left(\sqrt{1+v^2} - v\right) - \frac{\gamma}{2}\left(\sqrt{1+v^2} + v\right)\right\}$$

Our aim now is to integrate RHS of (1) so as to come up with a pdf. Thus we wish to evaluate

$$\int_{-\infty}^{\infty} \exp\left\{-\left(\frac{\phi}{2}\sqrt{1+v^2} - v\right) - \frac{\gamma}{2}\left(\sqrt{1+v^2} + v\right)\right\} dv$$

and then normalise it.

Let

$$x = \sqrt{1+v^2} + v$$

Then

$$
\begin{aligned}
(i)\ v &= -\infty \ \Rightarrow\ x = 0 \\
v &= \infty \quad \Rightarrow\ x = \infty \\
(ii)\ \frac{1}{x} &= \frac{1}{\sqrt{1+v^2}+v} = \frac{\sqrt{1+v^2}-v}{\left[\sqrt{1+v^2}+v\right]\left[\sqrt{1+v^2}-v\right]} \\
&= \frac{\sqrt{1+v^2}-v}{(1+v^2)-v^2} \\
&= \sqrt{1+v^2}-v \\
(iii)\ dx &= d\left[\left(1+v^2\right)^{\frac{1}{2}}+v\right] \\
&= \left(\frac{1}{2}\left(1+v^2\right)^{-\frac{1}{2}}(2v)+1\right)dv \\
&= \left[\frac{v}{\sqrt{1+v^2}}+1\right]dv = \frac{v+\sqrt{1+v^2}}{\sqrt{1+v^2}}dv
\end{aligned}
$$

but

$$x = \sqrt{1+v^2}+v \ \Rightarrow\ \sqrt{1+v^2} = x - v$$

which implies that

$$
\begin{aligned}
1+v^2 &= (x-v)^2 \\
&= x^2 - 2xv + v^2
\end{aligned}
$$

therefore

$$1 = x^2 - 2xv$$

and

$$
\begin{aligned}
2xv &= x^2 - 1 \\
v &= \frac{x^2-1}{2x}
\end{aligned}
$$

Therefore

$$\begin{aligned}
dx &= \frac{v+\sqrt{1+v^2}}{\sqrt{1+v^2}}dv \\
&= \frac{x}{\sqrt{1+v^2}}dv \\
&= \left(\frac{x}{x-v}\right)dv = \frac{x}{x-\left(\frac{x^2-1}{2x}\right)}dv \\
&= \frac{x}{x-\left[\frac{x}{2}-\frac{1}{2x}\right]}dv \\
&= \frac{x}{\frac{x}{2}+\frac{1}{2x}}dv \\
&= \frac{x}{\frac{x^2+1}{2x}}dv \\
&= \frac{2x^2}{x^2+1}dv
\end{aligned}$$

therefore

$$\begin{aligned}
dv &= \frac{x^2+1}{2x^2}dx \\
&= \frac{1}{2}\left[1+x^{-2}\right]dx
\end{aligned}$$

The integration

$$\begin{aligned}
&\int_{-\infty}^{\infty} \exp\left\{-\frac{\phi}{2}\left(\sqrt{1+v^2}-v\right)-\frac{\gamma}{2}\left(\sqrt{1+v^2}+v\right)\right\} dv \\
= \quad & \frac{1}{2}\int_{0}^{\infty} \left(1+x^{-2}\right)\exp\left\{-\frac{\phi}{2}\frac{1}{x}-\frac{\gamma}{2}x\right\} dx \\
= \quad & \frac{1}{2}\int_{0}^{\infty} \left(1+x^{-2}\right) e^{-\frac{\phi}{2}\frac{1}{x}-\frac{\gamma}{2}x} dx \\
= \quad & \frac{1}{2}\int_{0}^{\infty} \left(1+x^{-2}\right) e^{-\frac{\gamma}{2}x-\frac{\phi}{2x}} dx \\
= \quad & \frac{1}{2}\int_{0}^{\infty} \left(1+x^{-2}\right) e^{-\frac{\gamma}{2}\left(x+\frac{\phi}{\gamma x}\right)} dx
\end{aligned}$$

Let

$$x = \sqrt{\frac{\phi}{\gamma}}z \;\Rightarrow\; dx = \sqrt{\frac{\phi}{\gamma}}dz$$

Therefore

$$
\begin{aligned}
& \frac{1}{2}\int_{0}^{\infty}\left(1+x^{-2}\right)e^{-\frac{\gamma}{2}\left(x+\frac{\phi}{\gamma x}\right)}dx \\
= \;& \frac{1}{2}\int_{0}^{\infty}\left[1+\left(\sqrt{\frac{\phi}{\gamma}}z\right)^{-2}\right]e^{-\frac{\gamma}{2}\left[\sqrt{\frac{\phi}{\gamma}}z+\frac{\phi}{\gamma}\frac{1}{\sqrt{\frac{\phi}{\gamma}}z}\right]}\sqrt{\frac{\phi}{\gamma}}dz \\
= \;& \frac{1}{2}\int_{0}^{\infty}\left[\sqrt{\frac{\phi}{\gamma}}+\left(\sqrt{\frac{\phi}{\gamma}}\right)^{-1}z^{-2}\right]e^{-\frac{\gamma}{2}\left[\sqrt{\frac{\phi}{\gamma}}z+\frac{\phi}{\gamma}\frac{1}{\sqrt{\frac{\phi}{\gamma}}z}\right]}dz \\
= \;& \frac{1}{2}\int_{0}^{\infty}\left[\sqrt{\frac{\phi}{\gamma}}+\left(\sqrt{\frac{\phi}{\gamma}}\right)^{-1}z^{-2}\right]e^{-\frac{\sqrt{\gamma\phi}}{2}\left[z+\frac{1}{z}\right]}dz \\
= \;& \frac{1}{2}\int_{0}^{\infty}\sqrt{\frac{\phi}{\gamma}}e^{-\frac{\sqrt{\gamma\phi}}{2}\left[z+\frac{1}{z}\right]}dz+\frac{1}{2}\left(\sqrt{\frac{\phi}{\gamma}}\right)^{-1}\int_{0}^{\infty}z^{-1-1}e^{-\frac{\sqrt{\gamma\phi}}{2}\left[z+\frac{1}{z}\right]}dz \\
= \;& \frac{1}{2}\sqrt{\frac{\phi}{\gamma}}\int_{0}^{\infty}z^{1-1}e^{-\frac{\sqrt{\gamma\phi}}{2}\left[z+\frac{1}{z}\right]}dz+\frac{1}{2}\left(\sqrt{\frac{\phi}{\gamma}}\right)^{-1}\int_{0}^{\infty}z^{-1-1}e^{-\frac{\sqrt{\gamma\phi}}{2}\left[z+\frac{1}{z}\right]}dz \\
= \;& \frac{1}{2}\sqrt{\frac{\phi}{\gamma}}2\kappa_{1}\left(\sqrt{\gamma\phi}\right)+\frac{1}{2}\left(\sqrt{\frac{\phi}{\gamma}}\right)^{-1}2\kappa_{-1}\left(\sqrt{\gamma\phi}\right) \\
= \;& \sqrt{\frac{\phi}{\gamma}}\kappa_{1}\left(\sqrt{\gamma\phi}\right)+\left(\sqrt{\frac{\phi}{\gamma}}\right)^{-1}\kappa_{-1}\left(\sqrt{\gamma\phi}\right) \\
= \;& \left[\sqrt{\frac{\phi}{\gamma}}+\left(\sqrt{\frac{\phi}{\gamma}}\right)^{-1}\right]\kappa_{-1}\left(\sqrt{\gamma\phi}\right) \\
= \;& \left[\sqrt{\frac{\phi}{\gamma}}+\sqrt{\frac{\gamma}{\phi}}\right]\kappa_{-1}\left(\sqrt{\gamma\phi}\right) \\
= \;& \left[\frac{\sqrt{\phi}}{\sqrt{\gamma}}+\frac{\sqrt{\gamma}}{\sqrt{\phi}}\right]\kappa_{-1}\left(\sqrt{\gamma\phi}\right) \\
= \;& \frac{\phi+\gamma}{\sqrt{\phi\gamma}}\kappa_{-1}\left(\sqrt{\gamma\phi}\right) \\
= \;& \frac{\phi+\gamma}{\sqrt{\phi\gamma}}\kappa_{1}\left(\sqrt{\gamma\phi}\right) \\
= \;& \frac{\sqrt{\phi\gamma}\left(\phi+\gamma\right)}{\phi\gamma}\kappa_{1}\left(\sqrt{\gamma\phi}\right) \\
= \;& \sqrt{\phi\gamma}\left(\frac{1}{\gamma}+\frac{1}{\phi}\right)\kappa_{1}\left(\sqrt{\gamma\phi}\right) \\
= \;& \sqrt{\phi\gamma}\left(\gamma^{-1}+\phi^{-1}\right)\kappa_{1}\left(\sqrt{\gamma\phi}\right)
\end{aligned}
$$

where

$$K = \frac{1}{\left(\gamma^{-1} + \phi^{-1}\right)} \; and \; \omega = \sqrt{\phi\gamma}$$

Therefore

$$\frac{1}{2}\int_{0}^{\infty} \left(1 + x^{-2}\right) e^{-\frac{\gamma}{2}x - \frac{\phi}{2x}} dx = \frac{\omega}{K}\kappa_1(\omega)$$

normalising

$$\begin{aligned} \frac{1}{2}\int_{0}^{\infty} \frac{K}{\omega}\frac{\left(1 + x^{-2}\right)}{\kappa_1(\omega)} e^{-\frac{\gamma}{2}x - \frac{\phi}{2x}} dx &= 1 \\ \int_{0}^{\infty} \frac{1}{2}\frac{\frac{1}{\left(\gamma^{-1} + \phi^{-1}\right)}}{\sqrt{\phi\gamma}}\frac{\left(1 + x^{-2}\right)}{\kappa_1(\omega)} e^{-\frac{\gamma}{2}x - \frac{\phi}{2x}} dx &= 1 \end{aligned}$$

The pdf is given by

$$f(x) = \frac{\omega}{2(\phi + \gamma)}\frac{\left(1 + x^{-2}\right)}{\kappa_1(\omega)} e^{-\frac{\gamma}{2}x - \frac{\phi}{2x}}, \; x > 0$$

where is the modified Bessel function of the third kind with index one.

Chapter 5

PARAMETER ESTIMATION

5.1 Maximum-Likelihood Estimation

Assuming the independence of observation $x_i,\ i = 1, 2, \ldots, n,$ we maximize the log-likehood function:

$$\begin{aligned} & L_{GH}\left(\lambda, \alpha, \beta, \delta, \mu\right) \\ = \; & n \log\left\{a\left(\lambda, \alpha, \beta, \delta\right)\right\} + \left(\frac{\lambda}{2} - \frac{1}{4}\right) \sum_{i=1}^{n} \log\left\{\delta^2 + \left(x_i - \mu\right)^2\right\} \\ & + \sum_{i=1}^{n} \left[\log K_{\lambda - \frac{1}{2}}\left(\alpha \sqrt{\delta^2 + \left(x_i - \mu\right)^2} + \beta\left(x_i - \mu\right)\right)\right] \end{aligned} \tag{5.1}$$

Taking the first derivative of the log-likehood function respect to the five parameters, we obtain the following expression, in which the log-likehood function is denoted by L. see Prause (1999).

$$
\begin{aligned}
\frac{\partial}{\partial \lambda} L &= n\left\{\frac{1}{2}\ln\left(\frac{\alpha^2-\beta^2}{\alpha\delta}\right) - \frac{k_\lambda\left(\delta\sqrt{\alpha^2-\beta^2}\right)}{K_\lambda\left(\delta\sqrt{\alpha^2-\beta^2}\right)}\right\} \\
&\quad + \sum_{i=1}^{n}\left\{\begin{array}{c} \frac{1}{2}\ln\left\{\delta^2+(x_i-\mu)^2\right\} \\ +\frac{k_{\lambda-1/2}\left(\alpha\sqrt{\delta^2+(x_i-\mu)^2}\right)}{K_{\lambda-1/2}\left(\alpha\sqrt{\delta^2+(x_i-\mu)^2}\right)} \end{array}\right\}
\end{aligned} \tag{5.2}
$$

$$
\begin{aligned}
\frac{\partial}{\partial \alpha} L &= n\frac{\alpha\delta}{\sqrt{\alpha^2-\beta^2}} R_\lambda\left(\delta\sqrt{\alpha^2-\beta^2}\right) \\
&\quad - \sum_{i=1}^{n}\sqrt{\delta^2+(x_i-\mu)^2}\, R_{\lambda-1/2}\left(\alpha\sqrt{\delta^2+(x_i-\mu)^2}\right)
\end{aligned} \tag{5.3}
$$

$$
\frac{\partial}{\partial \beta} L = n\left\{-\frac{\delta\beta}{\sqrt{\alpha^2-\beta^2}} R_\lambda\left(\delta\sqrt{\alpha^2-\beta^2}\right) - \mu\right\} + \sum_{i=1}^{n} x_i \tag{5.4}
$$

$$
\begin{aligned}
\frac{\partial}{\partial \delta} L &= n\left\{-\frac{2\lambda}{\delta} + \sqrt{\alpha^2-\beta^2} R_\lambda\left(\delta\sqrt{\alpha^2-\beta^2}\right)\right\} \\
&\quad + \sum_{i=1}^{n}\left\{\frac{(2\lambda-1)\,\delta}{\delta^2+(x_i-\mu)^2} - \frac{\alpha\delta R_\lambda\left(\alpha\sqrt{\delta^2+(x_i-\mu)^2}\right)}{\sqrt{\delta^2+(x_i-\mu)^2}}\right\}
\end{aligned} \tag{5.5}
$$

$$
\begin{aligned}
\frac{\partial}{\partial \mu} L &= -n\beta + \sum_{i=1}^{n}\frac{x_i-\mu}{\sqrt{\delta^2+(x_i-\mu)^2}} \\
&\quad \times\left\{\frac{2\lambda-1}{\delta^2+(x_i-\mu)^2} - \alpha R_{\lambda-1/2}\left(\alpha\sqrt{\delta^2+(x_i-\mu)^2}\right)\right\}
\end{aligned} \tag{5.6}
$$

where

$$
\begin{aligned}
k_{\lambda}(x) &= \frac{d}{d\lambda} K_{\lambda}(x) \\
R_{\lambda}(x) &= \frac{K_{\lambda+1}(x)}{K_{\lambda}(x)}
\end{aligned}
$$

Set them to zero, we obtain a complicated nonlinear equation system. Theorectically, there is a solution to a system with five equations and five unknown parameters. However, in practice, the solution is very difficult to be acquired.

Different Algorithm have been proposed to solve the problem. (Press et al, 2002) proposed the Golden search method in one dimension. However the algorithm is not easy programmable.

In this dissertation, an easily programmable algorithm is considered in the next section.

5.2 EM ALGORITHM

5.2.1 Introduction

The EM algorithm (Dempster *et al.*,1977) is a powerful algorithm for ML estimation for data containing missing values or being considered as containg missing values.This formulation is particulaly suitable for distribution arising as mixtures since the mixing operation can be considered responsible for producing missing data (Karlis, 2000).

The EM algorithm can be optimised to estimate the parameters of GHD and it's subfamilies. The Univariate generalised hyperbolic distribution used in this case is parameterised as follows

$$
\begin{aligned}
& f_{GHD}\left(x;\lambda,\alpha,\beta,\delta,\mu\right) \\
= \; & \frac{\left(\alpha^2-\beta^2\right)^{\lambda/2} K_{\lambda-1/2}\left(\alpha\sqrt{\delta^2+\left(x-\mu\right)^2}\right)\exp\left[\beta\left(x-\mu\right)\right]}{\sqrt{2\pi}\alpha^{\lambda-1/2}\delta^{\lambda}K_{\lambda}\left(\delta\sqrt{\alpha^2-\beta^2}\right)\left(\sqrt{\delta^2+\left(x-\mu\right)^2}\right)^{1/2-\lambda}}
\end{aligned}
$$

The loglikehood function of the GHD is:

$$
\begin{aligned}
& \log L\left(\lambda,\alpha,\beta,\delta,\mu;X_1,X_2,\ldots,X_n\right) \\
= \; & \sum_{i=1}^{n}\log f_{GHD}\left(x_i;\lambda,\alpha,\beta,\delta,\mu\right) \\
= \; & -\frac{n}{2}\log\left(2\pi\right)-n\left(\lambda-1/2\right)\log\alpha-n\lambda\log\delta \\
& -n\log K_{\lambda}\left(\delta\sqrt{\alpha^2-\beta^2}\right)-n\left(1/2-\lambda\right) \\
& \times\log\left(\sqrt{\delta^2+\left(x-\mu\right)^2}\right)+\frac{n\lambda}{2}\log\left(\alpha^2-\beta^2\right) \\
& +n\log K_{\lambda-1/2}\left(\alpha\sqrt{\delta^2+\left(x-\mu\right)^2}\right) \\
& +\sum_{i=1}^{n}\beta\left(x_i-\mu\right) \qquad (5.7)
\end{aligned}
$$

Since the derivative of this function involve the Bessel function, direct maximization is not an easy task. The numerical maximazation method that can be used, such as the modification of the steepest descent method implemented in the HYP program developed by Blaesild and Sorensen (1992), possesses all the problems involved with numerical maximization of complicated function like non-convergence and need for good initial values as noted by Dennis and Schnabel, (1983). Morever the solution may not be in the admissible range and thus constraints must be imposed.

The EM algorithm is easily programmable, it surely converges to the maximum and provides interesting insight into the model. If the initial values are in the admissible range then the final solution belongs to the admissible range. In this chapter, we will describe an EM type algorithm for maximum likelihood estimation for mixed normal variance-mean distributions. The main achievement is that it reduces the problem of estimation to one of estimation of the mixing distribution which is usually easier. In addition, an important feature of the EM algorithm is that it is not merely a numerical technique but it also offers useful statistical insight.

Suppose...

5.2.2 EM-algorithm for the GHD parameter estimation

The joint density of X and Z is given by

$$f_{X,Z}(x,z) = f_{X|Z}(x \mid z)\, f_Z(z)$$

Therefore the log-likelihood

$$\begin{aligned} & \log L(\lambda,\alpha,\beta,\delta,\mu; X_1,X_2,\ldots,X_n,Z_1,Z_2,\ldots,Z_n) \\ = & \sum_{i=1}^{n} \log f_{X|Z}(x_i \mid z_i;\mu,\beta) + \sum_{i=1}^{n} \log f_Z(z_i;\lambda,\delta,\gamma) \\ = & \log L_1(\mu,\beta) + \log L_2(\lambda,\delta,\gamma) \end{aligned} \tag{5.8}$$

Therefore, based on the mixture representation, we have augmented the observed data $x_1,x_2,x_3,\ldots,x_n$, and the unobserved data $z_1,z_2,z_3,\ldots,z_n$. This then ensures that the log likelihood of the complete data (X_i,Z_i), $i = 1,2,3,\ldots,n$ factorises into two parts: $\log L_1(\mu,\beta)$ and $\log L_2(\lambda,\delta,\gamma)$ (Kostas, 2007) since

$$
\begin{aligned}
& f_{X|Z}\left(x_i \mid z_i; \mu, \beta\right) \\
= & \frac{1}{\sqrt{2\pi z_i}} \exp\left\{-\frac{1}{2}\left[\frac{\left(x_i - \mu - \beta z_i\right)^2}{z_i}\right]\right\}
\end{aligned}
$$

$$
\begin{aligned}
L_1\left(\mu, \beta\right) &= \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi z_i}} \exp\left\{-\frac{1}{2}\left[\frac{\left(x_i - \mu - \beta z_i\right)^2}{z_i}\right]\right\} \\
&= \left(2\pi\right)^{-n/2} \prod_{i=1}^{n} \left(z_i\right)^{-1/2} \exp\left\{-\frac{1}{2}\sum_{i=1}^{n}\left[\frac{\left(x_i - \mu - \beta z_i\right)^2}{z_i}\right]\right\} \qquad (5.9)
\end{aligned}
$$

therefore

$$
\log L_1\left(\mu, \beta\right) = -\frac{n}{2}\log\left(2\pi\right) - \frac{1}{2}\sum_{i=1}^{n}\log z_i - \frac{1}{2}\sum_{i=1}^{n}\frac{\left(x_i - \mu - \beta z_i\right)^2}{z_i} \qquad (5.10)
$$

Similarly

$$
f_Z\left(z_i; \lambda, \delta, \gamma\right) = \left(\frac{\gamma}{\delta}\right)^{\lambda} \frac{z_i^{\lambda-1}}{2K_\lambda\left(\delta\gamma\right)} \exp\left[-\frac{1}{2}\left(\frac{\delta^2}{z_i} + \gamma^2 z_i\right)\right]
$$

$$
\begin{aligned}
L_2\left(\lambda, \delta, \gamma\right) &= \prod_{i=1}^{n}\left(\frac{\gamma}{\delta}\right)^{\lambda} \frac{z_i^{\lambda-1}}{2K_\lambda\left(\delta\gamma\right)} \exp\left[-\frac{1}{2}\left(\frac{\delta^2}{z_i} + \gamma^2 z_i\right)\right] \\
&= \left(\frac{\gamma}{\delta}\right)^{n\lambda} \prod_{i=1}^{n} z_i^{\lambda-1}\left(2K_\lambda\left(\delta\gamma\right)\right)^{-n} \exp\left[-\frac{1}{2}\sum_{i=1}^{n}\left(\frac{\delta^2}{z_i} + \gamma^2 z_i\right)\right] \qquad (5.11)
\end{aligned}
$$

$$\log L_2(\lambda,\delta,\gamma) = n\lambda\log\frac{\gamma}{\delta} + (\lambda-1)\log z_i - n\log\left(2K_\lambda(\delta\gamma)\right) - \frac{\delta^2}{2}\sum_{i=1}^{n}\frac{1}{z_i} - \frac{\gamma^2}{2}\sum_{i=1}^{n} z_i \quad (5.12)$$

where $\gamma = \sqrt{\alpha^2-\beta^2}$

As noted earlier, the EM-algorithm consist in iterating two steps; the expectation step (E-step) and the maximization step (M-step).

M-step: In this step one start by maximising $log \log L_1(\mu,\beta)$ with respect to the parameters μ and β. At the kth iteration of the algorithm, the estimate for β and μ are simply the maximization ot the derivative of $\log L_1(\mu,\beta)$ with respect to β and μ respectively.

maximizing w.r.t β

$$\begin{aligned}
\frac{\partial}{\partial\beta}\left(\log L_1(\mu,\beta)\right) &= 0\\
\frac{\partial}{\partial\beta}\left(-\frac{n}{2}\log(2\pi) - \frac{1}{2}\sum_{i=1}^{n}\log z_i - \frac{1}{2}\sum_{i=1}^{n}\frac{(x_i-\mu-\beta z_i)^2}{z_i}\right) &= 0\\
-\frac{1}{2}\frac{\partial}{\partial\beta}\left(\sum_{i=1}^{n}\frac{(x_i-\mu-\beta z_i)^2}{z_i}\right) &= 0\\
\frac{1}{2}\sum_{i=1}^{n}\left(\frac{2z_i(x_i-\mu-\beta z_i)}{z_i}\right) &= 0\\
\sum_{i=1}^{n}(x_i-\mu-\beta z_i) &= 0
\end{aligned}$$

which implies that

$$\begin{aligned}
\sum_{i=1}^{n} x_i - n\mu - \beta \sum_{i=1}^{n} z_i &= 0 \\
-n\mu &= -\sum_{i=1}^{n} x_i + \beta \sum_{i=1}^{n} z_i \\
\mu &= \bar{x} - \beta \bar{z} \qquad (5.13)
\end{aligned}$$

maximizing w.r.t μ

$$\begin{aligned}
\frac{\partial}{\partial \mu} \left(\log L_1 (\mu, \beta) \right) &= 0 \\
\frac{\partial}{\partial \mu} \left(-\frac{n}{2} \log (2\pi) - \frac{1}{2} \sum_{i=1}^{n} \log z_i - \frac{1}{2} \sum_{i=1}^{n} \frac{(x_i - \mu - \beta z_i)^2}{z_i} \right) &= 0 \\
-\frac{1}{2} \frac{\partial}{\partial \mu} \left(\sum_{i=1}^{n} \frac{(x_i - \mu - \beta z_i)^2}{z_i} \right) &= 0 \\
\frac{1}{2} \sum_{i=1}^{n} \left(\frac{2 (x_i - \mu - \beta z_i)}{z_i} \right) &= 0 \\
\sum_{i=1}^{n} \frac{(x_i - \mu - \beta z_i)}{z_i} &= 0
\end{aligned}$$

which implies that

$$\begin{aligned}\sum_{i=1}^{n}\frac{x_i}{z_i}-\mu\sum_{i=1}^{n}\frac{1}{z_i}-n\beta &= 0\\ \sum_{i=1}^{n}\frac{x_i}{z_i}-(\bar{x}-\beta\bar{z})\sum_{i=1}^{n}\frac{1}{z_i}-n\beta &= 0\\ \sum_{i=1}^{n}\frac{x_i}{z_i}-\bar{x}\sum_{i=1}^{n}\frac{1}{z_i}+\beta\bar{z}\sum_{i=1}^{n}\frac{1}{z_i}-n\beta &= 0\\ \sum_{i=1}^{n}\frac{x_i}{z_i}-\bar{x}\sum_{i=1}^{n}\frac{1}{z_i} &= \beta\left(n-\bar{z}\sum_{i=1}^{n}\frac{1}{z_i}\right)\end{aligned}$$

which implies that

$$\beta=\frac{\sum_{i=1}^{n}\frac{x_i}{z_i}-\bar{x}\sum_{i=1}^{n}\frac{1}{z_i}}{n-\bar{z}\sum_{i=1}^{n}\frac{1}{z_i}} \tag{5.14}$$

Therefore at the *kth* iteration of the algorithm, the estimate for β and μ are

$$\beta^{(k+1)}=\frac{\sum_{i=1}^{n}\frac{x_i}{z_i}-\bar{x}\sum_{i=1}^{n}\frac{1}{z_i}}{n-\bar{z}\sum_{i=1}^{n}\frac{1}{z_i}} \text{ and } \mu^{(k+1)}=\bar{x}-\beta^{(k+1)}\bar{z} \tag{5.15}$$

Next, $\log L_2(\lambda,\delta,\gamma)$ is maximized with respect to the parameters λ, δ and γ. In the general case ,this maximization must be performed numerically. However, for certain values of λ, the estimate of δ and γ have closed form expressions. This is for instance the case for NIG distribution $\left(\lambda=\frac{1}{2}\right)$ as shown in section(**). In the general case, it suffices to find a value of λ that improves the $\log -likelihood$ and not necessarily the maximum one (Kostas, 2007). Since for given λ the other estimates improves the $\log -$ likelihood, a new λ that provides better $\log -$ likehood ensures the monotonicity property of the EM.

Now in practice, we do not know the values of the variables $z_1, z_2, \ldots, z_n$ (these are the “missing values”). Hence, when computing the estimates in the M-step, we must replace $z_i,\ z_i^{-1},\ \log z_i$ with $E\left(Z_i \mid X_i = x_i\right),\ E\left(Z_i^{-1} \mid X_i = x_i\right)$ and $E\left(\log Z_i \mid X_i = x_i\right)$, respectively. Performing the E-step amounts to estimating these quantities.

It was shown by (Barndorff-Nielsen, 1997) that the main feature of the GIGD is that it is conjugate for the Normal distribution. More specifically, if the prior distribution for z is a $GIG(\lambda, \delta\gamma)$ distribution, the posterior distribution of $z|x$,$\alpha, \beta, \mu, \delta$ is the $GIG\left(\lambda - \frac{1}{2}, \sqrt{\delta^2 + (x-\mu)^2}, \alpha\right)$ distribution. See the proof below.

5.2.3 The GIG conjugate for the Normal distribution.

(Barndorff-Nielsen,1977) shows that the main feature of the GIGD is that it is conjugate to the Normal distribution. In chapter 4 we should that,

$$f\left(x|z\right) = \frac{1}{\sqrt{2\pi z}} \exp\left\{-\frac{1}{2}\left[\frac{\left(x-\left(\mu+\beta z\right)\right)^2}{z}\right]\right\}$$

and

$$\begin{aligned} f\left(z;\lambda,\delta,\gamma\right) &= \frac{\left(\frac{\gamma}{\delta}\right)^{\lambda}}{2K_\lambda\left(\delta\gamma\right)}\left(z\right)^{\lambda-1}\exp\left\{-\frac{1}{2}\left(\delta^2\left(z\right)^{-1}+\gamma^2 z\right)\right\}, z>0 \\ &= \frac{\left(\frac{\alpha^2-\beta^2}{\delta^2}\right)^{\frac{\lambda}{2}}}{2K_\lambda\left(\delta\gamma\right)}\left(z\right)^{\lambda-1}\exp\left\{-\frac{1}{2}\left(\delta^2\left(z\right)^{-1}+\left(\alpha^2-\beta^2\right)z\right)\right\}, z>0 \end{aligned}$$

then

$$f\left(x\right) = \frac{\left(\alpha^2-\beta^2\right)^{\frac{\lambda}{2}}}{\delta^\lambda\sqrt{2\pi}\alpha^{\lambda-\frac{1}{2}}}\left[\sqrt{\delta^2+\left(x-\mu\right)^2}\right]^{\lambda-\frac{1}{2}}\frac{K_{\lambda-\frac{1}{2}}\left(\alpha\sqrt{\delta^2+\left(x-\mu\right)^2}\right)}{K_\lambda\left(\delta\sqrt{\left(\alpha^2-\beta^2\right)}\right)}e^{\beta(x-\mu)}$$

Therefore the posterior distribution:

$$
\begin{aligned}
f(z \mid x) &= \frac{f(x \mid z)\, g(z)}{f(x)} \\
&= \frac{\frac{1}{\sqrt{2\pi z}} \exp\left\{-\frac{1}{2}\left[\frac{(x-(\mu+\beta z))^2}{z}\right]\right\} \left(\frac{\alpha^2-\beta^2}{\delta^2}\right)^{\frac{\lambda}{2}} z^{\lambda-1}}{2K_\lambda\left(\delta\sqrt{\left(\alpha^2-\beta^2\right)}\right)\left(\alpha^2-\beta^2\right)^{\frac{\lambda}{2}} e^{\beta(x-\mu)}} \\
&\quad \times \frac{\exp\left\{-\frac{1}{2}\left(\delta^2 (z)^{-1} + \left(\alpha^2-\beta^2\right) z\right)\right\} \delta^\lambda \sqrt{2\pi}\alpha^{\lambda-\frac{1}{2}} K_\lambda\left(\delta\sqrt{\left(\alpha^2-\beta^2\right)}\right)}{\left[\sqrt{\delta^2+(x-\mu)^2}\right]^{\lambda-\frac{1}{2}} K_{\lambda-\frac{1}{2}}\left(\alpha\sqrt{\delta^2+(x-\mu)^2}\right)} \\
&= \frac{z^{\lambda-\frac{1}{2}-1}}{2K_{\lambda-\frac{1}{2}}\left(\alpha\sqrt{\delta^2+(x-\mu)^2}\right)}\left(\frac{\alpha}{\delta^2+(x-\mu)^2}\right)^{\lambda-\frac{1}{2}} \\
&\quad \times \exp\left\{-\frac{(x-\mu)^2}{2z} + \beta(x-\mu) - \frac{\beta^2 z}{2} + \beta(x-\mu) - \frac{\delta^2}{2z} - \frac{\alpha^2-\beta^2}{2} z\right\} \\
&= \frac{z^{\lambda-\frac{1}{2}-1}}{2K_{\lambda-\frac{1}{2}}\left(\alpha\sqrt{\delta^2+(x-\mu)^2}\right)}\left(\frac{\alpha}{\delta^2+(x-\mu)^2}\right)^{\lambda-\frac{1}{2}} \\
&\quad \times \exp\left\{-\frac{(x-\mu)^2}{2z} - \frac{\beta^2 z}{2} - \frac{\delta^2}{2z} - \frac{\alpha^2}{2} z + \frac{\beta^2}{2} z\right\} \\
&= \frac{z^{\lambda-\frac{1}{2}-1}}{2K_{\lambda-\frac{1}{2}}\left(\alpha\sqrt{\delta^2+(x-\mu)^2}\right)}\left(\frac{\alpha}{\delta^2+(x-\mu)^2}\right)^{\lambda-\frac{1}{2}} \\
&\quad \times \exp\left\{-\frac{(x-\mu)^2}{2z} - \frac{\delta^2}{2z} - \frac{\alpha^2}{2} z\right\} \\
&= \frac{z^{\lambda-\frac{1}{2}-1}}{2K_{\lambda-\frac{1}{2}}\left(\alpha\sqrt{\delta^2+(x-\mu)^2}\right)}\left(\frac{\alpha}{\delta^2+(x-\mu)^2}\right)^{\lambda-\frac{1}{2}} \\
&\quad \times \exp\left\{-\frac{(x-\mu)^2+\delta^2}{2}\frac{1}{z} - \frac{\alpha^2}{2} z\right\}
\end{aligned}
\tag{5.16}
$$

Which is a $GIGD\left(\lambda - \frac{1}{2}, \sqrt{\delta^2 + (x-\mu)^2}, \alpha\right)$.

Now

$$\begin{aligned}
\log L_2(\lambda, \delta, \gamma) &= n\lambda \log \frac{\gamma}{\delta} + (\lambda - 1)\log z_i - n\log(2K_\lambda(\delta\gamma)) \\
&\quad - \frac{\delta^2}{2}\sum_{i=1}^{n}\frac{1}{z_i} - \frac{\gamma^2}{2}\sum_{i=1}^{n} z_i
\end{aligned}$$

Differentiating w.r.t δ

First note that

$$\begin{aligned}
&\frac{\partial}{\partial\gamma}K_\lambda(\delta\gamma) \\
=\ & \frac{1}{2}\int_0^\infty x^{\lambda-1}\frac{\partial}{\partial\delta}e^{-\frac{\delta\gamma}{2}\left(x+\frac{1}{x}\right)}dx \\
=\ & \frac{1}{2}\int_0^\infty x^{\lambda-1}\left(-\frac{\delta}{2}\left(x+\frac{1}{x}\right)\right)e^{-\frac{\delta\gamma}{2}\left(x+\frac{1}{x}\right)}dx \\
=\ & -\frac{\delta}{2}\left\{\frac{1}{2}\int_0^\infty x^{\lambda+1-1}e^{-\frac{\delta\gamma}{2}\left(x+\frac{1}{x}\right)}dx + \frac{1}{2}\int_0^\infty x^{\lambda-1-1}e^{-\frac{\delta\gamma}{2}\left(x+\frac{1}{x}\right)}dx\right\} \\
=\ & -\frac{\delta}{2}\left[K_{\lambda+1}(\delta\gamma) + K_{\lambda-1}(\delta\gamma)\right]
\end{aligned}$$

Similarly

$$
\begin{aligned}
\frac{\partial}{\partial\delta}K_\lambda(\delta\gamma) &= \frac{1}{2}\int_0^\infty x^{\lambda-1}\frac{\partial}{\partial\gamma}e^{-\frac{\delta\gamma}{2}\left(x+\frac{1}{x}\right)}dx \\
&= \frac{1}{2}\int_0^\infty x^{\lambda-1}\left(-\frac{\gamma}{2}\left(x+\frac{1}{x}\right)\right)e^{-\frac{\delta\gamma}{2}\left(x+\frac{1}{x}\right)}dx \\
&= -\frac{\gamma}{2}\left\{\frac{1}{2}\int_0^\infty x^{\lambda+1-1}e^{-\frac{\delta\gamma}{2}\left(x+\frac{1}{x}\right)}dx+\frac{1}{2}\int_0^\infty x^{\lambda-1-1}e^{-\frac{\delta\gamma}{2}\left(x+\frac{1}{x}\right)}dx\right\} \\
&= -\frac{\gamma}{2}\left[K_{\lambda+1}(\delta\gamma)+K_{\lambda-1}(\delta\gamma)\right]
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&\frac{\partial}{\partial\delta}\left(n\lambda\log\frac{\gamma}{\delta}+(\lambda-1)\log z_i-n\log\left(2K_\lambda(\delta\gamma)\right)\right) \\
&-\frac{\partial}{\partial\delta}\left(\frac{\delta^2}{2}\sum_{i=1}^n\frac{1}{z_i}-\frac{\gamma^2}{2}\sum_{i=1}^n z_i\right) \\
=\;& -\frac{n\lambda}{\delta}-\frac{2n}{2K_\lambda(\delta\gamma)}\left\{-\frac{\gamma}{2}\left[K_{\lambda+1}(\delta\gamma)+K_{\lambda-1}(\delta\gamma)\right]\right\} \\
&-\delta\sum_{i=1}^n\frac{1}{z_i} \\
=\;& -\frac{n\lambda}{\delta}-\frac{n}{K_\lambda(\delta\gamma)}\left\{-\frac{\gamma}{2}\left[K_{\lambda+1}(\delta\gamma)+K_{\lambda-1}(\delta\gamma)\right]\right\} \\
&-\delta\sum_{i=1}^n\frac{1}{z_i}
\end{aligned}
$$

Equating to zero and solving for δ

$$
\begin{aligned}
\delta^2\sum_{i=1}^n\frac{1}{z_i}+\frac{n\delta}{K_\lambda(\delta\gamma)}\left\{-\frac{\gamma}{2}\left[K_{\lambda+1}(\delta\gamma)+K_{\lambda-1}(\delta\gamma)\right]\right\}+n\lambda &= 0 \\
a\delta^2+b\delta+c &= 0
\end{aligned}
$$

where:

$$a = \sum_{i=1}^{n} \frac{1}{z_i},\ b = \frac{n}{K_\lambda\left(\delta\gamma\right)}\left\{-\frac{\gamma}{2}\left[K_{\lambda+1}\left(\delta\gamma\right) + K_{\lambda-1}\left(\delta\gamma\right)\right]\right\},\ c = n\lambda$$

which is a quadratic equation. Since we have two solutions, we propose taking the minimum of the solutions.

Similarly for γ

$$\begin{aligned}
& \frac{\partial}{\partial\gamma}\left(n\lambda\log\frac{\gamma}{\delta} + (\lambda-1)\log z_i - n\log\left(2K_\lambda\left(\delta\gamma\right)\right)\right) \\
& -\frac{\partial}{\partial\gamma}\left(\frac{\delta^2}{2}\sum_{i=1}^{n}\frac{1}{z_i} - \frac{\gamma^2}{2}\sum_{i=1}^{n} z_i\right) \\
= & -\frac{n\lambda}{\gamma} - \frac{2n}{2K_\lambda\left(\delta\gamma\right)}\left\{-\frac{\delta}{2}\left[K_{\lambda+1}\left(\delta\gamma\right) + K_{\lambda-1}\left(\delta\gamma\right)\right]\right\} \\
& -\gamma\sum_{i=1}^{n} z_i \\
= & -\frac{n\lambda}{\gamma} - \frac{n}{K_\lambda\left(\delta\gamma\right)}\left\{-\frac{\delta}{2}\left[K_{\lambda+1}\left(\delta\gamma\right) + K_{\lambda-1}\left(\delta\gamma\right)\right]\right\} \\
& -\gamma\sum_{i=1}^{n} z_i
\end{aligned}$$

$$\begin{aligned}
\gamma^2\sum_{i=1}^{n} z_i + \frac{n\gamma}{K_\lambda\left(\delta\gamma\right)}\left\{-\frac{\delta}{2}\left[K_{\lambda+1}\left(\delta\gamma\right) + K_{\lambda-1}\left(\delta\gamma\right)\right]\right\} + n\lambda &= 0 \\
a\gamma^2 + b\gamma + c &= 0
\end{aligned}$$

where:

$$a = \sum_{i=1}^{n} z_i,\ b = \frac{n}{K_\lambda\left(\delta\gamma\right)}\left\{-\frac{\delta}{2}\left[K_{\lambda+1}\left(\delta\gamma\right) + K_{\lambda-1}\left(\delta\gamma\right)\right]\right\},\ c = n\lambda$$

which is a quadratic equation. Since we have two solutions, we propose taking the minimum of the solutions.

5.3 The Normal Inverse Gaussian Distribution

The mixing distribution is the inverse gaussian distribution with density $(Karlis,\ 2002)$

$$f(z) = \frac{\delta}{\sqrt{2\pi}} \exp(\delta\gamma) z^{-3/2} \exp\left\{-\frac{1}{2}\left(\frac{\delta^2}{z} + \gamma^2 z\right)\right\}$$

This implies that

$$\log L_2(\lambda, \delta, \gamma) = \log L_2(\delta, \gamma)$$

Likelihood function is therefore;

$$\begin{aligned} L &= \delta^n (2\pi)^{-n/2} \exp(n\delta\gamma) \prod_{i=1}^{n} z_i^{3/2} \\ &\quad \times \exp\left\{-\sum_{i=1}^{n} \frac{1}{2}\left(\frac{\delta^2}{z_i} + \gamma^2 z_i\right)\right\} \end{aligned} \tag{5.17}$$

The loglikelihood function is;

$$\begin{aligned} L &= n \log \delta - \frac{n}{2} \log(2\pi) + n\delta\gamma + \frac{3}{2} \sum_{i=1}^{n} \log z_i \\ &\quad - \frac{\delta^2}{2} \sum_{i=1}^{n} \frac{1}{z_i} - \frac{\gamma^2}{2} \sum_{i=1}^{n} z_i \end{aligned}$$

Therefore

$$\log L_2\left(\delta,\gamma\right) = n\log\delta + n\delta\gamma - \frac{\delta^2}{2}\sum_{i=1}^{n}\frac{1}{z_i} - \frac{\gamma^2}{2}\sum_{i=1}^{n}z_i$$

differentiating with respect to γ

$$\frac{\partial}{\partial\gamma}\log L_2\left(\delta,\gamma\right) = n\delta - \gamma\sum_{i=1}^{n}z_i$$

equating to zero and solving for γ

$$\begin{aligned} n\delta - \gamma\sum_{i=1}^{n}z_i &= 0 \\ \gamma\sum_{i=1}^{n}z_i &= n\delta \\ \gamma &= \frac{n\delta}{\sum_{i=1}^{n}z_i} \\ \gamma &= \frac{\delta}{\bar{z}} \end{aligned}$$

where

$$\bar{z} = \frac{\sum_{i=1}^{n}z_i}{n}$$

differentiating with respect to δ

$$\frac{\partial}{\partial\delta}\log L_2\left(\delta,\gamma\right) = \frac{n}{\delta} + n\gamma - \delta\sum_{i=1}^{n}\frac{1}{z_i}$$

equating to zero and solving for δ

$$\begin{aligned}
\frac{n}{\delta} + n\gamma - \delta \sum_{i=1}^{n} \frac{1}{z_i} &= 0 \\
n + n\delta \left(\frac{\delta}{\bar{z}} \right) - \delta^2 \sum_{i=1}^{n} \frac{1}{z_i} &= 0 \\
n + \frac{n\delta^2}{\bar{z}} - \delta^2 \sum_{i=1}^{n} \frac{1}{z_i} &= 0 \\
n + \delta^2 \left(\frac{n}{\bar{z}} - \sum_{i=1}^{n} \frac{1}{z_i} \right) &= 0 \\
\delta^2 \left(\sum_{i=1}^{n} \frac{1}{z_i} - \frac{n}{\bar{z}} \right) &= n \\
\delta &= \sqrt{\frac{n}{\sum_{i=1}^{n} \frac{1}{z_i} - \frac{n}{\bar{z}}}}
\end{aligned}$$

Therefore,

$$\delta^{(K+1)} = \sqrt{\frac{n}{\sum_{i=1}^{n} \frac{1}{z_i} - \frac{n}{\bar{z}}}} \text{ and } \gamma^{(k+1)} = \frac{\delta^{(k+1)}}{\bar{z}}$$

In this case, the posterior distribution is

$$GIGD\left(\lambda - \frac{1}{2}, \sqrt{\delta^2 + (x-\mu)^2}, \alpha \right) \ where, \ \lambda = -\frac{1}{2}$$

that is

$$Z \mid X \sim GIGD\left(-1, \sqrt{\delta^2 + (x-\mu)^2}, \alpha \right)$$

Since for $X \sim GIG(\lambda, \delta, \gamma)$

$$E(X) = \left(\frac{\delta}{\gamma} \right)^{\lambda} \frac{K_{\lambda+r}(\delta\gamma)}{K_{\lambda}(\delta\gamma)}$$

Therefore,

$$s_i = E(Z \mid X) = \left(\frac{\sqrt{\delta^2 + (x-\mu)^2}}{\alpha}\right) \frac{K_{-1+1}\left(\alpha\sqrt{\delta^2 + (x-\mu)^2}\right)}{K_{-1}\left(\alpha\sqrt{\delta^2 + (x-\mu)^2}\right)} \tag{5.17}$$

$$= \left(\frac{\sqrt{\delta^2 + (x-\mu)^2}}{\alpha}\right) \frac{K_0\left(\alpha\sqrt{\delta^2 + (x-\mu)^2}\right)}{K_1\left(\alpha\sqrt{\delta^2 + (x-\mu)^2}\right)}$$

and

$$w_i = E\left(Z^{-1} \mid X\right) = \left(\frac{\sqrt{\delta^2 + (x-\mu)^2}}{\alpha}\right)^{-1} \frac{K_{-1-1}\left(\alpha\sqrt{\delta^2 + (x-\mu)^2}\right)}{K_{-1}\left(\alpha\sqrt{\delta^2 + (x-\mu)^2}\right)} \tag{5.18}$$

$$= \left(\frac{\alpha}{\sqrt{\delta^2 + (x-\mu)^2}}\right) \frac{K_{-2}\left(\alpha\sqrt{\delta^2 + (x-\mu)^2}\right)}{K_{-1}\left(\alpha\sqrt{\delta^2 + (x-\mu)^2}\right)}$$

$$= \left(\frac{\alpha}{\sqrt{\delta^2 + (x-\mu)^2}}\right) \frac{K_2\left(\alpha\sqrt{\delta^2 + (x-\mu)^2}\right)}{K_1\left(\alpha\sqrt{\delta^2 + (x-\mu)^2}\right)}$$

If we let

$$\hat{M} = \sum_{i=1}^{n} s_i/n \text{ and } \hat{\Lambda} = n\left(\sum_{i=1}^{n}\left(w_i - \hat{M}^{-1}\right)\right)^{-1}$$

The parameters can be updated as follows

$$\delta^{(k+1)} = \hat{\Lambda}^{1/2} \tag{5.19}$$

$$\gamma^{(k+1)} = \delta^{(k+1)}/\hat{M} \tag{5.20}$$

$$\beta^{(k+1)} = \frac{\sum_{i=1}^{n} x_i w_i - \bar{x} \sum_{i=1}^{n} w_i}{n - \bar{s} \sum_{i=1}^{n} w_i} \tag{5.21}$$

$$\mu^{(k+1)} = \bar{x} - \beta^{(k+1)} \bar{s} \tag{5.22}$$

$$\alpha^{(k+1)} = \left[\left(\gamma^{(k+1)} \right)^2 - \left(\beta^{(k+1)} \right)^2 \right]^{1/2} \tag{5.23}$$

Chapter 6

Copulas

6.1 Introduction

6.2 Generating Copulas

There are various method of generating copulas. Different authors have diversified opinion as it regards this subject. For a detailed illustration, see Nelsen (2006), Pravin and David (2007) and Frabrizio and Carlo (2009). We shall briefly represent some of the method here of the most common copula which we shall use later.

6.2.1 Method of Inversion

Given Continuos random variables y_1 *and* y_2 with margins F_1 *and* F_2, then

$$F(y_1, y_2) = C(F_1(y_1), F_2(y_2)) \tag{5.24}$$

The corresponding copula is generated using the unique inverse transformations

$$y_1 = F_1^{-1}(u_1) \ \ and \ y_2 = F_2^{-1}(u_2) \tag{5.25}$$

where u_1 and u_2 are standard uniform variates.

Examples of copula generated by inversion

Example 1

Let

$$\begin{aligned} F(y_1, y_2) &= \exp\left\{-\left[e^{-y_1} + e^{-y_2} + \left(e^{-\theta y_1} + e^{-\theta y_2}\right)^{-\frac{1}{\theta}}\right]\right\} \\ -\infty &< y_1, y_2 < \infty,\ \theta \geq 0 \end{aligned}$$

Now,

$$\lim_{y_2 \to \infty} F(y_1, y_2) = F_1(y_1) = \exp\left[-e^{-y_1}\right] \equiv u_1$$

$$\lim_{y_1 \to \infty} F(y_1, y_2) = F_2(y_2) = \exp\left[-e^{-y_2}\right] \equiv u_2$$

Hence;

$$\begin{aligned} \exp\left[-e^{-y_1}\right] &= u_1 \\ e^{-y_1} &= -\log u_1 \\ y_1 &= -\log(-\log(u_1)) \end{aligned}$$

similarly;

$$\begin{aligned} \exp\left[-e^{-y_2}\right] &= u_2 \\ e^{-y_2} &= -\log u_2 \\ y_2 &= -\log(-\log(u_2)) \end{aligned}$$

Therefore:

$$
\begin{aligned}
C(u_1,u_2) &= \exp\left\{-\left[\begin{array}{c} e^{\log(-\log(u_1))}+e^{\log(-\log(u_2))} \\ -\left(e^{-\theta\log(-\log(u_1))}+e^{-\theta\log(-\log(u_2))}\right)^{-\frac{1}{\theta}} \end{array}\right]\right\} \\
&= \exp\left\{-\left[\begin{array}{c} -\log(u_1)-\log(u_2) \\ -\left(e^{\log(-\log(u_1))^{-\theta}}+e^{\log(-\log(u_2))^{-\theta}}\right)^{-\frac{1}{\theta}} \end{array}\right]\right\} \\
&= \exp\left\{\begin{array}{c} \log(u_1)+\log(u_2)+((-\log(u_1))^{-\theta} \\ +(-\log(u_2))^{-\theta})^{-\frac{1}{\theta}} \end{array}\right\} \\
&= u_1u_2\exp\left\{\begin{array}{c} [(-\log(u_1))^{-\theta} \\ +(-\log(u_2))^{-\theta}]^{-\frac{1}{\theta}} \end{array}\right\}
\end{aligned}
$$

Note: This expression can be rewritten as

$$
C(u_1,u_2)=u_1u_2\phi^{-1}\left\{[(-\phi(u_1))^{-\theta}+(-\phi(u_2))^{-\theta}]^{-\frac{1}{\theta}}\right\} \tag{2.26}
$$

which will be seen to be a member of the **Archimedean class**.
Example 2

$$
\begin{aligned}
F(y_1,y_2) &= \exp\left\{-\left[e^{-y_1}+e^{-y_2}+\left(e^{-\theta y_1}+e^{-\theta y_2}\right)^{-\frac{1}{\theta}}\right]\right\} \\
F(y_1,y_2) &= 1-\left(e^{-\theta y_1}+e^{-2\theta y_2}-e^{-\theta(y_1+2y_2)}\right)^{-\frac{1}{\theta}} \\
-\infty &< y_1,y_2<\infty,\ \theta\geq 0
\end{aligned}
$$

Now

$$
\lim_{y_2\to\infty}F(y_1,y_2)=F_1(y_1)=1-e^{-y_1}\equiv u_1
$$

$$
\lim_{y_1\to\infty}F(y_1,y_2)=F_2(y_2)=1-e^{-2y_2}\equiv u_2
$$

Therefore

$$\begin{aligned}1-e^{-y_1} &\equiv u_1\\ e^{-y_1} &= 1-u_1\\ y_1 &= -\ln(1-u_1)\end{aligned}$$

Similarly

$$\begin{aligned}1-e^{-2y_1} &\equiv u_1\\ e^{-2y_1} &= 1-u_1\\ y_1 &= (-\ln(1-u_1))/2\end{aligned}$$

Which implies that

$$\begin{aligned}C(u,v) &= 1-\left[e^{\theta\ln(1-u_1)}+e^{\theta\ln(1-u_2)}-e^{-\theta[-\ln(1-u_1)-\ln(1-u_2)]}\right]^{1/\theta}\\ &= 1-\left[(1-u_1)^{\theta}+(1-u_2)^{\theta}-e^{\theta\{\ln[(1-u_1)(1-u_2)]\}}\right]^{1/\theta}\\ &= 1-\left[(1-u_1)^{\theta}+(1-u_2)^{\theta}-[(1-u_1)(1-u_2)]^{\theta}\right]^{1/\theta}\\ &= 1-\left[(1-u_1)^{\theta}+(1-u_2)^{\theta}-(1-u_1)^{\theta}(1-u_2)^{\theta}\right]^{1/\theta}\\ &= 1-\left[(1-u_1)^{\theta}\left(1+(1-u_2)^{\theta}\right)-(1-u_2)^{\theta}\right]^{1/\theta}\end{aligned}$$

Example 3

$$F(y_1,y_2)=\exp\left\{-\left(e^{-\theta y_1}+e^{-\theta y_2}\right)^{1/\theta}\right\}\ ,\ \theta\geq 1$$

Now

$$\lim_{y_2\to\infty}F(y_1,y_2)=F_1(y_1)=\exp\left[-e^{-y_1}\right]\equiv u_1$$

$$\lim_{y_1\to\infty}F(y_1,y_2)=F_2(y_2)=\exp\left[-e^{-y_2}\right]\equiv u_2$$

Hence;

$$\begin{aligned} \exp\left[-e^{-y_1}\right] &= u_1 \\ e^{-y_1} &= -\log u_1 \\ y_1 &= -\log\left(-\log\left(u_1\right)\right) \end{aligned}$$

similarly;

$$\begin{aligned} \exp\left[-e^{-y_2}\right] &= u_2 \\ e^{-y_2} &= -\log u_2 \\ y_2 &= -\log\left(-\log\left(u_2\right)\right) \end{aligned}$$

This implies that

$$\begin{aligned} C\left(u,v\right) &= \exp\left\{-\left[e^{\theta\log(-\log(u_1))} + e^{\theta\log(-\log(u_2))}\right]^{1/\theta}\right\} \\ &= \exp\left\{-\left[\left(-\log\left(u_1\right)\right)^{\theta} + \left(-\log\left(u_2\right)\right)^{\theta}\right]^{1/\theta}\right\} \qquad (5.27) \end{aligned}$$

This parametric family of copulas is known as the $Gumbel-Hougaard\ family$ (Hutchison and Lai 1990) which is also in the **Archimedean class**.

Example 4

$$F\left(y_1,y_2\right) = \left(1 + e^{-y_1} + e^{-y_2}\right)^{-1}$$

which is the Gumbel bivariate logistic distribution.

Now

$$\lim_{y_2\to\infty} F\left(y_1,y_2\right) = F_1\left(y_1\right) = \left(1 + e^{-y_1}\right)^{-1} \equiv u_1$$

$$\lim_{y_1\to\infty} F\left(y_1,y_2\right) = F_2\left(y_2\right) = \left(1 + e^{-y_2}\right)^{-1} \equiv u_2$$

Hence;

$$
\begin{aligned}
\left(1+e^{-y_1}\right)^{-1} &= u_1 \\
1+e^{-y_1} &= (u_1)^{-1} \\
e^{-y_1} &= (u_1)^{-1}-1 \\
y_1 &= -\log\left((u_1)^{-1}-1\right) \\
&= \log\left(\frac{u_1}{1-u_1}\right)
\end{aligned}
$$

Similarly

$$
\begin{aligned}
\left(1+e^{-y_2}\right)^{-1} &= u_2 \\
1+e^{-y_2} &= (u_2)^{-1} \\
e^{-y_2} &= (u_2)^{-1}-1 \\
y_2 &= -\log\left((u_2)^{-1}-1\right) \\
&= \log\left(\frac{u_2}{1-u_2}\right)
\end{aligned}
$$

This implies that

$$\begin{aligned}
C\left(u_1,u_2\right) &= \left(1+e^{-\log\left(\frac{u_1}{1-u_1}\right)}+e^{-\log\left(\frac{u_2}{1-u_2}\right)}\right)^{-1} \\
&= \left(1+\frac{1-u_1}{u_1}+\frac{1-u_2}{u_2}\right)^{-1} \\
&= \left(\frac{u_1u_2+u_2\left(1-u_1\right)+u_1\left(1-u_2\right)}{u_1u_2}\right)^{-1} \\
&= \left(\frac{u_1u_2+u_2-u_1u_2+u_1-u_1u_2}{u_1u_2}\right)^{-1} \\
&= \left(\frac{u_2+u_1-u_1u_2}{u_1u_2}\right)^{-1} \\
&= \frac{u_1u_2}{u_2+u_1-u_1u_2} \qquad (5.28)
\end{aligned}$$

An unattractive feature of the inversion method is that the joint distribution is required to derive the copula (Pravin and David, 2007). This limits the usefulness of the inversion method for application in which the reasearcher does not know the joint distribution.

6.2.2 Algebraic Methods

The derivation of these copulas is based on modifying the relationship between marginals based on indepence by introducing a parameter to capture the dependence relationship. A good example of such copula is the Ali-Mikhail-Haq family.

Consider the Gumbel's bivariate logistic distribution

$$F\left(y_1,y_2\right)=\left(1+e^{-y_1}+e^{-y_2}\right)^{-1}$$

The bivariate survival odds ratio is given by

$$\frac{1-F\left(y_1,y_2\right)}{F\left(y_1,y_2\right)}$$

now

$$\begin{aligned} 1-F\left(y_1,y_2\right) &= 1-\left(1+e^{-y_1}+e^{-y_2}\right)^{-1} \\ &= \frac{1+e^{-y_1}+e^{-y_2}-1}{1+e^{-y_1}+e^{-y_2}} \\ &= \frac{e^{-y_1}+e^{-y_2}}{1+e^{-y_1}+e^{-y_2}} \end{aligned}$$

$$\begin{aligned} \frac{1-F\left(y_1,y_2\right)}{F\left(y_1,y_2\right)} &= \frac{e^{-y_1}+e^{-y_2}}{1+e^{-y_1}+e^{-y_2}} \times 1+e^{-y_1}+e^{-y_2} \\ &= e^{-y_1}+e^{-y_2} \\ &= \frac{1-F_1\left(y_1\right)}{F_1\left(y_1\right)}+\frac{1-F_2\left(y_2\right)}{F_2\left(y_2\right)} \end{aligned}$$

Where $F_1\left(y_1\right)$ and $F_2\left(y_2\right)$ are univariate marginals. Observe that in this case there is no explicit depennce parameter.

In the case of indepence, since $F\left(y_1,y_2\right)=F_1\left(y_1\right)F_2\left(y_2\right),$

$$\begin{aligned} \frac{1-F\left(y_1,y_2\right)}{F\left(y_1,y_2\right)} &= \frac{1-F_1\left(y_1\right)F_2\left(y_2\right)}{F_1\left(y_1\right)F_2\left(y_2\right)} \\ &= \frac{1-F_1\left(y_1\right)}{F_1\left(y_1\right)}+\frac{1-F_2\left(y_2\right)}{F_2\left(y_2\right)}+\frac{1-F_1\left(y_1\right)}{F_1\left(y_1\right)}\frac{1-F_2\left(y_2\right)}{F_2\left(y_2\right)} \end{aligned}$$

Noting the similarity between the bivariate odds ratio in the dependence and independence cases, Ali, Mikhail, and Haq proposed a modified or generalized bivariate ratio with depence parameter θ :

$$\begin{aligned} \frac{1-F\left(y_1,y_2\right)}{F\left(y_1,y_2\right)} &= \frac{1-F_1\left(y_1\right)}{F_1\left(y_1\right)}+\frac{1-F_2\left(y_2\right)}{F_2\left(y_2\right)} \\ &\quad +\left(1-\theta\right)\frac{1-F_1\left(y_1\right)}{F_1\left(y_1\right)}\frac{1-F_2\left(y_2\right)}{F_2\left(y_2\right)} \end{aligned}$$

Then, defining $u_1 = F_1(y_1)$, $u_2 = F_2(y_2)$,

$$\begin{aligned}
\frac{1-C(u_1,u_2)}{C(u_1,u_2)} &= \frac{1-u_1}{u_1} + \frac{1-u_2}{u_2} + (1-\theta)\frac{1-u_1}{u_1}\frac{1-u_2}{u_2} \\
&= \frac{u_2(1-u_1) + u_1(1-u_2) + (1-\theta)(1-u_1)(1-u_2)}{u_1u_2} \\
&= \frac{1-u_1u_2-\theta(1-u_1)(1-u_2)}{u_1u_2}
\end{aligned}$$

therefore

$$\begin{aligned}
u_1u_2(1-C(u_1,u_2)) &= C(u_1,u_2)(1-u_1u_2-\theta(1-u_1)(1-u_2)) \\
u_1u_2 - u_1u_2C(u_1,u_2) &= C(u_1,u_2)(1-u_1u_2-\theta(1-u_1)(1-u_2)) \\
u_1u_2 &= C(u_1,u_2)(u_1u_2+1-u_1u_2-\theta(1-u_1)(1-u_2)) \\
u_1u_2 &= C(u_1,u_2)(1-\theta(1-u_1)(1-u_2))
\end{aligned}$$

Therefore;

$$C(u_1,u_2) = \frac{u_1u_2}{1-\theta(1-u_1)(1-u_2)}, \quad -1 < \theta < 1 \tag{5.29}$$

6.2.3 Mixture Method

This family of copulas, is generated from existing copulas. Starting from a copula family $C_\theta(u_1, u_2/\theta)$ we may obtain a very rich new family of copulas if we allow the parameter θ to be itself a random variable with distribution function $G(\theta/\varphi)$ depending on a vector parameter φ. Then the unconditional copula is given by

$$C(u_1, u_2; \varphi) = \int_{-\infty}^{\infty} C_\theta(u_1, u_2/\theta)\, dG(\theta/\varphi)$$

we refer to $C_\theta(u_1, u_2; \varphi)$ as the mixed copula and $G(\theta/\varphi)$ the mixing distribution.

Example 1

Consider the AMH copula

$$C_\theta\left(u_1, u_2/\theta\right) = \frac{u_1 u_2}{1-\theta\left(1-u_1\right)\left(1-u_2\right)}$$

Applying parameter mixing to this copula using a $Beta\left(\alpha, \beta\right)$ mixing distribution, which has a *pdf*

$$f\left(x;\alpha,\beta\right) = \frac{x^{\alpha-1}\left(1-x\right)^{\beta-1}}{B\left(\alpha,\beta\right)},\ 0<x<1$$

Since the domain for the parameter (θ) and *Beta* distribution defer, a transformation is required.

Let

$$\theta = 2x-1,\ \ -1<\theta<1$$

Thus, the support of the transform $(2x-1)$ project onto the same range of values as that of the depence parameter θ.

Therefore

$$
\begin{aligned}
C\left(u_1, u_2; x\right) &= \int_0^1 \frac{u_1 u_2}{1-(2x-1)\left(1-u_1\right)\left(1-u_2\right)} \times \\
&\quad \frac{x^{\alpha-1}(1-x)^{\beta-1}}{B(\alpha,\beta)} dx \\
&= \frac{1}{B(\alpha,\beta)} \int_0^1 \frac{u_1 u_2}{1+\left(1-u_1\right)\left(1-u_2\right)} \times \\
&\quad \frac{\left(1+\left(1-u_1\right)\left(1-u_2\right)\right) x^{\alpha-1}(1-x)^{\beta-1}}{1+\left(1-u_1\right)\left(1-u_2\right)-2x\left(1-u_1\right)\left(1-u_2\right)} dx \\
&= \frac{u_1 u_2}{B(\alpha,\beta)\left[1+\left(1-u_1\right)\left(1-u_2\right)\right]} \times \\
&\quad \int_0^1 \left(\frac{1+\left(1-u_1\right)\left(1-u_2\right)-2x\left(1-u_1\right)\left(1-u_2\right)}{1+\left(1-u_1\right)\left(1-u_2\right)}\right)^{-1} \times \\
&\quad x^{\alpha-1}(1-x)^{(\alpha+\beta)-\alpha-1} dx \\
&= \frac{u_1 u_2}{B(\alpha,\beta)\left[1+\left(1-u_1\right)\left(1-u_2\right)\right]} \times \\
&\quad \int_0^1 \left(1-\frac{2\left(1-u_1\right)\left(1-u_2\right)}{1+\left(1-u_1\right)\left(1-u_2\right)} x\right)^{-1} x^{\alpha-1}(1-x)^{(\alpha+\beta)-\alpha-1} dx \\
&= \frac{u_1 u_2}{B(\alpha,\beta)\left[1+\left(1-u_1\right)\left(1-u_2\right)\right]} \int_0^1 (1-sx)^{-1} x^{\alpha-1}(1-x)^{(\alpha+\beta)-\alpha-1} dx
\end{aligned}
$$

Now

$$\int_0^1 (1-sx)^{-1} x^{\alpha-1} (1-x)^{(\alpha+\beta)-\alpha-1} dx$$

$$= \int_0^1 \left(1 + sx + \frac{(sx)^2}{2!} + \frac{(sx)^3}{3!} + \frac{(sx)^4}{4!}\right) \times$$
$$x^{\alpha-1} (1-x)^{(\alpha+\beta)-\alpha-1} dx$$

$$= \int_0^1 x^{\alpha-1} (1-x)^{(\alpha+\beta)-\alpha-1} dx + s \int_0^1 x^{\alpha} (1-x)^{(\alpha+\beta)-\alpha-1} dx +$$
$$\frac{s^2}{2!} \int_0^1 x^{\alpha+1} (1-x)^{(\alpha+\beta)-\alpha-1} dx + \frac{s^3}{3!} \int_0^1 x^{\alpha+2} (1-x)^{(\alpha+\beta)-\alpha-1} dx + \frac{s^4}{4!} \int_0^1 x^{\alpha+3} (1-x)^{(\alpha+\beta)-\alpha-1}$$

$$= B(\alpha,\beta) + sB(\alpha+1,\beta) + \frac{s^2}{2!} B(\alpha+2,\beta) + \frac{s^3}{3!} B(\alpha+3,\beta) + \frac{s^4}{4!} B(\alpha+4,\beta) + \ldots$$

Therefore

$$\begin{aligned} C(u_1,u_2;x) &= \frac{u_1 u_2}{B(\alpha,\beta)\left[1+(1-u_1)(1-u_2)\right]} \times \\ &\left(\begin{array}{c} B(\alpha,\beta) + sB(\alpha+1,\beta) + \frac{s^2}{2!} B(\alpha+2,\beta) + \frac{s^3}{3!} B(\alpha+3,\beta) \\ +\frac{s^4}{4!} B(\alpha+4,\beta) + \ldots \end{array}\right) \\ &= \frac{u_1 u_2}{1+(1-u_1)(1-u_2)} \left(\begin{array}{c} 1 + \frac{\alpha}{\alpha+\beta} s + \frac{\alpha(\alpha+1)}{(\alpha+\beta)(\alpha+\beta+1)} \frac{s^2}{2!} \\ + \frac{\alpha(\alpha+1)(\alpha+2)}{(\alpha+\beta)(\alpha+\beta+1)(\alpha+\beta+2)} \frac{s^3}{3!} + \ldots \end{array}\right) \\ &= \frac{u_1 u_2}{1+(1-u_1)(1-u_2)} \left({}_1F_1(\alpha,\alpha+\beta;s)\right) \qquad (5.30) \end{aligned}$$

where

$$s = \frac{2(1-u_1)(1-u_2)}{1+(1-u_1)(1-u_2)}$$

Example 2

Let

$$
\begin{aligned}
C\left(u_1, u_2/\theta\right) &= u_1 u_2 + \theta u_1 u_2 \left(1-u_1\right)\left(1-u_2\right) \\
&= u_1 u_2 \left(1+\theta\left(1-u_1\right)\left(1-u_2\right)\right)
\end{aligned}
$$

which is the $Farlie - Gumbel - Morgenstern\ family$, abbreviated as FGM copula.

Let the mixing distribution be Beta distribution:

$$
g\left(x/\alpha,\beta\right) = \frac{x^{\alpha-1}\left(1-x\right)^{\beta-1}}{B\left(\alpha,\beta\right)}
$$

following the same transformation as above

$$\begin{aligned}
C\left(u_1,u_2;\alpha,\beta\right) &= \int_0^\infty u_1u_2\left(1+\left(2x-1\right)\left(1-u_1\right)\left(1-u_2\right)\right)\times \\
&\quad \frac{x^{\alpha-1}\left(1-x\right)^{\beta-1}}{B\left(\alpha,\beta\right)}dx \\
&= \frac{u_1u_2}{B\left(\alpha,\beta\right)}\left(\int_0^\infty \left(2x-1\right)\left(1-u_1\right)\left(1-u_2\right)x^{\alpha-1}\left(1-x\right)^{\beta-1}dx+B\left(\alpha,\beta\right)\right) \\
&= \frac{u_1u_2}{B\left(\alpha,\beta\right)}2\int_0^\infty \left(1-u_1\right)\left(1-u_2\right)x^{\alpha+1-1}\left(1-x\right)^{\beta-1}dx- \\
&\quad \frac{u_1u_2}{B\left(\alpha,\beta\right)}\int_0^\infty \left(1-u_1\right)\left(1-u_2\right)x^{\alpha-1}\left(1-x\right)^{\beta-1}dx+u_1u_2 \\
&= \frac{2u_1u_2\left(1-u_1\right)\left(1-u_2\right)}{B\left(\alpha,\beta\right)}B\left(\alpha+1,\beta\right)- \\
&\quad \frac{u_1u_2\left(1-u_1\right)\left(1-u_2\right)}{B\left(\alpha,\beta\right)}B\left(\alpha,\beta\right)+u_1u_2 \\
&= u_1u_2\left(2\left(1-u_1\right)\left(1-u_2\right)\frac{\alpha}{\alpha+\beta}-\left(1-u_1\right)\left(1-u_2\right)+1\right) \\
&= u_1u_2\left(1+\left(1-u_1\right)\left(1-u_2\right)\left(\frac{2\alpha-\alpha-\beta}{\alpha+\beta}\right)\right) \\
&= u_1u_2\left(1+\left(\frac{\alpha-\beta}{\alpha+\beta}\right)\left(1-u_1\right)\left(1-u_2\right)\right) \\
&= u_1u_2\left(1+\mu\left(1-u_1\right)\left(1-u_2\right)\right) \qquad (5.31)
\end{aligned}$$

Where

$$\mu=\frac{\alpha-\beta}{\alpha+\beta}$$

The resultant mixture copula preserves the functional form, and thus the dependence stucture, of the parent FGM copula.

6.2.4 Generator method

This method nests the Archimedean copulas.

Recall the AMH copula

$$\begin{aligned} \frac{1-F(y_1,y_2)}{F(y_1,y_2)} &= \frac{1-F_1(y_1)}{F_1(y_1)} + \frac{1-F_2(y_2)}{F_2(y_2)} \\ &\quad + (1-\theta)\frac{1-F_1(y_1)}{F_1(y_1)}\frac{1-F_2(y_2)}{F_2(y_2)} \end{aligned}$$

with a little algebra, this can be rewritten as

$$\begin{aligned} 1+(1-\theta)\frac{1-F(y_1,y_2)}{F(y_1,y_2)} &= \left[1+(1-\theta)\frac{1-F_1(y_1)}{F_1(y_1)}\right] \times \\ &\quad \left[1+(1-\theta)\frac{1-F_2(y2)}{F_2(y_2)}\right] \end{aligned}$$

that is, $\lambda(F(y_1,y_2)) = \lambda F_1(y_1)\lambda F_2(y2)$, where $\lambda(t) = 1+(1+\theta)(1-t)/t$. Equivalently, whenever we can write $\lambda(F(y_1,y_2)) = \lambda F_1(y_1)\lambda F_2(y2)$ for a function λ (which must be a positive on the interval (0,1)) then on setting

$$\varphi(t) = -\ln\lambda(t)$$

we can also write F as a su of functions of the marginals F and G, $i.e$ $\varphi(F(y_1,y_2)) = \varphi(F_1(y_1)) + \varphi(F_2(y_2))$, or for Copulas,

$$\varphi(C(u_1,u_2)) = \varphi(u_1) + \varphi(u_2) \tag{5.32}$$

Properties of $\varphi(t)$

Examples 1

when

$$\varphi_\theta(t) = \ln\frac{1-\theta(1-t)}{t}$$

then

$$C\left(u_1, u_2; \theta\right) = \frac{u_1 u_2}{1 - \theta\left(1 - u_1\right)\left(1 - u_2\right)} \tag{5.33}$$

which is the AMH family of copulas.

Example 2

when

$$\varphi_\theta\left(t\right) = \left(-\ln t\right)^\theta$$

then

$$C\left(u_1, u_2; \theta\right) = \exp\left\{-\left[\left(-\ln u_1\right)^\theta + \left(-\ln u_2\right)^\theta\right]^{1/\theta}\right\} \tag{5.34}$$

which is the Gumbel family.

example 3

when

$$\varphi_\theta\left(t\right) = -\ln\frac{e^{-\theta t} - 1}{e^{-\theta} - 1}$$

then

$$C\left(u_1, u_2; \theta\right) = -\frac{1}{\theta}\ln\left(1 + \frac{\left(e^{-\theta u_1} - 1\right)\left(e^{-\theta u_2} - 1\right)}{e^{-\theta} - 1}\right) \tag{5.35}$$

which is the Frank family.

Example 4

when

$$\varphi_\theta\left(t\right) = \theta^{-1}\left(t^{-\theta} - 1\right);\ 0 < \theta < \infty$$

then

$$C\left(u_1, u_2; \theta\right) = \left(u_1^{-\theta} + u_2^{-\theta} - 1\right)^{-1/\theta} \tag{5.36}$$

which is the clayton copula.

Other common Copula

Gaussian (Normal) Copula

$$C(u_1,u_2;\theta) = \Phi_G\left(\Phi^{-1}(u_1),\Phi^{-1}(u_2)\right) = \int_{-\infty}^{\Phi^{-1}(u_1)}\int_{-\infty}^{\Phi^{-1}(u_2)} \frac{1}{2\pi\left(1-\theta^2\right)^{1/2}}\left\{\frac{-\left(s^2-2\theta st+t^2\right)}{2\left(1-\theta^2\right)}\right\} dsdt \quad (5.37)$$

Student's t-Copula

An example of copula with two dependence parameters is that for the bivariate t-distribution with ν degrees of freedom and correlation ρ,

$$C(u_1,u_2;\theta_1,\theta_2) = \int_{-\infty}^{t_{\theta_1}^{-1}(u_1)}\int_{-\infty}^{t_{\theta_2}^{-1}(u_2)} \frac{1}{2\pi\left(1-\theta_2^2\right)^{1/2}}\left\{1+\frac{\left(s^2-2\theta_2 st+t^2\right)}{\nu\left(1-\theta_2^2\right)}\right\}^{-(\theta_1+2)/2} dsdt \quad (5.38)$$

6.3 Parameter Estimation and Goodness of Fit

Parameter estimation and goodness of fit has been done using the CDVine Package using the R statistical software. The abbreviation and terminology in line with the package.

Chapter 7

Application

In this chapter, we consider the return of the s&p500 index and two energy investment companies: Range Resource Corporation (RRC) and Shares of Chevron Corporation (CVX). Both companies are among the 500 stocks of the s&p500 index. Weekly returns from 3/01/2000 to 1/07/2013, both the the index and companies are analysed using the establised literature. The first step entails finding the heavy tailed distribution that fits the data well. Then, the bivariate distribution between the company's return and the s&p500 index return and between the companies is established using the copulas constructed in the previous chapter.

7.1 Dataset

- s&p500 index

Standard and Poor's 500 Index is a capitalization-weighted index of 500 stocks. The index is designed to measure performance of the broad domestic economy through changes in the aggregate market value of 500 stocks representing all major industries. The index was developed with a base level of 10 for the 1941- 43 base period.

- Range Resource Corporation (RRC)

One of the top gainer in the energy sector.Range Resources is an independent oil and gas exploration and production company based in Fort Worth, Texas. Range is best known for its pioneering of the Devonian-aged Marcellus Shale in Pennsylvania, which is now the most productive natural gas field in the United States. Range has over $1 billion USD invested in southwestern Pennsylvania, while it also has operations in the Southwestern United States. Founded in 1976, the current President and Chief Executive Officer is Jeffrey L. Ventura.

- Shares of Chevron Corporation (CVX)

The company has a long, robust history, which began when a group of explorers and merchants established the Pacific Coast Oil Co. on Sept. 10, 1879. Since then, the company's name has changed more than once, but has always retained its founders' spirit, grit, innovation and perseverance. Chevron has taken many actions toward achieving incident-free performance, and has constantly strived to improve its operation excellence efforts.

7.2 Historical prices for the dataset

For these dataset, the historical weekly prices considered are from 3/01/2000 to 1/07/2013. The extract is as seen below:

7.3 Return for the dataset

Let $(P_t)_{t\geq 0}$ denote the price process of a security, in particular of a stock. In order to allow comparison of investments in different securities we shall investigate the rates of return defined by

$$X_t = \log P_t - \log P_{t-1} \tag{6.1}$$

The reason for this is that the return over n periods, for example n days, is then just the sum

$$X_t + X_{t+1} + X_{t+2} + \ldots + X_{t+n-1} = \log P_{t+n-1} - \log P_{t-1} \qquad (6.2)$$

which does not hold for Y_t defined by

$$Y_t = (P_t - P_{t-1}) / P_{t-1}$$

The extract tables for the weekly returns of the datasets can be found at the appendix.

7.4 Fitting returns to GHDs

7.4.1 s&p500 index

Fitting to the NIG

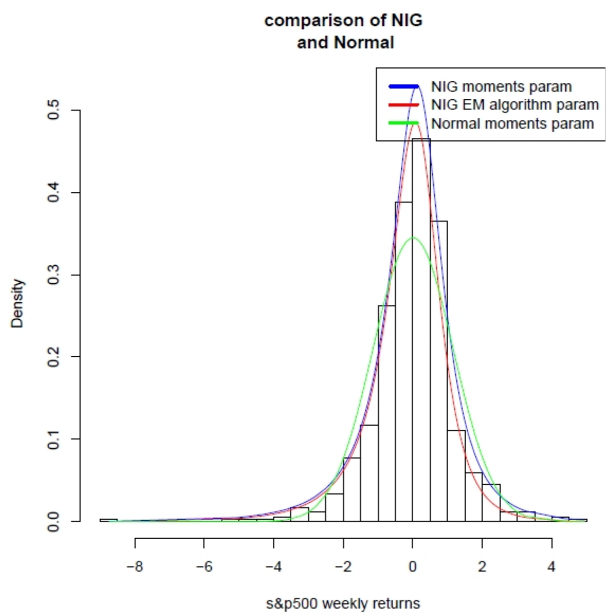
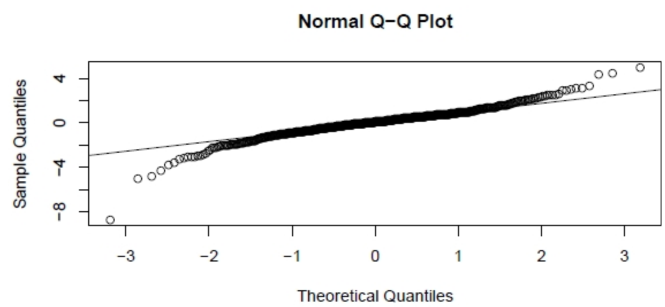
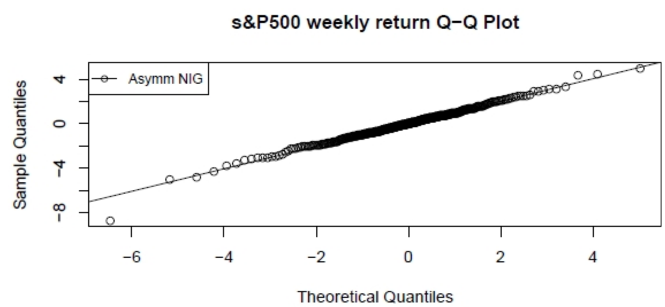


Figure 7.1: Fitting the NIG to s&P500 weekly returns

$Parameter$	$value$
α	0.7673208
β	-0.1299347
δ	0.9663424
μ	0.1727313





Fitting to the Variance Gamma

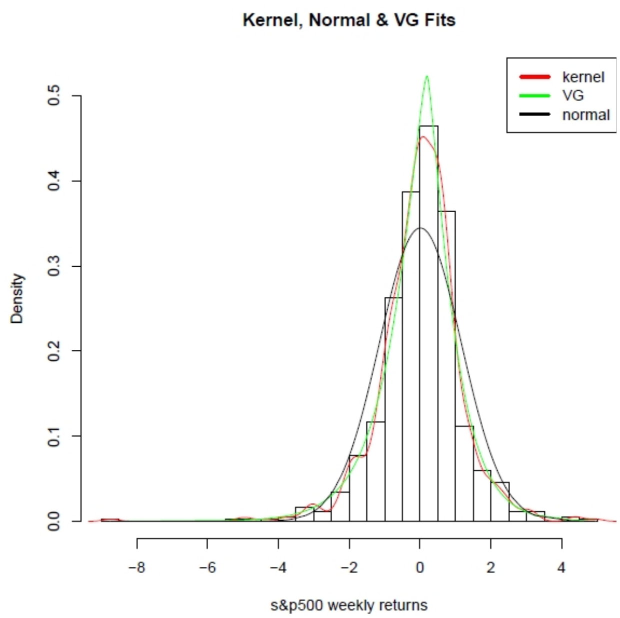
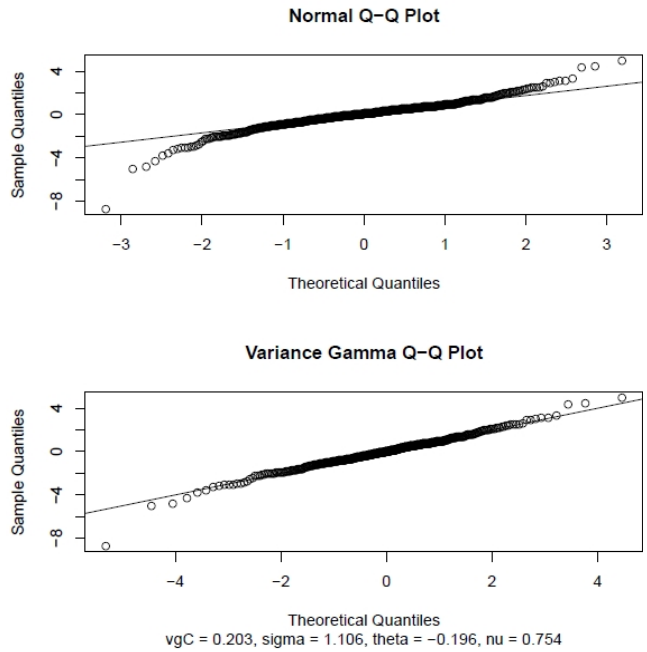


Figure 7.2: Fitting Variance Gamma to s&p500 weekly returns

$Parameter$	$value$
vgC	0.2028
$sigma$	1.1062
$theta$	-0.1963
nu	0.7543



Fitting to the Hyperbolic Skew Student's t distribution.

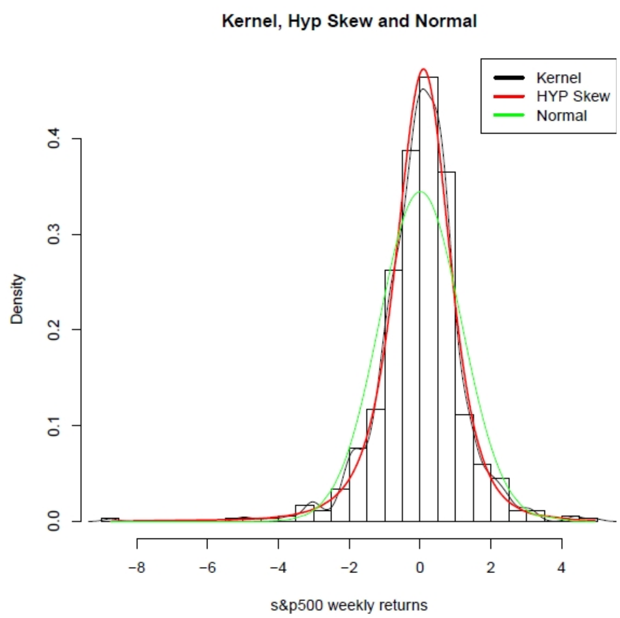


Figure 7.3: Fitting the hyperbolic skew Student's t distribution to s&p500 weekly returns

Parameter	*value*
β	-0.1044
δ	1.4983
μ	0.1462
nu	3.6248

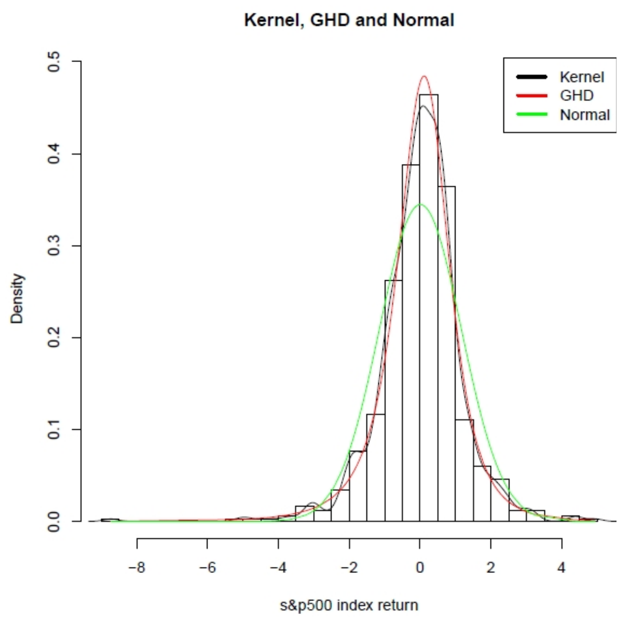
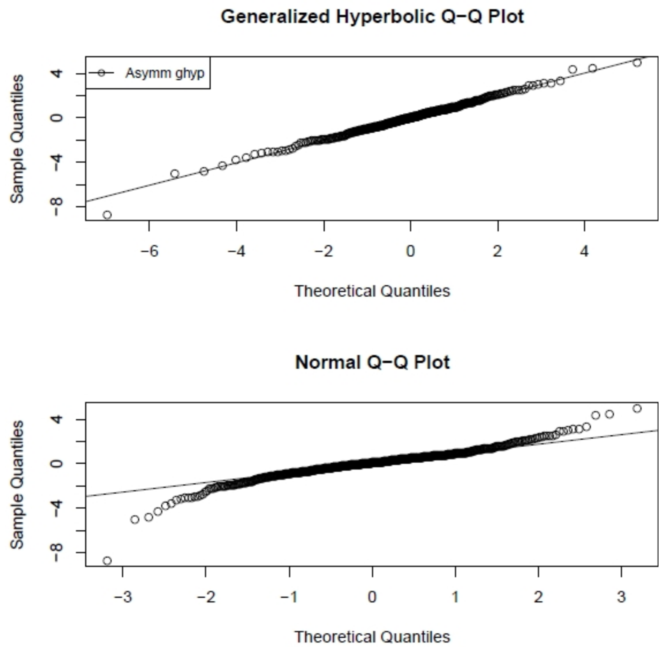


Figure 7.4: Fitting GHD to s&p500 weekly returns

Fitting GHD to s&p500 weekly returns

$Parameter$	$value$
λ	-1.0528006
$\bar{\alpha}$	0.6315397
μ	0.1628228
γ	-0.1556117
σ	1.1357878



Fitting Hyperbolic distribution to s&p500 weekly returns

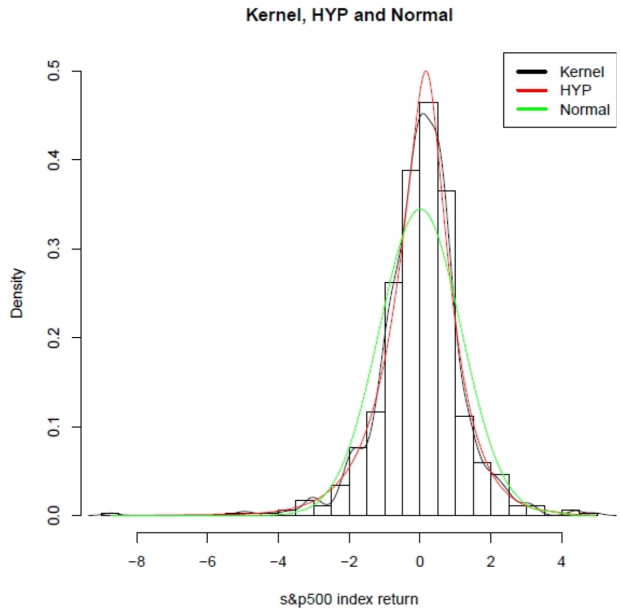
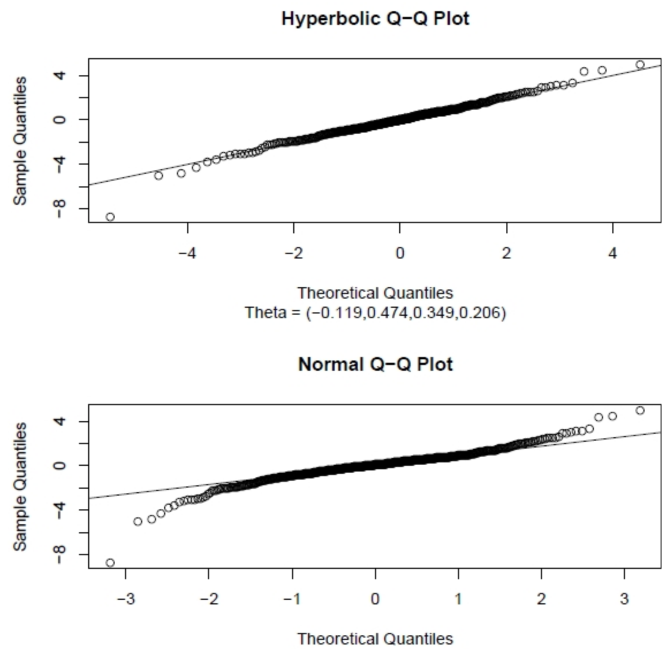


Figure 7.5: Fitting hyperbolic distribution to s&p500 weekly returns

Parameter	*value*
α	1.3687654
β	-0.1625062
δ	0.3490670
μ	0.2055611



Hyperbolic Q−Q Plot
Sample Quantiles
Theoretical Quantiles
Theta = (−0.119,0.474,0.349,0.206)
Normal Q−Q Plot
Sample Quantiles
Theoretical Quantiles

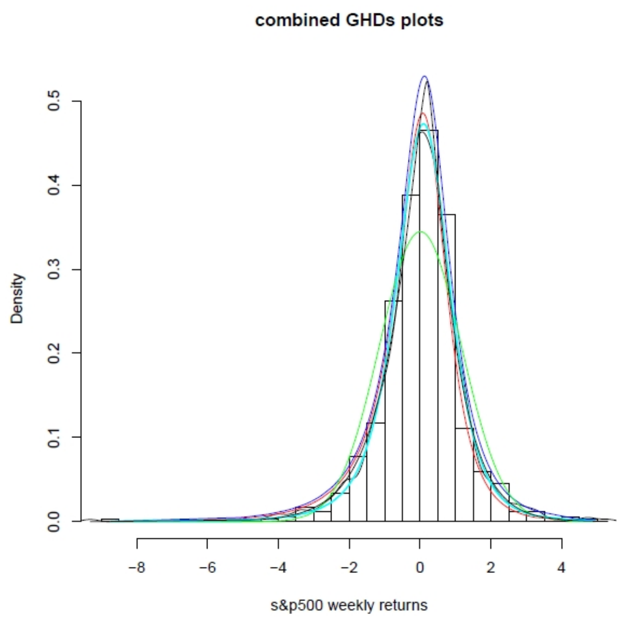


Figure 7.6: A Combined plot of GHDs fitted to s&p500 weekly returns

The Combined Plot of GHDs fitted to s&p500 weekly returns

The Selection creteria using the AIC

	$Model$	AIC
1	NIG	2079.158
2	t	2079.998
3	GHD	2080.806
4	HYP	2083.743
5	VG	2086.776
6	$Normal$	2201.019

Therefore, the model selected for the s&p500 index is the NIG distribution

$$f_{NIG}\left(x;\alpha,\beta,\delta,\mu\right)=e^{\delta\sqrt{\alpha^2-\beta^2}}\frac{\alpha\delta}{\pi\sqrt{\delta^2+\left(x-\mu\right)^2}}K_1\left(\alpha\sqrt{\delta^2+\left(x-\mu\right)^2}\right)e^{\beta(x-\mu)}$$

with parameters:

Parameter	*value*
α	0.7673208
β	−0.1299347
δ	0.9663424
μ	0.1727313

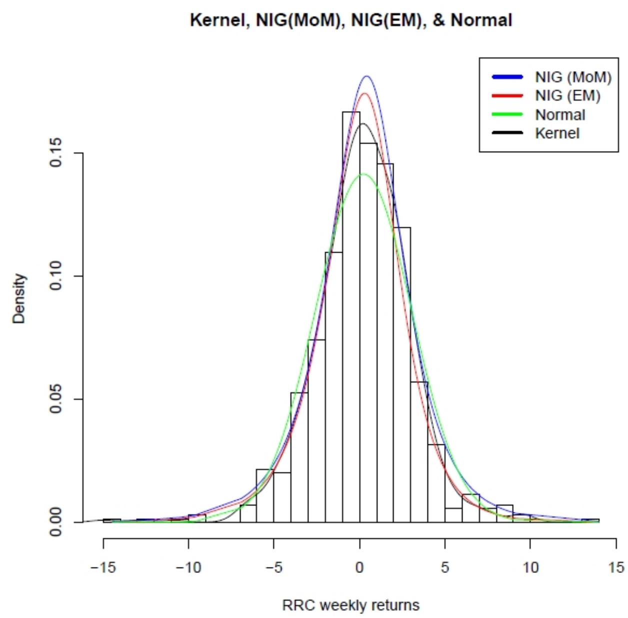
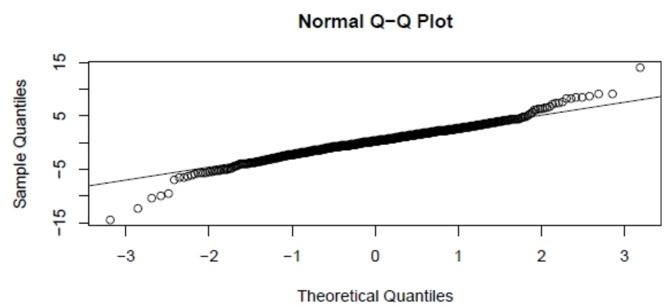


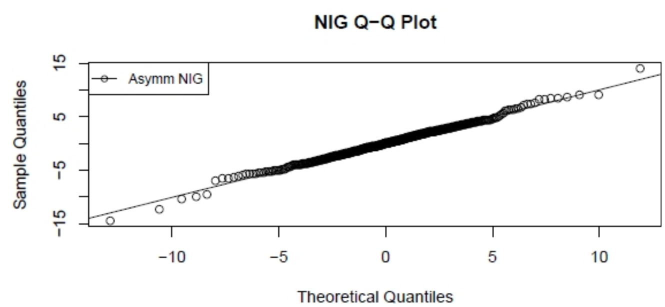
Figure 7.7: Fitting NIG to RRC weekly returns

7.4.2 Range Resourse Corporation (RRC)

Fitting NIG to RRC weekly returns

Parameter	*value*
α	0.42151452
β	-0.03585861
δ	3.28478285
μ	0.51377112





Fitting Hyperbolic distribution to RRC weekly returns

Parameter	*value*
α	0.60932548
β	−0.04041637
δ	1.93794355
μ	0.54568636

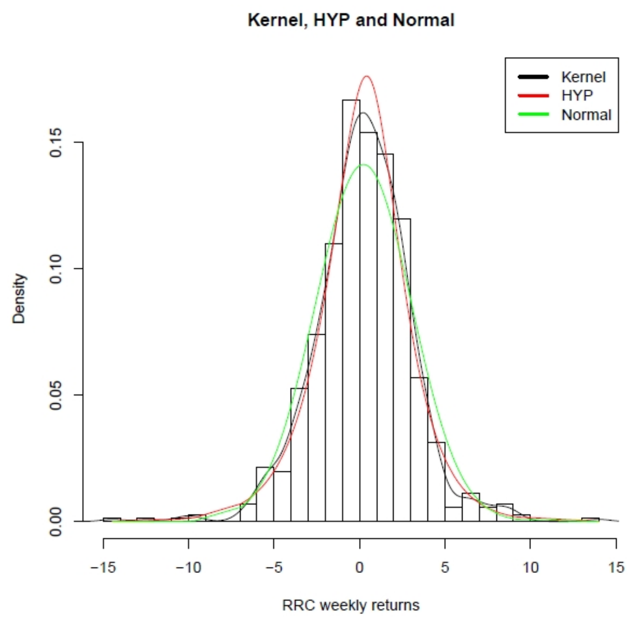


Figure 7.8: Fitting hyperbolic distribution to RRC weekly returns

Fitting Variance Gamma to RRC weekly returns

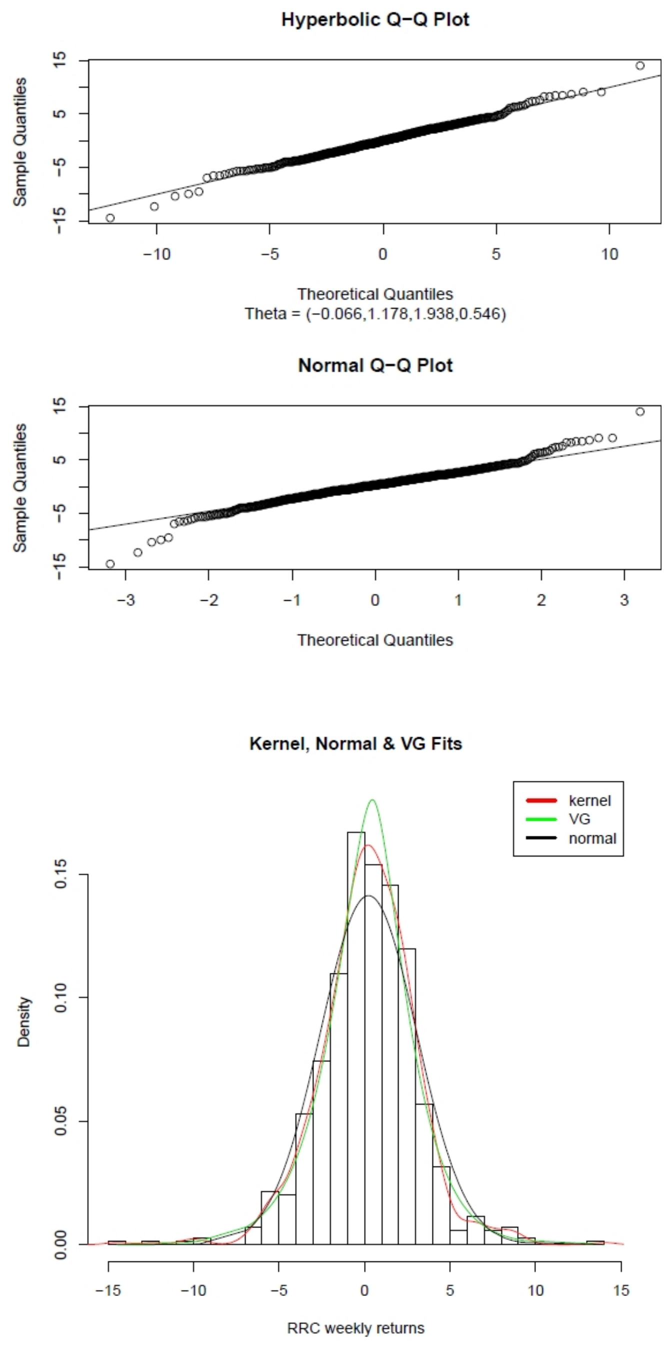


Figure 7.9: Fitting the Variance Gamma to RRC weekly returns

$Parameter$	$value$
vgC	0.5384
$sigma$	2.7776
$theta$	-0.3046
nu	0.5059

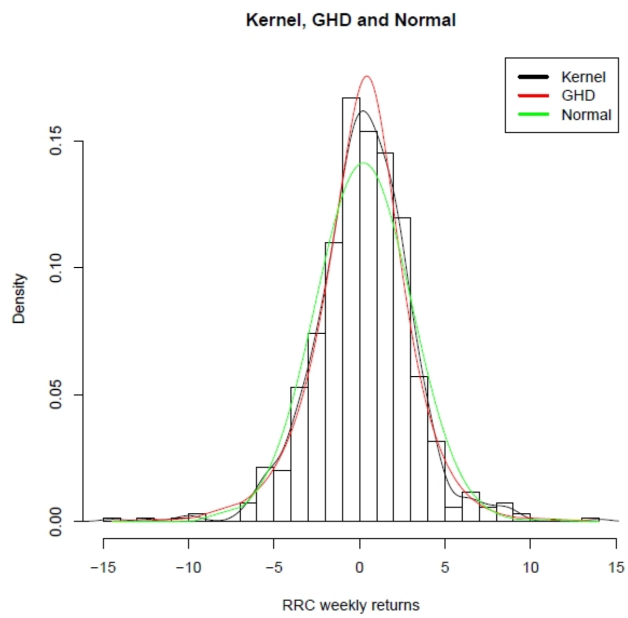
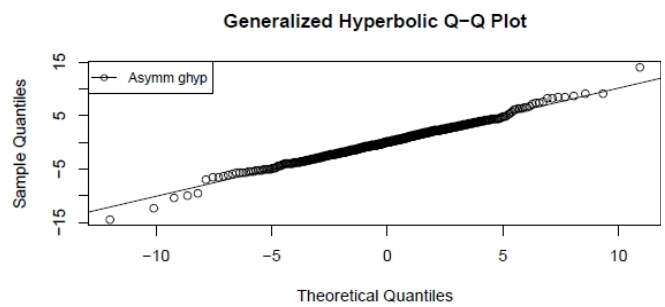
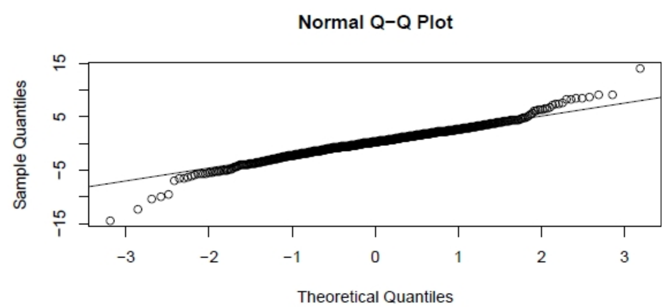


Figure 7.10: Fitting GHD to RRC weekly returns

Fitting the GHD to RRC weekly returns

Parameter	*value*
$\bar{\alpha}$	1.0212869
γ	-0.4216411
σ	2.7670216
μ	0.6015888
λ	1.4511491





Fitting the Skew Hyperbolic t-distribution to RRC weekly returns

$Parameter$	$value$
β	-0.02804
δ	4.99563
μ	0.45678
nu	5.12385

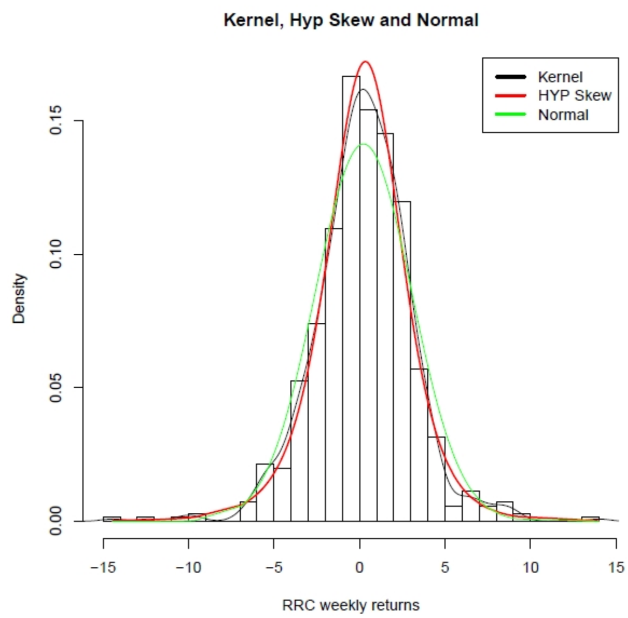
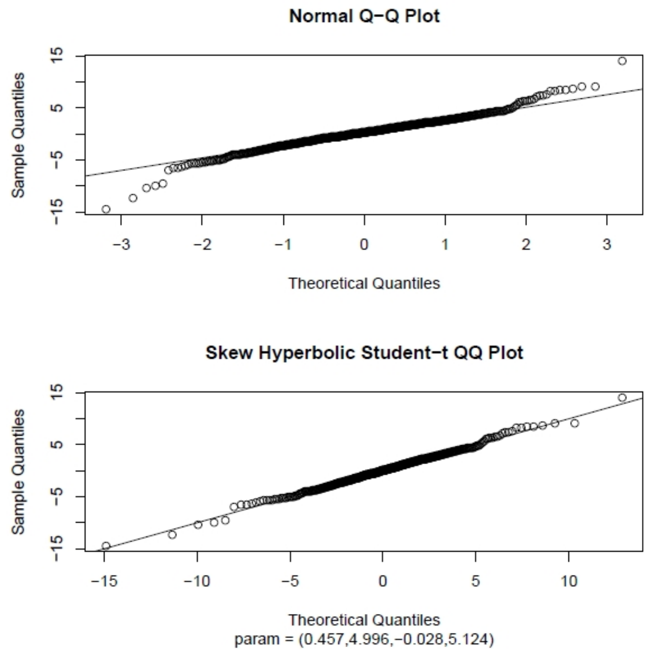


Figure 7.11: Fitting Skew Hyperbolic t-distribution to data.

Combined plot of GHDs Fitted to RRC weekly return data
AIC selection creteria

	$Model$	AIC
1	t	3398.739
2	NIG	3399.777
3	HYP	3401.587
4	VG	3403.541
5	GHD	3404.600
6	$Normal$	3453.124

The "best" model selected using the AIC is skew student t-distribution:



$$f_{GHD}\left(x;\lambda,\left|\beta\right|,\beta,\delta,\mu\right)=\frac{2^{\frac{-n+1}{2}}\delta^{n}}{\sqrt{2\pi}\Gamma\left(n/2\right)}\left(\frac{\sqrt{\delta^{2}+\left(x-\mu\right)^{2}}}{\left|\beta\right|}\right)^{-\frac{n+1}{2}}K_{-\frac{n+1}{2}}\left(\left|\beta\right|\sqrt{\left(x-\mu\right)^{2}+\delta^{2}}\right)e^{\beta(x-\mu}$$

with parameters

Parameter	*value*
β	-0.02804
δ	4.99563
μ	0.45678
nu	5.12385

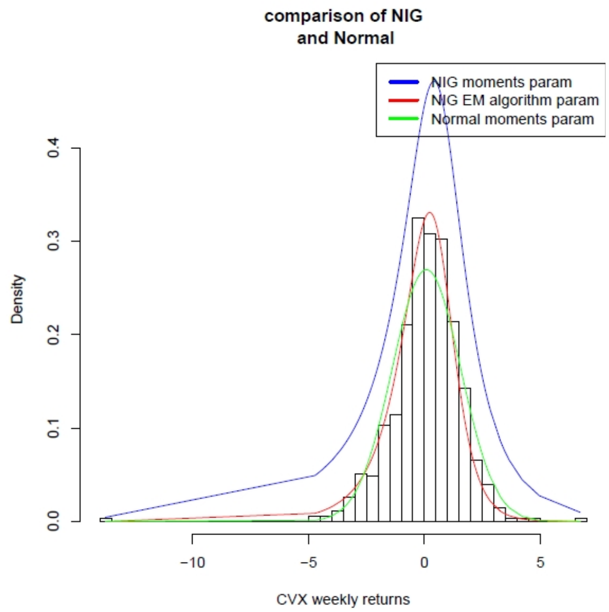
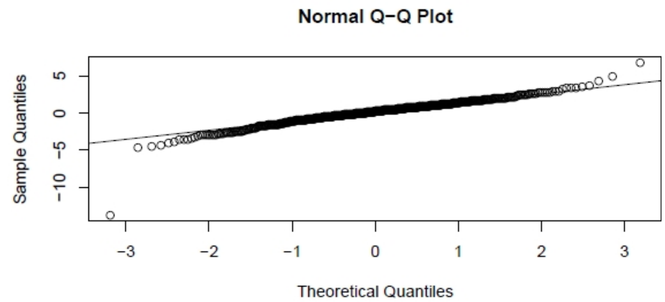


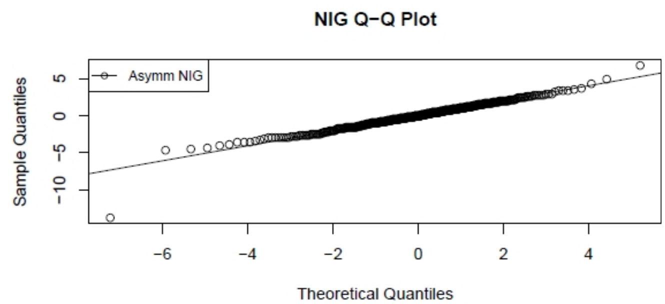
Figure 7.12: Fitting the NIG distribution of CVX weekly returns

7.4.3 Shares of Chevron Corporation

Fitting the NIG Distribution to CVX weekly returns

$Parameter$	$value$
β	-0.2438539
δ	1.736552
μ	0.5593046
α	0.9293567





Fitting the Hyperbolic Distribution

Parameter	*value*
β	-0.2618169
δ	1.0629097
μ	0.5892781
α	1.2768395

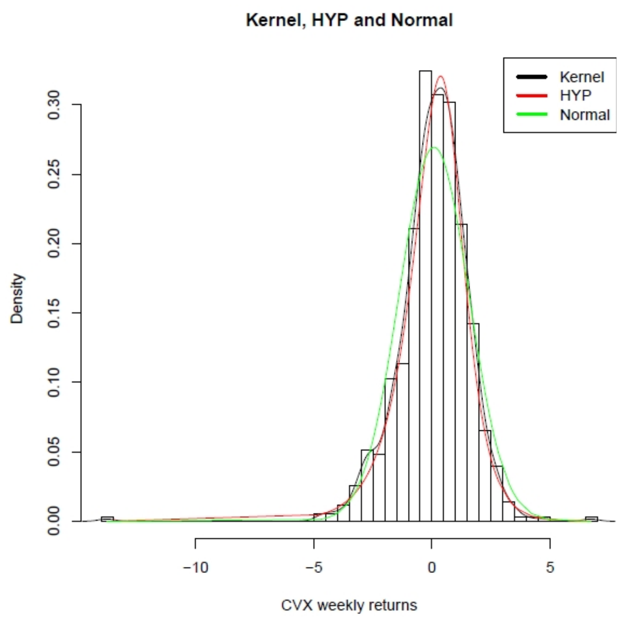
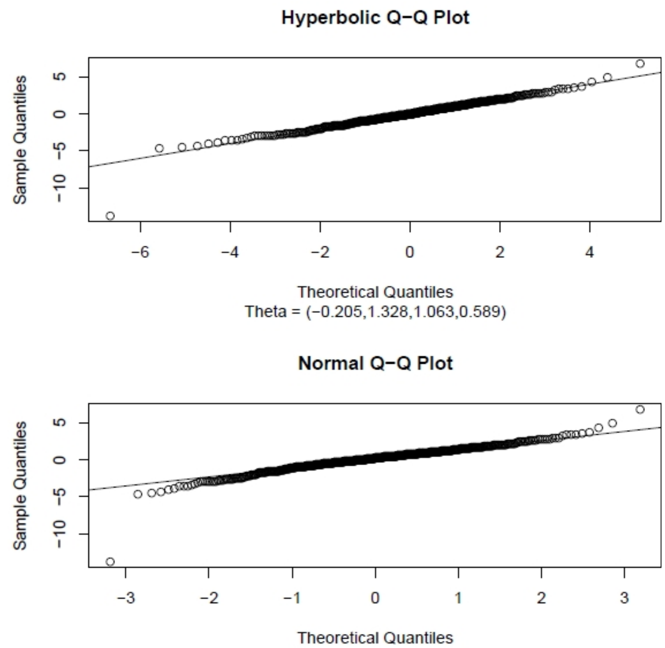
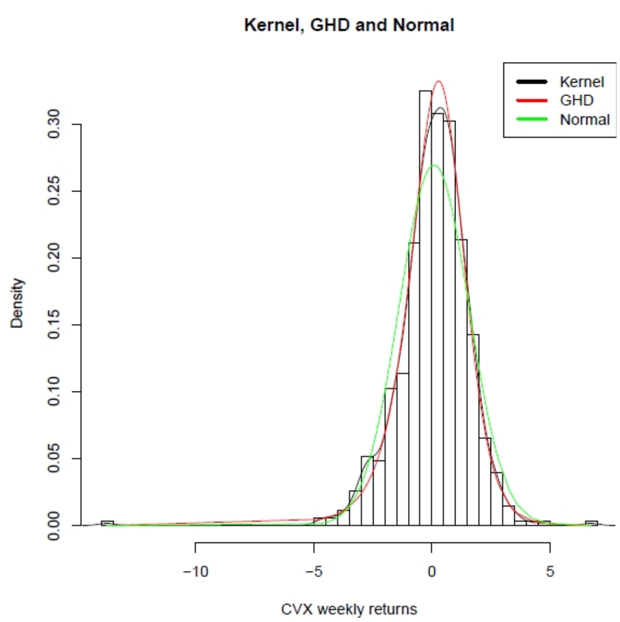


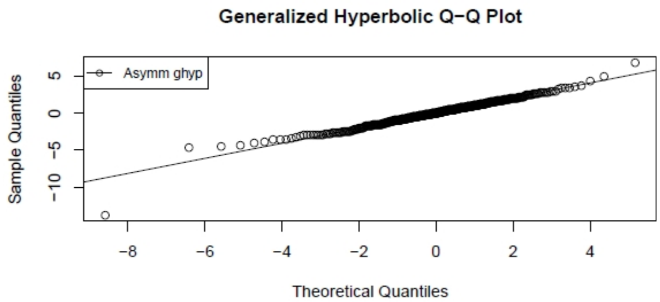
Figure 7.13: Fitting the hyperbolic distribution to CVX weekly returns

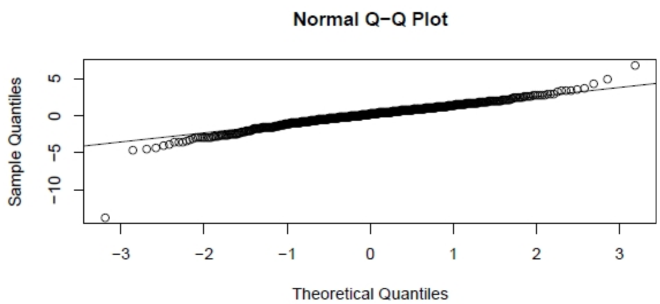
Fitting GHD to CVX weekly returns

Parameter	*value*
$\bar{\alpha}$	0.02941685
γ	-0.45361116
σ	1.39032668
μ	0.54272801
λ	-3.03935634


Hyperbolic Q−Q Plot
Sample Quantiles
Theoretical Quantiles
Theta = (−0.205,1.328,1.063,0.589)
Normal Q−Q Plot
Sample Quantiles
Theoretical Quantiles


Kernel, GHD and Normal
Kernel
GHD
Normal
Density
CVX weekly returns





Fitting Variance Gamma to CVX weekly returns

Parameter	*value*
vgC	0.5986
$sigma$	1.3863
$theta$	-0.5114
nu	0.4922

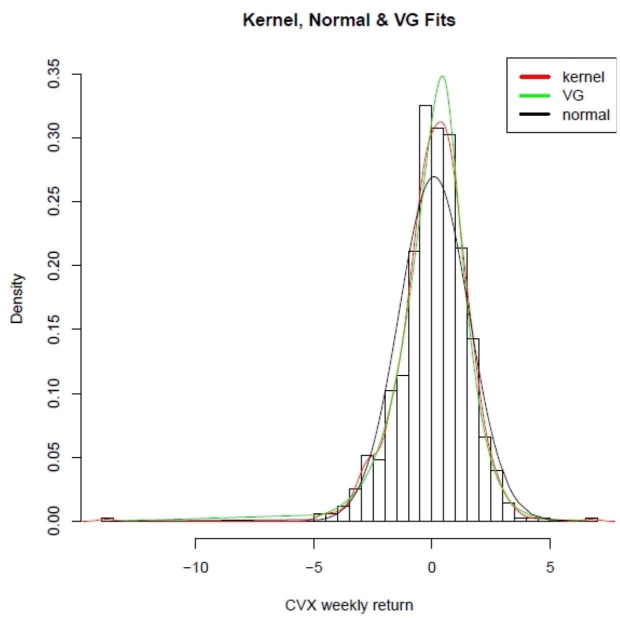
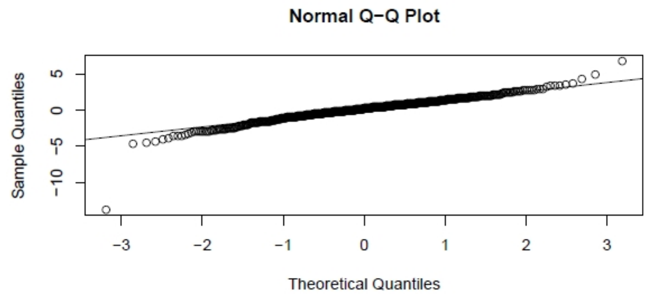


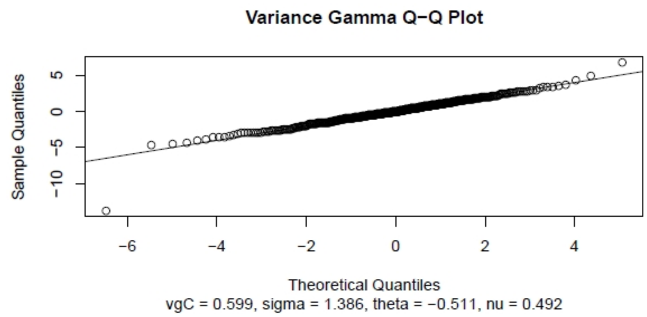
Figure 7.14: Fitting Variance Gamma to CVX weekly returns

The AIC selection creterion for CVX

	$Model$	AIC
1	t	2448.581
2	GHD	2450.583
2	NIG	2452.089
3	HYP	2454.513
4	VG	2456.783
6	$Normal$	2546.031

Therefore, based on the AIC creterion, the best model is skew Hyperbolic t-distribution:





$$f_{GHD}\left(x;\lambda,\left|\beta\right|,\beta,\delta,\mu\right)=\frac{2^{\frac{-n+1}{2}}\delta^{n}}{\sqrt{2\pi}\Gamma\left(n/2\right)}\left(\frac{\sqrt{\delta^{2}+\left(x-\mu\right)^{2}}}{\left|\beta\right|}\right)^{-\frac{n+1}{2}}K_{-\frac{n+1}{2}}\left(\left|\beta\right|\sqrt{\left(x-\mu\right)^{2}+\delta^{2}}\right)e^{\beta(x-\mu}$$

with parameters

$Parameter$	$value$
$\bar{\alpha}$	0.0000
σ	1.389178
μ	0.54063256
nu	−3.039712

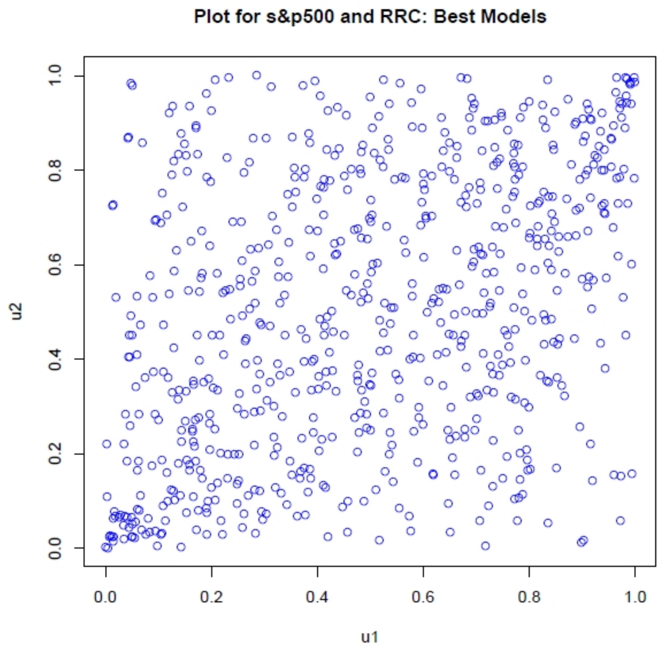
7.5 Bivariate cumulative return from Copulas

In this section, we Fit bivariate copulas to the cumulative returns. We first fit the selected copulas to the "best model" identified above and then proceed to fit assuming the univariate distribution are normally distributed.

7.6 RRC and s&p500 Index: Best Model

7.6.1 Estimation of parameters

The copula parameter estimation for RRC and $s\&p500$ $index$, for "best model" is as follows:



Gaussian Copula

$$C(u_1, u_2; \theta) = \Phi_G\left(\Phi^{-1}(u_1), \Phi^{-1}(u_2)\right) = \int_{-\infty}^{\Phi^{-1}(u_1)} \int_{-\infty}^{\Phi^{-1}(u_2)} \frac{1}{2\pi\left(1-\theta^2\right)^{1/2}} \left\{\frac{-\left(s^2 - 2\theta st + t^2\right)}{2\left(1-\theta^2\right)}\right\} dsdt$$

using the mle,

$$\hat{\theta} = 0.4221921$$

t-copula

$$C(u_1, u_2; \theta_1, \theta_2) = \int_{-\infty}^{t_{\theta_1}^{-1}(u_1)} \int_{-\infty}^{t_{\theta_2}^{-1}(u_2)} \frac{1}{2\pi\left(1-\theta_2^2\right)^{1/2}} \left\{1 + \frac{\left(s^2 - 2\theta_2 st + t^2\right)}{\nu\left(1-\theta_2^2\right)}\right\}^{-(\theta_1+2)/2} dsdt$$

using mle,

$$\hat{\theta}_1 = 0.4219336, \ \hat{\theta}_2 = 4.250148$$

Clayton Copula

$$C(u_1, u_2; \theta) = \left(u_1^{-\theta} + u_2^{-\theta} - 1\right)^{-1/\theta}$$

Using mle,

$$\hat{\theta} = 0.6383949$$

Gumbel Copula

$$C(u_1, u_2; \theta) = \exp\left\{-\left[(-\ln u_1)^{\theta} + (-\ln u_2)^{\theta}\right]^{1/\theta}\right\}$$

The maximum likelihood parameter estimate

$$\hat{\theta} = 1.354149$$

Frank Copula

$$C(u_1, u_2; \theta) = -\frac{1}{\theta} \ln\left(1 + \frac{\left(e^{-\theta u_1} - 1\right)\left(e^{-\theta u_2} - 1\right)}{e^{-\theta} - 1}\right)$$

The maximum likelihood parameter estimate

$$\hat{\theta} = 2.690227$$

Joe Copula

The maximum likelihood parameter estimate

$$\hat{\theta} = 1.419566$$

7.6.2 Goodness of Fit test

Gaussian Copula

	statistic	*p.value*
CvM	0.1907972	0.07
KS	1.084012	0.07

Clayton Copula

	statistic	*p.value*
CvM	0.1143522	0.15
KS	0.7912131	0.23

Gumbel Copula

	statistic	*p.value*
CvM	0.4703054	0
KS	1.433023	0

Frank Copula

	statistic	*p.value*
CvM	0.3072159	0
KS	1.335361	0

Joe Copula

	statistic	*p.value*
CvM	5.328536	1
KS	3.312072	1

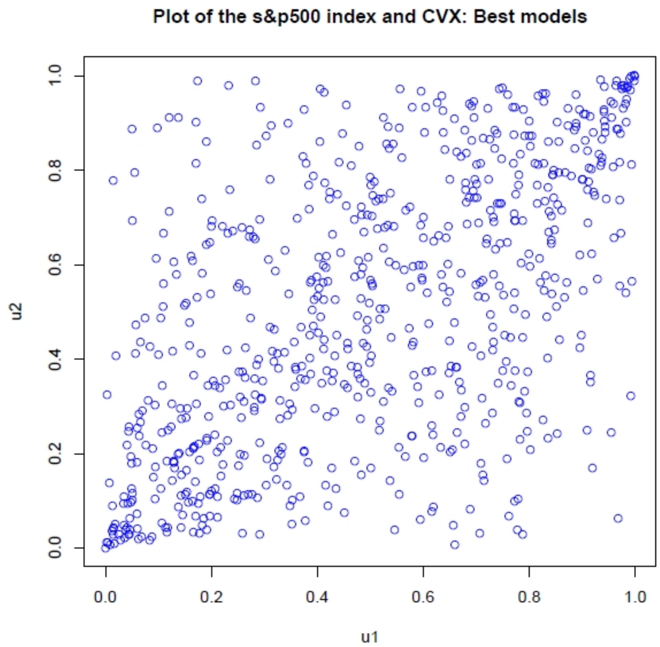
7.6.3 Selection creterion using AIC

The best copula obtained is the t-copula with parameters:

$$\hat{\theta}_1 = 0.4219336, \ \hat{\theta}_2 = 4.250148$$

7.7 CVX and s&p500 Index: Best Model

7.7.1 Estimation of parameters



The copula parameter estimation for CVX and $s\&p500$ $index$, for "best model" is as follows:

Gaussian Copula

$$\begin{aligned} C\left(u_1, u_2; \theta\right) &= \Phi_G\left(\Phi^{-1}\left(u_1\right), \Phi^{-1}\left(u_2\right)\right) \\ &= \int_{-\infty}^{\Phi^{-1}(u_1)} \int_{-\infty}^{\Phi^{-1}(u_2)} \frac{1}{2\pi\left(1-\theta^2\right)^{1/2}} \left\{\frac{-\left(s^2 - 2\theta st + t^2\right)}{2\left(1-\theta^2\right)}\right\} dsdt \end{aligned}$$

using the mle,

$$\hat{\theta} = 0.6014363$$

t-copula

$$C(u_1, u_2; \theta_1, \theta_2) = \int_{-\infty}^{t_{\theta_1}^{-1}(u_1)} \int_{-\infty}^{t_{\theta_2}^{-1}(u_2)} \frac{1}{2\pi\left(1-\theta_2^2\right)^{1/2}} \left\{1 + \frac{(s^2 - 2\theta_2 st + t^2)}{\nu\left(1-\theta_2^2\right)}\right\}^{-(\theta_1+2)/2} dsdt$$

using mle,

$$\hat{\theta}_1 = 0.5980452, \ \hat{\theta}_2 = 3.601428$$

Clayton Copula

$$C(u_1, u_2; \theta) = \left(u_1^{-\theta} + u_2^{-\theta} - 1\right)^{-1/\theta}$$

Using mle,

$$\hat{\theta} = 1.05697$$

Gumbel Copula

$$C(u_1, u_2; \theta) = \exp\left\{-\left[(-\ln u_1)^{\theta} + (-\ln u_2)^{\theta}\right]^{1/\theta}\right\}$$

The maximum likelihood parameter estimate

$$\hat{\theta} = 1.662317$$

Frank Copula

$$C(u_1, u_2; \theta) = -\frac{1}{\theta} \ln\left(1 + \frac{\left(e^{-\theta u_1} - 1\right)\left(e^{-\theta u_2} - 1\right)}{e^{-\theta} - 1}\right)$$

The maximum likelihood parameter estimate

$$\hat{\theta} = 4.305357$$

Joe Copula

The maximum likelihood parameter estimate

$$\hat{\theta} = 1.844623$$

7.7.2 Goodness of Fit test

Gaussian Copula

	statistic	*p.value*
CvM	0.08101083	0.37
KS	0.7321681	0.41

Clayton Copula

	statistic	*p.value*
CvM	0.3639415	0.01
KS	1.263143	0.01

Gumbel Copula

	statistic	*p.value*
CvM	0.3601119	0
KS	1.220172	0

Frank Copula

	statistic	*p.value*
CvM	0.2717979	0
KS	1.232988	0

Joe Copula

	statistic	*p.value*
CvM	8.074421	1
KS	4.065325	1

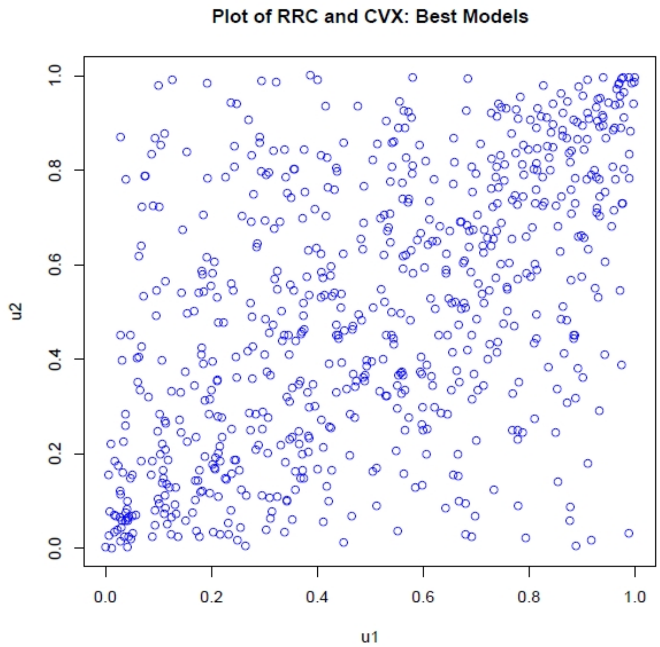
7.7.3 Selection creterion using AIC

The best copula obtained is the t-copula with parameters:

$$\hat{\theta}_1 = 0.5980452,\ \hat{\theta}_2 = 3.601428$$

7.8 CVX and RRC: Best Model

$$u_1 \Rightarrow CVX \ \&\ u_2 \Rightarrow RRC$$



7.8.1 Estimation of parameters

The copula parameter estimation for RRC and CVX, for "best model" is as follows:

Gaussian Copula

$$
\begin{aligned}
C(u_1, u_2; \theta) &= \Phi_G\left(\Phi^{-1}(u_1), \Phi^{-1}(u_2)\right) \\
&= \int_{-\infty}^{\Phi^{-1}(u_1)} \int_{-\infty}^{\Phi^{-1}(u_2)} \frac{1}{2\pi\left(1-\theta^2\right)^{1/2}} \left\{ \frac{-\left(s^2 - 2\theta st + t^2\right)}{2\left(1-\theta^2\right)} \right\} dsdt
\end{aligned}
$$

using the mle,

$$\hat{\theta} = 0.5339866$$

t-copula

$$C(u_1, u_2; \theta_1, \theta_2) = \int_{-\infty}^{t_{\theta_1}^{-1}(u_1)} \int_{-\infty}^{t_{\theta_2}^{-1}(u_2)} \frac{1}{2\pi\left(1-\theta_2^2\right)^{1/2}} \left\{1 + \frac{(s^2 - 2\theta_2 st + t^2)}{\nu\left(1-\theta_2^2\right)}\right\}^{-(\theta_1+2)/2} dsdt$$

using mle,

$$\hat{\theta}_1 = 0.5520239,\ \hat{\theta}_2 = 4.54858$$

Clayton Copula

$$C(u_1, u_2; \theta) = \left(u_1^{-\theta} + u_2^{-\theta} - 1\right)^{-1/\theta}$$

Using mle,

$$\hat{\theta} = 0.8237573$$

Gumbel Copula

$$C(u_1, u_2; \theta) = \exp\left\{-\left[(-\ln u_1)^{\theta} + (-\ln u_2)^{\theta}\right]^{1/\theta}\right\}$$

The maximum likelihood parameter estimate

$$\hat{\theta} = 1.549454$$

Frank Copula

$$C(u_1, u_2; \theta) = -\frac{1}{\theta}\ln\left(1 + \frac{\left(e^{-\theta u_1} - 1\right)\left(e^{-\theta u_2} - 1\right)}{e^{-\theta} - 1}\right)$$

The maximum likelihood parameter estimate

$$\hat{\theta} = 3.896887$$

Joe Copula

The maximum likelihood parameter estimate

$$\hat{\theta} = 1.698482$$

7.8.2 Goodness of Fit test

Gaussian Copula

	statistic	*p.value*
CvM	0.06299151	0.56
KS	0.6718177	0.60

Clayton Copula

	statistic	*p.value*
CvM	0.4788138	0
KS	1.492676	0

Gumbel Copula

	statistic	*p.value*
CvM	0.3159871	0
KS	1.271703	0

Frank Copula

	statistic	*p.value*
CvM	0.2159276	0
KS	1.127751	0

Joe Copula

	statistic	p.value
CvM	7.44012	1
KS	4.294077	1

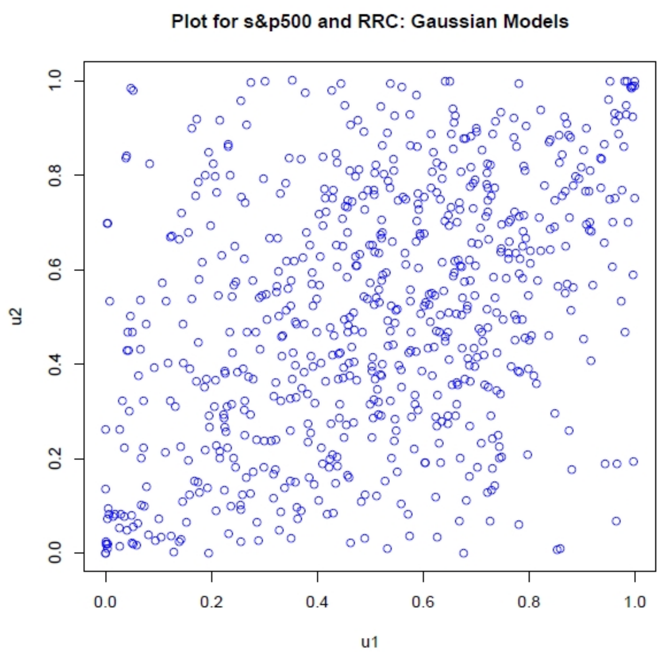
7.8.3 Selection creterion using AIC

The best copula obtained is the t-copula with parameters:

$$\hat{\theta}_1 = 0.5520239,\ \hat{\theta}_2 = 4.54858$$

7.9 s&p500 Index and RRC: Gaussian Models

$$u_1 \Rightarrow s\&p500\ index\ \&\ u_2 \Rightarrow RRC$$



7.9.1 Estimation of parameters

Gaussian Copula

$$C\left(u_1,u_2;\theta\right) = \Phi_G\left(\Phi^{-1}\left(u_1\right),\Phi^{-1}\left(u_2\right)\right) = \int_{-\infty}^{\Phi^{-1}(u_1)}\int_{-\infty}^{\Phi^{-1}(u_2)} \frac{1}{2\pi\left(1-\theta^2\right)^{1/2}}\left\{\frac{-\left(s^2-2\theta st+t^2\right)}{2\left(1-\theta^2\right)}\right\}dsdt$$

using the mle,

$$\hat{\theta} = 0.4478807$$

t-copula

$$C\left(u_1,u_2;\theta_1,\theta_2\right) = \int_{-\infty}^{t_{\theta_1}^{-1}(u_1)}\int_{-\infty}^{t_{\theta_2}^{-1}(u_2)} \frac{1}{2\pi\left(1-\theta_2^2\right)^{1/2}}\left\{1+\frac{\left(s^2-2\theta_2 st+t^2\right)}{\nu\left(1-\theta_2^2\right)}\right\}^{-(\theta_1+2)/2} dsdt$$

using mle,

$$\hat{\theta}_1 = 0.46491,\ \hat{\theta}_2 = 9.078517$$

Clayton Copula

$$C\left(u_1, u_2; \theta\right) = \left(u_1^{-\theta} + u_2^{-\theta} - 1\right)^{-1/\theta}$$

Using mle,

$$\hat{\theta} = 0.4188054$$

Gumbel Copula

$$C\left(u_1, u_2; \theta\right) = \exp\left\{-\left[\left(-\ln u_1\right)^{\theta} + \left(-\ln u_2\right)^{\theta}\right]^{1/\theta}\right\}$$

The maximum likelihood parameter estimate

$$\hat{\theta} = 1.549454$$

Frank Copula

$$C\left(u_1, u_2; \theta\right) = -\frac{1}{\theta}\ln\left(1 + \frac{\left(e^{-\theta u_1} - 1\right)\left(e^{-\theta u_2} - 1\right)}{e^{-\theta} - 1}\right)$$

The maximum likelihood parameter estimate

$$\hat{\theta} = 1.405334$$

Joe Copula

The maximum likelihood parameter estimate

$$\hat{\theta} = 3.472387$$

7.9.2 Goodness of Fit test

Gaussian Copula

	statistic	*p.value*
CvM	0.09335985	0.37
KS	0.8667505	0.22

Clayton Copula

	statistic	*p.value*
CvM	0.5142769	0
KS	1.270624	0.1

Gumbel Copula

	statistic	*p.value*
CvM	0.3809738	0
KS	1.324468	0

Frank Copula

	statistic	*p.value*
CvM	0.6684779	0
KS	1.527376	0

Joe Copula

	statistic	*p.value*
CvM	4.997613	1
KS	3.220799	1

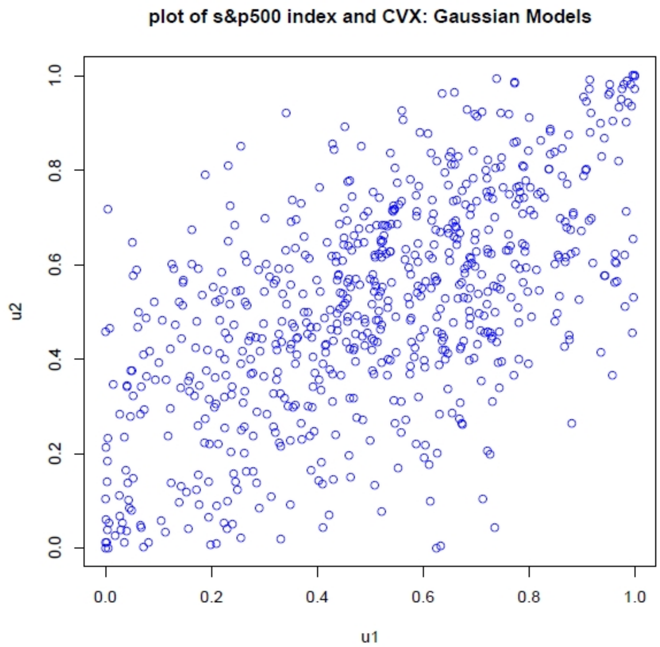
7.9.3 Selection creterion using AIC

The best copula obtained is the t-copula with parameters:

$$\hat{\theta}_1 = 0.46491, \ \hat{\theta}_2 = 9.078517$$

7.10 s&p500 Index and CVX: Gaussian Models

$$u_1 \Rightarrow s\&p500 \ index \ \& \ u_2 \Rightarrow CVX$$



7.10.1 Estimation of parameters

Gaussian Copula

$$\begin{aligned} C(u_1,u_2;\theta) &= \Phi_G\left(\Phi^{-1}(u_1),\Phi^{-1}(u_2)\right) \\ &= \int_{-\infty}^{\Phi^{-1}(u_1)}\int_{-\infty}^{\Phi^{-1}(u_2)} \frac{1}{2\pi\left(1-\theta^2\right)^{1/2}}\left\{\frac{-\left(s^2-2\theta st+t^2\right)}{2\left(1-\theta^2\right)}\right\}dsdt \end{aligned}$$

using the mle,

$$\hat{\theta} = 0.649695$$

t-copula

$$C(u_1,u_2;\theta_1,\theta_2) = \int_{-\infty}^{t_{\theta_1}^{-1}(u_1)}\int_{-\infty}^{t_{\theta_2}^{-1}(u_2)} \frac{1}{2\pi\left(1-\theta_2^2\right)^{1/2}}\left\{1+\frac{\left(s^2-2\theta_2 st+t^2\right)}{\nu\left(1-\theta_2^2\right)}\right\}^{-(\theta_1+2)/2} dsdt$$

using mle,

$$\hat{\theta}_1 = 0.6849407, \ \hat{\theta}_2 = 9.581754$$

Clayton Copula

$$C(u_1, u_2; \theta) = \left(u_1^{-\theta} + u_2^{-\theta} - 1\right)^{-1/\theta}$$

Using mle,

$$\hat{\theta} = 0.7207026$$

Gumbel Copula

$$C(u_1, u_2; \theta) = \exp\left\{-\left[(-\ln u_1)^{\theta} + (-\ln u_2)^{\theta}\right]^{1/\theta}\right\}$$

The maximum likelihood parameter estimate

$$\hat{\theta} = 1.905421$$

Frank Copula

$$C(u_1, u_2; \theta) = -\frac{1}{\theta}\ln\left(1 + \frac{\left(e^{-\theta u_1} - 1\right)\left(e^{-\theta u_2} - 1\right)}{e^{-\theta} - 1}\right)$$

The maximum likelihood parameter estimate

$$\hat{\theta} = 6.13902$$

Joe Copula

The maximum likelihood parameter estimate

$$\hat{\theta} = 2.188791$$

7.10.2 Goodness of Fit test

Gaussian Copula

	statistic	*p.value*
CvM	0.09409167	0.35
KS	0.6901983	0.46

Clayton Copula

	statistic	*p.value*
CvM	1.217899	0
KS	1.831002	0

Gumbel Copula

	statistic	*p.value*
CvM	0.2896533	0
KS	1.246939	0

Frank Copula

	statistic	*p.value*
CvM	0.9431915	0
KS	1.774227	0

Joe Copula

	statistic	*p.value*
CvM	10.92587	1
KS	4.558345	1

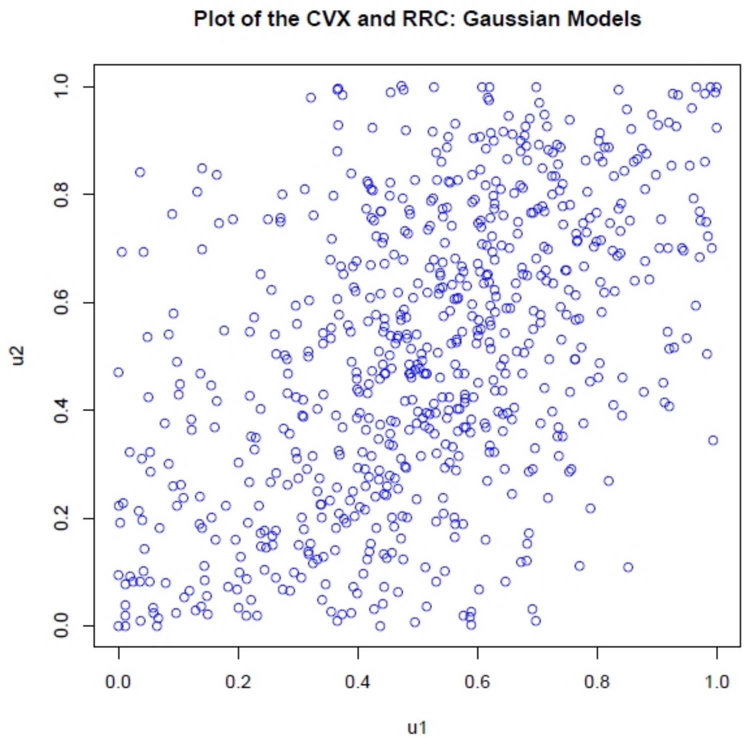
7.10.3 Selection creterion using AIC

The best copula obtained is the t-copula with parameters:

$$\hat{\theta}_1 = 0.6849407,\ \hat{\theta}_2 = 9.581754$$

7.11 RRC and CVX: Gaussian Models

$$u_1 \Rightarrow CVX\ index\ \&\ u_2 \Rightarrow RRC$$



7.11.1 Estimation of parameters

Gaussian Copula

$$\begin{aligned} C(u_1, u_2; \theta) &= \Phi_G\left(\Phi^{-1}(u_1), \Phi^{-1}(u_2)\right) \\ &= \int_{-\infty}^{\Phi^{-1}(u_1)} \int_{-\infty}^{\Phi^{-1}(u_2)} \frac{1}{2\pi\left(1-\theta^2\right)^{1/2}} \left\{ \frac{-\left(s^2 - 2\theta st + t^2\right)}{2\left(1-\theta^2\right)} \right\} dsdt \end{aligned}$$

using the mle,

$$\hat{\theta} = 0.5124679$$

t-copula

$$C\left(u_1,u_2;\theta_1,\theta_2\right)=\int\limits_{-\infty}^{t_{\theta_1}^{-1}(u_1)}\int\limits_{-\infty}^{t_{\theta_2}^{-1}(u_2)}\frac{1}{2\pi\left(1-\theta_2^2\right)^{1/2}}\left\{1+\frac{(s^2-2\theta_2 st+t^2)}{\nu\left(1-\theta_2^2\right)}\right\}^{-(\theta_1+2)/2}dsdt$$

using mle,

$$\hat{\theta}_1=0.5422314,\ \hat{\theta}_2=16.85843$$

Clayton Copula

$$C\left(u_1,u_2;\theta\right)=\left(u_1^{-\theta}+u_2^{-\theta}-1\right)^{-1/\theta}$$

Using mle,

$$\hat{\theta}=0.4298016$$

Gumbel Copula

$$C\left(u_1,u_2;\theta\right)=\exp\left\{-\left[\left(-\ln u_1\right)^{\theta}+\left(-\ln u_2\right)^{\theta}\right]^{1/\theta}\right\}$$

The maximum likelihood parameter estimate

$$\hat{\theta}=1.539072$$

Frank Copula

$$C\left(u_1,u_2;\theta\right)=-\frac{1}{\theta}\ln\left(1+\frac{\left(e^{-\theta u_1}-1\right)\left(e^{-\theta u_2}-1\right)}{e^{-\theta}-1}\right)$$

The maximum likelihood parameter estimate

$$\hat{\theta}=4.641425$$

Joe Copula

The maximum likelihood parameter estimate

$$\hat{\theta} = 1.606212$$

7.11.2 Goodness of Fit test

Gaussian Copula

	statistic	*p.value*
CvM	0.1598154	0.07
KS	1.043944	0.06

Clayton Copula

	statistic	*p.value*
CvM	1.918336	0
KS	2.170538	0

Gumbel Copula

	statistic	*p.value*
CvM	0.3416239	0
KS	1.323097	0

Frank Copula

	statistic	*p.value*
CvM	0.5180333	0
KS	1.536596	0

Joe Copula

	statistic	*p.value*
CvM	6.224195	1
KS	3.733411	1

7.11.3 Selection creterion using AIC

The best copula obtained is theFrank copula with parameters:

$$\hat{\theta} = 4.641425$$

Chapter 8

Conclusion and Reccommendation

8.1 Conclusion

Normal Variance-mean mixture are sufficient in modelling financial data. In particular, dependence modelling of financial data using Copulas, require a precise choice to be made of the marginal distribution. The choice of the marginal distribution determines the parameter(s) value as was the case when modelling the dependence structure of RRC and s&p500 index weekly returns. Similarly, the parameter(s) estimate was different when modelling the dependence structure of CVX and s&p500 index weekly returns using as one case, the "best" models and the other case, gaussian models. Its interesting to note that modelling the dependence structure of RRC and CVX weekly returns using as one case, the "best" models and the other case, gaussian models lead to a deferent choice of copula. In the "best" models case, a choice of GHDs to model the weekly asset return, leads to the t-copula while using the gaussian models lead to Frank copula. This then, will influence the financial decision to be made. For instance, since the two choices gives different values of financial risks calculated, Frank copula model will either underestimate the financial risk.

8.2 RECCOMMENDATION

We recommend the following:

- GHDs Models for Financial data
- Using Mixture copulas which adds parameters to enhance modelling

Illustration

$$\begin{aligned} C\left(u_1,u_2;x\right) &= \int_0^1 \frac{u_1u_2}{1-\left(2x-1\right)\left(1-u_1\right)\left(1-u_2\right)}\times \\ & \frac{x^{\alpha-1}\left(1-x\right)^{\beta-1}}{B\left(\alpha,\beta\right)}dx \\ &= \frac{u_1u_2}{1+\left(1-u_1\right)\left(1-u_2\right)}\left({}_1F_1\left(\alpha,\alpha+\beta;s\right)\right) \end{aligned}$$

where

$$s=\frac{2\left(1-u_1\right)\left(1-u_2\right)}{1+\left(1-u_1\right)\left(1-u_2\right)}$$

- To Model the β parameter using mixture mechanism to capture the skewness better.
- Normal Mixture Model where the mixing distribution is discrete to be compared with the continuous case.
- A comprehensive study of other the other parameterisation of the GHD.

REFERENCES

[1] M. Abramowitz and I. A. Stegun. Handbook of Mathematical Function. Dover, New York,1972.

[2] O. Barndorff-Nielsen. Exponentially decreasing distributions for the logarithm of particle size. Proc. R. Soc. Lond. A, 353:409–419, 1977.

[3] O. E. Barndorff-Nielsen. Normal inverse Gaussian distributions and stochastic volatility modelling. Scandinavian Journal of Statistics, 24:1–13, 1997.

[4] O. E. Barndorff-Nielsen and P. Blæsild. Hyperbolic distributions and ramifications: Contributions to theory and application. Statistical Distributions in Scientific Work, 4:19–44, 1981.

[5] O. E. Barndorff-Nielsen and N. Shepard. Normal modified stable processes. Theory of probability and Mathematical Statistics, 65:1–19, 2001.

[6] O. E. Barndorff-Nielsen and R. Stelzer. Absolute moments of generalized hyperbolic distribution sand approximate scaling of normal inverse gaussian l´evy processes. Submitted for publication, 2004.

[7] P. Blaesild. The two-dimensional hyperbolic distribution and related distributions, with applications to Johannsen's bean data. Biometrika, 68:251–263, 1981.

[8] E. Bolviken and F. E. Benth. Quantification of risk in Norwegian stocks via the normal inverse Gaussian distribution. In Proceedings of the AFIR 2000 Colloquium, Tromso, Norway, pages 87–98, 2000.

[9] A. P. Dempster, N. M. Laird, and D. Rubin. Maximum likelihood from incomplete data using the EM algorithm. Journal Roy. Statist. Soc. B, 39:1–38, 1977.

[10] E. Eberlein and U. Keller. Hyberbolic distributions in finance. Bernoulli,, 1(3):281–299, 1995.

[11] D. Karlis. An EM type algorithm for maximum likelihood estimation of the normal-inverse Gaussian distribution. Statistics & Probability Letters, 57:43–52, 2002.

[12] K. Prause. The generalized hyperbolic models: Estimation, financial derivatives and risk measurement, 1999. PhD Thesis, Mathematics Faculty, University of Freiburg.

[13] R Development Core Team. R: A language and environment for statistical computing. R Foundation for Statistical Computing, Vienna, Austria, 2004. URL http://www.R-project.org. ISBN 3-900051-00-3.

[14] T. H. Rydberg. The normal inverse Gaussian Levy process: Simulation and approximation. Commun. Statist.-Stochastic Models, 34:887–910, 1997.

[15] J. H. Venter and P. J. de Jongh. Risk estimation using the normal inverse Gaussian distribution. Journal of Risk, 4(2):1–24, 2002.

Chapter 9

Appendix

9.1 EM-algorithm R codes

9.2 Tables