ON SOME TRANSFORMS OF LINEAR OPERATORS IN A HILBERT SPACE.

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BY

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DECLARATION.

This dissertation is my original work and has not been presented for a degree award in any University.

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ABSTRACT.

Let B (H) be algebra of all bounded linear operator in a complex separable Hilbert space. For an operator $T \in B(H)$, let $\Delta(T) = |T|^{1/2} U |T|^{1/2}$, $\Gamma(T) = |T| U$, $C(T) = (T - iI)(T + iI)^{-1}$ be the Aluthge transform, Duggal transform and Cayley transform respectively with U being a partial isometry, $T = U|T|$ is the polar decomposition where $|T| = (T^*T)^{1/2}$ and U is the identity element.

For the Aluthge transform, we look at some properties on the range $R(\Delta) = \{\Delta(T) : T \in B(H)\}$, of Δ and we prove that $R(\Delta)$ neither closed nor dense in $B(H)$. We shall also discuss the properties of the spectrum and numerical range of Aluthge transform and their relationships and extend its iterated convergences of the Aluthge transform.

For the Duggal transform, we shall obtain the results about the polar decomposition of Duggal transform by giving the necessary and sufficient condition for the Duggal transform of T to have the polar decomposition for binormal operators and examine some complete contractivity of maps associated with the Duggal transform by exploring some relations between the operator T, the Aluthge transform of T and the Duggal transform of T by studying maps between the Riesz-Dunford algebras associated with the operators.

Finally under the Cayley transform, we define the Cayley transform of a linear relation directly by algebraic formula, its normal extension and the Quaternionic Cayley transform for bounded and unbounded operators and their inverses.

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DEDICATION.

This project is dedicated to my brother, in particular Dennis Murutu and my sister Valentine Murutu for both financial and moral support that has made me complete my graduate studies.

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Chapter 1

PRELIMINARIES

1.1 Introduction

The study of transforms originated from the study of the spectral theory which is one of the branches of Functional analysis which deals with classifying linear operators. Functional analysis has its origin in the theory of ordinary and partial differentiation which was used to solve several physical problems which included the work of Joseph Fourier (1768 – 1830) on the theory of heat in which he wrote differential equations as integral equations. His work triggered not only the development of trigonometric series which required mathematicians to consider what is a function and the meaning of convergence but also the Lebesque integral which could accommodate broader functions compared to the classical Riemannian integral and also development of other transforms like Aluthge transform by A. Aluthge (1900); Duggal transform by Prof.B.P Duggal; Cayley transform; Fourier transform and others. In the process, spectral theory was developed which is the central concept of Functional analysis. As functional analysis evolved in the 20^{th} century, it started to form a discipline of its own via integral equations which was first studied by the Swedish astronomer and mathematician called Fredholm in a series of papers in the year 1900 – 1903 in which he developed a theory of "determinant" for integral equations of the form $f(x) - \lambda \int_b^a K(x,y)f(y)dy = g(x)$ where $K(x,y)$ is the kernel function. His work got a lot of attention from many mathematicians all over the world where David Hilbert (1836 – 1943) in the year 1904 – 1906, published six papers on integral equation, in which he started transforming the integral equations to a finite system of equations under the restriction that the kernel function is symmetric. In the process of his study, he classified operators in terms of their spectral properties on a Hilbert space which refers to an infinite dimensional complete normed linear space which has an additional structure called an inner product. The inner product generalizes the scalar product of elementary Cartesian vector analysis. To begin with, let H be a complex Hilbert space, T to be an operator in a Hilbert space H and $B(H)$ be the algebra of all bounded linear operator. If $T = U \mid T \mid$ is the polar decomposition of a bounded linear operator on a Hilbert space H, then we have the following transformations in the respective chapters as follows :

In **chapter 1,** we start by the preliminaries by giving some definitions, notations and some examples.

In **Chapter 2,** we explore the Aluthge transformation by giving its basic definitions and its related properties. We look at the range $R(\Delta) = \{\widetilde{T} : T \in B(H)\}$ of Δ and prove that $R(\Delta)$ is neither closed nor dense in $B(H)$. Also we look at the spectrum and numerical range of Aluthge transform and their related properties. And also, we examine the convergence of iterated Aluthge transform and its Jordan structures.

In **Chapter 3,** we shall look at the polar decomposition of Duggal transformation, binormal operators of $\widehat{T}$ and the complete contractivity of maps associated to the Duggal and Aluthge transforms.

In **Chapter 4,** we study the Cayley transformation in the linear relations, its extension and Quaternionic Cayley transform. Finally, we shall give some conclusion in suggested research topics that arose during our study.

1.2 Notations and Terminologies.

Definition 1.2.1 *Let H be a vector space over a field of complex numbers $\mathbb{C}$. A mapping $<,>: H \times H \to K$ (where $K \in \mathbb{R}$ or $K \in \mathbb{C}$) which associates with every ordered pair $(x, y) \in H \times H$, a scalar denoted by $< x, y >$ is called an inner product on $H \times H$ if it satisfies the following properties :*

(*i*) $< x, x > \geq 0$ for all $x \in H$.

(*ii*) $< x, x >= 0$, if $x = \overline{0}$, for all $x \in H$.

(*iii*) $< x, y >= \overline{< y, x >}$ for all $x, y \in H$ (the bar denotes the complex conjugate).

(*iv*) $< \alpha x, y >= \alpha < x, y >$ for all $x, y \in H$, and $\alpha \in \mathbb{C}$.

(*v*) $< x + y, z >=< x, z > + < y, z >$for all $x, y, z \in H$.

Thus the vector space H together with the inner product function $<,>$ is called an inner product space over K. If $K = \mathbb{R}$, the inner product space is called a real inner product space and if $K = \mathbb{C}$, then the inner product space is called the complex inner product space. A space equipped with the inner product space is known as a pre-Hilbert space or inner product space.

Definition 1.2.2 *Let H_1 and H_2 be Hilbert spaces over a complex plane $\mathbb{C}$. A linear transformation $f : H_1 \to H_2$ is :*

(a) Injective (or one-to-one) if for all $x_1, x_2 \in H_1$,$x_1 \neq x_2 \Longrightarrow f(x_1) \neq f(x_2)$.

(*b*) Surjective (onto) if for all $y \in H_2$, $y = f(x)$ for all $x \in H_1$.

Thus a mapping that is both injective and surjective is said to be bijective.

Definition 1.2.3 *An operator $T \in B(H)$ is said to be :*

Involution if $T^2 = I$.
Unitary if $T^*T = TT^* = I$ i.e. $T^* = T^{-1}$.
Normal if $T^*T = TT^*$.
Self-adjoint if $T^* = T$.
Idempotent if $T^2 = T$.
Isometry if $T^*T = I$.
A co-isometry if $TT^* = I$.
A partial Isometry if $T = TT^*I$.
Binormal if T^*T and TT^* commutes.
Hyponormal if $T^*T \geq TT^*$.
$P-$Hyponormal if $(T^*T)^p \geq (TT^*)^p$ where $0 < p \leq 1$.
Semi-Hyponormal if $(T^*T)^{\frac{1}{2}} \geq (TT^*)^{\frac{1}{2}}$.
$Log-$Hyponormal if T is invertible and $logT^*T \geq logTT^*$.
Invertible if there exist an operator S such that $ST = TS = I$ (where I is identity).
(r,t) Weakly$-$Hyponormal if $| \check{T}_{r,t} | \geq | T | \geq | \check{T}_{r,t} |$.
Complex symmetric if there exist a conjugate- linear involution.
Quasiaffinity if T is one-to-one and has dense range.
Fredholm if the null space of T and T^* are finite dimensional and the range of T is closed.
Nilpotent if $T^n = 0$ for some positive integer n.
Quasinilpotent if $\sigma(T) = \{0\}$.
Centered if the following sequence , $.T^2(T^2)^*, TT^*, T^*T$, $(T^2)^*T^2, ..$is commutative.
Contraction if $\| T \| \leq 1$.

Definition 1.2.4 *Let H_1 and H_2 be Hilbert spaces over a complex plane $\mathbb{C}$. A function T which maps H_1 into H_2, i.e. $T : H_1 \rightarrow H_2$ is called a linear operator if for all $x, y \in H_1$ and $\alpha \in \mathbb{C}$, $T(x + y) = T(x) + T(y)$ and $T(\alpha x) = \alpha(T(x))$.*

Definition 1.2.5 *Let H_1 and H_2 be Hilbert spaces over a complex plane $\mathbb{C}$. A function T which maps H_1 into H_2 i.e. $T : H_1 \rightarrow H_2$ is called bounded if $Sup_{\|x\|=1} \| Tx \| < \infty$ and the norm of T written as $\| T \|$ is given as $\| T \| = Sup_{\|x\|=1} \| Tx \|$. Thus a bounded operator is a bounded linear transformation of a non-zero complex Hilbert space into itself.*

Definition 1.2.6 *Let $T \in B(H)$, then we have the following :*

(i) $KerT = N(T) = \{x \in H : Tx = 0\}$, the Kernel of T which is a subspace of H containing all the elements that have been mapped to the identity by the operator T (Null space).

(ii) $R(T)$ is the range or image of T.

Proposition 1.2.7 *Let H_1 and H_2 be Hilbert spaces over a complex plane $\mathbb{C}$. Let $T : H_1 \to H_2$ be a non-zero linear operator. Then the following are equivalent :*

(i) Range of T i.e. (Ran (T)) is a closed subspace of H.
(ii) T is a bounded linear operator.
(iii) Kernel of T (i.e. $N(T)$) is a closed subspace of H.

Lemma 1.2.8 *Let T be an operator such that for all $x \in H, \| Tx \| \geq c \| x \|$, where c is a positive constant. Then $Ran(T)$ is closed.*

Definition 1.2.9 *If $T \in B(H)$ then its adjoint T^* is the unique operator in $B(H)$ that satisfies $< Tx, y > = < x, T^*y > \forall\ x, y \in H$. An operator $T \in B(H)$ is called self-adjoint if $T^* = T$.*

Theorem 1.2.10 *Let $T \in B(H)$, then the following results hold :*

(i) $Ran(T)$ is closed iff $Ran(T^*T)$ is closed.
(ii) T^*T is a positive self-adjoint operator.
(iii) $\overline{Ran(T^*)} = N(T)^{\perp} = N(T^*T)^{\perp} = \overline{Ran(T^*T)}$.
(iv) $N(T$ and $N(T))^{\perp}$ is invariant under T^*T.

Definition 1.2.11 *Let H be a Hilbert space and $X \subset H$. Let R be a relation in X. Then R is said to be :*

(i) Reflective if $xRx, \forall\ x \in X$.
(ii) Symmetric if $xRy \Longrightarrow yRx, \forall\ x, y \in X$.
(iii) Antisymmetric if xRy,and $yRx \Longrightarrow x = y, x, y \in X$.
(iv) Transitive if xRy and $yRz \Longrightarrow xRz, \forall\ x, y, z \in X$.

If R is reflective, symmetric and transitive, then R is called an equivalence relation on X and if R is reflective, antisymmetric and transitive, then R is called a partial order on X.

Sometimes properties of an operator $T \in B(H)$ can be determined rather easily by considering simpler operators which are restrictions of T to certain subspaces of H, known as Invariant spaces.

Definition 1.2.12 *Let $T \in B(H)$, a subspace M of H is invariant under T (or $T-$ invariant) if $T(M) \subseteq M$ i.e. if $x \in M \Longrightarrow Tx \in M$ where*

$T(M) = \{Tx : x \in M\}$.

A subspace M of H reduces an operator T if it is invariant under both T and T^* (or if both M and $M^\perp$ are invariant under T). Thus every reducing subspace is an invariant subspace i.e. Reducing subspace $\subsetneq$ Invariant subspace

If M is invariant under T, then relative to the decomposition $H = M \oplus M^\perp$, T can be written as $T = \begin{pmatrix} T\mid_M & X \\ 0 & Y \end{pmatrix}$, for the operator $X \longrightarrow M^\perp \to M$ and $Y \longrightarrow M^\perp \to M^\perp$ where $T\mid_M: M \to M$ is a restriction of T to M and $X = 0$ iff M reduces T.

Definition 1.2.13 *A part of an operator is a restriction of it to an invariant subspace. Also a direct summand of an operator is restriction of it to a reducing subspace.*

Definition 1.2.14 *An operator $T \in B(H)$ is reducible if it has a non-trivial reducible subspace (equivalently, it has a proper non-zero direct summand), otherwise it is said to be irreducible.*

Example 1.2.15 *Let $H = \mathbb{C}^2$ and $T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$, then M which is a subspace of H is a reducing subspace.*

Proof. *Let $M = \left\{\begin{pmatrix} x \\ y \end{pmatrix} : x, y \in \mathbb{C}^2\right\}$, then $TM = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$. Let the basis for $M = \left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\} = \overline{e_1}\ \overline{e_2}$ be the orthonormal basis in $\mathbb{C}^2$. Then $T\ \overline{e_1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \subseteq \overline{e_1}$, $T\ \overline{e_2} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \subseteq \overline{e_2}$. Let $M_1 = span\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\}$ and $M_2 = span\left\{\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\}$ are invariant subspaces of T. Since $T^* = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, by the same computation, M is also invariant subspace of T^*. Hence M is a reducing subspace of T.* ■

Theorem 1.2.16 *Let H be a Hilbert space and M be a closed linear subspace of H. Let $T \in B(H)$ and P be a projection of H onto M. The M is invariant under T iff $PTP = TP$.*

Theorem 1.2.17 ***Theorem 1.2.18*** *Let H be a Hilbert space and $T \in B(H)$. A closed linear subspace M of H is invariant under T iff $M^\perp$ is invariant under T^*.*

Remark 1.2.19 *We note that there is a subtle difference between invariance and reducibility. We can have operators for which many proper subspaces are invariant but for which the only subspaces which reduce are H*

and $\{\overline{0}\}$. Also the subspaces H and $\{\overline{0}\}$ are invariant for any operator on H and these are called the trivial invariant subspaces. And for some linear operators, there is no non-trivial invariant.

Remark 1.2.20 *An operator is completely non-unitary if the restriction to any non-zero reducing subspace is not unitary. In particular, T has no non-zero direct summand.*

Definition 1.2.21 *(**Orthogonal projection**) Let H be a Hilbert space and M be a subspace of H. Let $H = M \oplus M^{\perp}$, then the map $P_M : H \to M$ defined by $P_M x = x'$ where $x = x' + x'', x' \in M, x'' \in M^{\perp}$ is called an orthogonal projection of H onto M and has the property that $P_M^* = P_M$ (Self-adjoint) and $P_M^2 = P_M$ (Idempotent) and $Ker(P) \perp Ran(P)$*

Example 1.2.22 *Let $P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ be a 3−dimensional Hilbert space $\mathbb{R}^3$. Then P is an orthogonal projection of $\mathbb{R}^3$ onto $\mathbb{R}$.*

Theorem 1.2.23 *For any element $x, y \in H$, the following properties hold:*

(i) $|< x, y >| \leq \| x \| \| y \|$ (Schwartz Inequality) .

(ii) $\| x + y \| \leq \| x \| + \| y \|$ (Triangular inequality).

(iii) $\| x + y \|^2 + \| x - y \|^2 = 2(\| x \|^2 + \| y \|^2)$ where $x \perp y$ (Parallelogram law).

(iv) Let X be a normed linear space with the norm $\| \, . \, \|$. Then there is a unique inner product function $<,>$ on $X \times X$,such that $< x, x > = \| x \|^2$ iff $\| x+y \|^2 + \| x-y \|^2 = 2(\| x \|^2 + \| y \|^2)$ $\forall x, y \in X$ (Von Neumann-Jordan Theorem).

Proposition 1.2.24 *Every separable Hilbert space has an orthonormal basis. Recall; Let M denote any subset of H. Then the set of vectors orthogonal to M is denoted by $M^{\perp}$, meaning $x \in M, y \in M^{\perp} \implies < x, y > = 0$.*

Theorem 1.2.25 *(**Projection Theorem**) Let H be a Hilbert space and M be a subspace of H. Then $M^{\perp}$ is a closed subspace and $H = M \oplus M^{\perp}$.*

The other notations used are :

$T = U \mid T \mid$; Polar decomposition of T where U is unitary.

$\tilde{T} = \Delta\,(T) = \mid T \mid^{\frac{1}{2}} U \mid T \mid^{\frac{1}{2}}$; Aluthge transform of T.

$\{\Delta^n(T)\}_{n=0}^{\infty}$; The Aluthge sequence.

$\Delta_\lambda\,(T) = \mid T \mid^{1-\lambda} U \mid T \mid^{\lambda}$; Generalized Aluthge transforms of T.

$\widehat{T} = \Gamma(T) = PU$ where $P = \mid T \mid = (T^*T)^{\frac{1}{2}}$; Duggal transform of T (U is a partial isometry satisfying the $kerU = kerT$ and $kerU^* = kerT^*$).

$C(T) = (T - iI)(T + iI)^{-1} = 1 - 2i(T + iI)^{-1}$; The Cayley transformation of T.

$Conv(T)$; The convex hull of T.

$W(T) = \{< Tx, x >: \| x \| = 1, x \in H\}$; Numerical range of T.

$w(T) = Sup\{\mid \lambda \mid; \lambda \in W(T)\}$; Numerical radius of T.

$W_q(T) = \{< Tx, x >: x, y \in \mathbb{C}^n, \| x \| = \| y \| = 1, < x, y >= q\}$; $q-$numerical range of T.

$\sigma(T) = \{\lambda \in \mathbb{C} : \lambda I - T$ is not invertible$\}$; Spectrum of T.

$r(T) = Sup\ \{\mid \lambda \mid : \lambda \in \sigma(T)\}$; Spectral radius of T.

$Hol(\sigma(T))$; The algebra of all complex-valued functions which are analytic on some neighborhood of $\sigma(T)$.

$\sigma_{ap}(T) = \{\lambda \in \mathbb{C} : \lambda I - T$ is not bounded below$\}$; Approximate point spectrum of T.

$\sigma_{cp}(T) = \left\{\lambda \in \mathbb{C} : \overline{R(T - \lambda)} \subsetneq H\right\}$; Compressive spectrum of T.

$\varphi(T\ ) = \{\lambda \in \mathbb{C} : \lambda I - T$ is invertible$\}$; Resolvent set of T.

$\mathbb{C}$; Space of complex numbers.

$<, >$; Inner product function.

$H^2 = H \oplus H$; Direct sum decomposition of H.

$\| \cdot \|$; Norm.

$\mathbb{F}$; closed unit disk in $\mathbb{C}$.

$LatT$; Lattice (collection) of all invariant subspace of T.

$A_T = \{f(T) : f \in Hol(\sigma(T))\}$; Riesz-Dunford algebra.

$M_r(\mathbb{C})$; Algebra of complex $r \times r$ matrices.

$GL_r(\mathbb{C})$; Group of all invertible elements in $M_r(\mathbb{C})$.

$U(r)$; Group of unitary operators.

$S(D)$; Similarity orbit of some diagonal operator D.

$D = diag(d_1, d_2, ...d_n)$; Diagonal matrix.

$L(H)$; The set of all bounded and every where defined linear operator on H.

$L_r(H)$; The set of linear relations on H .

$LR(H)$; The set of closed elements of $L_r(H)$.(i.e. those that have a closed graph in $H \oplus H$)

Chapter 2

THE ALUTHGE TRANSFORM.

2.1 Basic properties of Aluthge transform.

Let H be a complex Hilbert space and $B(H)$ be algebra on all bounded linear operator on H. Then we introduce the definition of Aluthge transform as follows:

Definition 2.1.1 *[19] Let T be an operator in a Hilbert space H. Then $T = U \mid T \mid$ is the polar decomposition of T with $\mid T \mid= (T^*T)^{\frac{1}{2}}$ and U is a partial isometry with the initial space the closure of the range of $\mid T \mid$ and the final space the closure of the range T. Then the Aluthge transform $\tilde{T}$ of T is defined as $\tilde{T} =\mid T \mid^{\frac{1}{2}} U \mid T \mid^{\frac{1}{2}}$, where $\tilde{T}$ is independent of the choice of the partial isometry U in the polar decomposition of T. Also we define $\Delta(T) = \tilde{T}$ for all $T \in B(H)$. Moreover, for each non-negative integer n, the n^{th} Aluthge transform $\Delta^n(T)$ of T is defined as $\Delta^n(T) = \Delta(\Delta^{n-1}(T))$, $\Delta^0(T) = T$. Thus we call the operator sequence $\{\Delta^n(T)\}_{n=0}^{\infty}$ the Aluthge sequence.*

Then Δ is a map defined on $B(H)$ and therefore we look at some properties on the Range $R(\Delta) = \{\tilde{T} : T \in B(H)\}$ of Δ.We prove that $R(\Delta)$ is neither closed nor dense in $B(H)$. However, $R(\Delta)$is strongly dense if H is infinite dimensional.

Let $F(H)$ and $K(H)$ denote the ideals of all finite rank and compact operators in $B(H)$ respectively, if H is infinite dimensional. If we consider a finite dimensional case, then let $dimH = p$. We identify $B(H)$ with the set of all $p \times p$ matrices M_p. Let $d_1, d_2, d_3, ..., d_p$ be $p-$complex numbers, we define $diag(d_1, d_2, ..., d_p)$ to be the diagonal matrix with diagonal $\{d_1, d_2, .., d_p\}$. Let $T \in M_p$ with the polar decomposition $T = U \mid T \mid$. Assuming that U is unitary, for the positive matrix $\mid T \mid^{\frac{1}{2}} \in M_p$, there are diagonal matrix $D = diag(d_1, d_2, .., d_p)$ and unitary matrix V such that $d_1 \geq d_2 \geq ..... \geq d_p$ and $V \mid T \mid^{\frac{1}{2}} V^* = D$.Then we have that $VTV^* = DWD$ where $W = VUV^*$ is unitary. For a matrix X, we denote $rank(X)$, the rank of X.

Lemma 2.1.2 [19] *Let* $p = 2$ *and* $A = \begin{pmatrix} x & y \\ -y & 0 \end{pmatrix}$, *where* x *and* y *are any non-zero complex number. Then* $A \notin R(\Delta)$

Lemma 2.1.3 *Assume that* $p \geq 2$, *and* $A \in M_p$ *is an idempotent such that* $Rank(A) = p - 1$ *and* $R(A) + R(A^*) = H$. *Then* A *is not in the closure* $\overline{R(\Delta)}$ *of* $R(\Delta)$.

Theorem 2.1.4 [19] *Let* $H = \mathbb{C}^p$ *for* $p \geq 2$, *neither* $R(\Delta)$ *is neither closed nor dense in* $B(H)$.

Proof. Firstly, we show that $R(\Delta)$ is not closed.

Case 1; $p = 2$. Let $D_n = \begin{pmatrix} \sqrt{n} & 0 \\ 0 & \sqrt{\frac{1}{n}} \end{pmatrix}, U_n = \begin{pmatrix} \frac{1}{n} & \sqrt{1 - \frac{1}{n^2}} \\ -\sqrt{1 - \frac{1}{n^2}} & \frac{1}{n} \end{pmatrix}$,

then $A_n = D_n U_n D_n \in R(\Delta)$ and $lim_{n\to\infty} A_n = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} \notin R(\Delta)$, by *Lemma* 2.1.2.

Case 2; $p = 3$. Let $B = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$. Then $B \notin R(\Delta)$ by *Proposition* 1.12 in [12] put $P_n = \begin{pmatrix} \frac{1}{n} & 0 & 0 \\ 0 & n & 0 \\ 0 & 0 & \frac{1}{n} \end{pmatrix}$ a $V_n = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$. Then $B = P_n V_n P_n \in R(\Delta)$ and $lim_{n\to\infty} B_n = B$, i.e. $B \in \overline{R(\Delta)}$.

Case 3; $p > 3$, we have $B \oplus I_{p-3} \notin R(\Delta)$ by *Proposition* 1.12 in [12] .However,we have $B \oplus I_{p-3} \in \overline{R(\Delta)}$, by the proof as above. Thus by *Lemma* 2.1 we know that $R(\Delta)$is not dense in $B(H)$. ∎

For the infinite dimensional case, consider the following.

Lemma 2.1.5 *Let* H *be complex separable infinite dimensional Hilbert space. Then* $F(H) \subset R(\Delta)$ *but* $K(H) \subsetneq R(\Delta)$.

Theorem 2.1.6 [[19] *Theorem* 5] *Let* H *be a complex separable infinite dimensional Hilbert space. Then* $R(\Delta)$ *is neither closed nor dense in norm topology but strongly dense in* $B(H)$.

Proof. By *Lemma* 2.1.5, we know that $R(\Delta)$ is not closed but strongly dense in $B(H)$. On the other hand, let G_r be the set of all right invertible operator in $B(H)$. Then we know that G_r is non-empty open subset in

$B(H)$. We have that $G_r \cap R(\Delta) = \phi$ by *Proposition* 1.12 in [26], i.e. $R(\Delta)$is not dense in $B(H)$. ■

The following *Proposition* contains the properties of Aluthge transform which follows from its definition.

Proposition 2.1.7 [26] *Let* $T \in B(H)$.*Then :*

(*i*) $\Delta(cT) = c\Delta(T)$ for all $c \in \mathbb{C}$.
(*ii*) $\Delta(VTV^*) = V\Delta(T)V^*$ for some V being unitary operator.
(*iii*) If $T = T_1 + T_2$, then $\Delta(T) = \Delta(T_1) + \Delta(T_2)$.
(*iv*) $\| \Delta(T) \|_2 \leq \| T \|_2$.
(*v*) T and $\Delta(T)$ have the same characteristic polynomial in particular $\sigma(\Delta(T)) = \sigma(T)$.

2.2 Spectral picture of Aluthge transform.

Definition 2.2.1 *Let* T *be a bounded linear operator on a Hilbert space* H. *The spectrum of* T*, defined by* $\sigma(T)$ *is the set given by*

$\sigma(T) = \{\lambda \in \mathbb{C} : T - \lambda$ is not invertible$\}$. Alternatively, if we consider the set of all $\lambda \in \mathbb{C}$, such that $(T - \lambda)$ is invertible and is bounded in H, it constitutes the regular value of T called the resolvent set of T denoted by $\varphi(T) = \{\lambda \in \mathbb{C} : \lambda I - T$ is invertible$\}$.

Thus the spectrum is defined as the complement of $\varphi(T)$ in H i.e. $\sigma(T) = (\varphi(T))^c$.

The spectrum of an operator T can be decomposed into the following subsets:

Continuous spectrum of T :

Denoted by $\sigma_c(T) = \lambda \in \mathbb{C} : Ker(\lambda I - T) \neq \{\bar{0}\}$, $\overline{Ran(\lambda I - T)} = H$ and $Ran(\lambda I - T) \neq H$ }, where $(\lambda I - T)^{-1}$ exists as a map which is bounded.

Residual spectrum of T :

Denoted by
$\sigma_r(T) = \left\{\lambda \in \mathbb{C} : Ker(\lambda I - T) = \{\bar{0}\} \text{ and } \overline{Ran(\lambda I - T)} \neq H\right\}$, where $(\lambda I - T)^{-1}$ exists as a map which may or may not be bounded.

Approximate point spectrum of T :

Denoted by $\sigma_{ap}(T) = \{\lambda \in \mathbb{C} : \lambda I - T$ is not bounded below$\}$.

Point spectrum of T :

Denoted by $\sigma_p(T) = \left\{\lambda \in \mathbb{C} : Ker(\lambda I - T) = \{\bar{0}\}\right\}$, which is the set of all eigenvalues of T.

Essential spectrum of T :

Denoted by $\sigma_e(T)$, is the set of complex number such that $(T - \lambda)$ is not Fredholm.

Spectral radius of T :

Denoted by $r(T) = Sup\{|\lambda| : \lambda \in \sigma(T)\}$

Continuous spectrum of T :

Denoted by $\sigma_{cp}(T) = \left\{\lambda \in \mathbb{C} : \overline{R(T-\lambda)} \subsetneqq H\right\}$

After discussing the various subclasses of the spectrum,we now give some results on the relationship between the various subclasses of the spectrum and their Aluthge transforms.

Proposition 2.2.2 *Let $T \in B(H)$, then $\sigma(T) = \sigma_p(T) \cup \sigma_c(T) \cup \sigma_r(T)$ holds, where $\sigma_p(T), \sigma_c(T)$ and $\sigma_r(T)$ are mutually disjoint parts of $\sigma(T)$.*

Proposition 2.2.3 *Let $T \in B(H)$, then $\sigma(T) = \sigma_{ap}(T) \cup \sigma_{cp}(T)$ holds, where $\sigma_{ap}(T)$ and $\sigma_{cp}(T)$ are not necessary disjoint parts of $\sigma(T)$. Also, $\sigma(T) = \sigma_r(T) \cup \sigma_{ap}(T)$ holds.*

Definition 2.2.4 [23] *Let $T \in B(T)$ and $\mu \in \sigma(T)$, where μ is the Lebesque measure. Then we denote :*

(i) $m(T, \mu)$, the algebraic multiplicity of eigenvalue μ for T.

(ii) $m_0(T, \mu) - Dim\ Ker(T - \mu I)$, the algebraic multiplicity of μ.

Proposition 2.2.5 [23] *Let $T \in B(T)$,*

(i) If $0 \in \sigma(T)$, then there exist $n \in \mathbb{N}$ such that $m(T, 0) = m_0(\Delta^n(T)), m_0(T, \mu) \leq m_0(\Delta(T), \mu)$.

(ii) For every $\mu \in \sigma(T)$, $m_0(T, \mu) \leq m_0(\Delta(T), \mu)$. This implies that if T is diagonalizable (i.e. $m_0(T, \mu)) = m(T, \mu)$ for every μ), then also $\Delta(T)$ is diagonalizable.

Theorem 2.2.6 [26] *Let $T = U \mid T \mid$be the polar decomposition in $B(H)$ and $T \in B(H)$ for a Hilbert space H and let $\Delta(T)$ denote the Aluthge transform of T.Then the following assertions holds :*

(i) The spectrum of T, $\sigma(T) = \sigma[\Delta(T)]$

(ii) The point spectrum of T, $\sigma_p(T) = \sigma_p[\Delta(T)]$

(iii) The approximate point spectrum of T, $\sigma_{ap}(T) = \sigma_{ap}[\Delta(T)]$

(iv) The essential spectrum of T, $\sigma_e(T) = \sigma_e[\Delta(T)]$

(v) The left essential spectrum of T, $\sigma_{le}(T) = \sigma_{le}[\Delta(T)]$

(vi) The right essential spectrum of T, $\sigma_{re}(T) = \sigma_{re}[\Delta(T)]$

(vii) $\| \Delta(T) \| \leq \| T^{\frac{1}{2}} \| \leq \| T \|$.

We now turn to the intimate connection between the invariant subspace lattice of an arbitrary operator T and its associate $\tilde{T}$. In particular, T is a quasiaffinity (i.e. T is one-to-one and has a dense range) if and only if $\mid T \mid$ is a quasiaffinity and U is a unitary operator, so $\tilde{T}$ is quasiaffinity if T is. Moreover, in this case, T and $\tilde{T}$ are quasisimilar. Furthermore, T is invertible iff $\check{T}$ is and in this case T and $\check{T}$ are similar. Based on the fact that if $T = U \mid T \mid$ (Polar decomposition) is an arbitrary operator in $B(H)$ and $\check{T} = \mid T \mid^{\frac{1}{2}} U \mid T \mid^{\frac{1}{2}}$is the Aluthge transform, then we have that:

$(i) \mid T \mid^{\frac{1}{2}} T = \check{T} \mid T \mid^{\frac{1}{2}}$, and

$(ii)\ T(U \mid T \mid^{\frac{1}{2}}) = (U \mid T \mid^{\frac{1}{2}})\check{T}$

As usual, we write $Lat(T)$ for the invariant subspace lattice of an arbitrary operator $T \in B(H)$. If $T \in B(H)$ is not a quasiaffinity, then $0 \in \sigma_p(T) \cup \sigma_p(T^*)$, so trivially T has a non-trivial invariant subspace. Thus when investigating the relation between $Lat(T)$ and $Lat(\check{T})$, it suffices to consider the case that T is a quasiaffinity.

Let $T = U \mid T \mid$(Polar decomposition) be an arbitrary quasiaffinity in $B(H)$. Then the following mapping $\phi : N \longrightarrow \overline{(\mid T \mid^{\frac{1}{2}} N)}$, $N \in Lat(T)$, maps $Lat(T)$ into $Lat(\check{T})$, and moreover if $\{0\} \neq N = H$, then $\{0\} \neq \phi(N) = \overline{(\mid T \mid^{\frac{1}{2}} N)} \neq H$. Moreover, the mapping $\psi : M \longrightarrow \overline{(U \mid T \mid^{\frac{1}{2}} M)}, M \in Lat(\check{T})$, maps $Lat(\check{T})$ into $Lat(T)$ and if$\{0\} \neq M \neq N$, then $\{0\} \neq \psi(M) = \overline{(U \mid T \mid^{\frac{1}{2}} M)} \neq H$. Consequently, $Lat(T)$ is a non-trivial iff $Lat(\check{T})$ is non-trivial.

2.3 Spectral Radius formula of Aluthge transform.

Let $T \in B(H)$ be an invertible operator on the complex Hilbert space H. For $0 < \lambda < 1$, we extend Yamazaki's formula of spectral radius in terms of $\lambda-$Aluthge transform i.e. $\Delta_\lambda(T) = \mid T \mid^\lambda U \mid T \mid^{1-\lambda}$ where $T = U \mid T \mid$ is the polar decomposition of T. Thus we prove that $lim_{n\to\infty} \||\ \Delta^n_\lambda(T) \||= r(T)$ where $r(T)$ is the spectral radius of T and $\||\cdot\||$ is the unitary invariant norm such that $(B(H), \||\cdot\||)$ is a Banach algebra with $\||\ I\ \||= 1$.

Lemma 2.3.1 *(Spectral radius formula)* [33.*p*.235] *For $T \in B(H)$ with respect to the norm $\||\cdot\||$ (not necessary unitary invariant),* $r(T) = lim_{k\to\infty} \||\ T^k\ \||^{\frac{1}{k}} = \inf_{k\in\mathbb{N}} \||\ T^k\ \||^{\frac{1}{k}}$.

In particular $\||\ T\ \|| \geq r(T)$, where $\||\ T\ \||$ is the spectral norm of T. For $B(H)$ to be a Banach algebra with respect to $\||\cdot\||$, the norm in *Lemma* 2.2.6 has to be submultiplicative i.e. $\||\ ST\ \||\leq\||\ S\ \||\||\ T\ \||$ [33.*p* .227] . The condition $\||\ I\ \||$ is inessential for the formula $r(T) = lim_{\ k\to\infty} \||\ T^k\ \||^{\frac{1}{k}}$ i.e. it is still valid even $\||\ I\ \|| > 1$. The formula $r(T) = inf_{k\in\mathbb{C}} \||\ T^k\ \||^{\frac{1}{k}}$ is valid for any normed algebra [8, *p*.236]

Lemma 2.3.2 *Suppose that $B(H)$ is a Banach algebra with respect to the unitary invariant norm $\||\ .\ \||$ and $\||\ I\ \||$. Let| $0 < \lambda < 1$. Let $s_k = \||\ (T^n)^k\ \||$and $s = s_1$. If $T \in B(H)$ is non-quasinilpotent, the $s > 0$. Moreover, if $T \in B(H)$ is invertible, then $s_k = s^k$ for each $k \in \mathbb{N}$.*

Proof. Let $T \in B(H)$. For each $k \in \mathbb{N}$, the sequence $\{||| (T_n)^k |||\}\, n \in \mathbb{N}$ is non-increasing so that $s_k = \lim_{n\to\infty} ||| (T_n)^k |||$ exist. By *Lemma* 2.3.1, $||| T_n ||| \geq r(T_n) = r(T)$. The spectrum $\sigma(T)$ of T is a compact non-empty set. If T is non-quasinilpotent i.e. $r(T) > 0$ so that $s = s_1 = \lim_{n\to\infty} ||| T_n ||| \geq r(T) > 0$. Now assuming that T is invertible, we proceed by induction to show that $s_k = s^k$ for all $k \in \mathbb{N}$. Where $k = 1$, the statement is trivial. Suppose that the statement is true for $1 \leq k \leq m$. Case 1; $0 < \lambda \leq \frac{1}{2}$, we have that $|||| T_n |^{1-2\lambda}||| \leq |||| T_n ||||^{1-2\lambda}$ since $||| I ||| = 1$ and $0 \leq 1 - 2\lambda < 1$. Since T is invertible, $| T_n |$ is also invertible and thus $| T_n |^{2\lambda-1}$ exists. So $\frac{|||(T_{n+1})^m|||}{||||T_n||||^{1-2\lambda}} \leq \frac{|||(T_{n+1})^m|||}{||||T_n|^{1-2\lambda}|||} \leq ||| (T_{n+1})^m \mid T_n |^{2\lambda-1}|||$, since $||| \cdot |||$ is submultiplicative $\leq ||| (T_n)^{m+1} |||^{\lambda} ||| (T_n)^{m-1} |||^{1-\lambda} \leq ||| (T_n)^m |||^{1-\lambda} ||| T_n |||^{1-\lambda} ||| (T_n)^{m-1} |||^{\lambda}$ *so* $\frac{s^m}{s^{2\lambda-1}} \leq s_{m+1}^{1-\lambda} s^{(m-1)\lambda} \leq s^{m(1-\lambda)} s^{(m-1)\lambda}$ which holds to $s_{m+1} = s^{m+1}$. ∎

Remark 2.3.3 [19] *Suppose that* $\lambda = \frac{1}{2}$. *In the proof of* [37, *Lemma* 4], *the possibility that* $s = 0$ *is not considered (the spectral* $\| \cdot \|$ *is the norm under consideration). It amounts to* $r(T) = 0$, *i.e.* T *is quasinilpotent* [20, *p.*50], [24, *p.*381] . *In the above induction proof if* $\lambda = \frac{1}{2}$, *we cannot deduce that* $s_{m+1} = s^{m+1}$ *for all* $||| \cdot |||$, *granted that* $s^m \leq s_{m+1}^{\frac{1}{2}} s^{(m-1)\frac{1}{2}} \leq s^m$. *However, if* $||| \cdot ||| = \| \cdot \|$ *and* $\lambda = \frac{1}{2}$, *then we have* $s_k = s^k$ *for all* $k \in \mathbb{N}$ *because* $\| T \| = ||| T |||$ *for any* $T \in B(H)$.

Theorem 2.3.4 [19] *Suppose that* $B(H)$ *is a Banach algebra with respect to the unitary invariant norm* $||| \cdot |||$ *and* $||| I ||| = 1$. *Let* $T \in B(H)$ *be invertible and* $0 < \lambda < 1$. *Then* $lim_{n\to\infty} ||| \Delta_\lambda^n(T) ||| = r(T)$.

Proof. *By Lemma* 2.2.7, *for each* $k \in \mathbb{N}$, *the sequence* $\left\{||| (T_n)^k |||^{\frac{1}{k}}\right\}_{n\in\mathbb{N}}$ *is non-increasing and converges to* $s = lim_{n\to\infty} ||| T_n |||$. *So for all* $n, k \in \mathbb{N}$, $s \leq ||| (T_n)^k |||^{\frac{1}{k}}$.*Suppose that* $r(T) < s$, *i.e.* $r(T_n) < s$ *for all* n. *Then for a fixed* $n \in \mathbb{N}$, *and sufficiently large* k, *by Lemma* 2.2.6, *we would have* $||| (T_n)^k |||^{\frac{1}{k}} < s$; *a contradiction. So* $r(T) = s$. ∎

Thus we summarize that *Theorem* 2.2.9 is true for non-invertible $T \in B(H)$ as well.

Remark 2.3.5 [12] *Of course the statement in Theorem* 2.2.9 *is valid for* $T \in B(H)$. *For* $\lambda = \frac{1}{2}$, *i.e.* $lim_{n\to\infty\infty\infty} \| \Delta_{\frac{1}{2}}^n(\| T) \| = r(T)$. *By Remark* 2.2.8, $lim_{n\to\infty} \| \Delta_{\frac{1}{2}}^n(T) \| = r(T)$ *is valid for non-quasinilpotent* $T \in B(H)$ *as the proof in Theorem* 2.2.9 *works for non-nilpotent* T, *is valid for any* $T \in B(H)$ *as* $s_k = s^k$ *for non-quasinilpotent by Remark* 2.2.6 . *If* T *is quasinilpotent, then consider the orthogonal sum* $T \oplus CL \in B(H \oplus H)$. *We may consider* $T \neq 0$. *Notice that* $T \oplus CL$ *is non-quasinilpotent if* $c > 0$. *Since* $\Delta_{\frac{1}{2}}^n(T \oplus CL) = \Delta_{\frac{1}{2}}^n(T) + \Delta_{\frac{1}{2}}^n(CL) = \Delta_{\frac{1}{2}}^n(T) \oplus CL$ *and* $r(T \oplus CL) = r(CL) = C$ *by Remark* 2.2.6, $max\left\{\| \Delta_{\frac{1}{2}}^n(T) \|, C\right\} = \| \Delta_{\frac{1}{2}}^n(T) \oplus CL \|$

$= \| \Delta^n_{\frac{1}{2}}(T \oplus CL) \|$

Definition 2.3.6 *Let $T \in B(H)$. Then the numerical range $W(T)$ of T is the subset of all complex numbers $\mathbb{C}$, given by:*

$W(T) = \{< Tx, x >: x \in H, \| x \|= 1\}$.

The numerical range of an operator T i.e. $W(T)$ is also called the field values of T and is the convex subset of a complex plane and it says a lot about the operator. The image of the unit ball is the union of all the closed segment that join the origin to the point of the numerical range; the entire range is the union of all the closed rays from the origin through the points of the numerical range. The numerical range can be divided into the following classes :

Classical numerical range.

This is just the ordinary numerical range of an operator T on a Hilbert space H and it is defined as $W(T) = \{< Tx, x >: x \in H, \| x \|= 1\}$. It is considered to be always convex according to the Toeplitz-Hausdorf theorem which states that for $T \in B(H)$, then the numerical range $W(T)$ of T is a convex subset of the complex plane $\mathbb{C}$. Toeplitz (1918) proved that the boundary of $W(T)$ is a convex curve, but left open the possibility that it had interior holes. Housdorff (1919) proved that it did not actually contain any holes.

Numerical radius of T.

Let $T \in B(H)$,then the numerical radius of T,denoted by $w(T)$ is defined as $w(T) = Sup\{| \lambda |: \lambda \in W(T)\}$.

Thus the numerical radius $w(T)$ of T is the radius of the smallest circle in the complex plane centred at the origin that encloses the numerical range of T. That is $w(T)$ is the greatest distance between any part in the numerical range and the origin. The following properties of numerical radius are well known :

(*i*) $w(T) \geq 0 \; \forall \; T \in B(H)$.

(*ii*) $w(T) > 0$ whenever $T \neq 0, T \in B(H)$.

(*iii*) $w(T) =| \alpha | w(T)$ for every $\alpha \in \mathbb{C}$.

(*iv*) $w(T + S) \leq w(T) + w(S) \quad \forall \quad T, S \in B(H)$.

(*v*) $w(T) = w(T^*)$ and $w(T^*T) = \| T \|^2 \; \forall \quad T \in B(H)$.

Spartial numerical range.

It is the union of the classical numerical range. Suppose that $T = \{T_1, T_2, ..., T_n\}$, then the Spartial numerical range of the given operator T is $W(T) = \cup_{i=1}^n W(T_i)$.

Joint Numerical range.

This refers to the set of the numerical range of a set of operator, that is, they can be all self-adjoint (Hermitian) or normal. Suppose that $T = \{T_1, T_2, ..., T_n\}$ is a set of self-adjoint operator, then the joint numerical range is given as

$W(T) \ = \{W(T_1), W(T_2), ..., W(T_n)\}$.

Essential numerical range.

Let $T \in B(H)$ and $K(H)$ be the set of compact operators on $B(H)$, then the essential numerical range of an operator T is given as

$W(T) = \overline{\cap_{n \in K(H)} \{W(T+K)\}}$, where K is compact operator.

C−numerical range

For $n \times n$ matrices C and T, the $C-$numerical range $W_C(T)$ of T is the compact subset of complex number $\mathbb{C}$, given by

$W_C(T) \ = \ \{tr(CU^*TU) : U$ is unitary matrix$\}$, where tr denotes the trace.

As a result of numerical range of the Aluthge transform of T i.e. $W(\tilde{T})$, the following results has been shown.

Theorem 2.3.7 [38] *Let T be a bounded linear operator, then the following inclusion relation holds* : $\overline{W(\tilde{T})} \ \subseteq \overline{W(T)}$, *and* $w(\tilde{T}) \leq w(T)$.

Remark 2.3.8 *Theorem* 2.3.2 *was firstly shown in* [25] *in case T is a* 2×2 *matrix (in this case, $W(\tilde{T})$ and $W(T)$are closed subset of the complex number $\mathbb{C}$). Then one of the authors in* [40] *proved that $\overline{W(\tilde{T})} \subset \overline{W(T)}$ holds if T admits a polar decomposition $T = U \mid T \mid$ for any isometry operator U. This condition is always satisfied if T is an $n \times n$ matrix or H is a finite dimensional. In* [40], *the relation $\overline{W(\tilde{T})} \subset \overline{W(T)}$ is shown by using the property of the numerical range $\overline{W(\tilde{T})} = \cap_{\lambda \in \mathbb{C}} \{z \in \mathbb{C} : \mid z - \lambda \mid \leq w(T - \lambda I)\}$, where $w(T)$ is the numerical radius of T i.e. $w(T) = Sup\{\mid \lambda \mid : \lambda \in W(T)\}$, and the following is the characterization of $w(T) \leq 1$ by Berger and Stampli* [7] *that $w(T) \leq 1$ iff $\parallel T - zI \parallel \leq 1 + \sqrt{1 + \mid z \mid^2}$ for all $z \in \ \mathbb{C}$.*

Now we introduce the following *Lemma* to look at the relationship between q-numerical range and its Aluthge transform.

Lemma 2.3.9 [27,*p*.389,*Theorem* 2.1] *Suppose that A is an $n \times n$ matrix and B is an $m \times m$ matrix .Then the following two conditions are mutually equivalent:*

(i) The inclusion $W_q(B) \subset W_q(A)$ holds for every $0 \ \leq q \ \leq 1$.

(ii) The inclusion $W(B) \subset W(A)$ and the inequality $max\{h : (z,h) \in W(B, B^*B)\} \leq max\{h : (z,h) \in W(A, A^*A)\}$,holds for every $z \in W(B)$.

Thus we state the following theorem.

Theorem 2.3.10 *Suppose that T is an $n \times n$ matrix and $f(z)$ is a polynomial in z, then the inclusion $W_q(f(\tilde{T})) \subset W_q(f(T))$, holds for every complex number q with $\mid q \mid \leq 1$. In particular, by putting $q = 1$ in Theorem* 2.4.5, *we have the following relation.*

Corollary 2.3.11 *If T is an $n \times n$ matrix, then $W(f(\tilde{T})) \subset W(f(T))$ holds for all polynomial f. Moreover, we obtain the inequality of the numerical radius and the spectral norm.*

Corollary 2.3.12 *Let T be an $n \times n$ matrix, then the following assertions holds:*

(i) $w(f(\tilde{T})) \leq w(f(T))$ for all polynomial f.
(ii) $\| f(\tilde{T}) \| \leq \| f(T) \|$ for all polynomial f, where $\| \cdot \|$ means the spectral norm.

Theorem 2.3.13 [39] *For any $T \in B(H)$, $\frac{1}{2} \| T \| \leq w(T) \leq \frac{1}{2} w(\tilde{T})$. The relation between operator norm and numerical radius is known as $w(T) \leq \| T \|$ i.e. $\frac{1}{2} \| T \| \leq w(T) \leq \| T \|$ and $r(T) \leq w(T) \leq \| T \|$.*

Thus *Theorem* 2.4.8 is more exact estimation than Kittaneh's result by the following inequality :
$w(T) \leq \frac{1}{2} \| T \| + \frac{1}{2}\, w(\check{T}) \leq \frac{1}{2} \| T \| + \frac{1}{2} \| \check{T} \| \leq \frac{1}{2} \| T \| + \frac{1}{2} \| \check{T} \| \leq \| T \|$.

2.4 Relationship between spectrum and Numerical range of Aluthge transform.

Definition 2.4.1 *A set B is convex if for any two points $x, y \in B$, we have $z = tx + (1-t)y \in B$, for all $t \in [0,1]$ and the convex hull of B denoted by $Conv(B)$, is the smallest convex set containing B.*

Recall the definition of spectrum and numerical range, then the following relations between the spectrum and numerical range is well known. $Conv(\sigma(T)) \subseteq \overline{W(T)}$, where $conv(x)$ is the convex hull of a subset $x \in \mathbb{C}$.

Definition 2.4.2 [3] *Let $T \in B(H)$. Then :*

(i) $\| T \| = Sup\{\| Tx \| : \| x \| = 1\}$ (operator norm).
(ii) $w(T) = Sup\{| \lambda | : \lambda \in W(T)\}$ (numerical radius).
(iii) $r(T) = Sup\{| \lambda | : \lambda \in \sigma(T)\}$ (spectral radius).
Then the following relation are well known $r(T) \leq w(T) \leq \| T \|$ and $\frac{1}{2} \| T \| \leq w(T) \leq \| T \|$.

Definition 2.4.3 *An operator $T \in B(H)$ is said to be :*

(i) Convexiod if $\overline{W(T)} = conv(\sigma(T))$.
(ii) Normaloid if $r(T) = \| T \|$.
(iii) Spectraloid if $w(T) = r(T)$.

Proposition 2.4.4 *Let $T \in B(H)$, then $\sigma_p \subseteq W(T)$.*

Proof. Suppose that $\lambda \in \sigma_p(T)$, then $\ni x \neq 0 \in H : \lambda x = Tx$. Therefore $\lambda = \lambda < x, x > = < \lambda x, x >= < Tx, x > \in W(T) \Rightarrow \lambda \in W(T)$. Therefore $\sigma_p(T) \subseteq W(T)$. ■

Corollary 2.4.5 *Let $T \in B(H)$, then $\sigma_p(T) \cup \sigma_r(T) \subseteq W(T)$.*

Proof. Suppose $\lambda \in \sigma_p(T) \Rightarrow \lambda \in W(T)$. If $\lambda \in \sigma_r(T)$, then $\overline{\lambda} \in \sigma_p(T^*) \Rightarrow \lambda \in W(T^*)$. Hence $\sigma_p(T) \cup \sigma_r(T) \subseteq W(T)$. ■

Proposition 2.4.6 *Let $T \in B(H)$, then $\sigma(T) \subseteq \overline{W(T)}$.*

Proof. Recall that $\sigma(T) = \sigma_r(T) \cup \sigma_{ap}(T)$. Suppose that $\lambda \in \sigma_{ap}(T)$, $\Rightarrow$ $0 \leq| \lambda - < Tx_n, x_n >| = \underline{| < (T - \lambda I)x_n, x_n > |} \leq \| (T - \lambda I)x_n \| \| x_n \|$ $\rightarrow 0$ as $n \rightarrow \infty \Rightarrow \lambda \in \overline{W(T)}$. Therefore $\sigma_{ap}(T) \subseteq \overline{W(T)} \Rightarrow \sigma(T) \subseteq$ $\overline{W(T)}$. ■

Theorem 2.4.7 *Let $T \in B(H)$, if T is normal then $\overline{W(T)} = Conv(\sigma(T))$, i.e. $\overline{W(T)}$ is completely determined by $\sigma(T)$ if T is a normal operator. However, there are normal operators with the same spectrum but different numerical range.*

Therefore Ando, gave a characterization of the convex hull of spectral radius as follows :

Theorem 2.4.8 [3] *For each $T \in B(H)$, $\cap_{n=1}^{\infty} \overline{w(\Delta^n(T))} = Conv(\sigma(T))$.*

Moreover, the following characterization is obtained.

Theorem 2.4.9 [17] *For each $T \in B(H)$, then $conv(\sigma(T)) = \overline{W(T)}$ is equivalent to $\overline{W(T)} = \overline{W(\Delta(T))}$.*

2.5 Convergence of iterated Aluthge transformation.

The research on some operators classes which include the class of normal operators on a complex Hilbert space has been developed by many authors, especially, the classes of normal, quasinormal, subnormal, hypornormal and paranormal operators are very famous. It is well known that every normal operators has the spectral decomposition and the structure of normal operator is well known. The structure of quasinormal operator is also known as a direct sum of normal and operator valued weighted shift in [1]. Also it is known that every subnormal operator has a non-trivial invariant subspace [11].

In 1990, A.Aluthge [10], defined an operator transform in the research on Hyponormal operator called Aluthge transform, which is a good tool for the study on the Hyponormal operators and we call the operator sequence of the iterated Aluthge transform of an operator Aluthge sequence.

Remark 2.5.1 *The Aluthge transform of an operator $T = U \mid T \mid$ does not depend on the partial isometry part U of the polar decomposition of an operator.*

Example 2.5.2 (*i*) *Let T be a unilateral weighted shift on $\ell^2(\mathbb{N})$ such that $T\ (f_1, f_2, f_3, ...) = (0, \alpha_1 f_1 \alpha_2 f_2, ...)$. Then*

$$\Delta(T) = (f_1, f_2, f_3, ..) = (0, \sqrt{\alpha_1\alpha_2}f_1, \sqrt{\alpha_2\alpha_3}f_2, ..).$$

(*ii*) Let T be a bilateral weighted shift on $\ell^2(\mathbb{Z})$ such that $T(.., f_{-1}, f_0, f_1, f_2, ..) = (.., \alpha_{-1}f_{-1}, \alpha_0 f_0, \alpha_1 f_1, ...)$.
Then $(.., f_{-1}, f_0, f_1, , ..) = (.., \sqrt{\alpha_{-1}\alpha_0}f_{-1}, \sqrt{\alpha_0\alpha_1}f_0, \sqrt{\alpha_1\alpha_2}f_1, ..)$.
Some of the operator classes which are important are :

Definition 2.5.3 *Let $T \in B(H)$. Then :*

(*i*) T is quasinormal iff $T^*TT = TT^*T$.
(*ii*) T is normal iff $T^*T = TT^*$.
(*iii*) T is subnormal iff T is a normal extension.
(*iv*) For $p > 0$, T is $p-$Hyponormal iff $(T^*T)^p \geq (TT^*)^p$.

Thus we call 1-Hyponormal simply hyponormal and also we call $\frac{1}{2}$-hyponormal semi-normal. The following inclusion relations are well known and they are proper: {normal} $\subset$ {quasinormal} $\subset$ {subnormal} $\subset$ {hyponormal} $\subset$ {semi-hyponormal}. Now we discuss on convergence of Aluthge sequence. Firstly, by *Example*2.5.3 above, we have that Aluthge sequence of Weighted shift converges in the strong operator topology if its weighted sequence $\{\alpha_n\}$ converges. Next, we have the following results :

Theorem 2.5.4 *For each 2×2 matrix T, there exist a normal matrix N such that $lim_{n\to\infty}\Delta^n(T) = N$ and $\sigma(T) = \sigma(N)$.*

Remark 2.5.5 *For the general operator, every Aluthge sequence converges,but there is a counter example in Example* 2.6.8.

Remark 2.5.6 *There exist an operator T such that the Aluthge sequence does not converge in the weak operator topology. Moreover, there exist a hyponormal operator whose Aluthge sequence converges in strong operator topology not norm topology as follows :*

Theorem 2.5.7 [13] *Let T be a hyponormal bilateral weighted shift on $\ell^2(\mathbb{Z})$ with a weight sequence $\{\alpha_n\}$. Let $a = sup\{\alpha_n\}$ and $b = inf\{\alpha_n\}$. Then the Aluthge sequence converges to a quasinormal operator in the norm topology iff $a = b$.*

Example 2.5.8 [9] *Let T a bilateral shift with weight sequence $\{\alpha_n\}$, where α_n is given by* $\alpha_n = \left\{ \begin{array}{ll} \frac{1}{2} & n < 0, \\ 1 & n \geq 0 \end{array} \right\}$. *Then the Aluthge sequence does not converge to a quasinormal operator in the norm topology but converges in the strong operator topology. Thus every Aluthge sequence of a hyponormal operator converges to a quasinormal operator in the strong operator topology. Thus we look at the convergence of iterated $\lambda-$Aluthge transformation for the diagonalizable matrix and $\lambda \in (0,1)$. Let $M_r(\mathbb{C})$ denotes the algebra of complex $r \times r$ matrices; $GL_r(\mathbb{C})$ denotes the group of all invertible elements in $M_r(\mathbb{C})$; $U(r)$ denotes the group of unitary operators ; $S(D)$ denotes the similarity orbit of some diagonal operator D; diagonal matrix $D = diag(d_1, d_2, ...d_n) \in GL_r(\mathbb{C})$ is fixed. For every $j \in \{1, 2, 3, ..., n\}$, let $d_j = e^{i\theta_j} \mid d_j \mid$ be the polar decomposition of d_j where $\theta_j \in [0, 2\Pi]$.*

Lemma 2.5.9 [23] *Given that $T \in M_r(\mathbb{C})$ and $\lambda \in (0,1)$, the limit point of the sequence $\{\Delta_\lambda^n(T)\}_{n\in\mathbb{N}}$ are normal. Moreover if L is a limit point, then $\sigma(L) = \sigma(T)$ with the same algebraic multiplicity. In particular, for each $\lambda \in (0,1)$, we have that $\Delta_\lambda(T) = T$ iff T is normal.*

Proof. Let $\{\Delta_\lambda^{n_k}(T)\}_{k\in\mathbb{N}}$ be a subsequence which converges in norm to a limit point L. By the continuity of Aluthge transform, $\Delta_\lambda^{n_k+1}(T) \longrightarrow \Delta_\lambda(T)$ as $k \to \infty$. Then $\| \Delta_\lambda(L) \|_2 = \lim_{k\to\infty} \| \Delta_\lambda^{n_k+1} \|_2 = \lim_{n\to\infty} \| \Delta_\lambda^n(T) \|_2$ $= \lim_{k\to\infty} \| \Delta_\lambda^{n_k}(T) \|_2 = \| L \|_2$. Hence L is the norm by the fact that if $\lambda \in (0,1)$, $1 \leq p \leq \infty$ and $T \in B^p(H)$, then $\| \Delta_\lambda(T) \|_p = \| T \|_p$iff T is normal. Then we prove that $\sigma(L) = \sigma(T)$ with the same algebraic multiplicity or equivalently that $tr(T^m) = tr(L^m)$ for every $m \in \mathbb{N}$. Indeed $trL^m = \lim_{k\to\infty} tr\Delta_\lambda^{n_k}(T)^m = trT^m, m \in \mathbb{N}$, because for every $k \in \mathbb{N}$, $\sigma(\Delta_\lambda^{n_k}(T)) = \sigma(T)$ (with algebraic multiplicity) and therefore $tr\Delta_\lambda^{n_k}(T)^m = trT^m$. ■

Thus consider the conjecture below.

Conjecture 2.5.10 [23] *The sequence of iterates $\{\Delta_\lambda^n(T)\}_{n\in\mathbb{N}}$ converges for every matrix T.*

Let us now consider the iterated Aluthge transforms in $M_2(\mathbb{C})$ and we look at the convergence of the sequence $\{\Delta_\lambda^n(T)\}$ where T is a 2×2 matrix. The convergence of this sequence for $n \times n$ matrices and $\lambda = \frac{1}{2}$ was conjectured by Jung, Ko and Pearcy in [23] for *Conjecture* 2.6.10. Thus we prove that the map which assigns to each pair (T, λ), the limit of the sequence $\{\Delta_\lambda^n(T)\}_{n\in\mathbb{N}}$ is continuous in both variables T and λ.

Lemma 2.5.11 [23] *Let $T \in M_2(\mathbb{C})$ and $\lambda \in (0,1)$. Suppose that $\sigma(T) = \{\mu_1, \mu_2\}$ with $\mu_1 \neq \mu_2$. Then there exists $\gamma(T, \lambda) \in (0,1)$ such that for all $n \in \mathbb{N}$, $\| \Delta_\lambda^n(T)^*\Delta_\lambda^n(T) - \Delta_\lambda^n(T)\Delta_\lambda^n(T)^* \|_2 \leq \gamma(T, \lambda)^n \| T^*T - TT^* \|_2$.*

Moreover, if $\alpha = min\{\lambda, (1-\lambda)\}$, the we can take $\gamma(T, \lambda) = \left(1 - \frac{2\alpha^2|\mu_1 - \mu_2|^2}{2|\mu_1\mu_2| + \|T\|_2^2}\right)^{\frac{1}{2}}$.

Theorem 2.5.12 [23] *Let $T \in M_2(\mathbb{C})$ and $\lambda \in (0,1)$. Then the sequence $\{\Delta_\lambda^n(T)\}_{n\in\mathbb{N}}$ converges.*

Proof. Suppose that $\sigma(T) = \{\mu_1, \mu_2\}$. Since we have proved (*Lemma* 2.6.9) that the limit points of the sequence $\{\Delta_\lambda^n(T)\} \longrightarrow CL$ as $n \to \infty$. Thus we consider the case in which $\mu_1 \neq \mu_2$ and we denote $T_n = \Delta_\lambda^n(T)$ for all $n \in \mathbb{N}$. Let $n \geq o$. If $T_n = U \mid T_n \mid$ is the polar decomposition of T_n, then $\mid T_n^* \mid^t = U \mid T_n \mid^t U^*$, for every $t > 0$. Therefore we obtain $(T_{n+1} - T_n)U_n^* = \mid T_n \mid^\lambda U_n \mid T_n \mid^{1-\lambda} U_n^* - U_n \mid T_n \mid = \mid T_n \mid^\lambda \mid T_n^* \mid^{1-\lambda} - \mid T_n^* \mid^\lambda = (\mid T_n \mid^\lambda - \mid T_n^* \mid^\lambda) \mid T_n^* \mid^{1-\lambda}$and then $\| T_{n+1} - T_n \|_2 \leq \||| T_n \mid^\lambda - \mid T_n^* \mid^\lambda \|_2 \||\, T_n^* \mid^{1-\lambda} \|_2 \leq 2 \||\, T_n \mid^\lambda - \mid T_n^* \||_2 \||\, T_n \mid^{1-\lambda} \| \leq 2 \||\, T_n \mid^\lambda - \mid T_n^* \mid^\lambda \|_2 \| T_n \| \leq 2 \||\, T_n \mid^\lambda - \mid T_n^* \mid^\lambda \|_2 \| T \|^{1-\lambda}$. And by Bhatia-Kittanech[13], which states that, given $A, B \in M_n(\mathbb{C})^+$ and $r \in [0,1]$, then $\| A^r - B^r \| \leq \| I \|^{1-r} \| A - B \|^r$ for every unitary invariant norm $\| \cdot \|$. Then $A = T_n^* T_n, B = T_n T_n^*$ and $r = \frac{\lambda}{2}$, since $\| I_2 \|_2^{1-\frac{\lambda}{2}} \leq 2$, we get $\| T_{n+1} - T_n \|_2 \leq 2 \||\, T_n \mid^\lambda - \mid T_n^* \mid^\lambda \|_2 \| T \|^{1-\lambda} \leq (4 \| T \|^{1-\lambda}) \| T_n^* T_n - T_n T_n^* \|^{\frac{\lambda}{2}}$. If $\gamma(T,\lambda) \in (0,1)$ is the constant of *Lemma* 2.6.9, and $a = \gamma(T,\lambda)^{\frac{\lambda}{2}} < 1$. Then $\| T_{n+1} - T_n \|_2 \leq (4 \| T \|^{1-\lambda}) \| T_n^* T_n - T_n T_n^* \|_2^{\frac{\lambda}{2}} \leq a^n (4 \| T \|^{1-\lambda} \| T^*T - TT^* \|_2^{\frac{\lambda}{2}})$. Denoting $N(T,\lambda) = 4 \| T \|^{1-\lambda} \| T^*T - TT^* \|_2^{\frac{\lambda}{2}}$. Then if $n, m \in \mathbb{N}$, with $n < m$, $\| T_m - T_n \|_2 \leq \sum_{k=n}^{m-1} \| T_{k=1} - T_k \|_2 \leq N(T,\lambda) \sum_{n=k}^{m-1} a^k \longrightarrow 0$ as $n, m \to \infty$, which shows that the $lim_{n\to\infty} T_n = lim_{n\to\infty} \Delta_\lambda^n(T)$ exists. ∎

2.6 The Jordan structure of Aluthge transform.

We look at some properties of the Jordan structure of the iterated Aluthge transforms. We claim that the proof of the convergence of iterated $\lambda-$Aluthge transform can be reduced to the invertible case. We show a reduction of *Conjecture* 2.6.10 on the convergence of the sequence $\{\Delta_\lambda^n(T)\}$ for $T \in M_r(\mathbb{C})$ to the invertible case. Also, we look at the behavior of the angles between the spectral subspace of iterates of Aluthge transforms for $T \in M_r(\mathbb{C})$. Indeed, let $T \in M_r(\mathbb{C})$ be diagonalizable matrix with polar decomposition $T = U \mid T \mid$. As $R(T)$ is a(oblique) compliment of $KerT = Ker \mid T \mid^\lambda$ and $R(U \mid T \mid^{1-\lambda}) = R(T)$, it holds that

$R(\Delta_\lambda(T)) = R(\mid T \mid^\lambda U \mid T \mid^{1-\lambda}) = R(\mid T \mid)$.

On the other hand, we know that $Ker\Delta_\lambda(T) = Ker \mid T \mid^{1-\lambda} = Ker \mid T \mid$, which is orthogonal to the $R(\mid T \mid)$. By *Proposition* 2.2.4, after one iteration, we get that $\Delta_\lambda(T) = \begin{pmatrix} T_1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} KerT^\perp \\ KerT \end{pmatrix}$, where T_1 is invertible and diagonalization on $KerT^\perp$. By *Proposition* 2.2.4 again,

$\Delta_\lambda^n(T) = \begin{pmatrix} \Delta_\lambda^{n-1}(T_1) & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} KerT^\perp \\ KerT \end{pmatrix}$, for every $n \in \mathbb{N}$. Hence,the convergence of $\{\Delta_\lambda^n(T)\}_{n\,\in\,\mathbb{N}}$ is equivalent to the convergence of $\{\Delta_\lambda^n(T_1)\}_{n\,\in\,\mathbb{N}}$.

Therefore the following results states a simple relation between the null space of polynomials in T and in $\Delta_\lambda(T)$. This relation has some consequences regarding multiplicity and Jordan structure of eigenvalues of T and $\Delta_\lambda(T)$.

Lemma 2.6.1 [23] *Let $T \in M_r(T)$ and $\lambda \in (0,1)$:*

(*i*) Given $p \in \mathbb{C}[x]$, then *dim* $N(p(T)) \leq$ *dim* $N(p(\Delta_\lambda(T)))$.
(*ii*) For $n \in \mathbb{N}$, $n \geq 2$, *dim* $N(T^n) =$ *dim* $N(\Delta_\lambda(T)^{n-1})$.

Proposition 2.6.2 *Let $T \in M_r(\mathbb{C})$ and $\lambda \in (0,1)$:*

(*i*) Suppose that $0 \in \sigma(T)$, then $m(T,0) = m_0(\Delta_\lambda^{r(T,0)-1}(T),0) = \dim N(\Delta_\lambda^{r(T,0)-1}(T))$. Therefore, after $r(T,0)-1$ iterations of the Aluthge transforms, we get a matrix whose Jordan structure for the eigenvalue 0 is trivial.

(*ii*) If $\mu \in \sigma(T)/\{0\}$, then $m_0(T,\mu) \leq m_0(\Delta_\lambda(T),\mu)$ and $r(T,\mu) \geq r(\Delta_\lambda(T),\mu)$.

Proof. (*i*) Denoting $r(T,0) = r$. Then by *Lemma* 2.7.1, $m(T,0) = dim\ N(T^r) = dimN(\Delta_\lambda(T)^{r-1}) = dimN(\Delta_\lambda^2(T)^{r-2}) = dimN(\Delta_\lambda^{r-2}(T)^2) = dimN(\Delta_\lambda^{r-1}(T))$.

(*ii*) Consider $p_m(x) = (x-\mu)^m \in \mathbb{C}[x], m \in \mathbb{N}$. Taking $m=1$, by *Lemma* 2.7.1, $m_0(T,\mu) = \dim N(T-\mu I) \leq \dim N(\Delta_\lambda(T)-\mu I) = m_0(\Delta_\lambda(T),\mu)$. Taking $m = r(T,\mu)$, again by *Lemma* 2.7.1, we have that $m(T,\mu) = dimN((T-\mu I)^{r(T,\mu)}) \leq \dim N((\Delta_\lambda(T)-\mu I)^{r(T,\mu)}) \leq m(\Delta_\lambda(T),\mu)$. since $m(\Delta_\lambda(T),\mu) = m(T,\mu)$, we get by its definition that $r(T,\mu) \geq r(\Delta_\lambda(T),\mu)$. ■

Remark 2.6.3 *In particular, Proposition 2.7.2 shows that if T is nilpotent of order n, then $\Delta_\lambda^{n-1}(T) = 0$. This result was proved by Jung,Ko and Pearcy in* [24].

Corollary 2.6.4 [23] *Let $\lambda \in (0,1)$. If the sequence $\{\Delta_\lambda^n(S)\}$ converges for every invertible matrix $S \in M_r(\mathbb{C})$ and every $r \in \mathbb{N}$, then the sequence $\{\Delta_\lambda^n(T)\}$ converges for all $T \in M_r(\mathbb{C})$ and every $r \in \mathbb{N}$.*

Proof. Let $T \in M_r(\mathbb{C})$. By *Proposition* 2.7.2 we can assume that $m(T,0) = m_0(T,0)$. We note that in this case , $N(\Delta_\lambda(T)) = N(T)$, since $N(T) \subseteq N(\Delta_\lambda(T))$ and $m(\Delta_\lambda(T),0) = m(T,0)$. On the other hand $R(\Delta_\lambda(T)) \subseteq R(\mid T \mid)$ so that $R(\Delta_\lambda(T))$ and $N(\Delta_\lambda(T))$ are orthogonal subspace. Thus , there exist a unitary matrix U such that $U\Delta_\lambda(T)U^* = \begin{pmatrix} S & 0 \\ 0 & 0 \end{pmatrix}$ where $S \in M_s(\mathbb{C})$ is invertible ($s = r - m(T,0)$). Since for every $n \geq 2$, $\Delta_\lambda^n(T) = U^* \begin{pmatrix} \Delta_\lambda^{n-1(s)} & 0 \\ 0 & 0 \end{pmatrix} U$, the sequence $\{\Delta_\lambda^n(T)\}$ converges because the sequence $\{\Delta_\lambda^{n-1}(S)\}$ converges by hypothesis. ■

Remark 2.6.5 [23] *If $T \in M_r(\mathbb{C})$ is invertible , then $\mid T \mid^{\lambda}$ is invertible , for every $\lambda \in (0,1)$ and $\Delta_\lambda(T) =\mid T \mid^{\lambda} U \mid T \mid^{-\lambda}$. Therefore T and $\Delta_\lambda^n(T)$ are similar matrices for every $n \in \mathbb{N}$, i.e. $\Delta_\lambda^n(T)$ and T have the same Jordan structure. This shows that T may be not similar. Thus the numerical experiences show that the rate of convergence of the sequence$\{\Delta_\lambda^n(T)\}$ is smaller for non-diagonalizable T, than for diagonalizable examples.*

Definition 2.6.6 *Given two subspaces M and N of $\mathbb{C}^n$ such that $M \cap N = \{0\}$, the angle between M and N is the angle in $[0, \frac{\pi}{2}]$ whose cosine is defined by $c[M,N] = sup\{\mid< x,y >\mid: x \in M, y \in N$ and $\parallel x \parallel=\parallel y \parallel= 1\} =\parallel P_M P_N \parallel$. The sine of this angle is $s\ [M,N] = \left(1 - c\ [M,N]^2\right)^{\frac{1}{2}}$. If $M \oplus N = \mathbb{C}^n$ and Q is the oblique projection with the range M and null space N, it is known that $\parallel Q \parallel= \frac{1}{(1-\parallel P_M P_N \parallel^2)^{\frac{1}{2}}} = \frac{1}{(1-c[M,N]^2)^{\frac{1}{2}}} = s\ [M,N]^{-1}$, where P_M denotes the orthogonal projection onto M.*

Chapter 3

THE DUGGAL TRANSFORM OF A BOUNDED LINEAR OPERATOR.

Introduction.

Let T be a bounded linear operator in a complex Hilbert space H, i.e. $T \in B(H)$. Then T is said to be positive (denoted $T \geq 0$) if $< Tx, x > \geq 0$ for all $x \in H$. Thus for every operator T, it can be decomposed into $T = U \mid T \mid$ with a partial isometry U and $\mid T \mid = (T^*T)^{\frac{1}{2}}$. U is determined uniquely by the kernel condition $Ker(U) = Ker(T)$, then this decomposition is called the polar decomposition. In [15], Foias, Jung, Ko and Pearcy defined a transformation $\Gamma(T) = \mid T \mid U$ called the Duggal transformation (named after Professor B.P.Duggal) .Thus for each non-negative integer n, the n^{th} Duggal transformation $\Gamma^n(T)$ can be defined as $\Gamma^n(T) = \Gamma(\Gamma^{n-1}(T))$ and $\Gamma^0(T) = T$.

Thus we obtain the results about the polar decomposition of Duggal transformation i.e. $\Gamma(T) = U \mid \Gamma(T) \mid$ and give the necessary and sufficient condition for $\Gamma(T)$ to have the polar decomposition $\Gamma(T) = \Gamma(U) \mid \Gamma(T) \mid$. As a consequence, we get $\Gamma(T) = \Gamma(U) \mid \Gamma(T) \mid$ to be polar decomposition of $\Gamma(T)$ if T is binormal.

3.1 Relationship between the Duggal and Aluthge transforms.

Definition 3.1.1 *An operator T is said to be binormal if $[\mid T \mid, \mid T^* \mid] = 0$ (i.e. $\mid T \mid\mid T^* \mid - \mid T^* \mid\mid T \mid = 0$) and T is said to be centered if the following sequence , ...$T^3(T^3)^*, T^2(T^2)^*, TT^*, T^*T$, $(T^2)^*T^2, (T^3)^*T^3$, ...is commu-*

tative. The relations among these and that of quasinormal operators are that quasinormal $\subset$ centered (normal).

Lemma 3.1.2 [34] *Let $T = U \mid T \mid$ be the polar decomposition of T. Then If T is invertible, then $\Gamma(T)$ are invertible and in that case $\Gamma(T) =\mid T \mid T \mid T \mid^{-1}$.*

Proof. Since T is invertible, then $\Gamma(T) =\mid T \mid U =\mid T \mid UI =\mid T \mid U \mid T \mid\mid T \mid^{-1}=\mid T \mid T \mid T \mid^{-1}$ since $T = U \mid T \mid$. ■

Lemma 3.1.3 *Let $T = U \mid T \mid$ be the polar decomposition of T. Then $\Gamma(T) = U^*TU$.*

Theorem 3.1.4 [22] *Let $T = U \mid T \mid$ be the polar decomposition of T. If T is binormal then $\Delta(T) = U^*UU \mid \Delta(T) \mid$ is the polar decomposition of $\Delta(T)$.*

Theorem 3.1.5 [34] *Let T be invertible and suppose that $T = U \mid T \mid$ is the polar decomposition of T. If T is binormal, then $\Delta(T) = U \mid \Delta(T) \mid$ is the polar decomposition of $\Delta(T)$.*

Proof. Since T is invertible and U is unitary,then by *Theorem* 3.1.4, the proof follows. ■

Theorem 3.1.6 *Let $T = U \mid T \mid$ be the polar decomposition of the operator T, and U is coisometry .Then T is binormal iff $\Gamma(T)$ is binormal.*

Proof. We know that $\Gamma(T) = U^*TU$ from *Lemma* 3.1.3. Therefore $(\Gamma(T))^*\Gamma(T) = U^* \mid T \mid^2 U \geq 0$ and $\mid \Gamma(T) \mid= U^* \mid T \mid U$. On the other hand, $\Gamma(T)(\Gamma(T))^* = U^* \mid T^* \mid^2 U \geq 0$ and $(\Gamma(T))^* = U^* \mid T^* \mid U$. Hence , if T is binormal, then $\Gamma(T)$ is also binormal.

Conversely, if $\Gamma(T)$ is binormal, then we have that $U^* \mid T^* \mid\mid T \mid U = U^* \mid T \mid\mid T^* \mid U$. Multiplying U and U^* with both sides, we obtain
$\mid T^* \mid\mid T \mid=\mid T \mid\mid T^* \mid$ i.e. T is binormal. ■

Note that *Theorem* 3.1.6 does not hold if U coisometry is replaced by U isometry.

Theorem 3.1.7 [34] *Let T be invertible. If T is binormal, then $\Gamma(\Delta(T)) = \Delta(\Gamma(T))$.*

The following is characterization of centered operators from the view point of polar decomposition and Aluthge transformation as seen in [19].

Theorem 3.1.8 [22] *Let T be an operator. Then $\Delta^n(T)$ is binormal for all $n \geq 0$ iff T is a centered operator.*

Theorem 3.1.9 [34] *Let T be invertible and centered. Then* $\Gamma(\Delta^n(T)) = \Delta^n(\Gamma(T))$ *for all* $n \geq 0$.

Proof. Since T is centered, then T is binormal. Therefore, the result is true for $n = 1$ by *Theorem* 3.1.7. Suppose that the result is true for $n = m - 1$, then $\Gamma(\Delta^{m-1}(T)) = \Delta^{m-1}(\Gamma(T))$. Now, $\Delta^{m-1}(T)$ is invertible since T is invertible. By *Theorem* 3.1.8, $\Delta^{m-1}(T)$ is binormal. Therefore by *Theorem* 3.1.7, $\Gamma[\Delta(\Delta^{m-1}(T))] = \Delta[\Gamma(\Delta^{m-1}(T))]$. Hence $\Gamma(\Delta^m(T)) = \Gamma[\Delta(\Delta^{m-1}(T))] = \Delta[\Gamma(\Delta^{m-1}(T))] = \Delta[\Delta^{m-1}(\Gamma(T))] = \Delta^m(\Gamma(T))$. ■

Theorem 3.1.10 *Let T be invertible and binormal. Then* $\Delta(\Gamma^n(T)) = \Gamma^n(\Delta(T))$ *for all* $n \geq 0$.

Proof. The result is true for $n = 1$, by *Theorem* 3.1.7. Suppose that the result is true for $n = m - 1$. Then $\Delta(\Gamma^{m-1}(T)) = \Gamma^{m-1}(\Delta(T))$. Since T is invertible and binormal, by *Lemma* 3.1.2and *Theorem* 3.1.6, $\Gamma^n(T)$ is invertible and binormal for all $n \geq 0$. Since $\Gamma^{m-1}(T)$ is invertible and binormal, by *Theorem* 3.1.7, $\Delta[\Gamma^{m-1}(T)] = \Gamma[\Delta(\Gamma^{m-1}(T))]$. Hence, $\Delta(\Gamma^m(T)) = \Delta[\Gamma(\Gamma^{m-1}(T))] = \Gamma[\Delta(\Gamma^{m-1}(T))] = \Gamma[\Gamma^{m-1}(\Delta(T))] = \Gamma^m(\Delta(T))$. ■

Theorem 3.1.11 [34] *Let T be invertible and centered, then* $\Gamma^m(\Delta^n(T)) = \Delta^n(\Gamma^m(T))$, *for all* $m, n \geq 0$.
Proof. *Every centered operator is binormal. Therefore the proof follows by Theorem* 3.1.7 *and Theorem* 3.1.10 ■

3.2 The Polar decomposition of Duggal transform.

The following results shows the polar decomposition of the product of two operators.

Theorem 3.2.1 [16] *Let* $T = U \mid T \mid$*and* $S = V \mid S \mid$ *be the polar decompositions. If T and S are doubly commutative(i.e.* $[T, S] = [T, S^*] = 0$*),then* $TS = UV \mid TS \mid$ *is the polar decomposition of* TS.

The following is a generalization of this result.

Theorem 3.2.2 [21] *Let* $\mathbf{T} = \mathbf{U} \mid \mathbf{T} \mid, S = V \mid S \mid$*and*$\mid T \mid\mid S^* \mid = W \mid\mid T \mid\mid S^* \mid\mid$*be the polar decompositions. Then* $TS = UWV \mid TS \mid$*is also the polar decomposition.*

Thus *Theorem* 3.2.2 can be used to obtain the the polar decomposition of Duggal transformation of an invertible operator as follows.

Theorem 3.2.3 [34] *Let T be invertible. If* $T = U \mid T \mid$ *is the polar decomposition of T, then* $\Gamma(T) = U \mid \Gamma(T) \mid$ *is the polar decomposition of* $\Gamma(T)$.

Lemma 3.2.4 [34] *If U is a partial isometry, then $\mid U \mid= U^*U$ and $U = U \mid U \mid$ is the polar decomposition of U. Also $\Gamma(U) = \Delta(U) = U^*UU$.*

Theorem 3.2.5 [21] *Let $T = U \mid T \mid$and $S = V \mid S \mid$ be the polar decompositions. Then $\mid T \mid\mid S^* \mid=\mid S^* \mid\mid T \mid$ iff $TS = UV \mid TS \mid$ is the polar decomposition.*

Therefore *Theorem* 3.2.5 is used to prove the following result on the polar decomposition of the Duggal transformation on an operator.

Theorem 3.2.6 *Let $T = U \mid T \mid$ be the polar decomposition of T. Then $\Gamma(T) = \Gamma(U) \mid \Gamma(T) \mid$ is the polar decomposition of $\Gamma(T)$ iff $\mid T \mid\mid U^* \mid=\mid U^* \mid\mid T \mid$.*

Theorem 3.2.7 *Let $T = U \mid T \mid$be the polar decomposition of T. If T is binormal, then $\Gamma(T) = \Gamma(U) \mid \Gamma(T) \mid$ is the polar decomposition of $\Gamma(T)$.*

Proof. Let $F = U^*U$. Then F is the support of T^*. If T is binormal, then $Ker(T^*)$ is invariant under $\mid T \mid$. Therefore , $(Ker \mid T \mid)^{\perp}$ is invariant under $\mid T \mid$. But $(Ker \mid T^* \mid)^{\perp} = (KerT^*)^{\perp} = RanF$. Therefore $F \mid T \mid F =\mid T \mid F$ and $\mid T \mid F = F \mid T \mid$. It follows that $\mid T \mid\mid U^* \mid=\mid U^* \mid\mid T \mid$ by *Theorem* 3.2.6, $\Gamma(T) = \Gamma(U) \mid \Gamma(T) \mid$ is the polar decomposition of $\Gamma(T)$. ∎

Theorem 3.2.8 [34] *Let $T = U \mid T \mid$ be the polar decomposition and E, F the initial and final projection respectively of the partial isometry U. If $\Gamma(T) = \Gamma(U) \mid \Gamma(T) \mid$ is the polar decomposition of $\Gamma(T)$, then $EF = FE$ are equivalently and U is binormal.*

Proof. We have that $E = U^*U$ and $F = UU^*$. If $\Gamma(T) = \Gamma(U) \mid \Gamma(T) \mid$ is the polar decomposition of $\Gamma(T)$, then by *Theorem* 3.2.6, $\mid T \mid\mid U^* \mid=\mid U^* \mid\mid T \mid$. Thus, $\mid T \mid F = F \mid T \mid$ and therefore $Ran \mid T \mid$ is invariant under F. Hence $\overline{Ran \mid T \mid}$ is invariant under F. But $\overline{Ran \mid T \mid} = (KerT)^{\perp} = RanE$. Hence $EF = FE$. Next, U is binormal iff $\mid U \mid\mid U^* \mid=\mid U^* \mid\mid U \mid$ iff $EF = FE$.
∎

Remark 3.2.9 [34] *Let T be an operator and $T = U \mid T \mid$, be the polar decomposition of T. Let E, F be the initial and final projections respectively of the partial isometry U. By Theorem 3.2.4, $\Gamma(U) = \Delta(U) = U^*UU$.*

Theorem 3.2.10 [*Theorem*3.2.1, [20]] *If U is a partial isometry, then the following assertions are mutually equivalent :*

(*i*) U is binormal.
(*ii*) $\Delta(U)$ is a partial isometry.
(*ii*) U^2 is a partial isometry.

Remark 3.2.11 *If* $\Gamma(T) = \Gamma(U) \mid \Gamma(T) \mid$ *is a polar decomposition of* $\Gamma(T)$, *then the following hold :*

(*i*) $\mid T \mid\mid U^* \mid = \mid U^* \mid\mid T \mid$.
(*ii*) $EF = FE$.
(*iii*) U is binormal.
(*iv*) $\Gamma(U)$ is a partial isometry.
(*v*) U^2 is a partial isometry.
On the other hand, if T is binormal, then $\Gamma(T) = \Gamma(U) \mid \Gamma(T) \mid$ is the polar decomposition of $\Gamma(T)$.

Theorem 3.2.12 [35] *Let* $T = U \mid T \mid$ *be the polar decomposition of* T. *If* $\Gamma^n(T) = \Gamma^n(U) \mid \Gamma^n(T) \mid$ *is the polar decomposition of* $\Gamma^n(T)$ *for all* $n = 0, 1, 2, 3, \ldots\ldots\ldots$, *then* $\mid U^* \mid$ *commutes with every* $\mid \Gamma^n(T) \mid$, *i.e.* $RanU$ *is invariant under every* $\mid \Gamma^n(T) \mid$.

3.3 Complete contractivity of maps associated with the Duggal transform.

For a bounded linear operator T in a complex Hilbert space H, we study the maps $\widehat{\phi} : f(T) \to f(\Gamma(T))$ and $\widetilde{\phi} : f(T) \to f(\Delta(T))$ where $\Gamma(T)$, $\Delta(T)$ are the Duggal and Aluthge transformation of T respectively and $f \in Hol(\sigma(T))$ where $\sigma(T)$ is the spectrum of T. Thus we show that the maps are completely contractive algebra homomorphisms. As applications,we obtain that every spectral set for T is also a spectral set for $\Gamma(T)$ and $\Delta(T)$ and also the inclusion $\overline{W(f(\tilde{T}))} \cup \overline{W(f(\Gamma(T)))} \subset \overline{W(f(T))}$ relating to the numerical range of $f(T)$, $f(\tilde{T})$ and $f(\Gamma(T))$. Therefore we explore various relations between the T, $\tilde{T}$ and $\Gamma(T)$, by studying the maps between the Riesz-Dunford algebras associated with these operators. From[25], we have that $\sigma(T) = \sigma(\Gamma(T)) = \sigma(\tilde{T})$ for all $T \in B(H)$ and when some $T \in B(H)$, in some consideration, we denote $Hol(\sigma(T))$-the algebra of all complex-valued functions which are analytic on some neighborhood of $\sigma(T)$.

Moreover, the Riesz-Dunford algebra $A_T \subset B(H)$ is defined as

$A_T = \{f(T) : f \in Hol(\sigma(T))\}$, where $f(T)$ is defined by the Riesz-Dunford functional calculus. Thus we study the maps between A_T, $A_{\tilde{T}}$ and $A_{\widehat{T}}$ where $\tilde{T} = \Delta(T)$ is Aluthge transform, and $\widehat{T} = \Gamma(T)$ is the Duggal transform.

Theorem 3.3.1 [15] *For every* $T \in B(H)$, *with* $\tilde{T}$, $\widehat{T}$, *and* $Hol(\sigma(T))$ *as defined above:*

(*a*) The maps $\widehat{\phi} : A_T \to A_{\widehat{T}}$ and $\widetilde{\phi} : A_T \to A_{\tilde{T}}$, defined by $\widehat{\phi}(f(T)) = f(\widehat{T})$ and $\widetilde{\phi}(f(T)) = f(\tilde{T})$, $f \in Hol(\sigma(T))$. i.e.

$$\max\left\{\| f(\widehat{T}) \|, \| f(\check{T}) \|\right\} \leq \| f(T) \|,\ f \in Hol(\sigma(T)).$$

(b) The maps $\widehat{\phi}$ and $\widetilde{\phi}$ in (a) above are completely contractive i.e. for every $n \in \mathbb{N}$ and every $n \times n$ matrix (f_{ij}) with entries from $Hol(\sigma(T))$, $max\left\{\| f_{ij}(\widehat{T}) \|, \| f_{ij}(\check{T}) \|\right\} \leq \| (f_{ij}(T)) \|$.

(c) If $T \in B(H)$, then $\overline{W(f(\check{T}))} \cup \overline{W(f(\Gamma(T)))} \subset \overline{W(f(T))}$, $f \in Hol(\sigma(T))$.

Lemma 3.3.2 [15] *For every $T = U \mid T \mid$ in $B(H)$, then we have that:*

(a) $\mid T \mid T = \Gamma(T) \mid T \mid$.
(b) $TU = U\ \Gamma(T)$.
(c) $\mid T \mid^{\frac{1}{2}} T = \check{T} \mid T \mid^{\frac{1}{2}}$.
(d) $\mid T \mid^{\frac{1}{2}} \check{T} = \Gamma(T) \mid T \mid^{\frac{1}{2}}$.

Lemma 3.3.3 *For every $T = U \mid T \mid$ in $B(H)$ and every $f \in Hol(\sigma(T))$, we have :*

(a) $\mid T \mid f(T) = f(\widehat{T}) \mid T \mid$.
(b) $f(T)U = U\ f(\widehat{T})$.
(c) $\mid T \mid^{\frac{1}{2}} f(T) = f(\check{T}) \mid T \mid^{\frac{1}{2}}$.
(d) $\mid T \mid^{\frac{1}{2}} f(\check{T}) = f(\widehat{T}) \mid T \mid^{\frac{1}{2}}$.

Lemma 3.3.4 *For $T = U \mid T \mid$ in $B(H)$ and every $f \in Hol(\sigma(T)), f(\widehat{T})$ is the (orthogonal) direct sum, $f(\widehat{T}) = EU^{*}f(T)UE/_{(KerT)^{\perp}} \oplus f(0)/_{KerT}$; where E is the (orthogonal) projection $U^{*}U$ on $(KerT)^{\perp}$ and consequently,*

$$\| f(\widehat{T}) \| \leq \| f(T) \|.$$

Proposition 3.3.5 [15] *For every $T \in B(H)$, $\overline{W(T)}$ is the intersection of all closed half planes H containing $W(T)$ such that H is a spectral set for T.*

Proof. Since $W(T)$ is convex and is thus the intersection of all closed half planes containing $W(T)$. It suffices to show that if H is any closed half plane containing $W(T)$, then H is a spectral set for$T$. By a Harmles rotation and translation, we may suppose that H is the closed right-half plane $\{z : Rez \geq 0\}$.Thus, writing $T = K + iL$, with K and L Hermitian(self-adjoint), we see that K is positive semi-definite, and therefore that the Cayley transformation of T, i.e. $C(T) = (T + iI)^{-1}(T - iI)$, is a contraction [21, p.167]. Hence by Von-Neuman's inequality (Let X be a normed linear space with the norm $\| \cdot \|$. Then there is a unique inner product function $< , >$ on $X \times X$, such that $< x, x >= \| x \|^2$ iff $\| x + y \|^2 + \| x - y \|^2 = 2(\| x \|^2 + \| y \|^2)\ \forall\, x, y \in X$) , the closed unit disk $\mathbb{F}$ in $\mathbb{C}$ is a spectral set for $C(T)$ and thus , by taking the inverse of Cayley transformation, we obtain that H is a spectral set for T,as defined. ■

Chapter 4

THE CAYLEY TRANSFORM OF THE LINEAR OPERATORS.

4.1 Introduction.

Let H be a complex Hilbert Space with scalar product $(.,.)$ and norm $\|\cdot\|$. $H \oplus H$ will denote the direct sum of H with itself, with its natural Hilbert space structure. The notion of linear relation on a normed space which was first introduced by R.Arens in [9, 1961], generalizes the notion of linear operator in (which is equivalent to the notion of multivalued operators and a general introduction is found in [5]). The set of such linear relations on H will be denoted by $L_r(H)$ while $LR(H)$ will denote the set of closed elements of $L_r(H)$. (i.e. those that have a closed graph in $H \oplus H$). The concept of symmetric and self-adjoint extensions of symmetric operators and the Von-Neumann *Theorem* about the extensions of symmetric operators can also be extended on $L_r(H)$ [27].Thus, we show that the Cayley transform of a linear relation can be defined directly (as for operator) by an algebraic formula and can therefore be extended to $L_r(H)$.

Definition 4.1.1 *In a finite dimensional and over $\mathbb{R}$, if $TI = -T$, then $I+T$ is invertible and the Cayley transform $C(T)$ of an operator T is defined as $C(T) = (I-T)(I+T)^{-1}$ is orthogonal.In this case, multiplcation is commutative, thus we can also write $C(T) = (I+T)^{-1}(I-T)$. Now, suppose that S is any orthogonal matrix which does not have -1 as an eigenvector, then $T = (I-S)(I+S)^{-1}$ is skew-symmetric which owns functional inverse, so that $C(T) = \left(C(T)^C\right)^C$, where " C " denotes the tranform and*

$$T = (T^C)^C.$$

Definition 4.1.2 *In the complex plane, the Cayley transform $C(T)$ of an operator T such that $T + iI$ has a trivial kernel is usually defined as*

$C(T) = (T - iI)(T + iI)^{-1} = 1 - 2i(T + iI)^{-1}$.

In complex analysis, the Cayley transform is a mapping of the complex plane to itself, given as $W : Z \longrightarrow \frac{z-i}{z+i}$ where W has the following properties:

(i) W maps the upper half plane of $\mathbb{C}$ conforming onto the unit disc of $\mathbb{C}$.

(ii) W maps the real line $\mathbb{R}$ injectively into the unit circle∂ $\mathbb{F}$(complex numbers of absolute values).

(iii) W maps the upper imajinary axis $i[0, \infty]$ bijectively onto the half-open interval $[-1, 1)$.

(iv) W maps 0 to -1.

(v) W maps the part of infinity to 1.

(vi) W maps $-i$ to the part at infinity (so W has a pole at $-i$).(vii) W maps -1 to i and it also maps both $\frac{1}{2}(1+\sqrt{3})(-1+i)$ and $\frac{1}{2}(1+\sqrt{3})(1-i)$ to themselves.

Thus the two expressions in *Definition* 4.1.2 coincide when T is an operator,but this in not in general the case when T is a linear relation as shown below in *Corollary* 4.3.3 and it turns out that it is the second expression that yields the correct definition. Also we deal with a small variation of the Cayley transform- the Z transform- which yields the additional and desirable property of being an involution : $Z(T) = -i(T) - 2(T + iI)^{-1}$.

4.2 Linear relations.

Let J and K denote the symmetric operator in $H \oplus H$ described by matrices :

$J = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, K = \begin{pmatrix} 0 & -iI \\ iI & 0 \end{pmatrix}$. we also use $U = \frac{(J+K)}{\sqrt{2}}$,

$P_1 = \frac{(I+J)}{2}$ and $P_2 = \frac{(I-J)}{2}$.Finally ,let $H_1 = H \oplus \{0\} = P_1(H \oplus H)$,and $H_2 = \{0\} + H = P_2(H \oplus H)$.

Proposition 4.2.1 $(JK) \times (KJ) = I$.

Definition 4.2.2 *Let $P(H)$ denote the set of all subsets of H. Then $P(H)$ can be given as a linear structure on $\mathbb{C}$ as follows: If $M, N \in P(H)$, set $M + N = \{z : z = m + n,\ m, n \in \mathbb{N}\}$ and $\forall\ \lambda \in \mathbb{C}$, set $\lambda M = \{\lambda z : z \in M\}$.*

Definition 4.2.3 *A linear relation T in H is a mapping of a linear subspace $\widetilde{D}(T)$ of H onto $P(H)$ such that $\forall\ x, y \in \widetilde{D}(T)$, $T(x + y) = T(x) + T(y)$, and $\forall\ \lambda \in \mathbb{C}$, $\forall\ x \in \widetilde{D}(T)$, $T(\lambda x) = \lambda T(x)$.*

Definition 4.2.4 *Let T be a linear relation in H. We define the domain of T by $D(T) = P_1(G(T)) = \tilde{D}(T) \oplus \{0\}$, the range of T by $R(T) = P_2(G(T)) = \{0\} + \tilde{R}(T)$ and the kernel of T by $N(T) = G(T) \cap H_1 = \tilde{N}(T) \oplus \{0\}$. Then the graph of T is the linear subspace of $H \oplus H$ given by*

$G(T) = \{(x, y) : x \in \tilde{D}(T), y \in T(x)\}$. Setting $G(T) \cap H_2 = \{0\} \oplus T(0)$, we call $T(0)$ the multivalued part of T and we note that if $T(0) = \{0\}$, then T is an operator. The set of all bounded and every where defined linear operator on H will be denoted by $L(H)$. When $T \in LR(T)$, we shall write P_T for the orthogonal projection in $H \oplus H$ onto $G(T)$.

Example 4.2.5 [30] *Let X, Y be two closed subspace of H such that $X + Y = H$. Then every $z \in H$ can be written as $z = x + y$, with$x \in X$ and $y \in Y$. Denote by P the linear mapping of H into $P(H)$ given by $P : z \longrightarrow \{x\} + X \cap Y$.Then $P \in LR(H)$.Thus, we notice that $\tilde{N}(P) = Y$ and $P(0) = X \cap Y$.*

Definition 4.2.6 *Let $T, S \in LR(H)$. Then $g(T, S) =\| P_T - P_S \|$, $LR(H)$ equipped with the metric g (the gap matrix) is complete.*

Definition 4.2.7 *Let $T \in L_r(H)$ and $\lambda \in \mathbb{C}$. Then λT is the linear relation such that its graph is $G(\lambda T) = \{\{x, \lambda y\} : \{x, y\} \in G(T)\}$.*

Definition 4.2.8 *Let $T, S \in L_r(H)$. Then $T + S$, the sum of T and S, is the linear relation with the graph $G = \{\{x, \mu\} : \exists y, z : \{x, y\} \in G(T); \{x, z\} \in G(S)$ and $\mu = y + z\}$. Clearly, $D(T + S) = D(T) \cap D(S)$ and $(T + S)(0) = T(0) + S(0)$.[see* [30]]

Remark 4.2.9 *The sum is associative and commutative .If 0 is the linear relation whose graph is H_1, then clearly $\forall\, T \in L_r(H)$, $0+T = T+0 = T$ so that 0 is neutral element for the sum. Finally we note that unless $T \in L(H)$, $T + (-T) \neq 0$ although $T + (-T) = T$ and $(-T) + T + (-T) = -T$ so that $-T$ is a generalized inverse of T for the sum.*

Definition 4.2.10 *Let $T \in L_r(H)$.Then the inverse T^{-1} is the linear relation with the graph $G(T^{-1}) = KJ(G(T)) = JK(G(T))$ (Since $JK = -KJ$ by Proposition* 4.1.1) *and clearly $(T^{-1})^{-1} = T$.*

Definition 4.2.11 *Let $T, S \in L_r(H)$.Then TS, the product of T on the left by S, is the linear relation with graph $G = \{\{x, y\} : \exists z \in H$ such that $\{x, z\} \in G(T), \{z, y\} \in G(S)\}$.*

Remark 4.2.12 *The product is associative and non-commutative. If I is the identity mapping on H, then clearly there exist $T \in L_r(H)$, $IT = TI = T$ so that T^{-1} is a generalized inverse of T for the product.*

Definition 4.2.13 *Let $T \in L_r(H)$.Then the adjoint T^* is the closed linear relation with gragh $G(T^*) = K(G(T)^\perp) = K(G(T))^\perp$.*

Proposition 4.2.14 [30] *Let $T \in L_r(H)$.Then $\tilde{D}(T^*) = E(0)^\perp$.*

Proposition 4.2.15 *Let $T \in LR(H)$. Then $(T^*)^{-1} = (T^{-1})^*$.*

Proof. $G((T^*)^{-1}) = JK(G(T^*)) = JKK(G(T)^\perp) = K[[KJ(G(T))]^\perp] = G((T^{-1})^*)$. ■

Proposition 4.2.16 [30] *Let $T \in LR(H)$ and $S \in L(H)$. Then $T^* + S^* = (T+S)^*$.*

4.3 The extended Cayley transform.

Definition 4.3.1 *Let T be the a linear relation. Then*

$Z(T) = -iI - 2(T+iI)^{-1}$ by the definition of the Cayley transform of T.

Proposition 4.3.2 *Let $T \in L_r(H)$ and let $S \in L(H)$ .Then*

$$G[(T+S^*)(T+S)^{-1}] = [I + (S^* - S)(T+S)^{-1}] = \{0\} + T(0).$$

Corollary 4.3.3 *Let $T \in L_r(H)$ and and let $S \in L(H)$. Then if $T+S$ is injective and T is not an operator, then*

$$G[(T+S^*)(T+S)^{-1}] \neq G[I + (S^* - S)(T+S)^{-1}].$$

Proof. Under the hypothesis, $I + (S^* - S)(T+S)^{-1})$ is an operator.(i.e. has a trivial multivalued part) while $T(0)$ is the nontrivial multivalued part of $(T+S^*)(+S)^{-1}$. ■

Corollary 4.3.4 *The transform Z is an isometry on $LR(H)$, i.e. for all $T, S \in LR(H), g(Z(T), Z(S)) = g(T, S)$.*

Proof. $g(Z(T), Z(S)) = \| P_{Z(T)} - P_{Z(S)} \| = \| U(P_T - P_S)U \| = \| P_T - P_S \| = g(T, S)$. ■

Proposition 4.3.5 *Let T be a linear relation (not necessary closed). Then $Z(Z(T)) = T$. This property is the reason for the modification (by a factor -1) of the usual definition of the Cayley transform.*

Proof. Let $S = Z(T)$. Then $S + iI = -2(T+iI)^{-1} \iff (S+iI)^{-1} = -(\frac{1}{2})(T+iI) \iff (T+iI) = -2(S+iI)^{-1} \iff T = Z(S)$, so that $T = Z(Z(T))$. ■

Theorem 4.3.6 [30] *Let T be a linear relation (not necessary closed). Then* $Z(T^{-1}) = (Z(T))^{-1}$.

Proof. Let $S = Z(T)$. Then we have $(u,v) \in G(S) \Longleftrightarrow v = -u + v$ and $\{u,v\} \in G(-2(T+iI)^{-1}) \Longleftrightarrow v = -iu + v$ and $\{-2u, v\} \in G((T = iI)^{-1}) \Longleftrightarrow v = -iu + v$ and $\{v, -2u\} \in G(T + iI) \Longleftrightarrow v = iu = v$ and $\{r, -2u - iv\} \in G(T) \Longleftrightarrow v = -iu + v$ and $\{-2u - iv, r\} \in G(T^{-1}) \Longleftrightarrow v = -iu + v$ and $\{-2u - iv, 2r - 2iu\} \in G(T^{-1} + iI) \Longleftrightarrow v = iu + r$ and $\{2r, -2u - ir\} \in G((T^{-1} + iI)^{-1}) \Longleftrightarrow u = 2u + ir - iv$ and $\{v, 2u + ir\} \in G(-2(T^{-1} + iI)^{-1}) \Longleftrightarrow \{u,v\} = \{v, 2u + ir - iv\} \in G(Z(T^{-1}))$ so that $\{u,v\} \in G((Z(T))^{-1}) \Longleftrightarrow \{u,v\} \in G(Z(T^{-1}))$ which establishes the *Theorem.* ∎

Thus we define the Cayley transform of a self-adjoint operator and show how to integrate measurable (but bounded) functions with respect to a resolution of the identity. Therefore we state a special case of a result.

Proposition 4.3.7 [2] *Let T be a symmetric operator on H. Then :*

(*a*) $\| (T \pm iI)x \|^2 = \| Tx \|^2 = \| x \|^2 \; \forall x \in D(T)$.

(*b*) $(T \pm iI)$ is one-to-one $(1-1)$.

(*c*) The map $(T \pm iI)x \longrightarrow (x, Tx)$ is an isometry from $R(T \pm iI)$ onto $G(T)$.

Thus $R(T \pm iI)$ is closed iff T is closed. Since $\| (T + iI)x \|^2 = \| (T - iI)x \|$, we define an isometry $U : R(T + iI) \longmapsto R(T - iI)$ by $U((T + iI)x) = (T - iI)x$ by $U((T + iI)x) = (T - iI)x$. Then $T + iI$ maps $D(T)$ in a 0ne-to-one fashion to $R(T + iI) = D(U)$, so we define $(T + iI)^{-1} : D(U) \longrightarrow D(T)$, and obtain $U = (T - iI)(T + iI)^{-1}$ which maps $D(U) = R(T + iI) \longrightarrow R(U) = R(T - iI)$. Thus U is called the Cayley transform of T.

Theorem 4.3.8 [2] *Let T be a symmetric operator and $U = (T - iI)(T + iI)^{-1}$ be the Cayley transform. Then:-*

(*a*) U is closed iff T is closed.

(*b*) $R(I - U) = D(T), I - U$ is one-to -one and $T = i(I + U)(I - U)^{-1}$,where $DimU = (A + iI)dimT$.

(*c*) U is unitary iff T is self-adjoint.

Conversely, if W is isometric on its domain and if $(I - W)$ is one-to-one, then W is the Cayley transform of a symmetric operator on H.

Applications of symmetric operators.

$< Tx, y > = < x, Ty >$ since bounded symmetric operators are Hermitian(sel-adjoint).

Example 4.3.9 *Consider the complex Hilbert space $\ell^2[0,1]$ and the differential operator $T = \frac{-d^2}{dx^2}$ defined on the subspace consisting of all complex-valued infinitely differentiable functions f on $[0,1]$ with the boundary condition: $f(0) = f(1) = 0$. Integration by parts shows that T is symmetric.Cayley transform can be used to find the self-adjoin extension of a symmetric operator.*

Theorem 4.3.10 *Suppose that T is symmetric operator. Then $\exists$ a unique linear operator $W(T) : Ran(T + iI) \longrightarrow Ran(T - iI)$ such that $W(T)(Tx + iIx) = Tx - ix, x \in Dom(T)$. Hence $W(T)$ is isometric.*

Conversely, given any isometric operator U, such that $I - U$ is dense, $\exists$ aunique operator $S(U) : Ran(I - U) \longrightarrow Ran(I + U)$ such that $S(U)(x - Ux) = i(x + Ux), x \in Dom(U)$. The operator $S(U)$ is symmetric and the mappings W and S are inverse of each other. W is a Cayley transform. It associates an isometry to a symmetric (self-adjoint) operator.

Theorem 4.3.11 *A necessary and sufficient condition for T to be self-adjoint is that its Cayley transform $C(T)$ is unitary. This gives a neccesary and sufficient condition for T to have a self-adjoint extension.*

Application of self-adjoint extensions.

Observable corresponds to the self-adjoint operators are the generation of unitary groups of time evolution operators. However, many physical problems are formulated as a time evolution equation involving differential operators for which the Hamiltonian is only symmetric.On such cases,either the Hamiltona is essentially self-adjoint (and not necessarily self-adjoint) in which are the physical problem has unique solutions or one attempts to find self-adjoint extension of the Hamiltonian correspondingto different types of boundary condition at infinity.

Example 4.3.12 (1) *Let M be a multiplicative operator on $\ell^2(\mathbb{R})$. Compute the Cayley transform of M and verify that it is unitary.*

(2) Let M be the multiplicative operator on $\ell^2[0,1]$.Compute the Cayley transform of M and verify that it is unitary.

(3) The right shift S on $\ell_2(\mathbb{N})$ is an isometry and $I - S$ is injective(or one-to-one), so S is the Cayley transform of a symmetric operator T. What are the deficiency indices of T ?

Definition 4.3.13 *Let T be an operator with $Dom(T)$ dense in H. The operator $C(T) = (T - iI)(T + iI)^{-1}$ defined on $DomC(T) = (T + iI)DomT$.*

This transform establishes a correspondence between the properties of operator T with operator $\sigma(T)$ " close " to the real line and operators with almost-unitary spectra (close to the circle $\{\zeta \in \mathbb{C} : |\zeta| = 1\}$) :

(i) If T is a disspative operator,then $C(T)$ is a contraction (i.e. $\| C(T)x \| \leq \| x \|, x \in DomT$) and $Ker(I - C(T)) = \{0\}$.

(ii) If A is a contraction, $Ker(I - A) = \{0\}$ and $(I - A)DomA$ is dense in H, then $A = C(T)$.

For some linear disspative operator T:

(iii) T is symmetric (self-adjoint) iff $C(T)$ is isometric (unitary).

(iv) $\sigma(T) = W(\sigma(C(T)))$, where $W(\zeta) = i(I+\zeta)(I-\zeta)^{-1}.T$ is bounded iff $1 \notin \sigma(C(T))$.

(v) If γ is an operator ideal in H, then $A - B \in \gamma \Longrightarrow C(A) - C(B) \in \gamma$, if A, B are bounded operator ; then the converse is also valid : $C(A) - C(B) \in \gamma \Longrightarrow A - B \in \gamma$

The Cayley transform also establishes a correspondence between certain other characterization of the operators T and $C(T)$ classifications of parts of the spectrum, multiplication, of spectra, structures of invariant subspaces, functional calculus, spectral decompositions. etc.

4.4 Quaternionic Cayley transform.

The classical Cayley transformation $K(t) = \frac{t-i}{t+i} = \frac{t^2-1-2it}{1+t^2}$,is a bijection map between the real line $\mathbb{R}$ and the set $\Bbbk \mid \{1\}$, where $\Bbbk$ is the unit circle in the complex plane $\mathbb{C}$. The formula can be extended to more general situations as for instance, that of (not necessary bounded) symmetric operator in Hilbert space. A Cayley transform can actually be defined for large classes of operators which are no longer symmetric,as well as for other objects, in particular for some linear relations as in 4.2.

In order to find a formulae of this type, valid for normal or formally normal operator, we consider a Quaternionic framework, where we extended this transform using the concept of quaternions by modifying the basic definition which allows us to get the properties of the Quaternionic Cayley transform directly from those of Neumann's Cayley transformation in [14]. Also, we shall look at the consideration of the Cayley transformation for some operators in Quaternionic context. We recall that the image of a (not necessary bounded) self-adjoint operator by the usual Cayley transform is a unitary operator U with the property $I - U$ is injective, where I is the identity and the converse is true from [31]. By this property, we shall describe the unitary lying in the range of Quaternionic Cayley transform which are images of some (not necessary bounded) normal operator.

Considering the normal extension, let D be a dense subspace in a Hilbert space H, T be a densely defined linear operator in H with the property that T and its adjoint T^* are both defined in D. Writing $T = A + iB$, with $A = \frac{(T+T^*)}{2}$ and $B = \frac{(T-T^*)}{2}$ and so A and B are symmetric operator on D . We can associate the operator T with the matrix operator $Q_T = \begin{pmatrix} A & B \\ -B & A \end{pmatrix}$.

And we know from ([31], *Theorem* 3.7), that T is normal in H if the operator Q_T is normal in the Hilbert space $H \oplus H$, because our aim is based on the Quaternionic Cayley transform, give the condition to ensure that the extension of a normal extension for a matrix operator resembling to Q_T.

Finally, we note that the Quaternionic algebra is intimately related also as the spectral theory of pairs of commuting operators as in [36]. Now let us look at the Cayley transforms in the algebra of Quaternionic.

4.5 Cayley transforms in the algebra of quaternions.

We present an approach to the Cayley transform in the algebra of Quaternions. Consider a 2 ×2 matrices
$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, K = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, L = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Let H be the identity with the $\mathbb{R}$−sub-algebra of the algebra M_2 of 2×2 matrices with the complex entries, generated by the matrices I, J, K and L. The embedding $\Re \subset M_2$ allows us to regard the element of $\Re$ as matrices and to perform some operations in M_2 rather than $\Re$. (Matrices $I, -iK$ and L are called *pauli matrices* in mathematical physics and they do not belong to $\Re$). If we put $Q(Z) = Q(z_1, z_2) = \begin{pmatrix} z_1 & z_2 \\ \overline{z_2} & \overline{z_1} \end{pmatrix}$ for every $Z = (z_1, z_2) \in \mathbb{C}^2$, the set$\{Q(Z) : Z \in \mathbb{C}^2\}$ is precisely the algebra of quaternion, because of the decomposition $Q(Z) = (Rez_1)I \; + i(Imz_1)J \; + (Rez_2)k \; + i(Imz_2)L$. We note that

$J^* = J,\, K^* = -K,\, L^* = L,\, J^2 = -K^2 = L^2 = I.$

$JK = L = -KJ,\, KL = J = -LK,\, JL = K = -LJ.$

When the adjoint are computed in the Hilbert space $\mathbb{C}^2$ (endowed with usual Euclidean norm). We note also that $Q(Z)Q^*(Z) = Q(Z)^*Q(Z) = \| Z \|^2 I$ for all $Z \in \mathbb{C}^2$ and so $Q(Z)$ is normal for each $Z \in \mathbb{C}^2$. Moreover, $\| Q(Z) \| = \| Z \|$ for all $Z \in \mathbb{C}^2$ and $Q(Z)^{-1} = \| Z \|^{-2} Q(Z)^*$ for all $Z \in \mathbb{C}^2 / \{0\}$ i.e. .every non null element of the element $\Re$ is invertible. Letting $E = iJ$, we have $E^* = -E$, $E^2 = -I$ and $(Z) = (\operatorname{Re} z_1)I + (\operatorname{Re} z_2)K + E(\operatorname{Im} z_1)I + (\operatorname{Im} z_2)K$ for every $Z = (z_1, z_2) \in \mathbb{C}^2$. Similarly, letting $F = iL$, we have $F^* = -F$, $F^2 = -I$ and $Q(Z) = (\operatorname{Re} z_1)I + (\operatorname{Re} z_2)K + ((\operatorname{Im} z_2)I + (Imz_1)K)F$ for every $Z = (z_1, z_2) \in \mathbb{C}^2$.

Definition 4.5.1 *Let $a, b, c \in \mathbb{R}$ and let*

$$S = S_{a,b,c} = \begin{pmatrix} a & b+ic \\ -b+ic & a \end{pmatrix} = aI + bK + icL.$$

Then the E−Cayley transform of S is the matrix $U = (S-E)(S+E)^{-1} \in H$. Let again $a, b, c \in \mathbb{R}$ and let

$$T = T_{a,b,c} = \begin{pmatrix} a+ic & b \\ -b & a-ic \end{pmatrix} = aI + bK + icJ \ .$$

The $F-$Cayley transform of T is the matrix $V = (T-F)(T+F)^{-1} \in \Re$.

Remark 4.5.2 (a) *The matrices U and V are well defined since $S \neq -E$ and $T \neq - F$.*

(b) The concept from *Definition* 4.5.1 have similar properties to those of the Quaternionic Cayley transform from [31], defined for matrices of the form $S = aI = bK$, i.e. the formula $(S-Q)(S+Q)^{-1}$, with $Q' = Q\left(i\sqrt{2}, i\frac{\sqrt{2}}{2}\right)$.

(c) Let $Q = aI + ibJ + cK + idL$, with $a, b, c, d \in \mathbb{R}$, we have that $b = 0$ iff $JQ = Q^*J$ and $d = 0$ iff $LQ = Q^*L$.

Proposition 4.5.3 [35] *Let $a, b, c \in \mathbb{R}$ and let $S = S_{a,b,c}$. Then matrix $U = (S-E)(S+E)^{-1}$ is unitary and $U = I$. Conversely,given a unitary matrix $U \in H$ with $U \neq I$, there are $a, b, c \in \mathbb{R}$ such that $S = S_{a,b,c}$ where $S = (I+U)(I-U)^{-1}E$. Moreover, the $E-$Cayley transform of the matrix S is unitary matrix U.*

Proof. The proof uses some properties of the Cayley transform of self-adjoint matrices in M_2. Let $S = S_{a,b,c}$, and let $U = (S-E)(S+E)^{-1}$. The matrix $A = JS$ is self-adjoint by *Remark* 4.3.2[3]. Therefore the matrix $W = (A-iI)(A+iI)^{-1}$ which is the Cayley transform of A, is unitary and $I - W$ is invertible. But we have $W = (JS - iJ^2)(JS + iJ^2) = J(S-E)(S+E)^{-1}J$. Consequently $U = JWJ$ is a unitary matrix. Moreover, $I - U = J(I-W)$ is invertible which in $\Re$ is equivalent to $U \neq I$.

Conversely, let $U \in H$ be unitary, with $U \neq I$. We set $W = JUJ$, which is a unitary matrix with $I - W$ invertible. Therefore, the matrix $A = i(I+W)(I-W)^{-1}$ is well defined and self-adjoint as an inverse Cayley transformation. Setting $S = (I+U)(I-U)^{-1}E$,we have $S \in \Re$ and $JS = J(J^2 + JWJ)(J^2 - JWJ)^{-1}E = A$. In particular, we have $S = S_{a,b,c}$ for some $a, b, c \in \mathbb{R}$, by *Remark* 4.3.2[26]. Finally, the equation $S = (I-U)(I+U)^2E = (I-U)^{-1}(I+U)E$ has a unique solution $U = (S-E)(S+E)^{-1}$, which is precisely the $E-$Cayley transform of S. ■

Remark 4.5.4 [35] *Let $a, b, c \in \mathbb{R}$, and let $S = S_{a,b,c}$. A direct calculation shows that the $E-$Cayley transform of S is given by*

$$U = (a^2+b^2+c^2+1)^{-1}((a^2+b^2+c^2-1)I - 2cK - 2aiJ + 2biL)$$
$$= \frac{1}{a^2+b^2+c^2+1} \begin{pmatrix} a^2+b^2+c^2-1-2ai & -2c+2bi \\ 2c+2bi & a^2+b^2+c^2-1+2ai \end{pmatrix}.$$

Conversely, we give a unitary matrix $U \in \Re$ such that $I \neq U$. In fact a unitary matrix $U \in \Re$ is necessary of the form $U = \begin{pmatrix} z_1 & z_2 \\ -\overline{z_2} & \overline{z_1} \end{pmatrix}$, with $z_1, z_2 \in \mathbb{C}$ as $\mid z_1 \mid + \mid z_2 \mid = 1$. As we also have $I \neq U$, and so $Rez_1 \neq 1$, the

matrix $S=(I+U)(I-U)^{-1}E$ is given by : $S=\frac{1}{\operatorname{Re} z_1-1}\begin{pmatrix} \operatorname{Im} z_1 & iz_2 \\ i\overline{z_2} & \operatorname{Im} z_1 \end{pmatrix}$.
This shows in particular that $S=S_{a,b,c}$ with $a=(Rez_1-1)^{-1}\ Imz_1$, $b=-(\operatorname{Re} z_1-1)^{-1}\operatorname{Im} z_2$ and $c=(Rez_1-1)^{-1}\operatorname{Re} z_2$. In fact, the matrix $S=(I+U)(I-U)^{-1}E$ may be called the inverse matrix E−Cayley transform of the unitary matrix U. For F−Cayley transformation, we have the following :

Proposition 4.5.5 [35] *Let $a,b,c\in\mathbb{R}$, and let $T=T_{a,b,c}$. The matrix $V=(T-F)(T+F)^{-1}$ is unitary and $V\neq I$. Conversely,given a unitary matrix $V\in\Bbbk$ with $V\neq I$, there are $a,b,c\in\mathbb{R}$ such that $T=T_{a,b,c}$ where $T=(I+V)(I-V)^{-1}F$. Moreover, the F−Cayley transform of the matrix T is the unitary matrix V.*

4.6 Quaternionic Cayley transform of unbounded operator.

Here we extend the Quaternionic Cayley transform(s) defined in the previous section to some classes of unbounded operators, acting on the Cartesian product of two Hilbert space. Thus we deal with the extension of E−Cayley transform, the properties of the corresponding extension of the F−Cayley transform being similar. Let H be a Hilbert space whose scalar product is denoted by $\langle *,*\rangle$ and whose norm is denoted by $\|*\|$. We work in the Hilbert space $H^2=H\oplus H$, whose scalar product naturally induced by that from H, denoted by $\langle *,*\rangle_2$ and whose norm is denoted by $\|*\|_2$. The matrices M_2 naturally act on H^2 simply by replacing their entries with the corresponding multiples of the identity on H. In particular, the matrices I, J, K, L, E, F, defined in the previous section, naturally act on H^2 and we have the relations:

$$J^*=J,\ K^*=-K,\ L^*=L,\ J^2=-K^2=L^2=I$$
$$JK=L=-KJ,\ KL=J=-LK,\ JL=K=-LJ.$$
$$E^*=-E,\ E^2=-I,\ F^*=-F,\ F^2=-I.$$

Thus we have also some notion and terminology for Hilbert space (always linear). For an operator T acting on H, we denote by $D(T)$, its domain of the definition, the range of T is denoted by $R(T)$, and the kernel of T is denoted by $N(T)$. If T is closable,the closure of T will be denoted by $\overline{T}$ and if T is densely defined, then let T^* be its adjoint and if T_2 extends T_1, we write $T_1\subset T_2$.

Lemma 4.6.1 *Let $S:D(S)\subset H^2\longmapsto H^2$. Suppose that the operator JS is symmetric. Then we have : $\|(S\pm E)x\|_2^2=\|Sx\|_2^2+\|x\|_2^2, x\in D(S)$. If in addition, $JD(S)\subset D(S)$, we have*

$$\|(S\pm E)x\|_2^2=\|Sx\|_2^2+\|x\|_2^2, x\in D(S), \text{ iff } \|SJx\|_2=\|Sx\|_2$$

for all $x \in D(S)$.

Proof. Note that $\| (S \pm E)x \|_2^2 = \| SX \|_2^2 + \langle Sx, \pm iJx \rangle_2 + \langle \pm iJx, Sx \rangle_2 + \| \pm Ex \|_2^2 = \| Sx \|_2^2 + \| x \|_2^2$ because JS is symmetric and E is unitary. Now, if in addition we have $JD(S) \subset D(S)$ and so $JD(S) = D(S)$ because $J^2 = I$, we can write as above $\| (S \pm E)Ex \|_2^2 = \| SEx \|_2^2 + \langle SEx, \pm iJEx \rangle_2 + \langle \pm iJEx, SEx \rangle_2 + \| x \|_2^2 = \| SJx \|_2^2 + \| x \|_2^2$, from which we derive easily the assertion. ■

Example 4.6.2 [35] (1) *Let* $A, B : D \subset H \longmapsto H$, *be symmetric operator. We put* $S = S_{A,B} = AI + BK$, *which is an operator in* H^2, *defined on* $D(S) = D^2 = D \oplus D$. *The operator* JS *is easily seen to be symmetric in* H^2. *Therefore*

$\| Ax + By \pm ix \|^2 + \| -Bx + Ay \pm iy \|^2 = \| Ax + By \|^2 + \| -Bx + Ay \|^2 + \| x \|^2 \| y \|^2$ for all $x.y \in D^2$, by *Lemma* 4.6.1.

(2) Let $L^2(\mathbb{R})$ and let $D \subset L^2(\mathbb{R})$ be the subset of all continuously differentiable function with compact support. Consider the operator $T = i\frac{d}{dt}I + \delta(t)K + i\tau(t)L$,defined on D^2; with values in H^2, where δ and τ are continuous real-valued functions on $\mathbb{R}$. We know that JK is symmetric. Moreover, JK has a self -adjoint extension called the *Dirac* operator [35], of course *Lemma* 4.6.1 applies to this operator T too and the operator T has an $E-$Cayley transform (defined in the next *Remark*).

Remark 4.6.3 [35] *Let* $S : D(S) \subset H^2 \longmapsto H^2$ *be such that* JS *is symmetric, them by Lemma* 4.6.1, *we define the operator* $V : R(S+E) \longmapsto R(S-E)$, $V(S+E)x = (S-E)x$, $x \in D(S)$ *which is a partial isometry. In other words ,* $V = (S-E)(S+E)^{-1}$, *defined on* $D(V) = R(S+E)$. *The operator* V *will be called the* $E-$*Cayley transform of* S. *Similarly, if* LS *is symmetric, the corresponding version of Lemma* 4.6.1 *leads to the definition of an operator* $W : R(S+E) \longmapsto R(S-F)$, $V(S+F)x = (S-F)x$, $x \in D(S)$ *which is again a partial isometry and* $W = (S-F)(S+F)^{-1}$ *defined on* $D(W) = R(S+F)$. *The operator* W *is called the* $F-$*Cayley transform of* S. *Because the two Cayley transform defined above are alike in the sequel, we shall mainly deal with the* $E-$*Cayley transform. For asymmetric operator, by Cayley transform, we always mean the classical concepts as defined by Von Neumann in* [29]. *Let* $V : D(V) \subset H^2 \longmapsto H^2$ *be a partial isometry. Then the inverse* V^{-1} *is well defined on the subspace* $D(V^{-1}) = R(V)$.

Lemma 4.6.4 *Let* $S : D(S) \subset H^2 \longmapsto H^2$ *be such that* JS *is symmetric and let* V *be the* $E-$*Cayley transform of* S. *We have the following :*

(a) The operator V is closed iff the operator S is closed and iff the space $R(S \pm E)$ are closed.

(b) The operator $I-V$ is injective. Moreover, the operator S is densely defined iff the space $R(I-V)$ is dense in H^2.

(c) If $SK \subset KS$, then $SK = KS$ and $V^{-1} = -KVK$.

(d) The operator JS is self-adjoint iff the operator V is unitary.

Proof. Let $A = JS$ which is symmetric. Then its Cayley transform W is a partial isometry from $R(A+iI)$ onto $R(A-iI)$. Moreover, $S \pm E = J(A \pm iI)$. Therefore$V = JWJ$.

(a) From the properties of the Cayley transform, it follows that A is closed iff W is closed and iff the space $R(A+iI)$ are closed. It is clear that S is closed iff A is closed and $R(S \pm E)$ are closed iff $R(A+iI)$ are closed, implying the assertion.

(b) The equality $I - V = 2E(S+E)^{-1}$ on $D(V)$ shows that $I-V$ is injective and that $R(I-V) = ED(S)$. The latter equality implies that the operator S is densely defined iff the space $R(I-V)$ is dense in H^2, because E is unitary.

(c) If $SK = KS$, then $KD(S) \subset D(S)$ implying $KN(S) = D(S)$, because $K^2 = -I$. Consequently, $SK = KS$, implying $K(S \pm E) = (S \mp E)K$. Hence, $V^{-1} = -(S+E)(S-E)^{-1} = -K(S-E)(S+E)^{-1}K = -KVK$.

'(d) The operator JS is self-adjoint iff its Cayley transform W is unitary and hence iff $V = JWJ$ is unitary. ∎

We now summarize the properties of the Quaternionic Cayley transform in the following results.

Theorem 4.6.5 [35]*The $E-$Cayley transform is an order preserving bijective map assigning to each operator S with $S : D(S) \subset H^2 \longmapsto H^2$ and JS is symmetric a partial isometry V in H^2 with $I-V$ injective. Moreover :*

(1) The operator V in H^2 with $I-V$ injective is closed.

(2) The equality $V^{-1} = -KVK$ holds iff the equality $SK = KS$ holds.

(3) The operator JS is self-adjoint iff V is unitary on H^2.

Remark 4.6.6 *Note that the class of operators having an $E-$Cayley transform consists of operator $S : D(S) \subset H^2 \longmapsto H^2$ such that JS is symmetric and $SK \subset KS$ (which implies that $KS = KS$).This is equivalent to saying that S is $(J,S)-$symmetric in* [32], *(i.e. JS, LS are symmetric and $KD(S) \subset D(S)$. Also the class of these $(J,L)-$symmetric operators having a normal extension is the main motivation of the introduction of the Quaternionic Cayley transform in* [32].

4.7 Unitary operator and the inverse Quaternionic Cayley transform.

Here, we deal with those unitary operators producing (unbounded) normal operators by inverse $E-$Cayley transform.

Lemma 4.7.1 *Let U be abounded operator on H^2.The operator U is unitary and has the property that $U^* = -KUK$ iff there are a bounded operator T and bounded self-adjoint operator A, B on H such that $T^*T + A^2 = I$, $T^*T + B^2 = I$, $AT = TB$ and $U = \begin{pmatrix} T & iA \\ iB & T^* \end{pmatrix}$, where I is the identity operator on H.*

Proof. If U is given by the matrix in the statement, it is easily checked that U is unitary operator on H^2 and we have$U^* = -KUK$.

Conversely, assuming that $U = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix}$, and we easily infer that $-KUK = \begin{pmatrix} U_{22} & -U_{21} \\ -U_{12} & U_{11} \end{pmatrix}$. The equality $U^* = -KUK$ leads to the equation $U_{11}^* = U_{22}U_{12}^* = -U_{12}$ and $U_{21}^* = -U_{21}$. Setting $T = -U_{11}$, $U_{12} = iA$ and $U_{22} = iB$, with A, B self-adjoint, the equation $U^*U = I$ and $UU^* = I$ and equivalent to the equation $T^*T + A^2 = I, T^*T + B^2 = I$ and $AT = TB$. ■

Example 4.7.2 [35] *Let T be a contraction operator on H. Setting $D_{T^*} = (I - TT^*)^{\frac{1}{2}}, D_T = (I - T^*T)^{\frac{1}{2}}$ and $U = \begin{pmatrix} T & iD_{T^*} \\ iD_T & T^* \end{pmatrix}$, the operator U is unitary on H^2 and satisfies the equation $U^* = -KUK$. Indeed, $A = D_{T^*}$ and $B = D_T$ satisfies all the functions from the previous Lemma* 4.6.1.

Lemma 4.7.3 *Let U be a unitary operator on H^2 such that $I - U$ is injective. If we set $S = (I + U)(I - U)^{-1}E$, we have that S is densely defined, closed and $S^* = E(I + U)(I - U)^{-1}$.*

Proof. The operator S is the inverse $E-$ Cayley transform of the unitary operator U. Therefore JS is self-adjoint by *Lemma* 4.6.4(d).This implies that S is densely defined and closed.Moreover,$(JS)^* = JS = S^*J$,hence $S^* = JSJ = E(I + U)(I - U)^{-1}$. ■

Lemma 4.7.4 *Let U be an operator on H^2 having the form $U = \begin{pmatrix} T & iA \\ iB & T^* \end{pmatrix}$, with T, $A = A^*$, $B = B^*$ bounded operator on H, such that $TT^* + A^2 = I$, $T^*T + B^2 = I$, $AT = TB$, we have the equality $(U + U^*)E = E(U + U^*)$ iff T is normal and $A = B$.*

Proof. We know that the operator U is unitary and we note that $J\,(U+U^{*}) = \begin{pmatrix} T+T^{*} & i(A-B) \\ i(A-B) & -(T+T^{*}) \end{pmatrix}$. Similarly, $(U+U^{*})\,J = \begin{pmatrix} T+T^{*} & -i(A-B) \\ -i(A-B) & -(T+T^{*}) \end{pmatrix}$. The equality $(U+U^{*})E = -E(U+U^{*})$ is equivalent to $A = B$, which clearly implies that $T^{*}T = TT^{*}$.

Conversely, if U has the matrix representation from the statement with T normal and $A = B$, then $U+U^{*} = \begin{pmatrix} T+T^{*} & 0 \\ 0 & T+T^{*} \end{pmatrix}$, $(U+U^{*})E = E(U+U^{*})$ follows. ∎

Remark 4.7.5 *It follows from Lemma 4.6.1 and Lemma 4.6.4 that an operator U on H^2 has the form $U = \begin{pmatrix} T & iA \\ iA & T^{*} \end{pmatrix}$ with T normal, A self-adjoint such that $T^{*}T + A^2 = I$ and $AT = TA$ iff U is unitary, $U^{*} = -KUK$ and $(U+U^{*})E = E(U+U^{*})$.*

Lemma 4.7.6 [35] *Let V be a partial isometry such that $V^{-1} = -KVK$ and $I - V$ is injective. Let S be the inverse $E-$Cayley transform of V. We have $JD(S) \subset D(S)$ and $\| SJx \|_2 = \| Sx \|_2$ for all $x \in D(S)$ iff there exist a surjective isometry $G : D(V) \longmapsto D(S)$ such that $E(I-V) = (I-V)G$.*

Proof. Assuming first that $JD(S) \subset D(S)$ and $\| SJx \|_2 = \| Sx \|_2$for all $x \in D(S)$. The inclusion $JD(S) \subset D(S)$ means the inclusion $R(I-V) \subset ER(I-V)$, by *Remark* 4.5[2]. But the inclusion is actually equality because $E^2 = -I$. Consequently, for every $u \in D(V)$, we can find a unique vector $v \in D(V)$, such that $(I-V)v = E(I-V)u$. Setting, $G_u = v$, we get a linear operator $G : D(V) \longmapsto D(V)$ such that $E(I-V) = (I-V)G$ which is clearly bijective. In fact, $G^{-1} = -(I-V)^{-1}E(I-V) = -G$. We now show that G is an isometry on $D(G) = D(V)$. Let $x \in D(S) = ED(S)$ and set $u = (S+E)xD(V)$. As we clearly have that $I - V = 2E(S+E)^{-1}$, we deduce that $(I-V)G_u = E(I-V)u = 2E^2(S+E)^{-1}u = -2x = 2E(S+E)^{-1}v = (I-V)v$, hence$G_u = v$. Moreover, $\| G_u \|_2 = \| (S+E)Ex \|_2 = \| (S+E)x \|_2 = \| u \|_2$by *Lemma* 4.7.1,showing that G is an isometry on $D(V)$.

Conversely,assuming that there exist a surjective isometry $G : D(V) \longmapsto D(V)$ such that $E(I-V) = (I-V)G$, we have in particular, $D(S) \subset R(I-V)$ implying $JD(S) \subset D(S)$. In addition, with notation from above, $\| (S+E)Ex \|_2 = \| G_u \|_2 = \| u \|_2 = \| (S+E)x \|_2$because G is an isometry , showing that $\| SJx \|_2 = \| Sx \|_2$for all $x \in D(S)$, by *Lemma* 4.7.1. ∎

Theorem 4.7.7 *Let U be a unitary operator on H^2 with the property $U^{*} = -KUK$ and such that $I - U$ is injective. Let also S be the inverse $E-$Cayley transform of U.The operator S is normal iff $(U+U^{*})E = E(U+U^{*})$.*

Proposition 4.7.8 *Let U be a unitary operator on H^2 with the property $U^* = -KUK$ and such that $I-U$ is injective. Let also S be inverse $E-$Cayley transform of U.The operator S is normal iff there exist a unitary operator G_U on H^2 such that $E(I-U) = (I-U)G_U$ and $(G_U)^* = -G'_U$.Also let $U \in \mho_C(H^2)$.Then the operator*

$i\begin{pmatrix} T^*T - \mathrm{Re}(T)\theta_T^{-1} & -iA(I-T^*)\theta_T^{-1} \\ -iA(I-T)\theta_T^{-1} & -(T^*T - \mathrm{Re}(T)\theta_T^{-1}) \end{pmatrix}$ is a densely defined isometry, where $\theta_T^{-1} = I = \mathrm{Re}(T)$ and its extension to H^2 equals to the unitary operator G_U.

Example 4.7.9 [35] *Let us compute the operator U and G_U from Proposition 4.7.8 in particular case, Let μ be a probability measure in the plane $\mathbb{R}^2$, having moments of all orders. In particular, if $(s,t)$is the variable in $\mathbb{R}^2$, the numbers $\gamma_{k,l} = \int s^k t^l du$ are well defined for all integers $k,l \geq 0$. Let P be the algebra of all polynomials in s,t with complex coefficients. The hypothesis implies that $P \in L^2(\mu)$. The linear operators given by $p(s,t) \longmapsto sp(s,t), p(t) \longmapsto tp(s,t), p \in P$ are symmetric on P. In fact those operators have natural self-adjoint extensions in $H = L^2(\mu)$, defined by similar formulas, whose joint domain of definition is given by $D_0 = \{f \in L^2(\mu) : sf, tf \in L^2(\mu)\}$ and those extensions commutes (i.e. their spectral measure commute). If we consider the multiplication operator with the corresponding (matrix of) functions, the matrix $N = \begin{pmatrix} s & t \\ -t & s \end{pmatrix}$ defined on $D = D_0 \oplus D_0$ is normal. T he $E-$Cayley transform U of N will have the form $U = \frac{1}{s^2+t^2+1}\begin{pmatrix} (s-i)^2+t^2 & 2ti \\ 2ti & (s+i)^2+t^2 \end{pmatrix}$ by Remark* 4.6.4.

4.8 Normal extension.

Remark 4.8.1 [35] *Let $T : D(T) \subset H^2 \longmapsto H^2$, with $D(T) = D_0 \oplus D_0, D_0 \subset H$. According to [[21], Lemma 1.2], the equality $D(T) = D_0 \oplus D_0$ is equivalent to the inclusions :*

(*i*) $JD(T) \subset D(S)$ and $KD(T) \subset D(T)$.

In order that T have a normal extension, $S \in N_{IC}(H^2)$ the following conditions are necessary :

(*ii*) JT is symmetric.

(*iii*) $TK = KT$.

(*iv*) $\| TJx \|_2 = \| Tx \|_2$for all $x \in D(T)$.

We denote $S_{IC}(H^2)$, the set of those operator $T : D(T) \subset H^2 \longmapsto H^2$ such that $(i)-(iv)$ hold. Let also $P_C(H^2)$ be the set of those partial isometries $V : D(V) \subset H^2 \longmapsto H^2$ such that :

(a) $V^{-1} = -KVK$.

(b) $I - V$ is injective.

(c) $ER(I-V) = R(I-V)$ and $(I-V)^{-1}E(I-V)$ is an isometry on $D(V)$.

It follows from *Lemma* 4.7.6 that the $E-$Cayley transform is bijective map from $S_{IC}(H^2)$ onto $P_C(H^2)$. And we have that $\mho_C(H^2) = P_C(H^2) \cap \mho(H^2)$ by *Theorem* 4.7.7 and *Proposition* 4.7.8. Thus the existence of the extension $S \in N_{IC}(H^2)$ of an operator $T \in S_{IC}(H^2)$ is equivalent to the description of those partial isometries in $P_C(H^2)$ having the extension in the family $\mho_C(H^2)$.

Proposition 4.8.2 [35] *Let $U \in \mho_C(H2)$ and let $D \subset H^2$ be a closed subspace with the property $KU(D) \in D$ and $E(I-U)(D) \subset (I-U)(D)$. If $V = U\mid_D$, $\varepsilon = D^{\perp}$ and $W = U\mid_{\varepsilon}$, then $V = V \oplus W$ and $V, W \in P_C(H^2)$.*

Proof. If $V = U\mid_D$,then V is a partial isometry from $D(V) = D$ onto $R(V) = U(D)$. Moreover, $V^{-1} = U^{-1}/R(V)$. Therefore, $V^{-1} = -KVK$, because of the equality $U^{-1} = -KUK$ and inclusion $KU(D) \subset D$. The injectivity of $I - U$ implies that $I - V$ is also injective. In addition, $E(I-V)(D) \subset (I-V)(D)$ as a direct consequence of the given inclusion, $E(I-V)(D) \in (I-V)(D)$. As $E^2 = -I$, we have in fact that $E(I-V)(D) = JD(T) = ED(T) = R(I-V) = ER(I-V) = D(T)$. We now show that $(I-V)^{-1}E(I-V)$ is an isometry on $D(V)$. Let S be the inverse of $E-$Cayley transform of U, which is an extension of T, because JT is self-adjoint by *Lemma* 4.6.4(d) and S is normal by *Theorem* 4.7.7,we have $\| TJx \|_2 = \| STx \|_2 = \| JS^*x \|_2 = \| S^*x \|_2 = \| Sx \|_2 = \| Tx \|_2$for all $x \in D(T)$.Then, $(I-V)^{-1}(I-V)$ is an isometry on $D(V)$ by *Lemma* 4.7.6. By properties $(a) - (c)$ from *Remark* 4.8.1 being verified, we have that $V \in P_C(H^2)$. Now let $\varepsilon = D^{\perp}$, and let $W = U\mid_{\varepsilon}$. We also have the inclusion $KU\mid_{\varepsilon} \subset \varepsilon$ because $(KU)^* = -KU$ as well as the inclusion and $E(I-U)(\varepsilon) \subset (I-U)(\varepsilon)$ because the operator $G_U = (I-U)^{-1}E(I+U))$ has the property $(G_U)^* = -G_U$. Therefore, the operator W is also a (closed) partial isometry in $P_C(H^2)$, by the first part of the proof. Hence, the equality $U = V \oplus W$. ■

Lemma 4.8.3 *Let $T \in S_{IC}(H^2)$ be densely defined. Then T is closable and its closure $\overline{T} \in S_{IC}(H^2)$.*

Proof. First, note that the operator T is closable. Indeed, as the operator JT is symmetric, assuming that $(x_n)_{n\geq 1}$ is a sequence from $D(T)$ such that $x_n \longrightarrow 0$ and $Tx_n \longrightarrow y$ as $n \to \infty$.For all$v \in D(T)$,we have $\langle y, v\rangle = \lim_{n\to\infty} \langle Tx_n, v\rangle = \lim_{n\to\infty} \langle x_n, JTJx\rangle = 0$showing that the closure of the graph of T is a graph. Let $\overline{T}$ be the closure of T, and let $\overline{x} \in D(\overline{T})$. Hence, $\overline{x} = \lim_{n\to\infty} x_n$ and $\overline{T}\overline{x} = \lim_{n\to\infty} Tx_n$ for some sequence $(x_n)_{n\geq 1}$

from $D(T)$. Condition (iv) from *Remark* 4.8.1 show that $((TJx_n))_{n\geq 1}$ is a Cauchy sequence, implying that $J\overline{x} \in D(\overline{T})$ and $\| \overline{T}J\overline{x} \|_2 = \| \overline{T}\overline{x} \|_2$. In other words, $JD(\overline{T}) \in D(\overline{T})$ and condition (iv) from *Remark* 4.7.1 holds for $\overline{T}$. As we also have $TKx_n = KTx_n$ for all $n \geq 1$. We infer $KD(\overline{T}) \subset D(\overline{T})$ and $\overline{T}K = K\overline{T}$, the latter being condition (iii) from *Remark* 4.8.1. Finally, let if $\overline{y} \in D(\overline{T})$ is another element with $\overline{y} = \lim_{n\to\infty} y_n$ and $\overline{T}\overline{y} = \lim_{n\to\infty} Ty_n$ for some sequence $(y_n)_{n\geq 1}$ from $D(T)$, then we have $\langle J\overline{T}\overline{x}, \overline{y}\rangle = \lim_{n\to\infty} \langle JTx_n, y_n\rangle = \lim_{n\to\infty} \langle x_n, JTy_n\rangle = \langle \overline{x}, J\overline{T}\overline{y}\rangle$, implying condition (iii) by *Remark* 4.8.1 for $\overline{T}$. Consequently, $\overline{T} \subset S_{IC}(H^2)$. ∎

Theorem 4.8.4 *Let $T \in S_{IC}(H^2)$ be densely defined .The operator T has an extension in $N_{IC}(H^2)$ iff there exists a $W \in P(H^2)$, with $D(W) = R(T+E)^{\perp}$.*

Proof. According to *Lemma* 4.8.3, with no loss of generality, we may assume that T is closed. If V is the $E-$Cayley transform of T, then as noticed in *Remark* 4.8.1, we have $V \in P_C(H^2)$. Moreover, V is closed by *Lemma* 4.6.4. In particular, $D(V) = R(T+E)$ and $R(V) = R(T-E)$ are closed in H^2. Assuming first that there exist a $W \in P_C(H^2)$, with$D(W) = R(T+E)^{\perp}$. Hence $R(W) = KD(W) = R(T-E)^{\perp}$. Put $U = V \oplus W$, which is a unitary operator on H^2, we show that $U \in \mho_C(H^2)$. Since $KD(V) = R(V)$ and $KD(W) = R(W)$, we clearly have $U^* = V^{-1} \oplus W^{-1} = -K(V \oplus W)K = -KUK$.Next, let $G_V : D(V) \longmapsto D(V)$ and $G_W : D(W) \longmapsto D(W)$ be surjective isometry given by $G_V = (I-V)^{-1}E(I-V)$ and $G_W = (I-E)^{-1}E(I-W)$. Then $G = G_V \oplus G_W$ is a unitary operator on H^2. In addition, if $x \in D(V)$ and $y \in D(W)$, are arbitrarily, then $E(I-U)(x \oplus y) = E(x - Vx) + E(y - Wy) = (G_Vx - VG_Vx) + (G_Wy - WG_Wy) = (I-V)G(x \oplus y)$. As from *Lemma* 4.7.4, the space $R(I-V)$ is dense in H^2 because the operator T is densely defined. Therefore, $R(I-V) \subset R(I-U)$ is dense in H^2, implying that $I-U$ is injective. Consequently, $U \in P_C(H^2)$ and because U is unitary, we actually have $U \in \mho(H^2)$ by *Proposition* 4.7.8. Clearly T has a normal extension in $N_{IC}(H^2)$, which is the inverse E-Cayley transform of U.

Conversely, if the operator T has a normal extension $S \in N_{IC}(H^2)$ and if $U \in \mho_C(H^2)$ is the $E-$Cayley transform of S and to find the operator $W \in P_C(H^2)$, we apply *Proposition* 4.8.2 to $D = D(V)$, where$U$ is the $E-$Cayley transform of T. ∎

Corollary 4.8.5 [33] *Let $T \in S_{IC}(H^2)$ be closed and let V be the $E-$Cayley transform of T. The operator T has an extension in $N_{IC}(H^2)$ iff there exists a $W \in P_C(H^2)$, there the properties $D(W) = R(T+E)^{\perp}$and $R(I-V) \cap R(I-W) = \{0\}$.*

Proof. We show that the unitary operator $U = V \oplus W$ is in $\mho_C(H^2)$, where $W \in P_C(H^2)$ which has the stated properties and we only proof that $I-U$ is injective. This is true because if $v \in D(V)$ and $w \in D(W)$ have the property $v \oplus w = U(v \oplus w)$, we infer that $v - Vv \in R(I-V) = Ww - w \in R(I-W)$, implying $v = w = 0$, because both $I - V$, $I - W$ are injective.

Conversely, by *Theorem* 4.8.4, we know that operator $U \in \mho_C(H^2)$ is the $E-$Cayley transform of a normal extension of T and $W \in P_C(H^2)$ by *Proposition* 4.8.2.Since U is an $E-$Cayley transform, then $I - U$ is injective. Choosing a vector $u \in R(I-V) \cap R(I-W)$, then we have $u = v - Vv = Ww - w$, with $v \in D(V)$ and $w \in D(W)$.Therefore $v \oplus w = U(v \oplus w)$, implying$v = w = u = 0$ and so $R(I-V) \cap R(I-W) = \{0\}$. ■

Remark 4.8.6 [35] *It follows Theorem 4.8.4 that if $T \in S_{CI}(H^2)$ is densely defined and the space $R(T+E)$ is dense in H^2, then T has an extension in $N_{IC}(H^2)$.*

Remark 4.8.6 can be applied in the following situation. Let A, B be a pair of linear operator having a joint domain of definition $D_0 \subset H$. As in the introduction, we associate this pair with a matrix operator $T = \begin{pmatrix} A & B \\ -B & A \end{pmatrix}$, defined on $D(T) = D_0 + D_0 \in H^2$. Then T is symmetric iff both A and B are symmetric.

Application of Cayley transformation.

In the theory of operator in Hilbert space, the Cayley transform , $C(T) = (T - iI)(T + iI)^{-1}$ is frequently used to switch from the study of closed but general dense unbounded linear operator T with dense domain $D(T)(\overline{D}(T) = H)$ to that of bounded operator $C(T)$. In the classical case, $K(t) = \frac{t-i}{t+i} = \frac{t^2-1-2it}{1+t^2}$, this transform converts the self-adjoint (symmetric) operator T into unitary (isometric) operator.

The Cayley transforms are used to turn the processes with a continuous time parameter into such with a discrete time e.g., dealing with the production theory of stationary stochastic process.

Also this transform is used in to obtain explicit and constructive representation of solutions of various evolution differential equations with operator coefficients where in fact the solutions with a continuous time parameter are represented in terms of those with discrete time.

Moreover, these representations can be considered as a method of separating time and "Spartial"variables. Thus, these representations serves as a basis for algorithms without accuracy saturation i.e. their accuracy increases automatically and unbounded with increasing smoothness of the solution.

4.9 Conclusion.

The study of transformations and its related operators in a Hilbert space gives some basic results on the structure of transforms in a Hilbert space. Since the classes of operators has not be studied extensively, we would like to suggest some possible areas that can be investigated in future.

Schrodinger Equation in form of a transform

Let H be a Hilbert space and B be a self-adjoint unbounded positive definite operator in H with the domain $D(B)$ and the spectrum $\sigma(B) = \{\lambda \in \mathbb{C} : \lambda I - B \text{ is not invertible}\}$ .Consider the initial value problem for the Schrodinger equation; $x^{'}(t) = iBx(t), x(0) = 0$.Thus we would like to study how to derive the explicit representation of the solution operator as a series where the time variable separated in Laguerre polynomial the ,"spatial " operator in the power of a Cayley transform and investigate whether its accuracy depends on the smoothness of the exact solutions.

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