The Title

The Author

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Dedication

I take this opportunity to dedicate this project to my family, starting with my wife Emma and son Darwin for their support and encouragement. To my Mum Judith for the continued inspiration for academic excellence. God bless you all.

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Finally I thank God for the far that he has taken me academically.

Notations

H: Hilbert space over the complex scalars $\mathbb{C}$

$B(H)$: Banach algebra of bounded linear operators on H

$<, >$: Inner products

$\langle x, y\rangle$: The inner product of x and y on the Hilbert space H

$\mid T \mid$: The square root of an operator T

$\| \,.\, \|$: The norm

$\| x \|$: The norm of vector x

$\|T\|$: The operators norm of T

$x \oplus y$: The direct sum of x and y

$M \oplus N$: The direct sum of subspaces M and N

$R(T)$: The range of an operator T

$\sigma(T)$: The spectrum of an operator T

$r(T)$: The spectral radius of an operator T

$W(T)$: The numerical range of an operator T

$w(T)$: The numerical radius of an operator T

$\rho(T)$: The resolvent of an operator T

$P\sigma(T)$: The point spectrum of an operator T

$C\sigma(T)$: The continuous spectrum of T

$\pi(T)$: The approximate spectrum of T

$R\sigma(T)$: The residual spectrum of T

$\mu(T)$: Multiplicity of T

$T \to 0$: Operator T converges to strongly 0

$T \nrightarrow 0$: Operator T does not converge to 0

$w.r.t.$: with respect to

Abstract

This study is on the characterization of the numerical ranges of a bounded linear operator T on a Hilbert space H.

In case of a bounded linear operator, the closure of the numerical range apart from including the spectrum of the operator turns out to be a convex subset of the complex plane. We provide essential expository material on numerical ranges and then proceed towards investigation of some significant aspects described below which are the highlights of the study.

First, we give a set of sufficient conditions for the numerical range of an operator to be closed.

For bounded linear operator T which is hyponormal we show that $Conv$ $\sigma(T) \subseteq \overline{W(T)}$ and $\pi(T) \subset \overline{W(T)}$.

Secondly, we provide a proof to show that the numerical range of an operator on a 2-dimensional complex Hilbert space is, in general, an ellipse.

Thirdly, We also consider some special points on the boundary of the numerical range and in this connection shows that: If the numerical range is closed and $\lambda \in W(T)$ such that the boundary of the numerical range is not a differentiable arc at λ, then λ belongs to the point spectrum; if λ is just a corner of the numerical range, then λ belongs to the spectrum. If $\lambda \in W(T)$ is such that $\lambda \in \delta\ D$ for a circular disc D of $\mathbb{C}$ and $Dn\ W(T) = \{\ \lambda\ \}$, then $\lambda \in P_\sigma(T)$

In our study and pursuit of these investigations, we expose many minor results and observations which are of interest in themselves.

Chapter 1

Introduction

With mathematics analysts' interest shifting from finite dimensional inner product spaces to infinite -dimensional Hilbert spaces and with consequent shift of interest from matrices to linear operators, their focus of attention changed from quadratic forms to numerical ranges of linear operators. In the case of a bounded linear operator, the closure of the numerical range apart from including the spectrum of the operator turns out to be a convex subset of the complex plane. It is this aspect that makes the study of the numerical range more appealing and worthy of the increasing attention currently directed towards it.

We shall in this chapter introduce some essential results in the setting of normal linear or inner product spaces, we are interested in complex Banach spaces and complex Hilbert spaces and we shall assume all fundamental topological notions particularly with regard to the topology generated by the norm of a normed linear space or by the norm obtained from the inner product function in an inner product space. As is well known, a strongly complete normed linear space (inner product space) is called a **Banach** (respectively, **Hilbert pace**).

We discuss some essential results on the numerical ranges and numerical radii of bounded operators in chapter 2 proving a simple proof to show that the numerical range of an operator $T \in B(H)$ is a convex subset of $\mathbb{C}$, this result was first proved by Toeplitz and Hausdorff (Hausdorff).

A normal operator $T \in B(H)$ has been an object of much study in operator theory and possesses many nice properties a few of which being that $\text{conv}\sigma(T) = \overline{W(T)}$, $r(T) = \|T\| = w(T)$. Analysts' have therefore tried to classify operators by having classes of operators satisfying some of these specific properties and such attempts have led to classes of non-normal operators imitating normal operators as regards some specific properties only. For example, we have quasinormal operators, subnormal operators, hyponormal operators and so on. In fact, we have a hierarchical relation among some of the classes so obtained.[The works of Halmos (1967), Fillmore (1970) are particular helpful in this connection]. Towards the end of the chapter we discuss a sufficient condition under which the numerical range W(T) is closed, De Barra et all. (1972)

In chapter 3, we investigate some special points on the boundary of $W(T)$. We begin by discussing the numerical range of an operator T on a 2-dimensional complex Hilbert space and the conclusions obtained are essential in that they not only provide an alternative proof for the convexity of the numerical range of a bounded T on an infinite-dimensional Hilbert space but also assist us to show some "special" points on the boundary of the numerical range turn out to be eigenvalues.

We also discuss the subnormal and hyponormal operators to the extent as is necessary for us to establish that

$\text{conv}\sigma(T) = \overline{W(T)}$, and $r(T) = \|T\|$.

1.1 Objectives of the study

The main objectives is to study the characterization of the numerical range of bounded linear operators.

specific objectives

1. To show that the closure of closure of the numerical range a part from including the spectrum of the operator turns out to be a complex subset of the complex plane.
2. To give a necessary and sufficient condition for the numerical range to be closed.
3. To show that the numerical range of an operator on a 2-dimensional complex plane is, in general an ellipse.
4. To show that some "special" points on the boundary of the numerical range turn out to be eigenvalues.

1.2 Convexity and Convex functionals

Definition 1.2.1 *A subset K of X is said to be convex if whenever $x, y \in K$, it follows that the segment connecting the vectors $x, y \in X$ given by*

$$z = (1 - \alpha)\, x + \alpha y$$

Where $0 \leq \alpha \leq 1$, also belongs to K.

Theorem 1.2.2 *Let X be a normed linear space and let K be a convex subset of X , then $\overline{K}$ is convex*

Proof. Let x, $y \in \overline{K}$ and let ϵ be an arbitrary positive real number. There exists elements $x_1, y_1 \in K$ such that

$$\|x - x_1\| < \epsilon$$

and

$$\|y - y_1\| < \epsilon$$

Let $\alpha, \beta \geq 0$ and $\alpha + \beta = 1$. Then

$$\|\alpha x + \beta y - (\alpha x_1 + \beta y_1)\| \leq \alpha \|x - x_1\| + \beta \|y - y_1\|$$

$$\leq (\alpha + \beta)\, \epsilon = \in$$

But

$$\alpha x_1 + \beta y_1 \in K$$

Since K is convex. Thus, since $\epsilon > 0$ is arbitrary, it follows that

$$\alpha x_1 + \beta y_1 \in \overline{K},$$

So $\overline{K}$ is convex also. ■

Theorem 1.2.3 *The intersection of any number of convex subsets of the vector space X is a convex subset.*

Definition 1.2.4 *Let S be be a subset of the vector space X.* ***The convex hull*** *of S is the intersection of all convex sets containing S.*

It is clear that the definition makes sense since there is a convex set containing S that is X itself. For that matter, it is clear that every subspace of X is a convex set. An alternate characterization of convex hull is given in the next theorem.

Theorem 1.2.5 *The convex hull of the subset S of the vector space X consists of all vectors of the form*

$$\alpha_1 x_1 + \alpha_2 x_2 + ... + \alpha_n x_n$$

where the $x_i \in S$, $\alpha_i \geq 0$, for $\geq i = 1, \geq 2, ..., n$
and

$$\sum_{i=1}^{n} \alpha_i = 1$$

Proof. Let K be a set of all vectors of the form given in the statement of the theorem. K is clearly a convex set from its very definition. Moreover,

$$K \supset S$$

it follows that

$$K \supset S_c$$

the convex hull of S. However, it is clear that any convex set containing S must contain K, in particular,

$$S_c \supset K$$

therefore

$$S_c = K$$

and this completes the proof. ■

Definition 1.2.6 *Let X be a normed linear space and let S be any subset of X. The closure of the convex hull of S is called the* ***closed convex*** *hull of S.*

Theorem 1.2.7 *Let $E = \cap \{K | K \supset S,\ K \text{ convex and closed}\}$,* that is, E is the intersection of all closed and convex sets containing S. Let S_c equal the convex hull of S. Then

$$E = S_c$$

the closed convex hull of S.

Proof. Since S_c is convex, $\overline{S_c}$ is convex and is of course, closed and $\overline{S_c} \supset S$. Therefore

$$E \subset \overline{S_c}$$

however,

$$E \supset S_c$$

since S_c is the intersection of all convex sets containing S , where E is just the intersection of the closed and convex sets containing S. But

$$E \subset S_c$$

implies that

$$E = \overline{E} \supset \overline{S_c}$$

this, combined with the preceding inclusion, gives

$$E = \overline{S_c}$$

■

1.3 Important Concepts and Definitions

Definition 1.3.1 *A linear transformation T of a normed linear space X into a normed linear space Y is bounded below if there exists a constant $\alpha > 0$ such that*

$$\alpha \|x\| \leq \|Tx\|$$

for every $x \in X$.

Note that $T \in B(X, Y)$ actually means that T is bounded above.

Theorem 1.3.2 *Let X be a Banach space, Y a normed linear space and $T \in B(X, Y)$. If T is bounded below, then its range R_T is bounded in Y.*

Definition 1.3.3 *Let T be a subset of X and consider a function $F : T \to X$, an element x of T is a fixed point of F (or F leaves x fixed) if $F(x) = x$. The function $J : T \to X$ defined by $J(x) = x$ for every $x \in T$ is the* ***inclusion map*** *(or the* ***embedding****, or the injection) of T into X (i.e. leaves each point of T fixed)*

the inclusion map of X into X is called the **identity map** on X.

Definition 1.3.4 *A function $F : X \to Y$ from a set X into a set Y has an* ***inverse*** *on its range R_F if there exists a (unique) function $F^{-1} : R_F \to X$ (called the inverse of F on R_F) such that $F^{-1}F = Ix$ where Ix stands for the identity on X(identity map). Moreover, F has an inverse or its range if and only if its injective.*

If F is injective and surjective, then it is called **invertible** and $F^{-1} : Y \to X$ is the inverse of F furthermore $F : X \to Y$ is invertible if and only if there exists a unique function $F^{-1} : Y \to X$ (the inverse of it) such that $F^{-1}F = Ix$ and $F^{-1}F = Iy$, where Iy stands for the identity on y.

Theorem 1.3.5 *Suppose X and Y are normed linear spaces, and D be a non-zero linear space of X and $T : D \to Y$ a linear operator. Then T has a bounded inverse on its range R_T if and only if T is bounded below(Halmos).*

Theorem 1.3.6 *Let X be a Banach space and Y a normed linear space. Let $T \in B(X, Y)$ then T is invertible if and only if R_T is dense in Y and T is bounded.*

Definition 1.3.7 *Let T be a linear transformation mapping the complex Hilbert space X into itself, T is called* ***positive*** *if;*

$$\langle Tx, x\rangle \geq 0$$

for all $x \in X$.

Theorem 1.3.8 *If T is a bounded linear transformation mapping the complex Hilbert space X into the complex space Y , the adjoint operator of T, T^* always exists and is a bounded linear transformation defined everywhere on Y. Moreover,*

$$\|T\| = \|T^*\|$$

Proof. Let X, Y be complex Hilbert space, $T \in B(X, Y)$ letting y be an arbitrary fixed element of Y.

We define;

$$f_y : X \to C$$

$x \to \langle Tx, y \rangle$ f_y is clearly a linear functional and the fact that it is bounded follows from the cauchy-schwartz inequality. Applying the inequality also yields

$$\|f_y\| \leq \|T\| \|y\|$$

Using the Riesz representation theorem, we now assert the existence of a unique vector $z \in X$ such that,

$$f_y(x) = \langle x, z \rangle \quad \text{for all } x............*$$

$$\|f_y\| = \|z\|$$

rewriting equation $*$ as

$$\langle Tx, y \rangle = \langle x, z \rangle$$

we see that

$$y \in D_{T^*} \text{ and } T^* y = z$$

since y was any vector in Y, we see that $D_{T^*} = Y$, Moreover,

$$\|T^* y\| = \|z\| = \|f_y\| \leq \|T\| \|y\|$$

for any $y \in D_{T^*} = Y$, which implies T^* is a bounded linear transformation and that

$$\|T^*\| \leq \|T\|.$$

By exactly the same reasoning, it can also be shown that T^{**} is a bounded linear transformation and that

$$\|T^{**}\| \leq T^*$$

since $T \in B(x, y)$, T is closed, therefore $T = T^{**}$substituting T for T^{**} and comparing the resulting inequality to conclude that

$$\|T\| = \|T^*\|.$$

■

Definition 1.3.9 *An operator T in $B(H)$ is self-adjoint if and only if,*

$$\langle Tx, y \rangle = \langle x, Ty \rangle$$

for every $x, y \in H$

Definition 1.3.10 *Let T be a linear transformation in a linear space $X \neq \{\overline{0}\}$, a scalar $\lambda \in \Bbbk$ (where $\Bbbk$ is the field over which X is defined is called an* ***eigenvalue*** *of T if there exists a nonzero $x \in D_T$ (domain of T) such that*

$$Tx = \lambda x$$

the vector x is then called an **eigenvector** of T corresponding to the **eigenvalue** λ.

Definition 1.3.11 *The* ***spectrum*** *$\sigma(T)$ of an operator T is the set of all complex numbers λ such that $\lambda I - T$ has no inverse.*

Definition 1.3.12 *The compliment (in $\mathbb{C}$) of $\sigma(T)$, that is, the set of all complex numbers λ such that $\lambda I - T$ has an inverse in $B(H)$, is called the* ***resolvent*** *set of T denoted by $\rho(T)$.*

Definition 1.3.13 *A* ***point spectrum*** *$P_\sigma(T)$ is the set of all complex numbers for which $\lambda I - T$ does not have an inverse.*

Definition 1.3.14 *A* ***continuos spectrum*** *$C_\sigma(T)$ is the set of all complex numbers λ for which $\lambda I - T$ has an unbounded inverse with domain dense in X i.e.*

$$R_{\lambda I - T} = X.$$

Definition 1.3.15 ***Residual spectrum*** *$R_\sigma(T)$ is the set of all complex numbers λ for which $\lambda I - T$ has an inverse (bounded or unbounded) whose domain is not dense in X i.e.*

$$\overline{R}_{\lambda I - T} \neq X$$

Definition 1.3.16 *An **approximate spectrum*** $\pi(T)$ *is the set of all such* $\lambda \in \mathbb{C}$ *for which* $\lambda I - T$ *is not bounded from below is called the approximate point spectrum of* T*.*

Theorem 1.3.17 $r(T) = \lim\limits_{n\to\infty} \|T^n\|^{\frac{1}{n}}$

Theorem 1.3.18 $\pi(T) \subset \sigma(T)$

Theorem 1.3.19 $P_\sigma(T) \subseteq \pi(T)$

Proof. Let $\lambda \in P_\sigma(T)$, hence $\pi_T(\lambda) \neq \{\overline{0}\}$ and consequently there are non-zero $x \in D_T$ such that,

$$\|(\lambda I - T)x\| = 0$$

This shows that for every real $K > 0$, there are elements $x \in D_T$ for which

$$\|(\lambda I - T)x\| < K\|x\|$$

thus there is a number $K > 0$ such that

$$\|(\lambda I - T)y\| \geq K\|y\|$$

for all $y \in D_T$ i.e. $\lambda I - T$ is not bounded from below. Thus $\lambda \in \pi(T)$, and

$$P\sigma(T) \subseteq \pi(T). \quad \blacksquare$$

Theorem 1.3.20 *Let* T *be a closed linear operator in a Banach space* X*. Then,*

$$C\sigma(T) \subseteq \pi(T)$$

Proof. Let $\lambda \in C\sigma(T)$, then,

$$(\lambda I - T)^{-1} \text{ exists for } \overline{R}_{\lambda I - x} = X$$

and

$$(\lambda I - T)^{-1}$$

is unbounded and

$$T : D_T \to Y$$

(D_T a nonzero linear subspace of X).

is a linear operator,then T has a bounded inverse on R_T if and only if T is bounded$\lambda I - T$ is not bounded from below i.e.

$$\lambda \in \pi(T).$$

Thus

$$C\sigma(T) \subseteq \pi(T)$$

■

Corollary 1.3.21 $P\sigma(T) \cup C\sigma(T) \subseteq \pi(T).$

Definition 1.3.22 *The sequence $\{x_n\}$ from the normed linear space X is said to converge weakly to $x \in X$, written as $x_n \to x$ if for every $f \in \widetilde{X}$, ($\widetilde{X}$ the conjugate space)*

$$f(x_n) \to f(x)$$

Definition 1.3.23 *Let X and Y be normed spaces, an operator $T : X \to Y$ is called compact linear operator (completely continuos linear operator) if T is linear and if for every bounded subset M of X; the image $T(M)$ that is the closure $\overline{TM}$ is compact.*

Definition 1.3.24 *A vector space X is said to be the direct sum of two subspaces Y and Z of X, written as*

$$X = Y \oplus Z$$

if each $x \in X$ has a unique representation

$$x = y \oplus z$$

$y \in Y$, $z \in Z$.

Definition 1.3.25 *For any two subspaces Y and Z of a vector space X, an orthogonal complement is the set of all vectors orthogonal to Y, such that*

$$Y^{\perp} = \{z \in H \ \langle y, x\rangle = 0, \ z \in Y\}$$

Definition 1.3.26 *x is said to be orthogonal to y if*

$$\langle x, y \rangle = 0$$

and we write $x \perp y$

Theorem 1.3.27 *Let N be a closed subspace of H and let P be an orthogonal projection onto N. Then*

(*i*) P is linear
(*ii*) $\|P\| = 1$ (unless $N = 0$)
(*iii*) $P^2 = P$
(*iv*) $P^* = P$

Definition 1.3.28 *An operator $T \in (B)$ is said to be*

Normal if $T^*T = TT^*$
Unitary if $T^*T = TT^* = I$
Isometry if $T^*T = I$

Theorem 1.3.29 *$T \in B(H, K)$ is unitary if and only if*

$$TT^* = I_K \text{ and } T^*T = I_H$$

Definition 1.3.30 *Linear operators $T \in B(H)$ and $S \in B(K)$ are* ***unitarily equivalent*** *(denoted by $T \cong S$), if there exists a unitary operator $U \in G(H, K)$ such that*

$$UT = SU$$

i.e. $T = U^*SU$ or equivalently $S = UTU^*$.

Theorem 1.3.31 ***Spectral mapping theorem***

Let $\sigma(T)$ be the spectrum of an operator T, and $p(t)$ be any polynomial of a complex number t. Then

$$\sigma(p(T)) = \quad p(\sigma(T))$$

Chapter 2

Numerical Range And Consequences

2.1 Definitions and consequences

We assume, unless otherwise stated, that H denotes a complex Hilbert space with the inner product function

$$\langle , \rangle : H \times H \to \mathbb{C}$$

Definition 2.1.1 *The **numerical range** $W(T)$ of an operator T on a Hilbert space H is defined by;*

$$W(T) = \{\langle Tx, x\rangle : x \in H \text{ and } \|x\| = 1\}$$

Definition 2.1.2 *The **numerical radius** $w(T)$ of an operator T is defined by*

$$w(T) = \sup\{|\lambda| : \lambda \in W(T)\}$$

Definition 2.1.3 *The spectral radius $r(T)$ of an operator T is defined by*

$$rT = \sup\{|\lambda| : \lambda \in \sigma(T)\}$$

Since T is bounded

$$\|Tx\| \leq \|Tx\| \|x\|$$

for all $x \in H$.Now by the Cauchy-schwartz inequality,

$$|\langle Tx, x\rangle| \leq \|Tx\| \, \|x\|$$

Consequently, if $\lambda \in W(T)$, then

$$|\lambda| \leq \|T\|$$

So $W(T)$ is a bounded subset of $\mathbb{C}$ and hence $w(T)$ is a non-negative real numbers and satisfies

$$w(T) \leq \|T\| \text{ for all } T \in B(H) \text{..................}(1.1)$$

If the bounded linear operator has its domain, the linear subspace $D(T)$ of H, then the numerical range $W(T)$ is just the set

$$\{\langle Tx, x\rangle : x \in D(H)\} \text{ and } \|x\| = 1.$$

Theorem 2.1.4 *Let T be a bounded linear operator on a Hilbert space G. Then the following properties hold.*

(i) $W(\alpha I + \beta T) = \alpha + \beta W(T)$ for all α, $\beta \in \mathbb{C}$
Here the set

$$\alpha + \beta W(T) \, (= \beta W(T) + \alpha)$$

is defined to be the set

$$\{\alpha + \beta\lambda : \lambda \in W(T)\}$$

(ii) $W(T^*) = \{\overline{\lambda} : \lambda \in W(T)\} = W(T^*)$
(iii) $|\langle Tx, x\rangle| \leq W(T) \|x\|^2$ for all $x \in H$
(iv) $W(U^*TU) = W(T)$ for all unitary operators $U \in B(H)$

Theorem 2.1.5 *Let T be a bounded linear operator on a Hilbert space H then;*

(i) $\sigma(T) \subseteq \overline{W(T)}$ (The closure of the numerical range of an operator includes the spectrum)
(ii) If

$$d = dist(\lambda, \overline{W(T)})$$

where $dist$ is the distance function derived from the modulus in $\mathbb{C}$,then

$$\lambda I - TM$$

has an inverse and

$$\left\|(\lambda I - T)^{-1}\right\| < \frac{1}{d}$$

(iii) $r(T) \leq w(T) \leq \|T\|$

Proof. (i) If $\lambda \notin \overline{W(T)}$, then, of course,

$$dist(\lambda, \overline{W(T)}) > 0$$

that is $d > 0$; and by the definition of distance

$$d \leq |\langle Tx, x\rangle - \lambda|$$

for all $x \in H$, such that ■

$$\|x\| = 1$$

This implies

$$d\|x\|^2 \leq |\langle (T - \lambda I)x, x\rangle|$$

for all $x \in H$

using cauchy-schwartz inequality,it is clear that

$$\|(T - \lambda I)x\| \geq d\|x\|$$

since $T - \lambda I$ is bounded from below, $(T - \lambda I)^{-1}$ exists on $R(T - \lambda I)$ and is bounded; moreover,

$$\left\|(T - \lambda I)^{-1} y\right\| \leq d^{-1}\|y\| \text{ for all } y \in R(T - \lambda I) \text{...............(1.6)}$$

hence there are only two possibilities, either $\lambda \in \rho(T)$ or $\lambda \in R\sigma(T)$ suppose

$$\lambda \in R\sigma(T)$$

since

$$\left\{\overline{R(T - \lambda I)}\right\}^{\perp} = \{R(T - \lambda I)\}^{\perp} = \ker(T^{\star} - \overline{\lambda} I)$$

If $\lambda \in R\sigma(T)$,

then

$$\left\{\overline{R\,(T-\lambda I)}\right\}^{\perp} \neq \{0\}$$

$$i.e.\,\ker(T^{\star}-\overline{\lambda}I) \neq \{0\}$$

and hence $\overline{\lambda}$ is an eigenvalue of $T^{\star}$.

If $x \in H$,

$$\|x\| = 1$$

and is such that

$$T^{\star}x = \overline{\lambda}x$$

then

$$\langle Tx, x\rangle = \langle x, T^{\star}x\rangle \quad = \left\langle x, \overline{\lambda}x\right\rangle \ = \lambda$$

which implies that

$$\lambda \in W\,(T)$$

a contradiction.

Hence $\lambda \notin W\,(T)$, then $\lambda \in \rho\,(T)$ this shows that

$$\sigma\,(T) \subseteq \overline{W\,(T)}$$

(ii) Next we show the first inequality of (ii) which will implies the second inequality of (ii) by (i) above,we know that

$$\frac{1}{d(\lambda,\sigma(T)} = \sup \frac{1}{|\sigma(T)-\lambda|} \text{ for any } \lambda \notin \sigma(T)$$

now by the spectral mapping theorem,where we know that

$$\sigma(P(T)) = P(\sigma(T))\text{.......................}\ (*)$$

and also

$$\sigma(T)^{-1} = \{\sigma(T)^{-1}\text{.......................}\ (**)$$

we have that

$$\frac{1}{d(\lambda,\sigma(T)} = \sup \frac{1}{|\sigma(T)-\lambda|} \ = \sup \frac{1}{|\sigma(T-\lambda)|}$$

by $(*) = sup\ |\sigma((T-\lambda)^{-1})|$ by $(**) = r((T-\lambda)^{-1})$

so that the first inequality of (ii) follows by equation(i) and

$$r((T-\lambda)^{-1}) \leq \left\|(T-\lambda)^{-1}\right\|$$

by equation (iii) hence the proof is complete.

(iii) the proof follows easily from theorem(...........) of chapter 2

Lemma 2.1.6 *Let $T \in B(H)$, then $Re(T) \geq 0$ iff for all $\lambda < 0$*

$$(T-\lambda I)^*(T-\lambda I) \geq \lambda^2 I \text{...}(1.7)$$

Proof. First we consider the following identity.

$$(T-\lambda I)^*(T-\lambda I) - \lambda^2 I = T^*T - \lambda(T+T^*)$$

Suppose now that

$$(T-\lambda I)^*(T-\lambda I) \geq \lambda^2 I$$

then it is clear that

$$T+T^* \geq \frac{1}{\lambda}T^*T \text{ for all } \lambda < 0$$

Let $\lambda \rightarrow -\infty$,then we get

$$T+T^* \geq 0. \text{ i.e. } Re(T) \geq 0$$

Conversely,suppose that

$$T+T^* \geq \frac{1}{\lambda}T^*T \text{ for all } \lambda < 0$$

the above identity implies the necessity. ■

Lemma 2.1.7 *Let $k_\circ = \{\lambda : \operatorname{Re}\lambda \geq 0\}$ if $T \in B(H)$, then $W(T) \subset K_\circ$ if and only if*

$$(T-\lambda I)^*(T-\lambda I) \geq (\operatorname{Re}\lambda)^2 I, \lambda \in K_\circ$$

Proof. First we note that from

$$Re\,\langle Tx, x\rangle = \langle (ReT)\,x, x\rangle \text{ for all } x \in H................(1.8)$$

it follows that

$$W(T) \subset K_\circ$$

if and only if $\operatorname{Re} T \geq 0$ to see (1.8) ,
let

$$\langle Tx, x\rangle = a + ib, \text{ where } a, b \in \mathbb{R}$$

then

$$\overline{\langle Tx, x\rangle} = a - ib$$

that is

$$\langle T^{*}x, x\rangle = a - ib$$

hence

$$\langle (T + T^{*})\,x, x\rangle = 2a$$

that is

$$< \frac{1}{2}\,(T + T^{*})\,x, x >= a$$

but

$$\frac{1}{2}(T + T^{*}) = \operatorname{Re} T$$

thus (1.8) is established. Now if $\lambda = a_1 + ib_1$, where $a_1, b_1 \in \mathbb{R}$ and

$$\lambda \notin K_\circ T - \lambda I = (T - ib) - a_1 I$$

thus

$$Re(T) = \frac{1}{2}(T + T^{*}) = \frac{(T - ib) + (T - ib)^{*}}{2} = \operatorname{Re}(T - ib)$$

and from lemma 2 we obtain

$$(T - \lambda I)^{*}(T - \lambda I) \geq a^2 I$$

if and only if

$$Re(T - ib) \geq 0$$

which is equivalent to $\operatorname{Re} T \geq 0$ and the lemma is proved. ■

Theorem 2.1.8 *If K is a closed half-plane, then $K \supset W(T)$ if and only if*

$$(T - \lambda I)^*(T - \lambda I) \geq [\ dist\ (\lambda, K)]^2\ I$$

Proof. First we note that lemma 3 is exactly this lemma for $K = K_\circ$.If we consider the function

$$f(\lambda) = a\lambda + b, \|a\| = 1$$

which transforms K onto $K_\circ$(for some a and b) and

$$f(T) = aT + bT$$

Then we have

$$f(W(T)) = \{f(\lambda) : \lambda \in W(T)\}$$

and thus $W(T) \subset K$ if and only if $f(W(T)) \subset K_\circ$
Now since

$$f(T) - f(\lambda)I = a(T - \lambda I)$$

and

$$[f(T) - f(\lambda)I]^* [f(T) - f(\lambda)I] = (T - \lambda I)^*(T - \lambda I)$$

using lemma 3 and the fact that

$$dist(f(\lambda), K_\circ) = dist(f(\lambda), f(K)) = dist(\lambda, K)$$

The assertion of the theorem follows. ■

Theorem 2.1.9 *Let $T \in B(H)$, then*

$$p\sigma(T) \subset W(T) \text{ and } \pi(T) \subset \overline{W(T)}$$

This theorem may be included and proved immediately following theorem 2

Proof. If $\lambda \in P\sigma(T)$, then there exists $x \in H$ such that

$$\|x\| = 1 \text{ and } Tx = \lambda x$$

then

$$\langle Tx, x\rangle = \langle \lambda x, x\rangle\ = \lambda \|x\|^2 = \lambda$$

thus, $\lambda \in W(T)$, so $\lambda \in P\sigma(T)$ and

$$\lambda \in W(T) \Rightarrow P\sigma(T) \subset W(T)$$

and since $\pi(T) \subset \sigma(T)$ and $\sigma(T) \subset \overline{W(T)}$ (see theorem 1). We have

$$\pi(T) \subset \overline{W(T)}$$

Alternatively, $\lambda \in \pi(T)$, implies that there is a sequence (x_n) of unit vectors in H such that

$$\begin{matrix} lim \\ n \to \infty \end{matrix} \quad \|(\lambda I - T)\| = 0$$

since, for such $\blacksquare$

$$x_n \left|\lambda - \langle Tx_n, x\rangle\right| = \left|\langle(\lambda I - T)x_n, x_n\rangle\right| \leq \|(\lambda I - T)x_n\|$$

we see at once that

$$\lambda \in \overline{W(T)}.$$

2.2 Toeplitz-Hausdorff Theorem(T-H)

The most important property of the numerical range is given by so called Toeplitz-Hausdorff Theorem which is a historical monument.

Theorem 2.2.1 *The numerical range $W(T)$ of an operator T is a convex set in the complex plane.*

Remark 2.2.2 *We found a lot of proofs of theorem T-H. We cite the following nice proofs of this famous theorem:* [*Gustafson 1970*] , [*Goldberg-straus 1979*] *and* [*Li 1994*] , *for the sake of convenience.*

Proof. Suppose $\lambda = \langle Tx, x\rangle$ and $\mu = \langle Ty, y\rangle$, where x and y are unit vectors in H. Our task is to prove every point of the segment joining λ and μ is in $W(T)$.

If $\lambda = \mu$, the problem is trivial. if $\lambda \neq \mu$, then there exists complex numbers α and β such that

$$\alpha\lambda + \beta = 1 \text{ and } \alpha\mu + \beta = 0.................(*)$$

it is sufficient to prove that the unit interval $[0, 1]$ is included in

$$W(\alpha T + \beta I) = (= \alpha W(T) + \beta)$$

this is because if

$$\alpha \langle TZ, Z\rangle = t$$

then

$$\alpha \langle TZ, Z\rangle + \beta = t(\alpha\lambda + \beta) + (1 - t)(\alpha\mu + \beta)$$

by

$$= \alpha(t\lambda + (1 - t)\mu) + \beta$$

as a consequence, we may assume without loss of generality that $\lambda = 1$ and $\mu = 0$ in the first place. Write

$$T = s_1 + is_2 \text{ with } s_1, s_2 \text{ self adjoint,}$$

since

$$\langle Tx, x\rangle (= 1) \text{ and } \langle Ty, y\rangle \ (= 0)$$

are real, it follows that

$$\langle s_2x, x\rangle \text{ and } \langle s_2y, y\rangle$$

varnish x is replaced by ηx where $|\eta| = 1$, then $\langle Tx, x\rangle$ remains unaltered and

$$\langle s_2x, y\rangle \text{ becomes } \eta \langle s_2x, y\rangle$$

hence there is no loss of generality in assuming that $\langle s_2x, y\rangle$ is purely imaginary. Put

$$h(t) = tx + (1 - t)y, 0 \leq t \leq 1$$

now $h(t)$ is never 0, infact the vectors x and y are linearly independent. This is a consequence of

$$\langle Tx, x\rangle \neq \langle Ty, y\rangle .$$

If $\{x, y\}$ were linearly dependent, then since they are unit vectors, one of them could be written as a multiple of the other. Since moreover, the factor would have to have absolute value 1, it would follow that

$$\langle Tx, x\rangle = \langle Ty, y\rangle ,$$

a contradiction since

$$\langle s_2h(t), h(t)\rangle = t_2\langle s_2x, x\rangle + t(1-t)(\langle s_2x, y\rangle + \overline{\langle s_2 x, y\rangle}) + (1-t)2\langle s_2y, y\rangle ;$$

the relations

$$\langle s_2x, x\rangle = \langle s_2y, y\rangle = 0 \text{ and } Re\,\langle s_2x, y\rangle = 0$$

imply that

$$\langle s_2h(t), h(t)\rangle = 0 \text{ for all } t$$

and hence that

$$\langle Th(t), h(t)\rangle \;\text{ is real for all } t$$

the function

$$t \to \frac{\langle Th(t), h(t)\rangle}{\|h(t)\|^2}$$

Is real-valued and continuos on the closed interval $[0,1]$ since its value at $t=0$ and $t=1$ are 0 and 1 respectively, hence the range of the function contains every number in the unit interval $[0,1]$ i.e. we obtain

$$[0,1] \subseteq W(T)$$

■

Remark 2.2.3 *Note that it follows that the closure $\overline{W(T)}$ is also convex by theorem 2. Since $\overline{W(T)}$ is convex and contains $\sigma(T)$, it contains the convex hull of $\sigma(T)$. Thus we have*

Theorem 2.2.4 *Let $T \in B(H)$, then*

$$con\sigma(T) \subseteq \overline{W(T)}$$

Now, for $T \in B(H)$ the spectrum $\sigma(T)$ is compact subset of $\mathbb{C}$, a non trivial fact of finite dimension Euclidian geometry is that the convex hull of a compact set is closed. The most useful formulation of this fact for the plane $\mathbb{C}$ is that the convex hull of a compact set is the intersection of all the closed half-planes that include it. [see Valentine convex sets MC Graw Hill,1964] .

The question now is: can the closure of the numerical range be very much larger than the spectrum?

2.3 Some results on the numerical radius

In this section we introduce numerical radius of T associated with the numerical range which is equivalent to the operator norm $\|T\|$. We first prove some basic results on the numerical radius $w(T)$ of a bounded linear operator $T \in B(H)$.

Theorem 2.3.1 *The function defined on* $B(H)$ *by*

$$T \to w(T)$$

is a norm which is equivalent to the standard norm

$$T \to \|T\| \,\text{of} B(H)$$

(i) For any $T \in B(H), w(T) = w(T^*)$
(ii) For any $T \in B(H), w(T^*T) = \|T\|^2$
(iii) For any $T \in B(H), r(T) \leq w(T)$
Proof. The properties

$$w(T+S) \leq w(T) + w(S) \text{ and } w(\lambda T) = |\lambda|\, w(T)$$

for any S, $T \in B(H)$ and for all $\lambda \in \mathbb{C}$, are obvious. We had already seen that

$$w(T) \leq \|T\|$$

Now for all $x, y \in H$

$$\langle Tx, y\rangle = \frac{1}{4}[\langle T(x+y), x+y\rangle - \langle T(x-y), x-y\rangle + i\,\langle T(x+iy), x+iy\rangle - i\,\langle T(x-iy), x-iy\rangle] \text{....(2.1}$$

Using theorem $1(iii)$ on the right hand side of (2.10) we have

$$\langle Tx, y\rangle \leq \frac{1}{4}w(T)[\|x+y\|^2 + \|x-y\|^2\,\|x+iy\|^2\,\|x-y\|^2]$$

Hence using the parallelogram law in H, we have

$$|\langle Tx, y\rangle| \leq \frac{1}{4}w\,(T)\,\{4(\|x\|^2 + \|y\|^2)\} = w(T)\{\|x\|^2 + \|y\|^2\}$$

$$= w(T)(1+1)$$

Now

$$\|T\| = \sup\{|\langle Tx, y\rangle| : x, y \in H$$

and

$$\|x\| = \|y\| = 1$$

Hence

$$\|T\| \leq 2w(T), \text{i.e.} \frac{1}{2}\|T\| \leq w(T)$$

Thus

$$\frac{1}{2}\|T\| \leq w(T) \leq \|T\| \text{..............}(2.11)$$

and the constant $\frac{1}{2}$ on the left side is the best possible constant. Hence we have proved(i).

(ii) $w(T^*) = sup\{|\langle T^*x, x\rangle| : x \in H \text{ and } \|x\| = 1\}$
$= sup\{|\langle x, Tx\rangle| : x \in H \text{ and } \|x\| = 1\}$
$= sup\{\left|\overline{\langle Tx, x\rangle}\right| : x \in H \text{ and } \|x\| = 1\}$

$$= sup\{|\langle Tx, x\rangle| : x \in H \text{ and } \|x\| = 1 = w(T)\}$$

(iii) $w(T^*T) = sup\{|\langle T^*x, x\rangle| : x \in H \text{ and } \|x\| = 1\}$
$= sup\{|\langle T^*Tx, x\rangle| : x \in H \text{ and } \|x\| = 1\}$
$= sup\{\|Tx\|^2 : x \in H \text{ and } \|x\| = 1\}$

$$= [sup\{\|Tx\| \, x \in H \text{ and } \|x\| = 1\}]^2 = \|T\|^2$$

(iv) Since $\sigma(T) \subset \overline{W(T)}$, by theorem 2 it is obvious that

$$r(T) \leq W(T).$$

■

Remark 2.3.2 *The existence of quasinilpotent (or nilpotent) operators i.e. operators $T \in B(H)$ for which*

$$\lim_{n\to\infty} \|T^n\|^{\frac{1}{n}} = 0 \text{ or } T^n = 0$$

for some $n \in N$, respectively shows that nothing like the reverse inequality could be true.

Theorem 2.3.3 *Let $T \in B(H)$.*

(a) if $w(I-T) < 1$, then T is invertible
(b) if $w(T) = \|T\|$, then $r(T) = \|T\|$

Proof. (a) If T is not invertible, then

$$0 \in \sigma(T)$$

so $1 \in \sigma(I-T)$, then follows from (2.12) that

$$1 \leq r(I-T) \leq w(I-T)$$

by taking the contrapositive of this, we obtain

$$w(I-T) < 1 \Rightarrow T$$

is invertible. ■

(b) We may assume without loss of generality that

$$\|T\| = 1$$

(All we have to do is to multiply by a suitable positive constant.)the hypothesis

$$w(T) = \|T\|$$

Then assumes the existence of a sequence (xn) of unit vectors such that

$$|\langle Tx, xn\rangle| \to 1 \text{as} n \to \infty$$

Proof. Again, assume without loss of generality that

$$\langle Txn, xn\rangle \to 1 \text{ as } n \to \infty$$

(All we have to do is to multiply by a suitable constant of modulus 1).

$$if\ \ \langle Txn, xn\rangle \to e^{i\theta} \text{ as } n \to \infty,$$

(take $e^{-i\theta}T$ instead of T). since

$$|\langle Txn, xn\rangle| \leq \|Txn\| \leq 1$$

and

$$\langle Txn, xn\rangle \to 1$$

It follows that

$$\|Txn\| \to 1 \text{ as } n \to \infty$$

then

$$\|Txn - xn\|^2 = \|Txn\|^2 - 2\operatorname{Re}\langle Txn, xn\rangle + 1 \to 0 \text{ as } n \to \infty$$

and consequently

$$1 \in \pi(T).$$

Now

$$\pi(T) \subseteq \sigma(T).\text{so } 1 \in \sigma(T)$$

thus

$$r(T) \geq 1$$

but

$$r(T) \leq \|T\| = 1 \text{ i.e. } r(T) \leq 1$$

this shows that

$$r(T) = 1$$

■

2.4 Further properties of the numerical range

The numerical range $W(T)$ of an operator T is not generally closed. In the finite dimensional case, however, the numerical range of an operator is a continuos image of a compact set. The unit sphere

$$S(H) = \{x : \|x\| = 1\}$$

and hence is necessarily a compact subset of $\mathbb{C}$ and is therefor closed. The next theorem is useful in that it helps us to construct examples of operators whose numerical range is not closed.

Theorem 2.4.1 *If $T \in B(H)$ and λ is a complex number such that $|\lambda| = \|T\|$ and $\lambda \in W(T)$, then λ is an eigenvalue of T.*

Proof. Let $T \in B(H)$; so there is an $x \in H$ with $\|x\| = 1$ such that

$$\lambda = \langle Tx, x \rangle$$

then since

$$|\lambda| = \|T\|$$

$$\|T\| = |\lambda| = |\langle Tx, x \rangle| \leq \|Tx\| \, \|x\| = \|Tx\| \leq \|T\| \, .$$

hence equality holds throughout in the last line and we have,

$$|\langle Tx, x \rangle| = \|Tx\| \, \|x\|$$

In the Cauchy-shwarz Inequality, therefore the set $\{Tx, x\}$ must be linearly dependent,

that is

$$Tx = \lambda_\circ x \text{ for some } \lambda_\circ \in \mathbb{C}$$

This in turn implies that

$$\lambda_\circ = \lambda_\circ \langle x, x \rangle = \langle \lambda_\circ x, x \rangle = \langle Tx, x \rangle = \lambda$$

And thus λ is an eigenvalue of T. ■

Remark 2.4.2 *It follows from the theorem that if $\lambda \in \overline{W(T)}$ such that $|\lambda| = \|T\|$ and if λ is not an eigenvalue of T, (and in particular if T has no eigen values) then λ does not belong to $W(T)$. In view of this remark it easy to construct examples of operators whose numerical range is* ***not*** *closed.*

Example 1 *By theorem 4 we see that the eigenvalue of every operator $T \in B(H)$ belong to $W(T)$. And if T is normal, then*

$$\|T\| = w(T)$$

(a consequence of lemma 7)

Hence since

$$sup\{|\lambda| : \lambda \in W(T)\} = w(T)$$

There always exists a $\lambda \in \overline{W(T)}$ such that

$$|\lambda| = \|T\|$$

It follows that if a normal operator has sufficiently many eigenvalues to approximate its norm, but does not have one whose modulus is as large as the norm, then its numerical range will not be closed. A concrete example is provided by a diagonal operator such that the modulus of the diagonal term does not attain its supremum.

Example 2 *Consider the operator $T : L^2(\mathbb{N}) \to L^2(\mathbb{N})$ defined by*

$$Tx = (x_1, \frac{1}{2}x_2, \frac{1}{3}x_3, ...)$$

for $x = (x_1, x_2, x_3, ...) \in L^2(\mathbb{N})$. Now $T \geq 0$ and $ker\ (T) = \{\overline{0}\}$, and $\|T\| = 1$ indeed, we note that

$$\|Tx\| = \|x\|$$

for all $x \in L^2(\mathbb{N})$, thus $\|T\| \leq 1$. If we take $x = (1, 0, 0, ...)$, then

$$Tx = (1, 0, 0, 0...)$$

so $\|Tx\| = 1$ and $\|x\| = 1$. Thus

$$\|T\| = 1$$

Since $W(T) \subseteq \mathbb{R}^+$(for all $\langle Ty, y\rangle \geq 0$ for all $y \in H$) and $w(T) \leq \|T\|$ we note that

$$W(T) \subset [0, 1]$$

for $x = (1, 0, 0, ...)$, we have

$$\langle Tx, x\rangle = 1$$

and hence $1 \in W(T)$. let $\in$ be any arbitrary positive number less than 1, that is , $0 < \in < 1$. Choose a $k \in \mathbb{N}$ such that $\frac{1}{k} <\in$ .Let $x = (x_n)$, where $x_k = 1$, and $x_n = 0$ if $n \neq k$.Then

$$\|x\| = 1 \text{ and } < Tx, x >= \sum_{n=1}^{\infty} \frac{1}{n} |x_n|^2 = \frac{1}{k} |x_k|^2 = \frac{1}{k} <\in$$

thus $\in\in W(T)$. It therefore follows using the fact that $W(T)$ is convex,

$$W(T) = (0, 1]$$

this example shows that $W(T)$ may fail to be closed even for compact operators in $B(H)$

Example 3 *Consider the unilateral shift* $U : L^2 \to L^2$ *defined by;*

$$U(x_\circ, x_1, x_2, ...) = (0, x_\circ, x_1, x_2, ...)$$

then

$$U^*(x_\circ, x_1, x_2) = (x_1, x_2, ...)$$

suppose λ *is an eigenvalue of* U^**,*

i.e.

$$U^* x = \lambda x$$

for any $x \neq \overline{0}(x \in L^2)$. Let

$$X = (x_\circ, x_1, x_2, ...)$$

then

$$(x_1, x_2, ...) = (\lambda x_\circ, \lambda x_1, \lambda_2, ...)$$

hence$X_{n+1} = \lambda x_n$,that is

$$X_{n+1} = \lambda^n x_\circ$$

for all $n \in \mathbb{N}$

Now if $x_\circ = 0$, then $x = \overline{0}$, otherwise, a neccesary and sufficient condition that the resulting $X_n's$ be the coordinates of a vector (that is, they may be square summable and hence $x \in L^2$) is $|\lambda| < 1$. So each λ in the open disc of $\mathbb{C}$ is a simple eigenvalue of U^*, that is each eigenvalue is of multiplicity 1. It therefore follows that the open unit disc $\{\lambda : |\lambda| < 1\}$ is contained in $W(U^*)$. Since $W(U^*)$ is always

$$W(U^*)(= \overline{\lambda} : \lambda \in W(U)$$

It follows that the open unit disc is contained in $W(U)$.

Since U has no eigenvalues, theorem 12 (proved above) implies that $W(U)$ cannot. It is contain any number of modulus 1. Hence $W(U)$ equals the open unit disc, we note that the number 0 plays a special role with respect to the spectrum of a compact operator. It is remarkable that it also plays a special role in regard to the numerical range of a compact operator,

the next theorem gives a sufficient condition for an operator to have a closed numerical range.

Theorem 2.4.3 *If* $T \in B(H)$ *is a compact operator and* $0 \in W(T)$*, then* $W(T)$ *is a closed set.*

Proof. If $\lambda \in \overline{W(T)}$, then there exists a sequence (x_n) of elements of H such that

$$\|x\| = 1$$

for all $n \in \mathbb{N}$ and

$$\underset{n\to\infty}{lim} \langle Tx_n, x_n \rangle = \lambda$$

We now use a result known in Hilbert space theorem: every bounded sequence (x_n) of elements of H contains a weekly convergent subsequence. In fact, the closed unit disc $\overline{N}(\overline{0}, 1)$ of H has the property that; every sequence of elements of $\overline{N}(\overline{0}, 1)$ converges weakly to a point in $\overline{N}(\overline{0}, 1)$. From this fact, we can choose a weak convergent subsequence and may assume, without loss of generality, that (x_n) is weakly convergent to x. Then by compactness of T

$$Tx_n \xrightarrow{s} x$$

since

$$|\langle Tx_n, x_n \rangle - \langle Tx, x \rangle| \leq |\langle Tx_n, x_n \rangle - \langle Tx, x_n \rangle| + |\langle Tx, x_n \rangle - \langle Tx, x \rangle|$$

$$\leq \|Tx_n - Tx\| \, \|x_n\| + |\langle (x_n - x), Tx \rangle| \, .$$

We see that from the observations that

(i) The first summand tends to 0 as $n \to \infty$ since $Tx_n \xrightarrow{s} Tx$

(ii) The second summand tends to 0 as $n \to \infty$ since $x_n \xrightarrow{w} x$ and that $\not{x} \in \overline{N}(\overline{0}, 1)$.

Thus

$$\lambda = \underset{n\to\infty}{lim} \langle Tx_n, x_n \rangle = \langle Tx, x \rangle$$

Now if $0 \in W(T)$, then

$$\langle Ty, y \rangle \in W(T)$$

for every $y \in \overline{N}(\overline{0}, 1)$ indeed,

if $\|z\| = 1$ and $0 \leq t \leq 1$ so that $z \in \overline{N}(\overline{0}, 1)$.

then;

$$\langle T(tz), tz \rangle = t^2 \langle Tz, z \rangle = t^2 \langle Tz, z \rangle + (1 - t^2).0 \in W(T)$$

by convexity, note that this particular argument does not need T to be copmact infact it holds for any operator T. Thus the element $x \in \overline{N}(\bar{0}, 1)$ obtained in the earlier part of the proof being such that

$$\langle Tx, x \rangle = \lambda$$

proves that

$$\lambda \in W(T) \text{ if } \lambda = 0$$

the assertion of the theorem is clear thus $W(T)$ is closed. Note: this result is due to G.De Barva,J.R.icircles,B.sins.on the numerical range of compact operators on a Hilbert space.(J.London math soc.5(1972) pg 704-706. ■

Chapter 3

Some special points on the Boundary of the Numerical Range of an Operator

3.1 Introduction

Besides giving a set inside which the point spectrum must lie, the numerical range can be used to prove that certain points are eigenvalues of the operator in the context. Hence we prove a useful general lemma about $W(T)$ where T is any two-by-two matrix.

Lemma 3.1.1 *Let T be a linear operator on a two-dimensional Hilbert space H_2. If the matrix of T (which is obviously a 2×2 matrix. Has distinct eigenvalues λ_1 and λ_2 and the corresponding eigenvectors x_1 and x_2 so normalized that*

$$\|x_1\| = \|x_2\| = 1$$

Then $W(T)$ is a closed elliptical disc with foci at $\lambda_1$and λ_2, if $r = |\langle x_1, x_2\rangle|$ and $\delta = \sqrt{1-r^2}$, then the minor axis is $\frac{r|\lambda_1-\lambda_2|}{\delta}$ and major axis is $\frac{|\lambda_1-\lambda_2|}{\delta}$

If T has only one eigenvalue λ, then $W(T)$ is the (circular) disc with center λ and radius.

$$\frac{1}{2}\|T - \lambda I\|.$$

Proof. Since H_2 has the unit disc $\{x : \|x\| = 1\}$ as a compact set and the function

$$x \to \langle Tx, x\rangle$$

is continuos it follows that $W(T)$ is a compact set. Suppose that T has only one eigenvalue λ. In this case

$$T_1 = T - \lambda I$$

has the property that

$$\sigma(T_1) = \{0\}$$

and also $T_1^2 = 0$. For the characteristic polynomial of the matrix of T is

$$P(t) = \alpha(t - \lambda)^2$$

for a non-zero $\alpha \in \mathbb{C}$. Hence

$$\alpha(T - \lambda I)^2 - 0$$

i.e. $T_1^2 = 0$) if $T_1 = 0$, we have $W(T_1) = (0)$ and thus

$$T(T) = \{\lambda\}$$

This clearly a circle with center λ and radii 0. If $T_1 \neq 0$, then there exists an orthogonal basis $\{e_1, e_2\}$ of H_2 such that $T_1 e_1 = ae_2$ and $T_1 e_2 = \bar{0}$ and $\|T_1\| = |a|$. To compute $W(T)$, we proceed in the same way and we obtain it as a circular closed disc centered at 0 with radius $\frac{|a|}{2}$. This implies that $W(T)$ is the closed circular disc with center λ and radius

$$= \frac{|a|}{2} = \frac{\|T\|}{2} = \frac{1}{2}\|T - \lambda I\|$$

Now if T has distinct eigenvalues λ_1, λ_2, the operator.

$$T_1 = \frac{1}{\lambda_1 - \lambda_2}(T - \lambda I)$$

has eigenvalue 0 and 1. Also from the definition of the numerical range, the set $W(T_1)$ is obtained from $w(T)$ by a rigid motion and homothetic transformation and conversely. Let $\{e_1, e_2\}$ be an orthonormal basis of H_2 such that $T_1 e_1 = 0$ and we may choose these such that

$$T_1 u = u, \|u\| = 1.$$

where

$$u = (\cos\phi)\, e_1 + (\sin\phi)\, e_2$$

and ϕ is the angle between u and e,

i.e.

$$cos\phi = |\langle e, u\rangle|\,, 0 \le \phi \le \frac{\pi}{2}$$

now since $T_1 = u$ we have

$$T_1\left((\cos\phi)\, e_1 + (\sin\phi)\, e_2\right) = (\cos\phi)\, e_1 + (\sin\phi)\, e_2$$

that is

$$T_1 e_2 = (\cot\phi)\, e_1 + e_2$$

take any $x = ae_1 + be_{2,}$ $\|x\| = 1$ (that is, $|a|^2 + |b|^2 = 1$) then

$$\langle T_1 x, x\rangle = \overline{a}b\cot\phi + |b|^2 = |b|^2 + |a|\,|b|\,e^{iw}\cot\phi$$

if w varies with $|a|\,, |b|$,fixed and $|a|^2 + |b|^2 = 1$,then the scalars $\langle T_1 x, x\rangle$ trace a circle with center at $(t, 0)$ and with radius

$$t\,(1-t)]^{\frac{1}{2}}\cot\phi$$

where $t = |b|^2$ and $W\,(T)$ is the union of all the circles.

$$(x-t)^2 + y^2 = \left(t - t^2\right)\cot^2\phi .... (2.17)$$

■

The envelop of this family of circles is obtained by treating 2.17 as a quadratic equation in t and equating its discriminant to 0. Doing this, we obtain

$$\left(2x + \cot^2\phi\right)^2 - 4\left(\cos ec^2\phi\right)\left(x^2 + y^2\right) = 0 .... (2.18)$$

which can be simplified to

$$\frac{\left(x - \frac{1}{2}\right)^2}{\left(\frac{1}{2}\cos ec\phi\right)^2} + \frac{y^2}{\left(\frac{\cot\phi}{2}\right)^2} = 1$$

which is an ellipse with foci at $(0, 0)$ and $(1, 0)$ and with eccentricity $sin\phi$. the center of this ellipse is the point $\left(\frac{1}{2}, 0\right)$ and its major and minor axes are of lenghts $cosec\phi$ and $cot\phi$, respectively. Consider the closed elliptic disc whose boundary is the elliptic (2.19). We show that $W(T)$ contains all the interior points of this elliptic disc, clearly, all points of the ellipse (2.19) It being the envelop of the family of circles (2.17)) belong to $w(T)$ to prove that any interior point $\lambda_\circ$ of the elliptic disc whose boundary is the ellipse (2.19) is in $w(T)$ we remark that there exists a circle $C_{t\circ}$ containing $\lambda_\circ$ which is tangent to the ellipse at the foot of the perpendicular from $(t_\circ, 0)$ the center of the circle $C_{t\circ}$ is $(t_\circ, 0)$ to the ellipse and it is exterior to one of the circles $C_\circ$ and C_1.Since these circles vary continuously, with it, it follows that there is a circle through $\lambda_\circ$ and this proves that $W(T_1)$ and hence $W(T)$ is a closed elliptic disc, using the relation see theorem1(i)

$$W(T_1) = W\left(\frac{T - \lambda_1 I}{\lambda_2 - \lambda_1}\right) = W\left(\frac{1}{\lambda_2 - \lambda_1} T - \frac{\lambda_1}{\lambda_2 - \lambda_1}\right) = \frac{1}{\lambda_2 - \lambda_1} W(T) - \frac{\lambda_1}{\lambda_2 - \lambda_1}$$

i.e.

$$W(T) = (\lambda_2 - \lambda_1)\{W(T_1) + \frac{\lambda_1}{\lambda_2 - \lambda_1}\}$$

the foci $(0,0)$ and $(0,1)$ of $w(T_1)$ are transformed to λ_1 and λ_2 respectively for $W(T)$. Other conclusions stated in statement of the theorem are obvious, the proof is complete.

Remark 3.1.2 *From this lemma, we can prove theorem 5 of chapter 1 in the following alternative fashion a and b are distinct points in $w(T)$ then there exists x and $y \in H$ such that*

$$a = \langle Tx, x\rangle, b = \langle Ty, y\rangle, \|x\| = \|y\| = 1$$

Let M be the subspace $[\{x, y\}]$ spanned by x and y. Hence M is a closed linear subspace of H of dimension 2 over $\mathbb{C}$. assume the contrary that $\{x, y\}$ is linearly dependent over $\mathbb{C}$ .Then

$$x = \alpha y$$

for some $\alpha \in \mathbb{C}$ with $|\alpha| = 1$, we then have

$$\langle Tx, x\rangle = \langle T(\alpha y), \alpha y\rangle = |\alpha|^2 \langle Ty, y\rangle = \langle Ty, y\rangle$$

i.e. $a = b$, a contradiction, hence $\{x, y\}$ must be linearly independent over $\mathbb{C}$ let $P_{x,y}$ be the orthogonal projector of H onto M take a $z \in M$ with $\|z\| = 1$. We have

$$P_{x,y} z = z$$

thus

$$TP_{x,y}z = Tz$$

Now T_z need not be in M ,however,

$$P_{x,y}Tz \in M$$

Consequently,

$$P_{x,y}TP_{x,y}z = P_{x,y}Tz.$$

Thus

$$\langle P_{x,y}TP_{x,y}z, z\rangle = \langle P_{x,y}Tz, z\rangle = \langle Tz, P_{x,y}z\rangle = \langle Tz, z\rangle$$

Now $\langle Tz, z\rangle \in W(T)$ and we thus obtain

$$W(P_{x,y}TP_{x,y})$$

is an elliptic(or circular disc,it follows that $W(T)$ is convex, hence the proof.

Theorem 3.1.3 *Let $T \in B(H)$ and $W(T)$ be a closed set, every point λ in the boundary of $w(T)$ at which the boundary is not a differential arc is an eigenvalue for T.*

Proof. It is well known that the boundary $w(T)$ being a convex function, is differentiable except perhaps at an utmost countable set of points. Let λ be a point of non-differentiability and x, $\|x\| = 1$, such that

$$\lambda = \langle Tx, x\rangle$$

also, at λ there exists a left and right tangents such that the angle between these tangents is smaller than π. Let Y be arbitrary in H and $P_{x,y}$ be the orthogonal projector on H onto the linear subspace $[\{x, y\}]$. The operator

$$T_1 = P_{x,y}TP_{x,y}$$

has a closed elliptical disc as its numerical range,and since no circle contained in $w(T)$ can pass through λ, it follows that the ellipse $w(T_1)$ is a line segment or a point, thus λ is an eigenvalue with x as an eigenvector. ■

3.2 Higher Dimensional Numerical Range

The numerical range of an operator T can be regarded as the one-dimensional course of a multi-dimensional concept. To see how that goes we see that an orthogonal projector P of rank 1 can be expressed in terms of a unit vector x in its range:

$$Py = \langle y, x \rangle x$$

for all $y \in H$, $x \in R_p$ and $\|x\| = 1$. If $T \in B(H)$, then PTP is an operator of rank 1, and therefore a finite-dimensional concept such as trace makes sense for it. The trace of PTP can be computed by finding the one-by-one matrix of the restriction of PTP to the range of P, with respect to the one-element basis $\{x\}$, since $px = x$, the value of that trace is

$$\langle PTPx, x \rangle = \langle TPx, Px \rangle = \langle Tx, x \rangle$$

These remarks can be summarized as follows: $W(T)$ is equal to the set of all complex numbers of the form $trPTP$, where P varies over all projections of rank 1, replace 1 by an arbitrary positive integer k, and obtain the $k-$numerical range of T, in symbols $W_k(T)$; the ordinary numerical range $W(T)$ is the $k-$numerical range with $k = 1$

Theorem 3.2.1 *For every operator $A \in B(H)$ and for every positive integer k, the $k-$numerical range $W_k(A)$ is convex.*

Proof. Suppose M and N are $k-$ dimensional Hilbert space and

$$T : M \to N$$

is a linear transformation. There is a useful sense in which T and T^* from N into M can be simultaneously diagonalized. The assertion is that there exists orthornomal basis $\{x_1, ..., x_n\}$ for M and $\{y_1, ..., y_n\}$ for N, and there exists positive (≥ 0)scalars $\alpha_1, ..., \alpha_k$ such that

$$Tx_i = \alpha_i y_i$$

and

$$T^* y_i = \alpha_i x_i; i = 1, ..., k.$$

■

To prove this, let UP be the polar decomposition of T, and diagonalize P, that is: find an orthonormal basis $\{x_1, ..., x_k\}$ for M, and find positive scalars $\alpha_1, ..., \alpha_k$ such that

$$Px_i = \alpha_i x_i$$

If the partial isometry U is not an isometry from M onto N, it can be replaced by one since $dimM = dimN = k$. Assume that has been done. Then put $y_i = Ux_i, i = 1, ..., k$. We then have the consequence:

$$Tx_i = UPx_i = U(\alpha_i x_i) = \alpha_i y_i$$

and

$$T^* y_i = PU^* y_i = Px_i = \alpha_i x_i; i = 1, ..., k.$$

So far, it is a lemma, now we go to the theorem, suppose that P and Q are projections of rank k, with respective ranges M and N if T is the restriction of QP to M, then the proceeding lemma is applicable. For each $i(= 1, ..., k)$, let L_i be the span of x_i and y_i we now assert that the subspaces L_i are pair wise orthogonal. Suppose, indeed that $i \neq j$; since $x_i \perp x_j$ and $y_i \perp y_j$,it is sufficient to prove that

$$x_i \perp y_j$$

for then $x_j \perp y_j$ follows by symmetry.

Now

$$\langle xi, yj \rangle = \langle Pxi, Qyj \rangle = \langle QPxi, yj \rangle = \langle \alpha iyi, yj \rangle$$

Next we go to the proof of the convexity, if $o \leq t \leq 1$, we use the classical Toeplitz-Hausdorff Theorem $k-$times to obtain a unit vector $z_i \in L_i$, so that

$$\langle Azi, zi \rangle = t \langle Axi, xi \rangle + (1-t) \langle Ayi, yi \rangle$$

since $\{h1, ..., hk\}$ is an orthogonal set, the projection R onto its span has rank k, and

$$ttrPAP + (1-t)trQAQ = t \sum_i \langle Axi, xi \rangle + (1-t) \sum_i \langle Ayi, yi \rangle$$

$$= \sum_i \langle Ahi, hi \rangle = trRAR$$

3.3 Continuity of the Numerical Range

Consider the Hausdorff metric for compact subsets of the plane. To define this metric, we write

$$M + (\in) = \{z + \alpha : z \in M, |\alpha| < \in\}$$

For each set M of complex numbers and each positive number $\in$ .If M and N are compact sets, the Hausdorff distance $d(M, N)$ between them is the infimum of all positive numbers $\in$ such that both

$$M \subset N + (\in) \text{ and } N \subset M + (\in)$$

Since the Hausdorff metric is defined for compact sets, we use $\overline{W}$ and not W. In what sense is the numerical range a continuous function of its argument? This question however has many interpretations as there are topologies for operators.

Is $\overline{W}$ weakly continuous? Strongly? Uniformly?.

The only thing that is immediate obvious is that if $\overline{W}$ is continuous with respect to any topology, then so is W, and, consequently, if W is discontinuous, then so is $\overline{W}$.

Theorem 3.3.1 *The function $\overline{W}$ is continuous with respect to uniform (norm) topology; if the underlying Hilbert space is infinite-dimensional, then the function W is discontinuous with respect to the strong topology and hence with respect to the weak topology.*

Proof. If $\|S - T\| < \in$ and if x is a unit vector, then$|\langle (S - T)\, x, x\rangle| < \in$ and hence

$$\langle Sx, x\rangle = \langle Bx, x\rangle + \langle (S - T)\, x, x\rangle \in W\,(T) + (\in)$$

it follows that

$$W\,(S) \subset W\,(T) + (\in)$$

symmetrically,

$$W\,(T) \subset W\,(S) + (\in)$$

This proves the first assertion, as for the second assertion, consider the unilateral shift U, the sequence (U_n^*) tends to 0 in the strong topology more and more Fourier Coefficients get lost as n increases but

$$W\,(U^{*n}) = 1$$

for all n. ■

3.4 Bare points, Semi Bare points and the Corner of the Numerical range.

Definition 3.4.1 *For a closed subset F of $\mathbb{C}$ a point λ is called a **bare point** if*

1) $\lambda \in F$ and
2) λ is on the boundary of a circular disc containing F

Definition 3.4.2 *A point λ of a closed set F in $\mathbb{C}$is called a **semi-bare** point if*

1) λ is on the boundary of a circular disc C
2) The circular disc C contains no there points of F

Definition 3.4.3 *If C is a closed convex subset of the complex plane, then C has a **corner** with vertex λ if*

1) $\lambda \in C$
2) if C is contained in an angle with vertex at λ and magnitude less than π radians

Lemma 3.4.4 *Let $T \in B(T)$ and*

(i) $1 \in W(T)$
(ii) $w(T) = 1$
Then

$$1 \in P\sigma(\mathrm{Re}(T))$$

Proof.

$$W(ReT) = \left\{ \left\langle \frac{1}{2}(T+T^*)x, x \right\rangle : \|x\| = 1 \right\}$$

$$= W(ReT) = \left\{ \frac{1}{2}\langle Tx, x\rangle + \frac{1}{2}\langle T^*x, x\rangle : \|x\| = 1 \right\}$$

$$= \left\{ \frac{1}{2}\langle Tx, x\rangle + \frac{1}{2}\overline{\langle x, Tx\rangle} : \|x\| = 1 \right\}$$

$$= \{\mathrm{Re}\langle Tx, x\rangle : \|x\| = 1\} = \mathrm{Re}(W(T))$$

■

i.e.

$$W(ReT) = \mathrm{Re}(W(T))$$

hence

$$1 \in W(T) \Longrightarrow 1 \in \mathrm{Re}\, W(T) = W(ReT)$$

Consequently,

$$1 = \langle ReTx, x\rangle$$

for some $x \in H$ with $\|x\| = 1$ this shows that

$$\|\mathrm{Re}\, Tx - x\|^2 = \|\mathrm{Re}\, Tx\|^2 - \langle \mathrm{Re}\, Tx, x\rangle - \langle x, \mathrm{Re}\, Tx\rangle + \|x\|^2$$

$$= \|\mathrm{Re}\, Tx\|^2 - 1 - 1 + 1$$

$$= \|\mathrm{Re}\, Tx\|^2 - 1$$

$$\leq \|\mathrm{Re}\, T\|^2 \|x\|^2 - 1 = \|\mathrm{Re}\, T\|^2 - 1.................. (2.20)$$

since

$$w(T) = 1sup\{|< Ty, y >: y \in H \text{ and } \|y\| = 1|\} = 1$$

hence

$$|\ \langle Ty, y\rangle| \leq 1$$

for all $y \in H$ satisfying $\|y\| = 1$ therefore,

$$|\ \mathrm{Re}\,\langle Ty, y\rangle| \leq 1$$

for all $y \in H$ satisfying

$$\|y\| = 1.$$

Since

$$\langle(\mathrm{Re}\, T)\, y, y\rangle = \mathrm{Re}\,\langle Ty, y\rangle$$

we get $w\,(\mathrm{Re}\, T) = w\,(T)$ hence

$$w\,(T) = 1 \Longrightarrow w(\mathrm{Re}\, T) = 1$$

Since ReT is a self-adjoint element in $B(H)$,

$$\|\operatorname{Re} T\| = \sup\{\langle(\operatorname{Re} T)\, y, y\rangle : y \in H \text{ and } \|y\| = 1\}$$

$$= w(\operatorname{Re} T) = 1$$

Substituting in equation (2.20), we get

$$\|\operatorname{Re} Tx - x\| = 0$$

that is,

$$ReTx = x$$

and this proves that 1 is an eigenvalue of ReT,

i.e.

$$1 \in P\sigma(\operatorname{Re} T).$$

Theorem 3.4.5 *If $T \in B(H)$ and $\lambda \in \overline{W(T)}$ is a bare point of $\overline{W(T)}$, then*

$$\left(e^{-i\phi}T + e^{i\phi}T^*\right)x = \left(e^{-i\phi}\lambda + e^{i\phi}\overline{\lambda}\right)x$$

for some $x \in H, \|x\| = 1$ and $0 \leq \phi \leq 2\pi$

Proof. Since λ is a bare point of $\overline{W(T)}$, we can find an $r > 0$ such that

$$W(T) \subset \{z : |z - \lambda_\circ| \leq r\}$$

for some $\lambda_\circ$ and

$$\lambda \in \overline{W(T)} \cap |z - \lambda_\circ| = r$$

Let$\lambda - \lambda_\circ = re^{i\phi}$with $0 \leq \phi \leq 2\pi$ and

$$T_1 = r^{-1}e^{-i\phi}(T - \lambda_\circ I)$$

In this case, we see that $W(T_1)$ is contained in the closed unit disc,and if $x \in H$,$\|x\| = 1$and$\lambda = \langle Tx, x\rangle$, we obtain

$$1 = \langle Tx, x\rangle \in W(T_1)$$

and by lemma 10

$$\frac{1}{2}(T_1 + T_1^*)\, x = x$$

i.e. $ReTx = x$ from the form of T_1 it follows that

$$\frac{1}{2}[r^{-1}e^{-i\phi}(T-\lambda_\circ I)+r^{-1}e^{i\phi}(T-\lambda_\circ I)^*]x = x$$

■

and

$$\frac{1}{2}[e^{-i\phi}T+e^{i\phi}T^*]x = \frac{1}{2}[e^{-i\phi}\lambda x+e^{i\phi}\overline{\lambda}x]x$$

and the theorem is proved. An interesting result was proved by D.F Donoghue Jr.(1957)

Theorem 3.4.6 *If $T \in B(T)$ and $\lambda \in W(T)$ and is a vertex of a corner of $\overline{W(T)}$, then λ is an eigenvalue of T. if λ is just a vertex of the corner of $W(T)$, then $\lambda \in \sigma(T)$*

Proof. By theorem $1(i)$ we may suppose that $\lambda = 0$ and that there exists a $\delta > 0$ such that

$$Re(W(e^{i\phi})) \subset \mathbb{R}^+ \ (-\delta \le \phi \le \delta) \ldots\ldots\ldots\ldots (2.21)$$

■

since

$$\langle Sx, x\rangle = 2Re\langle Tx, x\rangle$$

It follows that from 2.21 that $W(S) \subset \mathbb{R}^+$and hence $S \ge 0$ since $0 \in \overline{W(T)}$, there exists a sequence (x_n) of unit vectors such that

$$lim_{n\to\infty}\langle Tx_n, x_n\rangle = 0$$

by lemma 1 (see (2.21)), we have

$$\|Sx_n\|^2 \le w(S)\langle Sx_n, x_n\rangle$$

and hence $s - limSx_n = \overline{0}$, that is

$$s - lim\,(T+T^*)\,x_n = \overline{0}$$

by lemma (2.21), we may replace T by $e^{i\phi}T$ with $-\delta \le \phi \le \delta$ and so,

$$s - \lim_{n\to\infty}\left(e^{i\phi}T + e^{-i\phi}\,T^*\right)x_n = 0$$

for all such ϕ therefore $s - \lim_{n\to\infty} Tx_n = 0$ thus

$$0 \in \pi(T) \subset \sigma(T)$$

if also $0 \in W(T)$, the sequence (x_n) is replaced by x with $\|x\| = 1$ and $\langle Tx, x\rangle = 0$. Then we get

$$sx = \overline{0}$$

and hence $Tx = \overline{0}$ therefore this shows that

$$0 \in P\sigma(T)$$

alternatively, if λ is the vertex of a corner of $\overline{W(T)}$, then λ is a bare point of this set, hence we can find positive numbers r_1 and r_2 and two complex numbers $a_\circ$ and $b_\circ$ such that

$$a_\circ \neq b_\circ$$

and

$$W(T) \subset \{z : |z - a_\circ| \leq r_1\} = Dr_1$$

$$W(T) \subset \{z : |z - b_\circ| \leq r_2\} = Dr_2$$

$\lambda \in \overline{W(T)} \cap Dr_1 \cap Dr_2$ as in the proof of theorem 15, we find ϕ_1 and ϕ_2 such that ϕ_1, $\phi_2 \in (0, 2\pi)$,

$$|\phi_1 - \phi_2| < \pi\frac{1}{2}[e^{-i\phi j}T + e^{i\phi j}T^*] = \frac{1}{2}[e^{-i\phi j}\lambda + e^{i\phi j}\overline{\lambda}]x \;\; (j = 1, 2)$$

which gives

$$\frac{1}{2}[e^{-2i\phi_1} - e^{-2i\phi_2}]Tx = \frac{1}{2}[e^{-2i\phi_1} - e^{-2i\phi_2}]\lambda x$$

and since$\phi_1 \neq \phi_2$, it follows that $Tx = \lambda x$.

Theorem 3.4.7 *If $T \in B(H)$ is a compact operator and $0 \in W(T)$ is the vertex of a corner of $W(T)$, then $0 \in P\sigma(T)$*

Proof. Since $0 \in W(T)$ and T is compact, by theorem 13 we see that it was shown that if $T \in B(T)$ and is a compact operator and $0 \in W(T)$, then $W(T)$ is a closed set. So our $W(T)$ here is closed, theorem 17 showed that if $\lambda \in W(T)$ and is a vertex of a corner of $W(T)$, then λ is an eigenvalue. Thus in this case our λ is 0 hence 0 is an eigenvalue of T

$$\Longrightarrow 0 \in P\sigma(T).$$

■

Corollary 3.4.8 *Let $T \in B(H)$ and $W(T)$ be a closed polygonal region, then*

$$conv\sigma(T) = W(T)$$

Proof. If $\{\lambda_i\}$ are the vertices of the convex polygonal region, the λ_i are eigenvalues and thus

$$conv\sigma(T) \supset W(T) = \overline{W(T)}$$

■

since the reverse inclusion is true for all operators, the corollary follows. In the next corollary, we give a necessary and sufficient condition for a closed polygonal region to be the numerical range of a bounded linear operator on finite dimensional space.

Corollary 3.4.9 *If C is a closed polygonal region with M vertices, then C is the numerical range of an operator on a finite dimensional Hilbert space H of dimension n if and only if $M \leq n$.*

Proof. If C is the numerical range which is a closed polygonal region with vertices $\lambda_1, ..., \lambda_m$, then from theorem 16, it was shown that if $\lambda \in W(T)$ and is a vertex of the corner of $\overline{W(T)}$, then λ is an eigenvalue of T. This implies that in our case, each λ_i is an eigenvalue for T and this clearly implies

$$M \leq n$$

for T can have at most n distinct eigenvalues. Conversely, if $\lambda_1, ..., \lambda_m$ are the vertices of a polygonal region C, then the normal operator with the $n \times n$ matrix $[a_{ij}]$, where

$$a_{ij} = \left\{ \begin{array}{l} \lambda_i \sigma_{ij}, 1 \leq i \leq M \\ 0 \quad\ ,M < i \leq n \end{array} \right\}$$

has the property that $W(T) = C$.Here δ_{ij} is the Kronecker delta. ■

We can obtain an extension of these results using the notion of semi-bare point. It is easy to see that every barepoint is a semibare point.

We extend theorem 15 to the case of a semibare.

Theorem 3.4.10 *If $T \in B(T)$ and $\lambda \in W(T)$ is a semibare point of $W(T)$, then*

$$[e^{-i\phi}T + e^{i\phi}T^*]x = [e^{-i\phi}\lambda + e^{i\phi}\lambda]x$$

for some $x \in H$ and $\phi \in [0, 2\pi]$.

Proof. Since λ is a semibare point of $W(T)$, there exists $r > 0$ and $a_\circ \in \mathbb{C}$ such that

$$dist\,(a_\circ, W(T)) = r$$

■

and

$$W(T) \cap \{z\cap : |z - a_\circ| = r\} = \{\lambda\}$$

Let $\lambda - a_\circ = re^{i\phi}$ and consider the operator T

$$_1 = r^{-1}e^{-i\phi}(T - a_\circ I)$$

In this case

$$W(T_1) = \left\{r^{-1}e^{-i\phi}[\langle Tx, x\rangle - a_\circ] : \|x\| = 1\right\}$$

which shows that $W(T_1)$is contained in the unit disc. also, for

$$\lambda = \langle Tx, x\rangle, 1 \in W(T_1)$$

and lemma 10 implies that.

$$\in \operatorname{Re} T_1 x = x$$

as in theorem 15,we obtain our assertion.

using this result we can generalize the result obtained in theorem 16.

Theorem 3.4.11 *Let $T \in B(H)$ and $\lambda \in W(T)$ be a semibare point of $W(T)$, then λ is an eigenvalue of T*

Proof. Since λ is a semibare point of $W(T)$, we can find positive numbers r_1 and r_2 and two complex numbers $a_\circ$ and $b_\circ$ such that

$$W(T) \cap \{z : |z - a_\circ| < r_1\} = \{\lambda\}$$

■

$$W(T) \cap \{z : |z - b_\circ| < r_2\} = \{\lambda\}$$

we can find $\phi_i \in [0, 2\pi]$ such that

$$\frac{1}{2}[e^{-i\phi j}T + e^{i\phi j}T^*]x = \frac{1}{2}[e^{-i\phi j}\lambda + e^{i\phi j}\overline{\lambda}]x, (j = 1, 2)$$

and since $\phi_1 \neq \phi_2$, we obtain

$$Tx = \lambda x. \Longrightarrow \lambda$$

is an eigenvalue of T.

Chapter 4

Conclusion and summary

The conclusions made about the Numerical range are arrived at from the illustrations and the characterization of the research carried out. In the first place the main goal was to study the characterization of the numerical range of bounded linear operators. The closure of closure of the numerical range a part from including the spectrum of the operator turns out to be a complex subset of the complex plane. This is a result that was done by Toeplitz Hausdorff.

Theorem 4.0.12 ***Toeplitz-Hausdorff:*** *The numerical range $W(T)$ of an operator T is a convex set in the complex plane.*

This shows that apart from including the spectrum of the operator, the closure of the numerical range turns out to be a convex subset of the complex plane.

Now, for $T \in B(H)$ the spectrum $\sigma(T)$ is compact subset of $\mathbb{C}$, a non trivial fact of finite dimension Euclidian geometry is that the convex hull of a compact set is closed. The most useful formulation of this fact for the plane $\mathbb{C}$ is that the convex hull of a compact set is the intersection of all the closed half-planes that include it.

Also the inclusion of the origin is a necessary and sufficient condition for the numerical range to be closed. The numerical range $W(T)$ of an operator T is not generally closed. In the finite dimensional case, however, the numerical range of an operator is a continuos image of a compact set. The unit sphere

$$S(H) = \{x : \|x\| = 1\}$$

thus it is necessarily a compact subset of $\mathbb{C}$ and is therefor closed. The theorem used to arrive at is:

Theorem 4.0.13 *If $T \in B(H)$ is a compact operator and $0 \in W(T)$, then $W(T)$ is a closed set.*

It follows from the theorem that if $\lambda \in \overline{W(T)}$ such that $|\lambda| = \|T\|$ and if λ is not an eigenvalue of T, and in particular if T has no eigen values then λ does not belong to $W(T)$. In view of this remark it easy to construct examples of operators whose numerical range is **not** closed.

It follows that if a normal operator has sufficiently many eigenvalues to approximate its norm, but does not have one whose modulus is as large as the norm, then its numerical range will not be closed. A concrete example is provided by a diagonal operator such that the modulus of the diagonal term does not attain its supremum.

Now if $x_\circ = 0$, then $x = \overline{0}$, otherwise, a neccesary and sufficient condition that the resulting $X_n's$ be the coordinates of a vector that is, they may be square summable and hence $x \in L^2$ is $|\lambda| < 1$. So each λ in the open disc of $\mathbb{C}$ is a simple eigenvalue of U^*, that is each eigenvalue is of multiplicity 1. It therefore follows that the open unit disc $\{\lambda : |\lambda| < 1\}$ is contained in $W(U^*)$. Since $W(U^*)$ is always

$$W(U^*)(= \overline{\lambda} : \lambda \in W(U)$$

It follows that the open unit disc is contained in $W(U)$.

Since U has no eigenvalues, implies that $W(U)$ cannot have. It is contains any number of modulus 1. Hence $W(U)$ equals the open unit disc, we note that the number 0 plays a special role with respect to the spectrum of a compact operator. It is remarkable that it also plays a special role in regard to the numerical range of a compact operator, that is why it gives a sufficient condition for an operator to have a closed numerical range.

One of the other major task is to show that the numerical range of an operator on a 2-dimensional complex plane is, in general an ellipse. Besides giving a set inside which the point spectrum must lie, the numerical range can be used to prove that certain points are eigenvalues of the operator in the context. Hence we proved a useful general lemma about $W(T)$ where T is any two-by-two matrix.

Lemma 4.0.14 *Let T be a linear operator on a two-dimensional Hilbert space H_2. If the matrix of T (which is obviously a 2×2 matrix. Has distinct eigenvalues λ_1 and λ_2 and the corresponding eigenvectors x_1 and x_2 so normalized that*

$$\|x_1\| = \|x_2\| = 1$$

Then $W(T)$ is a closed elliptical disc with foci at $\lambda_1$and λ_2, if $r = |\langle x_1, x_2 \rangle|$ and $\delta = \sqrt{1 - r^2}$, then the minor axis is $\frac{r|\lambda_1 - \lambda_2|}{\delta}$ and major axis is $\frac{|\lambda_1 - \lambda_2|}{\delta}$

If T has only one eigenvalue λ, then $W(T)$ is the (circular) disc with center λ and radius.

$$\frac{1}{2} \|T - \lambda I\| .$$

It is well known that the boundary $w(T)$ being a convex function, is differentiable except perhaps at an utmost countable set of points. Let λ be a point of non-differentiability and x, $\|x\| = 1$, such that

$$\lambda = \langle Tx, x \rangle$$

also, at λ there exists a left and right tangents such that the angle between these tangents is smaller than π. Let Y be arbitrary in H and $P_{x,y}$ be the orthogonal projector on H onto the linear subspace $[\{x, y\}]$. The operator

$$T_1 = P_{x,y} T P_{x,y}$$

has a closed elliptical disc as its numerical range,and since no circle contained in $w(T)$ can pass through λ, it follows that the ellipse $w(T_1)$ is a line segment or a point, thus λ is an eigenvalue with x as an eigenvector. This is actually a proof to the result:

Theorem 4.0.15 *Let $T \in B(H)$ and $W(T)$ be a closed set, every point λ in the boundary of $w(T)$ at which the boundary is not a differential arc is an eigenvalue for T.*

The numerical range of an operator T can be regarded as the one-dimensional course of a multi-dimensional concept. To see how that goes we see that an orthogonal projector P of rank 1 can be expressed in terms of a unit vector x in its range:

$$Py = \langle y, x \rangle x$$

for all $y \in H$, $x \in R_p$ and $\|x\| = 1$. If $T \in B(H)$, then PTP is an operator of rank 1, and therefore a finite-dimensional concept such as trace makes sense for it. The trace of PTP can be computed by finding the one-by-one matrix of the restriction of PTP to the range of P, with respect to the one-element basis $\{x\}$, since $px = x$, the value of that trace is

$$\langle PTPx, x\rangle = \langle TPx, Px\rangle = \langle Tx, x\rangle$$

These remarks can be summarized as follows: $W(T)$ is equal to the set of all complex numbers of the form $trPTP$, where P varies over all projections of rank 1, replace 1 by an arbitrary positive integer k, and obtain the $k-$numerical range of T, in symbols $W_k(T)$; the ordinary numerical range $W(T)$ is the $k-$numerical range with $k = 1$

For each set M of complex numbers and each positive number $\in$ .If M and N are compact sets, the Hausdorff distance $d(M, N)$ between them is the infimum of all positive numbers $\in$ such that both

$$M \subset N + (\in) \text{ and } N \subset M + (\in)$$

Since the Hausdorff metric is defined for compact sets, we use $\overline{W}$ and not W. In what sense is the numerical range a continuous function of its argument? This question however has many interpretations as there are topologies for operators.

Is $\overline{W}$ weakly continuous? Strongly? Uniformly?.

The only thing that is immediate obvious is that if $\overline{W}$ is continuous with respect to any topology, then so is W, and, consequently, if W is discontinuous, then so is $\overline{W}$.

Theorem 4.0.16 *The function $\overline{W}$ is continuous with respect to uniform (norm) topology; if the underlying Hilbert space is infinite-dimensional, then the function W is discontinuous with respect to the strong topology and hence with respect to the weak topology.*

Lastly it is shown that some "special" points on the boundary of the numerical range turn out to be eigenvalues. This is realized by the result that:

Theorem 4.0.17 *Let $T \in B(H)$ and $\lambda \in W(T)$ be a semibare point of $W(T)$, then λ is an eigenvalue of T*

So all our goals and objectives of the study are achieved and even some more interesting questions came up which opens up this area for further research work in operator theory. So this area generally is rich with many sections of further research work to be done.

Bibliography

[1] Apostol C., et al, Topics in mordern operator theory, London mathematical society, 1998

[2] Carlos S. Kubrusly, Elements of Operator Theory, Birkhauser, Basel, Boston, 2001

[3] Carlos S. Kubrusly, Hilbert Space Operators: A Problem Solving Approach, Birkhauser, Basel, Boston, 2003

[4] Carlos S. Kubrusly, Spectral theory of operators on Hilbert spaces, Birkhauser, Basel, Boston, 2002

[5] Dan Amir, Characterization of inner product spaces: Operator theory, London mathematical society, 2001

[6] Erwin Kreyzig, Introductory Functional Analysis With Applications, New York, 1978

[7] F.F Bonsall, J. Duncan, Numerical ranges of operators on Normed spaces and of elements of Algebras, London mathematical Society, 2004

[8] G. Bachman and L. Narici, Functional Analysis, Academic Press, New york, 2000

[9] G. De Barra, J.R. Giles, and B. Sims, on the numerical range of compact operators on hilbert spaces, J. London Math. Soc (2), 5(1972), 704-706.

[10] Joachim Weidmann, Linear operators in Hilbert space,Springer-Verlag, New York,1980

[11] Juss behrndt, Karl-Heinz Forster, Heint Langer, Carsten Trunk, Spectral theory in inner product spaces and applications: 6th Berlin workshop on operator theory in krein spaces

[12] Julian Lopez-Gomez, Carlos mora-corral, Agebriac multiplicity of eigenvalues of linear operators, London mathematical society, 2002

[13] Joseph A. Ball, Yuli Eidelman, J. William Helton, Vadim Olshesky, James Rounyak, Recent advances in matrix and operator theory, London mathematical society, 2001

[14] Karl-Heinz forster, Peter Jonas, Heinz Langer, Operator theory in Krein spaces and nonlinear Eigenvalue problems, London mathematical society, 2000

[15] Kurt O. Friedrichs, Spectral theory of operators in Hilbert spaces, Springer-Verlag, New York, 1996

[16] Karl -Heinz forster, Peter Jonas, Heinz Langer, Carsten Trunk, Operator theory in inner product spaces, London mathematical society, 2001

[17] N.I. Akhiezer and I.M. Glazman, Theory of linear operators in Hilbert spaces ,Dover publications , New York,1993

[18] P.R. Halmos, Hilbert Space Problem Book, 2nd edition, Springer-Verlag, New York, 1982

[19] Serge Lang, Real and Functional analysis, Third Edition, Springer-Verlag, New York, 1997

[20] Takayuki Furuta,Invitation To Linear Operators,From matrices to bounded linear operators on a Hilbert space, New york, 2002

[21] Walter Rudin, Functional analysis, Second Edition, McGraw Hill,New york, 2000

[22] Y. A. Abramovich, Charambos, D. Aliprantis, An invitation to opertaor theory, Graduate studuies in mathematics V. 50, London mathematical society, 2002