POWER SERIES DISTRIBUTIONS AND ZERO-INFLATED MODELS

TUWEI KIPKORIR EDWIN

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Abstract

There are two main objectives in this project. First, to construct Power Series Distributions and obtain their properties. Specifically, to express the Power Series Distributions in explicit form, in recursive form (in Katz recursive form), special function form (in terms of Confluent and Gauss hypergeometric functions), expectation form (in terms of probability generating functions)

Secondly, to generalize Power Series Distributions by introducing an inflated parameter. Specifically,

(*i*) To construct an inflated Poisson, Binomial, Negative Binomial and Logarithmic distributions.

(*ii*) To obtain the moments and the maximum likelihood estimators of the zero inflated power series distributions.

(*iii*) To obtain Compound Poisson distributions for the inflated power series distributions.

(*iv*) Application of the inflated models described above to migration data.

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Chapter 1

General Introduction

1.1 Introduction

One way of constructing discrete probability distributions is based on power series expansions resulting into Poisson, Binomial, Negative Binomial and Logarithmic distributions forming a family of Power Series Distributions (PSDs).

The power series distributions have been generalized to what is called Generalized Power Series Distributions (GPSDs) and Modified Power Series Distributions (MPSDs)

By introducing a parameter to these distributions we have inflated PSDs, GPSDs and MPSDs. The inflated distributions can also be seen as finite mixtures consisting of a degenerate and a non-degenerate distribution.

1.2 Problem Statement

In applications involving count data, it is common to encounter the frequency of observed zeros significantly higher than predicted by the model based on the standard parametric family of discrete distributions. In such situations, zero-inflated Poisson and zero-inflated negative binomial distribution have been widely used in modelling the data, yet other models may be more appropriate in handling the data with excess zeros. The consequences of this is misspecifying the statistical model leading to erroneous conclusions and bring uncertainty into research and practice. Therefore the problem is to identify by constructing other alternatives, to the models already present in the literature that may be more appropriate for modelling data with excess zeros.

1.3 Objectives of the Study

There are two main objectives in this project. First, is to express Power Series Distributions in explicit form, in recursive form, special function form, and in expectation form. Specifically to express Poisson, Binomial, Negative Binomial and Logarithmic series distributions.

(i) In explicit form.

(ii) In Katz recursive form.

(iii) In terms of Confluent and Gauss hypergeometric functions.

(*iv*) In terms of probability generating functions (pgf).

Secondly, to generalize Power Series Distributions by introducing an inflated parameter. Specifically,

(*i*) To construct an inflated Poisson, Binomial, Negative Binomial and Logarithmic distributions.

(*ii*) To obtain the moments and maximum likelihood estimators of the zero-inflated power series distributions.

(*iii*) To obtain Compound Poisson distributions for the inflated power series distributions.

(*iv*) To apply the inflated models described above to migration data.

1.4 Literature Review

1.4.1 Power series distribution

A number of attempts have been made during the past decades to study power series Distributions, Noak (1950) considered a class of random variable with discrete distributions. He defined power series distributions and investigated its moment and cumulant properties. He also, constructed the special cases of important discrete distributions belong to this class with their moment and cumulant properties that is; the binomial, Poisson, negative binomial, and logarithmic series distributions.

Khatri (1959) on certain properties of power series distribution extended what was done by Noak (1950) to multivariate distributions. He established the recurrence relations for cumulants and factorial cumulant, which are utilized to show that any power series distribution in a single parameter is determined uniquely from its first two moments. He further derives the multivariate extensions of powers series distributions with the illustration of multinomial distributions and extended it to truncated powers series distributions.

Patil (1962) studied on certain properties of generalized power series distribution. He allowed the set of values that the variate can take to be any non-empty enumerable set S of non-negative integers and called this extended class generalized power series distributions (GPSDs). He also studied estimation and other properties of GPSDs. He noted that among the distributions of major importance belonging to this class are the binomial, Poisson, negative binomial, and logarithmic series distributions and related multivariate distributions. Furthermore, if a GPSD is truncated, then the truncated version is also a GPSD. He also noted that the sum of n mutually independent random variables each having the same GPSD, has a distribution of the same class, with series function $[\eta(\theta)]^n$.

1.4.2 Zero-Inflated models

A zero-inflated model is a statistical model based on a zero-inflated probability distribution. Zero-inflated models has become fairly popular in the research literature with a number of attempts being made during the past decades to study it, Katti and Rao

(1966) used a criterion for flexibility and showed that there is a distribution called the log-zero-Poisson distribution (l.z.P.) which has more flexibility than the Neyman type A, the negative binomial, and the Poisson binomial distribution. They derived some basic properties of the I.z.P. distribution and compared it with other distributions, fitted using the 35 sets of data given in Martin and Katti (1965). In their study they regard the l.z.P. as a 'one-distribution summary' of Neyman type A, the negative binomial, and the Poisson binomial distribution.

David Kemp and Adrienne Kemp (1988) examined the construction of rapid estimation procedures for discrete distributions, using the empirical probability generating function (epgf), mathematical approximations to the maximum likelihood equations, and bounds for the maximum likelihood estimators. They reviewed some standard rapid estimation procedures for discrete distributions, placed in the context of epgf estimation, and developed a new methods. They also, considered a number of distributions in depth and found that different distributions require different procedures of estimation as shown in their illustrative example of zero-inflated binomial distribution with large and small to moderate sized sample in their simulation.

Farewell and Sprott (1988) studied the use of a mixture model in the analysis of count data. They analyzed data on the effect of a drug with antiarrythmic properties on patients who experienced frequent premature ventricular contractions (PVCs). Where the number of PVCs per minute was recorded before and after the drug was administered. They noted that a wide range of counts were observed during the predrug measurements but that 7 of the 12 patients experienced no PVCs during the postdrug measurement. They assume that any non-zero count is considered abnormal. Individuals with no PVCs may be "cured," in which case their zero count is assured; otherwise, the observed zero is a sampling zero. The observations occur as paired data (x_i, y_i) which are the predrug and postdrug count, respectively, for the i^{th} patient. Assuming that x_i is a Poisson variate with mean λ_i and that for patients who are not cured y_i is independently Poisson with mean $\beta\lambda_i$, then the conditional distribution of y_i given $t_i = x_i + y_i$ is zero-inflated binomial distribution.

Lambert (1992) studied Zero-Inflated Poisson Regression, with an application to defects in manufacturing. She observed that when a reliable manufacturing process is in control, the number of defects on an item should be Poisson distributed. If the Poisson mean is λ, a large sample of n items should have about $ne^{-\lambda}$ items with no defects. Some times, however, there are many more items without defects than would be predicted from the numbers of defects on imperfect items. She gives an interpretation that slight, unobserved changes in the environment cause the process to move randomly back and forth between a perfect state in which defects are extremely rare and an imperfect state in which defects are possible but not inevitable.

Gupta et al., (1995) studied the zero inflated modified power series distributions (IMPSD) which include among others the generalized Poisson and the generalized negative binomial distributions and hence the Poisson, binomial and negative binomial distributions. They also considered the structural properties along with the distribution of the sum of independent IMPSD variables, the maximum likelihood estimators of the parameters of the model and obtained the variance-covariance matrix of the estimators. Finally they gave examples on the generalized Poisson distribution to illustrate the results. Murat and Szynal (1998) have extended the results of Gupta et al., (1995) to the discrete distributions inflated at any of the support points.

Gupta et al., (1996) considered a zero adjusted discrete model. Such a situation arises when the proportion of zeros in the data is higher (lower) than that predicted by the original model. They also, studied the effect of such an adjustment and compared the failure rates and the survival functions of the adjusted and the non-adjusted models. As an example, they studied an adjusted generalized poisson distribution and the three parameters of the model estimated by the maximum likelihood method. They recommended that in order to obtain more accurate results for zero-inflated data, the model should be adjusted for the number of zeros.

Dowling and Nakamura (1997) studied estimating parameters for discrete distributions via the empirical probability generating function. They considered parameter estimation for a family of discrete distributions characterized by probability generating functions (pgf's). Kemp (1988) suggest estimators based on the empirical probability generating function (epgf); the methods involve solving estimating equations obtained by equating functions of the epgf and pgf on a fixed, finite set of values. They provide an asymptotic theory for these epgf-based methods that allows computation of asymptotic efficiency in a unified setting, and suggests asymptotic estimates of standard errors. They considered some examples as used by Kemp (1988) and based on the theory, they gave graphical techniques that are shown to be useful for exploratory analysis.

Nikolai et al., (2001, 2002) studied an extension of the generalized power series distributions by including an additional parameter ρ. This parameter has a natural interpretation in terms of both "zero inflated" proportion and correlation coefficient, and because of this they called this family Inflated-parameter generalized power series distributions. They presented probability mass functions (pmf) and probability generating functions (pgf) of the corresponding inflated-parameter distributions, with two different representations of the corresponding pmf's of the r.v's belonging to the inflated parameter family of the generalized power series distributions. They successfully fitted a real frequency data using inflated-parameter poisson and inflated-parameter negative binomial models.

Jansakul et al., (2002) studied score tests for zero-Inflated Poisson Models. They considered a situation where the count data have a large proportion of zeros than specified by the Poisson model. Thus for data of this form they adopted the use of the zero-inflated Poisson (ZIPo) model. ZIPo model with a constant proportion of excess zeros to a standard Poisson regression model, was given by van den Broek (Biometrics, 51 (1995) 738–743). They extend this test to the more general situation where the zero probability is allowed to depend on covariates and evaluated the performance of this test using a simulation study. They also proposed a composite test to identify potentially important covariates in the zero-inflation model. lastly, they illustrated the use of the general score test and the composite procedure on some real datasets and showed that the composite tests are useful for suggesting appropriate models.

Inuwor (2004) considered a model that takes into account zero observation. He assumed the Poisson distribution for the number of clusters migrating, and that the number of migrants in a cluster follows each of the members of the class of one-Inflated power series distributions namely: the binomial, the Poisson, the negative binomial, the geometric, the log-series, and the mis-recorded Poisson. At least one person is expected to migrate in household is exposed to the risk of migration thus, the use of the one-inflated distributions. This is justified by the need to reduce the risk of underestimation of the probability that one person migrates in households are exposed to the risk of migration. Hence the use of zero-truncated distributions as proposed by

Yadava and Singh (1991) is not justifiable since the zeros are real zeros are real and observable as there is the possibility that nobody migrates in a cluster in a household.

Perakis and Xekalaki (2005) proposed process capability index useful for both the discrete and continuous processes. They further provided a process capability index for Poisson and attribute data. Notably these indices are based on maximum likelihood estimate of the Poisson parameter as well as on minimum variance unbiased estimator (MVUE). The simulation study performed by them reveals that indices based on maximum likelihood estimates perform better than the one based on MVUE. In the recent years, due to adoption of technology, production processes produce extremely good products. Thus, zero-inflated models have been found useful in statistical process control.

Patil and Shirke (2007) considered testing parameter of a zero inflated power series model. They provided three asymptotic tests for testing the parameters of power series distribution, using an unconditional (standard) likelihood approach, a conditional likelihood approach and a test based on sample mean, respectively. They illustrated this by using zero inflated Poisson distribution (ZIPo).

Perumean et al., (2012) studied Zero-inflated and overdispersed. They recommended that when there is evidence of overdispersion, other models (e.g. zero-inflated Poisson and zero-inflated negative binomial) may replace the Poisson model in handling excessive zeros in count data and that the modification of the Poisson or Negative binomial procedure is to avoid the incorrect estimation of the model parameters and standard errors and the incorrect specification of the distribution of the test statistic. If ignored these misspecifications may easily lead to erroneous conclusions about the data and bring uncertainty into research and practice. They utilized simulation methods in educating researchers on the importance of accounting for zero inflation in count data and the consequences of misspecifying the statistical model.

1.5 Significance of the Study

The study is significant to various fields in expanding on the previous research in addressing the problems encountered in data modelling. A brief description and an illustration on the significances of the study in various fields are given below:

The study will be beneficial to those in the field of demography in understanding population history of areas and drivers of regional change. One of the applications is in the analysis of migration data. Researchers studying migration have widely used probability models with the primary purpose of modelling being simplification and to reduce a confusing mass of numbers to a few intelligible basic parameters, to make possible an approximate representation of reality without its complexity. Moreover, by doing so bring policy makers and development planners to a new level of awareness in the formulation of their policies. Several studies have been proposed for modelling rural out migration at household level with Inuwor (2004) proposing a model that takes into account zero observation. He assumed the Poisson distribution for the number of clusters migrating, and that the number of migrants in a cluster follows each of the members of the class of one-Inflated power series distributions. He fitted the various distributions and tested their adequacy for various villages using data contained in Sharma (1995) and found that the distributions that takes into consideration variations in the probability of a person migrating in a cluster in a household (The log-series, and the geometric) performed better.

The study will also be useful to Actuaries in enriching their knowledge to develope and use statistical and financial models to inform financial decisions, pricing, establishing the amount of liabilities, and setting capital requirements for uncertain future events. In actuarial applications zero-inflated distributions have been widely used for modeling observed situations whose various characteristics as reflected by the data differ from those that would be anticipated under the simple component distribution. For example, observed data on the number of claims often exhibit a variance that noticeably exceeds their mean. Hence, assuming a Poisson form (or any other form that would imply equality of the mean to the variance) for the claim frequency distribution is not appropriate in such cases, (Karlis and Xekalaki, 2005). To have over-dispersion, then there is need to have models whose variance is greater than the mean. This is where zero-inflated models are adopted to handle the excess zeros in modelling.

Furthermore, the study will be helpful to analyst in environmental studies in their efforts to raise public awareness regarding the environment and conservation of natural resources. An application example is the use of zero-inflated models by Viviano et al (2004) to analyze environmental data sets with many zeroes. The first data set refers to a daily time series (1997-1999) data to study the effect of air pollution on health in Palermo. Their interest lies in estimating the effect of PM_{10} which is one of the major causes of health problems in air pollution studies, and the second data set referred to a study of bathing water quality in the district of Palermo. Their goal was to analyze the effect of some covariates (Month, Water Temperature, Oxygen, Sea Condition) on the response variable 'Number of Fecal Streptococcus' (counts in 100 ml of water), that ranges from 0 to 200 and presents a great percentage of zeroes ($\approx 54\%$). Results from the fitted models confirmed that that in the mortality data set, the classical Poisson model is the best choice, while in the second data set. Zero-inflated negative binomial is preferable.

Moreover, this study will be important to those in the field of engineering. One application is in the process of manufacturing items, the explanation offered by Lambert (1992) for the defects produced is that the system randomly changes its state from perfect to imperfect. The system is said to be perfect when there can be no defects manufactured. The system is said to be imperfect when defects are possible but not inevitable. Gupta et. al. (1995) have given another explanation to the above situation. Their explanation refers to the situation in which the items are manufactured by different machines out of which some machines do not produce any defectives; when the items are mixed up, the information is not available as to which item is produced by which machine. Thus is difficult to give a simple explanation for the excess zeros, hence the use of inflated model has been adopted to handle the excess zero and to effectively check on the quality of production thus, producing output according to specified requirements.

The study will also be of great importance for those in Biomedical field in adopting the recommended statistical models to evaluate the effectiveness of treatment and to diagnose disease. Their success in modern health care relies on the accuracy of the models and their efficiency. One of the applications is given by Böhning et al., (1999) in dental epidemiology the DMF-index is an important and well-known indicator and overall measure for the dental status of a person. It is a count number standing for the number of DECAY, MISSING, and FILLED Teeth (in which case it is called DMFT-Index) or Tooth Surfaces (in which case it is called DMFS-Index). They showed that

DMFT-index is a special mixture model having two classes, where the first class has a fixed value at 0. This class consists of children with no caries at all. In the case of the DMFT, this zero-class corresponds to those children showing no improvement. As an their application, they considered data coming from a prospective study of school children from an urban area of Belo Horizonte (Brazil) and from their study the zero-inflated Poisson distributions has been described to fit the data, since 90% of the over-dispersion is explained by the model.

Moreover, this study will benefit and help future researcher in Traffic Accident Research as a guide in modelling accident data, with the focus in their studies and numerous others of similar kind, is on evaluating public policy on how successful was past (traffic) safety legislation in reducing the number of accidents. Kuan et al., (1991), as one example considers data coming from the California Department of Motor Vehicles master driver license file. Here the variable of interest is the number of accidents per driver. From the data we see that there is excess number of zero counts and the frequency of X is greater than or equal to 3 is 21. Generally such data is modeled by Poisson distribution. But Poisson distribution does not fit well for the data. They fitted the above data for zero-inflated Poisson (ZIPo) and observed that the ZIPo provides the best fit.

Lastly, the output from this study will be helpful to the retail industry and business practitioners in marketing by using the probability models to understand and profile individual behavior, understand market-level patterns, and their origin in individual behaviors, provide norms or benchmarks for comparison and Prediction or forecasting of: Aggregate results beyond current observation period and Individual behavior, given knowledge of past actions.

Chapter 2

Power Series Distributions

2.1 Introduction

A numbers of researchers have worked on these class of power series distributions. Noak (1950) considered a class of random variable with discrete distribution. Where he defined power series distributions and showed that many important discrete distributions belong to this class. Khatri (1959) studied the multivariate extensions of powers series distributions with the illustration of multinomial distributions and extended it to truncated powers series distributions. Patil (1961, 1962) studied on certain properties of generalized power series distribution. Where he allowed the set of values that the variate can take to be any non-empty enumerable set T of non-negative integers and called this extended class generalized power series distributions (GPSDs). He also studied estimation and other properties of GPSDs.

In this chapter, we define Power Series Distributions (PSD) and provide an overview of some of its structural properties that includes; probability generating function, moments and their recurrence relation, central moments, the recurrence relation for cumulants and factorial cumulants. Lastly, the special cases of discrete probability distributions belonging to the class of power series distributions with their corresponding structural properties will be covered.

2.2 Definition

A large class of random variables with discrete probability distributions can be derived from certain power series.

Let

$$f(\theta) = \sum_{k=0}^{\infty} a_k \theta^k \tag{2.1}$$

be a power series.

Therefore,

$$1 = \sum_{k=0}^{\infty} \frac{a_k \theta^k}{f(\theta)}$$

That is

$$\Pr(X = k) = \frac{a_k \theta^k}{f(\theta)} \text{ for } k = 0, 1, 2, ... \text{ and } \theta > 0,\ a_k > 0 \tag{2.2}$$

is called a power series distribution (PSD). A power series distribution belongs to the exponential family, since

$$\begin{aligned}\Pr(X = k) &= \frac{a_k \theta^k}{f(\theta)} = \exp\left\{\ln\left[\frac{a_k \theta^k}{f(\theta)}\right]\right\} \\ &= \exp\left\{\ln a_k \theta^k - \ln f(\theta)\right\} \\ &= \exp\{\ln a_k + k \ln \theta - \ln f(\theta)\}\end{aligned} \tag{2.3}$$

The first and second derivatives of $f(\theta)$ are

$$f'(\theta) = \frac{df}{d\theta} = \sum_{k=0}^{\infty} k a_k \theta^{k-1} = \sum_{k=1}^{\infty} k a_k \theta^{k-1} \tag{2.4}$$

and

$$f''(\theta) = \frac{d^2 f}{d\theta^2} = \sum_{k=1}^{\infty} k(k-1) a_k \theta^{k-2} = \sum_{k=2}^{\infty} k(k-1) a_k \theta^{k-2} \tag{2.5}$$

Therefore it follows that

$$E(X) = \sum_{k=0}^{\infty} k \frac{a_k \theta^k}{f(\theta)} = \frac{\theta}{f(\theta)} \sum_{k=1}^{\infty} k a_k \theta^{k-1} = \frac{\theta}{f(\theta)} f'(\theta) \tag{2.6}$$

$$\begin{aligned}E[X(X-1] &= \sum_{k=0}^{\infty} k(k-1) \frac{a_k \theta^k}{f(\theta)} = \frac{\theta^2}{f(\theta)} \sum_{k=2}^{\infty} k(k-1) a_k \theta^{k-2} \\ &= \frac{\theta^2}{f(\theta)} f''(\theta)\end{aligned}$$

and

$$\begin{aligned}Var(X) &= E[X(X-1] + E(X) - [E(X)]^2 \\ &= \frac{\theta^2}{f(\theta)} f''(\theta) + \frac{\theta}{f(\theta)} f'(\theta) - \left[\frac{\theta}{f(\theta)} f'(\theta)\right]^2\end{aligned} \tag{2.7}$$

2.3 Probability Generating Function(pgf)

The pgf of X is given by

$$G(s) = E(s^X) = \sum_{k=0}^{\infty} p_k s^k = \sum_{k=0}^{\infty} \frac{a_k (\theta s)^k}{f(\theta)} = \frac{f(\theta s)}{f(\theta)} \tag{2.8}$$

See Feller (1968, Chap. XI and Chap. XII).

Thus,

$$G'(s) = \frac{dG}{ds} = \theta \frac{f'(\theta s)}{f(\theta)} \tag{2.9}$$

and

$$G''(s) = \frac{d^2G}{ds^2} = \theta^2 \frac{f''(\theta s)}{f(\theta)} \tag{2.10}$$

Hence the mean and the variance is given by

$$E(X) = G'(1) = \theta \frac{f'(\theta)}{f(\theta)} \tag{2.11}$$

and

$$\begin{aligned} Var(X) &= G''(1) + G'(1) - [G'(1)]^2 \\ &= \theta^2 \frac{f''(\theta)}{f(\theta)} + \theta \frac{f'(\theta)}{f(\theta)} - \left[\theta \frac{f'(\theta)}{f(\theta)}\right]^2 \end{aligned} \tag{2.12}$$

As given by Feller (1968: 266)

2.4 Moments and their recurrence relations

The r^{th} moment is defined by

$$\begin{aligned} \mu'_r = E(X^r) &= \sum_{k=0}^{\infty} k^r \Pr(X = k) = \sum_{k=0}^{\infty} k^r \frac{a_k \theta^k}{f(\theta)} \\ &= \frac{1}{f(\theta)} \sum_{k=0}^{\infty} k^r a_k \theta^k \end{aligned} \tag{2.13}$$

Therefore taking the derivative of μ'_r with respect to θ we obtain

$$\begin{aligned} \frac{d}{d\theta}\mu'_r &= \frac{1}{f(\theta)} \frac{d}{d\theta} \sum_{k=0}^{\infty} k^r a_k \theta^k + \left[\frac{d}{d\theta} \frac{1}{f(\theta)}\right] \sum_{k=0}^{\infty} k^r a_k \theta^k \\ &= \frac{1}{f(\theta)} \sum_{k=0}^{\infty} k^{r+1} a_k \theta^{k-1} + \left[-\frac{1}{[f(\theta)]^2} \frac{d}{d\theta} f(\theta)\right] \sum_{k=0}^{\infty} k^r a_k \theta^k \\ &= \frac{1}{f(\theta)} \sum_{k=0}^{\infty} k^{r+1} a_k \theta^{k-1} - \frac{f'(\theta)}{[f(\theta)]^2} \sum_{k=0}^{\infty} k^r a_k \theta^k \end{aligned}$$

multiplying $\frac{d}{d\theta}\mu'_r$ by θ we obtain

$$\theta\frac{d}{d\theta}\mu_r' = \frac{1}{f(\theta)}\sum_{k=0}^{\infty}k^{r+1}a_k\theta^k - \theta\frac{f'(\theta)}{[f(\theta)]}\cdot\frac{1}{f(\theta)}\sum_{k=0}^{\infty}k^r a_k\theta^k$$
$$= \sum_{k=0}^{\infty}k^{r+1}\frac{a_k\theta^k}{f(\theta)} - \theta\frac{f'(\theta)}{[f(\theta)]}\sum_{k=0}^{\infty}k^r\frac{a_k\theta^k}{f(\theta)}$$
$$= \mu_{r+1}' - \theta\frac{f'(\theta)}{[f(\theta)]}\mu_r' \tag{2.14}$$

but

$$E(X) = \sum_{k=0}^{\infty}k\frac{a_k\theta^k}{f(\theta)} = \mu_1'$$
$$= \sum_{k=0}^{\infty}k\frac{a_k\theta^k}{f(\theta)} = \frac{\theta}{f(\theta)}\sum_{k=1}^{\infty}ka_k\theta^{k-1} = \theta\frac{f'(\theta)}{f(\theta)}$$

Hence,

$$\theta\frac{f'(\theta)}{f(\theta)} = \mu_1' \tag{2.15}$$

equation (2.14) then becomes

$$\theta\frac{d}{d\theta}\mu_r' = \mu_{r+1}' - \mu_1'\mu_r'$$

Thus the recurrence relation for the r^{th} moments of a PSD is given as

$$\mu_{r+1}' = \theta\frac{d}{d\theta}\mu_r' + \mu_1'\mu_r' \tag{2.16}$$

The r^{th} central moment. i.e., the r^{th} moment about the mean is defined by

$$\mu_r = E\left[X - \mu_1'\right]^r = \sum_{k=0}^{\infty}(k-\mu_1')^r\frac{a_k\theta^k}{f(\theta)}$$
$$= \frac{1}{f(\theta)}\sum_{k=0}^{\infty}(k-\mu_1')^r a_k\theta^k \tag{2.17}$$

Therefore taking the derivative of μ_r with respect to θ we obtain

$$\frac{d}{d\theta}\mu_r = \left(\frac{d}{d\theta}\frac{1}{f(\theta)}\right)\sum_{k=0}^{\infty}(k-\mu_1')^r a_k\theta^k + \frac{1}{f(\theta)}\frac{d}{d\theta}\sum_{k=0}^{\infty}(k-\mu_1')^r a_k\theta^k$$
$$= -\frac{f'(\theta)}{[f(\theta)]^2}\sum_{k=0}^{\infty}(k-\mu_1')^r a_k\theta^k + \frac{1}{f(\theta)}\sum_{k=0}^{\infty}(k-\mu_1')^r a_k\frac{d}{d\theta}\theta^k$$
$$+ \frac{1}{f(\theta)}\sum_{k=0}^{\infty}\left[\frac{d}{d\theta}(k-\mu_1')^r\right]a_k\theta^k$$
$$= -\frac{f'(\theta)}{f(\theta)}\sum_{k=0}^{\infty}(k-\mu_1')^r\frac{a_k\theta^k}{f(\theta)} + \frac{1}{f(\theta)}\sum_{k=0}^{\infty}k(k-\mu_1')^r a_k\theta^{k-1}$$
$$+ \frac{1}{f(\theta)}\sum_{k=0}^{\infty}r(k-\mu_1')^{r-1}\left(-\frac{d}{d\theta}\mu_1'\right)a_k\theta^k$$
$$= -\frac{f'(\theta)}{f(\theta)}\mu_r + \frac{1}{f(\theta)}\sum_{k=0}^{\infty}(k-\mu_1'+\mu_1')(k-\mu_1')^r a_k\theta^{k-1}$$
$$- r\frac{d}{d\theta}\mu_1'\sum_{k=0}^{\infty}(k-\mu_1')^{r-1}\frac{a_k\theta^k}{f(\theta)}$$
$$= -\frac{f'(\theta)}{f(\theta)}\mu_r + \frac{1}{f(\theta)}\sum_{k=0}^{\infty}\left[(k-\mu_1')^{r+1} + \mu_1'(k-\mu_1')^r\right]a_k\theta^{k-1} - r\frac{d}{d\theta}\mu_1'\mu_{r-1}$$

multiplying $\frac{d}{d\theta}\mu_r$ by θ we obtain,

$$\theta\frac{d}{d\theta}\mu_r = -\theta\frac{f'(\theta)}{f(\theta)}\mu_r + \sum_{k=0}^{\infty}(k-\mu_1')^{r+1}\frac{a_k\theta^k}{f(\theta)} + \mu_1'\sum_{k=0}^{\infty}(k-\mu_1')^r\frac{a_k\theta^k}{f(\theta)} - r\theta\frac{d}{d\theta}\mu_1'\mu_{r-1}$$
$$= -\mu_1'\mu_r + \mu_{r+1} + \mu_1'\mu_r - r\theta\frac{d}{d\theta}\mu_1'\mu_{r-1}$$
$$= \mu_{r+1} - r\theta\frac{d}{d\theta}\mu_1'\mu_{r-1}$$

Therefore, it follows that the recurrence relation for the central moments of a PSD is given as

$$\mu_{r+1} = \theta\left[\frac{d}{d\theta}\mu_r + r\mu_{r-1}\frac{d}{d\theta}\mu_1'\right] \tag{2.18}$$

putting $r = 1$ in (2.18) we get

$$\mu_2 = \theta\left[\frac{d}{d\theta}\mu_1 + \mu_0\frac{d}{d\theta}\mu_1'\right]$$

but

$$\mu_0 = E(X-\mu_1')^0 = 1$$

and

$$\mu_1 = E(X-\mu_1') = E(X) - \mu_1' = \mu_1' - \mu_1' = 0$$

As a result,

$$\mu_2 = \theta\left[0 + \frac{d}{d\theta}\mu_1'\right]$$
$$= \theta\frac{d}{d\theta}\mu_1' \tag{2.19}$$

substituting (2.19) in (2.18)

$$\mu_{r+1} = \theta\left[\frac{d}{d\theta}\mu_r + r\mu_{r-1}\frac{\mu_2}{\theta}\right]$$
$$= \theta\frac{d}{d\theta}\mu_r + r\mu_{r-1}\mu_2 \tag{2.20}$$

Also from (2.19)

$$\begin{aligned}
\mu_2 = \theta\frac{d}{d\theta}\mu_1' &= \theta\frac{d}{d\theta}\theta\frac{f'(\theta)}{f(\theta)} \\
&= \theta\left\{1.\frac{f'(\theta)}{f(\theta)} + \theta\frac{d}{d\theta}\frac{f'(\theta)}{f(\theta)}\right\} \\
&= \theta\left\{\frac{f'(\theta)}{f(\theta)} + \theta\left[\frac{f(\theta)f''(\theta) - [f'(\theta)]^2}{[f(\theta)]^2}\right]\right\} \\
&= \theta\left\{\frac{f'(\theta)}{f(\theta)} + \theta\frac{f''(\theta)}{f(\theta)} - \theta\left[\frac{f'(\theta)}{f(\theta)}\right]^2\right\} \\
&= \theta\frac{f'(\theta)}{f(\theta)} + \theta^2\frac{f''(\theta)}{f(\theta)} - \left[\theta\frac{f'(\theta)}{f(\theta)}\right]^2
\end{aligned}$$

The variance of X will be given by

$$Var(X) = \mu_2 = \theta^2\frac{f''(\theta)}{f(\theta)} + \theta\frac{f'(\theta)}{f(\theta)} - \left[\theta\frac{f'(\theta)}{f(\theta)}\right]^2$$

2.5 Moment Generating Functions (mgf)

The mgf of X is given by

$$M_X(t) = E\left[e^{Xt}\right] = \sum_{k=0}^{\infty} e^{tk}p_k = \sum_{k=0}^{\infty}\frac{a_k(\theta e^t)^k}{f(\theta)}$$
$$= \frac{f(\theta e^t)}{f(\theta)} \tag{2.21}$$

To obtain the r^{th} moment

$$
\begin{aligned}
M_X(t) &= E\left[e^{tX}\right] \\
&= E\left[1 + \frac{(tX)}{1!} + \frac{(tX)^2}{2!} + \frac{(tX)^3}{3!} + \cdots\right] \\
&= E[1] + tE[X] + \frac{t^2}{2!}E\left[X^2\right] + \frac{t^3}{3!}E\left[X^3\right] + \cdots \\
&= 1 + t\mu_1' + \frac{t^2}{2!}\mu_2' + \frac{t^3}{3!}\mu_3' + \cdots \\
&= \sum_{r=0}^{\infty} \frac{t^r}{r!}\mu_r' \qquad (2.22)
\end{aligned}
$$

From (2.22) it is observed that μ_r' = coefficient of $\frac{t^r}{r!}$ in the expansion of $M_X(t)$. Also, the r^{th} moment is the r^{th} derivative of $M_X(t)$ w.r.t t and setting $t = 0$ i.e.

$$\mu_r' = \frac{d^r M_X(t)}{dt^r} \mid_{t=0}$$

2.6 Factorial Moments

r^{th} factorial moment is defined by

$$\mu_{[r]} = E\left[X(X-1)(X-2)\ldots(X-r+1)\right] \qquad (2.23)$$

when

$r = 1$ we have

$$\mu_{[1]} = E[X] = \mu_1'$$

$r = 2$

$$\mu_{[2]} = E[X(X-1)] = \mu_2' - \mu_1'$$

$r = 3$

$$
\begin{aligned}
\mu_{[3]} &= E[X(X-1)(X-2)] \\
&= E\left[X^3\right] - 3E\left[X^2\right] + 2E[X] \\
&= \mu_3' - 3\mu_2' + 2\mu_1'
\end{aligned}
$$

$r = 4$

$$
\begin{aligned}
\mu_{[4]} &= E[X(X-1)(X-2)(X-3)] \\
&= E\left[X^4\right] - 6E\left[X^3\right] + 11E\left[X^2\right] - 6E[X] \\
&= \mu_4' - 6\mu_3' + 11\mu_2' - 6\mu_1'
\end{aligned}
$$

2.7 Factorial Moment Generating Function (fmgf)

The fmgf of X is given by

$$
\begin{aligned}
M_{[X]}(t) &= E\left[1+t\right]^X \\
&= \sum_{k=0}^{\infty}\left[1+e\right]^k p_k = \sum_{k=0}^{\infty}\frac{a_k(\theta\left[1+e\right])^k}{f(\theta)} \\
&= \frac{f(\theta+\theta t)}{f(\theta)} \qquad (2.24)
\end{aligned}
$$

The r^{th} factorial moment is given by the r^{th} derivative of $M_{[X]}(t)$ w.r.t t and setting $t=0$

$$\mu_{[r]} = \frac{d^r M_{[X]}(t)}{dt^r}\,|_{t=0}$$

or the coefficient of $\frac{t^r}{r!}$ in the expansion of $M_{[X]}(t)$.

2.8 Cumulant and Cumulant Generating Function (cgf)

The cgf of X is given by

$$K_X(t) = \log M_X(t)$$

provided $E\left[e^{tX}\right]$ exist. Therefore to obtain the r^{th} cumulant

$$
\begin{aligned}
K_X(t) &= \log M_X(t) \\
&= \log\left\{E\left[1+tX+\frac{(tX)^2}{2!}+\frac{(tX)^3}{3!}+\cdots\right]\right\} \\
&= \log\left\{E\left[1\right]+tE\left[X\right]+\frac{t^2}{2!}E\left[X^2\right]+\frac{t^3}{3!}E\left[X^3\right]+\right\} \\
&= \log\left\{1+t\mu_1'+\frac{t^2}{2!}\mu_2'+\frac{t^3}{3!}\mu_3'+\cdots\right\} \\
&= \left[t\mu_1'+\frac{t^2}{2!}\mu_2'+\frac{t^3}{3!}\mu_3'+\cdots\right]-\frac{1}{2}\left[t\mu_1'+\frac{t^2}{2!}\mu_2'+\frac{t^3}{3!}\mu_3'+\cdots\right] \\
&+\frac{1}{3}\left[t\mu_1'+\frac{t^2}{2!}\mu_2'+\frac{t^3}{3!}\mu_3'+\cdots\right]-\cdots \\
&= t\mu_1'+\mu_2\frac{t^2}{2!}+\mu_3\frac{t^3}{3!}+\cdots \\
&= k_1 t+k_2\frac{t^2}{2!}+k_3\frac{t^3}{3!}+\cdots \\
&= \sum_{r=1}^{\infty}\frac{k_r t^r}{r!} \qquad (2.25)
\end{aligned}
$$

Thus, k_r(r^{th} cumulant of X) is the coefficient of $\frac{t^r}{r!}$ in the expansion of $K_X(t)$. The r^{th} cumulant can also be obtained from the r^{th} derivative of $K_X(t)$ w.r.t t and setting $t=0$. That is

$$k_r = \frac{d^r K_X(t)}{dt^r} |_{t=0}$$

2.9 Special Cases

2.9.1 Poisson Distribution

$$f(\theta) = e^\theta = \sum_{k=0}^{\infty} \frac{\theta^k}{k!}$$

Then

i.
$$\Pr(X = k) = \frac{e^{-\theta}\theta^k}{k!}, \ k = 0, 1, 2, ...$$
which is a Poisson Distribution.

ii.
$$a_k = \frac{1}{k!}$$

iii.
$$f'(\theta) = e^\theta$$

iv.
$$f''(\theta) = e^\theta$$

v. The mean is given by
$$E(X) = \theta \frac{f'(\theta)}{f(\theta)} = \theta$$

vi. The variance is given by
$$\begin{aligned} Var(X) &= \theta^2 \frac{f''(\theta)}{f(\theta)} + \theta \frac{f'(\theta)}{f(\theta)} - \left[\theta \frac{f'(\theta)}{f(\theta)}\right]^2 \\ &= \theta^2 + \theta - \theta^2 = \theta \end{aligned}$$

vii.
$$\mu_{r+1} = \theta \left[\frac{d}{d\theta}\mu_r + r\mu_{r-1}\frac{d}{d\theta}\mu_1'\right]$$
but
$$\mu_1' = E(X) = \theta$$
Hence the recurrence relation for the central moments of Poisson Distribution is given by
$$\mu_{r+1} = \theta \left[\frac{d}{d\theta}\mu_r + r\mu_{r-1}\right]$$
setting: $r = 1$
$$\begin{aligned} \mu_2 &= \theta \left[\frac{d}{d\theta}\mu_1 + \mu_0\right] \\ &= \theta [0 + 1] = \theta \end{aligned}$$

$r = 2$

$$\mu_3 = \theta\left[\frac{d}{d\theta}\mu_2 + 2\mu_1\right]$$
$$= \theta\left[\frac{d}{d\theta}\theta + 2\cdot 0\right] = \theta$$

$r = 3$

$$\mu_4 = \theta\left[\frac{d}{d\theta}\mu_3 + 3\mu_2\right]$$
$$= \theta\left[\frac{d}{d\theta}\theta + 3\cdot\theta\right]$$
$$= \theta\left(1 + 3\theta\right)$$

viii. Probability generating function for Poisson Distribution is given by

$$G(s) = \frac{f(\theta s)}{f(\theta)} = \frac{e^{\theta s}}{e^{\theta}}$$

Taking the first and second derivative w.r.t s and setting $s = 1,$ we obtain

$$G'(s) = \theta\frac{e^{\theta s}}{e^{\theta}}$$

$$G''(s) = \theta^2\frac{e^{\theta s}}{e^{\theta}}$$

$$G'(1) = \theta$$
$$G''(1) = \theta^2$$

Thus, to obtain the mean and variance

$$E\left(X\right) = G'(1) = \theta$$

$$Var(X) = G''(1) + G'(1) - \left[G'(1)\right]^2$$
$$= \theta^2 + \theta - \theta^2$$
$$= \theta$$

ix. The moment generating function of Poisson Distribution is given by

$$M_X\left(t\right) = \frac{f(\theta e^t)}{f(\theta)} = \frac{e^{\theta e^t}}{e^{\theta}} = e^{\theta\left(e^t-1\right)}$$

The r^{th} moment about the origin is obtained from the r^{th} derivative of $M_X(t)$ w.r.t t and setting $t = 0$

That is for $r = 1$, we have

$$\mu_1' = \frac{dM_X(t)}{dt}\mid_{t=0} = \frac{d}{dt}\left[e^{\theta\left(e^t-1\right)}\right] = \theta e^t e^{\theta\left(e^t-1\right)}\mid_{t=0} = \theta$$

for $r = 2$

$$\mu_2' = \frac{d}{dt}\left[\theta e^{\theta\left(e^t-1\right)+t}\right]|_{t=0} = \theta\left(\theta e^t+1\right)e^{\theta\left(e^t-1\right)+t}|_{t=0} = \theta^2+\theta$$

Also,

$$\mu_2 = \mu_2' - \mu_1'^2 = \theta^2 + \theta - \theta^2 = \theta$$

for $r = 3$

$$\begin{aligned}\mu_3' &= \frac{d}{dt}\left\{\theta\left(\theta e^t+1\right)\exp\left[t+\theta\left(e^t-1\right)\right]\right\}|_{t=0}\\ &= \left\{\theta^2 e^t \exp\left[t+\theta\left(e^t-1\right)\right] + \theta\left(\theta e^t+1\right)^2 \exp\left[t+\theta\left(e^t-1\right)\right]\right\}|_{t=0}\\ &= \theta + 3\theta^2 + \theta^3\end{aligned}$$

when $r = 4$

$$\begin{aligned}\mu_4' &= \frac{d}{dt}\left\{\theta^2 e^t \exp\left[t+\theta\left(e^t-1\right)\right] + \theta\left(\theta e^t+1\right)^2 \exp\left[t+\theta\left(e^t-1\right)\right]\right\}|_{t=0}\\ &= \left\{\begin{array}{c}\theta^2 e^t \exp\left[t+\theta\left(e^t-1\right)\right] + \theta\left(\theta e^t+1\right)^3 \exp\left[t+\theta\left(e^t-1\right)\right]\\ +3\theta^2 e^t\left(\theta e^t+1\right)\exp\left[t+\theta\left(e^t-1\right)\right]\end{array}\right\}|_{t=0}\\ &= \theta^2 + \theta\left(\theta+1\right)^3 + 3\theta^2\left(\theta+1\right)\\ &= \theta + 7\theta^2 + 6\theta^3 + \theta^4\end{aligned}$$

x. The factorial moment generating function of Poisson Distribution is given by

$$\begin{aligned}M_{[X]}(t) &= \frac{f(\theta+\theta t)}{f(\theta)} = e^{\theta t}\\ &= 1 + \theta t + \frac{\theta^2 t^2}{2!} + \frac{\theta^3 t^3}{3!} + \cdots\end{aligned}$$

The r^{th} factorial moment is the coefficient of $\frac{t^r}{r!}$ in the expansion of $M_{[X]}(t)$. Thus

$$\mu_{[r]} = \theta^r, \ r \geq 1$$

That is

$$\mu_{[1]} = \theta = \mu_1', \ \mu_{[2]} = \theta^2, \ \mu_{[3]} = \theta^3, \ \mu_{[4]} = \theta^4$$

The recursive relationship between factorial moments of Poisson Distribution is given by

$$\mu_{[r]} = \theta\mu_{[r-1]}$$

xi. The cumulant generating function of Poisson Distribution is given by

$$\begin{aligned}K_X(t) &= \log M_X(t) = \log\left[e^{\theta\left(e^t-1\right)}\right]\\ &= \theta\left(e^t-1\right)\\ &= \theta\left[1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots\right]\end{aligned}$$

The r^{th} cumulant of the distribution is the coefficient of $\frac{t^r}{r!}$ in the expansion of $K_X(t)$.

Thus

$$k_r = \theta \quad \text{for all} \quad r \geq 1$$

The recursion formula for the cumulants of Poisson Distribution is given by

$$k_r = \frac{d^r}{dt^r}\theta\left(e^t - 1\right)|_{t=0}$$

$$\frac{d}{d\theta}k_r = \frac{d^r}{dt^r}\left(e^t - 1\right)|_{t=0} \qquad (***)$$

Multiply equation $(***)$ by θ to obtain

$$\theta\frac{d}{d\theta}k_r = \frac{d^r}{dt^r}\theta\left(e^t - 1\right)|_{t=0}$$

Also,

$$\begin{aligned} k_{r+1} &= \frac{d^{r+1}}{dt^{r+1}}\theta\left(e^t - 1\right)|_{t=0} \\ &= \frac{d^r}{dt^r}\theta e^t|_{t=0} \end{aligned}$$

Subtracting $k_{r+1} - \theta\frac{d}{d\theta}k_r$ we obtain

$$k_{r+1} - \theta\frac{d}{d\theta}k_r = \frac{d^r}{dt^r}\left\{\theta e^t - \theta e^t + \theta\right\} = 0$$

Therefore it follows that

$$k_{r+1} = \theta\frac{d}{d\theta}k_r$$

2.9.2 Binomial Distribution

$$f(\theta) = (1+\theta)^n = \sum_{k=0}^{\infty}\binom{n}{k}\theta^k$$

Then

i.

$$\begin{aligned} \Pr\left(X = k\right) &= \binom{n}{k}\frac{\theta^k}{(1+\theta)^n} \\ &= \binom{n}{k}\left(\frac{\theta}{1+\theta}\right)^k\left(\frac{1}{1+\theta}\right)^{n-k}, \; k = 0, 1, 2, ..., n \end{aligned}$$

which is Binomial with parameters n and $\frac{\theta}{(1+\theta)}$.

ii.

$$a_k = \binom{n}{k}, \; k = 0, 1, 2, ..., n$$

iii.

$$f'(\theta) = n(1+\theta)^{n-1};\ n = 1, 2, ...$$

iv.

$$f''(\theta) = n(n-1)(1+\theta)^{n-2}; n = 2, 3, ...$$

v. The mean is given by

$$E(X) = \theta\frac{f'(\theta)}{f(\theta)} = \theta\frac{n(1+\theta)^{n-1}}{(1+\theta)^n} = n\frac{\theta}{1+\theta}$$

vi. And the variance

$$\begin{aligned}
Var\,(X) &= \theta^2\frac{f''(\theta)}{f(\theta)} + \theta\frac{f'(\theta)}{f(\theta)} - \left[\theta\frac{f'(\theta)}{f(\theta)}\right]^2 \\
&= \theta^2\frac{n(n-1)(1+\theta)^{n-2}}{(1+\theta)^n} + n\frac{\theta}{1+\theta} - \left[n\frac{\theta}{1+\theta}\right]^2 \\
&= \theta^2\frac{n(n-1)}{(1+\theta)^2} + n\frac{\theta}{1+\theta} - \frac{n^2\theta^2}{(1+\theta)^2} \\
&= \frac{\theta^2 n^2}{(1+\theta)^2} - \frac{n\theta^2}{(1+\theta)^2} + n\frac{\theta}{1+\theta} - \frac{n^2\theta^2}{(1+\theta)^2} \\
&= n\frac{\theta}{1+\theta}\left[1 - \frac{\theta}{1+\theta}\right] \\
&= n\left(\frac{\theta}{1+\theta}\right)\left(1 - \frac{\theta}{1+\theta}\right) \\
&= n\left(\frac{\theta}{1+\theta}\right)\left(\frac{1}{1+\theta}\right)
\end{aligned}$$

vii.

$$\mu_{r+1} = \theta\left[\frac{d}{d\theta}\mu_r + r\mu_{r-1}\frac{d}{d\theta}\mu_1'\right]$$

but

$$\mu_1' = E(X) = n\frac{\theta}{1+\theta}$$

Hence the recurrence relation for the central moments of Binomial Distribution is given by

$$\begin{aligned}
\mu_{r+1} &= \theta\left[\frac{d}{d\theta}\mu_r + r\mu_{r-1}\frac{d}{d\theta}n\frac{\theta}{1+\theta}\right] \\
&= \theta\left[\frac{d}{d\theta}\mu_r + nr\mu_{r-1}\left(\frac{(1+\theta)\cdot 1 - \theta\cdot 1}{(1+\theta)^2}\right)\right] \\
&= \theta\left[\frac{d}{d\theta}\mu_r + nr\mu_{r-1}\frac{1}{(1+\theta)^2}\right] \\
&= \theta\left[\frac{d}{d\theta}\mu_r + \frac{nr\mu_{r-1}}{(1+\theta)^2}\right]
\end{aligned}$$

As a result, when $r = 1$

$$\mu_2 = \theta \frac{d}{d\theta}\mu_1 + \frac{n\theta\mu_0}{(1+\theta)^2} = 0 + n\frac{\theta}{(1+\theta)^2} = n\left(\frac{\theta}{1+\theta}\right)\left(\frac{1}{1+\theta}\right)$$

$r = 2$

$$\begin{aligned}
\mu_3 &= \theta\left\{\frac{d}{d\theta}\mu_2 + \frac{2n\mu_1}{(1+\theta)^2}\right\} \\
&= \theta\frac{d}{d\theta}\left\{n\frac{\theta}{(1+\theta)^2}\right\} \\
&= \left\{\frac{n(\theta+1) - 2n\theta}{(\theta+1)^3}\right\}\theta \\
&= n\theta(1-\theta)\frac{1}{(\theta+1)^3}
\end{aligned}$$

$r = 3$

$$\begin{aligned}
u_4 &= \theta\left[\frac{d}{d\theta}\mu_3 + \frac{3n\mu_2}{(1+\theta)^2}\right] \\
&= \theta\left\{\frac{d}{d\theta}\left[\frac{n\theta(\theta+1) - 2n\theta^2}{(\theta+1)^3}\right] + \frac{3n\cdot n\frac{\theta}{(1+\theta)^2}}{(1+\theta)^2}\right\} \\
&= \theta\left\{\frac{(n-2n\theta)(\theta+1) - 3(n\theta - n\theta^2) + 3n^2\theta}{(\theta+1)^4}\right\} \\
&= \frac{n\theta - 4n\theta^2 + n\theta^3 + 3n^2\theta^2}{(\theta+1)^4}
\end{aligned}$$

viii. Probability generating function for Binomial Distribution is given by

$$G(s) = \frac{f(\theta s)}{f(\theta)} = \frac{(1+\theta s)^n}{(1+\theta)^n}$$

$$G'(s) = n\theta\frac{(1+\theta s)^{n-1}}{(1+\theta)^n}$$

$$G''(s) = n(n-1)\theta^2\frac{(1+\theta s)^{n-2}}{(1+\theta)^n}$$

setting $s = 1$ we obtain

$$G'(1) = n\frac{\theta}{(1+\theta)}$$

$$G''(1) = n(n-1)\frac{\theta^2}{(1+\theta)^2}$$

To obtain the mean and variance

$$E(X) = G'(1) = n\frac{\theta}{(1+\theta)}$$

$$
\begin{aligned}
Var(X) &= G''(1) + G'(1) - \left[G'(1)\right]^2 \\
&= n(n-1)\frac{\theta^2}{(1+\theta)^2} + n\frac{\theta}{(1+\theta)} - n^2\left[\frac{\theta}{(1+\theta)}\right]^2 \\
&= \frac{\theta^2 n^2}{(1+\theta)^2} - \frac{n\theta^2}{(1+\theta)^2} + n\frac{\theta}{1+\theta} - \frac{n^2\theta^2}{(1+\theta)^2} \\
&= n\left(\frac{\theta}{1+\theta}\right)\left(1-\frac{\theta}{1+\theta}\right) \\
&= n\left(\frac{\theta}{1+\theta}\right)\left(\frac{1}{1+\theta}\right)
\end{aligned}
$$

ix. The moment generating function for Binomial Distribution is given by

$$M_X(t) = \frac{f(\theta e^t)}{f(\theta)} = \frac{(1+\theta e^t)^n}{(1+\theta)^n}$$

The r^{th} moment about the origin is obtained from the r^{th} derivative of $M_X(t)$ w.r.t t and setting $t = 0$. That is

$$\mu_r' = \frac{d^r M_X(t)}{dt^r}\,|_{t=0}$$

Hence setting $r = 1$

$$
\begin{aligned}
\mu_1' &= \frac{dM_X(t)}{dt}\,|_{t=0} \\
&= \frac{d}{dt}\left[\left(\frac{1+\theta e^t}{1+\theta}\right)^n\right]|_{t=0} \\
&= n\left(\frac{1+\theta e^t}{1+\theta}\right)^{n-1}\frac{\theta e^t}{1+\theta}\,|_{t=0} \\
&= n\frac{\theta}{1+\theta}
\end{aligned}
$$

for $r = 2$

$$
\begin{aligned}
\mu_2' &= \frac{d}{dt}\left[n\left(\frac{1+\theta e^t}{1+\theta}\right)^{n-1}\frac{\theta e^t}{1+\theta}\right]|_{t=0} \\
&= \left\{n\,(n-1)\left(\frac{1+\theta e^t}{1+\theta}\right)^{n-2}\left(\frac{\theta e^t}{1+\theta}\right)^2 + \left(\frac{1+\theta e^t}{1+\theta}\right)^{n-1}\frac{n\theta e^t}{1+\theta}\right\}|_{t=0} \\
&= n\,(n-1)\left(\frac{\theta}{1+\theta}\right)^2 + n\frac{\theta}{1+\theta}
\end{aligned}
$$

Also,

$$
\begin{aligned}
\mu_2 &= n\,(n-1)\left(\frac{\theta}{1+\theta}\right)^2 + n\frac{\theta}{1+\theta} - \left(n\frac{\theta}{1+\theta}\right)^2 \\
&= \frac{n^2\theta^2}{(1+\theta)^2} - \frac{n\theta^2}{(1+\theta)^2} + \frac{n\theta}{1+\theta} - \frac{n^2\theta^2}{(1+\theta)^2} \\
&= n\left(\frac{\theta}{1+\theta}\right)\left(\frac{1}{1+\theta}\right)
\end{aligned}
$$

for $r = 3$

$$\mu_3' = n(n-1)\frac{d}{dt}\left\{\left(\frac{1+\theta e^t}{1+\theta}\right)^{n-2}\left(\frac{\theta e^t}{1+\theta}\right)^2 + n\left(\frac{1+\theta e^t}{1+\theta}\right)^{n-1}\frac{\theta e^t}{1+\theta}\right\}|_{t=0}$$

$$= \left\{\begin{array}{c} n(n-1)\left[\frac{2\theta^2 e^{2t}}{(\theta+1)^2}\left(\frac{\theta e^t+1}{\theta+1}\right)^{n-2} + \frac{\theta^3 e^{3t}(n-2)}{(\theta+1)^3}\left(\frac{\theta e^t+1}{\theta+1}\right)^{n-3}\right] \\ +n\left[\frac{\theta e^t}{\theta+1}\left(\frac{\theta e^t+1}{\theta+1}\right)^{n-1} + \frac{\theta^2 e^{2t}(n-1)}{(\theta+1)^2}\left(\frac{\theta e^t+1}{\theta+1}\right)^{n-2}\right] \end{array}\right\}|_{t=0}$$

$$= n(n-1)\left\{\frac{2\theta^2}{(\theta+1)^2} + \frac{(n-2)\theta^3}{(\theta+1)^3}\right\} + n\left\{\frac{1}{\theta+1} + \frac{(n-1)\theta^2}{(\theta+1)^2}\right\}$$

x. The factorial moment generating function of Binomial Distribution is given by

$$M_{[X]}(t) = \frac{f(\theta+\theta t)}{f(\theta)} = \left(\frac{1+(\theta+\theta t)}{1+\theta}\right)^n$$

The r^{th} factorial moment is obtained by from the r^{th} derivative of $M_{[X]}(t)$ w.r.t t and setting $t = 0$. i.e.,

$$\mu_{[r]} = \frac{d^r M_{[X]}(t)}{dt^r}|_{t=0}$$

When; $r = 1$,

$$\begin{aligned} \mu_{[1]} &= \frac{dM_{[X]}(t)}{dt}|_{t=0} = \frac{d}{dt}\left(\frac{1+(\theta+\theta t)}{1+\theta}\right)^n|_{t=0} \\ &= n\left(\frac{1+(\theta+\theta t)}{1+\theta}\right)^{n-1}\left(\frac{\theta}{1+\theta}\right)|_{t=0} \\ &= n\frac{\theta}{1+\theta} \end{aligned}$$

for $r = 2$

$$\begin{aligned} \mu_{[2]} &= n\left(\frac{\theta}{1+\theta}\right)\frac{d}{dt}\left\{\left(\frac{1+(\theta+\theta t)}{1+\theta}\right)^{n-1}\right\}|_{t=0} \\ &= n(n-1)\left(\frac{1+(\theta+\theta t)}{1+\theta}\right)^{n-2}\left(\frac{\theta}{1+\theta}\right)^2|_{t=0} \\ &= n(n-1)\left(\frac{\theta}{1+\theta}\right)^2 \end{aligned}$$

for $r = 3$

$$\begin{aligned} \mu_{[3]} &= n(n-1)\left(\frac{\theta}{1+\theta}\right)^2\frac{d}{dt}\left[\left(\frac{1+(\theta+\theta t)}{1+\theta}\right)^{n-2}\right]|_{t=0} \\ &= n(n-1)(n-2)\left(\frac{\theta}{1+\theta}\right)^3\left(\frac{1+(\theta+\theta t)}{1+\theta}\right)^{n-3}|_{t=0} \\ &= n(n-1)(n-2)\left(\frac{\theta}{1+\theta}\right)^3 \end{aligned}$$

and for $r = 4$

$$\mu_{[4]} = n\left(n-1\right)\left(n-2\right)\left(\frac{\theta}{1+\theta}\right)^{3}\frac{d}{dt}\left[\left(\frac{1+\left(\theta+\theta t\right)}{1+\theta}\right)^{n-3}\right]|_{t=0}$$
$$= n\left(n-1\right)\left(n-2\right)\left(n-3\right)\left(\frac{\theta}{1+\theta}\right)^{4}$$

The recursive relationship between the factorial moments of Binomial Distribution is given by

$$\mu_{[r]} = \left(n-r+1\right)\left(\frac{\theta}{1+\theta}\right)\mu_{[r-1]}$$

xi. The cumulant generating function of Binomial Distribution is given by

$$K_X(t) = \log M_X(t) = \log\left(\frac{1+\theta e^t}{1+\theta}\right)^n$$

The r^{th} cumulant of the distribution is the r^{th} derivative of $K_x(t)$ w.r.t t and setting $t = 0$

$$k_r = \frac{d^r K_X(t)}{dt^r}|_{t=0}$$

When $r = 1$, we have

$$k_1 = \frac{dK_X(t)}{dt}|_{t=0} = \frac{d}{dt}n\log\left(\frac{1+\theta e^t}{1+\theta}\right)|_{t=0}$$
$$= n\left(\frac{1+\theta}{1+\theta e^t}\right)\left(\frac{\theta e^t}{1+\theta}\right)|_{t=0}$$
$$= n\left(\frac{\theta}{1+\theta}\right)$$

for $r = 2$

$$k_2 = \frac{d}{dt}\left[n\left(\frac{1+\theta}{1+\theta e^t}\right)\left(\frac{\theta e^t}{1+\theta}\right)\right]|_{t=0}$$
$$= n\left\{-\frac{\theta e^t\left(1+\theta\right)}{\left(1+\theta e^t\right)^2}\left(\frac{\theta e^t}{1+\theta}\right)+\left(\frac{1+\theta}{1+\theta e^t}\right)\left(\frac{\theta e^t\left(1+\theta\right)}{\left(1+\theta\right)^2}\right)\right\}|_{t=0}$$
$$= -\frac{n\theta^2}{\left(1+\theta\right)^2}+\frac{\theta n}{\left(1+\theta\right)}$$
$$= n\left(\frac{\theta}{1+\theta}\right)\left\{1-\frac{\theta}{1+\theta}\right\}$$
$$= n\left(\frac{\theta}{1+\theta}\right)\left(\frac{1}{1+\theta}\right)$$

for $r = 3$

$$
\begin{aligned}
k_3 &= \frac{d}{dt} n \left\{ -\frac{\theta^2 e^{2t}}{(1+\theta e^t)^2} + \frac{\theta e^t}{1+\theta e^t} \right\} |_{t=0} \\
&= n \left\{ \frac{-2\theta^2 e^{2t} (1+\theta e^t)^2 + 2 (\theta^3 e^{3t}) (1+\theta e^t)}{(1+\theta e^t)^4} + \frac{\theta e^t (1+\theta e^t) - \theta^2 e^{2t}}{(1+\theta e^t)^2} \right\} \\
&= n \left\{ \frac{-2\theta^2 (1+\theta)^2 + 2\theta^3 (1+\theta)}{(1+\theta)^4} + \frac{\theta}{(1+\theta)^2} \right\} \\
&= n \left\{ \frac{-2\theta^2 - 2\theta^3 + 2\theta^3}{(1+\theta)^3} + \frac{\theta}{(1+\theta)^2} \right\} \\
&= n \left\{ \frac{-2\theta^2 + \theta (1+\theta)}{(1+\theta)^3} \right\} = n \left\{ \frac{-2\theta^2 + \theta + \theta^2}{(1+\theta)^3} \right\} \\
&= n\theta \frac{(1-\theta)}{(1+\theta)^3}
\end{aligned}
$$

and for $r = 4$

$$
\begin{aligned}
k_4 &= n \frac{d}{dt} \left\{ \frac{-2\theta^2 e^{2t} (1+\theta e^t)^2 + 2\theta^3 e^{3t} (1+\theta e^t)}{(1+\theta e^t)^4} + \frac{\theta e^t [(1+\theta e^t) - \theta e^t]}{(1+\theta e^t)^2} \right\} |_{t=0} \\
&= n \frac{d}{dt} \left\{ \frac{-2\theta^2 e^{2t} (1+\theta e^t)^2 + 2\theta^3 e^{3t} (1+\theta e^t)}{(1+\theta e^t)^4} + \frac{\theta e^t [\theta (1+\theta e^t) - \theta]}{(1+\theta e^t)^2} \right\} |_{t=0} \\
&= n \left\{ \frac{6\theta^3 e^{3t} - 4\theta^3 e^{3t} - 4\theta^2 e^{2t}}{4\theta e^t + 6\theta^2 e^{2t} + 4\theta^3 e^{3t} + \theta^4 e^{4t} + 1} + \left[\begin{array}{c} \frac{\left(\theta e^t \left(\theta e^t + 1\right) - 2\theta^2 e^{2t} + \theta^2 e^{2t}\right)}{(\theta e^t + 1)^2} \\ -\frac{2\theta e^t \left(\theta e^t \left(\theta e^t + 1\right) - \theta^2 e^{2t}\right)}{(\theta e^t + 1)^3} \end{array} \right] \right\} |_{t=0} \\
&= n \left\{ \frac{6\theta^3 - 4\theta^3 - 4\theta^2}{4\theta + 6\theta^2 + 4\theta^3 + \theta^4 + 1} + \frac{\theta}{(\theta+1)^2} - \frac{2\theta^2}{(\theta+1)^3} \right\} \\
&= n \left\{ \frac{2\theta^3 - 4\theta^2 - 2\theta^2 (\theta+1) + \theta (\theta+1)^2}{(\theta+1)^4} \right\} \\
&= \frac{n\theta^2 (\theta^2 - 4\theta + 1)}{(\theta+1)^4}
\end{aligned}
$$

The recursion formula for cumulants of Binomial Distribution is given by

$$
k_{r+1} = \theta \frac{d}{d\theta} k_r, \; r \geq 1
$$

2.9.3 Negative Binomial Distribution (NB)

$$
f(\theta) = (1-\theta)^{-\alpha} = \sum_{k=0}^{\infty} \binom{-\alpha}{k} (-\theta)^k
$$

Therefore

i.

$$\begin{aligned}
\Pr(X=k) &= \binom{-\alpha}{k}\frac{(-\theta)^k}{(1-\theta)^{-\alpha}} \\
&= (-1)^k\binom{-\alpha}{k}\theta^k(1-\theta)^{\alpha} \\
&= \binom{\alpha+k-1}{k}\theta^k(1-\theta)^{\alpha}\,, \text{ for } k=0,1,2,...
\end{aligned}$$

which is a Negative Binomial Distribution.

ii.

$$a_k = \binom{\alpha+k-1}{k}$$

iii.

$$f'(\theta) = \alpha(1-\theta)^{-\alpha-1}$$

iv.

$$f''(\theta) = \alpha(\alpha+1)(1-\theta)^{-\alpha-2}$$

v. The mean is given by

$$E(X) = \theta\frac{f'(\theta)}{f(\theta)} = \theta\alpha\frac{(1-\theta)^{-\alpha-1}}{(1-\theta)^{-\alpha}} = \alpha\frac{\theta}{1-\theta}, \qquad 0<\theta<1$$

vi. The variance is given by

$$\begin{aligned}
Var(X) &= \theta^2\frac{f''(\theta)}{f(\theta)} + \theta\frac{f'(\theta)}{f(\theta)} - \left[\theta\frac{f'(\theta)}{f(\theta)}\right]^2 \\
&= \theta^2\frac{\alpha(\alpha+1)(1-\theta)^{-\alpha-2}}{(1-\theta)^{-\alpha}} + \alpha\frac{\theta}{1-\theta} - \alpha^2\frac{\theta^2}{(1-\theta)^2} \\
&= \alpha(\alpha+1)\frac{\theta^2}{(1-\theta)^2} + \alpha\frac{\theta}{1-\theta} - \alpha^2\frac{\theta^2}{(1-\theta)^2} \\
&= \frac{\alpha^2\theta^2}{(1-\theta)^2} + \frac{\alpha\theta^2}{(1-\theta)^2} + \frac{\alpha\theta}{1-\theta} - \frac{\alpha^2\theta^2}{(1-\theta)^2} \\
&= \alpha\frac{\theta}{1-\theta}\left[1+\frac{\theta}{1-\theta}\right] \\
&= \alpha\frac{\theta}{1-\theta}\cdot\frac{1}{1-\theta} = \frac{\alpha\theta}{(1-\theta)^2}
\end{aligned}$$

vii.

$$\mu_{r+1} = \theta\left[\frac{d}{d\theta}\mu_r + r\mu_{r-1}\frac{d}{d\theta}\mu_1'\right]$$

but

$$\mu_1' = E(X) = \alpha\frac{\theta}{1-\theta}$$

Thus the recurrence relation for the central moments of NB is given by

$$\begin{aligned}
\mu_{r+1} &= \theta\left[\frac{d}{d\theta}\mu_r + r\mu_{r-1}\frac{d}{d\theta}\alpha\frac{\theta}{1-\theta}\right] \\
&= \theta\left[\frac{d}{d\theta}\mu_r + \alpha r\mu_{r-1}\left(\frac{(1-\theta)\cdot 1+\theta\cdot 1}{(1-\theta)^2}\right)\right] \\
&= \theta\left[\frac{d}{d\theta}\mu_r + \alpha r\mu_{r-1}\left(\frac{1}{(1-\theta)^2}\right)\right] \\
&= \theta\frac{d}{d\theta}\mu_r + \frac{\alpha r\theta\mu_{r-1}}{(1-\theta)^2}
\end{aligned}$$

setting $r = 1$

$$\begin{aligned}
\mu_2 &= \theta\left[\frac{d}{d\theta}\mu_1 + \frac{\alpha\mu_0}{(1-\theta)^2}\right] \\
&= \theta\left[0 + \alpha\frac{1}{(1-\theta)^2}\right] = \alpha\frac{\theta}{(1-\theta)^2}
\end{aligned}$$

setting $r = 2$

$$\begin{aligned}
\mu_3 &= \theta\left[\frac{d}{d\theta}\mu_2 + 2\alpha\mu_1\frac{1}{(1-\theta)^2}\right] \\
&= \theta\left[\frac{d}{d\theta}\frac{\alpha\theta}{(1-\theta)^2} + 2\alpha\cdot 0\cdot\frac{1}{(1-\theta)^2}\right] \\
&= \theta\alpha\frac{d}{d\theta}\frac{\theta}{(1-\theta)^2} = \theta\alpha\left[\frac{(1-\theta)^2\cdot 1 + 2\theta(1-\theta)}{(1-\theta)^4}\right] \\
&= \theta\alpha\left[\frac{(1-\theta)^2 + 2\theta - 2\theta^2)}{(1-\theta)^4}\right] \\
&= \frac{\theta\alpha\,(1-\theta)\,(1-\theta+2\theta)}{(1-\theta)^4} = \frac{\theta\alpha\,(1-\theta)\,(1+\theta)}{(1-\theta)^4} \\
&= \theta\alpha\frac{(1+\theta)}{(1-\theta)^3}
\end{aligned}$$

using (2.20), setting $r = 2$ we obtain

$$\begin{aligned}
\mu_3 &= \theta\frac{d}{d\theta}\mu_2 + 2\mu_1\mu_2 = \theta\frac{d}{d\theta}\mu_2 + 0 \\
&= \theta\alpha\frac{d}{d\theta}\frac{\theta}{(1-\theta)^2} \\
&= \theta\alpha\left[\frac{(1-\theta)^2\cdot 1 + 2\theta(1-\theta)}{(1-\theta)^4}\right] \\
&= \frac{\theta\alpha\,(1-\theta)\,(1-\theta+2\theta)}{(1-\theta)^4} \\
&= \theta\alpha\frac{(1+\theta)}{(1-\theta)^3}
\end{aligned}$$

and for $r = 3$, again using (2.20) we obtain

$$\begin{aligned}
\mu_4 &= \theta \frac{d}{d\theta} \mu_3 + 3\mu_2\mu_2 \\
&= \theta \frac{d}{d\theta} \theta\alpha \frac{(1+\theta)}{(1-\theta)^3} + 3\left[\alpha \frac{\theta}{(1-\theta)^2}\right]^2 \\
&= \alpha\theta \frac{d}{d\theta} \frac{(\theta+\theta^2)}{(1-\theta)^3} + \frac{3\alpha^2\theta^2}{(1-\theta)^4} \\
&= \alpha\theta \left[\frac{(1-\theta)^3 (1+2\theta) + 3(\theta+\theta^2)(1-\theta)^2}{(1-\theta)^6}\right] + \frac{3\alpha^2\theta^2}{(1-\theta)^4} \\
&= \alpha\theta \left[\frac{(1-\theta)(1+2\theta) + 3(\theta+\theta^2)}{(1-\theta)^4}\right] + \frac{3\alpha^2\theta^2}{(1-\theta)^4} \\
&= \frac{\alpha\theta}{(1-\theta)^4} \left[1 + 2\theta - \theta - 2\theta^2 + 3\theta + 3\theta^2) + 3(\theta+\theta^2)\right] + \frac{3\alpha^2\theta^2}{(1-\theta)^4} \\
&= \frac{\alpha\theta}{(1-\theta)^4} \left[1 + 4\theta + \theta^2\right] + \frac{3\alpha^2\theta^2}{(1-\theta)^4} \\
&= \frac{\alpha\theta}{(1-\theta)^4} \left[1 + 4\theta + \theta^2 + 3\alpha\theta\right]
\end{aligned}$$

viii. Probability generating function for NB is given by

$$G(s) = \frac{f(\theta s)}{f(\theta)} = \frac{(1-\theta s)^{-\alpha}}{(1-\theta)^{-\alpha}}$$

$$G'(s) = \alpha\theta \frac{(1-\theta s)^{-\alpha-1}}{(1-\theta)^{-\alpha}}$$

$$G''(s) = \alpha(\alpha+1)\theta^2 \frac{(1-\theta s)^{-\alpha-2}}{(1-\theta)^{-\alpha}}$$

setting $s = 1$ we obtain

$$G'(1) = \alpha \frac{\theta}{1-\theta}$$

$$G''(1) = \alpha(\alpha+1) \frac{\theta^2}{(1-\theta)^2}$$

To obtain the mean and variance

$$E(X) = G'(1) = \alpha \frac{\theta}{1-\theta}$$

$$
\begin{aligned}
Var(X) &= G''(1) + G'(1) - \left[G'(1)\right]^2 \\
&= \alpha(\alpha+1)\frac{\theta^2}{(1-\theta)^2} + \alpha\frac{\theta}{1-\theta} - \alpha^2\left[\alpha\frac{\theta}{1-\theta}\right]^2 \\
&= \frac{\alpha^2\theta^2}{(1-\theta)^2} + \frac{\alpha\theta^2}{(1-\theta)^2} + \frac{\alpha\theta}{1-\theta} - \frac{\alpha^2\theta^2}{(1-\theta)^2} \\
&= \alpha\frac{\theta}{1-\theta}\left[1 + \frac{\theta}{1-\theta}\right] \\
&= \frac{\alpha\theta}{(1-\theta)^2}
\end{aligned}
$$

ix. The moment generating function for NB is given by

$$M_X(t) = \frac{f(\theta e^t)}{f(\theta)} = \frac{(1-\theta e^t)^{-\alpha}}{(1-\theta)^{-\alpha}}$$

The r^{th} moment about the origin is obtained from the r^{th} derivative of $M_X(t)$ w.r.t t and setting $t = 0$. i.e.,

$$\mu_r' = \frac{d^r M_X(t)}{dt^r}\,|_{t=0}$$

Therefore setting $r = 1$

$$
\begin{aligned}
\mu_1' &= \frac{dM_X(t)}{dt}\,|_{t=0} \\
&= \frac{d}{dt}\left[\frac{(1-\theta e^t)^{-\alpha}}{(1-\theta)^{-\alpha}}\right]|_{t=0} \\
&= \alpha\theta e^t\frac{(1-\theta e^t)^{-\alpha-1}}{(1-\theta)^{-\alpha}}\,|_{t=0} = \alpha\frac{\theta}{1-\theta}
\end{aligned}
$$

for $r = 2$

$$
\begin{aligned}
\mu_2' &= \frac{d}{dt}\left[\alpha\theta e^t\frac{(1-\theta e^t)^{-\alpha-1}}{(1-\theta)^{-\alpha}}\right]|_{t=0} \\
&= \alpha\theta e^t\frac{(1-\theta e^t)^{-\alpha-1}}{(1-\theta)^{-\alpha}}\,|_{t=0} + \frac{\alpha(\alpha+1)\theta^2 e^{2t}(1-\theta e^t)^{-\alpha-2}}{(1-\theta)^{-\alpha}}\,|_{t=0} \\
&= \alpha(\alpha+1)\frac{\theta^2}{(1-\theta)^2} + \alpha\frac{\theta}{1-\theta}
\end{aligned}
$$

Also,

$$
\begin{aligned}
\mu_2 &= \mu_2' - \mu_1'^2 \\
&= \alpha(\alpha+1)\frac{\theta^2}{(1-\theta)^2} + \alpha\frac{\theta}{1-\theta} - \left(\alpha\frac{\theta}{1-\theta}\right)^2 \\
&= \frac{\alpha^2\theta^2}{(1-\theta)^2} + \frac{\alpha\theta^2}{(1-\theta)^2} + \frac{\alpha\theta}{1-\theta} - \frac{\alpha^2\theta^2}{(1-\theta)^2} \\
&= \alpha\frac{\theta}{1-\theta}\left[1 + \frac{\theta}{1-\theta}\right] \\
&= \frac{\alpha\theta}{(1-\theta)^2}
\end{aligned}
$$

for $r=3$

$$\mu_3' = \frac{d}{dt}\left\{\frac{\alpha\theta e^t\left(1-\theta e^t\right)^{-\alpha-1}}{\left(1-\theta\right)^{-\alpha}} + \frac{\alpha\left(\alpha+1\right)\theta^2 e^{2t}\left(1-\theta e^t\right)^{-\alpha-2}}{\left(1-\theta\right)^{-\alpha}}\right\}|_{t=0}$$

$$= \left\{\begin{array}{c} \frac{\theta\alpha e^t(1-\theta)^\alpha}{(1-\theta e^t)^{\alpha+1}} + \alpha\left(\alpha+1\right)\frac{2\theta^2 e^{2t}(1-\theta)^\alpha}{(1-\theta e^t)^{\alpha+2}} - \alpha\left(-\alpha-1\right)\frac{\theta^2 e^{2t}(1-\theta)^\alpha}{(1-\theta e^t)^{\alpha+2}} \\ -\alpha\left(\alpha+1\right)\left(-\alpha-2\right)e^{3t}\theta^3\frac{(1-\theta)^\alpha}{(1-\theta e^t)^{\alpha+3}} \end{array}\right\}|_{t=0}$$

$$= \left\{\frac{\theta\alpha}{\left(1-\theta\right)} + \frac{2\theta^2\alpha\left(\alpha+1\right)}{\left(1-\theta\right)^2} - \frac{\theta^2\alpha\left(-\alpha-1\right)}{\left(1-\theta\right)^2} - \frac{\theta^3\alpha\left(\alpha+1\right)\left(-\alpha-2\right)}{\left(1-\theta\right)^3}\right\}$$

$r=4$

$$\mu_4' = \frac{d}{dt}\left\{\begin{array}{c} \frac{\theta\alpha e^t(1-\theta)^\alpha}{(1-\theta e^t)^{\alpha+1}} + \alpha\left(\alpha+1\right)\frac{2\theta^2 e^{2t}(1-\theta)^\alpha}{(1-\theta e^t)^{\alpha+2}} - \alpha\left(-\alpha-1\right)\frac{\theta^2 e^{2t}(1-\theta)^\alpha}{(1-\theta e^t)^{\alpha+2}} \\ -\alpha\left(\alpha+1\right)\left(-\alpha-2\right)e^{3t}\theta^3\frac{(1-\theta)^\alpha}{(1-\theta e^t)^{\alpha+3}} \end{array}\right\}|_{t=0}$$

$$= \left\{\begin{array}{c} \frac{\theta\alpha}{(1-\theta)} + \frac{4\theta^2\alpha(\alpha+1)}{(1-\theta)^2} - \frac{2\theta^2\alpha(-\alpha-1)}{(1-\theta)^2} - \frac{\theta^2\alpha(-\alpha-1)}{(1-\theta)^2} - \frac{2\theta^3\alpha(\alpha+1)(-\alpha-2)}{(1-\theta)^3} \\ +\frac{\theta^3\alpha(-\alpha-1)(-\alpha-2)}{(1-\theta)^3} + \frac{\theta^4\alpha(\alpha+1)(-\alpha-2)(-\alpha-3)}{(1-\theta)^4} - \frac{3\theta^3\alpha(\alpha+1)(-\alpha-2)}{(1-\theta)^3} \end{array}\right\}$$

x. The factorial moment generating function of NB is given by

$$M_{[X]}\left(t\right) = \frac{f(\theta+\theta t)}{f(\theta)} = \left(\frac{1-\theta-\theta t}{1-\theta}\right)^{-\alpha}$$

The r^{th} factorial moment is obtained by from the r^{th} derivative of $M_{[X]}\left(t\right)$ w.r.t t and setting $t=0$, i.e.

$$\mu_{[r]} = \frac{d^r M_{[X]}\left(t\right)}{dt^r}\;|_{t=0}$$

When $r=1$

$$\mu_{[1]} = \frac{dM_{[X]}\left(t\right)}{dt}\;|_{t=0} = \frac{d}{dt}\left(\frac{1-\theta-\theta t}{1-\theta}\right)^{-\alpha}|_{t=0}$$

$$= -\alpha\left(\frac{1-\theta-\theta t}{1-\theta}\right)^{-\alpha-1}\left(\frac{-\theta}{1-\theta}\right)|_{t=0}$$

$$= \alpha\frac{\theta}{1-\theta}$$

for $r=2$

$$\mu_{[2]} = \alpha\frac{\theta}{1-\theta}\frac{d}{dt}\left(\frac{1-\theta-\theta t}{1-\theta}\right)^{-\alpha-1}|_{t=0}$$

$$= \alpha\frac{\theta}{1-\theta}\left\{\left(\alpha+1\right)\frac{\theta}{1-\theta}\left(\frac{1-\theta-\theta t}{1-\theta}\right)^{-\alpha-2}\right\}|_{t=0}$$

$$= \alpha\left(\alpha+1\right)\frac{\theta^2}{\left(1-\theta\right)^2}$$

for $r=3$

$$\begin{aligned}
\mu_{[3]} &= \alpha(\alpha+1)\frac{\theta^2}{(1-\theta)^2}\frac{d}{dt}\left(\frac{1-\theta-\theta t}{1-\theta}\right)^{-\alpha-2}|_{t=0} \\
&= \alpha(\alpha+1)\frac{\theta^2}{(1-\theta)^2}\left\{(\alpha+2)\frac{\theta}{1-\theta}\left(\frac{1-\theta-\theta t}{1-\theta}\right)^{-\alpha-3}\right\}|_{t=0} \\
&= \alpha(\alpha+1)(\alpha+2)\frac{\theta^3}{(1-\theta)^3}
\end{aligned}$$

and for $r=4$

$$\begin{aligned}
\mu_{[4]} &= \alpha(\alpha+1)(\alpha+2)\frac{\theta^3}{(1-\theta)^3}\frac{d}{dt}\left(\frac{1-\theta-\theta t}{1-\theta}\right)^{-\alpha-3}|_{t=0} \\
&= \alpha(\alpha+1)(\alpha+2)\frac{\theta^3}{(1-\theta)^3}\left\{(\alpha+3)\frac{\theta}{1-\theta}\left(\frac{1-\theta-\theta t}{1-\theta}\right)^{-\alpha-4}\right\}|_{t=0} \\
&= \alpha(\alpha+1)(\alpha+2)(\alpha+3)\frac{\theta^4}{(1-\theta)^4}
\end{aligned}$$

The recursive relationship between the factorial moments of NB is given by

$$\mu_{[r]} = (\alpha+r-1)\left(\frac{\theta}{1-\theta}\right)\mu_{[r-1]}$$

xi. The cumulant generating function of Negative Binomial Distribution is given by

$$K_X(t) = \log M_x(t) = \log\left(\frac{1-\theta e^t}{1-\theta}\right)^{-\alpha}$$

The r^{th} cumulant of the distribution is the r^{th} derivative of $K_x(t)$ w.r.t t and setting $t=0$

$$k_r = \frac{d^r K_X(t)}{dt^r}|_{t=0}$$

When $r=1$, we have

$$\begin{aligned}
k_1 &= \frac{dK_X(t)}{dt}|_{t=0} \\
&= -\alpha\frac{d}{dt}\log\left(\frac{1-\theta e^t}{1-\theta}\right)|_{t=0} \\
&= -\alpha\left(\frac{1-\theta}{1-\theta e^t}\right)\left(\frac{-\theta e^t}{1-\theta}\right)|_{t=0} \\
&= \alpha\left(\frac{\theta}{1-\theta}\right)
\end{aligned}$$

$r=2$

$$\begin{aligned}
k_2 &= \alpha\theta\frac{d}{dt}\left[\frac{e^t}{1-\theta e^t}\right]|_{t=0} \\
&= \alpha\theta\left\{\frac{e^t(1-\theta e^t)+\theta e^t(e^t)}{(1-\theta e^t)^2}\right\}|_{t=0} \\
&= \alpha\frac{\theta}{(1-\theta)^2}
\end{aligned}$$

$r = 3$

$$k_3 = \alpha\theta \frac{d}{dt} \left\{ \frac{e^t}{(1-\theta e^t)^2} \right\} |_{t=0}$$
$$= \alpha\theta \left\{ \frac{e^t (1-\theta e^t)^2 + 2\theta e^{2t} (1-\theta e^t)}{(1-\theta e^t)^4} \right\}$$
$$= \alpha\theta \left\{ \frac{(1-\theta)^2 + 2\theta (1-\theta)}{(1-\theta)^4} \right\}$$
$$= \alpha\theta \left\{ \frac{(1-\theta) + 2\theta}{(1-\theta)^3} \right\}$$
$$= \alpha\theta \frac{(1+\theta)}{(1-\theta)^3}$$

$r = 4$

$$k_4 = \alpha\theta \frac{d}{dt} \left\{ \frac{e^t (1-\theta e^t)^2 + 2\theta e^{2t} (1-\theta e^t)}{(1-\theta e^t)^4} \right\} |_{t=0}$$
$$= \alpha\theta \left\{ \frac{e^t + 4\theta e^{2t} + \theta^2 e^{3t}}{6\theta^2 e^{2t} - 4\theta e^t - 4\theta^3 e^{3t} + \theta^4 e^{4t} + 1} \right\} |_{t=0}$$
$$= \alpha\theta \left\{ \frac{1 + 4\theta + \theta^2}{6\theta^2 - 4\theta - 4\theta^3 + \theta^4 + 1} \right\}$$
$$= \frac{\alpha\theta (4\theta + \theta^2 + 1)}{(\theta - 1)^4}$$

The recursion formula for cumulants of Negative Binomial Distribution is given by

$$k_{r+1} = \theta \frac{d}{d\theta} k_r$$

xii. Re-parameterization

Writing,

$$\theta = \frac{\eta}{1+\eta}, \ \alpha = \frac{h}{\eta}, \ \eta > 0, \ h > 0$$

We get Polya-Eggenberger Distribution with

$$\Pr(X = k) = \binom{\frac{h}{\eta} + k - 1}{k} \left(\frac{\eta}{1+\eta} \right)^k \left(1 - \frac{\eta}{1+\eta} \right)^{\frac{h}{\eta}}; \ k = 0, 1, 2, ...$$
$$= \binom{\frac{h}{\eta} + k - 1}{k} \left(\frac{\eta}{1+\eta} \right)^k \left(\frac{1}{1+\eta} \right)^{\frac{h}{\eta}}; \ k = 0, 1, 2, ...$$

To obtain mean and variance

$$\mu_1' = E(X) = \alpha \frac{\theta}{1-\theta} = \frac{h}{\eta} \cdot \frac{\eta}{1+\eta} \cdot \frac{1}{\left(1 - \frac{\eta}{1+\eta}\right)}$$
$$= \frac{h}{1 + \eta - \eta} = h$$

$$\mu_2 = Var(X) = \frac{\alpha\theta}{(1-\theta)^2} = \frac{h}{\eta}\cdot\frac{\eta}{1+\eta}\cdot\frac{1}{\left(1-\frac{\eta}{1+\eta}\right)^2}$$
$$= \frac{h}{1+\eta}\frac{(1+\eta)^2}{(1+\eta-\eta)^2} = h(1+\eta)$$

$$\mu_3 = \theta\alpha\frac{(1+\theta)}{(1-\theta)^3} = \frac{h}{\eta}\cdot\frac{\eta}{1+\eta}\cdot\frac{\left(1+\frac{\eta}{1+\eta}\right)}{\left(1-\frac{\eta}{1+\eta}\right)^3} = \frac{h}{1+\eta}\cdot\frac{(1+2\eta)}{\left(\frac{1}{1+\eta}\right)^2}$$
$$= h\,(1+2\eta)\,(1+\eta) = h(1+\eta)\,(1+2\eta)$$

and

$$\begin{aligned}
\mu_4 &= \frac{\alpha\theta}{(1-\theta)^4}\left[1+4\theta+\theta^2+3\alpha\theta\right]\\
&= \frac{h}{\eta}\cdot\frac{\eta}{1+\eta}\cdot\frac{1}{\left(1-\frac{\eta}{1+\eta}\right)^4}\left[1+4\frac{\eta}{1+\eta}+\left(\frac{\eta}{1+\eta}\right)^2+3\frac{h}{\eta}\cdot\frac{\eta}{1+\eta}\right]\\
&= \frac{h}{1+\eta}(1+\eta)^4\left[1+4\frac{\eta}{1+\eta}+\left(\frac{\eta}{1+\eta}\right)^2+3\frac{h}{1+\eta}\right]\\
&= h(1+\eta)^3\left[\frac{(1+\eta)^2+4\eta(1+\eta)+\eta^2+3h(1+\eta)}{(1+\eta)^2}\right]\\
&= h(1+\eta)\left[(1+\eta)\,(1+\eta+4\eta+3h)+\eta^2\right]\\
&= h(1+\eta)\left[(1+\eta)\,(1+5\eta+3h)+\eta^2\right]\\
&= h(1+\eta)\left[(1+\eta)\,(1+5\eta+3h)+(\eta+1-1)^2\right]\\
&= h(1+\eta)\left[(1+\eta)\,(1+5\eta+3h)+(\eta+1)^2-2(\eta+1)+1\right]\\
&= h(1+\eta)\left[1+(1+\eta)\left\{1+5\eta+3h+\eta+1-2\right\}\right]\\
&= h(1+\eta)\left[1+(1+\eta)\left\{6\eta+3h\right\}\right]\\
&= h(1+\eta)\left[1+3(1+\eta)\left\{2\eta+h\right\}\right]
\end{aligned}$$

Next consider the recurrence relation (2.18)

$$\begin{aligned}
\mu_{r+1} &= \theta\left[\frac{d}{d\theta}\mu_r + r\mu_{r-1}\frac{d}{d\theta}\mu_1'\right]\\
&= \theta\frac{d}{d\theta}\mu_r + \alpha r\theta\mu_{r-1}.\frac{1}{(1-\theta)^2}
\end{aligned}$$

for Negative Binomial Distribution.

but

$$\frac{d}{d\theta}\mu_r = \frac{d\mu_r}{d\eta}\cdot\frac{d\eta}{d\theta}+\frac{d\mu_r}{dh}\cdot\frac{dh}{d\theta}$$

$$\theta = \frac{\eta}{1+\eta} \qquad \Rightarrow \theta+\theta\eta=\eta \qquad \Rightarrow \theta = (1-\theta)\,\eta \qquad \Rightarrow \eta = \frac{\theta}{1-\theta}$$

Therefore,

$$\frac{d\eta}{d\theta}=\frac{1-\theta+\theta}{(1-\theta)^2}=\frac{1}{(1-\theta)^2}$$

i.e.,

$$\frac{d\eta}{d\theta}=\frac{1}{\left(1-\frac{\eta}{1+\eta}\right)^2}=(1+\eta)^2$$

$$\alpha=\frac{h}{\eta} \qquad \Rightarrow \frac{1}{\eta}=\frac{\alpha}{h}$$

$$h=\alpha\eta=\alpha\frac{\theta}{1-\theta}$$

Thus,

$$\begin{aligned}\frac{dh}{d\theta}&=\frac{\alpha}{(1-\theta)^2}=\alpha\,(1+\eta)^2\\&=h\cdot\frac{1}{\eta}\,(1+\eta)^2\\&=\frac{h}{\eta}\,(1+\eta)^2\end{aligned}$$

so,

$$\begin{aligned}\frac{d}{d\theta}\mu_r&=\frac{d\mu_r}{d\eta}\cdot(1+\eta)^2+\frac{d\mu_r}{dh}\cdot\frac{h}{\eta}\,(1+\eta)^2\\&=(1+\eta)^2\left[\frac{d\mu_r}{d\eta}+\frac{h}{\eta}\frac{d\mu_r}{dh}\right]\end{aligned}$$

The recurrence relation for the central moments of Polya-Eggenberger Distribution is given by

$$\begin{aligned}\mu_{r+1}&=\theta\left[\frac{d}{d\theta}\mu_r+\frac{r\alpha}{(1-\theta)^2}\mu_{r-1}\right]\\&=\frac{\eta}{1+\eta}\left[(1+\eta)^2\frac{d\mu_r}{d\eta}+(1+\eta)^2\frac{h}{\eta}\frac{d\mu_r}{dh}+r\alpha(1+\eta)^2\mu_{r-1}\right]\\&=(1+\eta)\left[\eta\frac{d\mu_r}{d\eta}+h\frac{d\mu_r}{dh}+\eta r\alpha\mu_{r-1}\right]\\&=(1+\eta)\left[\eta\frac{d\mu_r}{d\eta}+h\frac{d\mu_r}{dh}+rh\mu_{r-1}\right]\end{aligned}$$

The pgf of Polya-Eggenberger Distribution is given by

$$\begin{aligned}G(s)&=\frac{f(\theta s)}{f(\theta)}=\frac{(1-\theta s)^{-\alpha}}{(1-\theta)^{-\alpha}}=\frac{\left[1-\frac{\eta}{1+\eta}s\right]^{-\frac{h}{\eta}}}{\left[1-\frac{\eta}{1+\eta}\right]^{-\frac{h}{\eta}}}\\&=[1+\eta-\eta s]^{-\frac{h}{\eta}}\end{aligned}$$

$$G'(s) = -\frac{h}{\eta}\left[1+\eta-\eta s\right]^{-\frac{h}{\eta}-1}(-\eta) = h\left[1+\eta-\eta s\right]^{-\frac{h}{\eta}-1}$$

$$\begin{aligned} G''(s) &= h\left(-\frac{h}{\eta}-1\right)\left[1+\eta-\eta s\right]^{-\frac{h}{\eta}-2}(-\eta) \\ &= h\eta\left(\frac{h}{\eta}+1\right)\left[1+\eta-\eta s\right]^{-\frac{h}{\eta}-2} \end{aligned}$$

setting $s = 1,$ we obtain

$$G'(1) = h$$

$$G''(1) = h\eta\left(\frac{h}{\eta}+1\right) = h^2 + \eta h$$

To obtain the mean and variance

$$E(X) = G'(1) = h$$

$$\begin{aligned} Var(X) &= G''(1) + G'(1) - \left[G'(1)\right]^2 \\ &= h^2 + \eta h + h - h^2 \\ &= h(\eta + 1) \end{aligned}$$

xiii. Special case. When $\alpha = 1$ we have

$$\Pr(X = k) = \theta^k(1-\theta), \text{ for } k = 0, 1, 2, ...$$

which is a Geometric Distribution.

The mean and the variance is given by

$$\mu_1' = \frac{\theta}{1-\theta}, \qquad 0 < \theta < 1$$

$$\mu_2 = \frac{\theta}{(1-\theta)^2}$$

The recurrence relation is given by

$$\mu_{r+1} = \theta\left[\frac{d}{d\theta}\mu_r + r\mu_{r-1}\frac{d}{d\theta}\mu_1'\right]$$

but

$$\mu_1' = E(X) = \frac{\theta}{1-\theta}$$

Thus the recurrence relation for the central moments of Geometric Distribution is given by

$$\begin{aligned} \mu_{r+1} &= \theta\left[\frac{d}{d\theta}\mu_r + r\mu_{r-1}\frac{d}{d\theta}\frac{\theta}{1-\theta}\right] \\ &= \theta\left[\frac{d}{d\theta}\mu_r + r\mu_{r-1}\left(\frac{(1-\theta)\cdot 1 + \theta\cdot 1}{(1-\theta)^2}\right)\right] \\ &= \theta\left[\frac{d}{d\theta}\mu_r + r\mu_{r-1}\left(\frac{1}{(1-\theta)^2}\right)\right] \\ &= \theta\frac{d}{d\theta}\mu_r + \frac{r\theta}{(1-\theta)^2}\mu_{r-1} \end{aligned}$$

setting $r = 1$

$$\mu_2 = \theta \left[\frac{d}{d\theta}\mu_1 + \frac{\alpha\mu_0}{(1-\theta)^2} \right]$$
$$= \theta \left[0 + \frac{1}{(1-\theta)^2} \right] = \frac{\theta}{(1-\theta)^2}$$

setting $r = 2$

$$\mu_3 = \theta \left[\frac{d}{d\theta}\mu_2 + 2\mu_1 \frac{1}{(1-\theta)^2} \right]$$
$$= \theta \left[\frac{d}{d\theta} \frac{\theta}{(1-\theta)^2} + 2 \cdot 0 \cdot \frac{1}{(1-\theta)^2} \right]$$
$$= \theta \frac{d}{d\theta} \frac{\theta}{(1-\theta)^2} = \theta \left[\frac{(1-\theta)^2 \cdot 1 + 2\theta(1-\theta)}{(1-\theta)^4} \right]$$
$$= \theta \left[\frac{(1-\theta)^2 + 2\theta - 2\theta^2)}{(1-\theta)^4} \right]$$
$$= \frac{\theta (1-\theta)(1-\theta+2\theta)}{(1-\theta)^4} = \frac{\theta (1-\theta)(1+\theta)}{(1-\theta)^4}$$
$$= \theta \frac{(1+\theta)}{(1-\theta)^3}$$

And for $r = 3$, using (2.20) we obtain

$$\mu_4 = \theta \frac{d}{d\theta}\mu_3 + 3\mu_2\mu_2$$
$$= \theta \frac{d}{d\theta} \theta \frac{(1+\theta)}{(1-\theta)^3} + 3 \left[\frac{\theta}{(1-\theta)^2} \right]^2$$
$$= \theta \frac{d}{d\theta} \frac{(\theta+\theta^2)}{(1-\theta)^3} + \frac{3\theta^2}{(1-\theta)^4}$$
$$= \theta \left[\frac{(1-\theta)^3 (1+2\theta) + 3(\theta+\theta^2)(1-\theta)^2}{(1-\theta)^6} \right] + \frac{3\theta^2}{(1-\theta)^4}$$
$$= \theta \left[\frac{(1-\theta)(1+2\theta) + 3(\theta+\theta^2)}{(1-\theta)^4} \right] + \frac{3\theta^2}{(1-\theta)^4}$$
$$= \frac{\theta}{(1-\theta)^4} \left[1 + 2\theta - \theta - 2\theta^2 + 3\theta + 3\theta^2) + 3(\theta+\theta^2) \right] + \frac{3\theta^2}{(1-\theta)^4}$$
$$= \frac{\theta}{(1-\theta)^4} \left[1 + 4\theta + \theta^2 \right] + \frac{3\theta^2}{(1-\theta)^4}$$
$$= \frac{\theta}{(1-\theta)^4} \left[1 + 7\theta + \theta^2 \right]$$

Probability generating function for Geometric Distribution is given by

$$G(s) = \frac{(1-\theta s)^{-1}}{(1-\theta)^{-1}} = \frac{(1-\theta)}{(1-\theta s)}$$

$$G'(s) = \theta \frac{(1-\theta s)^{-2}}{(1-\theta)^{-1}} = \theta \frac{(1-\theta)}{(1-\theta s)^2}$$

$$G''(s) = 2\theta^2 \frac{(1-\theta s)^{-3}}{(1-\theta)^{-1}} = 2\theta^2 \frac{(1-\theta)}{(1-\theta s)^3}$$

setting $s = 1$, we obtain

$$G'(1) = \frac{\theta}{1-\theta}$$

$$G''(1) = \frac{2\theta^2}{(1-\theta)^2}$$

To obtain the mean and variance

$$E(X) = G'(1) = \frac{\theta}{1-\theta}$$

$$\begin{aligned}
Var(X) &= G''(1) + G'(1) - \left[G'(1)\right]^2 \\
&= \frac{2\theta^2}{(1-\theta)^2} + \frac{\theta}{1-\theta} - \left[\frac{\theta}{1-\theta}\right]^2 \\
&= \frac{2\theta^2}{(1-\theta)^2} + \frac{\theta}{1-\theta} - \frac{\theta^2}{(1-\theta)^2} \\
&= \frac{\theta^2}{(1-\theta)^2} + \frac{\theta}{1-\theta} \\
&= \frac{\theta}{1-\theta}\left[1 + \frac{\theta}{1-\theta}\right] \\
&= \frac{\theta}{(1-\theta)^2}
\end{aligned}$$

The moment generating function for Geometric Distribution is given by

$$M_X(t) = \frac{(1-\theta)}{(1-\theta e^t)}$$

The r^{th} moment about the origin is obtained from the r^{th} derivative of $M_X(t)$ w.r.t t and setting $t = 0$ i.e.,

$$\mu'_r = \frac{d^r M_X(t)}{dt^r} |_{t=0}$$

Therefore setting $r = 1$

$$\begin{aligned}
\mu'_1 &= \frac{dM_X(t)}{dt} |_{t=0} \\
&= \frac{d}{dt}\left[\frac{(1-\theta e^t)^{-1}}{(1-\theta)^{-1}}\right] |_{t=0} \\
&= \theta e^t \frac{(1-\theta e^t)^{-2}}{(1-\theta)^{-1}} |_{t=0} = \frac{\theta}{1-\theta}
\end{aligned}$$

for $r = 2$

$$\mu_2' = \frac{d}{dt}\left[\theta e^t \frac{(1-\theta e^t)^{-2}}{(1-\theta)^{-1}}\right] |_{t=0}$$

$$= \alpha\theta e^t \frac{(1-\theta e^t)^{-2}}{(1-\theta)^{-1}} |_{t=0} + \frac{\alpha(\alpha+1)\theta^2 e^{2t}(1-\theta e^t)^{-\alpha-2}}{(1-\theta)^{-\alpha}} |_{t=0}$$

$$= 2\frac{\theta^2}{(1-\theta)^2} + \frac{\theta}{1-\theta}$$

Also,

$$\mu_2 = \mu_2' - \mu_1'^2$$

$$= 2\frac{\theta^2}{(1-\theta)^2} + \frac{\theta}{1-\theta} - \left(\frac{\theta}{1-\theta}\right)^2$$

$$= \frac{\theta}{(1-\theta)^2}$$

The factorial moment generating function of Geometric Distribution is given by

$$M_{[X]}(t) = \frac{f(\theta+\theta t)}{f(\theta)} = \left(\frac{1-\theta-\theta t}{1-\theta}\right)^{-1} = \frac{1-\theta}{1-\theta-\theta t}$$

The r^{th} factorial moment is obtained by from the r^{th} derivative of $M_{[X]}(t)$ w.r.t t and setting $t = 0$, i.e.

$$\mu_{[r]} = \frac{d^r M_{[X]}(t)}{dt^r} |_{t=0}$$

When $r = 1$

$$\mu_{[1]} = \frac{dM_{[X]}(t)}{dt} |_{t=0}$$

$$= \frac{d}{dt}\left(\frac{1-\theta-\theta t}{1-\theta}\right)^{-\alpha} |_{t=0}$$

$$= -\left(\frac{1-\theta-\theta t}{1-\theta}\right)^{-2}\left(\frac{-\theta}{1-\theta}\right) |_{t=0}$$

$$= \frac{\theta}{1-\theta}$$

$r = 2$

$$\mu_{[2]} = \frac{\theta}{1-\theta}\frac{d}{dt}\left(\frac{1-\theta-\theta t}{1-\theta}\right)^{-2} |_{t=0}$$

$$= \frac{\theta}{1-\theta}\left\{2\frac{\theta}{1-\theta}\left(\frac{1-\theta-\theta t}{1-\theta}\right)^{-3}\right\} |_{t=0}$$

$$= \frac{2\theta^2}{(1-\theta)^2}$$

$r = 3$

$$\mu_{[3]} = 2\frac{\theta^2}{(1-\theta)^2}\frac{d}{dt}\left(\frac{1-\theta-\theta t}{1-\theta}\right)^{-3}|_{t=0}$$

$$= 2\frac{\theta^2}{(1-\theta)^2}\left\{3\frac{\theta}{1-\theta}\left(\frac{1-\theta-\theta t}{1-\theta}\right)^{-4}\right\}|_{t=0}$$

$$= \frac{6\theta^3}{(1-\theta)^3}$$

$r = 4$

$$\mu_{[4]} = 6\frac{\theta^3}{(1-\theta)^3}\frac{d}{dt}\left(\frac{1-\theta-\theta t}{1-\theta}\right)^{-4}|_{t=0}$$

$$= 6\frac{\theta^3}{(1-\theta)^3}\left\{4\frac{\theta}{1-\theta}\left(\frac{1-\theta-\theta t}{1-\theta}\right)^{-5}\right\}|_{t=0}$$

$$= 24\frac{\theta^4}{(1-\theta)^4}$$

The recursive relationship between the factorial moments of Geometric Distribution is given by

$$\mu_{[r]} = r\left(\frac{\theta}{1-\theta}\right)\mu_{[r-1]}$$

The cumulant generating function of Geometric Distribution is given by

$$K_X(t) = \log M_x(t) = \log\left(\frac{1-\theta e^t}{1-\theta}\right)^{-1}$$

The r^{th} cumulant of the distribution is the r^{th} derivative of $K_x(t)$ w.r.t t and setting $t = 0$

$$k_r = \frac{d^r K_X(t)}{dt^r}|_{t=0}$$

When $r = 1$, we have

$$k_1 = \frac{dK_X(t)}{dt}|_{t=0} = -\frac{d}{dt}\log\left(\frac{1-\theta e^t}{1-\theta}\right)|_{t=0}$$

$$= -\left(\frac{1-\theta}{1-\theta e^t}\right)\left(\frac{-\theta e^t}{1-\theta}\right)|_{t=0}$$

$$= \left(\frac{\theta}{1-\theta}\right)$$

$r = 2$

$$k_2 = \theta\frac{d}{dt}\left[\frac{e^t}{1-\theta e^t}\right]|_{t=0}$$

$$= \theta\left\{\frac{e^t\left(1-\theta e^t\right)+\theta e^t(e^t)}{(1-\theta e^t)^2}\right\}|_{t=0}$$

$$= \frac{\theta}{(1-\theta)^2}$$

$r = 3$

$$\begin{aligned} k_3 &= \theta \frac{d}{dt}\left\{\frac{e^t}{\left(1-\theta e^t\right)^2}\right\}|_{t=0} \\ &= \theta\left\{\frac{e^t\left(1-\theta e^t\right)^2 + 2\theta e^{2t}\left(1-\theta e^t\right)}{\left(1-\theta e^t\right)^4}\right\} \\ &= \theta\left\{\frac{\left(1-\theta\right)^2 + 2\theta\left(1-\theta\right)}{\left(1-\theta\right)^4}\right\} \\ &= \theta\left\{\frac{\left(1-\theta\right) + 2\theta}{\left(1-\theta\right)^3}\right\} \\ &= \theta\frac{\left(1+\theta\right)}{\left(1-\theta\right)^3} \end{aligned}$$

$r = 4$

$$\begin{aligned} k_4 &= \theta \frac{d}{dt}\left\{\frac{e^t\left(1-\theta e^t\right)^2 + 2\theta e^{2t}\left(1-\theta e^t\right)}{\left(1-\theta e^t\right)^4}\right\}|_{t=0} \\ &= \theta\left\{\frac{e^t + 4\theta e^{2t} + \theta^2 e^{3t}}{6\theta^2 e^{2t} - 4\theta e^t - 4\theta^3 e^{3t} + \theta^4 e^{4t} + 1}\right\}|_{t=0} \\ &= \theta\left\{\frac{1 + 4\theta + \theta^2}{\left(\theta-1\right)^4}\right\} \end{aligned}$$

The recursion formula for cumulants of Geometric Distribution is given by

$$k_{r+1} = \theta\frac{d}{d\theta}k_r$$

2.9.4 Logarithmic Series Distribution

$$f(\theta) = -\log(1-\theta)$$

To obtain the power series of $-\log(1-\theta)$, we start by expanding $\left(1-\theta\right)^{-1}$. i.e.

$$\frac{1}{1-\theta} = 1 + \theta + \theta^2 + \cdots$$

Integrating both sides w.r.t θ we get

$$\int \frac{d\theta}{1-\theta} = \int \left[1 + \theta + \theta^2 + \cdots\right] d\theta$$

$$\begin{aligned} -\log(1-\theta) &= \theta + \frac{\theta^2}{2} + \frac{\theta^3}{3} + \cdots \\ &= \sum_{k=1}^{\infty}\frac{\theta^k}{k} \end{aligned}$$

Therefore,

$$1 = \sum_{k=1}^{\infty}\frac{\theta^k}{-k\log(1-\theta)}$$

i.

$$\Pr\left(X=k\right)=\frac{\theta^{k}}{-k\log(1-\theta)},\ \ k=1,2,\ldots$$

which is called Logarithmic Series Distribution.

ii.

$$a_k=\frac{1}{k}$$

iii.

$$f'(\theta)=\frac{1}{1-\theta}$$

iv.

$$f''(\theta)=\frac{1}{\left(1-\theta\right)^{2}}$$

v. The mean is given by

$$E(X)=\theta\frac{f'(\theta)}{f(\theta)}=\frac{\theta}{1-\theta}\cdot\frac{1}{-\log(1-\theta)}=\frac{\theta}{-\left(1-\theta\right)\log(1-\theta)}$$

vi. The variance is given by

$$\begin{aligned}
Var(X)&=\theta^{2}\frac{f''(\theta)}{f(\theta)}+\theta\frac{f'(\theta)}{f(\theta)}-\left[\theta\frac{f'(\theta)}{f(\theta)}\right]^{2}\\
&=\frac{-\theta^{2}}{\left(1-\theta\right)^{2}\log(1-\theta)}+\frac{-\theta}{\left(1-\theta\right)\log(1-\theta)}-\frac{\theta^{2}}{\left[\left(1-\theta\right)\log(1-\theta)\right]^{2}}\\
&=\frac{-\theta^{2}}{\left(1-\theta\right)^{2}\log(1-\theta)}+\frac{-\theta}{\left(1-\theta\right)\log(1-\theta)}-\frac{\theta^{2}}{\left[\left(1-\theta\right)\log(1-\theta)\right]^{2}}\\
&=\frac{-\theta^{2}\log(1-\theta)+\theta\left[-\left(1-\theta\right)\log(1-\theta)\right]-\theta^{2}}{\left[-\left(1-\theta\right)\log(1-\theta)\right]^{2}}\\
&=\frac{-\theta^{2}\log(1-\theta)-\theta\left(1-\theta\right)\log(1-\theta)-\theta^{2}}{\left[-\left(1-\theta\right)\log(1-\theta)\right]^{2}}\\
&=\frac{\left[\theta\log(1-\theta)\right]\left[-\theta-\left(1-\theta\right)\right]-\theta^{2}}{\left[-\left(1-\theta\right)\log(1-\theta)\right]^{2}}\\
&=\frac{-\theta\log(1-\theta)-\theta^{2}}{\left[\left(1-\theta\right)\log(1-\theta)\right]^{2}}\\
&=\frac{-\left[\theta^{2}+\theta\log(1-\theta)\right]}{\left[\left(1-\theta\right)\log(1-\theta)\right]^{2}}
\end{aligned}$$

vii. The recurrence relation for the central moments of Logarithmic series distribution is given by

$$\begin{aligned}
\mu_{r+1} &= \theta\left[\frac{d}{d\theta}\mu_r + r\mu_{r-1}\frac{d}{d\theta}\mu_1'\right] \\
&= \theta\left[\frac{d}{d\theta}\mu_r + r\mu_{r-1}\frac{d}{d\theta}\frac{\theta}{-(1-\theta)\log(1-\theta)}\right] \\
&= \theta\left\{\frac{d}{d\theta}\mu_r + r\mu_{r-1}\left[\frac{-(1-\theta)\log(1-\theta)+\theta\frac{d}{d\theta}(1-\theta)\log(1-\theta)}{(1-\theta)^2(\log(1-\theta))^2}\right]\right\} \\
&= \theta\left\{\frac{d}{d\theta}\mu_r + r\mu_{r-1}\left[\frac{-(1-\theta)\log(1-\theta)-\theta\left[\log(1-\theta)-\frac{1-\theta}{1-\theta}\right]}{(1-\theta)^2[\log(1-\theta)]^2}\right]\right\} \\
&= \theta\left\{\frac{d}{d\theta}\mu_r + r\mu_{r-1}\left[\frac{-(1-\theta)\log(1-\theta)-\theta\log(1-\theta)-\theta}{(1-\theta)^2[\log(1-\theta)]^2}\right]\right\} \\
&= \theta\left\{\frac{d}{d\theta}\mu_r + r\mu_{r-1}\left[\frac{(-1+\theta-\theta)\log(1-\theta)-\theta}{(1-\theta)^2[\log(1-\theta)]^2}\right]\right\} \\
&= \theta\left\{\frac{d}{d\theta}\mu_r + r\mu_{r-1}\left[\frac{-\log(1-\theta)-\theta}{(1-\theta)^2[\log(1-\theta)]^2}\right]\right\} \\
&= \theta\left\{\frac{d}{d\theta}\mu_r - r\left[\frac{\theta+\log(1-\theta)}{(1-\theta)^2[\log(1-\theta)]^2}\right]\mu_{r-1}\right\}
\end{aligned}$$

When $r = 1$ we obtain,

$$\begin{aligned}
\mu_2 &= \theta\left\{\frac{d}{d\theta}\mu_1 - \left[\frac{\theta+\log(1-\theta)}{(1-\theta)^2[\log(1-\theta)]^2}\right]\mu_0\right\} \\
&= \theta\left\{0 - \left[\frac{\theta+\log(1-\theta)}{(1-\theta)^2[\log(1-\theta)]^2}\right]\mu_0\right\} \\
&= -\left\{\frac{\theta^2+\theta\log(1-\theta)}{(1-\theta)^2[\log(1-\theta)]^2}\right\}
\end{aligned}$$

for $r = 2$

$$\begin{aligned}
\mu_3 &= \theta\left\{\frac{d}{d\theta}\mu_2 - 2\left[\frac{\theta+\log(1-\theta)}{(1-\theta)^2[\log(1-\theta)]^2}\right]\mu_1\right\} \\
&= \theta\left\{-\frac{d}{d\theta}\left\{\frac{\theta^2+\theta\log(1-\theta)}{(1-\theta)^2[\log(1-\theta)]^2}\right\} - 0\right\} \\
&= -\theta\left\{\frac{2\left[\theta^2+\theta\ln(1-\theta)\right]}{(1-\theta)^3\ln^2(1-\theta)} + \frac{2\left[\theta^2+\theta\ln(1-\theta)\right]}{(1-\theta)^3\ln^3(1-\theta)} + \frac{\left[2\theta+\ln(1-\theta)-\frac{\theta}{1-\theta}\right]}{(1-\theta)^2\ln^2(1-\theta)}\right\}
\end{aligned}$$

viii. Probability generating function for Logarithmic series distribution is given by

$$G(s) = \frac{f(\theta s)}{f(\theta)} = \frac{\log(1-\theta s)}{\log(1-\theta)}$$

$$G'(s) = \frac{-\theta}{1-\theta s}\cdot\frac{1}{\log(1-\theta)}$$

$$G''(s) = \frac{-\theta^2}{(1-\theta s)^2} \cdot \frac{1}{\log(1-\theta)}$$

setting $s = 1$ we obtain

$$G'(1) = \frac{-\theta}{1-\theta} \cdot \frac{1}{\log(1-\theta)} = \frac{\theta}{-(1-\theta)\log(1-\theta)}$$

$$G''(1) = \frac{-\theta^2}{(1-\theta)^2} \cdot \frac{1}{\log(1-\theta)}$$

To obtain the mean and variance

$$E(X) = G'(1) = \frac{\theta}{-(1-\theta)\log(1-\theta)}$$

$$\begin{aligned}
Var(X) &= G''(1) + G^{'}(1) - \left[G^{'}(1)\right]^2 \\
&= \frac{-\theta^2}{(1-\theta)^2\log(1-\theta)} - \frac{\theta}{(1-\theta)\log(1-\theta)} - \frac{\theta^2}{(1-\theta)^2\left[\log(1-\theta)\right]^2} \\
&= \frac{-\theta^2}{(1-\theta)^2\log(1-\theta)} - \frac{\theta}{(1-\theta)\log(1-\theta)} - \frac{\theta^2}{(1-\theta)^2\left[\log(1-\theta)\right]^2} \\
&= \frac{-\theta^2\log(1-\theta) - \theta(1-\theta)\log(1-\theta) - \theta^2}{(1-\theta)^2\left[\log(1-\theta)\right]^2} \\
&= \frac{-\theta^2\log(1-\theta) - \theta\log(1-\theta) + \theta^2\log(1-\theta) - \theta^2}{(1-\theta)^2\left[\log(1-\theta)\right]^2} \\
&= -\left(\frac{\theta^2 + \theta\log(1-\theta)}{(1-\theta)^2\left[\log(1-\theta)\right]^2}\right)
\end{aligned}$$

ix. The moment generating function for Logarithmic series distribution is given by

$$M_X(t) = \frac{f(\theta e^t)}{f(\theta)} = \frac{\log(1-\theta e^t)}{\log(1-\theta)}$$

The r^{th} moment about the origin is obtained from the r^{th} derivative of $M_X(t)$ w.r.t t and setting $t = 0$.That is

$$\mu_r^{\prime} = \frac{d^r M_X(t)}{dt^r} \mid_{t=0}$$

As a result when $r = 1$

$$\begin{aligned}
\mu_1^{\prime} &= \frac{dM_X(t)}{dt} \mid_{t=0} \\
&= \frac{1}{\log(1-\theta)} \frac{d}{dt} \log(1-\theta e^t) \mid_{t=0} \\
&= \frac{1}{-\log(1-\theta)} \cdot \frac{\theta e^t}{1-\theta e^t} \mid_{t=0} \\
&= \frac{\theta}{-(1-\theta)\log(1-\theta)}
\end{aligned}$$

$r = 2$

$$\mu_2' = \frac{\theta}{-\log(1-\theta)} \frac{d}{dt}\left[\frac{e^t}{1-\theta e^t}\right] |_{t=0}$$
$$= \frac{\theta}{-\log(1-\theta)}\left[\frac{e^t}{1-\theta e^t} + \frac{\theta e^{2t}}{(1-\theta e^t)^2}\right] |_{t=0}$$
$$= \frac{\theta}{-\log(1-\theta)}\left[\frac{1}{1-\theta} + \frac{\theta}{(1-\theta)^2}\right]$$
$$= \frac{\theta}{-(1-\theta)^2 \log(1-\theta)}$$

Also,

$$\mu_2 = \mu_2' - \mu_1'^2$$
$$= \frac{\theta}{-(1-\theta)^2\log(1-\theta)} - \left(\frac{\theta}{-(1-\theta)\log(1-\theta)}\right)^2$$
$$= \frac{-\theta\log(1-\theta) - \theta^2}{[(1-\theta)\log(1-\theta)]^2}$$
$$= \frac{-\left[\theta^2 + \theta\log(1-\theta)\right]}{[(1-\theta)\log(1-\theta)]^2}$$

$r = 3$

$$\mu_3' = \frac{\theta}{-\log(1-\theta)} \frac{d}{dt}\left[\frac{e^t}{1-\theta e^t} + \frac{\theta e^{2t}}{(1-\theta e^t)^2}\right] |_{t=0}$$
$$= \frac{\theta}{-\log(1-\theta)}\left\{\frac{e^t}{1-\theta e^t} + \frac{\theta e^{2t}}{(1-\theta e^t)^2} + \frac{2\theta^2 e^{2t}}{(1-\theta e^t)^2} + \frac{2\theta^2 e^{3t}}{(1-\theta e^t)^3}\right\} |_{t=0}$$
$$= \frac{\theta}{-\log(1-\theta)}\left\{\frac{1}{(1-\theta)^2} + \frac{2\theta}{(1-\theta)^3}\right\} = \frac{\theta(1+\theta)}{-(1-\theta)^3\log(1-\theta)}$$

x. The factorial moment generating function of Logarithmic series distribution is given by

$$M_{[X]}(t) = \frac{f(\theta+\theta t)}{f(\theta)} = \log\left(\frac{1-\theta-\theta t}{1-\theta}\right)$$

The r^{th} factorial moment is obtained by from the r^{th} derivative of $M_{[X]}(t)$ w.r.t t and setting $t = 0$

$$\mu_{[r]} = \frac{d^r M_{[X]}(t)}{dt^r} |_{t=0}$$

Thus when $r = 1$

$$\mu_{[1]} = \frac{dM_{[X]}(t)}{dt} |_{t=0} = \frac{d}{dt}\log\left(\frac{1-\theta-\theta t}{1-\theta}\right) |_{t=0}$$
$$= \left\{\frac{1}{-\log(1-\theta)} \cdot \frac{d}{dt}\left[-\log(1-\theta-\theta t)\right]\right\} |_{t=0}$$
$$= \left\{\frac{1}{-\log(1-\theta)} \cdot \frac{1}{1-\theta(1+t)} \cdot \theta\right\} |_{t=0}$$
$$= \frac{\theta}{-(1-\theta)\log(1-\theta)}$$

$r = 2$

$$\mu_{[2]} = \frac{\theta}{-\log(1-\theta)} \frac{d}{dt}\left\{\frac{1}{1-\theta(1+t)}\right\}|_{t=0}$$
$$= \frac{\theta}{-\log(1-\theta)}\left\{\frac{\theta}{\{1-\theta(1+t)\}^2}\right\}|_{t=0}$$
$$= \frac{\theta^2}{-(1-\theta)^2\log(1-\theta)}$$

$r = 3$

$$\mu_{[3]} = \frac{\theta^2}{-\log(1-\theta)} \frac{d}{dt}\left\{\frac{1}{[1-\theta(1+t)]^2}\right\}|_{t=0}$$
$$= \frac{\theta^2}{-\log(1-\theta)}\left\{\frac{2\theta}{[1-\theta(1+t)]^3}\right\}|_{t=0}$$
$$= \frac{2\theta^3}{-(1-\theta)^3\log(1-\theta)}$$

$r = 4$

$$\mu_{[4]} = \frac{2\theta^3}{-\log(1-\theta)} \frac{d}{dt}\left\{\frac{1}{[1-\theta(1+t)]^3}\right\}|_{t=0}$$
$$= \frac{2\theta^3}{-\log(1-\theta)}\left\{\frac{3\theta}{[1-\theta(1+t)]^4}\right\}$$
$$= \frac{6\theta^4}{-(1-\theta)^4\log(1-\theta)}$$

The recursive relationship between the factorial moments of Logarithmic series distribution is given by

$$\mu_{[r]} = \left(\frac{\theta}{1-\theta}\right)(r-1)\,\mu_{[r-1]}$$

xi. The cumulant generating function of Logarithmic series distribution is given by

$$K_X(t) = \log M_X(t) = \log\left\{\log\left(\frac{1-\theta e^t}{1-\theta}\right)\right\}$$

The r^{th} cumulant of the distribution is the r^{th} derivative of $K_x(t)$ w.r.t t and setting $t = 0$

$$k_r = \frac{d^r K_X(t)}{dt^r}|_{t=0}$$

When $r = 1$ we have

$$k_1 = \frac{dK_X(t)}{dt}|_{t=0}$$
$$= \frac{d}{dt}\log\left\{\log\left(\frac{1-\theta e^t}{1-\theta}\right)\right\}|_{t=0}$$
$$= \frac{\log(1-\theta)}{\log(1-\theta e^t)}\left\{\frac{\theta e^t}{-(1-\theta e^t)\log(1-\theta)}\right\}|_{t=0}$$
$$= \frac{\theta}{-(1-\theta)\log(1-\theta)}$$

$r = 2$

$$k_2 = \frac{d}{dt}\frac{\theta e^t}{-\left(1-\theta e^t\right)\log\left(1-\theta e^t\right)}\ |_{t=0}$$
$$= \frac{\theta e^t\left[-\left(1-\theta e^t\right)\log\left(1-\theta e^t\right)\right]-\theta e^t\left[\theta e^t+\theta e^t\log\left(1-\theta e^t\right)\right]}{\left[\left(1-\theta e^t\right)\log\left(1-\theta e^t\right)\right]^2}\ |_{t=0}$$
$$= \frac{-\theta\left(1-\theta\right)\log\left(1-\theta\right)-\theta\left[\theta+\theta\log\left(1-\theta\right)\right]}{\left[\left(1-\theta\right)\log\left(1-\theta\right)\right]^2}$$
$$= \frac{-\left[\theta^2+\theta\log\left(1-\theta\right)\right]}{\left[\left(1-\theta\right)\log\left(1-\theta\right)\right]^2}$$

$r = 3$

$$k_3 = \frac{d}{dt}\frac{-\theta e^t\log\left(1-\theta e^t\right)-\left(\theta e^t\right)^2}{\left[\left(1-\theta e^t\right)\log\left(1-\theta e^t\right)\right]^2}\ |_{t=0}$$
$$= \frac{\left\{\begin{array}{c}\left[-\theta e^t\log\left(1-\theta e^t\right)+\frac{\left(\theta e^t\right)^2}{\left(1-\theta e^t\right)}-2\left(\theta e^t\right)^2\right]\left[\left(1-\theta e^t\right)\log\left(1-\theta e^t\right)\right]^2\\ +2\theta e^t\left[\log\left(1-\theta e^t\right)+1\right]\left[\left(1-\theta e^t\right)\log\left(1-\theta e^t\right)\right]\end{array}\right\}}{\left[\left(1-\theta e^t\right)\log\left(1-\theta e^t\right)\right]^4}\ |_{t=0}$$
$$= \frac{\left\{\begin{array}{c}\left[-\theta\log\left(1-\theta\right)+\frac{\theta^2}{1-\theta}-2\theta^2\right]\left[\left(1-\theta\right)\log\left(1-\theta e^t\right)\right]^2\\ +2\left[\theta\log\left(1-\theta\right)+\theta\right]\left[\left(1-\theta\right)\log\left(1-\theta\right)\right]\end{array}\right\}}{\left[\left(1-\theta\right)\log\left(1-\theta\right)\right]^4}$$

The recursion formula for cumulants of Logarithmic series distribution is given by

$$k_{r+1} = \theta\frac{d}{d\theta}k_r$$

2.9.5 When

$$f\left(\theta\right) = \log(1+\theta)-\log(1-\theta)$$

$$f\left(\theta\right) = \log(1+\theta)-\log(1-\theta) = \log\left(\frac{1+\theta}{1-\theta}\right)$$

but

$$-\log(1-\theta)\ = \theta+\frac{\theta^2}{2}+\frac{\theta^3}{3}+\frac{\theta^4}{4}+\cdots$$

replace θ by $-\theta$ to obtain,

$$-\log(1+\theta)\ = -\theta+\frac{\theta^2}{2}-\frac{\theta^3}{3}+\frac{\theta^4}{4}-\cdots$$

Thus,

$$-\log(1-\theta)\ -\left[-\log(1+\theta)\right] = 2\theta+\frac{2\theta^3}{3}+\frac{2\theta^5}{5}+\cdots$$

i.e.

$$\log(1+\theta) - \log(1-\theta) = 2\left[\theta + \frac{\theta^3}{3} + \frac{\theta^5}{5} + \cdots\right]$$

$$\log\left(\frac{1+\theta}{1-\theta}\right) = 2\sum_{k=0}^{\infty} \frac{\theta^{2k+1}}{2k+1}$$

$$1 = 2\sum_{k=0}^{\infty} \frac{\theta^{2k+1}}{(2k+1)\log\left(\frac{1+\theta}{1-\theta}\right)}$$

Therefore

i.

$$\Pr(X = 2k+1) = \frac{2\theta^{2k+1}}{(2k+1)\log\left(\frac{1+\theta}{1-\theta}\right)}; k = 0, 1, 2, \ldots$$

ii.

$$a_k = \frac{2}{(2k+1)}$$

iii.

$$\begin{aligned} f'(\theta) &= \left(\frac{1-\theta}{1+\theta}\right)\frac{d}{d\theta}\left(\frac{1+\theta}{1-\theta}\right) \\ &= \left(\frac{1-\theta}{1+\theta}\right)\left(\frac{(1-\theta)+(1+\theta)}{(1-\theta)^2}\right) \\ &= \frac{1-\theta}{1+\theta}.\frac{2}{(1-\theta)^2} = \frac{2}{(1-\theta^2)} = 2\left(1-\theta^2\right)^{-1} \end{aligned}$$

iv.

$$f''(\theta) = -2\left(1-\theta^2\right)^{-1}(-2\theta) = \frac{4\theta}{(1-\theta^2)^2}$$

v. The mean is given by

$$E(X) = \theta\frac{f'(\theta)}{f(\theta)} = 2\theta\frac{(1-\theta^2)^{-1}}{\log\left(\frac{1+\theta}{1-\theta}\right)} = \frac{2\theta}{(1-\theta^2)}\cdot\frac{1}{\log\left(\frac{1+\theta}{1-\theta}\right)}$$

vi. The variance is given by

$$
\begin{aligned}
Var(X) &= \theta^2 \frac{f''(\theta)}{f(\theta)} + \theta \frac{f'(\theta)}{f(\theta)} - \left[\theta \frac{f'(\theta)}{f(\theta)}\right]^2 \\
&= \frac{4\theta^3}{\left(1-\theta^2\right)^2 \log\left(\frac{1+\theta}{1-\theta}\right)} + \frac{2\theta}{\left(1-\theta^2\right) \log\left(\frac{1+\theta}{1-\theta}\right)} - \frac{4\theta^2}{\left(1-\theta^2\right)^2 \left[\log\left(\frac{1+\theta}{1-\theta}\right)\right]^2} \\
&= \frac{4\theta^3}{\left(1-\theta^2\right)^2 \log\left(\frac{1+\theta}{1-\theta}\right)} + \frac{2\theta}{\left(1-\theta^2\right) \log\left(\frac{1+\theta}{1-\theta}\right)} - \frac{4\theta^2}{\left(1-\theta^2\right)^2 \left[\log\left(\frac{1+\theta}{1-\theta}\right)\right]^2} \\
&= \frac{2\theta\left\{2\theta^2 \log\left(\frac{1+\theta}{1-\theta}\right) + \left(1-\theta^2\right) \log\left(\frac{1+\theta}{1-\theta}\right) - 2\theta\right\}}{\left(1-\theta^2\right)^2 \left[\log\left(\frac{1+\theta}{1-\theta}\right)\right]^2} \\
&= 2\theta \left\{\frac{2\theta^2 \log\left(\frac{1+\theta}{1-\theta}\right) + \log\left(\frac{1+\theta}{1-\theta}\right) - \theta^2 \log\left(\frac{1+\theta}{1-\theta}\right) - 2\theta}{\left(1-\theta^2\right)^2 \left[\log\left(\frac{1+\theta}{1-\theta}\right)\right]^2}\right\} \\
&= 2\theta \left\{\frac{\theta^2 \log\left(\frac{1+\theta}{1-\theta}\right) + \log\left(\frac{1+\theta}{1-\theta}\right) - 2\theta}{\left(1-\theta^2\right)^2 \left[\log\left(\frac{1+\theta}{1-\theta}\right)\right]^2}\right\} \\
&= 2\theta \left\{\frac{\left(1+\theta^2\right) \log\left(\frac{1+\theta}{1-\theta}\right) - 2\theta}{\left(1-\theta^2\right)^2 \left[\log\left(\frac{1+\theta}{1-\theta}\right)\right]^2}\right\}
\end{aligned}
$$

vii.

$$
\begin{aligned}
\mu_{r+1} &= \theta \left[\frac{d}{d\theta}\mu_r + r\mu_{r-1}\frac{d}{d\theta}\mu_1'\right] \\
&= \theta \left[\frac{d}{d\theta}\mu_r + r\mu_{r-1}\frac{d}{d\theta}E\left(X\right)\right] \\
&= \theta \left\{\frac{d}{d\theta}\mu_r + r\mu_{r-1}\frac{d}{d\theta}\frac{2\theta}{\left(1-\theta^2\right)} \cdot \frac{1}{\log\left(\frac{1+\theta}{1-\theta}\right)}\right\} \\
&= \theta \left\{\frac{d}{d\theta}\mu_r + r\mu_{r-1}\left[\frac{2\left(1-\theta^2\right)\log\left(\frac{1+\theta}{1-\theta}\right) - 2\theta\frac{d}{d\theta}\left(1-\theta^2\right)\log\left(\frac{1+\theta}{1-\theta}\right)}{\left(1-\theta^2\right)^2\left[\log\left(\frac{1+\theta}{1-\theta}\right)\right]^2}\right]\right\}
\end{aligned}
$$

but

$$
\begin{aligned}
\frac{d}{d\theta}\left(1-\theta^2\right)\log\left(\frac{1+\theta}{1-\theta}\right) &= -2\theta\log\frac{1+\theta}{1-\theta} + \left(1-\theta^2\right)\frac{1-\theta}{1+\theta}\frac{d}{d\theta}\frac{1+\theta}{1-\theta} \\
&= -2\theta\log\frac{1+\theta}{1-\theta} + \frac{\left(1-\theta\right)\left(1-\theta^2\right)}{1+\theta} \cdot \frac{2}{\left(1+\theta\right)\left(1-\theta\right)^2} \\
&= -2\theta\log\frac{1+\theta}{1-\theta} + 2\frac{\theta^3-\theta^2-\theta+1}{\theta^3-\theta^2-\theta+1} \\
&= -2\theta\log\frac{1+\theta}{1-\theta} + 2
\end{aligned}
$$

Thus the recurrence relation for the central moments is given by

$$\mu_{r+1} = \theta\left\{\frac{d}{d\theta}\mu_r + r\mu_{r-1}\left[\frac{2\left(1-\theta^2\right)\log\left(\frac{1+\theta}{1-\theta}\right) - 2\theta\left(-2\theta\log\frac{1+\theta}{1-\theta}+2\right)}{\left(1-\theta^2\right)^2\left[\log\left(\frac{1+\theta}{1-\theta}\right)\right]^2}\right]\right\}$$

$$= \theta\left\{\frac{d}{d\theta}\mu_r + r\mu_{r-1}\left[\frac{\left(2-2\theta^2+4\theta^2\right)\log\left(\frac{1+\theta}{1-\theta}\right) - 4\theta}{\left(1-\theta^2\right)^2\left[\log\left(\frac{1+\theta}{1-\theta}\right)\right]^2}\right]\right\}$$

$$= \theta\left\{\frac{d}{d\theta}\mu_r + r\mu_{r-1}\left[\frac{\left(2+2\theta^2\right)\log\left(\frac{1+\theta}{1-\theta}\right) - 4\theta}{\left(1-\theta^2\right)^2\left[\log\left(\frac{1+\theta}{1-\theta}\right)\right]^2}\right]\right\}$$

$$= \theta\left\{\frac{d}{d\theta}\mu_r + 2r\frac{\left(1+\theta^2\right)\log\left(\frac{1+\theta}{1-\theta}\right) - 2\theta}{\left(1-\theta^2\right)^2\left[\log\left(\frac{1+\theta}{1-\theta}\right)\right]^2}\mu_{r-1}\right\}$$

putting $r = 1$

$$\mu_2 = \theta\left\{2\left[\frac{\left(1+\theta^2\right)\log\left(\frac{1+\theta}{1-\theta}\right) - 2\theta}{\left(1-\theta^2\right)^2\left[\log\left(\frac{1+\theta}{1-\theta}\right)\right]^2}\right]\right\}$$

$$= 2\theta\left[\frac{\left(1+\theta^2\right)\log\left(\frac{1+\theta}{1-\theta}\right) - 2\theta}{\left(1-\theta^2\right)^2\left[\log\left(\frac{1+\theta}{1-\theta}\right)\right]^2}\right]$$

viii. The probability generating function is given by

$$G(s) = \frac{f(\theta s)}{f(\theta)} = \frac{1}{\log\left(\frac{1+\theta}{1-\theta}\right)}\log\left(\frac{1+\theta s}{1-\theta s}\right)$$

$$= \frac{\log(1+\theta s) - \log(1-\theta s)}{\log(1+\theta) - \log(1-\theta)}$$

$$G'(s) = \left[\frac{\theta}{1+\theta s} + \frac{\theta}{1-\theta s}\right]\frac{1}{\log(1+\theta) - \log(1-\theta)}$$

$$G''(s) = \left[\frac{-\theta^2}{(1+\theta s)^2} + \frac{\theta^2}{(1-\theta s)^2}\right]\frac{1}{\log(1+\theta) - \log(1-\theta)}$$

setting $s = 1$, we obtain

$$G'(1) = \left[\frac{\theta}{1+\theta} + \frac{\theta}{1-\theta}\right]\frac{1}{\log(1+\theta) - \log(1-\theta)}$$

$$= \frac{2\theta}{1-\theta^2}\cdot\frac{1}{\log\left(\frac{1+\theta}{1-\theta}\right)}$$

$$G''(1) = \left[\frac{-\theta^2}{(1+\theta)^2} + \frac{\theta^2}{(1-\theta)^2}\right]\frac{1}{\log(1+\theta) - \log(1-\theta)}$$

$$= \frac{4\theta^3}{\left[1-\theta^2\right]^2}\cdot\frac{1}{\log\frac{1+\theta}{1-\theta}}$$

The mean and the variance is given by,

$$E(X) = G'(1) = \frac{2\theta}{1-\theta^2}\cdot\frac{1}{\log\left(\frac{1+\theta}{1-\theta}\right)}$$

and

$$
\begin{aligned}
Var(X) &= G''(1) + G'(1) - [G'(1)]^2 \\
&= \frac{4\theta^3}{[1-\theta^2]^2} \cdot \frac{1}{\log \frac{1+\theta}{1-\theta}} + \frac{2\theta}{1-\theta^2} \cdot \frac{1}{\log \frac{1+\theta}{1-\theta}} - \frac{4\theta^2}{\left[(1-\theta^2) \log\left(\frac{1+\theta}{1-\theta}\right)\right]^2} \\
&= \frac{4\theta^3 \log \frac{1+\theta}{1-\theta} + 2\theta\,(1-\theta^2) \log \frac{1+\theta}{1-\theta} - 4\theta^2}{(1-\theta^2)^2 \left[\log \frac{1+\theta}{1-\theta}\right]^2} \\
&= \frac{4\theta^3 \log \frac{1+\theta}{1-\theta} + 2\theta \log \frac{1+\theta}{1-\theta} - 2\theta^3 \log \frac{1+\theta}{1-\theta} - 4\theta^2}{(1-\theta^2)^2 \left[\log \frac{1+\theta}{1-\theta}\right]^2} \\
&= \frac{2\theta^3 \log \frac{1+\theta}{1-\theta} + 2\theta \log \frac{1+\theta}{1-\theta} - 4\theta^2}{(1-\theta^2)^2 \left[\log \frac{1+\theta}{1-\theta}\right]^2} \\
&= 2\theta \left[\frac{(1+\theta^2) \log \frac{1+\theta}{1-\theta} - 2\theta}{(1-\theta^2)^2 \left[\log \frac{1+\theta}{1-\theta}\right]^2}\right]
\end{aligned}
$$

ix. The moment generating function of the distribution is given by

$$M_X(t) = \frac{f(\theta e^t)}{f(\theta)} = \frac{\log(1+\theta e^t) - \log(1-\theta e^t)}{\log(1+\theta) - \log(1-\theta)}$$

The r^{th} moment about the origin is obtained from the r^{th} derivative of $M_X(t)$ w.r.t t and setting $t = 0$

That is for $r = 1$

$$
\begin{aligned}
\mu_1' &= \frac{dM_X(t)}{dt} \,\Big|_{t=0} \\
&= \frac{d}{dt}\left\{\frac{\log(1+\theta e^t) - \log(1-\theta e^t)}{\log(1+\theta) - \log(1-\theta)}\right\} \Big|_{t=0} \\
&= \frac{1}{\log\left(\frac{1+\theta}{1-\theta}\right)} \frac{d}{dt}\left\{\log\left(\frac{1+\theta e^t}{1-\theta e^t}\right)\right\} \Big|_{t=0} \\
&= \frac{1}{\log\left(\frac{1+\theta}{1-\theta}\right)} \left(\frac{1-\theta e^t}{1+\theta e^t}\right) \left\{\frac{\theta e^t - \theta^2 e^{2t} + \theta e^t + \theta^2 e^{2t}}{(1-\theta e^t)^2}\right\} \Big|_{t=0} \\
&= \frac{1}{\log\left(\frac{1+\theta}{1-\theta}\right)} \left(\frac{1-\theta e^t}{1+\theta e^t}\right) \left\{\frac{2\theta e^t}{(1-\theta e^t)^2}\right\} \Big|_{t=0} \\
&= \frac{1}{\log\left(\frac{1+\theta}{1-\theta}\right)} \left\{\frac{2\theta}{(1-\theta^2)}\right\}
\end{aligned}
$$

For $r = 2$

$$
\begin{aligned}
\mu_2' &= \frac{1}{\log\left(\frac{1+\theta}{1-\theta}\right)} \frac{d}{dt}\left\{\frac{2\theta e^t}{(1+\theta e^t)(1-\theta e^t)}\right\} \Big|_{t=0} \\
&= \frac{1}{\log\left(\frac{1+\theta}{1-\theta}\right)} \left\{\frac{2\theta e^t + 2\theta^3 e^{3t}}{\theta^4 e^{4t} - 2\theta^2 e^{2t} + 1}\right\} \Big|_{t=0} \\
&= \frac{1}{\log\left(\frac{1+\theta}{1-\theta}\right)} \left\{\frac{2\theta + 2\theta^3}{\theta^4 - 2\theta^2 + 1}\right\} \\
&= \frac{1}{\log\left(\frac{1+\theta}{1-\theta}\right)} \left\{\frac{2\theta + 2\theta^3}{(1-\theta^2)^2}\right\}
\end{aligned}
$$

Also,

$$\mu_2 = \mu_2' - \mu_1'^2$$

$$= \frac{1}{\log\left(\frac{1+\theta}{1-\theta}\right)} \left\{ \frac{2\theta + 2\theta^3}{\theta^4 - 2\theta^2 + 1} - \frac{1}{\log\left(\frac{1+\theta}{1-\theta}\right)} \left\{ \frac{2\theta}{(1-\theta^2)} \right\}^2 \right\}$$

$$= \frac{1}{\log\left(\frac{1+\theta}{1-\theta}\right)} \left\{ \frac{2\theta + 2\theta^3}{\theta^4 - 2\theta^2 + 1} - \frac{1}{\log\left(\frac{1+\theta}{1-\theta}\right)} \left[\frac{4\theta^2}{\theta^4 - 2\theta^2 + 1} \right] \right\}$$

$$= \frac{1}{\log\left(\frac{1+\theta}{1-\theta}\right)} \left\{ \frac{2\theta \log\left(\frac{1+\theta}{1-\theta}\right) + 2\theta^3 \log\left(\frac{1+\theta}{1-\theta}\right) - 4\theta^2}{\log\left(\frac{1+\theta}{1-\theta}\right) [\theta^4 - 2\theta^2 + 1]} \right\}$$

$$= \frac{2\theta}{\log\left(\frac{1+\theta}{1-\theta}\right)} \left\{ \frac{\log\left(\frac{1+\theta}{1-\theta}\right) + \theta^2 \log\left(\frac{1+\theta}{1-\theta}\right) - 2\theta}{\log\left(\frac{1+\theta}{1-\theta}\right) [\theta^4 - 2\theta^2 + 1]} \right\}$$

$$= 2\theta \left[\frac{(1+\theta^2) \log\left(\frac{1+\theta}{1-\theta}\right) - 2\theta}{(1-\theta^2)^2 \left[\log\left(\frac{1+\theta}{1-\theta}\right)\right]^2} \right]$$

$r = 3$

$$\mu_3' = \frac{1}{\log\left(\frac{1+\theta}{1-\theta}\right)} \frac{d}{dt} \left\{ \frac{2\theta e^t + 2\theta^3 e^{3t}}{\theta^4 e^{4t} - 2\theta^2 e^{2t} + 1} \right\} |_{t=0}$$

$$= \left\{ \frac{2\theta e^t + 6\theta^3 e^{3t}}{\theta^4 e^{4t} - 2\theta^2 e^{2t} + 1} - \frac{2\theta e^t + 2\theta^3 e^{3t}}{(\theta^4 e^{4t} - 2\theta^2 e^{2t} + 1)^2} \left(4\theta^4 e^{4t} - 4\theta^2 e^{2t}\right) \right\} |_{t=0}$$

$$= \frac{1}{\log\left(\frac{1+\theta}{1-\theta}\right)} \left\{ \frac{(2\theta + 6\theta^3)(\theta^4 - 2\theta^2 + 1) - (2\theta + 2\theta^3)(4\theta^4 - 4\theta^2)}{(\theta^4 - 2\theta^2 + 1)^2} \right\}$$

$$= \frac{1}{\log\left(\frac{1+\theta}{1-\theta}\right)} \left\{ \frac{-2\theta - 12\theta^3 - 2\theta^5}{3\theta^2 - 3\theta^4 + \theta^6 - 1} \right\}$$

$$= \frac{1}{\log\left(\frac{1+\theta}{1-\theta}\right)} \left\{ \frac{-2\theta - 12\theta^3 - 2\theta^5}{(1-\theta^2)^3} \right\}$$

$r = 4$

$$\mu_4' = \frac{1}{\log\left(\frac{1+\theta}{1-\theta}\right)} \frac{d}{dt} \left\{ \frac{\left[\begin{array}{c} (2\theta e^t + 6\theta^3 e^{3t})(\theta^4 e^{4t} - 2\theta^2 e^{2t} + 1) \\ -(2\theta e^t + 2\theta^3 e^{3t})(4\theta^4 e^{4t} - 4\theta^2 e^{2t}) \end{array} \right]}{(\theta^4 e^{4t} - 2\theta^2 e^{2t} + 1)^2} \right\} |_{t=0}$$

$$= \frac{1}{\log\left(\frac{1+\theta}{1-\theta}\right)} \left\{ \begin{array}{c} \frac{\left[\left(2\theta + 18\theta^3\right)\left(\theta^4 - 2\theta^2 + 1\right) - \left(2\theta + 2\theta^3\right)\left(16\theta^4 - 8\theta^2\right)\right]}{(\theta^4 - 2\theta^2 + 1)^2} \\ -\frac{2\left(4\theta^4 - 4\theta^2\right)\left[\left(2\theta + 6\theta^3\right)\left(\theta^4 - 2\theta^2 + 1\right) - \left(2\theta + 2\theta^3\right)\left(4\theta^4 - 4\theta^2\right)\right]}{(\theta^4 - 2\theta^2 + 1)^3} \end{array} \right\}$$

$$= \frac{1}{\log\left(\frac{1+\theta}{1-\theta}\right)} \left\{ \frac{2\theta + 30\theta^3 - 50\theta^5 - 14\theta^7}{6\theta^4 - 4\theta^2 - 4\theta^6 + \theta^8 + 1} - \frac{-16\theta^3 - 96\theta^5 - 16\theta^7}{6\theta^4 - 4\theta^2 - 4\theta^6 + \theta^8 + 1} \right\}$$

$$= \frac{1}{\log\left(\frac{1+\theta}{1-\theta}\right)} \left\{ \frac{2\theta + 30\theta^3 - 50\theta^5 - 14\theta^7 + 16\theta^3 + 96\theta^5 + 16\theta^7}{6\theta^4 - 4\theta^2 - 4\theta^6 + \theta^8 + 1} \right\}$$

$$= \frac{1}{\log\left(\frac{1+\theta}{1-\theta}\right)} \left\{ \frac{2\theta + 46\theta^3 + 46\theta^5 + 2\theta^7}{6\theta^4 - 4\theta^2 - 4\theta^6 + \theta^8 + 1} \right\}$$

$$= \frac{1}{\log\left(\frac{1+\theta}{1-\theta}\right)} \left\{ \frac{2\theta + 46\theta^3 + 46\theta^5 + 2\theta^7}{(1-\theta^2)^4} \right\}$$

x. The factorial moment generating function of the distribution is given by

$$M_{[X]}(t) = \frac{f(\theta+\theta t)}{f(\theta)} = \frac{\log(1+\theta+\theta t) - \log(1-\theta-\theta t)}{\log(1+\theta) - \log(1-\theta)}$$

The r^{th} factorial moment is obtained by from the r^{th} derivative of $M_{[X]}(t)$ w.r.t t and setting $t = 0$. i.e.,

$$\mu_{[r]} = \frac{d^r M_{[X]}(t)}{dt^r} \mid_{t=0}$$

When $r = 1$

$$\begin{aligned}
\mu_{[1]} &= \frac{dM_{[X]}(t)}{dt} \mid_{t=0} \\
&= \frac{1}{\log\left(\frac{1+\theta}{1-\theta}\right)} \frac{d}{dt} \{\log(1+\theta+\theta t) - \log(1-\theta-\theta t)\} \mid_{t=0} \\
&= \frac{1}{\log\left(\frac{1+\theta}{1-\theta}\right)} \left\{ \frac{\theta}{\theta+t\theta+1} + \frac{\theta}{1-t\theta-\theta} \right\} \mid_{t=0} \\
&= \frac{1}{\log\left(\frac{1+\theta}{1-\theta}\right)} \left\{ \frac{\theta}{\theta+1} + \frac{\theta}{1-\theta} \right\} \\
&= \frac{1}{\log\left(\frac{1+\theta}{1-\theta}\right)} \left\{ \frac{\theta(1-\theta)+\theta(1+\theta)}{(1+\theta)(1-\theta)} \right\} \\
&= \frac{1}{\log\left(\frac{1+\theta}{1-\theta}\right)} \left\{ \frac{2\theta}{(1-\theta^2)} \right\}
\end{aligned}$$

$r = 2$

$$\begin{aligned}
\mu_{[2]} &= \frac{1}{\log\left(\frac{1+\theta}{1-\theta}\right)} \left\{ \frac{d}{dt}\left(\frac{\theta}{\theta+t\theta+1}\right) + \frac{d}{dt}\left(\frac{\theta}{1-t\theta-\theta}\right) \right\} \mid_{t=0} \\
&= \frac{1}{\log\left(\frac{1+\theta}{1-\theta}\right)} \left\{ \frac{-\theta^2}{(\theta+1)^2} + \frac{\theta^2}{(\theta-1)^2} \right\} \mid_{t=0} \\
&= \frac{1}{\log\left(\frac{1+\theta}{1-\theta}\right)} \left\{ \frac{-\theta^2(\theta-1)^2 + \theta^2(\theta+1)^2}{\theta^4 - 2\theta^2 + 1} \right\} \\
&= \frac{1}{\log\left(\frac{1+\theta}{1-\theta}\right)} \left\{ \frac{4\theta^3}{\theta^4 - 2\theta^2 + 1} \right\} \\
&= \frac{1}{\log\left(\frac{1+\theta}{1-\theta}\right)} \left\{ \frac{4\theta^3}{(1-\theta^2)^2} \right\}
\end{aligned}$$

$r = 3$

$$\mu_{[3]} = \frac{1}{\log\left(\frac{1+\theta}{1-\theta}\right)} \frac{d}{dt}\left\{ \begin{array}{l} -\frac{\theta^2}{2\theta+2t\theta+\theta^2+2t\theta^2+t^2\theta^2+1} \\ +\frac{\theta^2}{\theta^2-2t\theta-2\theta+2t\theta^2+t^2\theta^2+1} \end{array} \right\} |_{t=0}$$

$$= \frac{1}{\log\left(\frac{1+\theta}{1-\theta}\right)} \cdot -\frac{d}{dt}\left\{\frac{\theta^2}{(\theta+t\theta+1)^2}\right\} + \frac{d}{dt}\left\{\frac{\theta^2}{(\theta+t\theta-1)^2}\right\} |_{t=0}$$

$$= \frac{1}{\log\left(\frac{1+\theta}{1-\theta}\right)} \left\{\frac{2\theta^3}{(\theta+1)^3} - \frac{2\theta^3}{(\theta-1)^3}\right\}$$

$$= \frac{1}{\log\left(\frac{1+\theta}{1-\theta}\right)} \left\{\frac{2\theta^3(\theta-1)^3 - 2\theta^3(\theta+1)^3}{(1-\theta^2)^3}\right\}$$

$$= \frac{1}{\log\left(\frac{1+\theta}{1-\theta}\right)} \left\{\frac{-4\theta^3 - 12\theta^5}{(1-\theta^2)^3}\right\}$$

$r = 4$

$$\mu_{[4]} = \frac{1}{\log\left(\frac{1+\theta}{1-\theta}\right)} \frac{d}{dt}\left\{-\frac{2\theta^3}{(\theta+t\theta-1)^3} + \frac{2\theta^3}{(\theta+t\theta+1)^3}\right\} |_{t=0}$$

$$= \frac{1}{\log\left(\frac{1+\theta}{1-\theta}\right)} \frac{d}{dt}\left[\frac{-2\theta^3}{(\theta+t\theta-1)^3} + \frac{2\theta^3}{(\theta+t\theta+1)^3}\right] |_{t=0}$$

$$= \frac{1}{\log\left(\frac{1+\theta}{1-\theta}\right)} \left\{\frac{6\theta^4(\theta+t\theta+1)^4 - 6\theta^4(\theta+t\theta-1)^4}{(\theta+t\theta-1)^4(\theta+t\theta+1)^4}\right\} |_{t=0}$$

$$= \frac{1}{\log\left(\frac{1+\theta}{1-\theta}\right)} \left\{\frac{6\theta^4(\theta+1)^4 - 6\theta^4(\theta-1)^4}{(\theta-1)^4(\theta+1)^4}\right\}$$

$$= \frac{1}{\log\left(\frac{1+\theta}{1-\theta}\right)} \left\{\frac{48\theta^5 + 48\theta^7}{(1-\theta^2)^4}\right\}$$

xi. The cumulant generating function of the distribution is given by

$$K_X(t) = \log M_X(t) = \log\left\{\frac{\log(1+\theta e^t) - \log(1-\theta e^t)}{\log(1+\theta) - \log(1-\theta)}\right\}$$

The r^{th} cumulant of the distribution is the r^{th} derivative of $K_x(t)$ w.r.t t and setting $t = 0$

$$k_r = \frac{d^r K_X(t)}{dt^r} |_{t=0}$$

When $r = 1$, we have

$$\begin{aligned}
k_1 &= \frac{dK_X(t)}{dt}\ |_{t=0} \\
&= \frac{d}{dt}\left\{\log\left\{\frac{\log(1+\theta e^t)-\log(1-\theta e^t)}{\log(1+\theta)-\log(1-\theta)}\right\}\right\}\ |_{t=0} \\
&= \left\{\frac{2\theta e^t}{\ln(\theta e^t+1)-\ln(1-\theta e^t)-\theta^2 e^{2t}\ln(\theta e^t+1)+\theta^2 e^{2t}\ln(1-\theta e^t)}\right\}\ |_{t=0} \\
&= \left\{\frac{2\theta}{\ln\left(\frac{\theta+1}{1-\theta}\right)-\theta^2\ln\left(\frac{\theta+1}{1-\theta}\right)}\right\} \\
&= \frac{1}{\log\left(\frac{1+\theta}{1-\theta}\right)}\left\{\frac{2\theta}{(1-\theta^2)}\right\}
\end{aligned}$$

$r = 2$

$$\begin{aligned}
k_2 &= \frac{d^2}{dt^2}\left\{\log\left\{\frac{\log(1+\theta e^t)-\log(1-\theta e^t)}{\log(1+\theta)-\log(1-\theta)}\right\}\right\}\ |_{t=0} \\
&= \frac{2\theta\ln(\theta+1)-4\theta^2-2\theta\ln(1-\theta)+2\theta^3\ln(\theta+1)-2\theta^3\ln(1-\theta)}{\begin{bmatrix}\ln^2(\theta+1)+\ln^2(1-\theta)-2\ln(\theta+1)\ln(1-\theta) \\ -2\theta^2\ln^2(\theta+1)+\theta^4\ln^2(\theta+1)-2\theta^2\ln^2(1-\theta)+ \\ \theta^4\ln^2(1-\theta)+4\theta^2\ln(\theta+1)\ln(1-\theta)-2\theta^4\ln(\theta+1)\ln(1-\theta)\end{bmatrix}} \\
&= \frac{2\theta\left\{\ln\left(\frac{\theta+1}{1-\theta}\right)-2\theta+\theta^2\ln\left(\frac{\theta+1}{1-\theta}\right)\right\}}{\{1-2\theta^2+\theta^4\}\left(\ln^2(\theta+1)-2\ln(\theta+1)\ln(1-\theta)+\ln^2(1-\theta)\right)} \\
&= \frac{2\theta\left\{\ln\left(\frac{\theta+1}{1-\theta}\right)-2\theta+\theta^2\ln\left(\frac{\theta+1}{1-\theta}\right)\right\}}{\{1-2\theta^2+\theta^4\}\left(\ln(1+\theta)-\ln(1-\theta)\right)^2} \\
&= 2\theta\left\{\frac{(1+\theta^2)\log\frac{1+\theta}{1-\theta}-2\theta}{(1-\theta^2)^2\left[\log\frac{1+\theta}{1-\theta}\right]^2}\right\}
\end{aligned}$$

2.9.6 Inverse Sine

$$f(\theta) = \sin^{-1}\theta.$$

To obtain the power series of $\sin^{-1}\theta$, we start by expanding $(1-\theta^2)^{-\frac{1}{2}}$ i.e.,

$$\begin{aligned}
\left(1-\theta^2\right)^{-\frac{1}{2}} &= \sum_{k=0}^{\infty}\binom{-\frac{1}{2}}{k}\left(-\theta^2\right)^k \\
&= \sum_{k=0}^{\infty}(-1)^k\binom{-\frac{1}{2}}{k}\theta^{2k}
\end{aligned}$$

Intergrating both sides w.r.t θ, we have

$$\int \left(1-\theta^2\right)^{-\frac{1}{2}} d\theta = \sum_{k=0}^{\infty} (-1)^k \binom{-\frac{1}{2}}{k} \int \theta^{2k} d\theta$$
$$= \sum_{k=0}^{\infty} (-1)^k \binom{-\frac{1}{2}}{k} \frac{\theta^{2k+1}}{2k+1}$$

Solving the left hand side(LHS)

$$LHS = \int \left(1-\theta^2\right)^{-\frac{1}{2}} d\theta$$

Let $\theta = \sin u \quad \therefore d\theta = \cos u du$ and $\sin^{-1}\theta = u$

$$\begin{aligned} LHS &= \int \left(1-\sin^2 u\right)^{-\frac{1}{2}} \cos u du \\ &= \int \left(\cos^2 u\right)^{-\frac{1}{2}} \cos u du \\ &= \int (\cos u)^{-1} \cos u du \\ &= \int 1 du \\ &= u \\ &= \sin^{-1}\theta \end{aligned}$$

Therefore,

$$\begin{aligned} \sin^{-1}\theta &= \sum_{k=0}^{\infty} (-1)^k \binom{-\frac{1}{2}}{k} \frac{\theta^{2k+1}}{2k+1} \qquad (2.26) \\ &= \theta + \sum_{k=1}^{\infty} (-1)^k \frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)\left(-\frac{1}{2}-1\right)\cdots\left[-\frac{1}{2}-(k-1)\right]}{1\cdot 2\cdot 3\cdots k} \frac{\theta^{2k+1}}{2k+1} \\ &= \theta + \sum_{k=1}^{\infty} (-1)^k \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\cdots\left(-\frac{2k-1}{2}\right)}{1\cdot 2\cdot 3\cdots k} \frac{\theta^{2k+1}}{2k+1} \\ &= \theta + \sum_{k=1}^{\infty} (-1)^k (-1)^k \frac{\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}\cdots\frac{2k-1}{2}}{1\cdot 2\cdot 3\cdots k} \frac{\theta^{2k+1}}{2k+1} \\ &= \theta + \sum_{k=1}^{\infty} (-1)^k (-1)^k \frac{1\cdot 3\cdot 5\cdots(2k-1)}{2\cdot 4\cdot 6\cdots(2k)} \frac{\theta^{2k+1}}{2k+1} \end{aligned}$$

implying that

$$1 = \frac{\theta}{\sin^{-1}\theta} + \sum_{k=1}^{\infty} (-1)^k (-1)^k \frac{1\cdot 3\cdot 5\cdots(2k-1)}{2\cdot 4\cdot 6\cdots(2k)\sin^{-1}\theta} \frac{\theta^{2k+1}}{2k+1}$$

Therefore

i.
$$\Pr(X=1)=\frac{\theta}{\sin^{-1}\theta}$$

and

$$\Pr(X=2k+1)=\frac{1\cdot 3\cdot 5\cdots(2k-1)}{2\cdot 4\cdot 6\cdots(2k)\sin^{-1}\theta}\frac{\theta^{2k+1}}{2k+1}; k=1,2,...\text{ and }0<\theta<1$$

Alternatively, from (2.21)

$$1=\sum_{k=0}^{\infty}(-1)^k\binom{-\frac{1}{2}}{k}\frac{1}{\sin^{-1}\theta}\frac{\theta^{2k+1}}{2k+1}$$

Therefore,

$$\Pr(X=2k+1)=(-1)^k\binom{-\frac{1}{2}}{k}\frac{1}{\sin^{-1}\theta}\frac{\theta^{2k+1}}{2k+1}; k=1,2,...\text{ and }0<\theta<1$$

ii.
$$a_k=(-1)^k\binom{-\frac{1}{2}}{k}\left(\frac{1}{2k+1}\right)$$

iii.
$$f(\theta)=\sum_{k=0}^{\infty}(-1)^k\binom{-\frac{1}{2}}{k}\frac{\theta^{2k+1}}{2k+1}=\sin^{-1}\theta$$

$$\begin{aligned}f'(\theta)&=\sum_{k=0}^{\infty}(-1)^k\binom{-\frac{1}{2}}{k}\theta^{2k}\\&=\sum_{k=0}^{\infty}\binom{-\frac{1}{2}}{k}\left(-\theta^2\right)^k\\&=\left(1-\theta^2\right)^{-\frac{1}{2}}\end{aligned}$$

iv.

$$\begin{aligned}f''(\theta)&=-\frac{1}{2}\left(1-\theta^2\right)^{-\frac{1}{2}-1}(-2\theta)\\&=\theta\left(1-\theta^2\right)^{-\frac{3}{2}}\end{aligned}$$

v. The mean is given by

$$\begin{aligned}E(X)&=\theta\frac{f'(\theta)}{f(\theta)}\\&=\theta\frac{(1-\theta^2)^{-\frac{1}{2}}}{\sin^{-1}\theta}=\frac{\theta}{\sqrt{1-\theta^2}\sin^{-1}\theta}\end{aligned}$$

vi. The variance is given by

$$
\begin{aligned}
Var(X) &= \theta^2 \frac{f''(\theta)}{f(\theta)} + \theta \frac{f'(\theta)}{f(\theta)} - \left[\theta \frac{f'(\theta)}{f(\theta)}\right]^2 \\
&= \frac{\theta^2 \cdot \theta \left(1-\theta^2\right)^{-\frac{3}{2}}}{\sin^{-1}\theta} + \frac{\theta}{\sqrt{1-\theta^2}\sin^{-1}\theta} - \frac{\theta^2}{\left(\sqrt{1-\theta^2}\right)^2 \left(\sin^{-1}\theta\right)^2} \\
&= \frac{\theta^3 \left(1-\theta^2\right)^{-\frac{3}{2}}}{\sin^{-1}\theta} + \frac{\theta}{\sqrt{1-\theta^2}\sin^{-1}\theta} - \frac{\theta^2}{\left(\sqrt{1-\theta^2}\right)^2 \left(\sin^{-1}\theta\right)^2} \\
&= \frac{\theta^3}{\left(\sqrt{1-\theta^2}\right)^3 \sin^{-1}\theta} + \frac{\theta}{\sqrt{(1-\theta^2)}\sin^{-1}\theta} - \frac{\theta^2}{\left(\sqrt{1-\theta^2}\right)^2 \left(\sin^{-1}\theta\right)^2} \\
&= \frac{\theta^3 \sin^{-1}\theta + \theta\left(\sqrt{1-\theta^2}\right)^2 \sin^{-1}\theta - \theta^2\sqrt{(1-\theta^2)}}{\left(\sqrt{1-\theta^2}\right)^3 \left(\sin^{-1}\theta\right)^2} \\
&= \frac{\theta^3 \sin^{-1}\theta + \theta\left(1-\theta^2\right) \sin^{-1}\theta - \theta^2\sqrt{(1-\theta^2)}}{\left(\sqrt{1-\theta^2}\right)^3 \left(\sin^{-1}\theta\right)^2} \\
&= \frac{\theta \sin^{-1}\theta - \theta^2\sqrt{(1-\theta^2)}}{\left(\sqrt{1-\theta^2}\right)^3 \left(\sin^{-1}\theta\right)^2} \\
&= \theta \left[\frac{\sin^{-1}\theta - \theta\sqrt{(1-\theta^2)}}{\left(\sqrt{1-\theta^2}\right)^3 \left(\sin^{-1}\theta\right)^2}\right].
\end{aligned}
$$

vii. From (2.18) the recurrence relation is given by

$$\mu_{r+1} = \theta\left[\frac{d}{d\theta}\mu_r + r\mu_{r-1}\frac{d}{d\theta}\mu_1'\right]$$

but

$$\mu_1' = E\left(X\right) = \frac{\theta}{\sqrt{1-\theta^2}\sin^{-1}\theta}$$

Differentiate w.r.t θ to obtain,

$$
\begin{aligned}
\frac{d}{d\theta}\mu_1' &= \frac{\sqrt{1-\theta^2}\sin^{-1}\theta - \theta\frac{d}{d\theta}\sqrt{1-\theta^2}\sin^{-1}\theta}{\left(\sqrt{1-\theta^2}\right)^2 \left(\sin^{-1}\theta\right)^2} \\
&= \frac{\sqrt{1-\theta^2}\sin^{-1}\theta - \theta\left\{\frac{(-2\theta)}{2}\left(1-\theta^2\right)^{-\frac{1}{2}}\sin^{-1}\theta + \sqrt{1-\theta^2}\frac{d}{d\theta}\sin^{-1}\theta\right\}}{\left(\sqrt{1-\theta^2}\right)^2 \left(\sin^{-1}\theta\right)^2}
\end{aligned}
$$

but

$$\frac{d}{d\theta}\sin^{-1}\theta = \frac{d}{d\theta}f\left(\theta\right) = f\left(\theta\right) = \left(1-\theta^2\right)^{-\frac{1}{2}}$$

Therefore,

$$\frac{d}{d\theta}\mu_1' = \frac{\sqrt{1-\theta^2}\sin^{-1}\theta - \theta\left\{\frac{(-2\theta)}{2}\left(1-\theta^2\right)^{-\frac{1}{2}}\sin^{-1}\theta + \sqrt{1-\theta^2}\left(1-\theta^2\right)^{-\frac{1}{2}}\right\}}{\left(\sqrt{1-\theta^2}\right)^2\left(\sin^{-1}\theta\right)^2}$$

$$= \frac{\sqrt{1-\theta^2}\sin^{-1}\theta + \theta^2\left(1-\theta^2\right)^{-\frac{1}{2}}\sin^{-1}\theta - \theta}{\left(\sqrt{1-\theta^2}\right)^2\left(\sin^{-1}\theta\right)^2}$$

$$= \frac{\sqrt{1-\theta^2}}{\sqrt{1-\theta^2}}\left[\frac{\sqrt{1-\theta^2}\sin^{-1}\theta + \theta^2\left(1-\theta^2\right)^{-\frac{1}{2}}\sin^{-1}\theta - \theta}{\left(\sqrt{1-\theta^2}\right)^2\left(\sin^{-1}\theta\right)^2}\right]$$

$$= \frac{\left(1-\theta^2\right)\sin^{-1}\theta + \theta^2\sin^{-1}\theta - \theta\sqrt{1-\theta^2}}{\left(\sqrt{1-\theta^2}\right)^3\left(\sin^{-1}\theta\right)^2}$$

$$= \frac{\sin^{-1}\theta - \theta\sqrt{1-\theta^2}}{\left(\sqrt{1-\theta^2}\right)^3\left(\sin^{-1}\theta\right)^2}$$

$\therefore$

$$\mu_{r+1} = \theta\left[\frac{d}{d\theta}\mu_r + r\left\{\frac{\sin^{-1}\theta - \theta\sqrt{1-\theta^2}}{\left(\sqrt{1-\theta^2}\right)^3\left(\sin^{-1}\theta\right)^2}\right\}\mu_{r-1}\right],$$

setting $r = 1$ we obtain,

$$\mu_2 = \theta\left[\frac{\sin^{-1}\theta - \theta\sqrt{1-\theta^2}}{\left(\sqrt{1-\theta^2}\right)^3\left(\sin^{-1}\theta\right)^2}\right]$$

viii. The pgf is given by

$$G(s) = \frac{f(\theta s)}{f(\theta)} = \frac{\sin^{-1}\theta s}{\sin^{-1}\theta}$$

$$G'(s) = \frac{\theta\frac{d}{ds}\sin^{-1}\theta s}{\sin^{-1}\theta}$$

$$= \frac{\theta\left(1-(\theta s)^2\right)^{-\frac{1}{2}}}{\sin^{-1}\theta}$$

and

$$G''(s) = \frac{-\frac{1}{2}\theta\left(1-(\theta s)^2\right)^{-\frac{1}{2}-1}\left(-2\theta\left(\theta s\right)\right)}{\sin^{-1}\theta}$$

$$= \frac{\theta^3 s\left(1-(\theta s)^2\right)^{-\frac{3}{2}}}{\sin^{-1}\theta}$$

Therefore,

$$G'(1) = \frac{\theta\left(1-\theta^2\right)^{-\frac{1}{2}}}{\sin^{-1}\theta} = \frac{\theta}{\sqrt{1-\theta^2}}\cdot\frac{1}{\sin^{-1}\theta}$$

$$G''(1) = \frac{\theta^3\left(1-\theta^2\right)^{-\frac{3}{2}}}{\sin^{-1}\theta}$$

To obtain mean and variance we have,

$$E(X) = G'(1) = \frac{\theta}{\sqrt{1-\theta^2}}\frac{1}{\sin^{-1}\theta}$$

and

$$\begin{aligned}
Var(X) &= G''(1) + G'(1) - [G'(1)]^2 \\
&= \frac{\theta^3\left(1-\theta^2\right)^{-\frac{3}{2}}}{\sin^{-1}\theta} + \frac{\theta}{\sqrt{1-\theta^2}\sin^{-1}\theta} - \frac{\theta^2}{\left(\sqrt{1-\theta^2}\right)^2\left(\sin^{-1}\theta\right)^2} \\
&= \frac{\theta\left(\sin^{-1}\theta - \theta\sqrt{(1-\theta^2)}\right)}{\left(\sqrt{1-\theta^2}\right)^3\left(\sin^{-1}\theta\right)^2}
\end{aligned}$$

ix. The moment generating function of the distribution is given by

$$M_X(t) = \frac{f(\theta e^t)}{f(\theta)} = \frac{\sum_{k=0}^{\infty}(-1)^k\binom{-\frac{1}{2}}{k}\frac{\left(\theta e^t\right)^{2k+1}}{2k+1}}{\sin^{-1}\theta}$$

The r^{th} moment about the origin is obtained from the r^{th} derivative of $M_X(t)$ w.r.t t and setting $t = 0$

That is for $r = 1$

$$\begin{aligned}
\mu_1' &= \frac{dM_X(t)}{dt}\Big|_{t=0} \\
&= \frac{\sum_{k=0}^{\infty}(-1)^k\binom{-\frac{1}{2}}{k}\frac{d}{dt}\left\{\frac{\left(\theta e^t\right)^{2k+1}}{2k+1}\right\}\Big|_{t=0}}{\sin^{-1}\theta} \\
&= \frac{\sum_{k=0}^{\infty}(-1)^k\binom{-\frac{1}{2}}{k}\left\{\theta e^t\left(\theta e^t\right)^{2k}\right\}\Big|_{t=0}}{\sin^{-1}\theta} \\
&= \frac{\sum_{k=0}^{\infty}(-1)^k\binom{-\frac{1}{2}}{k}\theta^{2k}\theta}{\sqrt{1-\theta^2}\sin^{-1}\theta} \\
&= \frac{\sum_{k=0}^{\infty}\binom{-\frac{1}{2}}{k}\left(-\theta^2\right)^k\theta}{\sin^{-1}\theta} \\
&= \frac{\theta}{\sqrt{1-\theta^2}\sin^{-1}\theta}
\end{aligned}$$

For $r = 2$

$$\begin{aligned}
\mu_2' &= \frac{\sum_{k=0}^{\infty}(-1)^k\binom{-\frac{1}{2}}{k}\frac{d}{dt}\left\{\theta e^t\left(\theta e^t\right)^{2k}\right\}\Big|_{t=0}}{\sin^{-1}\theta}\Big|_{t=0} \\
&= \frac{\sum_{k=0}^{\infty}(-1)^k\binom{-\frac{1}{2}}{k}\left\{\theta e^t\left(\theta e^t\right)^{2k} + 2k\theta e^t\left(\theta e^t\right)^{2k}\right\}}{\sin^{-1}\theta}\Big|_{t=0} \\
&= \frac{\sum_{k=0}^{\infty}(-1)^k\binom{-\frac{1}{2}}{k}\left\{\theta\left(\theta\right)^{2k} + 2k\theta\left(\theta\right)^{2k}\right\}}{\sin^{-1}\theta} \\
&= \frac{\sum_{k=0}^{\infty}(-1)^k\binom{-\frac{1}{2}}{k}\left\{\theta\left(\theta^2\right)^k + 2k\theta\left(\theta^2\right)^k\right\}}{\sin^{-1}\theta} = \frac{\theta\left[2k+1\right]}{\sqrt{1-\theta^2}\sin^{-1}\theta}
\end{aligned}$$

A summary of the univariate discrete probability distributions belonging to the class of power series distributions are illustrated in table 2.1 below.

	θ	a_k	$f(\theta)$	S	pmf	pgf
$Bin(n,p)$	$\frac{p}{1-p}$	$\binom{n}{k}$	$(1+\theta)^n$	$0,1,\ldots,n$	$\binom{n}{k}p^k(1-p)^{n-k}$	$[1-p+ps]^n$
$Po(\lambda)$	λ	$\frac{1}{k!}$	e^θ	$0,1,2,\ldots$	$\frac{e^{-\lambda}\lambda^k}{k!}$	$e^{\lambda(s-1)}$
$NB(\alpha,p)$	$1-p$	$\binom{\alpha+k-1}{k}$	$(1-\theta)^{-\alpha}$	$0,1,2,\ldots$	$\binom{\alpha+k-1}{k}p^\alpha(1-p)^k$	$\left(\frac{p}{(1-(1-p)s)}\right)^\alpha$
$LS(p)$	p	$\frac{1}{k}$	$-\log(1-\theta)$	$1,2,3,\ldots$	$\frac{p^k}{-k\log(1-p)}$	$\frac{\ln[1-ps]}{\ln(1-p)}$

Table 2.1 Summary of univariate discrete probability distributions

Chapter 3

Power Series Distributions in terms of Hypergeometric functions

3.1 Introduction

This chapter entails expressing of power series distribution interms of Hypergeometric functions, that is; the Confluent and Gauss Hypergeometric functions with their construction and properties of the discrete Hypergeometric functions. Further, A Generalized form of Hypergeometric function with it's construction and properties will also be covered. Lastly, Special cases of confluent and Gauss Hypergeometric distributions, that includes; Power Series Distributions based on exponential expansion, and those based on binomial expansions with their construction and properties will be covered.

3.2 Confluent Hypergeometric Distribution

3.2.1 Introduction

Confluent Hypergeometric (Kummers's) function denoted by the symbol ${}_1F_1\left(a;c;x\right)$ represents the series

$$\begin{aligned}
{}_1F_1\left(a;c;x\right) &= 1+\frac{a}{c}\frac{x}{1!}+\frac{a\left(a+1\right)}{c\left(c+1\right)}\frac{x^2}{2!}+\frac{a\left(a+1\right)\left(a+2\right)}{c\left(c+1\right)\left(c+2\right)}\frac{x^3}{3!}+\cdots \\
&= \sum_{k=0}^{\infty}\frac{a\left(a+1\right)\left(a+2\right)\cdots\left(a+k-1\right)}{c\left(c+1\right)\left(c+2\right)\cdots\left(c+k-1\right)}\frac{x^k}{k!} \quad for\ c\neq 0,-1,-2,\ldots \qquad (3.1)
\end{aligned}$$

$$\begin{aligned}
\frac{d}{dx}\left({}_1F_1\left(a;c;x\right)\right) &= \frac{a}{c}\frac{1}{1!}+\frac{a\left(a+1\right)}{c\left(c+1\right)}\frac{2x}{2!}+\frac{a\left(a+1\right)\left(a+2\right)}{c\left(c+1\right)\left(c+2\right)}\frac{3x^2}{3!}+\cdots \\
&= \frac{a}{c}+\frac{a\left(a+1\right)}{c\left(c+1\right)}\frac{x}{1!}+\frac{a\left(a+1\right)\left(a+2\right)}{c\left(c+1\right)\left(c+2\right)}\frac{x^2}{2!}+\cdots \\
&= \frac{a}{c}\left\{1+\frac{\left(a+1\right)}{\left(c+1\right)}\frac{x}{1!}+\frac{\left(a+1\right)\left(a+2\right)}{\left(c+1\right)\left(c+2\right)}\frac{x^2}{2!}+\cdots\right\} \\
&= \frac{a}{c}\,{}_1F_1\left(a+1;c+1;x\right) \qquad (3.2)
\end{aligned}$$

$$\frac{d^2}{dx^2}\left({}_1F_1\left(a;c;x\right)\right) = \frac{a}{c}\left\{0 + \frac{(a+1)}{(c+1)} + \frac{(a+1)(a+2)}{(c+1)(c+2)}\frac{x}{1!} + \cdots\right\}$$
$$= \frac{a}{c}\left\{\frac{(a+1)}{(c+1)} + \frac{(a+1)(a+2)}{(c+1)(c+2)}\frac{x}{1!} + \cdots\right\}$$
$$= \frac{a}{c}\frac{(a+1)}{(c+1)}\left\{1 + \frac{(a+2)}{(c+2)}\frac{x}{1!} + \cdots\right\}$$
$$= \frac{a}{c}\frac{(a+1)}{(c+1)}\ {}_1F_1\left(a+2;c+2;x\right) \tag{3.3}$$

In applied Mathematics, special functions are used in solving differential equations. Thus ${}_1F_1\left(a;c;x\right)$ is a solution to certain differential equation which is derived as follows:-

Let us use the operator

$$\delta = x\frac{d}{dx}$$

Therefore,

$$\delta x^k = x\frac{d}{dx}x^k = xkx^{k-1} = kx^k \tag{3.4}$$

$$\begin{aligned}(\delta + c - 1)\,x^k &= \delta x^k + (c-1)\,x^k\\ &= kx^k + (c-1)\,x^k\\ &= (k + c - 1)\,x^k\end{aligned} \tag{3.5}$$

$$\begin{aligned}\delta\,(\delta + c - 1)\,x^k &= \delta\,(k + c - 1)\,x^k\\ &= \delta x^k\,(k + c - 1)\\ &= kx^k\,(k + c - 1)\\ &= k\,(k + c - 1)\,x^k\end{aligned} \tag{3.6}$$

$$\begin{aligned}(\delta + a)\,(\delta + b)\,x^k &= (\delta + a)\left(\delta x^k + bx^k\right)\\ &= (\delta + a)\left(kx^k + bx^k\right)\\ &= (\delta + a)\,kx^k + (\delta + a)\,bx^k\\ &= k\delta x^k + akx^k + b\delta x^k + abx^k\\ &= k^2x^k + akx^k + bkx^k + abx^k\\ &= x^k\left(ab + ak + bk + k^2\right)\\ &= x^k\left[k\,(k+a) + b\,(k+a)\right]\\ &= (k+a)\,(k+b)\,x^k\end{aligned} \tag{3.7}$$

Let $\delta(\delta+c-1)\,{}_1F_1(a;c;x) = AB$. Therefore,

$$
\begin{aligned}
AB &= \delta(\delta+c-1)\sum_{k=0}^{\infty}\frac{a(a+1)(a+2)\cdots(a+k-1)}{c(c+1)(c+2)\cdots(c+k-1)}\frac{x^k}{k!}\\
&= \sum_{k=0}^{\infty}\frac{a(a+1)(a+2)\cdots(a+k-1)}{c(c+1)(c+2)\cdots(c+k-1)}\delta(\delta+c-1)\frac{x^k}{k!}\\
&= \sum_{k=0}^{\infty}\frac{a(a+1)(a+2)\cdots(a+k-1)}{c(c+1)(c+2)\cdots(c+k-1)}k(k+c-1)\frac{x^k}{k!}\\
&= \sum_{k=0}^{\infty}\frac{a(a+1)(a+2)\cdots(a+k-1)}{c(c+1)(c+2)\cdots(c+k-2)}\frac{x^k}{(k-1)!}\\
&= x\sum_{k=0}^{\infty}\frac{a(a+1)(a+2)\cdots(a+k-1)}{c(c+1)(c+2)\cdots(c+k-2)}\frac{x^{k-1}}{(k-1)!}\\
&= x\sum_{k=0}^{\infty}\frac{a(a+1)(a+2)\cdots(a+k)}{c(c+1)(c+2)\cdots(c+k-1)}\frac{x^k}{k!}\ \text{by replacing } k \text{ with } (k+1)\\
&= x\sum_{k=0}^{\infty}\frac{a(a+1)(a+2)\ldots(a+k-1)}{c(c+1)(c+2)\ldots(c+k-1)}(a+k)\frac{x^k}{k!}\\
&= x\sum_{k=0}^{\infty}\frac{a(a+1)(a+2)\ldots(a+k-1)}{c(c+1)(c+2)\ldots(c+k-1)}(a+\delta)\frac{x^k}{k!}\\
&= x(a+\delta)\sum_{k=0}^{\infty}\frac{a(a+1)(a+2)\cdots(a+k-1)}{c(c+1)(c+2)\cdots(c+k-1)}\frac{x^k}{k!} \qquad (3.8\text{ a})
\end{aligned}
$$

Thus replacing the value of AB in (3.8 a) we obtain

$$\delta(\delta+c-1)\,{}_1F_1(a;c;x) = x(a+\delta)\,{}_1F_1(a;c;x) \qquad (3.8\text{ b})$$

$$\text{i.e., } \delta(\delta+c-1)\,y = x(a+\delta)\,y \quad \text{where } y = {}_1F_1(a;c;x) \qquad (3.8\text{ c})$$

$\therefore$

$$
\begin{aligned}
\delta^2 y + \delta(c-1)\,y &= x(a+\delta)\,y\\
\delta^2 y + [\delta(c-1) - x(a+\delta)]\,y &= 0\\
\delta^2 y + \delta(c-1)\,y - xay - x\delta y &= 0\\
\delta^2 y + (c-1-x)\,\delta y - xay &= 0\\
\delta(\delta y) + (c-1-x)\,\delta y - xay &= 0\\
\delta\left(x\frac{d}{dx}y\right) + (c-1-x)\,x\frac{d}{dx}y - xay &= 0\\
x\frac{d}{dx}\left(x\frac{d}{dx}y\right) + (c-1-x)\,x\frac{d}{dx}y - xay &= 0\\
x\frac{d^2y}{dx^2} + \frac{dy}{dx} + (c-1-x)\frac{dy}{dx} - ay &= 0\\
x\frac{d^2y}{dx^2} + (c-x)\frac{dy}{dx} - ay &= 0 \qquad (3.9)
\end{aligned}
$$

Which is the differential equation.

3.2.2 Construction and Properties of Discrete Confluent Hypergeometric Distribution

By definition,

$$_1F_1\left(a;c;\theta\right)=\sum_{k=0}^{\infty}\frac{a\left(a+1\right)\left(a+2\right)\cdots\left(a+k-1\right)}{c\left(c+1\right)\left(c+2\right)\cdots\left(c+k-1\right)}\frac{\theta^k}{k!}\quad \text{is a power series}$$

Therefore,

$$1=\sum_{k=0}^{\infty}\frac{a\left(a+1\right)\left(a+2\right)\cdots\left(a+k-1\right)}{c\left(c+1\right)\left(c+2\right)\cdots\left(c+k-1\right)k!}\frac{\theta^k}{_1F_1\left(a;c;\theta\right)}$$

Therefore

i.

$$\Pr\left(X=k\right)=\frac{a\left(a+1\right)\left(a+2\right)\cdots\left(a+k-1\right)}{c\left(c+1\right)\left(c+2\right)\cdots\left(c+k-1\right)k!}\frac{\theta^k}{_1F_1\left(a;c;\theta\right)};k=0,1,2,\ldots \tag{3.10}$$

This is the Confluent Hypergeometric Probability Mass Function. It belongs to the class of a power series distribution given by

$$\Pr\left(X=k\right)=\frac{a_k\theta^k}{f(\theta)}\quad \text{for } k=0,1,2,... \text{ and } \theta>0,\ a_k>0$$

In this case

ii.

$$a_k=\frac{a\left(a+1\right)\left(a+2\right)\cdots\left(a+k-1\right)}{c\left(c+1\right)\left(c+2\right)\cdots\left(c+k-1\right)} \tag{3.11}$$

iii.

$$f\left(\theta\right)=\ _1F_1\left(a;c;\theta\right) \tag{3.12}$$

iv.

$$f'\left(\theta\right)=\frac{a}{c}\ _1F_1\left(a+1;c+1;\theta\right)\quad \text{as in (3.2)}$$

v.

$$f''\left(\theta\right)=\frac{a\left(a+1\right)}{c\left(c+1\right)}\ _1F_1\left(a+2;c+2;\theta\right)\ \text{ as in (3.3)}$$

vi.

$$E(X)=\frac{\theta}{f(\theta)}f'(\theta)=\frac{\theta a}{c}\frac{_1F_1\left(a+1;c+1;\theta\right)}{_1F_1\left(a;c;\theta\right)} \tag{3.13}$$

vii. To obtain $Var\left(X\right)$, consider the differential equation (3.9) by letting $x=\theta$ and $y=f(\theta)=\ _1F_1\left(a;c;\theta\right)$

$$\theta f''\left(\theta\right)+\left(c-\theta\right)f'\left(\theta\right)-af\left(\theta\right)=0$$

$\therefore$

$$\theta f''\left(\theta\right)=af\left(\theta\right)-\left(c-\theta\right)f'\left(\theta\right) \tag{**}$$

multiply equation $(**)$ with θ to obtain

$$\theta^2 f''(\theta) = \theta a f(\theta) - \theta (c-\theta) f'(\theta)$$
$$\frac{\theta^2 f''(\theta)}{f(\theta)} = \theta a - \theta (c-\theta) \frac{f'(\theta)}{f(\theta)}$$

$\therefore$

$$\begin{aligned} Var(X) &= \frac{\theta^2 f''(\theta)}{f(\theta)} + \theta \frac{f'(\theta)}{f(\theta)} - \left[\theta \frac{f'(\theta)}{f(\theta)}\right]^2 \\ &= \theta a - \theta (c-\theta) \frac{f'(\theta)}{f(\theta)} + \theta \frac{f'(\theta)}{f(\theta)} - \left[\theta \frac{f'(\theta)}{f(\theta)}\right]^2 \\ &= \theta a + [1-(c-\theta)]\, \theta \frac{f'(\theta)}{f(\theta)} - \left[\theta \frac{f'(\theta)}{f(\theta)}\right]^2 \\ &= \theta a + [1-c+\theta] \frac{\theta a}{c} \frac{{}_1F_1(a+1;c+1;\theta)}{{}_1F_1(a;c;\theta)} - \left[\frac{\theta a}{c} \frac{{}_1F_1(a+1;c+1;\theta)}{{}_1F_1(a;c;\theta)}\right]^2 \end{aligned} \tag{3.14}$$

viii. Probability Generating Function (pgf) of X is given by

$$G(s) = \frac{f(\theta s)}{f(\theta)} = \frac{{}_1F_1(a;c;\theta s)}{{}_1F_1(a;c;\theta)} \tag{3.15}$$

To derive the differential equation whose solution is $G(s)$ given in (3.15) let

$$\delta = s\frac{d}{ds} \text{ an operator}$$

Then

$$\begin{aligned}
\delta\left(\delta+c-1\right)G\left(s\right) &= \delta\left(\delta+c-1\right)\sum_{k=0}^{\infty}p_k s^k \\
&= \sum_{k=0}^{\infty}p_k\delta\left(\delta+c-1\right)s^k \\
&= \sum_{k=0}^{\infty}p_k\left\{\delta\left[\delta s^k+\left(c-1\right)s^k\right]\right\} \\
&= \sum_{k=0}^{\infty}p_k\left\{\delta\left[s\frac{d}{ds}s^k+\left(c-1\right)s^k\right]\right\} \\
&= \sum_{k=0}^{\infty}p_k\left\{\delta\left[sks^{k-1}+\left(c-1\right)s^k\right]\right\} \\
&= \sum_{k=0}^{\infty}p_k\left\{\delta\left[ks^k+\left(c-1\right)s^k\right]\right\} \\
&= \sum_{k=0}^{\infty}p_k\left\{\delta\left[k+c-1\right]s^k\right\} \\
&= \sum_{k=0}^{\infty}p_k\left\{\left[k+c-1\right]\delta s^k\right\} \\
&= \sum_{k=0}^{\infty}p_k\left\{\left[k+c-1\right]s\frac{d}{ds}s^k\right\} \\
&= \sum_{k=0}^{\infty}p_k\left\{\left[k+c-1\right]sks^{k-1}\right\} \\
&= \sum_{k=0}^{\infty}p_k\left\{\left[k+c-1\right]ks^k\right\} \\
&= \sum_{k=0}^{\infty}k\left[k+c-1\right]p_k s^k \qquad (3.16^*)
\end{aligned}$$

Therefore (3.16*) becomes

$$\begin{aligned}
\delta\left(\delta+c-1\right)G\left(s\right) &= \sum_{k=0}^{\infty}k\left[k+c-1\right]\frac{a\left(a+1\right)\left(a+2\right)\cdots\left(a+k-1\right)}{c\left(c+1\right)\left(c+2\right)\cdots\left(c+k-1\right)}\frac{\theta^k s^k}{f\left(\theta\right)k!} \\
&= \sum_{k=0}^{\infty}\frac{a\left(a+1\right)\left(a+2\right)\cdots\left(a+k-1\right)}{c\left(c+1\right)\left(c+2\right)\cdots\left(c+k-2\right)}\frac{\left(\theta s\right)^k}{f\left(\theta\right)\left(k-1\right)!} \\
&= \theta s\sum_{k=0}^{\infty}\frac{a\left(a+1\right)\left(a+2\right)\cdots\left(a+k-1\right)}{c\left(c+1\right)\left(c+2\right)\cdots\left(c+k-2\right)}\frac{\left(\theta s\right)^{k-1}}{\left(k-1\right)!}\frac{1}{f\left(\theta\right)}
\end{aligned}$$

Replace k by $k+1$ to obtain

$$\delta\left(\delta+c-1\right)G\left(s\right) = \theta s\sum_{k=0}^{\infty}\frac{a\left(a+1\right)\left(a+2\right)\cdots\left(a+k\right)}{c\left(c+1\right)\left(c+2\right)\cdots\left(c+k-1\right)}\frac{\left(\theta s\right)^k}{k!}\frac{1}{f\left(\theta\right)} \qquad (3.16**)$$

but

$$
\begin{aligned}
(a+\delta)(\theta s)^k &= a(\theta s)^k + \delta(\theta s)^k \\
&= a(\theta s)^k + s\frac{d}{ds}(\theta s)^k \\
&= a(\theta s)^k + sk\theta(\theta s)^{k-1} \\
&= a(\theta s)^k + k(\theta s)^k \\
&= (a+k)(\theta s)^k
\end{aligned}
$$

Therefore $(3.16**)$ becomes

$$
\begin{aligned}
\delta(\delta+c-1)G(s) &= \theta s\sum_{k=0}^{\infty}\frac{a(a+1)(a+2)\cdots(a+k-1)}{c(c+1)(c+2)\cdots(c+k-1)}\frac{\left[(a+k)(\theta s)^k\right]}{k!f(\theta)} \\
&= \theta s\sum_{k=0}^{\infty}\frac{a(a+1)(a+2)\cdots(a+k-1)}{c(c+1)(c+2)\cdots(c+k-1)}\frac{\left[(a+\delta)(\theta s)^k\right]}{k!f(\theta)} \\
&= \theta s(a+\delta)\sum_{k=0}^{\infty}\frac{a(a+1)(a+2)\cdots(a+k-1)}{c(c+1)(c+2)\cdots(c+k-1)}\frac{(\theta s)^k}{k!}\frac{1}{f(\theta)} \\
&= \theta s(a+\delta)\frac{f(\theta s)}{f(\theta)} \\
&= \theta s(a+\delta)G(s) \qquad (3.16)
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\delta[\delta G(s)+(c-1)G(s)] &= \theta saG(s)+\theta s\delta G(s) \\
\delta[\delta G(s)]+(c-1)\delta G(s) &= \theta saG(s)+\theta s\delta G(s) \\
\delta\left[s\frac{d}{ds}G(s)\right]+(c-1)s\frac{d}{ds}G(s) &= \theta saG(s)+\theta s\cdot s\frac{d}{ds}G(s)
\end{aligned}
$$

$$
\begin{aligned}
\delta\{sG'(s)\}+(c-1)sG'(s)-\theta s\cdot sG'(s)-\theta saG(s) &= 0 \\
\delta\{sG'(s)\}+(c-1)sG'(s)-\theta s^2G'(s)-\theta saG(s) &= 0 \\
s\frac{d}{ds}\{sG'(s)\}+(c-1)sG'(s)-\theta s^2G'(s)-\theta saG(s) &= 0 \\
\frac{d}{ds}\{sG'(s)\}+(c-1)G'(s)-\theta sG'(s)-\theta aG(s) &= 0 \\
\frac{d}{ds}\{sG'(s)\}+(c-1)G'(s)-\theta sG'(s)-\theta aG(s) &= 0 \\
sG''(s)+G'(s)+cG'(s)-G'(s)-\theta sG'(s)-\theta aG(s) &= 0 \\
sG''(s)+(c-\theta s)G'(s)-\theta aG(s) &= 0 \qquad (3.17)
\end{aligned}
$$

setting $s=1$, we obtain

$$G''(1)+(c-\theta)G'(1)-\theta aG(1)=0$$

$$G''(1)=\theta a-(c-\theta)G'(1) \qquad (3.18)$$

$$E(X) = G'(1) = \frac{\theta}{f(\theta)} f'(\theta) = \frac{\theta a}{c} \frac{{}_1F_1(a+1; c+1; \theta)}{{}_1F_1(a; c; \theta)} \quad \text{as given in (3.13)}$$

$$\begin{aligned} Var(X) &= G''(1) + G'(1) - [G'(1)]^2 \\ &= \theta a - (c-\theta) G'(1) + G'(1) - [G'(1)]^2 \\ &= \theta a + [1-c+\theta] G'(1) - [G'(1)]^2 \\ &= \theta a + [1-c+\theta] \frac{\theta a}{c} \frac{{}_1F_1(a+1; c+1; \theta)}{{}_1F_1(a; c; \theta)} - \left[\frac{\theta a}{c} \frac{{}_1F_1(a+1; c+1; \theta)}{{}_1F_1(a; c; \theta)} \right]^2 \end{aligned} \tag{3.19}$$

3.3 Gauss Hypergeometric Distribution

3.3.1 Introduction

Gauss Hypergeometric function denoted by the symbol ${}_2F_1(a, b; c; x)$ represents the series

$${}_2F_1(a, b; c; x) = 1 + \frac{ab}{c} \frac{x}{1!} + \frac{a(a+1)b(b+1)}{c(c+1)} \frac{x^2}{2!} + \cdots \text{ for } c \neq 0, -1, -2, \ldots \tag{3.20}$$

$$\begin{aligned} \frac{d}{dx} {}_2F_1(a, b; c; x) &= \frac{ab}{c} + \frac{a(a+1)b(b+1)}{c(c+1)} x + \frac{a(a+1)(a+2)b(b+1)(b+2)}{c(c+1)(c+2)} \frac{x^2}{2!} + \cdots \\ &= \frac{ab}{c} \left\{ 1 + \frac{(a+1)(b+1)}{(c+1)} x + \frac{(a+1)(a+2)(b+1)(b+2)}{(c+1)(c+2)} \frac{x^2}{2!} + \cdots \right\} \\ &= \frac{ab}{c} {}_2F_1(a+1, b+1; c+1; x) \end{aligned} \tag{3.21}$$

$$\begin{aligned} \frac{d^2}{dx^2} {}_2F_1(a, b; c; x) &= \frac{ab}{c} \left\{ \frac{(a+1)(b+1)}{(c+1)} + \frac{(a+1)(a+2)(b+1)(b+2)}{(c+1)(c+2)} \frac{x}{1!} + \cdots \right\} \\ &= \frac{a(a+1)b(b+1)}{c(c+1)} \left\{ 1 + \frac{(a+2)(b+2)}{(c+2)} x + \cdots \right\} \\ &= \frac{a(a+1)b(b+1)}{c(c+1)} {}_2F_1(a+1, b+1; c+1; x) \end{aligned} \tag{3.22}$$

we now wish to derive a differential equation whose solution is ${}_2F_1(a, b; c; x)$

Let us use the operator

$$\delta = x \frac{d}{dx}$$

Then

Let $\delta(\delta + c - 1)\, {}_2F_1(a, b; c; x) = AC$. Therefore,

$$\begin{aligned}
AC &= \sum_{k=0}^{\infty} \frac{a(a+1)\cdots(a+k-1)\,b(b+1)\cdots(b+k-1)}{c(c+1)\cdots(c+k-1)} \delta(\delta+c-1)\,\frac{x^k}{k!} \\
&= \sum_{k=0}^{\infty} \frac{a(a+1)\cdots(a+k-1)\,b(b+1)\cdots(b+k-1)\,k(k+c-1)}{c(c+1)\cdots(c+k-1)} \frac{x^k}{k!} \\
&= \sum_{k=0}^{\infty} \frac{a(a+1)\cdots(a+k-1)\,b(b+1)\cdots(b+k-1)}{c(c+1)\cdots(c+k-2)} \frac{x^k}{(k-1)!} \\
&= x\sum_{k=0}^{\infty} \frac{a(a+1)\cdots(a+k-1)\,b(b+1)\cdots(b+k-1)}{c(c+1)\cdots(c+k-2)} \frac{x^{k-1}}{(k-1)!} \\
AC &= x\sum_{k=0}^{\infty} \frac{a(a+1)\cdots(a+k-1)\,b(b+1)\cdots(b+k-1)\,(a+k)(b+k)}{c(c+1)\cdots(c+k-1)} \frac{x^k}{k!} \\
&= x\sum_{k=0}^{\infty} \frac{a(a+1)\cdots(a+k-1)\,b(b+1)\cdots(b+k-1)}{c(c+1)\cdots(c+k-1)}\,(a+\delta)(b+\delta)\,\frac{x^k}{k!} \\
&= x(a+\delta)(b+\delta)\sum_{k=0}^{\infty} \frac{a(a+1)\cdots(a+k-1)\,b(b+1)\cdots(b+k-1)}{c(c+1)\cdots(c+k-1)} \frac{x^k}{k!} \qquad (AC*)
\end{aligned}$$

Therefore replacing the value of AC in equation $(AC*)$ above we obtain

$$\delta(\delta+c-1)\ {}_2F_1(a,b;c;x) = x(a+\delta)(b+\delta)\ {}_2F_1(a,b;c;x) \tag{3.23}$$

Therefore,

$$\delta(\delta+c-1)\,F = x(a+\delta)(b+\delta)\,F \quad \text{where } F = {}_2F_1(a,b;c;x) \text{ and } \delta = x\frac{d}{dx}$$

Therefore

$$\begin{aligned}
\delta\{\delta F + (c-1)F\} &= x(a+\delta)(bF+\delta F) \\
\Rightarrow \delta\left\{x\frac{d}{dx}F + (c-1)F\right\} &= x(a+\delta)\left\{bF + x\frac{d}{dx}F\right\}
\end{aligned}$$

$$\begin{aligned}
x\frac{d}{dx}\left[x\frac{d}{dx}F\right] + (c-1)\,x\frac{d}{dx}F &= x(a+\delta)\left\{bF + x\frac{d}{dx}F\right\} \\
\frac{d}{dx}\left[x\frac{d}{dx}F\right] + (c-1)\frac{d}{dx}F &= (a+\delta)\left\{bF + x\frac{d}{dx}F\right\} \\
x\frac{d^2}{dx^2}F + \frac{d}{dx}F + c\frac{d}{dx}F - \frac{d}{dx}F &= abF + ax\frac{d}{dx}F + \delta bF + \delta x\frac{d}{dx}F
\end{aligned}$$

$$\begin{aligned}
x\frac{d^2}{dx^2}F + c\frac{d}{dx}F &= abF + ax\frac{d}{dx}F + xb\frac{d}{dx}F + x\frac{d}{dx}\left[x\frac{d}{dx}F\right] \\
&= abF + (ax+bx)\frac{d}{dx}F + x\left[x\frac{d^2}{dx^2}F + \frac{d}{dx}F\right] \\
&= abF + (a+b)\,x\frac{d}{dx}F + x\left[x\frac{d^2}{dx^2}F + \frac{d}{dx}F\right] \qquad (3.24)
\end{aligned}$$

Therefore (3.24) becomes

$$x(1-x)\frac{d^2}{dx^2}F + [c - ax - bx - x]\,\frac{d}{dx}F - abF = 0$$
$$x(1-x)\frac{d^2}{dx^2}F + [c - (a+b+1)\,x]\,\frac{d}{dx}F - abF = 0 \tag{3.25}$$

Which is the differential equation.

3.3.2 Construction and properties of Gauss Hypergeometric Distribution

By definition

$$_2F_1\,(a,b;c;\theta) = \sum_{k=0}^{\infty} \frac{a\,(a+1)\cdots(a+k-1)\,b\,(b+1)\cdots(b+k-1)}{c\,(c+1)\cdots(c+k-1)}\frac{\theta^k}{k!}$$

$$1 = \sum_{k=0}^{\infty} \frac{a\,(a+1)\cdots(a+k-1)\,b\,(b+1)\cdots(b+k-1)}{c\,(c+1)\cdots(c+k-1)\ _2F_1\,(a,b;c;\theta)}\frac{\theta^k}{k!}$$

Therefore

i.

$$\Pr\,(X=k) = \frac{a\,(a+1)\cdots(a+k-1)\,b\,(b+1)\cdots(b+k-1)}{c\,(c+1)\cdots(c+k-1)\ _2F_1\,(a,b;c;\theta)}\frac{\theta^k}{k!};\, k = 0,1,2,\ldots \tag{3.26}$$

which is the Gauss Hypergeometric probability mass function. It belongs to the class of power series distributions given by

$$\Pr\,(X=k) = \frac{a_k\theta^k}{f(\theta)} \text{ for } k = 0,1,2,\ldots \text{ and } \theta > 0,\ a_k > 0.$$

This implies that

ii.

$$a_k = \frac{a\,(a+1)\cdots(a+k-1)\,b\,(b+1)\cdots(b+k-1)}{c\,(c+1)\cdots(c+k-1)} \tag{3.27}$$

iii.

$$f(\theta) = \ _2F_1\,(a,b;c;\theta) \tag{3.28}$$

iv.

$$f'(\theta) = \frac{ab}{c}\ _2F_1\,(a+1,b+1;c+1;\theta) \text{ as given in (3.21)}$$

v.

$$f''(\theta) = \frac{a\,(a+1)\,b\,(b+1)}{c\,(c+1)}\ _2F_1\,(a+2,b+2;c+2;\theta) \text{ as given in (3.22)}$$

vi.

$$E(X) = \frac{\theta}{f(\theta)}f'(\theta) = \theta\frac{ab}{c}\frac{_2F_1\,(a+1,b+1;c+1;\theta)}{_2F_1\,(a,b;c;\theta)} \tag{3.29}$$

vii. To obtain $Var\,(X)$, consider the differential equation (3.25), i.e.

$$x(1-x)\frac{d^2}{dx^2}F + [c-(a+b+1)\,x]\,\frac{d}{dx}F - abF = 0 \tag{3.30}$$

Replace $x = \theta$ and $F = f(\theta) = {}_2F_1\,(a,b;c;\theta)$ in (3.30) to obtain

$\therefore$

$$\theta(1-\theta)f''(\theta) + [c-(a+b+1)\,\theta]\,f'(\theta) - ab \times f(\theta) = 0$$

$\therefore$

$$\frac{f''(\theta)}{f(\theta)} = \frac{ab}{\theta(1-\theta)} - \frac{[c-(a+b+1)\,\theta]}{\theta(1-\theta)}\frac{f'(\theta)}{f(\theta)}$$

$$\begin{aligned}
Var\,(X) &= \frac{\theta^2 f''\,(\theta)}{f\,(\theta)} + \theta\frac{f'\,(\theta)}{f\,(\theta)} - \left[\theta\frac{f'\,(\theta)}{f\,(\theta)}\right]^2 \\
&= \theta^2\left\{\frac{ab}{\theta(1-\theta)} - \frac{[c-(a+b+1)\,\theta]}{\theta(1-\theta)}\frac{f'(\theta)}{f(\theta)}\right\} + \theta\frac{f'\,(\theta)}{f\,(\theta)} - \left[\theta\frac{f'\,(\theta)}{f\,(\theta)}\right]^2 \\
&= \frac{\theta ab}{(1-\theta)} - \theta\frac{[c-(a+b+1)\,\theta]}{(1-\theta)}\frac{f'(\theta)}{f(\theta)} + \theta\frac{f'\,(\theta)}{f\,(\theta)} - \left[\theta\frac{f'\,(\theta)}{f\,(\theta)}\right]^2 \\
&= \frac{\theta ab}{(1-\theta)} + \left\{1 - \frac{[c-(a+b+1)\,\theta]}{(1-\theta)}\right\}\theta\frac{f'(\theta)}{f(\theta)} - \left[\theta\frac{f'\,(\theta)}{f\,(\theta)}\right]^2 \\
&= \frac{\theta ab}{(1-\theta)} + \left\{\frac{1-\theta-[c-(a+b+1)\,\theta]}{(1-\theta)}\right\}\theta\frac{f'(\theta)}{f(\theta)} - \left[\theta\frac{f'\,(\theta)}{f\,(\theta)}\right]^2 \\
&= \frac{\theta ab}{(1-\theta)} + \{1-\theta-c+(a+b)\,\theta+\theta\}\,\frac{\theta}{(1-\theta)}\frac{f'(\theta)}{f(\theta)} - \left[\theta\frac{f'\,(\theta)}{f\,(\theta)}\right]^2 \\
&= \frac{\theta ab}{(1-\theta)} + \{(1-c)+(a+b)\,\theta\}\,\frac{\theta}{(1-\theta)}\frac{f'(\theta)}{f(\theta)} - \left[\theta\frac{f'\,(\theta)}{f\,(\theta)}\right]^2
\end{aligned}$$

Therefore,

$$\begin{aligned}
Var(X) &= \frac{\theta}{(1-\theta)}\frac{ab}{c}c + \{(1-c)+(a+b)\,\theta\}\,\frac{\theta}{(1-\theta)}\frac{ab}{c}\times \\
&\quad \frac{{}_2F_1\,(a+1,b+1;c+1;\theta)}{{}_2F_1\,(a,b;c;\theta)} - \left\{\theta\frac{ab}{c}\frac{{}_2F_1\,(a+1,b+1;c+1;\theta)}{{}_2F_1\,(a,b;c;\theta)}\right\}^2 \\
&= \frac{\theta}{(1-\theta)}\frac{ab}{c}\left\{\begin{array}{c} c+\{(1-c)+(a+b)\,\theta\}\,\frac{{}_2F_1(a+1,b+1;c+1;\theta)}{{}_2F_1(a,b;c;\theta)} \\ -\theta\frac{ab}{c}(1-\theta)\left[\frac{{}_2F_1(a+1,b+1;c+1;\theta)}{{}_2F_1(a,b;c;\theta)}\right]^2 \end{array}\right\}
\end{aligned} \tag{3.31}$$

as obtained by Noack (1950)

viii. Probability Generating Function(pgf)

The pgf of X is given by

$$G(s) = \frac{f(\theta s)}{f(\theta)} = \frac{{}_2F_1\,(a,b;c;\theta s)}{{}_2F_1\,(a,b;c;\theta)} \tag{3.32}$$

To derive the differential equation whose solution is $G(s)$ given in (3.32), consider,

$$\delta\left(\delta+c-1\right)G\left(s\right)=\delta\left(\delta+c-1\right)\sum_{k=0}^{\infty}p_{k}s^{k}$$

where $\delta = s\frac{d}{ds}$

Therefore

$$\begin{aligned}
&\delta\left(\delta+c-1\right)G\left(s\right)\\
&=\sum_{k=0}^{\infty}p_{k}\delta\left(\delta+c-1\right)s^{k}\\
&=\sum_{k=0}^{\infty}p_{k}k\left(k+c-1\right)s^{k}\\
&=\sum_{k=0}^{\infty}\frac{a\left(a+1\right)\cdots\left(a+k-1\right)b\left(b+1\right)\cdots\left(b+k-1\right)k\left(k+c-1\right)}{c\left(c+1\right)\cdots\left(c+k-1\right)\ {}_{2}F_{1}\left(a,b;c;\theta\right)}\frac{\theta^{k}}{k!}s^{k}\\
&=\sum_{k=0}^{\infty}\frac{a\left(a+1\right)\cdots\left(a+k-1\right)b\left(b+1\right)\cdots\left(b+k-1\right)}{c\left(c+1\right)\left(c+k-2\right)\ {}_{2}F_{1}\left(a,b;c;\theta\right)}\frac{\theta^{k}}{\left(k-1\right)!}s^{k}\\
&=\theta s\sum_{k=0}^{\infty}\frac{a\left(a+1\right)\cdots\left(a+k-1\right)b\left(b+1\right)\cdots\left(b+k-1\right)}{c\left(c+1\right)\left(c+k-2\right)\ {}_{2}F_{1}\left(a,b;c;\theta\right)}\frac{\left(\theta s\right)^{k-1}}{\left(k-1\right)!}\\
&=\theta s\sum_{k=0}^{\infty}\frac{a\left(a+1\right)\cdots\left(a+k-1\right)b\left(b+1\right)\cdots\left(b+k-1\right)\left(a+k\right)\left(b+k\right)}{c\left(c+1\right)\left(c+k-1\right)\ {}_{2}F_{1}\left(a,b;c;\theta\right)}\frac{\left(\theta s\right)^{k}}{k!}\\
&=\theta s\sum_{k=0}^{\infty}\frac{a\left(a+1\right)\cdots\left(a+k-1\right)b\left(b+1\right)\cdots\left(b+k-1\right)\left(a+\delta\right)\left(b+\delta\right)}{c\left(c+1\right)\left(c+k-1\right)\ {}_{2}F_{1}\left(a,b;c;\theta\right)}\frac{\left(\theta s\right)^{k}}{k!}\\
&=\theta s\left(a+\delta\right)\left(b+\delta\right)\frac{{}_{2}F_{1}\left(a,b;c;\theta s\right)}{{}_{2}F_{1}\left(a,b;c;\theta\right)}\\
&=\theta s\left(a+\delta\right)\left(b+\delta\right)G\left(s\right) \qquad (3.33)
\end{aligned}$$

$\therefore$ from (3.33) we have

$$\begin{aligned}
\delta\left\{\delta G\left(s\right)+\left(c-1\right)G\left(s\right)\right\}&=\theta s\left(a+\delta\right)\left[bG\left(s\right)+\delta G\left(s\right)\right]\\
\delta\left\{sG'\left(s\right)+\left(c-1\right)G\left(s\right)\right\}&=\theta s\left(a+\delta\right)\left[bG\left(s\right)+sG'\left(s\right)\right]\\
s\frac{d}{ds}\left\{sG'\left(s\right)\right\}+\left(c-1\right)\delta G\left(s\right)&=\left(\theta sa+\theta s\delta\right)\left[bG\left(s\right)+sG'\left(s\right)\right]\\
s\left[sG''\left(s\right)+G'\left(s\right)\right]+\left(c-1\right)\delta G\left(s\right)&=\left\{\begin{array}{c}\theta sabG\left(s\right)+\theta s^{2}aG'\left(s\right)\\+\theta s\delta bG\left(s\right)+\theta s\delta\left[sG'\left(s\right)\right]\end{array}\right\}\\
s^{2}G''\left(s\right)+sG'\left(s\right)+\left(c-1\right)sG'\left(s\right)&=\left\{\begin{array}{c}\theta sabG\left(s\right)+\theta s^{2}aG'\left(s\right)\\+\theta s^{2}bG'\left(s\right)+\theta s^{2}\frac{d}{ds}\left[sG'\left(s\right)\right]\end{array}\right\}\\
sG''\left(s\right)+G'\left(s\right)+\left(c-1\right)G'\left(s\right)&=\left\{\begin{array}{c}\theta abG\left(s\right)+\theta saG'\left(s\right)\\+\theta sbG'\left(s\right)+\theta s\left[sG''\left(s\right)+G'\left(s\right)\right]\end{array}\right\}\\
sG''\left(s\right)+G'\left(s\right)+\left(c-1\right)G'\left(s\right)&=\left\{\begin{array}{c}\theta abG\left(s\right)+\theta saG'\left(s\right)+\theta sbG'\left(s\right)\\+\theta s^{2}G''\left(s\right)+\theta sG'\left(s\right)\end{array}\right\}
\end{aligned}$$

$$\left\{s-\theta s^2\right\} G''(s)+\{c-1+1-\theta sa-\theta s-\theta sb\} G'(s)-\theta abG(s)=0$$

$$s\{1-\theta s\} G''(s)+\{c-[a+b+1]\theta s\} G'(s)-\theta abG(s)=0 \quad (3.34)$$

setting $s=1$ we obtain

$$(1-\theta) G''(1)+(c-[a+1+b]\theta) G'(1)-\theta ab=0$$

Therefore,

$$G''(1)=\frac{\theta ab}{(1-\theta)}-\frac{[c-(a+b+1)\theta]}{(1-\theta)}G'(1) \quad (3.35)$$

$$E(X)=G'(1)=\frac{\theta f'(\theta)}{f(\theta)}=\theta\frac{ab}{c}\frac{{}_2F_1(a+1,b+1;c+1;\theta)}{{}_2F_1(a,b;c;\theta)}$$

$$\begin{aligned}
Var(X) &= G''(1)+G'(1)-[G'(1)]^2\\
&= \frac{\theta ab}{(1-\theta)}+\frac{[-c+(a+b+1)\theta]}{(1-\theta)}G'(1)+G'(1)-[G'(1)]^2\\
&= \frac{\theta ab}{(1-\theta)}+\frac{[1-\theta-c+(a+b+1)\theta]}{(1-\theta)}G'(1)-[G'(1)]^2\\
&= \frac{\theta ab}{(1-\theta)}+\frac{[1-\theta-c+(a+b)\theta+\theta]}{(1-\theta)}G'(1)-[G'(1)]^2\\
&= \frac{\theta ab}{(1-\theta)}+\frac{[1-c+(a+b)\theta]}{(1-\theta)}G'(1)-[G'(1)]^2\\
&= \frac{\theta ab}{(1-\theta)}+\frac{[1-c+(a+b)\theta]}{(1-\theta)}\theta\frac{ab}{c}\frac{{}_2F_1(a+1,b+1;c+1;\theta)}{{}_2F_1(a,b;c;\theta)}\\
&\quad -\left[\theta\frac{ab}{c}\frac{{}_2F_1(a+1,b+1;c+1;\theta)}{{}_2F_1(a,b;c;\theta)}\right]^2
\end{aligned}$$

Therefore,

$$\begin{aligned}
Var(X) &= \frac{\theta}{(1-\theta)}\frac{ab}{c}c+\frac{[1-c+(a+b)\theta]}{(1-\theta)}\theta\frac{ab}{c}\frac{{}_2F_1(a+1,b+1;c+1;\theta)}{{}_2F_1(a,b;c;\theta)}\\
&\quad -\left[\theta\frac{ab}{c}\frac{{}_2F_1(a+1,b+1;c+1;\theta)}{{}_2F_1(a,b;c;\theta)}\right]^2\\
&= \frac{\theta}{(1-\theta)}\frac{ab}{c}\left\{\begin{array}{c} c+[1-c+(a+b)\theta]\frac{{}_2F_1(a+1,b+1;c+1;\theta)}{{}_2F_1(a,b;c;\theta)}\\ -(1-\theta)\theta\frac{ab}{c}\left[\frac{{}_2F_1(a+1,b+1;c+1;\theta)}{{}_2F_1(a,b;c;\theta)}\right]^2\end{array}\right\}
\end{aligned}$$

3.4 Generalized Hypergeometric Distribution

3.4.1 Introduction

A Generalized Hypergeometric Distribution is defined as;

$$_pF_q(a_1,\cdots,a_p;b_1,\cdots,b_q;x)=\sum_{k=0}^{\infty}\frac{(a_1)_k(a_2)_k\cdots(a_p)_k}{(b_1)_k\cdots(b_q)_k}\frac{x^k}{k!} \quad (3.36)$$

$$\textit{where } b_i\neq 0,-1,-2,\ldots \textit{ and } i=1,2,\ldots,q$$

$$\begin{aligned}
&\frac{d}{dx}\ {}_pF_q\left(a_1,\cdots,a_p;b_1,\cdots,b_q;x\right)\\
&=\frac{a_1a_2\cdots a_p}{b_1b_2\cdots b_q}\ {}_pF_q\left(a_1+1,\cdots,a_p+1;b_1+1,\cdots,b_q+1;x\right)
\end{aligned} \tag{3.37}$$

$$\begin{aligned}
&\frac{d^2}{dx^2}\ {}_pF_q\left(a_1,\cdots,a_p;b_1,\cdots,b_q;x\right)\\
&=\frac{a_1\left(a_1+1\right)\cdots a_p\left(a_p+1\right)}{b_1\left(b_1+1\right)\cdots b_q\left(b_q+1\right)}\ {}_pF_q\left(\begin{array}{ccc} a_1+2, & \cdots, & a_p+2 \ ; \\ b_1+2, & \cdots, & b_q+2 \ ; \end{array} x\right)
\end{aligned} \tag{3.38}$$

Theorem

$$y=\ {}_pF_q\left(\begin{array}{cccc} a_1, & a_2, & \cdots, & a_p \ ; \\ b_1, & b_2, & \cdots, & b_q \ ; \end{array} x\right)$$

is the solution of the differential equation.

Proof

$$\delta\left(\delta+b_1-1\right)\left(\delta+b_2-1\right)\cdots\left(\delta+b_q-1\right)y=x\left(\delta+a_1\right)\left(\delta+a_2\right)\cdots\left(\delta+a_p\right)y \tag{3.39}$$

where

$$\delta=x\frac{d}{dx}$$

Therefore

$$\delta x^k=x\frac{d}{dx}x^k=xkx^{k-1}=kx^k \tag{3.40}$$

$$\left(\delta+b_1-1\right)x^k=\delta x^k+\left(b_1-1\right)x^k=\left(k+b_1-1\right)x^k \tag{3.41}$$

$$\begin{aligned}
\left(\delta+b_1-1\right)\left(\delta+b_2-1\right)x^k&=\left(\delta+b_1-1\right)\left(k+b_2-1\right)x^k\\
&=\left(k+b_1-1\right)\left(k+b_2-1\right)x^k
\end{aligned} \tag{3.42}$$

Consider

$$
\begin{aligned}
&\delta\left(\delta+b_1-1\right)\left(\delta+b_2-1\right)\cdots\left(\delta+b_q-1\right)\ {}_pF_q\left(\begin{array}{cccc} a_1, & a_2, & \cdots, & a_p \ ; \\ b_1, & b_2, & \cdots, & b_q \ ; \end{array} x\right)\\
&=\delta\left(\delta+b_1-1\right)\left(\delta+b_2-1\right)\cdots\left(\delta+b_q-1\right)\sum_{k=0}^{\infty}\frac{\left(a_1\right)_k\left(a_2\right)_k\cdots\left(a_p\right)_k}{\left(b_1\right)_k\cdots\left(b_q\right)_k}\frac{x^k}{k!}\\
&=\sum_{k=0}^{\infty}\frac{\left(a_1\right)_k\left(a_2\right)_k\cdots\left(a_p\right)_k}{\left(b_1\right)_k\cdots\left(b_q\right)_k}\delta\left(\delta+b_1-1\right)\left(\delta+b_2-1\right)\cdots\left(\delta+b_q-1\right)\frac{x^k}{k!}\\
&=\sum_{k=0}^{\infty}\frac{\left(a_1\right)_k\left(a_2\right)_k\cdots\left(a_p\right)_k}{\left(b_1\right)_k\cdots\left(b_q\right)_k}k\left(k+b_1-1\right)\left(k+b_2-1\right)\cdots\left(k+b_q-1\right)\frac{x^k}{k!}\\
&=\sum_{k=0}^{\infty}\frac{\left(a_1\right)_k\left(a_2\right)_k\cdots\left(a_p\right)_k}{\left(b_1\right)_{k-1}\cdots\left(b_q\right)_{k-1}}\frac{x^k}{(k-1)!}\\
&=x\sum_{k=1}^{\infty}\frac{\left(a_1\right)_k\left(a_2\right)_k\cdots\left(a_p\right)_k}{\left(b_1\right)_{k-1}\cdots\left(b_q\right)_{k-1}}\frac{x^{k-1}}{(k-1)!}\\
&=x\sum_{k=1}^{\infty}\frac{\left(a_1\right)_{k+1}\left(a_2\right)_{k+1}\cdots\left(a_p\right)_{k+1}}{\left(b_1\right)_k\cdots\left(b_q\right)_k}\frac{x^k}{k!}\ \text{(by replacing } k \text{ by } k+1\text{)}\\
&=x\sum_{k=0}^{\infty}\frac{\left(a_1\right)_k\cdots\left(a_p\right)_k\left(a_1+k\right)\left(a_2+k\right)\cdots\left(a_p+k\right)}{\left(b_1\right)_k\cdots\left(b_q\right)_k}\frac{x^k}{k!}\\
&=x\sum_{k=0}^{\infty}\frac{\left(a_1\right)_k\cdots\left(a_p\right)_k}{\left(b_1\right)_k\cdots\left(b_q\right)_k}\left(a_1+\delta\right)\left(a_2+\delta\right)\cdots\left(a_p+\delta\right)\frac{x^k}{k!}\\
&=x\left(a_1+\delta\right)\left(a_2+\delta\right)\cdots\left(a_p+\delta\right)\sum_{k=0}^{\infty}\frac{\left(a_1\right)_k\cdots\left(a_p\right)_k}{\left(b_1\right)_k\cdots\left(b_q\right)_k}\frac{x^k}{k!}\\
&=x\left(a_1+\delta\right)\left(a_2+\delta\right)\cdots\left(a_p+\delta\right)\ {}_pF_q\left(\begin{array}{cccc} a_1 & a_2 & \cdots & a_p \ ; \\ b_1 & b_2 & \cdots & b_q \ ; \end{array} x\right) \qquad (3.43)
\end{aligned}
$$

3.4.2 Construction and properties of a generalized Hypergeometric distribution

By definition,

$$
{}_pF_q\left(\begin{array}{cccc} a_1, & a_2, & \cdots, & a_p \ ; \\ b_1, & b_2, & \cdots, & b_q \ ; \end{array} \theta\right)=\sum_{k=0}^{\infty}\frac{\left(a_1\right)_k\left(a_2\right)_k\cdots\left(a_p\right)_k}{\left(b_1\right)_k\cdots\left(b_q\right)_k}\frac{\theta^k}{k!}
$$

Therefore,

$$
1=\sum_{k=0}^{\infty}\frac{\left(a_1\right)_k\left(a_2\right)_k\cdots\left(a_p\right)_k}{\left(b_1\right)_k\cdots\left(b_q\right)_k\ {}_pF_q\left(a_1,\cdots,a_p;b_1,\cdots,b_q;\theta\right)}\frac{\theta^k}{k!}
$$

i.

$$
P_k=\Pr\left(X=k\right)=\frac{\left(a_1\right)_k\left(a_2\right)_k\cdots\left(a_p\right)_k}{\left(b_1\right)_k\cdots\left(b_q\right)_k}\frac{1}{{}_pF_q\left(a_1,\cdots,a_p;b_1,\cdots,b_q;\theta\right)}\frac{\theta^k}{k!} \qquad (3.44)
$$

for $k = 0, 1, 2, 3, \ldots$ is a generalized hypergeometric probability mass function. It belongs to the class of power series distributions is defined by

$$\Pr(X = k) = \frac{a_k \theta^k}{f(\theta)} \text{ for } k = 0, 1, 2, ... \text{ and } \theta > 0, \; a_k > 0,$$

ii.

$$a_k = \frac{(a_1)_k (a_2)_k \cdots (a_p)_k}{(b_1)_k \cdots (b_q)_k} \tag{3.45}$$

iii.

$$f(\theta) = {}_pF_q(a_1, \cdots, a_p; b_1, \cdots, b_q; \theta) \tag{3.46}$$

iv.

$$f'(\theta) = \frac{a_1 a_2 \cdots a_p}{b_1 b_2 \cdots b_q} \; {}_pF_q\left(\begin{matrix} a_1+1, & a_2+1, & \cdots, & a_p+1 & ; \\ b_1+1, & b_2+1, & \cdots, & b_q+1 & ; \end{matrix} \theta \right) \tag{3.47}$$

v.

$$f''(\theta) = \frac{a_1(a_1+1)\cdots a_p(a_p+1)}{b_1(b_1+1)\cdots b_q(b_q+1)} \; {}_pF_q\left(\begin{matrix} a_1+2, & a_2+2, & \cdots, & a_p+2 & ; \\ b_1+2, & b_2+2, & \cdots, & b_q+2 & ; \end{matrix} \theta \right) \tag{3.48}$$

vi.

$$E(X) = \frac{\theta f'(\theta)}{f(\theta)} = \theta \frac{\frac{a_1 a_2 \cdots a_p}{b_1 b_2 \cdots b_q} \; {}_pF_q\left(\begin{matrix} a_1+1, & a_2+1, & \cdots, & a_p+1 & ; \\ b_1+1, & b_2+1, & \cdots, & b_q+1 & ; \end{matrix} \theta \right)}{{}_pF_q(a_1, \cdots, a_p; b_1, \cdots, b_q; \theta)} \tag{3.49}$$

vii.

$$Var(X) = \frac{\theta^2 f''(\theta)}{f(\theta)} + \theta \frac{f'(\theta)}{f(\theta)} - \left[\theta \frac{f'(\theta)}{f(\theta)} \right]^2$$

viii. Probability Generating Function(pgf)

The pgf of X is given by

$$G(s) = \frac{f(\theta s)}{f(\theta)} = \frac{{}_pF_q(a_1, \cdots, a_p; b_1, \cdots, b_q; \theta s)}{{}_pF_q(a_1, \cdots, a_p; b_1, \cdots, b_q; \theta)} \tag{3.50}$$

Theorem: the pgf above satisfies the differential equation

$$\delta(\delta + b_1 - 1)\cdots(\delta + b_q - 1)\, G(s) = \theta s (\delta + a_1)\cdots(\delta + a_p)\, G(s)$$

Proof:

To derive the differential equation whose solution is $G(s)$ given in (3.50) let us use the operator.

$$\delta = s\frac{d}{ds}$$

Then

$$
\begin{aligned}
&\delta\left(\delta+b_1-1\right)\cdots\left(\delta+b_q-1\right)G\left(s\right)\\
&=\delta\left(\delta+b_1-1\right)\cdots\left(\delta+b_q-1\right)\sum_{k=0}^{\infty}p_k s^k\\
&=\sum_{k=0}^{\infty}p_k\delta\left(\delta+b_1-1\right)\cdots\left(\delta+b_q-1\right)s^k\\
&=\sum_{k=0}^{\infty}\frac{\left(a_1\right)_k\left(a_2\right)_k\cdots\left(a_p\right)_k}{\left(b_1\right)_k\cdots\left(b_q\right)_k}\delta\left(\delta+b_1-1\right)\cdots\left(\delta+b_q-1\right)\frac{1}{f\left(\theta\right)}\frac{\left(\theta s\right)^k}{k!}\\
&=\sum_{k=0}^{\infty}\frac{\left(a_1\right)_k\left(a_2\right)_k\cdots\left(a_p\right)_k k\left(k+b_1-1\right)\cdots\left(k+b_q-1\right)}{\left(b_1\right)_k\cdots\left(b_q\right)_k}\frac{1}{f\left(\theta\right)}\frac{\left(\theta s\right)^k}{k!}\\
&=\sum_{k=0}^{\infty}\frac{\left(a_1\right)_k\cdots\left(a_p\right)_k}{\left(b_1\right)_{k-1}\cdots\left(b_q\right)_{k-1}}\frac{1}{f\left(\theta\right)}\frac{\left(\theta s\right)^k}{\left(k-1\right)!}\\
&=\theta s\sum_{k=0}^{\infty}\frac{\left(a_1\right)_k\cdots\left(a_p\right)_k}{\left(b_1\right)_{k-1}\cdots\left(b_q\right)_{k-1}}\frac{1}{f\left(\theta\right)}\frac{\left(\theta s\right)^{k-1}}{\left(k-1\right)!}\\
&=\theta s\sum_{k=0}^{\infty}\frac{\left(a_1\right)_{k+1}\cdots\left(a_p\right)_{k+1}}{\left(b_1\right)_k\cdots\left(b_q\right)_k}\frac{1}{f\left(\theta\right)}\frac{\left(\theta s\right)^k}{k!}\\
&\text{by replacing } k \text{ by } k+1\\
&=\theta s\sum_{k=0}^{\infty}\frac{\left(a_1\right)_k\cdots\left(a_p\right)_k\left(a_1+k\right)\cdots\left(a_p+k\right)}{\left(b_1\right)_k\cdots\left(b_q\right)_k}\frac{1}{f\left(\theta\right)}\frac{\left(\theta s\right)^k}{k!}\\
&=\theta s\sum_{k=0}^{\infty}\frac{\left(a_1\right)_k\cdots\left(a_p\right)_k}{\left(b_1\right)_k\cdots\left(b_q\right)_k}\left(a_1+\delta\right)\cdots\left(a_p+\delta\right)\frac{1}{f\left(\theta\right)}\frac{\left(\theta s\right)^k}{k!}\\
&=\frac{\theta s\left(a_1+\delta\right)\cdots\left(a_p+\delta\right)}{{}_pF_q\left(a_1,\cdots,a_p;b_1,\cdots,b_q;\theta\right)}\sum_{k=0}^{\infty}\frac{\left(a_1\right)_k\cdots\left(a_p\right)_k}{\left(b_1\right)_k\cdots\left(b_q\right)_k}\frac{\left(\theta s\right)^k}{k!}\\
&=\theta s\left(a_1+\delta\right)\cdots\left(a_p+\delta\right)\frac{f\left(\theta s\right)}{f\left(\theta\right)}\\
&=\theta s\left(a_1+\delta\right)\cdots\left(a_p+\delta\right)G\left(s\right) \qquad (3.51)
\end{aligned}
$$

3.5 Special cases of confluent and Gauss Hypergeometric Distributions

3.5.1 Power Series Distributions Based on Exponential Expansion

Poisson Distribution

$$\begin{aligned} e^{\theta} &= \sum_{k=0}^{\infty} \frac{\theta^k}{k!} \\ &= \sum_{k=0}^{\infty} \frac{a(a+1)(a+2)\cdots(a+k-1)}{a(a+1)(a+2)\cdots(a+k-1)} \frac{\theta^k}{k!} \\ &= \ {}_1F_1(a;a;\theta) \end{aligned}$$

Alternatively,

$$\begin{aligned} e^{\theta} &= 1 + \frac{\theta}{1!} + \frac{\theta^2}{2!} + \frac{\theta^3}{3!} + \cdots \\ &= 1 + \frac{n}{1!}\left(\frac{\theta}{n}\right) + \frac{n^2}{2!}\left(\frac{\theta}{n}\right)^2 + \frac{n^3}{3!}\left(\frac{\theta}{n}\right)^3 + \cdots \\ &= \lim_{n \longrightarrow \infty} \left[1 + \frac{n}{1!}\left(\frac{\theta}{n}\right) + \frac{n^2}{2!}\left(\frac{\theta}{n}\right)^2 + \frac{n^3}{3!}\left(\frac{\theta}{n}\right)^3 + \cdots\right] \\ &= \lim_{n \longrightarrow \infty} \left[1 + \frac{n \cdot 1}{1}\left(\frac{\theta}{n}\right)\frac{1}{1!} + \frac{n(n+1)}{1 \cdot 2} 1 \cdot 2 \left(\frac{\theta}{n}\right)^2 \frac{1}{2!} + \cdots\right] \\ &= \lim_{n \longrightarrow \infty} {}_2F_1\left(n, 1; 1; \frac{\theta}{n}\right) \end{aligned}$$

The pmf is given by

$$\Pr(X = k) = \frac{\theta^k}{k! e^{\theta}} = \frac{e^{-\theta}\theta^k}{k!}, \ k = 0, 1, 2, ...$$

which is a Poisson Distribution with parameter θ.

Probability Generating Function(pgf)

$$G(s) = \frac{f(\theta s)}{f(\theta)} = \frac{e^{\theta s}}{e^{\theta}} \text{ i.e. } G(s) = \frac{{}_1F_1(a;a;\theta s)}{{}_1F_1(a;a;\theta)} \ \Rightarrow c = a$$

$$\begin{aligned} G'(1) = E(X) &= \frac{\theta a}{c} \frac{{}_1F_1(a+1;c+1;\theta)}{{}_1F_1(a;c;\theta)} \\ &= \theta \frac{{}_1F_1(a+1;a+1;\theta)}{{}_1F_1(a;a;\theta)} \end{aligned}$$

$$\begin{aligned} G''(1) &= \theta a - (c - \theta) G'(1) \ \text{ from (3.18)} \\ &= \theta a - (a - \theta)\theta \frac{{}_1F_1(a+1;a+1;\theta)}{{}_1F_1(a;a;\theta)} \end{aligned}$$

Therefore the variance is given by

$$
\begin{aligned}
Var(X) &= G''(1) + G'(1) - \left[G'(1)\right]^2 \\
&= \theta a + (1 - a + \theta)\,\theta \frac{{}_1F_1\left(a+1; a+1; \theta\right)}{{}_1F_1\left(a; a; \theta\right)} - \left[\theta \frac{{}_1F_1\left(a+1; a+1; \theta\right)}{{}_1F_1\left(a; a; \theta\right)}\right]^2
\end{aligned}
$$

3.5.2 Power Series Distributions Based on Binomial Expansions

a) Binomial distribution

$$
\begin{aligned}
&(1+\theta)^n \\
&= \sum_{k=0}^{n} \binom{n}{k} \theta^k \\
&= 1 + \binom{n}{1}\theta + \binom{n}{2}\theta^2 + \binom{n}{3}\theta^3 + \cdots + \binom{n}{n}\theta^n \\
&= 1 + \frac{n}{1}\theta + \frac{n\left(n-1\right)}{1\cdot 2}\theta^2 + \frac{n\left(n-1\right)\left(n-2\right)}{3!}\theta^3 + \cdots + n\left(n-1\right)\cdots 2\cdot 1\frac{\theta^n}{n!} \\
&= 1 + \left(-1\right)n\left(-1\right)\frac{\theta}{1} + \left(-1\right)^2 n\left(n-1\right)\left(-1\right)^2\frac{\theta^2}{2!} + \left(-1\right)^3 n\left(n-1\right)\left(n-2\right)\left(-1\right)^3\frac{\theta^3}{3!} \\
&+ \cdots + \left(-1\right)^n n\left(n-1\right)\cdots 2.1\left(-1\right)^n\frac{\theta^n}{n!} \\
&= 1 + \frac{\left(-n\right)\cdot 1}{1}\frac{\left(-\theta\right)}{1!} + \frac{\left(-n\right)\left(-n+1\right)\cdot 1\cdot 2}{1\cdot 2}\frac{\left(-\theta\right)^2}{2!} + \cdots \\
&+ \frac{\left(-n\right)\left(-n+1\right)\cdots\left(-n+n-1\right)}{1\cdot 2\cdot 3\cdots n}1\cdot 2\cdot 3\ldots n\frac{\left(-\theta\right)^n}{n!} \\
&= {}_2F_1\left(-n, 1; 1; -\theta\right)
\end{aligned}
$$

The pmf is given by

$$
\begin{aligned}
\Pr\left(X=k\right) &= \binom{n}{k}\frac{\theta^k}{\left(1+\theta\right)^n} \\
&= \binom{n}{k}\left(\frac{\theta}{1+\theta}\right)^k\left(\frac{1}{1+\theta}\right)^{n-k}\ ;\ k = 0, 1, 2, ..., n
\end{aligned}
$$

which is Binomial distribution with parameters n and $\frac{\theta}{(1+\theta)}$.

Probability generating function for Binomial distribution is given by

$$
\begin{aligned}
G(s) &= \frac{\left(1+\theta s\right)^n}{\left(1+\theta\right)^n} \\
&= \frac{{}_2F_1\left(-n, 1; 1; -\theta s\right)}{{}_2F_1\left(-n, 1; 1; -\theta\right)}
\end{aligned}
$$

$$
G'(1) = E\left(X\right) = \frac{ab}{c}\theta\frac{{}_2F_1\left(a+1, b+1; c+1; \theta\right)}{{}_2F_1\left(a, b; c; \theta\right)} \text{ refer to (3.29)}
$$

Put $a = -n$, $b = 1$, and $c = 1$

$\therefore$

$$G'(1) = E(X) = -n\theta \frac{{}_2F_1(-n+1,2;2;-\theta)}{{}_2F_1(-n,1;1;-\theta)}$$

$$G''(1) = \frac{\theta ab}{(1-\theta)} - \frac{[c-(a+1+b)\theta]}{(1-\theta)} G'(1) \quad \text{refer to (3.35)}$$

Therefore,

$$\begin{aligned}
G''(1) &= \frac{-\theta(-n)\cdot 1}{(1+\theta)} - \frac{[1-(-n+1+1)(-\theta)]}{(1+\theta)} G'(1) \\
&= \frac{n\theta}{(1+\theta)} - \frac{[1+(2-n)\theta]}{(1+\theta)} G'(1) \\
&= \frac{n\theta}{(1+\theta)} - \frac{[1+(2-n)\theta]}{(1+\theta)} \left[-n\theta \frac{{}_2F_1(-n+1,2;2;-\theta)}{{}_2F_1(-n,1;1;-\theta)}\right] \\
&= \frac{n\theta}{(1+\theta)} \left\{1 + [1+(2-n)\theta] \frac{{}_2F_1(-n+1,2;2;-\theta)}{{}_2F_1(-n,1;1;-\theta)}\right\}
\end{aligned}$$

Therefore the variance is given by

$$\begin{aligned}
Var(X) &= G''(1) + G'(1) - [G'(1)]^2 \\
&= \frac{n\theta}{(1+\theta)} + \left[\frac{[1+(2-n)\theta] n\theta}{(1+\theta)} - n\theta\right] \frac{{}_2F_1(-n+1,2;2;-\theta)}{{}_2F_1(-n,1;1;-\theta)} \\
&- \left[-n\theta \frac{{}_2F_1(-n+1,2;2;-\theta)}{{}_2F_1(-n,1;1;-\theta)}\right]^2 \\
&= \frac{n\theta}{(1+\theta)} + \left[\frac{[1+(2-n)\theta] n\theta - n\theta(1+\theta)}{(1+\theta)}\right] \frac{{}_2F_1(-n+1,2;2;-\theta)}{{}_2F_1(-n,1;1;-\theta)} \\
&- \left[n\theta \frac{{}_2F_1(-n+1,2;2;-\theta)}{{}_2F_1(-n,1;1;-\theta)}\right]^2 \\
&= \frac{n\theta}{(1+\theta)} \left\{ \begin{array}{c} 1 + (1-n)\theta \frac{{}_2F_1(-n+1,2;2;-\theta)}{{}_2F_1(-n,1;1;-\theta)} \\ -n\theta(1+\theta)\left[\frac{{}_2F_1(-n+1,2;2;-\theta)}{{}_2F_1(-n,1;1;-\theta)}\right]^2 \end{array} \right\}
\end{aligned}$$

b) Negative Binomial Distribution

$$
\begin{aligned}
(1-\theta)^{-\alpha} &= \sum_{k=0}^{\infty} \binom{-\alpha}{k} (-\theta)^k \qquad \alpha > 0 \qquad (3.35)\\
&= \sum_{k=0}^{\infty} (-1)^k \binom{-\alpha}{k} \theta^k \\
&= \sum_{k=0}^{\infty} \binom{\alpha+k-1}{k} \theta^k \\
&= \sum_{k=0}^{\infty} \frac{\Gamma\alpha+k}{\Gamma\alpha} \frac{\theta^k}{k!} \\
&= \sum_{k=0}^{\infty} (\alpha+k-1)(\alpha+k-2)\cdots(\alpha+k-k) \frac{\Gamma\alpha}{\Gamma\alpha} \frac{\theta^k}{k!} \\
&= \sum_{k=0}^{\infty} (\alpha+k-1)(\alpha+k-2)\cdots\alpha \frac{\theta^k}{k!} \\
&= \sum_{k=0}^{\infty} \frac{\alpha(\alpha+1)\cdots(\alpha+k-1)\, b(b+1)\cdots(b+k-1)}{b(b+1)\cdots(b+k-1)} \frac{\theta^k}{k!} \\
&= {}_2F_1(\alpha, b; b; \theta)
\end{aligned}
$$

The pmf is given by

$$
\Pr(X=k) = \binom{\alpha+k-1}{k} \theta^k (1-\theta)^{\alpha}, \text{ for } k = 0, 1, 2, ...
$$

which is a Negative Binomial Distribution.

Probability generating function for NB is given by

$$
\begin{aligned}
G(s) &= \frac{(1-\theta s)^{-\alpha}}{(1-\theta)^{-\alpha}} \\
&= \frac{{}_2F_1(\alpha, b; b; \theta s)}{{}_2F_1(\alpha, b; b; \theta)}
\end{aligned}
$$

$$
G'(1) = E(X) = \frac{ab}{c} \theta \frac{{}_2F_1(a+1, b+1; c+1; \theta)}{{}_2F_1(a, b; c; \theta)} \text{ refer to (3.29)}
$$

putting $a = \alpha$, $b = b$ and $c = b$.

$\therefore$

$$
G'(1) = E(X) = \alpha\theta \frac{{}_2F_1(\alpha+1, b+1; b+1; \theta)}{{}_2F_1(\alpha, b; b; \theta)}
$$

$$
G''(1) = \frac{a\theta b}{(1-\theta)} - \frac{[b-(a+1+b)\theta]}{(1-\theta)} G'(1) \quad \text{refer to (3.35)}
$$

Therefore,

$$\begin{aligned}
G''(1) &= \frac{\alpha\theta b}{(1-\theta)} - \frac{[b-(\alpha+1+b)\theta]}{(1-\theta)}G'(1) \\
&= \frac{\alpha\theta b}{(1-\theta)} - \frac{[b-(\alpha+1+b)\theta]}{(1-\theta)}\alpha\theta\frac{{}_2F_1(\alpha+1,b+1;b+1;\theta)}{{}_2F_1(\alpha,b;b;\theta)} \\
&= \frac{\alpha\theta b}{(1-\theta)}\left\{1-[b-(\alpha+1+b)\theta]\frac{1}{b}\frac{{}_2F_1(\alpha+1,b+1;b+1;\theta)}{{}_2F_1(\alpha,b;b;\theta)}\right\}
\end{aligned}$$

Therefore the variance is given by

$$\begin{aligned}
Var(X) &= G''(1)+G'(1)-[G'(1)]^2 \\
&= \frac{\alpha\theta b}{(1-\theta)}\left\{1+\left\{\begin{array}{c}1-\theta-b\\+(\alpha+1+b)\theta\end{array}\right\}\frac{1}{b}\frac{{}_2F_1(\alpha+1,b+1;b+1;\theta)}{{}_2F_1(\alpha,b;b;\theta)}\right\}-[G'(s)]^2 \\
&= \frac{\alpha\theta b}{(1-\theta)}\left\{\begin{array}{c}1+[1-b+(\alpha+b)\theta]\frac{1}{b}\frac{{}_2F_1(\alpha+1,b+1;b+1;\theta)}{{}_2F_1(\alpha,b;b;\theta)}\\ -\frac{\alpha\theta(1-\theta)}{b}\left[\frac{{}_2F_1(\alpha+1,b+1;b+1;\theta)}{{}_2F_1(\alpha,b;b;\theta)}\right]^2\end{array}\right\}
\end{aligned}$$

c) Logarithmic Series Distribution

$$\frac{1}{1-\theta} = 1+\theta+\theta^2+\cdots$$

which is obtained by putting $\alpha = 1$ in (3.35).

Therefore by Integrating both sides w.r.t θ we get

$$\int\frac{d\theta}{1-\theta} = \int\left[1+\theta+\theta^2+\cdots\right]d\theta$$

$$\begin{aligned}
-\log(1-\theta) &= \theta+\frac{\theta^2}{2}+\frac{\theta^3}{3}+\frac{\theta^4}{4}+\frac{\theta^5}{5}+\cdots \qquad (3.36)\\
&= \theta\left\{1+\frac{\theta}{2}+\frac{\theta^2}{3}+\frac{\theta^3}{4}+\frac{\theta^4}{5}+\cdots\right\} \\
&= \theta\left\{1+\frac{1!}{2}\frac{\theta}{1!}+\frac{2!}{3}\frac{\theta^2}{2!}+\frac{3!}{4}\frac{\theta^3}{3!}+\frac{4!}{5}\frac{\theta^4}{4!}+\cdots\right\} \\
&= \theta\left\{1+\frac{1\cdot 1}{2}\frac{\theta}{1!}+\frac{1\cdot 2}{3}\frac{\theta^2}{2!}+\frac{1\cdot 2\cdot 3}{4}\frac{\theta^3}{3!}+\frac{1\cdot 2\cdot 3\cdot 4}{5}\frac{\theta^4}{4!}+\cdots\right\} \\
&= \theta\left\{1+\frac{1\cdot 1}{2}\frac{\theta}{1!}+\frac{1\cdot 2\cdot 1\cdot 2}{1\cdot 2\cdot 3}\frac{\theta^2}{2!}+\frac{1\cdot 2\cdot 3\cdot 1\cdot 2\cdot 3}{1\cdot 2\cdot 3\cdot 4}\frac{\theta^3}{3!}+\cdots\right\} \\
&= \theta\left\{1+\frac{1\cdot 1}{2}\frac{\theta}{1!}+\frac{1\cdot 2\cdot 1\cdot 2}{2\cdot 3}\frac{\theta^2}{2!}+\frac{1\cdot 2\cdot 3\cdot 1\cdot 2\cdot 3}{2\cdot 3\cdot 4}\frac{\theta^3}{3!}+\cdots\right\} \\
&= \theta{}_2F_1(1,1;2;\theta)
\end{aligned}$$

$\therefore$

$$-\log(1-\theta) = \theta{}_2F_1(1,1;2;\theta)$$

The pmf is given by

$$\Pr(X=k) = \frac{\theta^k}{-k\log(1-\theta)},\ \ k=1,2,\ldots;0<\theta<1$$

which is called Logarithmic Series Distribution.

Probability generating function for Logarithmic Series Distribution is given by

$$\begin{aligned}
G(s) &= \frac{\log(1-\theta s)}{\log(1-\theta)} \\
&= \frac{\theta s {}_2F_1(1,1;2;\theta s)}{\theta {}_2F_1(1,1;2;\theta)} \\
&= s\frac{{}_2F_1(1,1;2;\theta s)}{{}_2F_1(1,1;2;\theta)}
\end{aligned}$$

$$G'(1) = E(X) = \frac{ab}{c}\theta\frac{{}_2F_1(a+1,b+1;c+1;\theta)}{{}_2F_1(a,b;c;\theta)} \quad \text{refer to (3.29)}$$

setting $a = 1$, $b = 1$ and $c = 2$

$\therefore$

$$G'(1) = E(X) = \frac{1}{2}\theta\frac{{}_2F_1(2,2;3;\theta)}{{}_2F_1(1,1;2;\theta)}$$

$$G''(1) = \frac{a\theta b}{(1-\theta)} - \frac{[c-(a+1+b)\theta]}{(1-\theta)}G'(1) \quad \text{refer to (3.35)}$$

Therefore,

$$\begin{aligned}
G''(1) &= \frac{\theta}{(1-\theta)} - \frac{[2-3\theta]}{(1-\theta)}G'(1) \\
&= \frac{\theta}{(1-\theta)} - \frac{[2-3\theta]}{(1-\theta)}\frac{1}{2}\theta\frac{{}_2F_1(2,2;3;\theta)}{{}_2F_1(1,1;2;\theta)} \\
&= \frac{\theta}{2(1-\theta)}\left\{2-[2-3\theta]\frac{{}_2F_1(2,2;3;\theta)}{{}_2F_1(1,1;2;\theta)}\right\}
\end{aligned}$$

Therefore the variance is given by

$$\begin{aligned}
Var(X) &= G''(1) + G'(1) - [G'(1)]^2 \\
&= \frac{\theta}{(1-\theta)} - \left\{\frac{(2-3\theta-1+\theta)\theta}{2(1-\theta)}\right\}\frac{{}_2F_1(2,2;3;\theta)}{{}_2F_1(1,1;2;\theta)} - \left[\frac{1}{2}\theta\frac{{}_2F_1(2,2;3;\theta)}{{}_2F_1(1,1;2;\theta)}\right]^2 \\
&= \frac{\theta}{2(1-\theta)}\left\{2-(1-2\theta)\frac{{}_2F_1(2,2;3;\theta)}{{}_2F_1(1,1;2;\theta)} - \frac{\theta(1-\theta)}{2}\left[\frac{{}_2F_1(2,2;3;\theta)}{{}_2F_1(1,1;2;\theta)}\right]^2\right\}
\end{aligned}$$

d) Replacing θ by $-\theta$ in c

we obtain

$$\begin{aligned}
-\log(1+\theta) &= -\theta + \frac{(-\theta)^2}{2} + \frac{(-\theta)^3}{3} + \frac{(-\theta)^4}{4} + \frac{(-\theta)^5}{5} + \cdots \\
&= -\theta + \frac{\theta^2}{2} - \frac{\theta^3}{3} + \frac{\theta^4}{4} - \frac{\theta^5}{5} + \cdots
\end{aligned}$$

$$
\begin{aligned}
\log(1+\theta) &= \theta - \frac{\theta^2}{2} + \frac{\theta^3}{3} - \frac{\theta^4}{4} + \frac{\theta^5}{5} - \cdots \\
&= \theta\left\{1 - \frac{\theta}{2} + \frac{\theta^2}{3} - \frac{\theta^3}{4} + \frac{\theta^4}{5} - \cdots\right\} \\
&= \theta\left\{1 + \frac{1}{2}(-\theta) + \frac{1}{3}(-\theta)^2 + \frac{1}{4}(-\theta)^3 + \frac{1}{5}(-\theta)^4 + \cdots\right\} \\
&= \theta\left\{1 + \frac{1!}{2}\frac{(-\theta)}{1!} + \frac{2!}{3}\frac{(-\theta)^2}{2!} + \frac{3!}{4}\frac{(-\theta)^3}{3!} + \frac{4!}{5}\frac{(-\theta)^4}{4!} + \cdots\right\} \\
&= \theta\left\{1 + \frac{1\cdot 1}{2}\frac{(-\theta)}{1!} + \frac{1\cdot 2}{3}\frac{(-\theta)^2}{2!} + \frac{1\cdot 2\cdot 3}{4}\frac{(-\theta)^3}{3!} + \frac{1\cdot 2\cdot 3\cdot 4}{5}\frac{(-\theta)^4}{4!} + \cdots\right\} \\
&= \theta\left\{1 + \frac{1\cdot 1}{2}\frac{(-\theta)}{1!} + \frac{1\cdot 2\cdot 1\cdot 2}{1\cdot 2\cdot 3}\frac{(-\theta)^2}{2!} + \frac{1\cdot 2\cdot 3\cdot 1\cdot 2\cdot 3}{1\cdot 2\cdot 3\cdot 4}\frac{(-\theta)^3}{3!} + \cdots\right\} \\
&= \theta\left\{1 + \frac{1\cdot 1}{2}\frac{(-\theta)}{1!} + \frac{1\cdot 2\cdot 1\cdot 2}{2\cdot 3}\frac{(-\theta)^2}{2!} + \frac{1\cdot 2\cdot 3\cdot 1\cdot 2\cdot 3}{2\cdot 3\cdot 4}\frac{(-\theta)^3}{3!} + \cdots\right\} \\
&= \theta \; {}_2F_1(1,1;2;-\theta)
\end{aligned}
$$

$\therefore$

$$
-\log(1+\theta) = \theta \; {}_2F_1(1,1;2;-\theta)
$$

since

$$
\begin{aligned}
-\log(1+\theta) &= -\theta + \frac{(-\theta)^2}{2} + \frac{(-\theta)^3}{3} + \frac{(-\theta)^4}{4} + \frac{(-\theta)^5}{5} + \cdots \\
&= \sum_{k=1}^{\infty} \frac{(-\theta)^k}{k}
\end{aligned}
$$

Then

$$
\begin{aligned}
1 &= \sum_{k=1}^{\infty} \frac{(-\theta)^k}{-k\log(1+\theta)} \\
&= \sum_{k=1}^{\infty} (-1)^{k-1} \frac{\theta^k}{k\log(1+\theta)}
\end{aligned}
$$

Thus

$$
\begin{aligned}
\Pr(X=k) &= (-1)^{k-1}\frac{\theta^k}{k\log(1+\theta)};\; k = 1,2,3,\ldots \\
&= \frac{(-\theta)^k}{-k\log(1+\theta)}
\end{aligned}
$$

Probability generating function for the distribution is given by

$$
\begin{aligned}
G(s) &= \frac{f(\theta s)}{f(\theta)} = \frac{\log(1+\theta s)}{\log(1+\theta)} \\
&= \frac{\theta s {}_2F_1(1,1;2;-\theta s)}{\theta {}_2F_1(1,1;2;-\theta)} \\
&= s\frac{{}_2F_1(1,1;2;-\theta s)}{{}_2F_1(1,1;2;-\theta)}
\end{aligned}
$$

$$G'(1) = E(X) = \frac{ab}{c}\theta\frac{{}_2F_1(a+1, b+1; c+1; \theta)}{{}_2F_1(a, b; c; \theta)} \quad \text{refer to (3.29)}$$

setting $a = 1$, $b = 1$, and $c = 2$

$\therefore$

$$G'(1) = E(X) = \frac{1}{2}\theta\frac{{}_2F_1(2, 2; 3; -\theta)}{{}_2F_1(1, 1; 2; -\theta)}$$

$$G''(1) = \frac{a\theta b}{(1-\theta)} - \frac{[c-(a+1+b)\theta]}{(1-\theta)}G'(1) \quad \text{refer to (3.35)}$$

Therefore,

$$\begin{aligned} G''(1) &= \frac{\theta}{(1-\theta)} - \frac{[2-3\theta]}{(1-\theta)}G'(1) \\ &= \frac{\theta}{(1-\theta)} - \frac{[2-3\theta]}{(1-\theta)}\frac{1}{2}\theta\frac{{}_2F_1(2, 2; 3; -\theta)}{{}_2F_1(1, 1; 2; -\theta)} \\ &= \frac{\theta}{2(1-\theta)}\left\{2 - [2-3\theta]\frac{{}_2F_1(2, 2; 3; -\theta)}{{}_2F_1(1, 1; 2; -\theta)}\right\} \end{aligned}$$

Therefore the variance is given by

$$\begin{aligned} Var(X) &= G''(1) + G'(1) - [G'(1)]^2 \\ &= \frac{\theta}{(1-\theta)} - \left\{\frac{(2-3\theta-1+\theta)\theta}{2(1-\theta)}\right\}\frac{{}_2F_1(2, 2; 3; -\theta)}{{}_2F_1(1, 1; 2; -\theta)} - \left[\frac{1}{2}\theta\frac{{}_2F_1(2, 2; 3; -\theta)}{{}_2F_1(1, 1; 2; -\theta)}\right]^2 \\ &= \frac{\theta}{2(1-\theta)}\left\{2 - \{1-2\theta\}\frac{{}_2F_1(2, 2; 3; -\theta)}{{}_2F_1(1, 1; 2; -\theta)} - \frac{\theta(1-\theta)}{2}\left[\frac{{}_2F_1(2, 2; 3; -\theta)}{{}_2F_1(1, 1; 2; -\theta)}\right]^2\right\} \end{aligned}$$

e)

$$f(\theta) = \log\left(\frac{1+\theta}{1-\theta}\right) = \log(1+\theta) - \log(1-\theta)$$

$$
\begin{aligned}
f(\theta) &= \log(1+\theta) - \log(1-\theta) \\
&= \left\{\theta - \frac{\theta^2}{2} + \frac{\theta^3}{3} - \frac{\theta^4}{4} + \cdots\right\} + \left\{\theta + \frac{\theta^2}{2} + \frac{\theta^3}{3} + \frac{\theta^4}{4} + \cdots\right\} \\
&= 2\theta + \frac{2\theta^3}{3} + \frac{2\theta^5}{5} + \cdots \\
&= 2\left[\theta + \frac{\theta^3}{3} + \frac{\theta^5}{5} + \cdots\right] \\
&= 2\sum_{k=0}^{\infty} \frac{\theta^{2k+1}}{2k+1} \\
&= 2\theta\left[1 + \frac{\theta^2}{3} + \frac{\theta^4}{5} + \frac{\theta^6}{7} + \cdots\right] \\
&= 2\theta\left[1 + \frac{1\cdot 1}{3}\frac{\theta^2}{1!} + \frac{1\cdot 2!}{5}\frac{\left(\theta^2\right)^2}{2!} + \frac{1\cdot 3!}{7}\frac{\left(\theta^2\right)^3}{3!} + \cdots\right] \\
&= 2\theta\left[1 + \frac{1\cdot 1}{3}\cdot\frac{2}{2}\cdot\frac{\theta^2}{1!} + \frac{1\cdot 2}{5}\frac{\left(\theta^2\right)^2}{2!} + \frac{1\cdot 2\cdot 3}{7}\frac{\left(\theta^2\right)^3}{3!} + \cdots\right] \\
&= 2\theta\left[1 + \frac{\frac{1}{2}\cdot 1}{\frac{3}{2}}\frac{\theta^2}{1!} + \frac{\frac{1}{2}\cdot\frac{3}{2}\cdot 1\cdot 2}{\frac{3}{2}\cdot\frac{5}{2}}\frac{\left(\theta^2\right)^2}{2!} + \frac{\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}\cdot 1\cdot 2\cdot 3}{\frac{3}{2}\cdot\frac{5}{2}\cdot\frac{7}{2}}\frac{\left(\theta^2\right)^3}{3!} + \cdots\right] \\
&= 2\theta\ {}_2F_1\left(\frac{1}{2}, 1; \frac{3}{2}; \theta^2\right)
\end{aligned}
$$

$$
\Pr(X=k) = \frac{2\theta^{2k+1}}{(2k+1)\log\left(\frac{1+\theta}{1-\theta}\right)}; k = 0, 1, 2, \ldots
$$

Probability generating function for Distribution is given by

$$
\begin{aligned}
G(s) &= \frac{\log(1+\theta s) - \log(1-\theta s)}{\log(1+\theta) - \log(1-\theta)} \\
&= s\frac{{}_2F_1\left(\frac{1}{2}, 1; \frac{3}{2}; \theta^2 s\right)}{{}_2F_1\left(\frac{1}{2}, 1; \frac{3}{2}; \theta^2\right)}
\end{aligned}
$$

$$
G'(1) = E(X) = \frac{ab}{c}\theta\frac{{}_2F_1(a+1, b+1; c+1; \theta)}{{}_2F_1(a, b; c; \theta)} \quad \text{refer to (3.29)}
$$

setting $a = \frac{1}{2}$, $b = 1$, and $c = \frac{3}{2}$

$\therefore$

$$
G'(1) = E(X) = \frac{2}{6}\theta\frac{{}_2F_1\left(\frac{3}{2}, 2; \frac{5}{2}; \theta^2\right)}{{}_2F_1\left(\frac{1}{2}, 1; \frac{3}{2}; \theta^2\right)}
$$

$$
G''(1) = \frac{a\theta b}{(1-\theta)} - \left[\frac{[c - (a+1+b)\theta]}{(1-\theta)}\right]G'(1) \quad \text{refer to (3.35)}
$$

Therefore,

$$
\begin{aligned}
G''(1) &= \frac{1}{2}\frac{\theta}{1-\theta} - \frac{\left[\frac{3}{2} - \frac{5}{2}\theta\right]}{(1-\theta)}G'(1) \\
&= \frac{\theta}{2(1-\theta)} - \frac{2\left[\frac{3}{2} - \frac{5}{2}\theta\right]}{6(1-\theta)}\theta\frac{{}_2F_1\left(\frac{3}{2}, 2; \frac{5}{2}; \theta^2\right)}{{}_2F_1\left(\frac{1}{2}, 1; \frac{3}{2}; \theta^2\right)}
\end{aligned}
$$

Therefore the variance is given by

$$\begin{aligned}
Var(X) &= G''(1) + G'(1) - [G'(1)]^2 \\
&= \frac{\theta}{2(1-\theta)} - \frac{2\left\{\frac{3}{2} - \frac{5}{2}\theta - 1 + \theta\right\}}{6(1-\theta)}\theta\frac{{}_2F_1\left(\frac{3}{2}, 2; \frac{5}{2}; \theta^2\right)}{{}_2F_1\left(\frac{1}{2}, 1; \frac{3}{2}; \theta^2\right)} - \left[\frac{2}{6}\theta\frac{{}_2F_1\left(\frac{3}{2}, 2; \frac{5}{2}; \theta^2\right)}{{}_2F_1\left(\frac{1}{2}, 1; \frac{3}{2}; \theta^2\right)}\right]^2 \\
&= \frac{\theta}{2(1-\theta)}\left\{1 - \frac{(1-3\theta)}{3(1-\theta)}\frac{{}_2F_1\left(\frac{3}{2}, 2; \frac{5}{2}; \theta^2\right)}{{}_2F_1\left(\frac{1}{2}, 1; \frac{3}{2}; \theta^2\right)} - \frac{4\theta}{18}\left[\frac{{}_2F_1\left(\frac{3}{2}, 2; \frac{5}{2}; \theta^2\right)}{{}_2F_1\left(\frac{1}{2}, 1; \frac{3}{2}; \theta^2\right)}\right]^2\right\}
\end{aligned}$$

3.5.3 Power series distributions based on trigonometric inverses

a) Inverse sine

From chapter two subsection 2.9.6

We have shown that

$$\begin{aligned}
\sin^{-1}\theta &= \sum_{k=0}^{\infty}(-1)^k\binom{-\frac{1}{2}}{k}\frac{\theta^{2k+1}}{2k+1}\\
&= \theta+\sum_{k=1}^{\infty}(-1)^k\binom{-\frac{1}{2}}{k}\frac{\theta^{2k+1}}{2k+1}\\
&= \theta+\sum_{k=1}^{\infty}(-1)^k\frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)\left(-\frac{1}{2}-1\right)\cdots\left[-\frac{1}{2}-(k-1)\right]}{1\cdot 2\cdot 3\cdots k}\frac{\theta^{2k+1}}{2k+1}\\
&= \theta+\sum_{k=1}^{\infty}(-1)^k\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\cdots\left[-\frac{(2k-1)}{2}\right]}{1\cdot 2\cdot 3\cdot\ldots\cdot k}\frac{\theta^{2k+1}}{2k+1}\\
&= \theta+\sum_{k=1}^{\infty}(-1)^k(-1)^k\frac{\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}\cdot\ldots\cdot\frac{2k-1}{2}}{1\cdot 2\cdot 3\cdot\ldots\cdot k}\frac{\theta^{2k+1}}{2k+1}\\
&= \theta+\theta\sum_{k=1}^{\infty}\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}\cdot\ldots\cdot\frac{2k-1}{2}\frac{1}{2k+1}\frac{\theta^{2k}}{k!}\\
&= \theta\left\{1+\sum_{k=1}^{\infty}\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}\cdot\ldots\cdot\frac{2k-1}{2}\frac{1}{2k+1}\frac{\theta^{2k}}{k!}\right\}\\
&= \theta\left\{1+\frac{1}{2}\cdot\frac{1}{3}\frac{\theta^2}{1!}+\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{1}{5}\frac{\left(\theta^2\right)^2}{2!}+\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}\cdot\frac{1}{7}\frac{\left(\theta^2\right)^3}{3!}+\cdots\right\}\\
&= \theta\left\{1+\frac{1}{2}\cdot\frac{1}{3}\cdot\frac{2}{2}\frac{\theta^2}{1!}+\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{1}{5}\frac{\left(\theta^2\right)^2}{2!}+\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}\cdot\frac{1}{7}\frac{\left(\theta^2\right)^3}{3!}+\cdots\right\}\\
&= \theta\left\{1+\frac{\frac{1}{2}\cdot\frac{1}{2}}{\frac{1}{2}\cdot 3}\frac{\theta^2}{1!}+\frac{\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\cdot\frac{3}{2}}{5\cdot\frac{1}{2}\cdot\frac{3}{2}}\frac{\left(\theta^2\right)^2}{2!}+\frac{\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}\cdot\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}}{\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}\cdot 7}\frac{\left(\theta^2\right)^3}{3!}+\cdots\right\}\\
&= \theta\left\{1+\frac{\frac{1}{2}\cdot\frac{1}{2}}{\frac{3}{2}}\frac{\theta^2}{1!}+\frac{\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\cdot\frac{3}{2}}{\frac{3}{2}\cdot\frac{5}{2}}\frac{\left(\theta^2\right)^2}{2!}+\frac{\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}\cdot\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}}{\frac{3}{2}\cdot\frac{5}{2}\cdot\frac{7}{2}}\frac{\left(\theta^2\right)^3}{3!}+\cdots\right\}\\
&= \theta\ {}_2F_1\left(\frac{1}{2},\frac{1}{2};\frac{3}{2};\theta^2\right)
\end{aligned}$$

Therefore

$$\Pr\left(X=2k+1\right)=(-1)^k\binom{-\frac{1}{2}}{k}\frac{1}{\sin^{-1}\theta}\frac{\theta^{2k+1}}{2k+1},\ \ k=1,2,3\text{...and } 0<\theta<1$$

The Probability generating function for Distribution is given by

$$\begin{aligned}
G\left(s\right) &= \frac{f\left(\theta s\right)}{f\left(\theta\right)}\\
&= \frac{s{}_2F_1\left(\frac{1}{2},\frac{1}{2};\frac{3}{2};\theta^2 s\right)}{{}_2F_1\left(\frac{1}{2},\frac{1}{2};\frac{3}{2};\theta^2\right)}
\end{aligned}$$

b) Inverse tan

Expanding $\left(1+\theta^2\right)^{-1}$

we obtain

$$\left(1+\theta^2\right)^{-1} = \sum_{k=0}^{\infty} \binom{-1}{k} \left(\theta^2\right)^k$$

Integrating both sides with respect to θ we get

$$\int \left(1+\theta^2\right)^{-1} d\theta = \sum_{k=0}^{\infty} \binom{-1}{k} \int \left(\theta^2\right)^k d\theta$$

Let $\theta = \tan x \Rightarrow \tan^{-1}\theta = x$ and $d\theta = \sec^2 x dx$

$\therefore$

$$LHS = \int \left(1+\theta^2\right)^{-1} d\theta = \int \frac{\sec^2 x}{1+\tan^2 x} dx = \int dx = x$$

$$RHS = \sum_{k=0}^{\infty} \binom{-1}{k} \frac{\theta^{2k+1}}{2k+1}$$

$\therefore$

$$x = \sum_{k=0}^{\infty} \binom{-1}{k} \frac{\theta^{2k+1}}{2k+1}$$

i.e.

$$\tan^{-1}\theta = \sum_{k=0}^{\infty} \binom{-1}{k} \frac{\theta^{2k+1}}{2k+1}$$

$\therefore$

$$1 = \sum_{k=0}^{\infty} \binom{-1}{k} \frac{1}{\tan^{-1}\theta} \frac{\theta^{2k+1}}{2k+1}; k = 0, 1, 2, \ldots$$

$\therefore$

$$\Pr\left(X = 2k+1\right) = \binom{-1}{k} \frac{1}{\tan^{-1}\theta} \frac{\theta^{2k+1}}{2k+1}; k = 0, 1, 2, \ldots$$

Now

$$\begin{aligned}
\tan^{-1}\theta &= \theta - \frac{\theta^3}{3} + \frac{\theta^5}{5} - \frac{\theta^7}{7} \pm \cdots \\
&= \theta\left\{1 - \frac{\theta^2}{3} + \frac{\theta^4}{5} - \frac{\theta^6}{7} \pm \cdots\right\} \\
&= \theta\left\{1 + \frac{(-\theta^2)}{3} + \frac{(-\theta^2)^2}{5} + \frac{(-\theta^2)^2}{7} + \cdots\right\} \\
&= \theta\left\{1 + \frac{1\cdot 1}{3}\frac{(-\theta^2)}{1!} + \frac{1\cdot 2!}{5}\frac{(-\theta^2)^2}{2!} + \frac{1\cdot 3!}{7}\frac{(-\theta^2)^2}{3!} + \cdots\right\} \\
&= \theta\left\{1 + \frac{\frac{1}{2}\cdot 1}{3\cdot\frac{1}{2}}\frac{(-\theta^2)}{1!} + \frac{1\cdot 2!}{5}\frac{(-\theta^2)^2}{2!} + \frac{1\cdot 3!}{7}\frac{(-\theta^2)^2}{3!} + \cdots\right\} \\
&= \theta\left\{1 + \frac{\frac{1}{2}\cdot 1}{\frac{1}{2}\cdot 3}\frac{(-\theta^2)}{1!} + \frac{\frac{1}{2}\cdot\frac{3}{2}\cdot 1\cdot 2}{\frac{1}{2}\cdot\frac{3}{2}\cdot 5}\frac{(-\theta^2)^2}{2!} + \frac{\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}\cdot 1\cdot 2\cdot 3}{\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}\cdot 7}\frac{(-\theta^2)^2}{3!} + \cdots\right\} \\
&= \theta\left\{1 + \frac{\frac{1}{2}\cdot 1}{\frac{3}{2}}\frac{(-\theta^2)}{1!} + \frac{\frac{1}{2}\cdot\frac{3}{2}\cdot 1\cdot 2}{\frac{3}{2}\cdot\frac{5}{2}}\frac{(-\theta^2)^2}{2!} + \frac{\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}\cdot 1\cdot 2\cdot 3}{\frac{3}{2}\cdot\frac{5}{2}\cdot\frac{7}{2}}\frac{(-\theta^2)^2}{3!} + \cdots\right\} \\
&= \theta\ {}_2F_1\left(\frac{1}{2}, 1; \frac{3}{2}; -\theta^2\right)
\end{aligned}$$

The Probability generating function for Distribution is given by

$$\begin{aligned}
G(s) &= \frac{f(\theta s)}{f(\theta)} \\
&= \frac{s\ {}_2F_1\left(\frac{1}{2}, 1; \frac{3}{2}; -\theta^2 s\right)}{{}_2F_1\left(\frac{1}{2}, 1; \frac{3}{2}; -\theta^2\right)}
\end{aligned}$$

3.5.4 Power series distributions based on special functions

a) Legendre Polynomials $P_l(x)$

Generating Function is given by:

$$\left(1 - 2xt + t^2\right)^{-\frac{1}{2}} = \sum_{l=0}^{\infty} P_l(x)\, t^l; |t| < 1, |x| \le 1$$

The expression function $P_l(x)$:

$$P_l(x) = \frac{1}{2^l}\sum_{v=0}^{\frac{l}{2}} \frac{(-1)^v (2l - 2v)!}{v!\,(l - v)!\,(l - 2v)!} x^{l-2v}$$

Differential equation:

$$\left(1 - x^2\right) P_l''(x) - 2xP_l'(x) + l(l+1) P_l(x) = 0$$

Recurrence relation:

$$lP_{l-1} - (2l+1)\,xP_l + (l+1)\,P_l\,(x) = 0$$

Examples:

$$P_0 = 1, \quad P_1 = x, \quad P_2 = \frac{1}{2}\left(3x^2 - 1\right), \quad P_3 = \frac{1}{2}x\left(5x^2 - 3\right)$$

Hypergeometric Functions:

$$P_l\,(x) = {}_2F_1\left(-l, l+1; 1; \frac{1-x}{2}\right)$$

b) Bessel Function of the First Kind: $J_n\,(x)$

Generating Function is given by:

$$e^{\frac{x}{2}\left(t-\frac{1}{t}\right)} = \sum_{n=-\infty}^{\infty} J_n\,(x)\,t^n$$

Expression for $J_n\,(x)$:

$$J_n\,(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,(n+k)!}\left(\frac{x}{2}\right)^{n+2k}$$
$$J_{-n}\,(x) = (-1)^n\,J_n$$

Differential equation:

$$x^2 J_n''\,(x) + xJ_n'\,(x) + \left(x^2 - n^2\right) J_n\,(x) = 0$$

Recurrence relations:

$$\frac{d}{dx}\left[x^n J_n\,(x)\right] = x^n J_{n-1}\,(x)\,;$$
$$J_{n-1}\,(x) + J_{n+1}\,(x) = \frac{2n}{x} J_n\,(x)\,;$$

$$\begin{aligned} J_n'\,(x) &= J_{n-1}\,(x) - \frac{n}{x} J_n\,(x) \\ &= \frac{n}{x} J_n\,(x) - J_{n+1}\,(x) \\ &= \frac{1}{2}\left\{J_{n-1}\,(x) - J_{n+1}\,(x)\right\} \end{aligned}$$

Hypergeometric functions:

$$\begin{aligned} J_n\,(x) &= \left(\frac{x}{2}\right)^n \frac{1}{n!}\,{}_1F_2\left(a; a, n+1; -\left(\frac{x}{2}\right)^2\right) \\ &= \left(\frac{x}{2}\right)^n \frac{1}{n!}\,{}_0F_1\left(n+1; -\left(\frac{x}{2}\right)^2\right) \end{aligned}$$

Alternatives

$$J_n\,(x) = \frac{\left(\frac{x}{2}\right)^n}{n!} e^{-it}\,{}_1F_1\left(n + \frac{1}{2}; 2n+1; 2ix\right)$$

c) Hermite Polynomials $H_n(x)$

Generating function:

$$e^{-t^2+2tx} = \sum_{n=0}^{\infty} H_n(x)\frac{t^n}{n!}$$

Expressions for $H_n(x)$:

$$H_n(-x) = (-1)^n H_n(x);$$
$$H_n(x) = (-1)^n e^{x^2}\left(\frac{d}{dx}\right)^n H_n(x);$$

$$H_n(x) = (-1)^{\frac{n}{2}} n! \sum_{k=0}^{\frac{n}{2}} (-1)^k \frac{(2x)^{2k}}{(2k)!\left(\frac{1}{2}n-k\right)!}; \text{if } n \text{ is even}$$

$$H_n(x) = (-1)^{\frac{n-1}{2}} n! \sum_{k=0}^{\frac{n-1}{2}} (-1)^k \frac{(2x)^{2k+1}}{(2k+1)!\left(\frac{n-1}{2}-k\right)!}; \text{if } n \text{ is odd}$$

Differential equation:

$$H_n''(x) - 2xH_n'(x) + 2nH_n(x) = 0$$

or

$$\frac{d^2}{dx^2}H_n(x)e^{-\frac{1}{2}x^2} + \left(2n - x^2 + 1\right)H_n(x)e^{-\frac{1}{2}x^2} = 0$$

Recurrence relations:

$$\frac{d^m}{dx^m}H_n(x) = \frac{2^m n!}{(n-m)!}H_{n-1}(x)$$

$$xH_n(x) = \frac{1}{2}H_{n+1}(x) + nH_{n-1}(x)$$
$$H_n(x) = \left(2x - \frac{d}{dx}\right)H_{n-1}(x)$$

Examples:

$$H_0(x) = 1, \qquad H_1(x) = 2x, \qquad H_2(x) = 4x^2 - 2$$

Hypergeometric functions:

$$H_{2n}(x) = (-1)^n \frac{(2n)!}{n!}\ {}_1F_1\left(-n; \frac{3}{2}; x^2\right)$$

$$H_{2n+1}(x) = (-1)^n \frac{2(2n+1)!}{n!}\ {}_1F_1\left(-n; \frac{3}{2}; x^2\right)$$

d) Laguerre Polynomials $L_n(x)$

Generating function:

$$\frac{1}{1-z}e^{-\frac{xz}{1-z}} = \sum_{n=0}^{\infty} L_n(x)\, z^n$$

Where

$$L_n(x) = \frac{e^x}{n!}\left(\frac{d}{dx}\right)^n \left(x^n e^{-x}\right)$$

$$L_n(x) = \frac{(-1)^n}{n!}\left(x^n - \frac{n^2 x^{n-1}}{1!} + \frac{n^2(n-1)^2 x^{n-2}}{2!} - \cdots + (-1)^n n!\right)$$

Differential equation:

$$xL_n''(x) + (1-x)\,L_n'(x) + nL_n(x) = 0$$

Recurrence relation:

$$(1+2n-x)\,L_n(x) - nL_{n-1}(x) - (n+1)\,L_{n+1}(x) = 0$$

$$xL_n'(x) = nL_n(x) - nL_{n-1}(x)$$

Examples:

$$L_n(x) = 1, L_n(x) = 1-x, L_n(x) = \frac{1}{2!}\left(x^2 - 4x + 2\right)$$

Hypergeometric function:

$$L_n(x) = {}_1F_1(-n; 1; x)$$

e) Associated Laguerre Polynomials $L_n^k(x)$

Generating function:

$$\frac{1}{(1-z)^{k+1}}e^{-\frac{xz}{1-z}} = \sum_{n=0}^{\infty} L_n^k(x)\, z^n$$

$$\sum_{k=0}^{\infty}\sum_{n=k}^{\infty} \frac{L_n^k(x)\, z^n u^k}{k!} = \frac{1}{1-z}\exp\left(\frac{-xz+u}{1-z}\right)$$

Expression for L_n^k

$$L_n^k(x) = (-1)^k \left(\frac{d}{dx}\right)^k L_{n+k}(x)$$

$$L_n^k(x) = \frac{e^x x^{-x}}{n!}\left(\frac{d}{dx}\right)^n \left(x^{n+k} eh - x\right)$$

Differential equation:

$$L_n^{k\prime\prime}(x) + (k+1-x)\,L_n^{k\prime}(x) + nL_n^k(x) = 0$$

Recurrence relation:

$$L_{n-1}^{k}(x)+L_{n}^{k-1}(x)=L_{n}^{k}(x);$$
$$xL_{n}^{k\prime}(x)=nL_{n}^{k}(x)-(n+k)\,L_{n-1}^{k}(x)$$

Examples:

$$L_{0}^{k}(x)=1;\;\; L_{1}^{k}(x)=-x+k+1$$
$$L_{2}^{k}(x)=\frac{1}{2}\left[x^{2}-2(k+2)\,x+(k+1)(k+2)\right]$$
$$L_{3}^{k}(x)=\frac{1}{6}\left[-x^{3}+3(k+3)\,x^{2}-3(k+2)(k+3)\,x+(k+1)(k+2)(k+3)\right]$$

Hypergeometric function:

$$L_{n}^{k}(x)=\frac{\Gamma(n+k+1)}{n!\Gamma(k+1)}\;{}_1F_1(-n;k+1;x)$$

f) Tschebyscheff polynomials $T_n(x)$

Generating functions:

$$\frac{1-xy}{1-2xy+y^{2}}=\sum_{n=0}^{\infty}T_{n}(x)\,y^{n}$$

Symmetry relation $T_n(x)=T_{-n}(x)$

Expression for T_n

$$T_{n}(x)=\frac{1}{2}\left[\left\{x+i\sqrt{1-x^{2}}\right\}^{n}+\left\{x-i\sqrt{1-x^{2}}\right\}^{n}\right]$$

Differential equation:

$$\left(1-x^{2}\right)\frac{d^{2}}{dx^{2}}T_{n}(x)-x\frac{d}{dx}T_{n}(x)+n^{2}T_{n}(x)=0$$

Recurrence relation:

$$T_{n+1}-2xT_{n}+T_{n-1}=0$$
$$\left(1-x^{2}\right)T_{n}'(x)=-nxT_{n}(x)+nT_{n-1}(x)=0$$

Examples:

$$T_{n}(x)=1,T_{n}(x)=x,T_{n}(x)=2x^{2}-1,T_{n}(x)=4x^{3}-3x$$

Hypergeometric function:

$$T_{n}(x)={}_2F_{1}\left(-n,n;\frac{1}{2};\frac{1-x}{2}\right)$$

Chapter 4

Zero-Inflated Power Series Distributions (ZIPSD)

4.1 Introduction

A zero-inflated model is a statistical model based on a zero-inflated probability distribution. It arises when probability mass at point zero exceeds the one allowed under the standard parametric family of discrete distributions. A numbers of researchers have worked on these family of zero inflated models. Gupta et al., (1995) have studied Zero-Inflated Modified Power Series distribution with the structural properties, in particular for zero-inflated Poisson distribution. Murat and Szynal (1998) extended the results of Gupta et al., (1995) to the distributions inflated at any of the support point 's'.

In this chapter, we provide an overview of the concept of zero-inflated distribution. Examples of situations giving rise to zero-inflated distribution and an overview of its structural properties that includes; probability generating function, moments and their recurrence relation, central moments, the recurrence relation for cumulants and factorial cumulants. Lastly, the special cases of zero inflated power series distributions with their corresponding structural properties will be covered.

4.2 The concept of Zero-Inflated Distributions

In applications involving discrete data we come across data having frequency of an observation 'zero' significantly higher than the one predicted by the assumed models. This situation is often called zero inflation because the data set contains an excess number of zero counts.

Consider the following uniform discrete distributions with $0, 1, 2, 3$ and 4 counts.

X	0	1	2	3	4
Frequency (Counts)	2	1	3	1	1
Proportion	$\frac{2}{8}$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{1}{8}$	$\frac{1}{8}$

Suppose that three more zeros are introduced to the initial data given in the following table.

Y	0	1	2	3	4
Frequency (Counts)	5	1	3	1	1
Proportion	$\frac{5}{11}$	$\frac{1}{11}$	$\frac{3}{11}$	$\frac{1}{11}$	$\frac{1}{11}$

Let the proportion of extra zeros be ρ. Then $\rho = \frac{3}{11}$

$$\begin{aligned}\Pr(Y=0) &= \rho + (1-\rho)\,p_0 \\ &= \frac{3}{11} + \frac{8}{11}\left(\frac{2}{8}\right) = \frac{5}{11}\end{aligned}$$

$$\begin{aligned}\Pr(Y=1) &= (1-\rho)\,p_1 \\ &= \frac{8}{11}\left(\frac{1}{8}\right) = \frac{1}{11}\end{aligned}$$

$$\begin{aligned}\Pr(Y=2) &= (1-\rho)\,p_2 \\ &= \frac{8}{11}\left(\frac{3}{8}\right) = \frac{3}{11}\end{aligned}$$

$$\begin{aligned}\Pr(Y=3) &= (1-\rho)\,p_3 \\ &= \frac{8}{11}\left(\frac{1}{8}\right) = \frac{1}{11}\end{aligned}$$

$$\Pr(Y=k) = (1-\rho)\,p_k,\ \ k = 1,2,3,4$$

Hence,

$$\Pr(Y=k) = \begin{cases} \rho + (1-\rho)\,p_0 & for \quad k=0 \\ (1-\rho)\,p_k & for \quad k=1,2,3,\ldots \end{cases} \quad 0<\rho<1$$

As in given (Gupta et al., 1996: 208, Dianliang D. 2000: 564)

4.3 Definition

If the data set contains excess number of zero counts, a mixture assigning a mass of ρ to the extra zeros and a mass of $(1-\rho)$ to the power series distribution, leads to the Zero-Inflated Power Series Distribution (ZIPSD). A discrete random variable Y is said to have Zero-Inflated Power Series Distribution (ZIPSD), if the probability mass function of Y is given by

$$\Pr(Y=k) = \begin{cases} \rho + (1-\rho)\,\frac{a_0}{f(\theta)} & \text{for} \quad k=0 \\ (1-\rho)\,\frac{a_k\theta^k}{f(\theta)} & \text{for} \quad k=1,2,3,\ldots \quad a_k>0 \text{ and } 0<\rho<1 \end{cases} \tag{4.1}$$

4.4 Examples of situations that give rise to Inflated Distribution

Example 1: (*Fetal movement data*). Leroux and Puterman (1992) Consider the Fetal movement data by a model consisting of a mixture of finite number of Poisson components. This data set was collected in a study of breathing and body movements in fetal lambs designed to examine the possible changes in the amount of pattern of fetal activity during the last two thirds of the gestation period. The numbers of movements by a fetal lamb observed through ultrasound were recorded and counts are given below:

Number of movements	0	1	2	3	4	5	6	7
Number of intervals	182	41	12	2	2	0	0	1

In this case the number of Fetal movement with zero intervals is inflated.

Example 2: (Ridout et al., (2001)) Consider the data Table below consisting of the number of roots produced by 270 micro-propogated shoots of the columnar apple cultivar Trajan. The roots had been produced under an 8-h or 16-h photoperiod in culture systems that utilized one of four different concentrations of the cytokinin BAP in culture medium. For illustration, we have merged the data on four concentrations into one group. Let Group I (Gr II) consist of the data produced under 8 hour period and Group II (Gr II) consist of the data produced under produced under 16 hour photo period.

Number of roots	Obs. fr. (Gr I)	Obs. fr. (Gr II)	fr.
0	2	62	64
1	3	7	10
2	6	7	13
3	7	8	15
4	13	8	21
5	12	6	18
6	14	10	24
7	17	4	21
8	21	2	23
9	14	7	21
10	13	4	17
11	10	2	12
12	2	3	5
13	2	0	2
14	3	0	3
15	0	0	0
16	0	0	0
17	1	0	1
Total	140	130	270

where

- fr.=Frequency

- Obs.=observed

In this case the number of roots with zero observations is inflated.

Example 3: (Yip, P. (1998)) A sample of n leavesof a particular plant is examined and x_i insects are found on the i^{th} leaf for $i = 1, 2, 3, ..., n$ The number of insects per leaf is assumed to be a Poisson variate, except that some leaves have no insects because they are unsuitable for feeding and not merely because of the chance variation allowed for by the Poisson distribution. In this case the number with no insect (i.e. zero observations) is inflated.

Example 4: (Gupta P. L., Gupta R. L. and Tripathi R. C. (1995)) Consider two machines, one of which (machine I) is perfect and does not produce any defective item and the other (machine II) produces defectives according to a Poisson distribution (with parameter, say λ). We observe data from the joint output of the two machines without knowing whether the item has been produced by machine I or by machine II. In this case the observed number of non-defectives (zero observations) produced is inflated.

4.5 The mean and variance of Zero-Inflated Power Series Distributions

Zero inflation is a special case of over dispersion that contradicts the relationship between the mean and variance in a one-parameter exponential family. One way to address this is to use a two-parameter distribution so that the extra parameter permits a larger variance.

The first and second derivatives of $f(\theta)$ are given by,

$$f'(\theta) = \frac{df}{d\theta} = \sum_{k=0}^{\infty} k a_k \theta^{k-1} = \sum_{k=1}^{\infty} k a_k \theta^{k-1} \tag{4.2}$$

and

$$f''(\theta) = \frac{d^2 f}{d\theta^2} = \sum_{k=1}^{\infty} k(k-1) a_k \theta^{k-2} = \sum_{k=2}^{\infty} k(k-1) a_k \theta^{k-2} \tag{4.3}$$

Therefore,

$$\begin{aligned} E(Y) &= \sum_{k=0}^{\infty} k \Pr(Y=k) = \sum_{k=1}^{\infty} k \Pr(Y=k) \\ &= \sum_{k=1}^{\infty} k(1-\rho) \frac{a_k \theta^k}{f(\theta)} = \frac{\theta}{f(\theta)} (1-\rho) \sum_{k=1}^{\infty} k a_k \theta^{k-1} \\ &= (1-\rho) \frac{\theta f'(\theta)}{f(\theta)} \end{aligned} \tag{4.4}$$

$$\begin{aligned} E[Y(Y-1)] &= \sum_{k=0}^{\infty} k(k-1) \Pr(Y=k) = \theta^2 \sum_{k=2}^{\infty} k(k-1)(1-\rho) \frac{a_k \theta^{k-2}}{f(\theta)} \\ &= (1-\rho) \frac{\theta^2}{f(\theta)} f''(\theta) \end{aligned}$$

and

$$
\begin{aligned}
Var(Y) &= E\left[Y(Y-1\right] + E(Y) - \left[E(Y)\right]^2 \\
&= (1-\rho)\,\frac{\theta^2}{f(\theta)} f''(\theta) + (1-\rho)\,\frac{\theta f'(\theta)}{f(\theta)} - \left[(1-\rho)\,\frac{\theta f'(\theta)}{f(\theta)}\right]^2 \\
&= (1-\rho)\,\theta \left\{\theta \frac{f''(\theta)}{f(\theta)} + \frac{f'(\theta)}{f(\theta)} - (1-\rho)\,\theta \left[\frac{f'(\theta)}{f(\theta)}\right]^2\right\} \qquad (4.5)
\end{aligned}
$$

4.6 Probability Generating Function of Zero-Inflated Power Series Distribution

The pgf of Y is given by

$$
\begin{aligned}
G_Y(s) &= \sum_{k=0}^{\infty} \Pr(Y=k)s^k = \Pr(Y=0) + \sum_{k=1}^{\infty} \Pr(Y=k)s^k \\
&= \rho + (1-\rho)\,\frac{a_0}{f\,(\theta)} + \sum_{k=1}^{\infty} (1-\rho)\,\frac{a_k\,(\theta s)^k}{f\,(\theta)} \\
&= \rho + (1-\rho)\,\frac{a_0}{f\,(\theta)} + \sum_{k=1}^{\infty} (1-\rho)\,\frac{a_k\,(\theta s)^k}{f\,(\theta)} \\
&= \rho + (1-\rho)\,\frac{a_0}{f\,(\theta)} + \frac{(1-\rho)}{f\,(\theta)}\,(f\,(\theta s) - a_0) \\
&= \rho + (1-\rho)\,\frac{f\,(\theta s)}{f\,(\theta)} \qquad (4.6)
\end{aligned}
$$

but

$$\frac{f\,(\theta s)}{f\,(\theta)} = G_X\,(s)$$

Therefore, (4.6) becomes;

$$G_Y(s) = \rho + (1-\rho)\,G_X\,(s)$$

$$G'_Y(s) = \frac{dG}{ds} = (1-\rho)\,\theta \frac{f'(\theta s)}{f(\theta)}, \qquad (4.7)$$

and

$$G''_Y(s) = \frac{d^2G}{ds^2} = (1-\rho)\,\theta^2 \frac{f''(\theta s)}{f(\theta)} \qquad (4.8)$$

Hence the mean and the variance of ZIPSD is given by

$$E(Y) = G'_Y(1) = (1-\rho)\,\theta \frac{f'(\theta)}{f(\theta)} \qquad (4.9)$$

and

$$
\begin{aligned}
Var(Y) &= G''_Y(1) + G'_Y(1) - [G'_Y(1)]^2 \\
&= (1-\rho)\,\theta^2 \frac{f''(\theta)}{f(\theta)} + (1-\rho)\,\theta \frac{f'(\theta)}{f(\theta)} - \left[(1-\rho)\,\theta \frac{f'(\theta)}{f(\theta)}\right]^2 \\
&= (1-\rho)\,\theta \left\{\theta \frac{f''(\theta)}{f(\theta)} + \frac{f'(\theta)}{f(\theta)} - (1-\rho)\,\theta \left[\frac{f'(\theta)}{f(\theta)}\right]^2\right\} \qquad (4.10)
\end{aligned}
$$

4.7 Moments and their recurrence relations of zero-inflated power series distributions

The r^{th} moment is defined by

$$
\begin{aligned}
\mu'_r = E\left(Y^r\right) &= \sum_{k=0}^{\infty} k^r \Pr(Y=k) = \sum_{k=1}^{\infty} k^r \,(1-\rho)\, a_k \frac{\theta^k}{f(\theta)} \\
&= \frac{(1-\rho)}{f(\theta)} \sum_{k=1}^{\infty} k^r a_k \theta^k \qquad (4.11)
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\frac{d}{d\theta}\mu'_r &= \frac{(1-\rho)}{f(\theta)} \frac{d}{d\theta} \sum_{k=1}^{\infty} k^r a_k \theta^k + (1-\rho)\left[\frac{d}{d\theta}\frac{1}{f(\theta)}\right] \sum_{k=1}^{\infty} k^r a_k \theta^k \\
&= \frac{(1-\rho)}{f(\theta)} \sum_{k=1}^{\infty} k^{r+1} a_k \theta^{k-1} + \left[-\frac{(1-\rho)}{[f(\theta)]^2}\frac{d}{d\theta} f(\theta)\right] \sum_{k=1}^{\infty} k^r a_k \theta^k \\
&= \frac{(1-\rho)}{f(\theta)} \sum_{k=1}^{\infty} k^{r+1} a_k \theta^{k-1} - (1-\rho)\,\frac{f'(\theta)}{[f(\theta)]^2} \sum_{k=1}^{\infty} k^r a_k \theta^k
\end{aligned}
$$

Multiplying $\frac{d}{d\theta}\mu'_r$ by θ, it becomes

$$
\begin{aligned}
\theta \frac{d}{d\theta}\mu'_r &= \frac{(1-\rho)}{f(\theta)} \sum_{k=1}^{\infty} k^{r+1} a_k \theta^k - (1-\rho)\,\theta \frac{f'(\theta)}{[f(\theta)]} \cdot \frac{1}{f(\theta)} \sum_{k=1}^{\infty} k^r a_k \theta^k \\
&= \sum_{k=1}^{\infty} k^{r+1}\,(1-\rho)\,\frac{a_k \theta^k}{f(\theta)} - \theta \frac{f'(\theta)}{[f(\theta)]} \sum_{k=1}^{\infty} k^r \,(1-\rho)\,\frac{a_k \theta^k}{f(\theta)} \\
&= \mu'_{r+1} - \theta \frac{f'(\theta)}{[f(\theta)]}\mu'_r, \qquad (4.12)
\end{aligned}
$$

but

$$
\begin{aligned}
E(Y) &= \sum_{k=0}^{\infty} k\,(1-\rho)\,\frac{a_k \theta^k}{f(\theta)} = (1-\rho)\,\frac{\theta}{f(\theta)} \sum_{k=1}^{\infty} k a_k \theta^{k-1} \\
&= (1-\rho)\,\theta \frac{f'(\theta)}{f(\theta)} = \mu'_1
\end{aligned}
$$

Hence,

$$
\theta \frac{f'(\theta)}{f(\theta)} = \frac{\mu'_1}{(1-\rho)} \qquad (4.13)
$$

Equation (4.13) then becomes

$$\theta \frac{d}{d\theta} \mu'_r = \mu'_{r+1} - \frac{\mu'_1}{(1-\rho)} \mu'_r$$

Thus we have the recurrence relation for the r^{th} moments of a ZIPSD as

$$\mu'_{r+1} = \theta \frac{d}{d\theta} \mu'_r + \frac{\mu'_1}{(1-\rho)} \mu'_r \tag{4.14}$$

The r^{th} central moment, i.e., the r^{th} moment about the mean is defined by

$$\begin{aligned} \mu_r &= E\left[Y - \mu'_1\right]^r \\ &= \sum_{k=0}^{\infty} \left(k - \mu'_1\right)^r \Pr(Y = k) \end{aligned} \tag{4.15}$$

but

$$\Pr\left(Y = k\right) = \begin{cases} \rho + (1-\rho) \frac{a_0}{f(\theta)} & for \quad k = 0 \\ \\ (1-\rho) \frac{a_k \theta^k}{f(\theta)} & for \quad k = 1, 2, 3, \ldots \quad a_k > 0 \text{ and } 0 < \rho < 1 \end{cases} \tag{4.16}$$

Let $s = 0$, thus (4.16) becomes

$$\Pr\left(Y = k\right) = \begin{cases} \rho + (1-\rho) \frac{a_s \theta^s}{f(\theta)} & for \quad k = s \\ \\ (1-\rho) \frac{a_k \theta^k}{f(\theta)} & for \quad k > s \quad a_k > 0 \text{ and } 0 < \rho < 1 \end{cases} \tag{4.17}$$

Therefore The r^{th} central moment as given in (4.15) becomes

$$u_r = \rho \left(s - \mu'_1\right)^r + (1-\rho)\left(s - \mu'_1\right)^r \frac{a_s \theta^s}{f\left(\theta\right)} + \sum_{k \neq s}^{\infty} (1-\rho)\left(k - \mu'_1\right)^r \frac{a_k \theta^k}{f\left(\theta\right)} \tag{4.18}$$

Hence

$$\frac{d}{d\theta}\mu_r = \rho\frac{d}{d\theta}\left((s-\mu_1')^r\right) + (1-\rho)\frac{d}{d\theta}\left((s-\mu_1')^r\frac{a_s\theta^s}{f(\theta)}\right)$$
$$+ (1-\rho)\left(\frac{d}{d\theta}\frac{1}{f(\theta)}\right)\sum_{k\neq s}^{\infty}(k-\mu_1')^r a_k\theta^k + \frac{(1-\rho)}{f(\theta)}\frac{d}{d\theta}\sum_{k\neq s}^{\infty}(k-\mu_1')^r a_k\theta^k$$
$$= -\rho r(s-\mu_1')^{r-1}\frac{d}{d\theta}\mu_1' - \frac{f'(\theta)}{f(\theta)}(1-\rho)(s-\mu_1')^r\frac{a_s\theta^s}{f(\theta)} + (1-\rho)(s-\mu_1')^r s\frac{a_s\theta^{s-1}}{f(\theta)} \tag{4.19}$$
$$- \frac{(1-\rho)}{f(\theta)}a_s\theta^s r(s-\mu_1')^{r-1}\frac{d}{d\theta}\mu_1' - \frac{f'(\theta)}{f(\theta)}(1-\rho)\sum_{k\neq s}^{\infty}(k-\mu_1')^r\frac{a_k\theta^k}{f(\theta)}$$
$$+ \frac{(1-\rho)}{f(\theta)}\sum_{k\neq s}^{\infty}k(k-\mu_1')^r a_k\theta^{k-1} - r\frac{(1-\rho)}{f(\theta)}\sum_{k\neq s}^{\infty}(k-\mu_1')^{r-1}a_k\theta^k\frac{d}{d\theta}\mu_1'$$
$$= -r\frac{d}{d\theta}\mu_1'\left\{\rho(s-\mu_1')^{r-1} + (1-\rho)(s-\mu_1')^{r-1}\frac{a_s\theta^s}{f(\theta)} + \sum_{k\neq s}^{\infty}(k-\mu_1')^{r-1}\frac{a_k\theta^k}{f(\theta)}\right\}$$
$$- \frac{f'(\theta)}{f(\theta)}\left\{(1-\rho)(s-\mu_1')^r\frac{a_s\theta^s}{f(\theta)} + \sum_{k\neq s}^{\infty}(1-\rho)(k-\mu_1')^r\frac{a_k\theta^k}{f(\theta)}\right\}$$
$$+ (s-\mu_1')^{r+1}(1-\rho)\frac{a_s\theta^{s-1}}{f(\theta)} + \mu_1'(1-\rho)(s-\mu_1')^r\frac{a_s\theta^{s-1}}{f(\theta)}$$
$$+ \sum_{k>s}^{\infty}(k-\mu_1')^{r+1}\frac{a_k\theta^{k-1}}{f(\theta)} + \mu_1'\sum_{k\neq s}^{\infty}(k-\mu_1')^r\frac{a_k\theta^{k-1}}{f(\theta)} \tag{4.20}$$

but, by definition u_r is given by (4.18). Thus by replacing the values of u_r in (4.19) we obtain

$$\frac{d}{d\theta}\mu_r = -r\frac{d}{d\theta}\mu_1'\mu_{r-1} - \frac{f'(\theta)}{f(\theta)}\{\mu_r - \rho(s-\mu_1')^r\} + (s-\mu_1')^{r+1}(1-\rho)\frac{a_s\theta^{s-1}}{f(\theta)}$$
$$+ \sum_{k\neq s}^{\infty}(k-\mu_1')^{r+1}\frac{a_k\theta^{k-1}}{f(\theta)} + \mu_1'\left\{(1-\rho)(s-\mu_1')^r\frac{a_s\theta^{s-1}}{f(\theta)} + \sum_{k\neq s}^{\infty}(k-\mu_1')^r\frac{a_k\theta^{k-1}}{f(\theta)}\right\} \tag{4.21}$$

Multiplying $\frac{d}{d\theta}\mu_r$ by θ in (4.20) and replacing the values of u_r, $\theta\frac{d}{d\theta}\mu_r$ then becomes

$$\theta\frac{d}{d\theta}\mu_r = -r\theta\frac{d}{d\theta}\mu_1'\mu_{r-1} - \theta\frac{f'(\theta)}{f(\theta)}\{\mu_r - \rho(s-\mu_1')^r\} + (s-\mu_1')^{r+1}(1-\rho)\frac{a_s\theta^s}{f(\theta)}$$
$$+ \sum_{k\neq s}^{\infty}(k-\mu_1')^{r+1}\frac{a_k\theta^k}{f(\theta)} + \mu_1'\left\{(1-\rho)(s-\mu_1')^r\frac{a_s\theta^s}{f(\theta)} + \sum_{k\neq s}^{\infty}(k-\mu_1')^r\frac{a_k\theta^k}{f(\theta)}\right\}$$
$$= -r\theta\frac{d}{d\theta}\mu_1'\mu_{r-1} - \theta\frac{f'(\theta)}{f(\theta)}(\mu_r - \rho(s-\mu_1')^r) + \mu_{r+1} - \rho(s-\mu_1')^{r+1} \tag{4.22}$$
$$+ \mu_1'(\mu_r - \rho(s-\mu_1')^r) \tag{4.23}$$

but

$$\theta \frac{f'(\theta)}{f(\theta)} = \frac{\mu_1'}{(1-\rho)}$$

Therefore,

$$\theta \frac{d}{d\theta}\mu_r = -r\theta \frac{d}{d\theta}\mu_1'\mu_{r-1} + \left(\mu_r - \rho\left(s-\mu_1'\right)^r\right)\left(\mu_1' - \frac{\mu_1'}{(1-\rho)}\right) + \mu_{r+1} - \rho\left(s-\mu_1'\right)^{r+1} \tag{4.24}$$

From (4.22) , we obtain

$$\mu_{r+1} = \theta\left(\frac{d}{d\theta}\mu_r + r\mu_{r-1}\frac{d}{d\theta}\mu_1'\right) + \rho\left(s-\mu_1'\right)^{r+1} - \left(\mu_r - \rho\left(s-\mu_1'\right)^r\right)\left(\mu_1' - \frac{\mu_1'}{(1-\rho)}\right)$$

but $s = 0$.

Thus we obtain the recurrence relation for central moments of ZIPSD as

$$\mu_{r+1} = \theta\left[\frac{d}{d\theta}\mu_r + r\mu_{r-1}\frac{d}{d\theta}\mu_1'\right] + \rho\left(-\mu_1'\right)^{r+1} - \left[\mu_r - \rho\left(-\mu_1'\right)^r\right]\left[\mu_1' - \frac{\mu_1'}{(1-\rho)}\right] \tag{4.25}$$

setting $r = 1$ in (4.23) we get

$$\mu_2 = \theta\left(\frac{d}{d\theta}\mu_0 + \mu_0\frac{d}{d\theta}\mu_1'\right) + \rho(\mu_1')^2 - (\mu_1 + \rho\mu_1')\left(\mu_1' - \frac{\mu_1'}{(1-\rho)}\right)$$

but

$$\mu_0 = E(Y-\mu_1')^0 = 1$$

and

$$\mu_1 = E(Y-\mu_1') = E(Y) - \mu_1' = \mu_1' - \mu_1' = 0$$

As a result,

$$\begin{aligned}\mu_2 &= \theta\left[0 + \frac{d}{d\theta}\mu_1'\right] + \rho(\mu_1')^2 - \rho(\mu_1')^2 + \frac{\rho(\mu_1')^2}{(1-\rho)}\\ &= \theta\frac{d}{d\theta}\mu_1' + \frac{\rho(\mu_1')^2}{(1-\rho)}\end{aligned} \tag{4.26}$$

Also from (4.24)

$$\begin{aligned}
\mu_2 &= \theta\frac{d}{d\theta}\mu_1' + \frac{\rho(\mu_1')^2}{(1-\rho)} \\
&= \theta\,(1-\rho)\,\frac{d}{d\theta}\left[\frac{\theta f'(\theta)}{f(\theta)}\right] + \rho\,(1-\rho)\left[\frac{\theta f'(\theta)}{f(\theta)}\right]^2 \\
&= \theta\,(1-\rho)\left[\frac{f'(\theta)}{f(\theta)} + \theta\left\{\frac{f(\theta)f''(\theta)-[f'(\theta)]^2}{[f(\theta)]^2}\right\}\right] + \rho\,(1-\rho)\,\theta^2\left[\frac{f'(\theta)}{f(\theta)}\right]^2 \\
&= \theta\,(1-\rho)\left[\frac{f'(\theta)}{f(\theta)} + \theta\frac{f''(\theta)}{f(\theta)} - \theta\left[\frac{f'(\theta)}{f(\theta)}\right]^2\right] + \rho\,(1-\rho)\,\theta^2\left[\frac{f'(\theta)}{f(\theta)}\right]^2 \\
&= \theta\,(1-\rho)\,\frac{f'(\theta)}{f(\theta)} + \theta^2\,(1-\rho)\,\frac{f''(\theta)}{f(\theta)} - (1-\rho)\,\theta^2\left[\frac{f'(\theta)}{f(\theta)}\right]^2 + \rho\,(1-\rho)\,\theta^2\left[\frac{f'(\theta)}{f(\theta)}\right]^2 \\
&= -\theta^2\left[\frac{f'(\theta)}{f(\theta)}\right]^2(1-\rho)^2 + \theta\,(1-\rho)\,\frac{f'(\theta)}{f(\theta)} + \theta^2\,(1-\rho)\,\frac{f''(\theta)}{f(\theta)} \\
&= \theta\,(1-\rho)\left\{\theta\frac{f''(\theta)}{f(\theta)} + \frac{f'(\theta)}{f(\theta)} - (1-\rho)\,\theta\left[\frac{f'(\theta)}{f(\theta)}\right]^2\right\} \qquad (3.25)
\end{aligned}$$

The variance of Y will be given by

$$Var(Y) = \mu_2 = \theta\,(1-\rho)\left\{\theta\frac{f''(\theta)}{f(\theta)} + \frac{f'(\theta)}{f(\theta)} - (1-\rho)\,\theta\left[\frac{f'(\theta)}{f(\theta)}\right]^2\right\}$$

4.8 Moment Generating Functions (mgf) of Zero-Inflated Power Series Distribution

The mgf of Y is given by

$$\begin{aligned}
M_Y(t) = E\left[e^{tY}\right] &= \sum_{k=0}^{\infty} e^{tk}p_k = \Pr(Y=0) + \sum_{k=1}^{\infty}\Pr(Y=k)e^{tk} \\
&= \rho + (1-\rho)\,\frac{a_0}{f(\theta)} + \sum_{k=1}^{\infty}(1-\rho)\,\frac{a_k\left(\theta e^t\right)^k}{f(\theta)} \\
&= \rho + (1-\rho)\,\frac{a_0}{f(\theta)} + \frac{(1-\rho)}{f(\theta)}\sum_{k=1}^{\infty}a_k\left(\theta e^t\right)^k \\
&= \rho + (1-\rho)\,\frac{a_0}{f(\theta)} + \frac{(1-\rho)}{f(\theta)}\left[f\left(\theta e^t\right)^k - a_0\right] \\
&= \rho + (1-\rho)\,\frac{f\left(\theta e^t\right)}{f(\theta)} \qquad (4.27)
\end{aligned}$$

The r^{th} moment is obtained from the r^{th} derivative of $M_Y(t)$ w.r.t t and setting $t=0$ i.e.

$$\mu_r' = \frac{d^r M_Y(t)}{dt^r}\,|_{t=0}$$

4.9 Factorial Moment Generating Function (fmgf) of Zero-Inflated Power Series Distribution

The fmgf of Y is given by

$$\begin{aligned}
M_{[Y]}(t) &= E\left[1+t\right]^{Y} \\
&= \sum_{k=0}^{\infty}\left[1+t\right]^{k} p_k = \sum_{k=0}^{\infty}\frac{a_k(\theta\left[1+t\right])^k}{f(\theta)} \\
&= \rho + (1-\rho)\frac{a_0}{f(\theta)} + (1-\rho)\sum_{k=1}^{\infty}\frac{a_k\left(\theta+\theta t\right)^k}{f(\theta)} \\
&= \rho + (1-\rho)\frac{a_0}{f(\theta)} + \frac{(1-\rho)}{f(\theta)}\sum_{k=1}^{\infty}a_k\left(\theta+\theta t\right)^k \\
&= \rho + (1-\rho)\frac{a_0}{f(\theta)} + \frac{(1-\rho)}{f(\theta)}\left[f\left(\theta+\theta t\right)^k - a_0\right] \\
&= \rho + (1-\rho)\frac{f(\theta+\theta t)}{f(\theta)} \qquad (4.28)
\end{aligned}$$

The r^{th} factorial moment is obtained from the r^{th} derivative of $M_{[Y]}(t)$ w.r.t t and setting $t=0$

$$\mu_{[r]} = \frac{d^r M_{[Y]}(t)}{dt^r}\,|_{t=0}$$

4.10 Cumulant and Cumulant Generating Function (cgf) of Zero-Inflated Power Series Distribution

The cgf of Y is given by

$$K_Y(t) = \log M_Y(t)$$

Thus the r^{th} cumulant of Y.is obtained from the r^{th} derivative of $K_Y(t)$ w.r.t t and setting $t=0$. That is

$$k_r = \frac{d^r K_Y(t)}{dt^r}\,|_{t=0}$$

4.11 Special Cases

4.11.1 Zero-Inflated Poisson Distribution (ZIPo)

$$f(\theta) = e^{\theta} = \sum_{k=0}^{\infty}\frac{\theta^k}{k!}$$

From this we obtain

$$\Pr\left(X=k\right)=\frac{e^{-\theta}\theta^{k}}{k!},\ \ k=0,1,2,...$$

which is a Poisson Distribution.

By definition the probability mass function of ZIPSD is given by,

$$\Pr\left(Y=k\right)=\begin{cases}\rho+\left(1-\rho\right)p_0 & for \quad\quad k=0\\ \\ \left(1-\rho\right)p_k & for \quad k=1,2,3,\ldots\end{cases}$$

Therefore

i.

$$\Pr\left(Y=k\right)=\begin{cases}\rho+\left(1-\rho\right)e^{-\theta} & for \quad\quad k=0\\ \\ \left(1-\rho\right)\frac{e^{-\theta}\theta^{k}}{k!} & for \quad k=1,2,3,\ldots\end{cases}$$

Which is the probability mass function of ZIPo and confirms with (Bohning et al 1999: 198; Chin-Shang, Li et al 199: 30; Jansakul 2002: 77)

ii.

$$f'(\theta)=e^{\theta}$$

iii.

$$f''(\theta)=e^{\theta}$$

iv. The mean is given by

$$E(Y)=\left(1-\rho\right)\theta\frac{f'(\theta)}{f(\theta)}=\left(1-\rho\right)\theta\frac{e^{\theta}}{e^{\theta}}=\left(1-\rho\right)\theta$$

v. The variance is given by

$$\begin{aligned}Var\left(Y\right)&=\left(1-\rho\right)\theta^{2}\frac{f''(\theta)}{f(\theta)}+\left(1-\rho\right)\theta\frac{f'(\theta)}{f(\theta)}-\left(1-\rho\right)^{2}\theta^{2}\left[\frac{f'(\theta)}{f(\theta)}\right]^{2}\\&=\left(1-\rho\right)\theta\left[\theta+1-\theta+\theta\rho\right]\\&=\left(1-\rho\right)\theta+\rho\left(1-\rho\right)\theta^{2}\\&=\left(1-\rho\right)\theta\left(1+\rho\theta\right)\end{aligned}$$

vi.

$$\mu_{r+1}=\theta\left[\frac{d}{d\theta}\mu_r+r\mu_{r-1}\frac{d}{d\theta}\mu_1'\right]+\rho\left(-\mu_1'\right)^{r+1}-\left[\mu_r-\rho\left(-\mu_1'\right)^{r}\right]\left[\mu_1'-\frac{\mu_1'}{(1-\rho)}\right]$$

but

$$\mu_1'=E(Y)=\left(1-\rho\right)\theta,$$

As a result the recurrence relation for the central moments of ZIPo is given by

$$\mu_{r+1}=\theta\left[\frac{d}{d\theta}\mu_r+\left(1-\rho\right)r\mu_{r-1}\right]+\rho\left(-\left(1-\rho\right)\theta\right)^{r+1}-\rho\theta\left(\mu_r-\rho\left(-\left(1-\rho\right)\theta\right)^{r}\right)$$

setting $s = 1$

$$\begin{aligned} u_2 &= (1-\rho)\,\theta + \rho\left((1-\rho)\,\theta\right)^2 - \rho^2\,(1-\rho)\,\theta^2 \\ &= (1-\rho)\,\theta + \rho\,(1-\rho)\,\theta^2\left((1-\rho)-\rho\right) \\ &= (1-\rho)\,\theta + \rho\,(1-\rho)\,\theta^2 \\ &= (1-\rho)\,\theta\,(1+\rho\theta) \end{aligned}$$

vii. The probability generating function for ZIPo is given by

$$\begin{aligned} G_Y(s) &= \rho + (1-\rho)\,\frac{f(\theta s)}{f(\theta)} \\ &= \rho + (1-\rho)\,e^{\theta(s-1)} \end{aligned}$$

$$G'_Y(s) = (1-\rho)\,\theta\frac{e^{\theta s}}{e^{\theta}}$$

$$G''_Y(s) = (1-\rho)\,\theta^2\frac{e^{\theta s}}{e^{\theta}}$$

setting $s = 1,$ we obtain

$$G'_Y(1) = (1-\rho)\,\theta G''_Y(1) = (1-\rho)\,\theta^2$$

To obtain the mean and variance

$$E(Y) = G'_Y(1) = (1-\rho)\,\theta$$

$$\begin{aligned} Var(Y) &= G''_Y(1) + G'_Y(1) - \left[G'_Y(1)\right]^2 \\ &= (1-\rho)\,\theta^2 + (1-\rho)\,\theta - \left[(1-\rho)\,\theta\right]^2 \\ &= (1-\rho)\,\theta\,(1+\rho\theta) \end{aligned}$$

viii. The moment generating function of ZIPo is given by

$$\begin{aligned} M_Y(t) &= \rho + (1-\rho)\,\frac{f\left(\theta e^t\right)}{f(\theta)} \\ &= \rho + (1-\rho)\,e^{\left(\theta e^t - \theta\right)} \\ &= \rho + (1-\rho)\,e^{\theta\left(e^t - 1\right)} \end{aligned}$$

The r^{th} moment is obtained from the r^{th} derivative of $M_Y(t)$ w.r.t t and setting $t = 0$ i.e.

$$\mu'_r = \frac{d^r M_Y(t)}{dt^r}\,|_{t=0}$$

For $r = 1$, we have

$$\begin{aligned} \mu'_1 &= \frac{d}{dt}\left[\rho + (\,1-\rho)\,e^{\left(\,\theta e^t - \theta\right)}\,\right]|_{t=0} \\ &= (\,1-\rho)\,\theta e^t e^{\left(\,\theta e^t - \theta\right)}\,\,|_{t=0} \\ &= (\,1-\rho)\,\theta \end{aligned}$$

$r = 2$

$$\begin{aligned}\mu_2' &= (\ 1-\rho)\,\theta \frac{d}{dt} e^{\theta\left(e^t-1\right)+t}\ |_{t=0} \\ &= (\ 1-\rho)\,\theta\left(\theta e^t+1\right) e^{\theta\left(e^t-1\right)+t}\ |_{t=0} \\ &= (\ 1-\rho)\,\theta\,(\theta+1) = (\ 1-\rho)\left(\theta^2+\theta\right) \\ &= (\ 1-\rho)\,\theta^2 + (\ 1-\rho)\,\theta\end{aligned}$$

The variance is given by

$$\begin{aligned}\mu_2 &= \mu_2' - \mu_1'^2 \\ &= (\ 1-\rho)\,\theta^2 + (\ 1-\rho)\,\theta - (\ 1-\rho)^2\,\theta^2 \\ &= (\ 1-\rho)\,\theta\,[\theta+1-\theta+\rho\theta] \\ &= (\ 1-\rho)\,\theta\,[1+\rho\theta]\end{aligned}$$

ix. Factorial moment generating function of ZIPo is given by

$$\begin{aligned}M_{[Y]}(t) &= \rho + (1-\rho)\,\frac{f(\theta+\theta t)}{f(\theta)} \\ &= \rho + (1-\rho)\,e^{\theta t}\end{aligned}$$

The r^{th} factorial moment is obtained from the r^{th} derivative of $M_{[Y]}(t)$ w.r.t t and setting $t = 0$

$$\mu_{[r]} = \frac{d^r M_{[Y]}(t)}{dt^r}\ |_{t=0}$$

setting $r = 1$

$$\begin{aligned}\mu_{[1]} &= \frac{d}{dt}\left(\rho + (1-\rho)\,e^{\theta t}\right)\ |_{t=0} \\ &= (1-\rho)\,\theta e^{\theta t}\ |_{t=0} \\ &= (1-\rho)\,\theta\end{aligned}$$

$r = 2$

$$\begin{aligned}\mu_{[2]} &= \frac{d}{dt}\,(1-\rho)\,\theta e^{\theta t}\ |_{t=0} \\ &= (1-\rho)\,\theta^2 e^{\theta t}\ |_{t=0} \\ &= (1-\rho)\,\theta^2\end{aligned}$$

$r = 3$

$$\begin{aligned}\mu_{[3]} &= \frac{d}{dt}\,(1-\rho)\,\theta^2 e^{\theta t}\ |_{t=0} \\ &= (1-\rho)\,\theta^3 e^{\theta t}\ |_{t=0} \\ &= (1-\rho)\,\theta^3\end{aligned}$$

$r = 4$

$$\mu_{[4]} = \frac{d}{dt}(1-\rho)\,\theta^3 e^{\theta t}\;|_{t=0}$$
$$= (1-\rho)\,\theta^4 e^{\theta t}\;|_{t=0}$$
$$= (1-\rho)\,\theta^4$$

The recursive relationship between factorial moments of ZIPo is given by

$$\mu_{[r]} = \theta\mu_{[r-1]} \quad r \geq 1$$

x. The cumulant generating function of ZIPo is given by

$$K_Y(t) = \log M_Y(t)$$
$$= \log\left\{\rho + (1-\rho)\,e^{\theta\left(e^t-1\right)}\right\}$$

The r^{th} cumulant of the distribution is obtained from the r^{th} derivative of $K_Y(t)$ w.r.t t and setting $t = 0$.

That is

$$k_r = \frac{d^r K_Y(t)}{dt^r}\;|_{t=0}$$

When $r = 1$, we have

$$k_1 = \frac{dK_Y(t)}{dt}\;|_{t=0}$$
$$= \frac{d}{dt}\log\left(\rho + (1-\rho)\,e^{\theta\left(e^t-1\right)}\right)\;|_{t=0}$$
$$= \frac{(1-\rho)\,\theta e^{\theta\left(e^t-1\right)+t}}{\rho + (1-\rho)\,e^{\theta\left(e^t-1\right)}}\;|_{t=0}$$
$$= \frac{(1-\rho)\,\theta}{(\rho+1-\rho)} = (1-\rho)\,\theta$$

$r = 2$

$$k_2 = \frac{d}{dt}\left\{\frac{(1-\rho)\,\theta e^{\theta\left(e^t-1\right)+t}}{\rho + (1-\rho)\,e^{\theta\left(e^t-1\right)}}\right\}\;|_{t=0}$$
$$= \left\{\frac{(1-\rho)\,\theta\left(\theta e^t+1\right)e^{\theta\left(e^t-1\right)+t}\left\{\rho + (1-\rho)\,e^{\theta\left(e^t-1\right)}\right\} - \left\{(1-\rho)\,\theta e^{\theta\left(e^t-1\right)+t}\right\}^2}{\left(\rho + (1-\rho)\,e^{\theta\left(e^t-1\right)}\right)^2}\right\}\;|_{t=0}$$
$$= \frac{(1-\rho)\,\theta\,(\theta+1)\,(\rho+1-\rho) - (1-\rho)^2\,\theta^2}{(\rho+(1-\rho))^2}$$
$$= (1-\rho)\,\theta\,(\theta+1) - (1-\rho)^2\,\theta^2$$
$$= (1-\rho)\,\theta^2 + (1-\rho)\,\theta - (1-\rho)^2\,\theta^2$$
$$= (1-\rho)\,\theta\,\{\theta + 1 - \theta + \rho\theta\}$$
$$= (1-\rho)\,\theta\,[1+\rho\theta]$$

4.11.2 Zero-Inflated Binomial Distribution (ZIBin)

$$f(\theta) = (1+\theta)^n = \sum_{k=0}^{\infty} \binom{n}{k} \theta^k$$

From this we obtain

$$\begin{aligned} \Pr(X=k) &= \binom{n}{k} \frac{\theta^k}{(1+\theta)^n} \\ &= \binom{n}{k} \left(\frac{\theta}{1+\theta}\right)^k \left(\frac{1}{1+\theta}\right)^{n-k}, \; k = 0,1,2,...,n \end{aligned}$$

Which is Binomial with parameters n and $\frac{\theta}{1+\theta}$.

By definition the probability mass function of ZIPSD is given by,

$$\Pr(Y=k) = \begin{cases} \rho + (1-\rho)\, p_0 & for \quad k=0 \\ (1-\rho)\, p_k & for \quad k=1,2,3,\ldots \end{cases}$$

Hence

i.
$$\Pr(Y=k) = \begin{cases} \rho + (1-\rho)\left(\frac{1}{1+\theta}\right)^n & for \quad k=0 \\ (1-\rho)\binom{n}{k}\left(\frac{\theta}{1+\theta}\right)^k \left(\frac{1}{1+\theta}\right)^{n-k} & for \quad k=1,2,3,\ldots,n \end{cases}$$

Which is the probability mass function of ZIBin with parameters n and $\frac{\theta}{1+\theta}$.

ii.
$$f'(\theta) = n(1+\theta)^{n-1}; \; n = 1,2,...$$

iii.
$$f''(\theta) = n(n-1)(1+\theta)^{n-2}; n = 2,3,...$$

iv. The mean is given by

$$E(Y) = (1-\rho)\,\theta \frac{f'(\theta)}{f(\theta)} = (1-\rho)\,\theta \frac{n(1+\theta)^{n-1}}{(1+\theta)^n} = (1-\rho)\, n \frac{\theta}{1+\theta}$$

v. The variance is given by

$$
\begin{aligned}
Var\,(Y) &= (1-\rho)\,\theta^2\frac{f''(\theta)}{f(\theta)} + (1-\rho)\,\theta\frac{f'(\theta)}{f(\theta)} - \left[(1-\rho)\,\theta\frac{f'(\theta)}{f(\theta)}\right]^2 \\
&= (1-\rho)\left\{\theta^2\frac{f''(\theta)}{f(\theta)} + \theta\frac{f'(\theta)}{f(\theta)} - (1-\rho)\left[\theta\frac{f'(\theta)}{f(\theta)}\right]^2\right\} \\
&= (1-\rho)\left\{\theta^2\frac{n(n-1)}{(1+\theta)^2} + n\frac{\theta}{1+\theta} - (1-\rho)\left[n\frac{\theta}{1+\theta}\right]^2\right\} \\
&= (1-\rho)\left\{\frac{\theta^2 n^2}{(1+\theta)^2} - \frac{n\theta^2}{(1+\theta)^2} + n\frac{\theta}{1+\theta} - (1-\rho)\frac{n^2\theta^2}{(1+\theta)^2}\right\} \\
&= (1-\rho)\left\{\frac{\theta^2 n^2}{(1+\theta)^2}(1-1+\rho) + n\frac{\theta}{1+\theta}\left[1-\frac{\theta}{1+\theta}\right]\right\} \\
&= (1-\rho)\left\{\frac{\theta^2 n^2}{(1+\theta)^2}\rho + n\frac{\theta}{1+\theta}\left[\frac{1}{1+\theta}\right]\right\}
\end{aligned}
$$

vi.

$$
\mu_{r+1} = \theta\left[\frac{d}{d\theta}\mu_r + r\mu_{r-1}\frac{d}{d\theta}\mu_1'\right] + \rho\,(-\mu_1')^{r+1} - \left[\mu_r - \rho\,(-\mu_1')^r\right]\left[\mu_1' - \frac{\mu_1'}{1-\rho}\right]
$$

but

$$
\mu_1' = E(Y) = (1-\rho)\,n\frac{\theta}{1+\theta}
$$

Hence the recurrence relation for the central moments of ZIBin is given by

$$
\begin{aligned}
\mu_{r+1} &= \theta\left[\frac{d}{d\theta}\mu_r + r\mu_{r-1}\,(1-\rho)\,n\frac{d}{d\theta}\frac{\theta}{1+\theta}\right] + \rho\left(-(1-\rho)\,n\frac{\theta}{1+\theta}\right)^{r+1} \\
&- \left[\mu_r - \rho\left(-(1-\rho)\,n\frac{\theta}{1+\theta}\right)^r\right]\left[(1-\rho)\,n\frac{\theta}{1+\theta} - \frac{(1-\rho)\,n\frac{\theta}{1+\theta}}{(1-\rho)}\right] \\
&= \theta\left[\frac{d}{d\theta}\mu_r + (1-\rho)\frac{nr\mu_{r-1}}{(1+\theta)^2}\right] + \rho\left(-(1-\rho)\,n\frac{\theta}{1+\theta}\right)^{r+1} \\
&- \left[\mu_r - \rho\left(-(1-\rho)\frac{n\theta}{1+\theta}\right)^r\right]\left[(1-\rho)\frac{n\theta}{1+\theta} - \frac{n\theta}{1+\theta}\right]
\end{aligned}
$$

setting $r = 1$ we have

$$
\begin{aligned}
\mu_2 &= (1-\rho)\frac{n\theta}{(1+\theta)^2} + \rho\,(1-\rho)^2\frac{n^2\theta^2}{(1+\theta)^2} - \rho\,(1-\rho)^2\frac{n^2\theta^2}{(1+\theta)^2} \\
&+ \rho\,(1-\rho)\frac{n^2\theta^2}{(1+\theta)^2} \\
&= (1-\rho)\frac{n\theta}{(1+\theta)^2} + \rho\,(1-\rho)\frac{n^2\theta^2}{(1+\theta)^2} \\
&= (1-\rho)\left\{\frac{n\theta}{(1+\theta)}\left[\frac{1}{1+\theta}\right] + \rho\frac{n^2\theta^2}{(1+\theta)^2}\right\}
\end{aligned}
$$

vii. The probability generating function of ZIBin is given by

$$\begin{aligned} G_Y(s) &= \rho + (1-\rho)\,\frac{f\,(\theta s)}{f\,(\theta)} \\ &= \rho + (1-\rho)\,\frac{(1+\theta s)^n}{(1+\theta)^n} \end{aligned}$$

$$G'_Y(s) = (1-\rho)\,n\theta\frac{(1+\theta s)^{n-1}}{(1+\theta)^n}$$

$$G''_Y(s) = (1-\rho)\,n(n-1)\theta^2\frac{(1+\theta s)^{n-2}}{(1+\theta)^n}$$

setting $s = 1,$ we obtain

$$G'_Y(1) = (1-\rho)\,\frac{n\theta}{1+\theta}$$

$$G''_Y(s) = (1-\rho)\,n(n-1)\theta^2\frac{(1+\theta s)^{n-2}}{(1+\theta)^n}$$

To obtain the mean and variance of ZIBin using pgf

$$E(Y) = G'_Y(1) = (1-\rho)\,\frac{n\theta}{1+\theta}$$

$$\begin{aligned} Var(Y) &= G''_Y(1) + G'_Y(1) - [G'_Y(1)]^2 \\ &= (1-\rho)\,\frac{n(n-1)\theta^2}{(1+\theta)^2} + (1-\rho)\,\frac{n\theta}{1+\theta} - \left[(1-\rho)\,\frac{n\theta}{1+\theta}\right]^2 \\ &= (1-\rho)\left\{\frac{\theta^2 n^2}{(1+\theta)^2} - \frac{n\theta^2}{(1+\theta)^2} + n\frac{\theta}{1+\theta} - (1-\rho)\,\frac{n^2\theta^2}{(1+\theta)^2}\right\} \\ &= (1-\rho)\left\{\frac{\theta^2 n^2}{(1+\theta)^2}\,(1-1+\rho) + n\frac{\theta}{1+\theta}\left[1-\frac{\theta}{1+\theta}\right]\right\} \\ &= (1-\rho)\left\{\frac{\theta^2 n^2}{(1+\theta)^2}\rho + n\frac{\theta}{1+\theta}\left[\frac{1}{1+\theta}\right]\right\} \end{aligned}$$

viii. The moment generating function of ZIBin is given by

$$\begin{aligned} M_Y(t) &= \rho + (1-\rho)\,\frac{f\,(\theta e^t)}{f(\theta)} \\ &= \rho + (1-\rho)\left(\frac{1+\theta e^t}{1+\theta}\right)^n \end{aligned}$$

The r^{th} moment is obtained from the r^{th} derivative of $M_Y(t)$ w.r.t t and setting $t = 0$ i.e.

$$\mu'_r = \frac{d^r M_Y(t)}{dt^r}\,|_{t=0}$$

For $r = 1$, we have

$$\mu_1' = \frac{d}{dt}\left\{\rho + (1-\rho)\left(\frac{1+\theta e^t}{1+\theta}\right)^n\right\}|_{t=0}$$
$$= (\ 1-\rho)\, n\left(\frac{1+\theta e^t}{1+\theta}\right)^{n-1}\left(\frac{\theta e^t}{1+\theta}\right)|_{t=0}$$
$$= (\ 1-\rho)\, n\left(\frac{\theta}{1+\theta}\right)$$

$r = 2$

$$\mu_2' = (\ 1-\rho)\, n\frac{d}{dt}\left(\frac{1+\theta e^t}{1+\theta}\right)^{n-1}\left(\frac{\theta e^t}{1+\theta}\right)|_{t=0}$$
$$= (\ 1-\rho)\, n\left\{\begin{array}{c}(n-1)\left(\frac{1+\theta e^t}{1+\theta}\right)^{n-2}\left(\frac{\theta e^t}{1+\theta}\right)^2\\ +\left(\frac{1+\theta e^t}{1+\theta}\right)^{n-1}\left(\frac{\theta e^t}{1+\theta}\right)\end{array}\right\}|_{t=0}$$
$$= (\ 1-\rho)\, n\,(n-1)\left(\frac{\theta}{1+\theta}\right)^2 + (\ 1-\rho)\, n\left(\frac{\theta}{1+\theta}\right)$$

The variance is given by

$$\mu_2 = \mu_2' - \mu_1'^2$$
$$= (\ 1-\rho)\left\{n\,(n-1)\left(\frac{\theta}{1+\theta}\right)^2 +\ n\left(\frac{\theta}{1+\theta}\right) - (1-\rho)\frac{n^2\theta^2}{(1+\theta)^2}\right\}$$
$$= (\ 1-\rho)\left\{\frac{\theta^2 n^2}{(1+\theta)^2} - \frac{n\theta^2}{(1+\theta)^2} + n\frac{\theta}{1+\theta} - (1-\rho)\frac{n^2\theta^2}{(1+\theta)^2}\right\}$$
$$= (1-\rho)\left\{\frac{\theta^2 n^2}{(1+\theta)^2}(1-1+\rho) + n\frac{\theta}{1+\theta}\left[1-\frac{\theta}{1+\theta}\right]\right\}$$
$$= (1-\rho)\left\{\frac{\theta^2 n^2}{(1+\theta)^2}\rho + n\frac{\theta}{1+\theta}\left[\frac{1}{1+\theta}\right]\right\}$$

ix. Factorial moment generating function of ZIBin is given by

$$M_{[Y]}(t) = \rho + (1-\rho)\frac{f(\theta+\theta t)}{f(\theta)}$$
$$= \rho + (1-\rho)\left[\frac{1+\theta+\theta t}{1+\theta}\right]^n$$

The r^{th} factorial moment is obtained from the r^{th} derivative of $M_{[Y]}(t)$ w.r.t t and setting $t = 0$

$$\mu_{[r]} = \frac{d^r M_{[Y]}(t)}{dt^r}|_{t=0}$$

setting $r = 1$

$$\mu_{[1]} = \frac{d}{dt}\left\{\rho + (1-\rho)\left[\frac{1+\theta+\theta t}{1+\theta}\right]^n\right\}|_{t=0}$$
$$= (1-\rho)\, n\left[\frac{1+\theta+\theta t}{1+\theta}\right]^{n-1}\left(\frac{\theta}{1+\theta}\right)|_{t=0}$$
$$= (1-\rho)\, n\left(\frac{\theta}{1+\theta}\right)$$

$r = 2$

$$\mu_{[2]} = (1-\rho)\, n \frac{d}{dt} \left\{ \left[\frac{1+\theta+\theta t}{1+\theta}\right]^{n-1} \left(\frac{\theta}{1+\theta}\right) \right\} |_{t=0}$$
$$= (1-\rho)\, n\,(n-1) \left[\frac{1+\theta+\theta t}{1+\theta}\right]^{n-2} \left(\frac{\theta}{1+\theta}\right)^2 |_{t=0}$$
$$= (1-\rho)\, n\,(n-1) \left(\frac{\theta}{1+\theta}\right)^2$$

$r = 3$

$$\mu_{[3]} = (1-\rho)\, n\,(n-1) \frac{d}{dt} \left[\frac{1+\theta+\theta t}{1+\theta}\right]^{n-2} \left(\frac{\theta}{1+\theta}\right)^2 |_{t=0}$$
$$= (1-\rho)\, n\,(n-1)\,(n-2) \left[\frac{1+\theta+\theta t}{1+\theta}\right]^{n-3} \left(\frac{\theta}{1+\theta}\right)^3 |_{t=0}$$
$$= (1-\rho)\, n\,(n-1)\,(n-2) \left(\frac{\theta}{1+\theta}\right)^3$$

$r = 4$

$$\mu_{[4]} = (1-\rho)\, n\,(n-1)\,(n-2) \frac{d}{dt} \left[\frac{1+\theta+\theta t}{1+\theta}\right]^{n-3} \left(\frac{\theta}{1+\theta}\right)^3 |_{t=0}$$
$$= (1-\rho)\, n\,(n-1)\,(n-2)\,(n-3) \left[\frac{1+\theta+\theta t}{1+\theta}\right]^{n-4} \left(\frac{\theta}{1+\theta}\right)^4 |_{t=0}$$
$$= (1-\rho)\, n\,(n-1)\,(n-2)\,(n-3) \left(\frac{\theta}{1+\theta}\right)^4$$

Therefore the recursive relationship between factorial moments of ZIBin is given by

$$\mu_{[r]} = (n-r+1) \left(\frac{\theta}{1+\theta}\right) \mu_{[r-1]}$$

x. The cumulant generating function of ZIBin is given by

$$K_Y(t) = \log M_Y(t)$$
$$= \log \left\{ \rho + (1-\rho) \left(\frac{1+\theta e^t}{1+\theta}\right)^n \right\}$$

The r^{th} cumulant of the distribution is obtained from the r^{th} derivative of $K_Y(t)$ w.r.t t and setting $t = 0$.

That is,

$$k_r = \frac{d^r K_Y(t)}{dt^r} |_{t=0}$$

When $r=1$, we have

$$\begin{aligned}
k_1 &= \frac{dK_Y(t)}{dt}\,|_{t=0} \\
&= \frac{d}{dt}\log\left\{\rho+(1-\rho)\left(\frac{1+\theta e^t}{1+\theta}\right)^n\right\}|_{t=0} \\
&= \frac{1}{\rho+(1-\rho)\left(\frac{1+\theta e^t}{1+\theta}\right)^n}\cdot(1-\rho)\,n\left(\frac{1+\theta e^t}{1+\theta}\right)^{n-1}\left(\frac{\theta e^t}{1+\theta}\right)|_{t=0} \\
&= (1-\rho)\,n\frac{\theta}{1+\theta}
\end{aligned}$$

$r=2$

$$\begin{aligned}
k_2 &= \frac{d}{dt}\left\{\frac{(1-\rho)\,n\left(\frac{1+\theta e^t}{1+\theta}\right)^{n-1}\left(\frac{\theta e^t}{1+\theta}\right)}{\rho+(1-\rho)\left(\frac{1+\theta e^t}{1+\theta}\right)^n}\right\}|_{t=0} \\
&= \left\{\frac{\begin{array}{c}(1-\rho)\,n\,(n-1)\left(\frac{1+\theta e^t}{1+\theta}\right)^{n-2}\left(\frac{\theta e^t}{1+\theta}\right)^2 \\ +\,(1-\rho)\,n\left(\frac{1+\theta e^t}{1+\theta}\right)^{n-1}\left(\frac{\theta e^t}{1+\theta}\right)-(1-\rho)^2\,n^2\left(\frac{\theta e^t}{1+\theta}\right)^2\end{array}}{\left(\rho+(1-\rho)\left(\frac{1+\theta e^t}{1+\theta}\right)^n\right)^2}\right\}|_{t=0} \\
&= \frac{(1-\rho)\,n\,(n-1)\left(\frac{\theta}{1+\theta}\right)^2+(1-\rho)\,n\left(\frac{\theta}{1+\theta}\right)-(1-\rho)^2\,n^2\left(\frac{\theta}{1+\theta}\right)^2}{(\rho+1-\rho)^2} \\
&= (1-\rho)\left\{n\,(n-1)\left(\frac{\theta}{1+\theta}\right)^2+n\left(\frac{\theta}{1+\theta}\right)-(1-\rho)\,n^2\left(\frac{\theta}{1+\theta}\right)^2\right\} \\
&= (\;1-\rho)\left\{\frac{\theta^2n^2}{(1+\theta)^2}-\frac{n\theta^2}{(1+\theta)^2}+n\frac{\theta}{1+\theta}-(1-\rho)\,\frac{n^2\theta^2}{(1+\theta)^2}\right\} \\
&= (1-\rho)\left\{\frac{\theta^2n^2}{(1+\theta)^2}\,(1-1+\rho)+n\frac{\theta}{1+\theta}\left[1-\frac{\theta}{1+\theta}\right]\right\} \\
&= (1-\rho)\left\{\frac{\theta^2n^2}{(1+\theta)^2}\rho+n\frac{\theta}{1+\theta}\left[\frac{1}{1+\theta}\right]\right\}
\end{aligned}$$

$r=3$

$$\begin{aligned}
k_3 &= (1-\rho)\,n\frac{d}{dt}\left\{\frac{\begin{array}{c}(n-1)\left(\frac{1+\theta e^t}{1+\theta}\right)^{n-2}\left(\frac{\theta e^t}{1+\theta}\right)^2+\left(\frac{1+\theta e^t}{1+\theta}\right)^{n-1}\left(\frac{\theta e^t}{1+\theta}\right) \\ -\,(1-\rho)\,n\left(\frac{\theta e^t}{1+\theta}\right)^2\end{array}}{\left(\rho+(1-\rho)\left(\frac{1+\theta e^t}{1+\theta}\right)^n\right)^2}\right\}|_{t=0} \\
&= (1-\rho)\,n\left\{\begin{array}{c}(n-1)\,(n-2)\left(\frac{\theta}{1+\theta}\right)^3+(n-1)\left(\frac{\theta}{1+\theta}\right)^2+(n-1)\left(\frac{\theta}{1+\theta}\right)^2+2\left(\frac{\theta}{1+\theta}\right) \\ -2\,(1-\rho)\,n\left(\frac{\theta}{1+\theta}\right)^2\end{array}\right\}
\end{aligned}$$

4.11.3 Zero-Inflated Negative Binomial Distribution (ZINB)

$$f(\theta) = (1-\theta)^{-\alpha} = \sum_{k=0}^{\infty} \binom{-\alpha}{k} (-\theta)^k$$

From this we obtain

$$\begin{aligned} \Pr(X=k) &= \binom{-\alpha}{k} \frac{(-\theta)^k}{(1-\theta)^{-\alpha}} \\ &= (-1)^k \binom{-\alpha}{k} \theta^k (1-\theta)^\alpha \\ &= \binom{\alpha+k-1}{k} \theta^k (1-\theta)^\alpha, \text{ for } k = 0, 1, 2, ... \end{aligned}$$

Which is a Negative Binomial Distribution. By definition the probability mass function of ZIPSD is given by,

$$\Pr(Y=k) = \begin{cases} \rho + (1-\rho) p_0 & for \quad k = 0 \\ (1-\rho) p_k & for \quad k = 1, 2, 3, \ldots \end{cases}$$

Therefore

i.
$$\Pr(Y=k) = \begin{cases} \rho + (1-\rho)(1-\theta)^\alpha & for \quad k = 0 \\ (1-\rho) \binom{\alpha+k-1}{k} \theta^k (1-\theta)^\alpha & for \quad k = 1, 2, 3, \ldots \end{cases}$$

which is the probability mass function of Zero Inflated Negative Binomial Distribution

ii.
$$f'(\theta) = \alpha (1-\theta)^{-\alpha-1}$$

iii.
$$f''(\theta) = \alpha(\alpha+1)(1-\theta)^{-\alpha-2}$$

iv. The mean is given by

$$E(Y) = (1-\rho)\theta\alpha \frac{(1-\theta)^{-\alpha-1}}{(1-\theta)^{-\alpha}} = (1-\rho)\alpha\frac{\theta}{1-\theta}, \; 0 < \theta < 1$$

v. Variance is given by

$$\begin{aligned} Var(Y) &= (1-\rho)\theta^2 \frac{f''(\theta)}{f(\theta)} + (1-\rho)\theta\frac{f'(\theta)}{f(\theta)} - \left[(1-\rho)\theta\frac{f'(\theta)}{f(\theta)}\right]^2 \\ &= (1-\rho)\left\{\theta^2 \frac{\alpha(\alpha+1)(1-\theta)^{-\alpha-2}}{(1-\theta)^{-\alpha}} + \alpha\frac{\theta}{1-\theta} - (1-\rho)\alpha^2 \frac{\theta^2}{(1-\theta)^2}\right\} \\ &= (1-\rho)\left\{\frac{\alpha^2\theta^2}{(1-\theta)^2} + \frac{\alpha\theta^2}{(1-\theta)^2} + \frac{\alpha\theta}{1-\theta} - (1-\rho)\frac{\alpha^2\theta^2}{(1-\theta)^2}\right\} \\ &= (1-\rho)\left\{\frac{\alpha^2\theta^2}{(1-\theta)^2}(1-1+\rho) + \frac{\alpha\theta}{1-\theta}\left(1+\frac{\theta}{1-\theta}\right)\right\} \\ &= (1-\rho)\left\{\frac{\alpha^2\theta^2}{(1-\theta)^2}\rho + \frac{\alpha\theta}{1-\theta}\left(\frac{1}{1-\theta}\right)\right\} \end{aligned}$$

vi.

$$\mu_{r+1} = \theta\left[\frac{d}{d\theta}\mu_r + r\mu_{r-1}\frac{d}{d\theta}\mu_1'\right] + \rho\left(-\mu_1'\right)^{r+1} - \left[\mu_r - \rho\left(-\mu_1'\right)^r\right]\left[\mu_1' - \frac{\mu_1'}{(1-\rho)}\right]$$

but

$$\mu_1' = E(Y) = (1-\rho)\,\alpha\frac{\theta}{1-\theta}$$

Thus, the recurrence relation for the central moments of ZINB is given by

$$\begin{aligned}
\mu_{r+1} &= \theta\left[\frac{d}{d\theta}\mu_r + r\mu_{r-1}(1-\rho)\,\alpha\frac{d}{d\theta}\frac{\theta}{1-\theta}\right] + \rho\left(-(1-\rho)\,\alpha\frac{\theta}{1-\theta}\right)^{r+1} \\
&- \left[\mu_r - \rho\left(-(1-\rho)\,\alpha\frac{\theta}{1-\theta}\right)^r\right]\left[(1-\rho)\,\alpha\frac{\theta}{1-\theta} - \alpha\frac{\theta}{1-\theta}\right] \\
&= \theta\left[\frac{d}{d\theta}\mu_r + (1-\rho)\,\frac{\alpha r\mu_{r-1}}{(1-\theta)^2}\right] + \rho\left(-(1-\rho)\,\alpha\frac{\theta}{1-\theta}\right)^{r+1} \\
&- \left[\mu_r - \rho\left(-\frac{(1-\rho)\,\alpha\theta}{1-\theta}\right)^r\right]\left[\frac{(1-\rho)\,\alpha\theta}{1-\theta} - \alpha\frac{\theta}{1-\theta}\right]
\end{aligned}$$

setting $r = 1$

$$\begin{aligned}
\mu_2 &= \theta\left[\frac{d}{d\theta}\mu_1 + (1-\rho)\,\frac{\alpha\mu_0}{(1-\theta)^2}\right] + \rho\left(-(1-\rho)\,\alpha\frac{\theta}{1-\theta}\right)^2 \\
&- \left[\mu_1 - \rho\left(-\frac{(1-\rho)\,\alpha\theta}{1-\theta}\right)\right]\left[\frac{(1-\rho)\,\alpha\theta}{1-\theta} - \alpha\frac{\theta}{1-\theta}\right] \\
&= (1-\rho)\,\frac{\alpha\theta}{(1-\theta)^2} + \rho(1-\rho)^2\alpha^2\left(\frac{\theta}{1-\theta}\right)^2 - \rho(1-\rho)^2\alpha^2\left(\frac{\theta}{1-\theta}\right)^2 \\
&+ \rho\frac{(1-\rho)\,\alpha^2\theta^2}{(1-\theta)^2} \\
&= (1-\rho)\left\{\frac{\alpha\theta}{(1-\theta)^2} + \rho\frac{\alpha^2\theta^2}{(1-\theta)^2}\right\} \\
&= (1-\rho)\left\{\frac{\alpha\theta}{(1-\theta)}\left[\frac{1}{(1-\theta)}\right] + \rho\frac{\alpha^2\theta^2}{(1-\theta)^2}\right\}
\end{aligned}$$

vii. Probability generating function for ZINB is given by

$$\begin{aligned}
G_Y(s) &= \rho + (1-\rho)\,\frac{f(\theta s)}{f(\theta)} \\
&= \rho + (1-\rho)\,\frac{(1-\theta s)^{-\alpha}}{(1-\theta)^{-\alpha}}
\end{aligned}$$

$$G_Y'(s) = (1-\rho)\,\alpha\theta\frac{(1-\theta s)^{-\alpha-1}}{(1-\theta)^{-\alpha}}$$

$$G_Y''(s) = (1-\rho)\,\alpha(\alpha+1)\theta^2\frac{(1-\theta s)^{-\alpha-2}}{(1-\theta)^{-\alpha}}$$

setting $s = 1$ we obtain

$$G'_Y(1) = (1-\rho)\,\frac{\alpha\theta}{(1-\theta)}$$

$$G''_Y(1) = (1-\rho)\,\frac{\alpha(\alpha+1)\theta^2}{(1-\theta)^2}$$

To obtain the mean and variance

$$E(Y) = G'_Y(1) = (1-\rho)\,\frac{\alpha\theta}{(1-\theta)}$$

$$\begin{aligned}
Var(Y) &= G''_Y(1) + G'_Y(1) - [G'_Y(1)]^2 \\
&= (1-\rho)\,\frac{\alpha(\alpha+1)\theta^2}{(1-\theta)^2} + (1-\rho)\,\frac{\alpha\theta}{(1-\theta)} - \left[(1-\rho)\,\frac{\alpha\theta}{(1-\theta)}\right]^2 \\
&= (1-\rho)\left\{\frac{\theta^2\alpha^2}{(1-\theta)^2} - \frac{\theta^2\alpha}{(1-\theta)^2} + \frac{\alpha\theta}{(1-\theta)} - (1-\rho)\,\frac{\alpha^2\theta^2}{(1-\theta)^2}\right\} \\
&= (1-\rho)\left\{\frac{\theta^2\alpha^2}{(1-\theta)^2}(1-1+\rho) + \alpha\frac{\theta}{1-\theta}\left[1-\frac{\theta}{1-\theta}\right]\right\} \\
&= (1-\rho)\left\{\frac{\theta^2\alpha^2}{(1-\theta)^2}\rho + \alpha\frac{\theta}{1-\theta}\left[\frac{1}{1-\theta}\right]\right\}
\end{aligned}$$

viii. The moment generating function of ZINB is given by

$$\begin{aligned}
M_Y(t) &= \rho + (1-\rho)\,\frac{f(\theta e^t)}{f(\theta)} \\
&= \rho + (1-\rho)\left(\frac{1-\theta e^t}{1-\theta}\right)^{-\alpha}
\end{aligned}$$

The r^{th} moment is obtained from the r^{th} derivative of $M_Y(t)$ w.r.t t and setting $t = 0$ i.e.

$$\mu'_r = \frac{d^r M_Y(t)}{dt^r}\,|_{t=0}$$

For $r = 1$, we have

$$\begin{aligned}
\mu'_1 &= \frac{d}{dt}\left\{\rho + (1-\rho)\left(\frac{1-\theta e^t}{1-\theta}\right)^{-\alpha}\right\}|_{t=0} \\
&= (\,1-\rho)\,(-\alpha)\left(\frac{1-\theta e^t}{1-\theta}\right)^{-\alpha-1}\left(\frac{-\theta e^t}{1-\theta}\right)|_{t=0} \\
&= (\,1-\rho)\,\alpha\left(\frac{\theta}{1-\theta}\right)
\end{aligned}$$

$r = 2$

$$\mu_2' = (\ 1-\rho)\,\alpha\frac{d}{dt}\ \left(\frac{1-\theta e^t}{1-\theta}\right)^{-\alpha-1}\left(\frac{\theta e^t}{1-\theta}\right)|_{t=0}$$

$$= \left\{\begin{array}{c} (\ 1-\rho)\,\alpha\,(\alpha+1)\left(\frac{\theta e^t}{1-\theta}\right)^2\left(\frac{1-\theta e^t}{1-\theta}\right)^{-\alpha-2} \\ +(\ 1-\rho)\,\alpha\left(\frac{1-\theta e^t}{1-\theta}\right)^{-\alpha-1}\left(\frac{\theta e^t}{1-\theta}\right) \end{array}\right\}|_{t=0}$$

$$= (\ 1-\rho)\,\alpha\,(\alpha+1)\left(\frac{\theta}{1-\theta}\right)^2 + (\ 1-\rho)\,\alpha\left(\frac{\theta}{1-\theta}\right)$$

$$= (\ 1-\rho)\left\{\frac{\alpha^2\theta^2}{(1-\theta)^2} + \frac{\alpha\theta^2}{(1-\theta)^2}\ + \alpha\frac{\theta}{1-\theta}\right\}$$

The variance is given by

$$\mu_2 = \mu_2' - \mu_1'^2$$

$$= (\ 1-\rho)\left\{\frac{\alpha^2\theta^2}{(1-\theta)^2} + \frac{\alpha\theta^2}{(1-\theta)^2}\ + \alpha\frac{\theta}{1-\theta} - (1-\rho)\,\frac{\alpha^2\theta^2}{(1-\theta)^2}\right\}$$

$$= (\ 1-\rho)\left\{\frac{\alpha^2\theta^2}{(1-\theta)^2}\rho + \alpha\frac{\theta}{1-\theta}\left[\frac{1}{1-\theta}\right]\right\}$$

ix. Factorial moment generating function of ZINB is given by

$$M_{[Y]}(t) = \rho + (1-\rho)\,\frac{f(\theta+\theta t)}{f(\theta)}$$

$$= \rho + (1-\rho)\left[\frac{1-\theta-\theta t}{1-\theta}\right]^{-\alpha}$$

The r^{th} factorial moment is obtained from the r^{th} derivative of $M_{[Y]}(t)$ w.r.t t and setting $t = 0$

$$\mu_{[r]} = \frac{d^r M_{[Y]}(t)}{dt^r}\ |_{t=0}$$

setting $r = 1$

$$\mu_{[1]} = \frac{d}{dt}\left\{\rho + (1-\rho)\left[\frac{1-\theta-\theta t}{1-\theta}\right]^{-\alpha}\right\}|_{t=0}$$

$$= (1-\rho)\,\alpha\left[\frac{1-\theta-\theta t}{1-\theta}\right]^{-\alpha-1}\left(\frac{\theta}{1-\theta}\right)|_{t=0}$$

$$= (1-\rho)\,\alpha\left(\frac{\theta}{1-\theta}\right)$$

$r = 2$

$$\mu_{[2]} = (1-\rho)\,\alpha\left(\frac{\theta}{1-\theta}\right)\frac{d}{dt}\left[\frac{1-\theta-\theta t}{1-\theta}\right]^{-\alpha-1}|_{t=0}$$

$$= (1-\rho)\,\alpha\,(\alpha+1)\left[\frac{1-\theta-\theta t}{1-\theta}\right]^{-\alpha-2}\left(\frac{\theta}{1-\theta}\right)^2|_{t=0}$$

$$= (1-\rho)\,\alpha\,(\alpha+1)\left(\frac{\theta}{1-\theta}\right)^2$$

$r = 3$

$$\mu_{[3]} = (1-\rho)\,\alpha\,(\alpha+1)\left(\frac{\theta}{1-\theta}\right)^2 \frac{d}{dt}\left[\frac{1-\theta-\theta t}{1-\theta}\right]^{-\alpha-2} |_{t=0}$$
$$= (1-\rho)\,\alpha\,(\alpha+1)\,(\alpha+2)\left(\frac{\theta}{1-\theta}\right)^3 \left[\frac{1-\theta-\theta t}{1-\theta}\right]^{-\alpha-3} |_{t=0}$$
$$= (1-\rho)\,\alpha\,(\alpha+1)\,(\alpha+2)\left(\frac{\theta}{1-\theta}\right)^3$$

$r = 4$

$$\mu_{[4]} = (1-\rho)\,\alpha\,(\alpha+1)\,(\alpha+2)\left(\frac{\theta}{1-\theta}\right)^3 \frac{d}{dt}\left[\frac{1-\theta-\theta t}{1-\theta}\right]^{-\alpha-3} |_{t=0}$$
$$= (1-\rho)\,\alpha\,(\alpha+1)\,(\alpha+2)\,(\alpha+3)\left[\frac{1-\theta-\theta t}{1-\theta}\right]^{-\alpha-4}\left(\frac{\theta}{1-\theta}\right)^4 |_{t=0}$$
$$= (1-\rho)\,\alpha\,(\alpha+1)\,(\alpha+2)\,(\alpha+3)\left(\frac{\theta}{1-\theta}\right)^4$$

Therefore the recursive relationship between factorial moments of ZINB is given by

$$\mu_{[r]} = (n+r-1)\left(\frac{\theta}{1-\theta}\right)\mu_{[r-1]}$$

x. The cumulant generating function of ZINB is given by

$$K_Y(t) = \log M_Y(t)$$
$$= \log\left\{\rho + (1-\rho)\left(\frac{1-\theta e^t}{1-\theta}\right)^{-\alpha}\right\}$$

The r^{th} cumulant of the distribution is obtained from the r^{th} derivative of $K_Y(t)$ w.r.t t and setting $t = 0$.

That is,

$$k_r = \frac{d^r K_Y(t)}{dt^r} |_{t=0}$$

When $r = 1$, we have

$$k_1 = \frac{dK_Y(t)}{dt} |_{t=0}$$
$$= \frac{d}{dt}\log\left\{\rho + (1-\rho)\left(\frac{1-\theta e^t}{1-\theta}\right)^{-\alpha}\right\} |_{t=0}$$
$$= \frac{1}{\rho + (1-\rho)\left(\frac{1-\theta e^t}{1-\theta}\right)^{-\alpha}} \cdot (1-\rho)\,(-\alpha)\left(\frac{1-\theta e^t}{1-\theta}\right)^{-\alpha-1}\left(\frac{-\theta e^t}{1-\theta}\right) |_{t=0}$$
$$= (1-\rho)\,\alpha\frac{\theta}{1-\theta}$$

$r = 2$

$$k_2 = \frac{d}{dt}\left\{\frac{(1-\rho)\,\alpha\left(\frac{1-\theta e^t}{1-\theta}\right)^{-\alpha-1}\left(\frac{\theta e^t}{1-\theta}\right)}{\rho+(1-\rho)\left(\frac{1-\theta e^t}{1-\theta}\right)^{-\alpha}}\right\}|_{t=0}$$

$$= \left\{\frac{\begin{array}{c}(1-\rho)\,\alpha\,(\alpha+1)\left(\frac{1-\theta e^t}{1-\theta}\right)^{-\alpha-2}\left(\frac{\theta e^t}{1-\theta}\right)^2+(1-\rho)\,\alpha\left(\frac{1-\theta e^t}{1-\theta}\right)^{-\alpha-1}\left(\frac{\theta e^t}{1-\theta}\right)\\ -(1-\rho)^2\,\alpha^2\left(\frac{\theta e^t}{1-\theta}\right)^2\end{array}}{\left(\rho+(1-\rho)\left(\frac{1-\theta e^t}{1-\theta}\right)^{-\alpha}\right)^2}\right\}|_{t=0}$$

$$= \frac{(1-\rho)\,\alpha\,(\alpha+1)\left(\frac{\theta}{1-\theta}\right)^2+(1-\rho)\,\alpha\left(\frac{\theta}{1-\theta}\right)-(1-\rho)^2\,\alpha^2\left(\frac{\theta}{1-\theta}\right)^2}{(\rho+1-\rho)^2}$$

$$= (1-\rho)\left\{\alpha\,(\alpha+1)\left(\frac{\theta}{1-\theta}\right)^2+\alpha\frac{\theta}{1-\theta}-(1-\rho)\,\alpha^2\left(\frac{\theta}{1-\theta}\right)^2\right\}$$

$$= (\ 1-\rho)\left\{\frac{\alpha^2\theta^2}{(1-\theta)^2}+\frac{\alpha\theta^2}{(1-\theta)^2}+\alpha\frac{\theta}{1-\theta}-(1-\rho)\,\frac{\alpha^2\theta^2}{(1-\theta)^2}\right\}$$

$$= (1-\rho)\left\{\frac{\alpha^2\theta^2}{(1-\theta)^2}(1-1+\rho)+\alpha\frac{\theta}{1-\theta}\left[1-\frac{\theta}{1-\theta}\right]\right\}$$

$$= (1-\rho)\left\{\frac{\alpha^2\theta^2}{(1-\theta)^2}\rho+\alpha\frac{\theta}{1-\theta}\left[\frac{1}{1-\theta}\right]\right\}$$

xii Special case when $\alpha = 1$ we have

$$\Pr(Y=k) = \begin{cases}\rho+(1-\rho)(1-\theta) & \text{for} \quad k=0\\ (1-\rho)\,\theta^k\,(1-\theta) & \text{for} \quad k=1,2,3,\ldots\end{cases}$$

which is the probability mass function of Zero Inflated Geometric Distribution (ZIGD)

The mean is given by

$$E(Y) = (1-\rho)\,\frac{\theta}{1-\theta}, \qquad 0<\theta<1$$

Variance is given by

$$Var\,(Y) = (1-\rho)\left\{\frac{\theta^2}{(1-\theta)^2}\rho+\frac{\theta}{1-\theta}\left(\frac{1}{1-\theta}\right)\right\}$$

The recurrence relation for the central moments of ZIGD is given by

$$\mu_{r+1} = \theta\left[\frac{d}{d\theta}\mu_r+(1-\rho)\,\frac{r\mu_{r-1}}{(1-\theta)^2}\right]+\rho\left(-(1-\rho)\,\frac{\theta}{1-\theta}\right)^{r+1}$$
$$-\left[\mu_r-\rho\left(-\frac{(1-\rho)\,\theta}{1-\theta}\right)^r\right]\left[\frac{(1-\rho)\,\theta}{1-\theta}-\frac{\theta}{1-\theta}\right]$$

Probability generating function for ZIGD is given by

$$G_Y(s) = \rho + (1-\rho)\,\frac{(1-\theta s)^{-1}}{(1-\theta)^{-1}}$$

$$G'_Y(s) = (1-\rho)\,\alpha\theta\frac{(1-\theta s)^{-2}}{(1-\theta)^{-1}}$$

$$G''_Y(s) = (1-\rho)\,2\theta^2\frac{(1-\theta s)^{-3}}{(1-\theta)^{-1}}$$

setting $s = 1$ we obtain

$$G'_Y(1) = (1-\rho)\,\frac{\theta}{(1-\theta)}$$

$$G''_Y(1) = (1-\rho)\,\frac{2\theta^2}{(1-\theta)^2}$$

To obtain the mean and variance

$$E(Y) = G'_Y(1) = (1-\rho)\,\frac{\theta}{(1-\theta)}$$

$$\begin{aligned} Var(Y) &= G''_Y(1) + G'_Y(1) - [G'_Y(1)]^2 \\ &= (1-\rho)\left\{\frac{\theta^2}{(1-\theta)^2}\rho + \frac{\theta}{1-\theta}\left[\frac{1}{1-\theta}\right]\right\} \end{aligned}$$

The moment generating function of ZIGD is given by

$$M_Y(t) = \rho + (1-\rho)\left(\frac{1-\theta e^t}{1-\theta}\right)^{-1}$$

The r^{th} moment is obtained from the r^{th} derivative of $M_Y(t)$ w.r.t t and setting $t = 0$ i.e.

$$\mu'_r = \frac{d^r M_Y(t)}{dt^r}\,|_{t=0}$$

For $r = 1$, we have

$$\mu'_1 = (\ 1-\rho)\left(\frac{\theta}{1-\theta}\right)$$

$r = 2$

$$\mu'_2 = (\ 1-\rho)\left\{\frac{\theta^2}{(1-\theta)^2} + \frac{\theta^2}{(1-\theta)^2} + \frac{\theta}{1-\theta}\right\}$$

The variance is given by

$$\begin{aligned} \mu_2 &= \mu'_2 - \mu'^2_1 \\ &= (\ 1-\rho)\left\{\frac{\theta^2}{(1-\theta)^2} + \frac{\theta^2}{(1-\theta)^2} + \frac{\theta}{1-\theta} - (1-\rho)\,\frac{\theta^2}{(1-\theta)^2}\right\} \\ &= (\ 1-\rho)\left\{\frac{\theta^2}{(1-\theta)^2}\rho + \frac{\theta}{1-\theta}\left[\frac{1}{1-\theta}\right]\right\} \end{aligned}$$

Factorial moment generating function of is given by

$$M_{[Y]}(t) = \rho + (1-\rho)\,\frac{f(\theta+\theta t)}{f(\theta)}$$
$$= \rho + (1-\rho)\left[\frac{1-\theta-\theta t}{1-\theta}\right]^{-\alpha}$$

The r^{th} factorial moment is obtained from the r^{th} derivative of $M_{[Y]}(t)$ w.r.t t and setting $t=0$

$$\mu_{[r]} = \frac{d^r M_{[Y]}(t)}{dt^r}\,|_{t=0}$$

setting $r=1$

$$\mu_{[1]} = (1-\rho)\left(\frac{\theta}{1-\theta}\right)$$

$r=2$

$$\mu_{[2]} = (1-\rho)\,2\left(\frac{\theta}{1-\theta}\right)^2$$

$r=3$

$$\mu_{[3]} = (1-\rho)\,6\left(\frac{\theta}{1-\theta}\right)^3$$

$r=4$

$$\mu_{[4]} = (1-\rho)\,24\left(\frac{\theta}{1-\theta}\right)^4$$

Therefore the recursive relationship between factorial moments of ZIGD is given by

$$\mu_{[r]} = r\left(\frac{\theta}{1-\theta}\right)\mu_{[r-1]}$$

The cumulant generating function of ZIGD is given by

$$K_Y(t) = \log M_Y(t)$$
$$= \log\left\{\rho + (1-\rho)\left(\frac{1-\theta e^t}{1-\theta}\right)^{-1}\right\}$$

The r^{th} cumulant of the distribution is obtained from the r^{th} derivative of $K_Y(t)$ w.r.t t and setting $t=0$.

That is,

$$k_r = \frac{d^r K_Y(t)}{dt^r}\,|_{t=0}$$

When $r=1$, we have

$$k_1 = (1-\rho)\,\alpha\frac{\theta}{1-\theta}$$

$r=2$

$$k_2 = (1-\rho)\left\{\frac{\alpha^2\theta^2}{(1-\theta)^2}\rho + \alpha\frac{\theta}{1-\theta}\left[\frac{1}{1-\theta}\right]\right\}$$

4.11.4 Zero-Modified Logarithmic Series Distribution (ZILS)

$$f(\theta) = -\log(1-\theta)$$

From this we obtain

$$\Pr\left(X=k\right) = \frac{\theta^k}{-k\log(1-\theta)},\ \ k=1,2,\ldots$$

And by definition the probability mass function of ZIPSD is given by,

$$\Pr\left(Y=k\right) = \begin{cases} \rho + (1-\rho)\,p_0 & for \quad k=0 \\ (1-\rho)\,p_k & for \quad k=1,2,3,\ldots \end{cases}$$

Therefore

i.

$$\Pr\left(Y=k\right) = \begin{cases} \rho & for \quad k=0 \\ (1-\rho)\,\frac{\theta^k}{-k\log(1-\theta)} & for \quad k=1,2,3,\ldots \end{cases}$$

which is the probability mass function of Zero-Modified Logarithmic Series Distribution

ii.

$$f'(\theta) = \frac{1}{1-\theta}$$

iii.

$$f''(\theta) = \frac{1}{(1-\theta)^2}$$

iv. The mean is given by

$$\begin{aligned} E(Y) &= (1-\rho)\,\theta\frac{f'(\theta)}{f(\theta)} = (1-\rho)\,\theta\left(\frac{1}{1-\theta}\right)\cdot\frac{1}{-\log(1-\theta)} \\ &= \frac{-\left(1-\rho\right)\theta}{(1-\theta)\log(1-\theta)} \end{aligned}$$

v. Variance is given by

$$\begin{aligned} Var\left(Y\right) &= (1-\rho)\,\theta^2\frac{f''(\theta)}{f(\theta)} + (1-\rho)\,\theta\frac{f'(\theta)}{f(\theta)} - \left[(1-\rho)\,\theta\frac{f'(\theta)}{f(\theta)}\right]^2 \\ &= (1-\rho)\begin{Bmatrix} \frac{-\theta^2}{(1-\theta)^2\log(1-\theta)} - \frac{\theta}{(1-\theta)\log(1-\theta)} \\ -\left(1-\rho\right)\left[\frac{\theta}{(1-\theta)\log(1-\theta)}\right]^2 \end{Bmatrix} \\ &= (1-\rho)\left\{\frac{-\theta^2\log(1-\theta) - \theta\left(1-\theta\right)\log(1-\theta) - (1-\rho)\,\theta^2}{(1-\theta)^2\left[\log(1-\theta)\right]^2}\right\} \\ &= (1-\rho)\left\{\frac{-\theta\log(1-\theta) - (1-\rho)\,\theta^2}{(1-\theta)^2\left[\log(1-\theta)\right]^2}\right\} \\ &= -\left(1-\rho\right)\left\{\frac{\theta\log(1-\theta) + (1-\rho)\,\theta^2}{(1-\theta)^2\left[\log(1-\theta)\right]^2}\right\} \end{aligned}$$

vi.

$$\mu_{r+1} = \theta\left[\frac{d}{d\theta}\mu_r + r\mu_{r-1}\frac{d}{d\theta}\mu_1'\right] + \rho\left(-\mu_1'\right)^{r+1} - \left[\mu_r - \rho\left(-\mu_1'\right)^r\right]\left[\mu_1' - \frac{\mu_1'}{(1-\rho)}\right]$$

but

$$\mu_1' = E(Y) = \frac{-(1-\rho)\,\theta}{(1-\theta)\log(1-\theta)}$$

Thus, the recurrence relation for the central moments of ZILS is given by

$$\begin{aligned}
&\mu_{r+1}\\
&= \theta\left[\frac{d}{d\theta}\mu_r + r\mu_{r-1}(1-\rho)\frac{d}{d\theta}\left(\frac{\theta}{(1-\theta)\log(1-\theta)}\right)\right] + \rho\left(\frac{-(1-\rho)\,\theta}{(1-\theta)\log(1-\theta)}\right)^{r+1}\\
&- \left[\mu_r - \rho\left(\frac{-(1-\rho)\,\theta}{(1-\theta)\log(1-\theta)}\right)^r\right]\left[\frac{-(1-\rho)\,\theta}{(1-\theta)\log(1-\theta)} + \frac{\theta}{(1-\theta)\log(1-\theta)}\right]\\
&= \theta\left[\frac{d}{d\theta}\mu_r - (1-\rho)\,r\mu_{r-1}\frac{\log(1-\theta)+\theta}{[(1-\theta)\log(1-\theta)]^2}\right] + \rho\left(\frac{-(1-\rho)\,\theta}{(1-\theta)\log(1-\theta)}\right)^{r+1}\\
&- \left[\mu_r - \rho\left(\frac{-(1-\rho)\,\theta}{(1-\theta)\log(1-\theta)}\right)^r\right]\left[\frac{-(1-\rho)\,\theta}{(1-\theta)\log(1-\theta)} + \frac{\theta}{(1-\theta)\log(1-\theta)}\right]
\end{aligned}$$

setting $r = 1$

$$\begin{aligned}
\mu_2 &= \theta\left[\frac{d}{d\theta}\mu_1 - (1-\rho)\,r\mu_0\frac{\log(1-\theta)+\theta}{[(1-\theta)\log(1-\theta)]^2}\right] + \rho\left(\frac{-(1-\rho)\,\theta}{(1-\theta)\log(1-\theta)}\right)^2\\
&- \left[\mu_1 - \rho\left\{\frac{-(1-\rho)\,\theta}{(1-\theta)\log(1-\theta)}\right\}\right]\left[\frac{-(1-\rho)\,\theta}{(1-\theta)\log(1-\theta)} + \frac{\theta}{(1-\theta)\log(1-\theta)}\right]\\
&= -(1-\rho)\left\{\frac{\theta\log(1-\theta)+\theta^2}{[(1-\theta)\log(1-\theta)]^2}\right\} + \frac{\rho\theta^2(1-\rho)^2}{[(1-\theta)\log(1-\theta)]^2} + \frac{\rho^2\theta^2(1-\rho)}{[(1-\theta)\log(1-\theta)]}\\
&= -(1-\rho)\left\{\frac{\theta\log(1-\theta)+\theta^2+\rho\theta^2(1-\rho)+\rho^2\theta^2}{[(1-\theta)\log(1-\theta)]^2}\right\}\\
&= -(1-\rho)\left\{\frac{\theta\log(1-\theta)+\theta^2+\rho\theta^2[1-\rho+\rho]}{[(1-\theta)\log(1-\theta)]^2}\right\}\\
&= -(1-\rho)\left\{\frac{\theta\log(1-\theta)+\theta^2(1+\rho)}{[(1-\theta)\log(1-\theta)]^2}\right\}
\end{aligned}$$

vii. Probability generating function for ZILS is given by

$$\begin{aligned}
G_Y(s) &= \rho + (1-\rho)\frac{f(\theta s)}{f(\theta)}\\
&= \rho + (1-\rho)\frac{\log(1-\theta s)}{\log(1-\theta)}
\end{aligned}$$

$$G_Y'(s) = \frac{-(1-\rho)\,\theta}{(1-\theta s)\log(1-\theta)}$$

$$G_Y''(s) = \frac{-(1-\rho)\,\theta^2}{(1-\theta s)^2\log(1-\theta)}$$

setting $s = 1$ we obtain

$$G'_Y(1) = \frac{-(1-\rho)\,\theta}{(1-\theta)\log(1-\theta)}$$

$$G''_Y(1) = \frac{-(1-\rho)\,\theta^2}{(1-\theta)^2\log(1-\theta)}$$

To obtain the mean and variance

$$E(Y) = G'_Y(1) = \frac{-(1-\rho)\,\theta}{(1-\theta)\log(1-\theta)}$$

$$\begin{aligned}
Var(Y) &= G''_Y(1) + G'_Y(1) - \left[G'_Y(1)\right]^2 \\
&= \frac{-(1-\rho)\,\theta^2}{(1-\theta)^2\log(1-\theta)} - \frac{(1-\rho)\,\theta}{(1-\theta)\log(1-\theta)} - \left[\frac{-(1-\rho)\,\theta}{(1-\theta)\log(1-\theta)}\right]^2 \\
&= (1-\rho)\left\{\begin{array}{c} \frac{-\theta^2}{(1-\theta)^2\log(1-\theta)} - \frac{\theta}{(1-\theta)\log(1-\theta)} \\ -\frac{(1-\rho)\theta^2}{[(1-\theta)\log(1-\theta)]^2} \end{array}\right\} \\
&= (1-\rho)\left\{\frac{-\theta^2\log(1-\theta) - \theta\log(1-\theta) + \theta^2\log(1-\theta) - (1-\rho)\,\theta^2}{\left[(1-\theta)\log(1-\theta)\right]^2}\right\} \\
&= (1-\rho)\left\{\frac{-\theta\log(1-\theta) - (1-\rho)\,\theta^2}{\left[(1-\theta)\log(1-\theta)\right]^2}\right\} \\
&= -(1-\rho)\left\{\frac{\theta\log(1-\theta) + (1-\rho)\,\theta^2}{\left[(1-\theta)\log(1-\theta)\right]^2}\right\}
\end{aligned}$$

viii. The moment generating function of ZILS is given by

$$\begin{aligned}
M_Y(t) &= \rho + (1-\rho)\,\frac{f(\theta e^t)}{f(\theta)} \\
&= \rho + (1-\rho)\,\frac{\log(1-\theta e^t)}{\log(1-\theta)}
\end{aligned}$$

The r^{th} moment is obtained from the r^{th} derivative of $M_Y(t)$ w.r.t t and setting $t = 0$ i.e.

$$\mu'_r = \frac{d^r M_Y(t)}{dt^r}\,|_{t=0}$$

For $r = 1$, we have

$$\begin{aligned}
\mu'_1 &= \frac{d}{dt}\left\{\rho + (1-\rho)\,\frac{\log(1-\theta e^t)}{\log(1-\theta)}\right\}|_{t=0} \\
&= \frac{(1-\rho)}{\log(1-\theta)}\frac{d}{dt}\log\left(1-\theta e^t\right)|_{t=0} \\
&= \frac{-(1-\rho)\,\theta e^t}{(1-\theta e^t)\log(1-\theta)}\,|_{t=0} \\
&= \frac{-(1-\rho)\,\theta}{(1-\theta)\log(1-\theta)}
\end{aligned}$$

$r = 2$

$$\begin{aligned}
\mu_2' &= \frac{-\,(\,1-\rho)\,\theta}{\log(1-\theta)}\frac{d}{dt}\;\frac{e^t}{(1-\theta e^t)}\;|_{t=0} \\
&= \frac{-\,(\,1-\rho)\,\theta}{\log(1-\theta)}\left\{\frac{e^t\,(1-\theta e^t)+\theta e^{2t}}{(1-\theta e^t)^2}\right\}|_{t=0} \\
&= \frac{-\,(\,1-\rho)\,\theta}{(1-\theta)^2\log(1-\theta)}
\end{aligned}$$

The variance is given by

$$\begin{aligned}
\mu_2 &= \mu_2' - \mu_1'^2 \\
&= -\,(\,1-\rho)\left\{\frac{\theta}{(1-\theta)^2\log(1-\theta)} + \frac{(\,1-\rho)\;\theta^2}{[(1-\theta)\log(1-\theta)]^2}\right\} \\
&= -\,(\,1-\rho)\left\{\frac{\theta\log(1-\theta)+(\,1-\rho)\;\theta^2}{[(1-\theta)\log(1-\theta)]^2}\right\}
\end{aligned}$$

ix. Factorial moment generating function of ZILS is given by

$$\begin{aligned}
M_{[Y]}(t) &= \rho + (1-\rho)\,\frac{f(\theta+\theta t)}{f(\theta)} \\
&= \rho + (1-\rho)\,\frac{\log(1-\theta-\theta t)}{\log(1-\theta)}
\end{aligned}$$

The r^{th} factorial moment is obtained from the r^{th} derivative of $M_{[Y]}(t)$ w.r.t t and setting $t = 0$

$$\mu_{[r]} = \frac{d^r M_{[Y]}(t)}{dt^r}\;|_{t=0}$$

setting $r = 1$

$$\begin{aligned}
\mu_{[1]} &= \frac{d}{dt}\left\{\rho + (1-\rho)\log\left(\frac{1-\theta-\theta t}{1-\theta}\right)\right\}|_{t=0} \\
&= \frac{(1-\rho)}{\log(1-\theta)}\cdot\frac{-\theta}{(1-\theta-\theta t)}\;|_{t=0} \\
&= \frac{-\,(1-\rho)\,\theta}{(1-\theta)\log(1-\theta)}
\end{aligned}$$

$r = 2$

$$\begin{aligned}
\mu_{[2]} &= \frac{-\,(1-\rho)}{\log(1-\theta)}\cdot\frac{d}{dt}\left\{\frac{\theta}{(1-\theta-\theta t)}\right\}|_{t=0} \\
&= \frac{-\,(1-\rho)}{\log(1-\theta)}\left\{\frac{\theta^2}{(1-\theta-\theta t)^2}\right\}|_{t=0} \\
&= \frac{-\,(1-\rho)\,\theta^2}{(1-\theta)^2\log(1-\theta)}
\end{aligned}$$

$r = 3$

$$\mu_{[3]} = \frac{-(1-\rho)\,\theta^2}{\log(1-\theta)} \cdot \frac{d}{dt}\frac{1}{(1-\theta-\theta t)^2} \,|_{t=0}$$
$$= \frac{-2(1-\rho)\,\theta^3}{(1-\theta-\theta t)^3\log(1-\theta)} \,|_{t=0}$$
$$= \frac{-2(1-\rho)\,\theta^3}{(1-\theta)^3\log(1-\theta)}$$

$r = 4$

$$\mu_{[4]} = \frac{-2(1-\rho)\,\theta^3}{\log(1-\theta)} \frac{d}{dt}\frac{1}{(1-\theta-\theta t)^3} \,|_{t=0}$$
$$= \frac{-6(1-\rho)\,\theta^4}{(1-\theta-\theta t)^4\log(1-\theta)} \,|_{t=0}$$
$$= \frac{-6(1-\rho)\,\theta^4}{(1-\theta)^4\log(1-\theta)}$$

Therefore the recursive relationship between factorial moments of ZILS is given by

$$\mu_{[r]} = (r-1)\left(\frac{\theta}{1-\theta}\right)\mu_{[r-1]}$$

x. The cumulant generating function of ZILS is given by

$$K_Y(t) = \log M_Y(t)$$
$$= \log\left\{\rho + (1-\rho)\,\frac{\log(1-\theta e^t)}{\log(1-\theta)}\right\}$$

The r^{th} cumulant of the distribution is obtained from the r^{th} derivative of $K_Y(t)$ w.r.t t and setting $t = 0$.

That is,

$$k_r = \frac{d^r K_Y(t)}{dt^r} \,|_{t=0}$$

When $r = 1$, we have

$$k_1 = \frac{dK_Y(t)}{dt} \,|_{t=0}$$
$$= \frac{d}{dt}\log\left\{\rho + (1-\rho)\log\left(\frac{1-\theta e^t}{1-\theta}\right)\right\} \,|_{t=0}$$
$$= \frac{1}{\rho + (1-\rho)\log\left(\frac{1-\theta e^t}{1-\theta}\right)} \cdot \frac{(1-\rho)}{\log(1-\theta)}\left(\frac{-\theta e^t}{1-\theta e^t}\right) \,|_{t=0}$$
$$= \frac{-(\ 1-\rho)\,\theta}{(1-\theta)\log(1-\theta)}$$

$r = 2$

$$k_2 = -(1-\rho)\frac{d}{dt}\left\{\frac{\frac{(1-\rho)}{\log(1-\theta)}\left(\frac{-\theta e^t}{1-\theta e^t}\right)}{\rho+(1-\rho)\log\left(\frac{1-\theta e^t}{1-\theta}\right)}\right\}|_{t=0}$$

$$= \frac{-\frac{(1-\rho)\theta}{\log(1-\theta)}\left\{\frac{e^t}{(1-\theta e^t)^2}\left[\rho+(1-\rho)\log\left(\frac{1-\theta e^t}{1-\theta}\right)\right]+\frac{(1-\rho)}{\log(1-\theta)}\cdot\frac{\theta e^{2t}}{(1-\theta e^t)^2}\right\}}{\left\{\rho+(1-\rho)\log\left(\frac{1-\theta e^t}{1-\theta}\right)\right\}^2}|_{t=0}$$

$$= -(1-\rho)\theta\left\{\frac{1}{(1-\theta)^2\log(1-\theta)}+\frac{(1-\rho)\theta}{(1-\theta)^2[\log(1-\theta)]^2}\right\}$$

$$= -(1-\rho)\left\{\frac{(1-\rho)\theta^2+\theta\log(1-\theta)}{(1-\theta)^2[\log(1-\theta)]^2}\right\}$$

A summary of ZIPSD distributions with their corresponding pmf and pgf are given in the table (3.2) below

	$\Pr(Y=0)$	$\Pr(Y=k),\ S-\{0\}$	pgf
$ZIPo(\lambda,\rho)$	$\rho+(1-\rho)e^{-\lambda}$	$(1-\rho)\frac{e^{-\lambda}\lambda^k}{k!}$	$\rho+(1-\rho)e^{\lambda(s-1)}$
$ZIBin(n,p,\rho)$	$\rho+(1-\rho)(1-p)^n$	$(1-\rho)\binom{n}{k}p^k(1-p)^{n-k}$	$\rho+(1-\rho)[1-p+ps]^n$
$ZINB(\alpha,p,\rho)$	$\rho+(1-\rho)p^\alpha$	$(1-\rho)\binom{\alpha+k-1}{k}p^\alpha(1-p)^k$	$\rho+(1-\rho)\left\{\frac{p}{(1-(1-p)s)}\right\}^\alpha$
$ZILS(p,\rho)$	ρ	$(1-\rho)\frac{p^k}{-k\log(1-p)}$	$\rho+(1-\rho)\frac{\ln[1-ps]}{\ln(1-p)}$

Table 3.2 Summary of zero-inflated discrete distributions

Chapter 5

Estimation of the Parameters of Zero-Inflated Power Series Distribution

5.1 Introduction

This chapter entails the estimation of Zero-Inflated Power Series Distribution parameters based on two methods; the maximum likelihood and the method of moments. Two parameters will be estimated, the first parameter indicating inflates of zero ρ and the other parameter θ is that of power series distribution. Also, the special cases of the Zero-Inflated Power Series Distributions that includes; Zero-Inflated Poisson, zero-inflated binomial, zero-inflated negative binomial and zero-modified logarithmic series distribution will also be covered.

5.2 Moment Estimator of ZIPSD

A random sample of size n, taking the values x_1, x_2, ..., x_n from ZIPSD defined as

$$\Pr\left(X = x_i\right) = \begin{cases} \rho + (1-\rho)\,\frac{a_0}{f(\theta)} & for \qquad x_i = 0 \\ \\ (1-\rho)\,\frac{a_{x_i}\theta^{x_i}}{f(\theta)} & for \quad x_i = 1,2,3,\ldots \quad a_{x_i} > 0 \text{ and } 0 < \rho < 1 \end{cases}$$

with unknown ρ and θ. The estimates of the parameters θ and ρ is obtained by the method of moment by equating the first r sample moments about zero, to the corresponding population (distribution) moment. That is,

Let

$$m_k' = \frac{\sum_{i=1}^{n} x_i^k}{n},\ k = 1,2,\ldots,r$$

be the r^{th} sample moment about the origin.

and let

$$\mu_k' = E\left(X^k\right),\ k = 1,2,\ldots,r$$

be the corresponding r^{th} distribution moment. Where r is the number of unknown parameters. The method of moments is based on matching the sample moments with the corresponding distribution moments and is founded on the assumption that sample moments should provide good estimates of the corresponding population moments. Because the population moments u'_k are often functions of the population parameters.

For ZIPSD the number of unknown parameters is two ($r = 2$); ρ and θ. Therefore we take the first and second distribution moment given by u'_1 and μ'_2 respectively

$$u'_1 = E\left(X\right) = \left(1-\rho\right)\frac{\theta f'(\theta)}{f(\theta)}$$

and

$$\mu'_2 = E\left(X^2\right) = \left(1-\rho\right)\theta\left\{\frac{\theta}{f(\theta)}f''(\theta) + \frac{f'(\theta)}{f(\theta)}\right\}$$

Also, the first and second sample moments m'_1 and m'_2 are respectively

$$m'_1 = \frac{\sum_{i=1}^{n} x_i}{n} = \bar{x}$$

and

$$m'_2 = \frac{\sum_{i=1}^{n} x_i^2}{n}$$

equating the distribution moments with the sample moments, we get

$$\bar{x} = \left(1-\rho\right)\frac{\theta f'(\theta)}{f(\theta)}$$

and

$$\frac{\sum_{i=1}^{n} x_i^2}{n} = \left(1-\rho\right)\theta\left\{\frac{\theta}{f(\theta)}f''(\theta) + \frac{f'(\theta)}{f(\theta)}\right\}$$

We obtain moment estimators of ρ and θ by solving the above simultaneous equations.

The method is fairly simple and yields consistent estimators though these estimators are often biased. Thus the estimates obtained by the method of moments will be used as the initial estimate to the solutions of the likelihood equations, and successive improved approximation of $\hat{\theta}$ may then be found by the Newton-Raphson iteration method.

5.3 Estimation of the Parameters Using Maximum Likelihood Function

A random sample of size n, taking the values x_1, x_2, ..., x_n from ZIPSD defined as

$$\Pr\left(X = x_i\right) = \begin{cases} \rho + \left(1-\rho\right)\frac{a_0}{f(\theta)} & for \qquad x_i = 0 \\ \\ \left(1-\rho\right)\frac{a_{x_i}\theta^{x_i}}{f(\theta)} & for \quad x_i = 1, 2, 3, \ldots \quad a_{x_i} > 0 \text{ and } 0 < \rho < 1 \end{cases}$$

with unknown ρ and θ. The estimates of the parameters θ and ρ are often found by the method of maximum likelihood. Where the random sample x_1, x_2, ..., x_n are considered to be known. The joint probability mass function (pmf) of the sample is the product of individual pmf of x's, called the likelihood function defined in (5.1) and, it is a function of parameter θ and ρ. We find the values of θ and ρ that maximizes it. This is equivalent to maximizing the logarithm of the likelihood function(the log of the likelihood function) with respect to θ and ρ respectively. Mathematically it is easier to maximize the logarithm of the likelihood function than the likelihood function, as it replaces the product by the sums. Moreover, it allows the use of the central limit theorem when studying the properties of maximum likelihood estimator.

Consider the likelihood function defined by

$$L(\theta, \rho; \underline{x}) = \prod_{i=1}^{n} \left\{ \rho + (1-\rho) \frac{a_0}{f(\theta)} \right\}^{1-b_i} \left\{ (1-\rho) \frac{a_{x_i} \theta^{x_i}}{f(\theta)} \right\}^{b_i} \qquad \theta,\ \rho > 0 \tag{5.1}$$

where $b_i = 0$ if $x_i = 0$ and $b_i = 1$ if $x_i = 1, 2, 3, \ldots$.

Maximum likelihood estimators (mles) of θ and ρ is obtained by maximizing $\log L(\theta, \rho; \underline{x})$ with respect to θ and ρ respectively, where

$$\log L(\theta, \rho; \underline{x}) = \left\{ \begin{array}{c} n_0 \log \left\{ \rho + (1-\rho) \frac{a_0}{f(\theta)} \right\} + \sum_{i=1}^{n} b_i \log(1-\rho) \\ + \sum_{i=1}^{n} b_i \log a_{x_i} + \sum_{i=1}^{n} b_i x_i \log\theta - \sum_{i=1}^{n} b_i \log f(\theta) \end{array} \right\} \tag{5.2}$$

Where n_0 denotes the number of observations that are zeros in the sample. Differentiating $\log L(\theta, \rho; \underline{x})$ w.r.t ρ and θ, we get

$$\frac{\partial \log L}{\partial \rho} = \frac{n_0 \left\{ 1 - \frac{a_0}{f(\theta)} \right\}}{\rho + (1-\rho) \frac{a_0}{f(\theta)}} - \frac{\sum_{i=1}^{n} b_i}{(1-\rho)} \tag{5.3}$$

$$\frac{\partial \log L}{\partial \theta} = \frac{n_0 \left\{ \frac{-(1-\rho) a_0 f'(\theta)}{f(\theta)^2} \right\}}{\rho + (1-\rho) \frac{a_0}{f(\theta)}} + \frac{\sum_{i=1}^{n} b_i x_i}{\theta} - \frac{\sum_{i=1}^{n} b_i f'(\theta)}{f(\theta)} \tag{5.4}$$

To show that $\hat{\rho}$ and $\hat{\theta}$ are maxima we take the second derivative of the log likelihood function w.r.t ρ and θ

That is,

$$\begin{aligned} \frac{\partial^2 \log L}{\partial \rho^2} &= \frac{\partial}{\partial \rho} \left\{ \frac{n_0 \left\{ 1 - \frac{a_0}{f(\theta)} \right\}}{\rho + (1-\rho) \frac{a_0}{f(\theta)}} - \frac{\sum_{i=1}^{n} b_i}{(1-\rho)} \right\} \\ &= \frac{\partial}{\partial \rho} \left\{ \frac{n_0 \left(1 - \frac{a_0}{f(\theta)} \right)}{\rho + (1-\rho) \frac{a_0}{f(\theta)}} \right\} - \frac{\partial}{\partial \rho} \left\{ \frac{\sum_{i=1}^{n} b_i}{(1-\rho)} \right\} \\ &= -n_0 \frac{\left(1 - \frac{1}{f\theta} a_0 \right)^2}{\left(\rho + \frac{1}{f\theta} a_0 (1-\rho) \right)^2} - \frac{\sum_{i=1}^{n} b_i}{(1-\rho)^2} \end{aligned}$$

$$
\begin{aligned}
\frac{\partial^2 \log L}{\partial \theta^2} &= \frac{\partial}{\partial \theta}\left\{\frac{n_0\left(\frac{-(1-\rho)a_0 f'(\theta)}{f(\theta)^2}\right)}{\rho+(1-\rho)\frac{a_0}{f(\theta)}}+\frac{\sum_{i=1}^n b_i x_i}{\theta}-\frac{\sum_{i=1}^n b_i f'(\theta)}{f(\theta)}\right\} \\
&= \frac{\partial}{\partial \theta}\left\{\frac{n_0\left(\frac{-(1-\rho)a_0 f'(\theta)}{f(\theta)^2}\right)}{\rho+(1-\rho)\frac{a_0}{f(\theta)}}\right\}+\frac{\partial}{\partial \theta}\left\{\frac{\sum_{i=1}^n b_i x_i}{\theta}\right\}-\frac{\partial}{\partial \theta}\left\{\frac{\sum_{i=1}^n b_i f'(\theta)}{f(\theta)}\right\} \\
&= \frac{\sum_{i=1}^n b_i f'(\theta)}{f(\theta)}-\frac{\sum_{i=1}^n b_i\left[f'(\theta)\right]^2}{f(\theta)^2}-\frac{n_0(1-\rho)a_0 f''(\theta)}{f(\theta)^2\rho+(1-\rho)f(\theta)a_0} \\
&\quad -\frac{\left(-n_0(1-\rho)a_0 f'(\theta)\right)\left(2f(\theta)\rho+(1-\rho)f'(\theta)a_0\right)}{\left(f(\theta)^2\rho+(1-\rho)f(\theta)a_0\right)^2}-\frac{1}{\theta^2}\sum_{i=1}^n b_i x_i
\end{aligned}
$$

Since

$$
\frac{\partial^2 \log L}{\partial \rho^2}<0 \qquad \text{and} \qquad \frac{\partial^2 \log L}{\partial \theta^2}<0
$$

Then it has a local maximum at ρ and θ respectively.

Equating (5.3) and (5.4) to zero and solving for ρ and θ we get

$$
\frac{\sum_{i=1}^n b_i}{(1-\rho)}=\frac{n_0\left\{1-\frac{a_0}{f(\theta)}\right\}}{\rho+(1-\rho)\frac{a_0}{f(\theta)}}
$$

$$
\sum_{i=1}^n b_i\rho+\sum_{i=1}^n b_i(1-\rho)\frac{a_0}{f(\theta)}=n_0\left\{1-\frac{a_0}{f(\theta)}\right\}-\rho n_0\left\{1-\frac{a_0}{f(\theta)}\right\} \tag{5.5}
$$

but

$$
(n-n_0)=\sum_{i=1}^n b_i
$$

substituting this in (5.5). We obtain

$$
(n-n_0)\rho+(n-n_0)\frac{a_0}{f(\theta)}-(n-n_0)\rho\frac{a_0}{f(\theta)}=n_0\left\{1-\frac{a_0}{f(\theta)}\right\}-\rho n_0\left\{1-\frac{a_0}{f(\theta)}\right\}
$$

$$
\rho\left\{(n-n_0)+(n-n_0)\frac{a_0}{f(\theta)}-n_0\left(1-\frac{a_0}{f(\theta)}\right)\right\}=-(n-n_0)\frac{a_0}{f(\theta)}+n_0\left\{1-\frac{a_0}{f(\theta)}\right\}
$$

$$
\begin{aligned}
\rho &= \frac{-(n-n_0)a_0+n_0(f(\theta)-a_0)}{(n-n_0)f(\theta)-(n-n_0)a_0-n_0(f(\theta)-a_0)} \\
\rho &= \frac{-(n-n_0)a_0+n_0(f(\theta)-a_0)}{(n-n_0)(f(\theta)-a_0)-n_0(f(\theta)-a_0)} \\
\rho &= \frac{n_0(f(\theta)-a_0)-(n-n_0)a_0}{n(f(\theta)-a_0)} \\
\hat{\rho} &= \frac{n_0 f\left(\hat{\theta}\right)-na_0}{n\left(f\left(\hat{\theta}\right)-a_0\right)} \qquad (5.6)
\end{aligned}
$$

and

$$\frac{\sum_{i=1}^{n} b_i x_i}{\hat{\theta}} = \left\{ \frac{n_0 (1-\hat{\rho}) a_0 f'\left(\hat{\theta}\right)}{f\left(\hat{\theta}\right)^2 \left[\hat{\rho} + (1-\hat{\rho}) \frac{a_0}{f(\hat{\theta})}\right]} \right\} + \frac{\sum_{i=1}^{n} b_i f'\left(\hat{\theta}\right)}{f\left(\hat{\theta}\right)} \tag{5.7}$$

Where $\hat{\rho}$ and $\hat{\theta}$ are the mles of ρ and θ respectively.
Substituting the value of $\hat{\rho}$ in (5.7), we obtain

$$\begin{aligned}
\frac{\sum_{i=1}^{n} b_i x_i}{\hat{\theta}} &= \left\{ \frac{n_0 (1-\hat{\rho}) a_0 f'\left(\hat{\theta}\right)}{f\left(\hat{\theta}\right)^2 \left[\hat{\rho} + (1-\hat{\rho}) \frac{a_0}{f(\hat{\theta})}\right]} \right\} + \frac{\sum_{i=1}^{n} b_i f'\left(\hat{\theta}\right)}{f\left(\hat{\theta}\right)} \\
&= \frac{n_0 \left(\frac{[n-n_0] f(\hat{\theta})}{n\left(f(\hat{\theta})-a_0\right)}\right) a_0 f'\left(\hat{\theta}\right)}{f\left(\hat{\theta}\right)^2 \frac{n_0}{n}} + \frac{[n-n_0] f'\left(\hat{\theta}\right)}{f\left(\hat{\theta}\right)} \\
&= \left\{ \frac{[n-n_0] a_0 f'\left(\hat{\theta}\right)}{f\left(\hat{\theta}\right)\left(f\left(\hat{\theta}\right)-a_0\right)} \right\} + \frac{[n-n_0] f'\left(\hat{\theta}\right)}{f\left(\hat{\theta}\right)} \\
\frac{\sum_{i=1}^{n} b_i x_i}{\hat{\theta}} &= [n-n_0] \left\{ \frac{a_0 f'\left(\hat{\theta}\right)}{f\left(\hat{\theta}\right)\left(f\left(\hat{\theta}\right)-a_0\right)} + \frac{f'\left(\hat{\theta}\right)}{f\left(\hat{\theta}\right)} \right\} \\
\frac{\sum_{i=1}^{n} b_i x_i}{[n-n_0]} &= \hat{\theta} \left\{ \frac{a_0 f'\left(\hat{\theta}\right)}{f\left(\hat{\theta}\right)\left(f\left(\hat{\theta}\right)-a_0\right)} + \frac{f'\left(\hat{\theta}\right)}{f\left(\hat{\theta}\right)} \right\} \\
&= \hat{\theta} \frac{f'\left(\hat{\theta}\right)}{f\left(\hat{\theta}\right)} \left\{ \frac{a_0}{f\left(\hat{\theta}\right)-a_0} + 1 \right\} \\
&= \hat{\theta} \frac{f'\left(\hat{\theta}\right)}{f\left(\hat{\theta}\right)-a_0}
\end{aligned} \tag{5.7 a}$$

but $\frac{\sum_{i=1}^{n} b_i x_i}{[n-n_0]} = \bar{x}$ (the sample mean of the positive observations). Therefore replacing this in $(5.7a)$ above we obtain

$$\bar{x} = \hat{\theta} \frac{f'\left(\hat{\theta}\right)}{f\left(\hat{\theta}\right)-a_0}$$

Which is non-linear equation in θ, a numerical procedure like Newton-Raphson method can be used to find $\hat{\theta}$.

To obtain $\hat{\theta}$ using The Newton-Raphson Iteration. Let r be a root of the equation $f(\hat{\theta}) = 0$. We start with an initial estimate $\hat{\theta}_0$ of r. From initial estimate $\hat{\theta}_0$(preferably obtained by method of moment), we produce an improved estimate $\hat{\theta}_1$. From $\hat{\theta}_1$, we produce a new estimate $\hat{\theta}_2$. From $\hat{\theta}_2$, we produce a new estimate $\hat{\theta}_3$. We go on until we are 'close enough' to r or until it becomes clear that we are getting nowhere. The general style of proceeding is, if $\hat{\theta}_r$ is the current estimate, then the next estimate $\hat{\theta}_{r+1}$ is given by

$$\hat{\theta}_{r+1} = \bar{x} \left\{ \frac{f\left(\hat{\theta}_r\right) - a_0}{f'\left(\hat{\theta}_r\right)} \right\}$$

and substitute the value of $\hat{\theta}$ in (5.6) to obtain $\hat{\rho}$.

5.4 Special Cases

5.4.1 Zero-Inflated Poisson Distribution

Moment Estimator of Zero-Inflated Poisson Distribution

The first and second r^{th} moment for ZIPo are given by

$$u_1' = E(X) = (1-\rho)\theta$$
$$u_2' = E\left(X^2\right) = (1-\rho)\theta^2 + (1-\rho)\theta$$

and the first and second r^{th} sample moment are given by

$$m_1' = \frac{\sum_{i=1}^n x_i}{n} = \bar{x}$$

and

$$m_2' = \frac{\sum_{i=1}^n x_i^2}{n}$$

Equating the distribution moments with the sample moments, we get

$$\bar{x} = (1-\rho)\theta \tag{5.8}$$

and

$$\frac{\sum_{i=1}^n x_i^2}{n} = (1-\rho)\theta^2 + (1-\rho)\theta \tag{5.9}$$

Solving (5.8) and (5.9) simultaneously to get the moment estimators of ρ and θ. We have

$$\frac{\sum_{i=1}^n x_i^2}{n} = \bar{x}\theta + \bar{x}$$
$$\sum_{i=1}^n x_i^2 = \bar{x}\theta n + \bar{x}n$$
$$\sum_{i=1}^n x_i^2 - \bar{x}n = \bar{x}\theta n$$

That is

$$\hat{\theta} = \frac{\sum_{i=1}^n x_i^2 - \bar{x}n}{\bar{x}n}$$
$$= \frac{\sum_{i=1}^n x_i^2}{\bar{x}n} - 1$$

substituting the value of $\hat{\theta}$ in (5.8) we obtain

$$\bar{x} = (1-\rho)\,\theta$$

That is

$$\begin{aligned}\hat{\rho} &= 1 - \frac{\bar{x}}{\hat{\theta}} \\ &= 1 - \frac{\bar{x}}{\frac{\sum_{i=1}^{n} x_i^2}{\bar{x}n} - 1} \\ &= 1 - \frac{\bar{x}^2 n}{\sum_{i=1}^{n} x_i^2 - \bar{x}n}\end{aligned}$$

Estimation of Zero-Inflated Poisson Distribution Parameters Using Full Likelihood Function

A random sample of size n, taking the values x_1, x_2, ..., x_n observed from ZIPo distribution. The likelihood function is given by

$$L(\theta, \rho; \underline{x}) = \prod_{i=1}^{n} \left\{\rho + (1-\rho)\,e^{-\theta}\right\}^{1-b_i} \left\{(1-\rho)\,\frac{e^{-\theta}\theta^{x_i}}{x_i!}\right\}^{b_i} \qquad \theta, \rho > 0$$

where $b_i = 0$ if $x_i = 0$ and $b_i = 1$ if $x_i = 1, 2, 3, \ldots$.

The corresponding log likelihood function is given by

$$\begin{aligned}\log L(\theta, \rho; \underline{x}) = n_0 \log\left(\rho + (1-\rho)\,e^{-\theta}\right) + \sum_{i=1}^{n} b_i \log(1-\rho) \\ - \theta \sum_{i=1}^{n} b_i + \sum_{i=1}^{n} b_i x_i \log(\theta) - \sum_{i=1}^{n} b_i \log x_i!\end{aligned}$$

where n_0 = number of x_i's equal to zero in the sample.

Therefore Maximum likelihood estimators (mles) of θ and ρ is obtained by maximizing $\log L(\theta, \rho; \underline{x})$ with respect to θ and ρ respectively. That is

$$\frac{\partial \log L}{\partial \rho} = \frac{n_0\left\{1 - e^{-\theta}\right\}}{\rho + (1-\rho)\,e^{-\theta}} - \frac{\sum_{i=1}^{n} b_i}{(1-\rho)} \tag{5.10}$$

$$\frac{\partial \log L}{\partial \theta} = \frac{n_0\left\{-(1-\rho)\,e^{-\theta}\right\}}{\rho + (1-\rho)\,e^{-\theta}} + \frac{\sum_{i=1}^{n} b_i x_i}{\theta} - \sum_{i=1}^{n} b_i \tag{5.11}$$

Equating (5.10) and (5.11) to zero and solving for ρ and θ we get

$$\frac{\sum_{i=1}^{n} b_i}{(1-\rho)} = \frac{n_0\left\{1 - e^{-\theta}\right\}}{\rho + (1-\rho)\,e^{-\theta}}$$

$$\sum_{i=1}^{n} b_i \rho + \sum_{i=1}^{n} b_i (1-\rho)\,e^{-\theta} = n_0\left\{1 - e^{-\theta}\right\} - \rho n_0 \left\{1 - e^{-\theta}\right\} \tag{5.12}$$

but

$$n - n_0 = \sum_{i=1}^{n} b_i$$

substituting this in (5.12) . We obtain

$$(n - n_0)\,\rho + (n - n_0)\,e^{-\theta} - (n - n_0)\,\rho e^{-\theta} = n_0\left\{1 - e^{-\theta}\right\} - \rho n_0\left\{1 - e^{-\theta}\right\}$$

$$\rho\left\{n - ne^{-\theta}\right\} = -ne^{-\theta} + n_0$$

$$\begin{aligned}\hat{\rho} &= \frac{n_0 - ne^{-\hat{\theta}}}{n\left(1 - e^{-\hat{\theta}}\right)} \\ &= \frac{n_0 e^{\hat{\theta}} - n}{n\left(e^{\hat{\theta}} - 1\right)} \end{aligned} \tag{5.13}$$

and

$$\frac{\sum_{i=1}^{n} b_i x_i}{\hat{\theta}} = \frac{n_0\,(1 - \hat{\rho})\,e^{-\hat{\theta}}}{\hat{\rho} + (1 - \hat{\rho})\,e^{-\hat{\theta}}} + \sum_{i=1}^{n} b_i \tag{5.14}$$

Where $\hat{\rho}$ and $\hat{\theta}$ are the mles of ρ and θ respectively. Substituting the value of $\hat{\rho}$ in (5.14) .We obtain

$$\begin{aligned}\frac{\sum_{i=1}^{n} b_i x_i}{\hat{\theta}} &= \frac{n_0\left(\frac{n-n_0}{n\left(1-e^{-\hat{\theta}}\right)}\right)e^{-\hat{\theta}}}{\frac{-ne^{-\hat{\theta}}+n_0}{n\left(1-e^{-\hat{\theta}}\right)} + \left(\frac{n-n_0}{n\left(1-e^{-\hat{\theta}}\right)}\right)e^{-\hat{\theta}}} + (n - n_0) \\ &= \frac{n_0\,(n - n_0)\,e^{-\hat{\theta}}}{-ne^{-\hat{\theta}} + n_0 + ne^{-\hat{\theta}} - n_0 e^{-\hat{\theta}}} + (n - n_0) \\ &= \frac{(n - n_0)\,e^{-\hat{\theta}}}{\left(1 - e^{-\hat{\theta}}\right)} + (n - n_0) \\ \frac{\sum_{i=1}^{n} b_i x_i}{\hat{\theta}} &= \frac{(n - n_0)\,e^{-\hat{\theta}}}{1 - e^{-\hat{\theta}}} + (n - n_0) \\ \frac{\sum_{i=1}^{n} b_i x_i}{(n - n_0)} &= \hat{\theta}\left\{\frac{e^{-\hat{\theta}}}{1 - e^{-\hat{\theta}}} + 1\right\} \\ \bar{x} &= \frac{\hat{\theta}}{1 - e^{-\hat{\theta}}} \\ &= \frac{\hat{\theta}e^{\hat{\theta}}}{e^{\hat{\theta}} - 1}\end{aligned}$$

where $\bar{x}$ is the sample mean of the positive observations.

To obtain $\hat{\theta}$ using The Newton-Raphson Iteration. Let r be a root of the equation $f(\hat{\theta}) = 0$. We start with an initial estimate $\hat{\theta}_0$ of r. From initial estimate $\hat{\theta}_0$ (Preferably the estimate obtained from the method of moment), we produce an improved estimate

$\hat{\theta}_1$. From $\hat{\theta}_1$, we produce a new estimate $\hat{\theta}_2$, and the general style of proceeding is, if $\hat{\theta}_r$ is the current estimate, then the next estimate $\hat{\theta}_{r+1}$ is given by

$$\hat{\theta}_{r+1} = \bar{x}\left\{\frac{e^{\hat{\theta}_r} - 1}{e^{\hat{\theta}_r}}\right\}$$

and substitute the value of $\hat{\theta}$ in (5.13) to obtain $\hat{\rho}$.

5.4.2 Zero-Inflated Binomial Distribution

Moment Estimator of Zero-Inflated Binomial Distribution

The first and second r^{th} moment for ZIBin are given by

$$u_1' = E(X) = (1-\rho)\, n\frac{\theta}{1+\theta}$$
$$u_2' = E\left(X^2\right) = (1-\rho)\left\{\theta^2\frac{n(n-1)}{(1+\theta)^2} + n\frac{\theta}{1+\theta}\right\}$$

Equating the distribution moments above with the sample moments, we get

$$\bar{x} = (1-\rho)\, n\frac{\theta}{1+\theta} \tag{5.15}$$

and

$$\frac{\sum_{i=1}^{n} x_i^2}{n} = (1-\rho)\left\{\theta^2\frac{n(n-1)}{(1+\theta)^2} + n\frac{\theta}{1+\theta}\right\} \tag{5.16}$$

solving (5.15) and (5.16) to get the moment estimators of ρ and θ. We have

$$\frac{\sum_{i=1}^{n} x_i^2}{n} = (1-\rho)\,\theta^2\frac{n(n-1)}{(1+\theta)^2} + (1-\rho)\, n\frac{\theta}{1+\theta}$$
$$(1+\theta)\sum_{i=1}^{n} x_i^2 = \bar{x}n\,(n-1)\,\theta + \bar{x}n\,(1+\theta)$$
$$\sum_{i=1}^{n} x_i^2 + \theta\sum_{i=1}^{n} x_i^2 = \bar{x}n^2\theta - \bar{x}n\theta + \bar{x}n + \bar{x}n\theta$$
$$\bar{x}n^2\theta + \bar{x}n = \sum_{i=1}^{n} x_i^2 + \theta\sum_{i=1}^{n} x_i^2$$
$$\theta\left\{\bar{x}n^2 - \sum_{i=1}^{n} x_i^2\right\} = \sum_{i=1}^{n} x_i^2 - \bar{x}n$$
$$\hat{\theta} = \frac{\sum_{i=1}^{n} x_i^2 - \bar{x}n}{\bar{x}n^2 - \sum_{i=1}^{n} x_i^2}$$

That is

$$\hat{\theta} = \frac{\sum_{i=1}^{n} x_i^2 - \bar{x}n}{\bar{x}n^2 - \sum_{i=1}^{n} x_i^2}$$

substituting the value of $\hat{\theta}$ in (5.15) we obtain

$$\begin{aligned}\bar{x} &= (1-\rho)\, n\frac{\theta}{1+\theta} \\ &= n\frac{\theta}{1+\theta} - n\rho\frac{\theta}{1+\theta} \\ \bar{x} + \bar{x}\theta &= n\theta - n\rho\theta \\ \bar{x} + \bar{x}\theta - n\theta &= -n\rho\theta\end{aligned}$$

Therefore

$$\begin{aligned}\hat{\rho} &= 1 - \frac{\bar{x}}{n\hat{\theta}} - \frac{\bar{x}}{n} \\ &= 1 - \frac{\bar{x}}{n\left\{\frac{\sum_{i=1}^{n} x_i^2 - \bar{x}n}{\bar{x}n^2 - \sum_{i=1}^{n} x_i^2}\right\}} - \frac{\bar{x}}{n} \\ &= 1 - \frac{\bar{x}\left(\bar{x}n^2 - \sum_{i=1}^{n} x_i^2\right)}{n\sum_{i=1}^{n} x_i^2 - \bar{x}n^2} - \frac{\bar{x}}{n}\end{aligned}$$

Estimation of Zero-Inflated Binomial Distribution Parameters Using Full Likelihood Function

A random sample of size n, taking the values x_1, x_2, ..., x_n observed from ZIBin distribution. The likelihood function is given by

$$L(\theta, \rho; \underline{x}) = \prod_{i=1}^{n}\left\{\rho + (1-\rho)\left(\frac{1}{(1+\theta)}\right)^{n}\right\}^{1-b_i}\left\{(1-\rho)\binom{n}{x_i}\theta^{x_i}(1+\theta)^{-n}\right\}^{b_i} \quad \theta, \rho > 0$$

where $b_i = 0$ if $x_i = 0$ and $b_i = 1$ if $x_i = 1, 2, 3, \ldots$.

The corresponding log likelihood function is given by

$$\begin{aligned}\log L(\theta, \rho; \underline{x}) = n_0 \log\left\{\rho + (1-\rho)(1+\theta)^{-n}\right\} + \sum_{i=1}^{n} b_i \log(1-\rho) + \sum_{i=1}^{n} b_i \log\binom{n}{x_i} \\ + \sum_{i=1}^{n} b_i x_i \log(\theta) - \sum_{i=1}^{n} b_i n \log(1+\theta)\end{aligned}$$

where n_0 = number of x_i's equal to zero in the sample.

Maximum likelihood estimators (mles) of θ and ρ is obtained by maximizing $\log L(\theta, \rho; \underline{x})$ with respect to θ and ρ respectively. That is

$$\frac{\partial \log L}{\partial \rho} = \frac{n_0\left\{1 - (1+\theta)^{-n}\right\}}{\rho + (1-\rho)(1+\theta)^{-n}} - \frac{\sum_{i=1}^{n} b_i}{(1-\rho)} \tag{5.17}$$

$$\frac{\partial \log L}{\partial \theta} = \frac{n_0\left\{-(1-\rho)\, n\,(1+\theta)^{-n-1}\right\}}{\rho + (1-\rho)(1+\theta)^{-n}} + \frac{\sum_{i=1}^{n} b_i x_i}{\theta} - \frac{\sum_{i=1}^{n} b_i n}{(1+\theta)} \tag{5.18}$$

Equating (5.17) and (5.18) to zero and solving for ρ and θ we get

$$\frac{\sum_{i=1}^{n} b_i}{(1-\rho)} = \frac{n_0\left[1-(1+\theta)^{-n}\right]}{\rho+(1-\rho)(1+\theta)^{-n}}$$

$$\sum_{i=1}^{n} b_i\rho + \sum_{i=1}^{n} b_i(1-\rho)(1+\theta)^{-n} = n_0\left[1-(1+\theta)^{-n}\right] - \rho n_0\left[1-(1+\theta)^{-n}\right] \quad (5.19)$$

but

$$n - n_0 = \sum_{i=1}^{n} b_i$$

substituting this in (5.19) , we obtain

$$\begin{aligned}
(n-n_0)\rho + (n-n_0)(1-\rho)(1+\theta)^{-n} &= n_0\left[1-(1+\theta)^{-n}\right] \\
&\quad - \rho n_0\left[1-(1+\theta)^{-n}\right] \\
(n-n_0)\rho + (n-n_0)(1+\theta)^{-n} - \rho(n-n_0)(1+\theta)^{-n} &= n_0 - n_0(1+\theta)^{-n} \\
&\quad - \rho n_0 + \rho n_0(1+\theta)^{-n} \\
\rho\left\{(n-n_0) - (n-n_0)(1+\theta)^{-n} + n_0 - n_0(1+\theta)^{-n}\right\} &= n_0 - n_0(1+\theta)^{-n} \\
&\quad - (n-n_0)(1+\theta)^{-n} \\
\rho\left(n - n(1+\theta)^{-n}\right) &= n_0 - n(1+\theta)^{-n}
\end{aligned}$$

$$\begin{aligned}
\hat{\rho} &= \frac{n_0 - n\left(1+\hat{\theta}\right)^{-n}}{n\left(1-\left(1+\hat{\theta}\right)^{-n}\right)} \\
&= \frac{n_0\left(1+\hat{\theta}\right)^{n} - n}{n\left[\left(1+\hat{\theta}\right)^{n} - 1\right]} \quad (5.20)
\end{aligned}$$

and

$$\frac{\sum_{i=1}^{n} b_i x_i}{\hat{\theta}} = \frac{n_0\left\{(1-\hat{\rho})\, n\left(1+\hat{\theta}\right)^{-n-1}\right\}}{\hat{\rho}+(1-\hat{\rho})\left(1+\hat{\theta}\right)^{-n}} + \frac{\sum_{i=1}^{n} b_i n}{1+\hat{\theta}} \quad (5.21)$$

Where $\hat{\rho}$ and $\hat{\theta}$ are the mles of ρ and θ respectively. Substituting the value of $\hat{\rho}$ in (5.21) .We obtain

$$\frac{\sum_{i=1}^{n} b_i x_i}{\hat{\theta}} = \frac{n_0\left(\frac{n-n_0}{n\left(1-\left(1+\hat{\theta}\right)^{-n}\right)}\right) n\left(1+\hat{\theta}\right)^{-n-1}}{\frac{n_0-n\left(1+\hat{\theta}\right)^{-n}}{n\left(1-\left(1+\hat{\theta}\right)^{-n}\right)} + \left(\frac{n-n_0}{n\left(1-\left(1+\hat{\theta}\right)^{-n}\right)}\right)\left(1+\hat{\theta}\right)^{-n}} + \frac{(n-n_0)\,n}{1+\hat{\theta}}$$

$$= \frac{n_0\,(n-n_0)\,n\left(1+\hat{\theta}\right)^{-n-1}}{n_0 - n\left(1+\hat{\theta}\right)^{-n} + (n-n_0)\left(1+\hat{\theta}\right)^{-n}} + \frac{(n-n_0)\,n}{1+\hat{\theta}}$$

$$= \frac{n_0\,(n-n_0)\,n\left(1+\hat{\theta}\right)^{-n-1}}{n_0 - n_0\left(1+\hat{\theta}\right)^{-n}} + \frac{(n-n_0)\,n}{1+\hat{\theta}}$$

$$\frac{\sum_{i=1}^{n} b_i x_i}{\hat{\theta}} = \frac{(n-n_0)\,n\left(1+\hat{\theta}\right)^{-n-1}}{1-\left(1+\hat{\theta}\right)^{-n}} + \frac{(n-n_0)\,n}{1+\hat{\theta}}$$

$$\frac{\sum_{i=1}^{n} b_i x_i}{\hat{\theta}} = (n-n_0)\left\{\frac{n\left(1+\hat{\theta}\right)^{-n-1}}{1-\left(1+\hat{\theta}\right)^{-n}} + \frac{n}{1+\hat{\theta}}\right\}$$

$$\frac{\sum_{i=1}^{n} b_i x_i}{(n-n_0)} = \frac{\hat{\theta} n}{\left(1+\hat{\theta}\right)}\left\{\frac{\left(1+\hat{\theta}\right)^{-n}}{1-\left(1+\hat{\theta}\right)^{-n}} + 1\right\}$$

$$= \frac{n}{\left(1+\hat{\theta}\right)}\left\{\frac{\hat{\theta}}{1-\left(1+\hat{\theta}\right)^{-n}}\right\}$$

$$\bar{x} = \frac{n}{\left(1+\hat{\theta}\right)}\left\{\frac{\hat{\theta}}{1-\left(1+\hat{\theta}\right)^{-n}}\right\}$$

$$= n\left\{\frac{\hat{\theta}\left(1+\hat{\theta}\right)^{n}}{\left(1+\hat{\theta}\right)^{n+1} - \left(1+\hat{\theta}\right)}\right\}$$

where $\bar{x}$ is the sample mean of the positive observations.

To obtain $\hat{\theta}$ using Newton-Raphson Iteration. Let r be a root of the equation $f(\hat{\theta}) = 0$. We start with an initial estimate $\hat{\theta}_0$ of r. From initial estimate $\hat{\theta}_0$ (Preferably the estimate obtained from the moment estimation method), we produce an improved estimate $\hat{\theta}_1$. From $\hat{\theta}_1$, we produce a new estimate $\hat{\theta}_2$, and the general style of proceeding is, if $\hat{\theta}_r$ is the current estimate, then the next estimate $\hat{\theta}_{r+1}$ is given by

$$\hat{\theta}_{r+1} = \bar{x}\left\{\frac{(1+\hat{\theta}_r)^n - 1}{n(1+\hat{\theta}_r)^{n-1}}\right\}$$

and substitute the value of $\hat{\theta}$ in (5.20) to obtain $\hat{\rho}$.

5.4.3 Zero-Inflated Negative Binomial Distribution

Moment Estimator of Zero-Inflated Negative Binomial Distribution

The first and second r^{th} moment for ZINB are given by

$$u_1' = E(X) = (1-\rho)\,\alpha\frac{\theta}{1-\theta}$$

$$u_2' = E\left(X^2\right) = (\ 1-\rho)\left\{\frac{\alpha^2\theta^2}{(1-\theta)^2} + \frac{\alpha\theta^2}{(1-\theta)^2} \ + \alpha\frac{\theta}{1-\theta}\right\}$$

Equating the distribution moments above with the sample moments, we get

$$\bar{x} = (1-\rho)\,\alpha\frac{\theta}{1-\theta} \tag{5.22}$$

and

$$\frac{\sum_{i=1}^{n} x_i^2}{n} = (\ 1-\rho)\left\{\frac{\alpha^2\theta^2}{(1-\theta)^2} + \frac{\alpha\theta^2}{(1-\theta)^2} \ + \alpha\frac{\theta}{1-\theta}\right\} \tag{5.23}$$

Solving (5.22) and (5.23) to get the moment estimators of ρ and θ. We have

$$\frac{\sum_{i=1}^{n} x_i^2}{n} = (\ 1-\rho)\left\{\frac{\alpha^2\theta^2}{(1-\theta)^2} + \frac{\alpha\theta^2}{(1-\theta)^2} \ + \alpha\frac{\theta}{1-\theta}\right\}$$

$$(1-\theta)\sum_{i=1}^{n} x_i^2 = \frac{(\ 1-\rho)\,n\alpha^2\theta^2}{(1-\theta)^2} + \frac{(\ 1-\rho)\,n\alpha\theta^2}{(1-\theta)^2} \ + (\ 1-\rho)\,n\alpha\frac{\theta\,(1-\theta)}{(1-\theta)}$$

$$\sum_{i=1}^{n} x_i^2 - \theta\sum_{i=1}^{n} x_i^2 = \bar{x}n\theta\alpha + \bar{x}n\theta + \bar{x}n - \bar{x}n\theta$$

$$\bar{x}n\theta\alpha + \bar{x}n = \sum_{i=1}^{n} x_i^2 - \theta\sum_{i=1}^{n} x_i^2$$

$$\theta\left\{\bar{x}n\alpha + \sum_{i=1}^{n} x_i^2\right\} = \sum_{i=1}^{n} x_i^2 - \bar{x}n$$

$$\hat{\theta} = \frac{\sum_{i=1}^{n} x_i^2 - \bar{x}n}{\bar{x}n\alpha + \sum_{i=1}^{n} x_i^2}$$

That is

$$\hat{\theta} = \frac{\sum_{i=1}^{n} x_i^2 - \bar{x}n}{\bar{x}n\alpha + \sum_{i=1}^{n} x_i^2}$$

Substituting the value of $\hat{\theta}$ in (5.22) we obtain

$$\bar{x} = (1-\rho)\,\alpha\frac{\theta}{1-\theta}$$

$$= \alpha\frac{\theta}{1-\theta} - \alpha\rho\frac{\theta}{1-\theta}$$

$$\bar{x} - \bar{x}\theta = \alpha\theta - \alpha\rho\theta$$

$$\bar{x} - \bar{x}\theta - \alpha\theta = -\alpha\rho\theta$$

Therefore

$$\begin{aligned}\hat{\rho} &= 1 - \frac{\bar{x}}{\alpha\hat{\theta}} + \frac{\bar{x}}{\alpha} \\ &= 1 - \frac{\bar{x}}{\alpha\left\{\frac{\sum_{i=1}^{n} x_i^2 - \bar{x}n}{\bar{x}n\alpha + \sum_{i=1}^{n} x_i^2}\right\}} + \frac{\bar{x}}{\alpha} \\ &= 1 - \frac{\bar{x}\left\{\bar{x}n\alpha + \sum_{i=1}^{n} x_i^2\right\}}{\alpha\left\{\sum_{i=1}^{n} x_i^2 - \bar{x}n\right\}} + \frac{\bar{x}}{\alpha}\end{aligned}$$

Estimation of Zero-Inflated Negative Binomial Distribution Parameters Using Full Likelihood Function

A random sample of size n, taking the values x_1, x_2, ..., x_n observed from ZINB distribution. The likelihood function is given by

$$L\left(\theta, \rho; \underline{x}\right) = \prod_{i=1}^{n} \left\{\rho + (1-\rho)(1-\theta)^{\alpha}\right\}^{1-b_i} \left\{(1-\rho)\binom{\alpha + x_i - 1}{x_i}\theta^{x_i}(1-\theta)^{\alpha}\right\}^{b_i}$$

where $b_i = 0$ if $x_i = 0$ and $b_i = 1$ if $x_i = 1, 2, 3, \ldots$.

The corresponding log likelihood function is given by

$$\begin{aligned}\log L\left(\theta, \rho; \underline{x}\right) &= n_0 \log\left\{\rho + (1-\rho)(1-\theta)^{\alpha}\right\} + \sum_{i=1}^{n} b_i \log(1-\rho) \\ &+ \sum_{i=1}^{n} b_i \log\binom{\alpha + x_i - 1}{x_i} + \sum_{i=1}^{n} b_i x_i \log(\theta) + \sum_{i=1}^{n} b_i \alpha \log(1-\theta)\end{aligned}$$

Where n_0 = number of x_i's equal to zero in the sample.

Maximum likelihood estimators (mles) of θ and ρ can be obtained by maximizing $\log L\left(\theta, \rho; \underline{x}\right)$ with respect to θ and ρ respectively. That is

$$\frac{\partial \log L}{\partial \rho} = n_0 \frac{1 - (1-\theta)^{\alpha}}{\rho + (1-\rho)(1-\theta)^{\alpha}} - \frac{\sum_{i=1}^{n} b_i}{(1-\rho)} \tag{5.24}$$

$$\frac{\partial \log L}{\partial \theta} = -\frac{\alpha n_0 (1-\rho)(1-\theta)^{\alpha-1}}{\rho + (1-\rho)(1-\theta)^{\alpha}} + \frac{\sum_{i=1}^{n} b_i x_i}{\theta} - \frac{\sum_{i=1}^{n} b_i \alpha}{1-\theta} \tag{5.25}$$

Equating (5.24) and (5.25) to zero and solving for ρ and θ we get

$$\frac{\sum_{i=1}^{n} b_i}{(1-\rho)} = n_0 \frac{1 - (1-\theta)^{\alpha}}{\rho + (1-\rho)(1-\theta)^{\alpha}}$$

$$\sum_{i=1}^{n} b_i \rho + \sum_{i=1}^{n} b_i (1-\rho)(1-\theta)^{\alpha} = n_0\left[1 - (1-\theta)^{\alpha}\right] - \rho n_0\left[1 - (1-\theta)^{\alpha}\right] \tag{5.26}$$

but

$$n - n_0 = \sum_{i=1}^{n} b_i$$

Substituting this in (5.26) . We obtain

$$\begin{aligned}
(n-n_0)\,\rho+(n-n_0)\,(1-\rho)\,(1-\theta)^{\alpha} &= n_0\,[1-(1-\theta)^{\alpha}] \\
&\quad -\rho n_0\,[1-(1-\theta)^{\alpha}] \\
(n-n_0)\,\rho+(n-n_0)\,(1-\theta)^{\alpha}-\rho\,(n-n_0)\,(1-\theta)^{\alpha} &= n_0-n_0\,(1-\theta)^{\alpha} \\
&\quad -\rho n_0+\rho n_0\,(1-\theta)^{\alpha} \\
\rho\,(n-n_0-(n-n_0)\,(1-\theta)^{\alpha}+n_0-n_0\,(1-\theta)^{\alpha}) &= n_0-n_0\,(1-\theta)^{\alpha} \\
&\quad -(n-n_0)\,(1-\theta)^{\alpha} \\
\rho\,(n-n\,(1-\theta)^{\alpha}) &= n_0-n\,(1-\theta)^{\alpha}
\end{aligned}$$

$$\begin{aligned}
\hat{\rho} &= \frac{n_0-n\left(1-\hat{\theta}\right)^{\alpha}}{n\left(1-\left(1-\hat{\theta}\right)^{\alpha}\right)} \\
&= \frac{n_0\left(1-\hat{\theta}\right)^{-\alpha}-n}{n\left[\left(1-\hat{\theta}\right)^{-\alpha}-1\right]}
\end{aligned} \tag{5.27}$$

and

$$\frac{\sum_{i=1}^{n} b_i x_i}{\hat{\theta}} = \frac{\alpha n_0\,(1-\hat{\rho})\left(1-\hat{\theta}\right)^{\alpha-1}}{\hat{\rho}+(1-\hat{\rho})\left(1-\hat{\theta}\right)^{\alpha}} + \frac{\sum_{i=1}^{n} b_i \alpha}{1-\hat{\theta}} \tag{5.28}$$

Where $\hat{\rho}$ and $\hat{\theta}$ are the mles of ρ and θ respectively. Substituting the value of $\hat{\rho}$ in (5.28) .We obtain

$$\frac{\sum_{i=1}^{n} b_i x_i}{\hat{\theta}} = \frac{\alpha n_0 \left(\frac{n-n_0}{n\left(1-\left(1-\hat{\theta}\right)^{\alpha}\right)}\right)\left(1-\hat{\theta}\right)^{\alpha-1}}{\frac{n_0-n\left(1-\hat{\theta}\right)^{\alpha}}{n\left(1-\left(1-\hat{\theta}\right)^{\alpha}\right)}+\left(\frac{n-n_0}{n\left(1-\left(1-\hat{\theta}\right)^{\alpha}\right)}\right)\left(1-\hat{\theta}\right)^{\alpha}} + \frac{(n-n_0)\,\alpha}{1-\hat{\theta}}$$

$$= \frac{\alpha n_0 (n-n_0)\left(1-\hat{\theta}\right)^{\alpha-1}}{n_0 - n\left(1-\hat{\theta}\right)^{\alpha} + (n-n_0)\left(1-\hat{\theta}\right)^{\alpha}} + \frac{(n-n_0)\,\alpha}{1-\hat{\theta}}$$

$$= \frac{\alpha n_0 (n-n_0)\left(1-\hat{\theta}\right)^{\alpha-1}}{n_0 - n_0\left(1-\hat{\theta}\right)^{\alpha}} + \frac{(n-n_0)\,\alpha}{1-\hat{\theta}}$$

$$= \frac{\alpha n_0 (n-n_0)\left(1-\hat{\theta}\right)^{\alpha-1}}{n_0\left(1-\left(1-\hat{\theta}\right)^{\alpha}\right)} + \frac{(n-n_0)\,\alpha}{1-\hat{\theta}}$$

$$\frac{\sum_{i=1}^{n} b_i x_i}{\hat{\theta}} = \frac{(n-n_0)\,\alpha}{1-\hat{\theta}}\left\{\frac{\left(1-\hat{\theta}\right)^{\alpha}}{1-\left(1-\hat{\theta}\right)^{\alpha}}+1\right\}$$

$$\frac{\sum_{i=1}^{n} b_i x_i}{(n-n_0)} = \frac{\alpha\hat{\theta}}{1-\hat{\theta}}\left\{\frac{1}{1-\left(1-\hat{\theta}\right)^{\alpha}}\right\}$$

$$\bar{x} = \frac{\alpha\hat{\theta}}{1-\hat{\theta}}\left\{\frac{1}{1-\left(1-\hat{\theta}\right)^{\alpha}}\right\}$$

$$\bar{x} = \alpha\left\{\frac{\hat{\theta}\left(1-\hat{\theta}\right)^{-\alpha-1}}{\left(1-\hat{\theta}\right)^{-\alpha}-1}\right\}$$

where $\bar{x}$ is the sample mean of the positive observations only.

Using Newton-Raphson method first we find $\hat{\theta}$ as follows. Let r be a root of the equation $f(\hat{\theta}) = 0$. We start with an initial estimate $\hat{\theta}_0$ of r. From initial estimate $\hat{\theta}_0$(the estimate obtained from the moment estimation method), we produce an improved estimate $\hat{\theta}_1$. From $\hat{\theta}_1$, we produce a new estimate $\hat{\theta}_2$, and the general style of proceeding is, if $\hat{\theta}_r$ is the current estimate, then the next estimate $\hat{\theta}_{r+1}$ is given by

$$\hat{\theta}_{r+1} = \bar{x}\left\{\frac{\left(1-\hat{\theta}_r\right)^{-\alpha}-1}{\alpha\left(1-\hat{\theta}_r\right)^{-\alpha-1}}\right\}$$

and substitute the value of $\hat{\theta}$ in (5.27) to obtain $\hat{\rho}$.

5.4.4 Zero-Modified Logarithmic Series Distribution

Moment Estimator of Zero-Modified Logarithmic Series Distribution

The first and second r^{th} moment for ZILS are given by

$$u_1' = E(X) = \frac{-(1-\rho)\theta}{(1-\theta)\log(1-\theta)}$$

$$u_2' = E\left(X^2\right) = (1-\rho)\left\{\frac{-\theta^2}{(1-\theta)^2\log(1-\theta)} - \frac{\theta}{(1-\theta)\log(1-\theta)}\right\}$$

Equating the distribution moments above with the sample moments, we get

$$\bar{x} = \frac{-(1-\rho)\theta}{(1-\theta)\log(1-\theta)} \tag{5.29}$$

and

$$\frac{\sum_{i=1}^{n} x_i^2}{n} = (1-\rho)\left\{\frac{-\theta^2}{(1-\theta)^2\log(1-\theta)} - \frac{\theta}{(1-\theta)\log(1-\theta)}\right\} \tag{5.30}$$

We obtain moment estimators of ρ and θ by solving Eq (5.29) and Eq (5.30) to obtain

$$\begin{aligned}
\frac{\sum_{i=1}^{n} x_i^2}{n} &= \frac{-(1-\rho)\theta^2}{(1-\theta)^2\log(1-\theta)} + \frac{-(1-\rho)\theta}{(1-\theta)\log(1-\theta)} \\
\frac{\sum_{i=1}^{n} x_i^2}{n} &= \frac{\bar{x}\theta}{(1-\theta)} + \bar{x} \\
\sum_{i=1}^{n} x_i^2 - \theta\sum_{i=1}^{n} x_i^2 &= \bar{x}n\theta - \bar{x}n\theta + \bar{x}n \\
\theta\sum_{i=1}^{n} x_i^2 &= \sum_{i=1}^{n} x_i^2 - \bar{x}n \\
\hat{\theta} &= \frac{\sum_{i=1}^{n} x_i^2 - \bar{x}n}{\sum_{i=1}^{n} x_i^2}
\end{aligned}$$

Therefore

$$\hat{\theta} = 1 - \frac{\bar{x}n}{\sum_{i=1}^{n} x_i^2}$$

Substituting the value of $\hat{\theta}$ in (5.29). We obtain

$$\begin{aligned}
\bar{x} &= \frac{-(1-\rho)\theta}{(1-\theta)\log(1-\theta)} \\
\bar{x}(1-\theta)\log(1-\theta) &= -(1-\rho)\theta \\
\frac{\bar{x}(1-\theta)\log(1-\theta)}{\theta} &= -1-\rho
\end{aligned}$$

Hence

$$\hat{\rho} = \frac{\bar{x}^2 n\log\left\{\frac{\bar{x}n}{\sum_{i=1}^{n} x_i^2}\right\}}{\sum_{i=1}^{n} x_i^2 - \bar{x}n} - 1$$

Estimation of Zero-Modified Logarithmic Series Distribution Parameters Using Full Likelihood Function

A random sample of size n, taking the values x_1, x_2, ..., x_n observed from ZILS distribution. The likelihood function is given by

$$L(\theta,\rho;\underline{x}) = \prod_{i=1}^{n} \{\rho\}^{1-b_i} \left\{(1-\rho)\frac{\theta^{x_i}}{-x_i \log(1-\theta)}\right\}^{b_i} \qquad \theta,\rho > 0$$

where $b_i = 0$ if $x_i = 0$ and $b_i = 1$ if $x_i = 1, 2, 3, \ldots$.

The corresponding log likelihood function is given by

$$\log L(\theta,\rho;\underline{x}) = n_0 \log(\rho) + \sum_{i=1}^{n} b_i \log(1-\rho) + \sum_{i=1}^{n} b_i \log\frac{1}{x_i} + \sum_{i=1}^{n} b_i x_i \log(\theta) - \sum_{i=1}^{n} b_i \log(-\log(1-\theta))$$

Where n_0 = number of x_i's equal to zero in the sample.

Maximum likelihood estimators (mles) of θ and ρ can be obtained by maximizing $\log L(\theta,\rho;\underline{x})$ with respect to θ and ρ respectively. That is

$$\frac{\partial \log L}{\partial \rho} = n_0 \frac{1}{\rho} - \frac{\sum_{i=1}^{n} b_i}{(1-\rho)} \tag{5.31}$$

$$\frac{\partial \log L}{\partial \theta} = \frac{\sum_{i=1}^{n} b_i x_i}{\theta} - \frac{\sum_{i=1}^{n} b_i}{-(1-\theta)\ln(1-\theta)} \tag{5.32}$$

Equating (5.31) and (5.32) to zero and solving for ρ and θ we get

$$\frac{\sum_{i=1}^{n} b_i}{(1-\rho)} = n_0 \frac{1}{\rho}$$

$$\sum_{i=1}^{n} b_i \rho = n_0 - \rho n_0 \tag{5.33}$$

but

$$n - n_0 = \sum_{i=1}^{n} b_i$$

Substituting this in (5.33). We obtain

$$(n-n_0)\rho = n_0 - \rho n_0$$
$$(n-n_0+n_0)\rho = n_0$$
$$\rho = \frac{n_0}{n}$$

$$\hat{\rho} = \frac{n_0}{n} \tag{5.34}$$

and

$$\frac{\sum_{i=1}^{n} b_i x_i}{\hat{\theta}} = \frac{\sum_{i=1}^{n} b_i}{-\left(1-\hat{\theta}\right)\ln\left(1-\hat{\theta}\right)} \tag{5.35}$$

$$\frac{\sum_{i=1}^{n} b_i x_i}{\hat{\theta}} = \frac{(n-n_0)}{-\left(1-\hat{\theta}\right)\ln\left(1-\hat{\theta}\right)}$$

$$\frac{\sum_{i=1}^{n} b_i x_i}{(n-n_0)} = \frac{\hat{\theta}}{-\left(1-\hat{\theta}\right)\ln\left(1-\hat{\theta}\right)}$$

$$\bar{x} = \frac{\hat{\theta}}{-\left(1-\hat{\theta}\right)\ln\left(1-\hat{\theta}\right)}$$

Where $\hat{\rho}$ and $\hat{\theta}$ are the mles of ρ and θ respectively.

Using Newton-Raphson method first we find $\hat{\theta}$ as follows. Let r be a root of the equation $f(\hat{\theta}) = 0$. We start with an initial estimate $\hat{\theta}_0$ of r. From initial estimate $\hat{\theta}_0$(the estimate obtained from the moment estimation), we produce an improved estimate $\hat{\theta}_1$. From $\hat{\theta}_1$, we produce a new estimate $\hat{\theta}_2$, and the general style of proceeding is, if $\hat{\theta}_r$ is the current estimate, then the next estimate $\hat{\theta}_{r+1}$ is given by

$$\hat{\theta}_{r+1} = \bar{x}\left\{\left(1-\hat{\theta}_r\right)\left[-\ln(1-\hat{\theta}_r)\right]\right\}$$

5.4.5 Zero-Inflated Geometric Distribution

Moment Estimator of Zero-Inflated Geometric Distribution

The first and second r^{th} moment for ZINB are given by

$$u_1' = E\left(X\right) = (1-\rho)\frac{\theta}{1-\theta}$$

$$u_2' = E\left(X^2\right) = (\ 1-\rho)\left\{\frac{\theta^2}{(1-\theta)^2} + \frac{\theta^2}{(1-\theta)^2} \ + \frac{\theta}{1-\theta}\right\}$$

Equating the distribution moments above with the sample moments, we get

$$\bar{x} = (1-\rho)\frac{\theta}{1-\theta} \tag{5.36}$$

and

$$\frac{\sum_{i=1}^{n} x_i^2}{n} = (\ 1-\rho)\left\{\frac{\theta^2}{(1-\theta)^2} + \frac{\theta^2}{(1-\theta)^2} \ + \frac{\theta}{1-\theta}\right\} \tag{5.37}$$

Solving (5.36) and (5.37) to get the moment estimators of ρ and θ. We have

$$\frac{\sum_{i=1}^{n} x_i^2}{n} = (\ 1-\rho)\left\{\frac{\theta^2}{(1-\theta)^2} + \frac{\theta^2}{(1-\theta)^2} + \frac{\theta}{1-\theta}\right\}$$

$$(1-\theta)\sum_{i=1}^{n} x_i^2 = \frac{(\ 1-\rho)\,n\theta^2}{(1-\theta)^2} + \frac{(\ 1-\rho)\,n\theta^2}{(1-\theta)^2} + (\ 1-\rho)\,n\frac{\theta\,(1-\theta)}{(1-\theta)}$$

$$\sum_{i=1}^{n} x_i^2 - \theta\sum_{i=1}^{n} x_i^2 = \bar{x}n\theta + \bar{x}n\theta + \bar{x}n - \bar{x}n\theta$$

$$\bar{x}n\theta + \bar{x}n = \sum_{i=1}^{n} x_i^2 - \theta\sum_{i=1}^{n} x_i^2$$

$$\theta\left\{\bar{x}n + \sum_{i=1}^{n} x_i^2\right\} = \sum_{i=1}^{n} x_i^2 - \bar{x}n$$

$$\hat{\theta} = \frac{\sum_{i=1}^{n} x_i^2 - \bar{x}n}{\bar{x}n + \sum_{i=1}^{n} x_i^2}$$

That is

$$\hat{\theta} = \frac{\sum_{i=1}^{n} x_i^2 - \bar{x}n}{\bar{x}n + \sum_{i=1}^{n} x_i^2} \tag{5.38}$$

Substituting the value of $\hat{\theta}$ in (5.36) we obtain

$$\bar{x} = (1-\rho)\,\frac{\theta}{1-\theta}$$

$$= \frac{\theta}{1-\theta} - \rho\frac{\theta}{1-\theta}$$

$$\bar{x} - \bar{x}\theta = \theta - \rho\theta$$

$$\bar{x} - \bar{x}\theta - \theta = -\rho\theta$$

Therefore

$$\hat{\rho} = 1 - \frac{\bar{x}}{\hat{\theta}} + \bar{x}$$

$$= 1 - \frac{\bar{x}}{\left\{\frac{\sum_{i=1}^{n} x_i^2 - \bar{x}n}{\bar{x}n + \sum_{i=1}^{n} x_i^2}\right\}} + \bar{x}$$

$$= 1 - \frac{\bar{x}\left\{\bar{x}n + \sum_{i=1}^{n} x_i^2\right\}}{\left\{\sum_{i=1}^{n} x_i^2 - \bar{x}n\right\}} + \bar{x} \tag{5.39}$$

Estimation of Zero-Inflated Geometric Distribution Parameters Using Full Likelihood Function

A random sample of size n, taking the values $x_1,\ x_2,\ ...,\ x_n$ observed from ZINB distribution. The likelihood function is given by

$$L\left(\theta,\rho;\underline{x}\right) = \prod_{i=1}^{n}\left\{\rho + (1-\rho)\,(1-\theta)\right\}^{1-b_i}\left\{(1-\rho)\,\theta^{x_i}\,(1-\theta)\right\}^{b_i} \qquad \theta,\rho > 0$$

where $b_i = 0$ if $x_i = 0$ and $b_i = 1$ if $x_i = 1, 2, 3, \ldots$.

The corresponding log likelihood function is given by

$$\log L(\theta, \rho; \underline{x}) = n_0 \log\{\rho + (1-\rho)(1-\theta)\} + \sum_{i=1}^{n} b_i \log(1-\rho) + \sum_{i=1}^{n} b_i x_i \log(\theta) + \sum_{i=1}^{n} b_i \log(1-\theta)$$

Where n_0 = number of x_i's equal to zero in the sample.

Maximum likelihood estimators (mles) of θ and ρ can be obtained by maximizing $\log L(\theta, \rho; \underline{x})$ with respect to θ and ρ respectively. That is

$$\frac{\partial \log L}{\partial \rho} = n_0 \frac{\theta}{\rho + (1-\rho)(1-\theta)} - \frac{\sum_{i=1}^{n} b_i}{(1-\rho)} \tag{5.40}$$

$$\frac{\partial \log L}{\partial \theta} = -\frac{n_0(1-\rho)}{\rho + (1-\rho)(1-\theta)} + \frac{\sum_{i=1}^{n} b_i x_i}{\theta} - \frac{\sum_{i=1}^{n} b_i}{1-\theta} \tag{5.41}$$

Equating (5.40) and (5.41) to zero and solving for ρ and θ we get

$$\frac{\sum_{i=1}^{n} b_i}{(1-\rho)} = n_0 \frac{\theta}{\rho + (1-\rho)(1-\theta)}$$

$$\sum_{i=1}^{n} b_i \rho + \sum_{i=1}^{n} b_i (1-\rho)(1-\theta) = n_0\theta - \rho n_0 \theta \tag{5.42}$$

but

$$n - n_0 = \sum_{i=1}^{n} b_i$$

Substituting this in (5.42). We obtain

$$(n-n_0)\rho + (n-n_0)(1-\rho)(1-\theta) = n_0\theta - \rho n_0\theta$$

$$n - n\theta - n_0 + n\theta\rho + \theta n_0 - \theta\rho n_0 = n_0\theta - \theta\rho n_0$$

$$\hat{\rho} = \frac{n_0 - n + n\hat{\theta}}{n\hat{\theta}} \tag{5.43}$$

and

$$\frac{\sum_{i=1}^{n} b_i x_i}{\hat{\theta}} = \frac{n_0(1-\hat{\rho})}{\hat{\rho} + (1-\hat{\rho})\left(1-\hat{\theta}\right)} + \frac{\sum_{i=1}^{n} b_i}{1-\hat{\theta}} \tag{5.44}$$

Where $\hat{\rho}$ and $\hat{\theta}$ are the mles of ρ and θ respectively. Substituting the value of $\hat{\rho}$ in (5.44). We obtain

$$\begin{aligned}
\frac{\sum_{i=1}^{n} b_i x_i}{\hat{\theta}} &= \frac{n_0\left(\frac{n-n_0}{n\hat{\theta}}\right)}{\frac{n_0-n\left(1-\hat{\theta}\right)}{n\hat{\theta}}+\left(\frac{n-n_0}{n\hat{\theta}}\right)\left(1-\hat{\theta}\right)}+\frac{(n-n_0)}{1-\hat{\theta}} \\
&= \frac{n_0\,(n-n_0)}{n_0-n\left(1-\hat{\theta}\right)+(n-n_0)\left(1-\hat{\theta}\right)}+\frac{(n-n_0)}{1-\hat{\theta}} \\
&= \frac{n_0\,(n-n_0)}{n_0-n_0\left(1-\hat{\theta}\right)}+\frac{(n-n_0)}{1-\hat{\theta}} \\
&= \frac{(n-n_0)}{\hat{\theta}}+\frac{(n-n_0)}{1-\hat{\theta}} \\
\frac{\sum_{i=1}^{n} b_i x_i}{\hat{\theta}} &= (n-n_0)\left\{\frac{1}{\hat{\theta}}+\frac{1}{1-\hat{\theta}}\right\} \\
\frac{\sum_{i=1}^{n} b_i x_i}{(n-n_0)} &= \hat{\theta}\left\{\frac{1}{\hat{\theta}\left(1-\hat{\theta}\right)}\right\} \\
\bar{x} &= \frac{1}{1-\hat{\theta}} \\
1-\hat{\theta} &= \frac{1}{\bar{x}} \qquad (5.45)
\end{aligned}$$

and substitute the value of in $\hat{\theta}$ (5.43) to obtain $\hat{\rho}$

Chapter 6

Katz-Panjer Class of Recursive Relations

6.1 Introduction

The objective of this chapter is to re-examine the work of katz (1965) on recursive relation in probabilities. A class of probability of distributions and moments on the model have been re-derived. Lastly, highlights on modifications and extensions of the Katz model have also been made.

6.2 Probability Distributions

Pearson difference equation is given by

$$\frac{f(x+1)}{f(x)} = \frac{P(x)}{Q(x)} \tag{6.0}$$

Where $f(\cdot)$ is the discrete probability distribution; $P(x)$ and $Q(x)$ are polynomials. Katz (1965) considered the difference equation

$$\frac{f(x+1)}{f(x)} = \frac{\alpha+\beta x}{1+x}; x = 0, 1, 2, ... \tag{6.1a}$$

where

$$f(x) = \Pr(X = x) \geqslant 0$$

and

$$\sum_{x=0}^{\infty} f(x) = 1$$

Re-arranging (6.1a) we have

$$f(x+1) = \frac{\alpha+\beta x}{1+x} f(x) \tag{6.1b}$$

and

$$(1+x) f(x+1) = (\alpha+\beta x) f(x) \tag{6.1c}$$

We want to identify all probability distributions generated by the Katz model by considering various cases of α and β

where $f(\cdot)$ is a discrete probability function; α and β are constants *a*) **When** $\beta = 0$ **and** $\alpha \neq 0$, then (6.1*a*) becomes

$$f(x+1) = \frac{\alpha}{1+x} f(x); x = 0, 1, 2, ... \tag{6.2}$$

Thus by iterative method,

$$\begin{aligned} x = 0 \quad & \Rightarrow f(1) = \alpha f(0) \\ x = 1 \quad & \Rightarrow f(2) = \frac{\alpha}{2} f(1) = \frac{\alpha^2}{2!} f(0) \\ x = 2 \quad & \Rightarrow f(3) = \frac{\alpha}{3} f(2) = \frac{\alpha^3}{3!} f(0) \end{aligned}$$

Therefore,

$$x = k-1 \Rightarrow f(k) = \frac{\alpha}{k} f(k-1) = \frac{\alpha^k}{k!} f(0); k = 1, 2, \ldots \tag{6.3}$$

$\therefore$

$$\sum_{k=0}^{\infty} f(k) = f(0) + \sum_{k=1}^{\infty} \frac{\alpha^k}{k!} f(0)$$

i.e.

$$\begin{aligned} 1 &= \sum_{k=0}^{\infty} \frac{\alpha^k}{k!} f(0) \\ &= f(0) \sum_{k=0}^{\infty} \frac{\alpha^k}{k!} \\ &= f(0) e^{\alpha} \end{aligned}$$

Therefore,

$$f(0) = e^{-\alpha} \tag{6.4}$$

From (6.3) and (6.4)

$$f(k) = \frac{e^{-\alpha}\alpha^k}{k!}; k = 0, 1, 2, \ldots$$

Which is Poisson with parameters α.

b) **When** $\alpha \neq 0$ **and** $\beta \neq 0$, then

$$f(x+1) = \frac{\alpha + \beta x}{1+x} f(x); x = 0, 1, 2, ...$$

$$\begin{aligned} x = 0 \quad & \Rightarrow f(1) = \alpha f(0) \\ x = 1 \quad & \Rightarrow f(2) = \frac{\alpha+\beta}{2} f(1) = \frac{\alpha(\alpha+\beta)}{1 \cdot 2} f(0) \\ x = 2 \quad & \Rightarrow f(3) = \frac{\alpha+2\beta}{3} f(2) = \frac{\alpha(\alpha+\beta)(\alpha+2\beta)}{1 \cdot 2 \cdot 3} f(0) \end{aligned}$$

Therefore

$$x = k-1 \ \Rightarrow f(k) = \alpha(\alpha+\beta)(\alpha+2\beta) \cdots (\alpha+(k-1)\beta) \frac{f(0)}{k!}$$

$$f(k)=\beta^k\frac{\alpha}{\beta}\left(\frac{\alpha}{\beta}+1\right)\left(\frac{\alpha}{\beta}+2\right)\cdots\left(\frac{\alpha}{\beta}+k-1\right)\frac{f(0)}{k!}$$

Therefore

$$f(k)=\beta^k\frac{\alpha}{\beta}\left(\frac{\alpha}{\beta}+1\right)\left(\frac{\alpha}{\beta}+2\right)\cdots\left(\frac{\alpha}{\beta}+k-1\right)\frac{f(0)}{k!};k=1,2,3,\ldots \quad (6.6)$$

Case (i) Let $m=\frac{\alpha}{\beta}$ is a positive integer. Then

$$\begin{aligned}f(k)&=\beta^k m(m+1)(m+2)\cdots(m+k-1)\frac{f(0)}{k!}\\&=\beta^k\binom{m+k-1}{k}f(0);k=1,2,3,\ldots \quad (6.7)\end{aligned}$$

To determine $f(0)$ we use,

$$\begin{aligned}1=\sum_{k=0}^{\infty}f(k)&=f(0)+\sum_{k=1}^{\infty}\beta^k\binom{m+k-1}{k}f(0)\\&=f(0)\sum_{k=0}^{\infty}\beta^k\binom{m+k-1}{k}\end{aligned}$$

Therefore

$$f(0)=\frac{1}{\sum_{k=0}^{\infty}\beta^k\binom{m+k-1}{k}} \quad (6.8)$$

From (6.7) and (6.8).

Hence,

$$f(k)=\frac{\beta^k\binom{m+k-1}{k}}{\sum_{k=0}^{\infty}\beta^k\binom{m+k-1}{k}};k=0,1,2,\ldots$$

i.e.

$$f(k)=\frac{\beta^k\binom{\frac{\alpha}{\beta}+k-1}{k}}{\sum_{k=0}^{\infty}\beta^k\binom{\frac{\alpha}{\beta}+k-1}{k}};k=0,1,2,\ldots \quad (6.9)$$

which is a Negative Binomial Distribution with parameters $\frac{\alpha}{\beta}$ and $0<\beta<1$. Since $\frac{\alpha}{\beta}$ is a positive integer and $0<\beta<1$ then $\alpha>0$.

Case (ii) Let $m=\frac{\alpha}{\beta}$ is a negative integer.

Let $m=-r$ where r is a positive integer. Then (6.7) becomes

$$\begin{aligned}f(k)&=\beta^k(-r)(-r+1)(-r+2)\cdots(-r+k-1)\frac{f(0)}{k!};k=1,2,3,\ldots\\&=(-\beta)^k r(r-1)(r-2)\cdots(r-(k-1))\frac{f(0)}{k!}\\&=(-\beta)^k\binom{r}{k}f(0);k=1,2,3,\ldots r \quad (6.10)\end{aligned}$$

Therefore to obtain $f(0)$

$$\begin{aligned}1=\sum_{k=0}^{\infty}f(k)&=f(0)\sum_{k=0}^{r}(-\beta)^k\binom{r}{k}\\&=f(0)\left[1+(-\beta)\right]^r\end{aligned}$$

$\therefore$

$$f(0) = \frac{1}{(1-\beta)^r} \tag{6.11}$$

From (6.10) and (6.11)

$$\begin{aligned} f(k) &= \frac{(-\beta)^k \binom{r}{k}}{(1-\beta)^r} \\ &= \binom{r}{k} \left(\frac{-\beta}{1-\beta}\right)^k \left(\frac{1}{1-\beta}\right)^{r-k} ; k = 0, 1, 2, 3, \ldots r \end{aligned} \tag{6.12}$$

which is Binomial with parameters $\left(-\frac{\alpha}{\beta}, \frac{-\beta}{1-\beta}\right)$ where $\frac{\alpha}{\beta}$ is a negative integer and $\beta < 0$. This implies that $\alpha > 0$.

We can therefore summarize the discussion by the following theorem:
Theorem (6.1)
Let

$$f(x+1) = \left[\frac{\alpha + \beta x}{x+1}\right] f(x) \text{ for } x = 0, 1, 2, 3, \ldots$$

where $f(\cdot)$ is a discrete probability function, α and β are constants

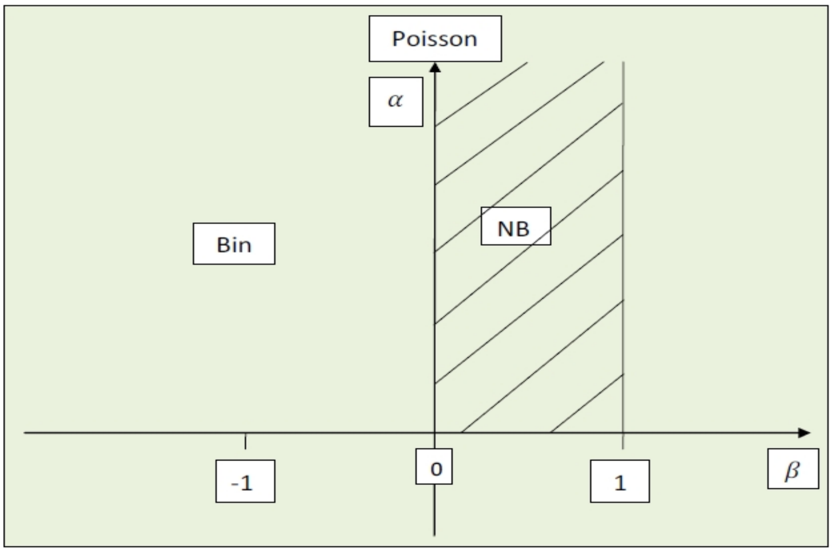
a) **When** $\beta = 0$ **and** $\alpha \neq 0$, then $f(x)$ is Poisson with parameter α.

b) **when** $\alpha \neq 0$ **and** $\beta \neq 0$, then

(*i*) $f(x)$ is NB$\left(\frac{\alpha}{\beta}, \beta\right)$ for $\frac{\alpha}{\beta}$ = a positive integer, $0 < \beta < 1$ and $\alpha > 0$

(*ii*) $f(x)$ is Bin$\left(-\frac{\alpha}{\beta}, -\frac{\beta}{1-\beta}\right)$ for $\frac{\alpha}{\beta}$ = a negative integer, $\beta < 0$ and $\alpha > 0$

Graphically we have the following figure (6.1)



Proof for **Theorem** (**6**.**1**) using the pgf technique is as follows, multiply $(6.1a)$ by $(x+1)\,s^x$ and sum the results over x. Thus

$$\sum_{x=0}^{\infty}(x+1)\,f\,(x+1)\,s^x = \alpha\sum_{x=0}^{\infty}f\,(x)\,s^x + \beta\sum_{x=0}^{\infty}xf\,(x)\,s^x$$

$$\Rightarrow \quad G'(s) = \alpha G(s) + \beta s G'(s)$$

Therefore

$$(1-\beta s)\,G'(s) = \alpha G(s)$$
$$\Rightarrow \frac{G'(s)}{G(s)} = \frac{\alpha}{1-\beta s} \tag{6.13}$$

This implies that,

$$\begin{aligned}
\frac{d}{ds}\ln G(s) &= \frac{\alpha}{1-\beta s}\\
\ln G(s) &= \int \frac{\alpha}{1-\beta s}ds + c_1\\
\ln G(s) &= \frac{\alpha}{-\beta}\ln(1-\beta s) + \ln C_1 \quad \text{let } k = \ln C_1\\
G(s) &= k\,(1-\beta s)^{-\frac{\alpha}{\beta}}\\
1 = G(1) &= k\,(1-\beta)^{-\frac{\alpha}{\beta}}\\
k &= (1-\beta)^{\frac{\alpha}{\beta}}
\end{aligned}$$

Therefore

$$G(s) = \left(\frac{1-\beta}{1-\beta s}\right)^{\frac{\alpha}{\beta}} \tag{6.14}$$

when $\alpha \neq 0$ **and** $\beta \neq 0$

Case (i) when $\frac{\alpha}{\beta}$ is a positive interger and $0<\beta<1$, then $G(s)$ is a pgf of a NB

Case (ii) when $\frac{\alpha}{\beta}$ is a negative integer $= -r$, say where r is a positive integer. Then,

$$\begin{aligned}
G(s) &= \left(\frac{1-\beta}{1-\beta s}\right)^{-r}\\
&= \left(\frac{1-\beta s}{1-\beta}\right)^{r}\\
&= \left(\frac{1}{1-\beta} + \frac{-\beta}{1-\beta}s\right)^{r}
\end{aligned}$$

which is the pgf of $\text{Bin}\left(r, \frac{-\beta}{1-\beta}\right)$. Where $r = -\frac{\alpha}{\beta}$, $\beta<0$ and $\alpha>0$.

When $\beta = 0$

Case (iii) Equation (6.13) becomes

$$\frac{G'(s)}{G(s)} = \alpha \Rightarrow \frac{d}{ds}\ln G(s) = \alpha$$

Therefore

$$\begin{aligned}\ln G(s) &= \alpha s + c\\ G(s) &= e^{c}e^{\alpha s}\end{aligned} \qquad (*)$$

setting $s = 1$ to obtain e^c we have,

$$\begin{aligned}1 = G(1) &= e^{c}e^{\alpha}\\ \therefore \quad e^{c} &= e^{-\alpha}\end{aligned}$$

Thus equation $(*)$ above becomes

$$\begin{aligned}G(s) &= e^{-\alpha}e^{\alpha s}\\ &= e^{-\alpha(1-s)}\end{aligned}$$

which is the pgf of a Poisson distribution with parameter α.

6.2.1 Panjer's model

Let $n = x + 1 \;\Rightarrow x = n - 1$. Then $(6.1a)$ becomes

$$f(n) = \frac{\alpha + \beta(n-1)}{n} f(n-1)$$

By replacing $f(n)$ and $f(n-1)$ by p_n and p_{n-1} respectively, we obtain

$$\begin{aligned}p_n &= \left(\frac{\alpha + \beta(n-1)}{n}\right) p_{n-1}\\ &= \left(\frac{\alpha - \beta + \beta n}{n}\right) p_{n-1}\\ &= \left(\beta + \frac{\alpha - \beta}{n}\right) p_{n-1} \text{ for } n = 1, 2, \ldots\end{aligned} \qquad (6.16)$$

Let $\beta = a$ and $\alpha - \beta = b$

This implies that, $a + b = \alpha$

Therefore

$$p_n = \left(a + \frac{b}{n}\right) p_{n-1} \text{ for } n = 1, 2, \ldots \text{ and } p_0 > 0.$$

Which is the Panjer's model for recursive relation.

So **Theorem** $(\mathbf{6.1})$ can be restated by putting $\alpha = a + b,\ \beta = a$ and replacing $f(x+1)$ by p_n as follows:

Theorem $(\mathbf{6.2})$

Let

$$p_n = \left(a + \frac{b}{n}\right) p_{n-1} \text{ for } n = 1, 2, \ldots \text{ and } p_0 > 0.$$

Its differential equation is given by

$$\frac{G'(s)}{G(s)} = \frac{a + b}{1 - as}$$

This implies that,

$$
\begin{aligned}
\frac{d}{ds}\ln G(s) &= \frac{a+b}{1-as} \\
\ln G(s) &= \int \frac{a+b}{1-as} ds + c_1 \\
\ln G(s) &= \frac{a+b}{-a}\ln(1-as) + \ln C_1 \quad \text{let } k = \ln C_1 \\
G(s) &= k(1-as)^{-\frac{a+b}{a}} \\
1 = G(1) &= k(1-a)^{-\frac{a+b}{a}} \\
k &= (1-a)^{\frac{a+b}{a}}
\end{aligned}
$$

Therefore

$$
G(s) = \left(\frac{1-a}{1-as}\right)^{\frac{a+b}{a}}
$$

a) (*i*) $G(s) = 1$ for $a = 0$ and $b = 0$; implying that $p_n = 1$ for $n = 0$ and $p_n = 0$ for $n > 0$.

(*ii*) $G(s) = e^{-b(1-s)}$ when $a = 0$ and $b \neq 0$. Which is the pgf of a Poisson distribution with parameter b, i.e.,

$$
p_n = \frac{e^{-b}b^n}{n!} \text{ for } n = 0, 1, 2, \ldots
$$

(*iii*) $G(s) = \frac{1-a}{1-as}$ when $a \neq 0$ and $b = 0$. Which is the pgf of a geometric distribution with probability $(1-a)$, i.e.,

$$
p_n = a^n(1-a); n = 0, 1, 2, \ldots \text{ for } 0 < a < 1
$$

b) **When** $a \neq 0$, $b \neq 0$

(*i*)

$$
G(s) = \left(\frac{1-a}{1-as}\right)^m
$$

Which is the pgf of a negative binomial distribution with parameters $m = \frac{a+b}{a}$, is a positive integer and $0 < a < 1$
Thus

$$
p_n = \binom{m+n-1}{n} a^n (1-a)^m \text{ for } n = 0, 1, 2, \ldots
$$

(*ii*)

$$
G(s) = \left(\frac{1-as}{1-a}\right)^\alpha = \left(\frac{1}{1-a} - \frac{a}{1-a}s\right)^\alpha
$$

Which is the pgf of a binomial distribution with parameters $\alpha = -\left(\frac{a+b}{a}\right)$ is a positive integer and $0 < \frac{-a}{1-a} < 1$, where $a < 0$ and $a + b > 0$.
Thus

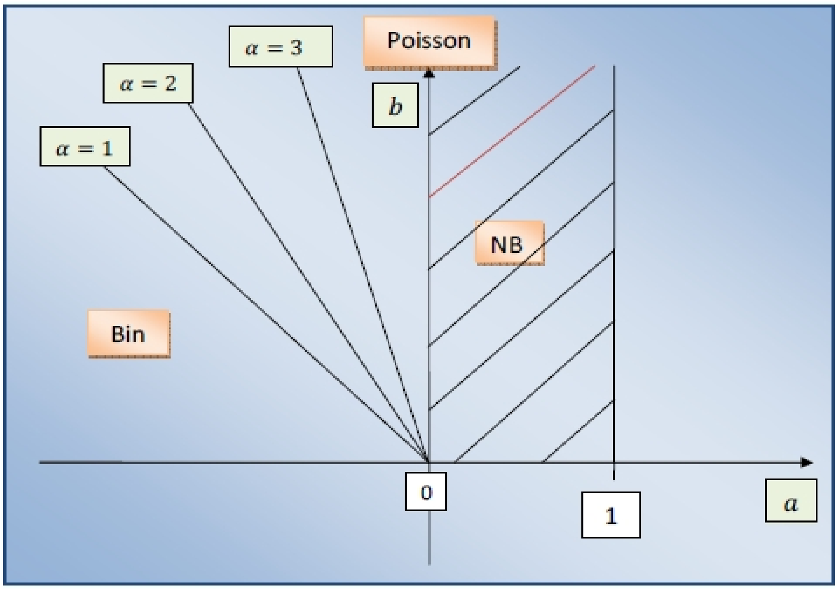
$$
p_n = \binom{\alpha}{n}\left(\frac{-a}{1-a}\right)^n \left(\frac{1}{1-a}\right)^{\alpha-n} \text{ for } n = 0, 1, 2, \ldots \alpha
$$

Remark :

Thus, when

$$a<0,\ p_n \text{ is Bin}\left(-\frac{a+b}{a},\frac{-a}{1-a}\right)$$
$$a=0,\ p_n \text{ is Po}\left(b\right)$$
$$1>a>0,\ p_n \text{ is NB}\left(\frac{a+b}{a},a\right)$$

Graphically we have figure (6.2)



6.3 Moments

6.3.1 Moments Based on Katz model

Katz (1965) also found moments as follows:

Multiply

$$(x+1)\,f\,(x+1)=(\alpha+\beta x)\,f\,(x)\,;x=0,1,2,\ldots$$

By $(x+1)^p$ and sum the results over x. That is,

$$\sum_{x=0}^{\infty}(x+1)^{p+1}\,f\,(x+1)=\sum_{x=0}^{\infty}(x+1)^{p}\,(\alpha+\beta x)\,f\,(x) \tag{6.17}$$

Let $p=0$, then

$$\sum_{x=0}^{\infty}(x+1)\,f\,(x+1)=\sum_{x=0}^{\infty}(\alpha+\beta x)\,f\,(x)$$

This implies that,

$$\begin{aligned}\mu&=\alpha\sum_{x=0}^{\infty}f\,(x)+\beta\sum_{x=0}^{\infty}xf\,(x)\\&=\alpha+\beta\mu\end{aligned}$$

Where

$$\mu = E(X) = \sum_{x=0}^{\infty} xf(x)$$

Therefore,

$$\mu = \frac{\alpha}{1-\beta} \tag{6.18}$$

When $p = 1$, then

$$\sum_{x=0}^{\infty} (x+1)^2 f(x+1) = \sum_{x=0}^{\infty} (x+1)(\alpha+\beta x) f(x)$$

$$\begin{aligned}
E\left(X^2\right) &= \sum_{x=0}^{\infty} \left(\alpha + x\alpha + x\beta + \beta x^2\right) f(x) \\
&= \sum_{x=0}^{\infty} \left[\alpha + x(\alpha+\beta) + \beta x^2\right] f(x) \\
&= \alpha + (\alpha+\beta)\mu + \beta E\left(X^2\right)
\end{aligned}$$

$$\begin{aligned}
(1-\beta) E\left(X^2\right) &= \alpha + (\alpha+\beta)\mu \\
E\left(X^2\right) &= \frac{\alpha + (\alpha+\beta)\mu}{(1-\beta)}
\end{aligned}$$

Hence to obtain the variance, we have

$$\begin{aligned}
\sigma^2 &= E\left(X^2\right) - [E(X)]^2 \\
&= \frac{\alpha + (\alpha+\beta)\mu}{(1-\beta)} - \mu^2 \\
&= \mu + \frac{(\alpha+\beta)\mu}{(1-\beta)} - \mu^2 \\
&= \mu + \frac{1}{\alpha}(\alpha+\beta)\mu^2 - \mu^2 \\
&= \mu + \frac{\beta}{\alpha}\mu^2
\end{aligned}$$

This implies that,

$$\begin{aligned}
\frac{\sigma^2}{\mu} &= 1 + \frac{\beta}{\alpha}\mu \\
&= 1 + \frac{\beta}{\alpha} \cdot \frac{\alpha}{1-\beta} = 1 + \frac{\beta}{1-\beta} \\
&= \frac{1}{1-\beta}
\end{aligned} \tag{6.19}$$

and

$$\sigma^2 = \frac{\mu}{1-\beta} = \frac{\alpha}{(1-\beta)^2} \tag{6.20}$$

In general for any positive integer p, (6.17) can be written as

$$\sum_{x=0}^{\infty} (x+1)^{p+1} f(x+1) = \sum_{x=0}^{\infty} (x+1)^p (\alpha+\beta x) f(x)$$

$$
\begin{aligned}
E\left(X^{p+1}\right) &= \alpha \sum_{x=0}^{\infty} (x+1)^p f(x) + \beta \sum_{x=0}^{\infty} (x+1)^p x f(x) \\
&= \alpha \sum_{x=0}^{\infty} (x+1)^p f(x) + \beta \sum_{x=0}^{\infty} (x+1)^p (x+1-1) f(x) \\
&= \alpha \sum_{x=0}^{\infty} (x+1)^p f(x) + \beta \sum_{x=0}^{\infty} (x+1)^{p+1} f(x) - \beta \sum_{x=0}^{\infty} (x+1)^p f(x) \\
&= (\alpha-\beta) \sum_{x=0}^{\infty} (x+1)^p f(x) + \beta \sum_{x=0}^{\infty} (x+1)^{p+1} f(x) \\
&= (\alpha-\beta) \sum_{x=0}^{\infty} \left\{ \sum_{j=0}^{p} \binom{p}{j} x^j \right\} f(x) + \beta \sum_{x=0}^{\infty} \left\{ \sum_{j=0}^{p+1} \binom{p+1}{j} x^j \right\} f(x) \\
&= (\alpha-\beta) \sum_{j=0}^{p} \left\{ \binom{p}{j} \sum_{x=0}^{\infty} x^j f(x) \right\} + \beta \sum_{j=0}^{p+1} \left\{ \binom{p+1}{j} \sum_{x=0}^{\infty} x^j f(x) \right\} \\
&= (\alpha-\beta) \sum_{j=0}^{p} \binom{p}{j} E\left(X^j\right) + \beta \sum_{j=0}^{p+1} \binom{p+1}{j} E\left(X^j\right) \\
&= (\alpha-\beta) \sum_{j=0}^{p} \binom{p}{j} E\left(X^j\right) + \beta \sum_{j=0}^{p} \binom{p+1}{j} E\left(X^j\right) + \beta \binom{p+1}{p+1} E\left(X^{p+1}\right)
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
(1-\beta) E\left(X^{p+1}\right) &= (\alpha-\beta) \sum_{j=0}^{p} \binom{p}{j} E\left(X^j\right) + \beta \sum_{j=0}^{p} \binom{p+1}{j} E\left(X^j\right) \\
&= \sum_{j=0}^{p} \left\{ (\alpha-\beta) \binom{p}{j} + \beta \binom{p+1}{j} \right\} E\left(X^j\right) \\
&= \sum_{j=0}^{p} \left\{ (\alpha-\beta) \binom{p}{j} + \beta \left[\binom{p}{j} + \binom{p}{j-1} \right] \right\} E\left(X^j\right) \\
&= \sum_{j=0}^{p} \left\{ \alpha \binom{p}{j} - \beta \binom{p}{j} + \beta \binom{p}{j} + \beta \binom{p}{j-1} \right\} E\left(X^j\right) \\
&= \sum_{j=0}^{p} \left\{ \alpha \binom{p}{j} + \beta \binom{p}{j-1} \right\} E\left(X^j\right) \qquad (6.21)
\end{aligned}
$$

Hence when

$$
p = 0 \;\Rightarrow (1-\beta) E(X) = \alpha \;\Rightarrow \mu = E(X) = \frac{\alpha}{1-\beta} \qquad (6.22)
$$

$$
p = 1 \;\Rightarrow (1-\beta) E\left(X^2\right) = \alpha + (\alpha+\beta) E(X)
$$

This implies that,

$$\begin{aligned} E\left(X^2\right) &= \frac{\alpha + (\alpha+\beta)\,\mu}{(1-\beta)} \\ &= \mu + \left(\mu + \frac{\beta}{1-\beta}\right)\mu \\ &= \left(1 + \frac{\beta}{1-\beta}\right)\mu + \mu^2 \end{aligned}$$

and the variance is

$$\begin{aligned} \sigma^2 &= E\left(X^2\right) - \left[E\left(X\right)\right]^2 \\ &= E\left(X^2\right) - \mu^2 \\ &= \frac{1}{1-\beta}\mu \end{aligned} \tag{6.23}$$

From (6.22) and (6.23) we obtain

$$\sigma^2 = \frac{\alpha}{(1-\beta)^2} \tag{6.24}$$

and from (6.23) we get

$$c = \frac{\sigma^2}{\mu} = \frac{1}{1-\beta} \tag{6.25}$$

Next

$$p = 2 \ \Rightarrow (1-\beta)\,E\left(X^3\right) = \sum_{j=0}^{2} \left\{\alpha\binom{2}{j} + \beta\binom{2}{j-1}\right\} E\left(X^j\right)$$

$$\begin{aligned} (1-\beta)\,E\left(X^3\right) &= \alpha + \left\{\alpha\binom{2}{1} + \beta\binom{2}{0}\right\} E\left(X\right) + \left\{\alpha\binom{2}{2} + \beta\binom{2}{1}\right\} E\left(X^2\right) \\ &= \alpha + (2\alpha+\beta)\,E\left(X\right) + (\alpha+2\beta)\,E\left(X^2\right) \\ &= \alpha + (2\alpha+\beta)\,\mu + (\alpha+2\beta)\left(\sigma^2+\mu^2\right) \\ &= \alpha + 2\alpha\mu + \beta\mu + \alpha\sigma^2 + 2\sigma^2\beta + \alpha\mu^2 + 2\beta\mu^2 \end{aligned}$$

Therefore,

$$(1-\beta)\,E\left(X^3\right) = \alpha\left[1 + 2\mu + \sigma^2 + \mu^2\right] + \beta\left[\mu + 2\sigma^2 + 2\mu^2\right]$$

$$E\left(X^3\right) = \frac{\alpha}{(1-\beta)}\left[1 + 2\mu + \sigma^2 + \mu^2\right] + \frac{\beta}{(1-\beta)}\left[\mu + 2\sigma^2 + 2\mu^2\right]$$

but

$$\frac{\alpha}{1-\beta} = \mu \text{ from } (6.22)$$

$$\begin{aligned} \frac{1}{1-\beta} &= \frac{\sigma^2}{\mu} \text{ from } (6.25) \\ \frac{1}{1-\beta} &= \frac{\sigma^2}{\mu} \ \Rightarrow \frac{\mu}{\sigma^2} = 1-\beta \\ &\Rightarrow \beta = 1 - \frac{\mu}{\sigma^2} \end{aligned}$$

Hence,

$$\begin{aligned}
E\left(X^3\right) &= \mu\left[1+2\mu+\sigma^2+\mu^2\right]+\beta\frac{\sigma^2}{\mu}\left[\mu+2\sigma^2+2\mu^2\right] \\
&= \mu^3+2\mu^2+\left(1+\sigma^2\right)\mu+\left(1-\frac{\mu}{\sigma^2}\right)\frac{\sigma^2}{\mu}\left[\mu+2\sigma^2+2\mu^2\right] \\
&= \mu^3+2\mu^2+\left(1+\sigma^2\right)\mu+\left(\frac{\sigma^2}{\mu}-1\right)\left(\mu+2\sigma^2+2\mu^2\right) \\
&= \mu^3+2\mu^2+\left(1+\sigma^2\right)\mu+2\sigma^2\mu+\sigma^2+2c\sigma^2-2\mu^2-\mu-2\sigma^2 \\
&= \mu^3+\left(1+\sigma^2+2\sigma^2-1\right)\mu+\left(\sigma^2+2c\sigma^2-2\sigma^2\right) \\
&= \mu^3+3\sigma^2\mu+(2c-1)\,\sigma^2 \qquad (6.26)
\end{aligned}$$

Where $c=\frac{\sigma^2}{\mu}$.

Now consider,

$$\begin{aligned}
\mu_3 &= E\left(X-\mu\right)^3 \\
&= E\left\{X^3-\mu^3+3X\mu^2-3X^2\mu\right\} \\
&= E\left(X^3\right)-3\mu E\left(X^2\right)+3\mu^3-\mu^3 \\
&= E\left(X^3\right)-3\mu\left[\sigma^2+\mu^2\right]+2\mu^3 \\
&= \left[\mu^3+3\sigma^2\mu+(2c-1)\,\sigma^2\right]-3\mu\left[\sigma^2+\mu^2\right]+2\mu^3 \\
&= \mu^3+3\sigma^2\mu+(2c-1)\,\sigma^2-3\mu\sigma^2-3\mu^3+2\mu^3 \\
&= (2c-1)\,\sigma^2 \qquad (6.27)
\end{aligned}$$

For $p=3$ we have

$$(1-\beta)\,E\left(X^4\right)=\sum_{j=0}^{3}\left\{\alpha\binom{3}{j}+\beta\binom{3}{j-1}\right\}E\left(X^j\right)$$

$$\begin{aligned}
(1-\beta)\,E\left(X^4\right) &= \alpha+\left[\alpha\binom{3}{1}+\beta\binom{3}{0}\right]E\left(X\right)+\left[\alpha\binom{3}{2}+\beta\binom{3}{1}\right]E\left(X^2\right) \\
&+\left[\alpha\binom{3}{3}+\beta\binom{3}{1}\right]E\left(X^3\right) \\
&= \alpha+(3\alpha+\beta)\,\mu+(3\alpha+3\beta)\left(\sigma^2+\mu^2\right) \\
&+(\alpha+3\beta)\left(\mu^3+3\sigma^2\mu+(2c-1)\,\sigma^2\right) \\
&= \alpha+(3\alpha+\beta)\,\mu+(3\alpha+3\beta)\left(\sigma^2+\mu^2\right) \\
&+(\alpha+3\beta)\left(\mu^3+3\sigma^2\mu+(2c-1)\,\sigma^2\right) \\
&= \alpha\left\{\begin{array}{c}1+3\mu+3\sigma^2+3\mu^2+\mu^3 \\ +3\sigma^2\mu+(2c-1)\,\sigma^2\end{array}\right\}+\beta\left\{\begin{array}{c}\mu+3\sigma^2+3\mu^2+3\mu^3 \\ +9\sigma^2\mu+3\,(2c-1)\,\sigma^2\end{array}\right\} \\
&= \alpha\left\{\begin{array}{c}\mu^3+3\mu^2+\left(3+3\sigma^2\right)\mu \\ +1+3\sigma^2+(2c-1)\,\sigma^2\end{array}\right\}+\beta\left\{\begin{array}{c}3\mu^3+3\mu^2+\left(1+9\sigma^2\right)\mu \\ +3\sigma^2+6c\sigma^2-3\sigma^2\end{array}\right\} \\
&= \alpha\left\{\begin{array}{c}\mu^3+3\mu^2+\left(3+3\sigma^2\right)\mu+ \\ 2c\sigma^2-\sigma^2+1+3\sigma^2\end{array}\right\}+\beta\left\{\begin{array}{c}3\mu^3+3\mu^2 \\ +\left(1+9\sigma^2\right)\mu+6c\sigma^2\end{array}\right\} \\
&= \alpha\left\{\begin{array}{c}\mu^3+3\mu^2+\left(3+3\sigma^2\right)\mu+ \\ 2c\sigma^2+2\sigma^2+1\end{array}\right\}+\beta\left\{\begin{array}{c}3\mu^3+3\mu^2 \\ +\left(1+9\sigma^2\right)\mu+6c\sigma^2\end{array}\right\}
\end{aligned}$$

Hence,

$$E\left(X^4\right)=\frac{\alpha}{(1-\beta)}\left\{\begin{array}{c}\mu^3+3\mu^2+\left(3+3\sigma^2\right)\mu\\+2c\sigma^2+2\sigma^2+1\end{array}\right\}+\frac{\beta}{(1-\beta)}\left\{\begin{array}{c}3\mu^3+3\mu^2\\+\left(1+9\sigma^2\right)\mu+6c\sigma^2\end{array}\right\} \tag{6.28}$$

but

$$\frac{\alpha}{(1-\beta)}=\mu \text{ and } \frac{\beta}{(1-\beta)}=\frac{\sigma^2}{\mu}\left(1-\frac{\mu}{\sigma^2}\right)=\frac{\sigma^2}{\mu}-1=c-1.$$

replacing this values in (6.28) we obtain

$$\begin{aligned}E\left(X^4\right)&=\mu\left\{\begin{array}{c}\mu^3+3\mu^2+\left(3+3\sigma^2\right)\mu\\+2c\sigma^2+2\sigma^2+1\end{array}\right\}+(c-1)\left\{\begin{array}{c}3\mu^3+3\mu^2\\+\left(1+9\sigma^2\right)\mu+6c\sigma^2\end{array}\right\}\\
&=\left\{\begin{array}{c}\mu^4+3\mu^3+\left(3+3\sigma^2\right)\mu^2\\+\left(2c\sigma^2+2\sigma^2+1\right)\mu\end{array}\right\}+\left\{\begin{array}{c}3c\mu^3+3c\mu^2+\left(1+9\sigma^2\right)c\mu+6c^2\sigma^2\\-3\mu^3-3\mu^2-\left(1+9\sigma^2\right)\mu-6c\sigma^2\end{array}\right\}\\
&=\left\{\mu^4+3\sigma^2\mu^2+2c\sigma^2\mu+2\mu\sigma^2+\mu\right\}+\left\{\begin{array}{c}3\sigma^2\mu^2+3\sigma^2\mu+\left(1+9\sigma^2\right)\sigma^2\\+6c^2\sigma^2-\mu-9\sigma^2\mu-6c\sigma^2\end{array}\right\}\\
&=\mu^4+6\sigma^2\mu^2+2\sigma^4+\left(2\sigma^2+1+3\sigma^2-1-9\sigma^2\right)\mu+\sigma^2+9\sigma^4+6c^2\sigma^2-6c\sigma^2\\
&=\mu^4+6\sigma^2\mu^2+\left(-4\sigma^2\right)\mu+\sigma^2+11\sigma^4+6c^2\sigma^2-6c\sigma^2\\
&=\mu^4+6\sigma^2\mu^2-4\sigma^2\mu+11\sigma^4+\left(6c^2-6c+1\right)\sigma^2\end{aligned} \tag{6.29}$$

Thus, the fourth central moment is given by,

$$\begin{aligned}\mu_4&=E\left(X-\mu\right)^4\\
&=E\left(X^4+\mu^4-4X\mu^3-4X^3\mu+6X^2\mu^2\right)\\
&=E\left(X^4\right)-4\mu E\left(X^3\right)+6\mu^2E\left(X^2\right)-3\mu^4\\
&=\mu^4+6\sigma^2\mu^2-4\sigma^2\mu+11\sigma^4+\left(6c^2-6c+1\right)\sigma^2\\
&-4\mu\left(\mu^3+3\sigma^2\mu+(2c-1)\sigma^2\right)+6\mu^2\left(\sigma^2+\mu^2\right)-3\mu^4\\
&=\mu^4+6\sigma^2\mu^2-4\sigma^2\mu+11\sigma^4+\left(6c^2-6c+1\right)\sigma^2-4\mu^4\\
&-12\sigma^2\mu^2-8\sigma^4+4\mu\sigma^2+6\mu^2\sigma^2+6\mu^4-3\mu^4\\
&=3\sigma^4+\left(6c^2-6c+1\right)\sigma^2\end{aligned} \tag{6.30}$$

6.3.2 The probability generating function technique of obtaining mean and variance

Multiply

$$f\left(x+1\right)=\frac{\alpha+\beta x}{1+x}f\left(x\right)$$

by $(x+1)\,s^x$ and sum the results over x we obtain,

$$\sum_{x=0}^{\infty}(x+1)\,f\left(x+1\right)s^x=\alpha\sum_{x=0}^{\infty}f\left(x\right)s^x+\beta\sum_{x=0}^{\infty}xf\left(x\right)s^x$$

$$G'\left(s\right)=\alpha G\left(s\right)+\beta sG'\left(s\right) \text{ where } G\left(s\right)=\sum_{x=0}f\left(x\right)s^x$$

Therefore

$$
\begin{aligned}
(1-\beta s) G'(s) &= \alpha G(s) \\
\frac{G'(s)}{G(s)} &= \frac{\alpha}{1-\beta s}
\end{aligned}
$$

This implies that,

$$
\begin{aligned}
\frac{d}{ds} \ln G(s) &= \frac{\alpha}{1-\beta s} \\
&= \int \frac{\alpha}{1-\beta s} ds + c_1 \\
\ln G(s) &= \frac{\alpha}{-\beta} \ln(1-\beta s) + \ln C_1 \quad \text{let } k = \ln C_1 \\
G(s) &= k(1-\beta s)^{-\frac{\alpha}{\beta}} \\
1 = G(1) &= k(1-\beta)^{-\frac{\alpha}{\beta}} \\
k &= (1-\beta)^{\frac{\alpha}{\beta}}
\end{aligned}
$$

Therefore the pgf is given by

$$
G(s) = \left(\frac{1-\beta}{1-\beta s}\right)^{\frac{\alpha}{\beta}}
$$

and

$$
\begin{aligned}
G'(s) &= \frac{\alpha}{1-\beta s} G(s) \\
G''(s) &= \frac{d}{ds}\left(\frac{\alpha}{1-\beta s} G(s)\right) \\
&= \frac{\alpha G'(s)}{1-\beta s} + \frac{\beta \alpha G(s)}{(1-\beta s)^2}
\end{aligned}
$$

setting $s = 1$ we obtain

$$
\begin{aligned}
G'(1) = E(X) &= \frac{\alpha}{1-\beta} \\
G''(1) &= \frac{\alpha G'(1)}{1-\beta} + \frac{\beta \alpha G(1)}{(1-\beta)^2} \\
&= \frac{\alpha^2 + \alpha\beta}{(1-\beta)^2} = \frac{\alpha(\alpha+\beta)}{(1-\beta)^2}
\end{aligned}
$$

Therefore the mean and variance is given by

$$
G'(1) = E(X) = \frac{\alpha}{1-\beta}
$$

$$\begin{aligned}
\sigma^2 &= G''(1) + G'(1) - [G'(1)]^2 \\
&= \frac{\alpha(\alpha+\beta)}{(1-\beta)^2} + \frac{\alpha}{1-\beta} - \frac{\alpha^2}{(1-\beta)^2} \\
&= \frac{\alpha\beta}{(1-\beta)^2} + \frac{\alpha}{1-\beta} = \frac{\alpha\beta + \alpha(1-\beta)}{(1-\beta)^2} \\
&= \frac{\alpha}{(1-\beta)^2}
\end{aligned}$$

Table 6.1: Distributions of the Katz class

Distribution	α	β	$f(x)$	$G(s)$
$Po(\lambda)$	λ	0	$\frac{e^{-\lambda}\lambda^k}{k!}$	$e^{-\lambda(1-s)}$
$NB(r,p)$	$r(1-p)$	$1-p$	$\binom{r+n-1}{n}p^r(1-p)^n$	$\left(\frac{p}{1-(1-p)s}\right)^r$
$Bin(n,p)$	$\frac{np}{1-p}$	$\frac{-p}{1-p}$	$\binom{n}{k}p^k(1-p)^{n-k}$	$(1-p+ps)^n$

Table 6.2: Moments of distributions in Katz class

Distribution	$\mu = \frac{\alpha}{1-\beta}$	$\sigma^2 = \frac{\alpha}{(1-\beta)^2}$	$c = \frac{\sigma^2}{\mu}$
$Po(\lambda)$	λ	λ	1
$NB(r,p)$	$r\left(\frac{1-p}{p}\right)$	$r\frac{(1-p)}{p^2}$	$\frac{1}{p}$
$Bin(n,p)$	np	$np(1-p)$	$1-p$

Hence,

$$\begin{aligned}
\beta = 0 \quad &\Rightarrow \frac{\sigma^2}{\mu} = 1 \text{ implying Po} \\
\beta < 0 \quad &\Rightarrow 0 < \frac{\sigma^2}{\mu} < 1 \text{ implying Bin} \\
0 < \beta < 1 \quad &\Rightarrow \frac{\sigma^2}{\mu} > 0 \text{ implying NB}
\end{aligned}$$

Table 6.3: Higher moments of distributions in the Katz class		
Distribution	$\mu_3 = (2c-1)\sigma^2$	$\mu_4 = 3\sigma^4 + (6c^2 - 6c + 1)\sigma^2$
$Po(\lambda)$	λ	$3\lambda(\lambda+1)$
$NB(r,p)$	$r\left(\frac{2-3p+p^2}{p^3}\right)$	$\frac{3r^2(1-p)^2 + r(6-12p+7p^2+p^3)}{p^4}$
$Bin(n,p)$	$np - 3np^2 + 2np^3$	$3n^2p^2 - 6n^2p^3 + 3n^2p^4 + 18np^3 - 12np^2 - 6np^4$

6.3.3 Moments based on Panjer model

Given

$$p_n = \left(a + \frac{b}{n}\right) p_{n-1} \text{ for } n = 1, 2, \ldots \text{ and } p_0 > 0.$$

Let

$$\begin{aligned} M_j &= \sum_{n=1}^{\infty} n^j p_n \\ &= \sum_{n=1}^{\infty} n^j \left(a + \frac{b}{n}\right) p_{n-1} \\ &= \sum_{n=1}^{\infty} \left(an^j + bn^{j-1}\right) p_{n-1} \\ &= \sum_{n=1}^{\infty} \left\{a\,(n-1+1)^j + b\,(n-1+1)^{j-1}\right\} p_{n-1} \end{aligned}$$

Therefore

$$\begin{aligned} M_j &= \sum_{n=1}^{\infty} \left\{ a \sum_{i=0}^{j} \binom{j}{i} (n-1)^i + b \sum_{i=0}^{j-1} \binom{j-1}{i} (n-1)^i \right\} p_{n-1} \\ &= a \sum_{i=0}^{j} \left\{ \binom{j}{i} \sum_{n=1}^{\infty} (n-1)^i p_{n-1} \right\} + b \sum_{i=0}^{j-1} \left\{ \binom{j-1}{i} \sum_{n=1}^{\infty} (n-1)^i p_{n-1} \right\} \\ &= a \sum_{i=0}^{j} \binom{j}{i} M_i + b \sum_{i=0}^{j-1} \binom{j-1}{i} M_i \\ &= a \sum_{i=0}^{j-1} \binom{j}{i} M_i + aM_j + b \sum_{i=0}^{j-1} \binom{j-1}{i} M_i \end{aligned}$$

$$
\begin{aligned}
(1-a)\,M_j &= \sum_{i=0}^{j-1} \left\{ a\binom{j}{i} M_i + b\binom{j-1}{i} \right\} M_i \\
&= \sum_{i=0}^{j-1} \left\{ a\left[\binom{j-1}{i} + \binom{j-1}{i-1}\right] M_i + b\binom{j-1}{i} \right\} M_i \\
&= \sum_{i=0}^{j-1} \left\{ (a+b)\binom{j-1}{i} + a\binom{j-1}{i-1} \right\} M_i \text{ for } j = 1, 2, \ldots \qquad (6.31)
\end{aligned}
$$

For $j = 1$, we have

$$
(1-a)\,M_1 = (a+b) \quad \Rightarrow \mu = M_1 = \frac{a+b}{1-a} \qquad (6.32)
$$

For $j = 2$, we have

$$
\begin{aligned}
(1-a)\,M_2 &= \sum_{i=0}^{1} \left\{ (a+b)\binom{j-1}{i} + a\binom{j-1}{i-1} \right\} M_i \\
&= \sum_{i=0}^{1} \left\{ (a+b)\binom{1}{1} + a\binom{1}{i-1} \right\} M_i \\
&= (a+b) + [(a+b)+a]\,M_1
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
M_2 &= \frac{a+b}{1-a} + \left[\frac{a+b}{1-a} + \frac{a}{1-a}\right] M_1 \\
&= M_1 + \left[M_1 + \frac{a}{1-a}\right] M_1 \\
&= M_1 + M_1^2 + \frac{a}{1-a} M_1 \\
&= \left(1 + \frac{a}{1-a}\right) M_1 + M_1^2 = \frac{M_1}{1-a} + M_1^2
\end{aligned}
$$

The variance is given by

$$
\sigma^2 = M_2 - M_1^2 = \left(1 + \frac{a}{1-a}\right) M_1 = \frac{1}{1-a} M_1
$$

That is,

$$
\sigma^2 = \frac{\mu}{1-a} = \frac{a+b}{(1-a)^2} \qquad (6.33)
$$

$$
\Rightarrow c = \frac{\sigma^2}{\mu} = \frac{1}{1-a} \qquad (6.34)
$$

For $j = 3$, we have

$$
\begin{aligned}
(1-a)\,M_3 &= \sum_{i=0}^{2} \left\{ (a+b)\binom{2}{i} + a\binom{2}{i-1} \right\} M_i \\
&= (a+b) + \left[(a+b)\binom{2}{1} + a\binom{2}{0}\right] M_1 + \left[(a+b)\binom{2}{2} + a\binom{2}{1}\right] M_2 \\
& (a+b) + [2\,(a+b)+a]\,M_1 + [(a+b)+2a]\,M_2 \\
&= (a+b)\,[1 + 2M_1 + M_2] + a\,[M_1 + 2M_2]
\end{aligned}
$$

Hence,

$$\begin{aligned} M_3 &= \frac{a+b}{1-a}\left[1+2M_1+M_2\right]+\frac{a}{1-a}\left[M_1+2M_2\right] \\ &= M_1\left[1+2M_1+\left(\sigma^2+M_1^2\right)\right]+\frac{a}{1-a}\left[M_1+2\sigma^2+2M_1^2\right] \qquad (**) \end{aligned}$$

but

$$\frac{\sigma^2}{M_1}=\frac{1}{1-a}, \text{ then } 1-a=\frac{M_1}{\sigma^2}\Rightarrow a=1-\frac{M_1}{\sigma^2}$$

Therefore

$$\frac{a}{1-a}=\left(1-\frac{M_1}{\sigma^2}\right)\frac{\sigma^2}{M_1}=\frac{\sigma^2}{M_1}-1 \qquad (**1)$$

replacing $(**1)$ in equation$(**)$ above, we obtain

$$\begin{aligned} M_3 &= M_1\left\{1+2M_1+\sigma^2+M_1^2\right\}+\left(\frac{\sigma^2}{M_1}-1\right)\left(M_1+2\sigma^2+2M_1^2\right) \\ &= M_1+2M_1^2+\sigma^2M_1+M_1^3+\sigma^2+\frac{2\sigma^4}{M_1}+2\sigma^2M_1-M_1-2\sigma^2-2M_1^2 \\ &= M_1^3+3\sigma^2\mu+2\sigma^2c-\sigma^2 \\ &= M_1^3+3\sigma^2\mu+\left(2c-1\right)\sigma^2 \end{aligned}$$

Thus,

$$E\left(X^3\right)=\mu^3+3\sigma^2\mu+\left(2c-1\right)\sigma^2 \text{ as in formula } (6.26)$$

and μ_3 is

$$\begin{aligned} \mu_3 &= E\left(X-\mu\right)^3 \\ &= \left(2c-1\right)\sigma^2 \text{ as in } (6.27) \end{aligned}$$

6.3.4 Factorial moments by pgf technique

$$G\left(s\right)=\sum_{n=0}^{\infty}p_ns^n$$

$$\Rightarrow \frac{dG}{ds}=\sum_{n=0}^{\infty}np_ns^{n-1}=\sum_{n=1}^{\infty}np_ns^{n-1}$$

$$\frac{d^2G}{ds^2}=\sum_{n=2}^{\infty}n\left(n-1\right)p_ns^{n-2}$$

$$\begin{aligned} \frac{d^3G}{ds^3} &= \sum_{n=3}^{\infty}n\left(n-1\right)\left(n-2\right)p_ns^{n-3} \\ &= 3!\sum_{n=3}^{\infty}\binom{n}{3}p_ns^{n-3} \end{aligned}$$

In general

$$\frac{d^lG}{ds^l} = l!\sum_{n=l}^{\infty}\binom{n}{l}p_n s^{n-l} \text{ for } l = 1, 2, \ldots \tag{6.35}$$

but from (6.16)

$$p_n = \left(\beta + \frac{\alpha-\beta}{n}\right)p_{n-1}; n = 1, 2, \ldots$$

Substituting this in (6.35) we have,

$$\begin{aligned}
\frac{1}{l!}\frac{d^lG}{ds^l} &= \sum_{n=l}^{\infty}\binom{n}{l}\left(\beta + \frac{\alpha-\beta}{n}\right)p_{n-1}s^{n-l} \\
&= \sum_{n=l}^{\infty}\left\{\beta\binom{n}{l} + \frac{\alpha-\beta}{n}\binom{n}{l}\right\}p_{n-1}s^{n-l} \\
&= \sum_{n=l}^{\infty}\left\{\beta\binom{n}{l} + \frac{\alpha-\beta}{l}\binom{n-1}{l-1}\right\}p_{n-1}s^{n-l} \\
&= \sum_{n=l}^{\infty}\left\{\beta\left[\binom{n-1}{l} + \binom{n-1}{l-1}\right] + \frac{\alpha-\beta}{l}\binom{n-1}{l-1}\right\}p_{n-1}s^{n-l} \\
&= \sum_{n=l}^{\infty}\left\{\beta\binom{n-1}{l} + \left(\beta + \frac{\alpha-\beta}{l}\right)\binom{n-1}{l-1}\right\}p_{n-1}s^{n-l} \\
&= \sum_{n=l}^{\infty}\beta\binom{n-1}{l}p_{n-1}s^{n-l} + \left(\beta + \frac{\alpha-\beta}{l}\right)\sum_{n=l}^{\infty}\binom{n-1}{l-1}p_{n-1}s^{n-l} \\
&= \beta\sum_{n=l}^{\infty}\binom{n-1}{l}p_{n-1}s^{n-l} + \left(\frac{l\beta+\alpha-\beta}{l}\right)\sum_{n=l}^{\infty}\binom{n-1}{l-1}p_{n-1}s^{n-l} \\
&= \beta\sum_{n=l}^{\infty}\binom{n-1}{l}p_{n-1}s^{n-l} + \left(\frac{l\beta+\alpha-\beta}{l}\right)\frac{1}{(l-1)!}\frac{d^{(l-1)}G}{ds^{(l-1)}} \\
&= \beta\sum_{n=l}^{\infty}\binom{n-1}{l}p_{n-1}s^{n-l} + \left(\frac{l\beta+\alpha-\beta}{l!}\right)\frac{d^{(l-1)}G}{ds^{(l-1)}}
\end{aligned}$$

Let $j = n - 1 \ \Rightarrow n = j + 1$

Hence,

$$\begin{aligned}
\beta\sum_{n=l}^{\infty}\binom{n-1}{l}p_{n-1}s^{n-l} &= \beta\sum_{j=l-1}^{\infty}\binom{j}{l}p_j s^{j-l+1} \\
&= \beta s\sum_{j=l-1}^{\infty}\binom{j}{l}p_j s^{j-l} \\
&= \beta s\sum_{j=l}^{\infty}\binom{j}{l}p_j s^{j-l} \\
&= \frac{\beta s}{l!}\frac{d^lG}{ds^l}
\end{aligned}$$

Therefore,

$$\frac{1}{l!}\frac{d^lG}{ds^l} = \frac{\beta s}{l!}\frac{d^lG}{ds^l} + \left(\frac{l\beta+\alpha-\beta}{l!}\right)\frac{d^{(l-1)}G}{ds^{(l-1)}}$$

$$(1-\beta s)\frac{d^l G}{ds^l} = (l\beta+\alpha-\beta)\frac{d^{(l-1)}G}{ds^{(l-1)}} \text{ for } l = 1, 2, \ldots \tag{6.36a}$$

$$\frac{G^{(l)}(s)}{G^{(l-1)}(s)} = \frac{l\beta+\alpha-\beta}{1-\beta s} \tag{6.36b}$$

$$\frac{d}{ds}\ln G^{(l-1)}(s) = \frac{l\beta+\alpha-\beta}{1-\beta s} \tag{6.36c}$$

This implies,

$$\begin{aligned}\ln G^{(l-1)}(s) &= \int \frac{l\beta+\alpha-\beta}{1-\beta s}ds \\ &= \frac{l\beta+\alpha-\beta}{-\beta}\ln(1-\beta s) + \ln C_1 \quad \text{let } \ln C_1 = C\end{aligned}$$

$$\therefore G^{(l-1)}(s) = C\left[1-\beta s\right]^{-\left(l-1+\frac{\alpha}{\beta}\right)}$$

setting $l = 1$ we obtain

$$\begin{aligned}G^{(0)}(s) &= C\left[1-\beta s\right]^{-\frac{\alpha}{\beta}} \\ G(s) &= C\left[1-\beta s\right]^{-\frac{\alpha}{\beta}}\end{aligned}$$

Also, setting $s = 1$ to obtain C

$$\begin{aligned}1 = G(1) &= C\left[1-\beta\right]^{-\frac{\alpha}{\beta}} \\ C &= \left[1-\beta\right]^{\frac{\alpha}{\beta}}\end{aligned}$$

Hence,

$$G^{(l-1)}(s) = \left[1-\beta\right]^{\frac{\alpha}{\beta}}\left[1-\beta s\right]^{-\left(l-1+\frac{\alpha}{\beta}\right)} \tag{6.37}$$

$$\begin{aligned}G^{(l)}(s) = \frac{d}{ds}G^{(l-1)}(s) &= -\left(l-1+\frac{\alpha}{\beta}\right)(-\beta)\left[1-\beta\right]^{\frac{\alpha}{\beta}}\left[1-\beta s\right]^{-l-\frac{\alpha}{\beta}} \\ &= (l\beta-\beta+\alpha)\left[1-\beta\right]^{\frac{\alpha}{\beta}}\left[1-\beta s\right]^{-l-\frac{\alpha}{\beta}} \qquad (6.38)\end{aligned}$$

$$\begin{aligned}G^{(l+1)}(s) &= (l\beta+\alpha)(l\beta+\alpha-\beta)\left[1-\beta\right]^{\frac{\alpha}{\beta}}\left[1-\beta s\right]^{-\left(l+\frac{\alpha}{\beta}+1\right)} \\ G^{(l+2)}(s) &= (l\beta+\alpha+\beta)(l\beta+\alpha)(l\beta+\alpha-\beta)\left[1-\beta\right]^{\frac{\alpha}{\beta}}\left[1-\beta s\right]^{-\left(l+\frac{\alpha}{\beta}+2\right)} \\ G^{(l+3)}(s) &= \left\{(l\beta+\alpha+2\beta)(l\beta+\alpha+\beta)(l\beta+\alpha)(l\beta+\alpha-\beta)\left[1-\beta\right]^{\frac{\alpha}{\beta}}\left[1-\beta s\right]^{-\left(l+\frac{\alpha}{\beta}+3\right)}\right\}\end{aligned}$$

$$\begin{aligned}G^{(l+4)}(s) &= \left\{\begin{array}{c}(l\beta+\alpha+3\beta)(l\beta+\alpha+2\beta)(l\beta+\alpha+\beta) \\ (l\beta+\alpha)(l\beta+\alpha-\beta)\left[1-\beta\right]^{\frac{\alpha}{\beta}}\left[1-\beta s\right]^{-\left(l+\frac{\alpha}{\beta}+4\right)}\end{array}\right\} \\ &= \left\{\begin{array}{c}\beta^5\left(\frac{\alpha}{\beta}+l+3\right)\left(\frac{\alpha}{\beta}+l+2\right)\left(\frac{\alpha}{\beta}+l+1\right) \\ \left(\frac{\alpha}{\beta}+l\right)\left(\frac{\alpha}{\beta}+l-1\right)\left[1-\beta\right]^{\frac{\alpha}{\beta}}\left[1-\beta s\right]^{-\left(l+\frac{\alpha}{\beta}+4\right)}\end{array}\right\} \\ &= 5!\beta^5\binom{\frac{\alpha}{\beta}+l+3}{5}\left[1-\beta\right]^{\frac{\alpha}{\beta}}\left[1-\beta s\right]^{-\left(\frac{\alpha}{\beta}+l+4\right)}\end{aligned}$$

Therefore,

$$G^{(l+m)}(s) = (m+1)!\beta^{m+1}\binom{\frac{\alpha}{\beta}+l+m-1}{m+1}[1-\beta]^{\frac{\alpha}{\beta}}[1-\beta s]^{-\left(l+\frac{\alpha}{\beta}+m\right)};m=0,1,2,\ldots$$

$$\Rightarrow G^{(l+m-1)}(s) = m!\beta^{m}\binom{\frac{\alpha}{\beta}+l+m-2}{m}[1-\beta]^{\frac{\alpha}{\beta}}[1-\beta s]^{-\left(l+\frac{\alpha}{\beta}+m-1\right)};m=0,1,2,\ldots \tag{6.39}$$

Therefore the m^{th} factorial moment is obtained by putting $l = 1$ and $s = 1$ in (6.39) .i.e.,

$$G^{(m)}(1) = m!\beta^{m}\binom{\frac{\alpha}{\beta}+m-1}{m}\left(\frac{1}{1-\beta}\right)^{m} \tag{6.40}$$

Hence when

$$m=0 \quad \Rightarrow \quad G^{(0)}(1) = G(1) = 1$$

$$m=1 \quad \Rightarrow \quad G'(1) = \beta\binom{\frac{\alpha}{\beta}}{1}\left(\frac{1}{1-\beta}\right) = \frac{\alpha}{1-\beta}$$

$$m=2 \quad \Rightarrow \quad G''(1) = 2!\beta^{2}\binom{\frac{\alpha}{\beta}+1}{2}\left(\frac{1}{1-\beta}\right)^{2}$$

$$\begin{aligned}G''(1) &= \beta^{2}\left(\frac{\alpha}{\beta}+1\right)\frac{\alpha}{\beta}\left(\frac{1}{1-\beta}\right)^{2}\\ &= \frac{(\alpha+\beta)\alpha}{(1-\beta)^{2}}\end{aligned}$$

This implies that,

$$\mu = E(X) = G'(1) = \frac{\alpha}{1-\beta}$$

and

$$\begin{aligned}\sigma^{2} &= G''(1) + G'(1) - [G'(1)]^{2}\\ &= \frac{(\alpha+\beta)\alpha}{(1-\beta)^{2}} + \frac{\alpha}{1-\beta} - \frac{\alpha^{2}}{(1-\beta)^{2}}\\ &= \frac{\beta\alpha}{(1-\beta)^{2}} + \frac{\alpha}{1-\beta}\\ &= \frac{\beta\alpha + \alpha(1-\beta)}{(1-\beta)^{2}} = \frac{\alpha}{(1-\beta)^{2}}\end{aligned}$$

$$m=3 \quad \Rightarrow \quad G'''(1) = 3!\beta^{3}\binom{\frac{\alpha}{\beta}+2}{3}\left(\frac{1}{1-\beta}\right)^{3}$$

$$\begin{aligned}G'''(1) &= \beta^{3}\left(\frac{\alpha}{\beta}+2\right)\left(\frac{\alpha}{\beta}+1\right)\frac{\alpha}{\beta}\frac{1}{(1-\beta)^{3}}\\ &= \frac{\alpha(\alpha+\beta)(\alpha+2\beta)}{(1-\beta)^{3}}\\ &= \frac{\alpha}{1-\beta}\left[\frac{\alpha}{1-\beta}+\frac{\beta}{1-\beta}\right]\left[\frac{\alpha}{1-\beta}+\frac{2\beta}{1-\beta}\right]\end{aligned}$$

but

$$\mu = \frac{\alpha}{1-\beta} \text{ and } \sigma^2 = \frac{\alpha}{(1-\beta)^2}$$

Therefore

$$c = \frac{\sigma^2}{\mu} = \frac{1}{1-\beta} \text{ and } \beta = 1 - \frac{\mu}{\sigma^2}$$
$$\Rightarrow \frac{\beta}{1-\beta} = \frac{\sigma^2}{\mu} - 1 = c - 1$$

Hence,

$$\begin{aligned}
G'''(1) &= \mu\left[\mu + (c-1)\right]\left[\mu + 2(c-1)\right] \\
&= \left[\mu^2 + \mu(c-1)\right]\left[\mu + 2(c-1)\right] \\
&= \left[\mu^2 + \sigma^2 - \mu\right]\left[\mu + 2(c-1)\right] \\
&= \mu^3 + 2\mu^2(c-1) + \sigma^2\mu + 2\sigma^2(c-1) - \mu^2 - 2\mu(c-1) \\
&= \mu^3 + 2\mu\sigma^2 - 2\mu^2 + \sigma^2\mu + 2\sigma^2(c-1) - \mu^2 - 2\sigma^2 + 2\mu \\
&= \mu^3 - 3\mu^2 + \left(3\sigma^2 + 2\right)\mu + 2\sigma^2 c - 4\sigma^2 \qquad (6.41)
\end{aligned}$$

Also,

$$\begin{aligned}
G'''(1) &= E\left[X(X-1)(X-2)\right] \\
&= E\left[X^3 - 3X^2 + 2X\right] \\
&= E\left(X^3\right) - 3E\left(X^2\right) + 2E(X) \\
&= E\left(X^3\right) - 3\left[\sigma^2 + \mu^2\right] + 2\mu \\
&= E\left(X^3\right) - 3\sigma^2 - 3\mu^2 + 2\mu \qquad (6.42)
\end{aligned}$$

Therefore

$$\mu^3 - 3\mu^2 + \left(3\sigma^2 + 2\right)\mu + 2\sigma^2 c - 4\sigma^2 = E\left(X^3\right) - 3\sigma^2 - 3\mu^2 + 2\mu$$
$$E\left(X^3\right) - 3\sigma^2 = \mu^3 + 3\sigma^2\mu + 2\sigma^2 c - 4\sigma^2$$

$$\begin{aligned}
E\left(X^3\right) &= \mu^3 + 3\sigma^2\mu + 2\sigma^2 c - \sigma^2 \\
&= \mu^3 + 3\sigma^2\mu + (2c-1)\sigma^2
\end{aligned}$$

As shown earlier in (6.26)

For $m = 4$

$$
\begin{aligned}
G^{iv}(1) &= 4!\beta^4 \binom{\frac{\alpha}{\beta}+3}{4} \left(\frac{1}{1-\beta}\right)^4 \\
&= \beta^4 \left(\frac{\alpha}{\beta}+3\right)\left(\frac{\alpha}{\beta}+2\right)\left(\frac{\alpha}{\beta}+1\right)\frac{\alpha}{\beta}\frac{1}{(1-\beta)^4} \\
&= \frac{\alpha(\alpha+\beta)(\alpha+2\beta)(\alpha+3\beta)}{(1-\beta)^4} \\
&= \frac{\alpha}{1-\beta}\left[\frac{\alpha}{1-\beta}+\frac{\beta}{1-\beta}\right]\left[\frac{\alpha}{1-\beta}+\frac{2\beta}{1-\beta}\right]\left[\frac{\alpha}{1-\beta}+\frac{3\beta}{1-\beta}\right] \\
&= \mu\left[\mu+(c-1)\right]\left[\mu+2(c-1)\right]\left[\mu+3(c-1)\right] \\
&= \left[\mu^2+\sigma^2-\mu\right]\left[\mu^2+5\mu(c-1)+6(c-1)^2\right] \\
&= \left[\mu^2+\sigma^2-\mu\right]\left[\mu^2-5\mu+5\sigma^2+6(c-1)^2\right] \\
&= \mu^4-5\mu^3+\left[5\sigma^2+6(c-1)^2\right]\mu^2-\mu^3+5\mu^2+\left[-5\sigma^2-6(c-1)^2\right]\mu \\
&+\sigma^2\mu^2-5\sigma^2\mu+\left[5\sigma^4+6\sigma^2(c-1)^2\right] \\
&= \mu^4-6\mu^3+\left[5\sigma^2+6(c-1)^2+5+\sigma^2\right]\mu^2+\left[-10\sigma^2-6(c-1)^2\right]\mu \\
&+\left[5\sigma^4+6\sigma^2(c-1)^2\right] \\
&= \mu^4-6\mu^3+\left[6\sigma^2+6(c-1)^2+5\right]\mu^2+\left[-10\sigma^2-6(c-1)^2\right]\mu \\
&+\left[5\sigma^4+6\sigma^2(c-1)^2\right] \\
&= \mu^4-6\mu^3+\left(6\sigma^2+5\right)\mu^2+6\left[\mu(c-1)\right]^2+\left[-10\sigma^2\mu\right]+\left[-6(c-1)^2\mu\right] \\
&+\left[5\sigma^4+6(c-1)^2\sigma^2\right] \\
&= \mu^4-6\mu^3+\left(6\sigma^2+5\right)\mu^2+6\left[\sigma^2-\mu\right]^2-10\sigma^2\mu-6(c\mu-\mu)(c-1) \\
&+\left[5\sigma^4+6(c-1)^2\sigma^2\right] \\
&= \mu^4-6\mu^3+\left(6\sigma^2+5\right)\mu^2+6\left[\sigma^4-2\mu\sigma^2+\mu^2\right]-10\sigma^2\mu-6\left(\sigma^2-\mu\right)(c-1) \\
&+\left[5\sigma^4+6(c-1)^2\sigma^2\right] \\
&= \mu^4-6\mu^3+\left(6\sigma^2+11\right)\mu^2+\left[-22\sigma^2+6c-6\right]\mu+6\sigma^4-6\sigma^2(c-1) \\
&+5\sigma^4+6(c-1)^2\sigma^2 \\
&= \mu^4-6\mu^3+\left(6\sigma^2+11\right)\mu^2+\left[6c-22\sigma^2-6\right]\mu+11\sigma^4+6(c-1)(c-1-1)\sigma^2 \\
&= \mu^4-6\mu^3+\left(6\sigma^2+11\right)\mu^2+\left[6c-22\sigma^2-6\right]\mu+11\sigma^4+6(c-1)(c-2)\sigma^2
\end{aligned} \tag{6.43}
$$

Also.

$$
\begin{aligned}
G^{iv}(1) &= E\left[X(X-1)(X-2)(X-3)\right] \\
&= E\left[11X^2-6X-6X^3+X^4\right] \\
&= E\left(X^4\right)-6E\left(X^3\right)+11E\left(X^2\right)-6E(X)
\end{aligned} \tag{6.44}
$$

Therefore

$$
\left\{\begin{array}{c} E\left(X^4\right)-6E\left(X^3\right) \\ +11E\left(X^2\right)-6E(X) \end{array}\right\} = \mu^4-6\mu^3+\left(6\sigma^2+11\right)\mu^2+\left[6c-22\sigma^2-6\right]\mu
$$
$$
+11\sigma^4+6(c-1)(c-2)\sigma^2
$$

$$
\begin{aligned}
E\left(X^4\right) &= \left\{6E\left(X^3\right) - 11E\left(X^2\right) + 6E\left(X\right)\right\} + \mu^4 - 6\mu^3 + \left(6\sigma^2 + 11\right)\mu^2 \\
&+ \left[6c - 22\sigma^2 - 6\right]\mu + 11\sigma^4 + 6\left(c-1\right)\left(c-2\right)\sigma^2 \\
&= 6\left\{\mu^3 + 3\sigma^2\mu + \left(2c-1\right)\sigma^2\right\} - 11\left\{\sigma^2 + \mu^2\right\} + 6\mu + \mu^4 - 6\mu^3 \\
&+ \left(6\sigma^2 + 11\right)\mu^2 + \left[6c - 22\sigma^2 - 6\right]\mu + 11\sigma^4 + 6\left(c-1\right)\left(c-2\right)\sigma^2 \\
&= \mu^4 + 18\sigma^2\mu + 6\left(2c-1\right)\sigma^2 - 11\sigma^2 - 11\mu^2 + 6\mu + 6\sigma^2\mu^2 + 11\mu^2 \\
&+ \left[6c - 22\sigma^2 - 6\right]\mu + 11\sigma^4 + 6\left(c-1\right)\left(c-2\right)\sigma^2 \\
&= \mu^4 + 6\sigma^2\mu^2 + \left[18\sigma^2 + 6 + 6c - 22\sigma^2 - 6\right]\mu \\
&+ 6\left(2c-1\right)\sigma^2 - 11\sigma^2 + 11\sigma^4 + 6\left(c-1\right)\left(c-2\right)\sigma^2 \\
&= \mu^4 + 6\sigma^2\mu^2 + \left[6c - 4\sigma^2\right]\mu + 12c\sigma^2 - 6\sigma^2 - 11\sigma^2 \\
&+ 11\sigma^4 + 6\left(c-1\right)\left(c-2\right)\sigma^2 \\
&= \mu^4 + 6\sigma^2\mu^2 + 6\sigma^2 - 4\sigma^2\mu + 12c\sigma^2 - 6\sigma^2 - 11\sigma^2 + 11\sigma^4 \\
&+ 6\left(c^2 - 3c + 2\right)\sigma^2 \\
&= \mu^4 + 6\sigma^2\mu^2 - 4\sigma^2\mu + 12c\sigma^2 - 11\sigma^2 + 11\sigma^4 + 6c^2\sigma^2 - 18c\sigma^2 + 12\sigma^2 \\
&= \mu^4 + 6\sigma^2\mu^2 - 4\sigma^2\mu + 12c\sigma^2 + \sigma^2 + 11\sigma^4 + 6c^2\sigma^2 - 18c\sigma^2 \\
&= \mu^4 + 6\sigma^2\mu^2 - 4\sigma^2\mu - 6c\sigma^2 + \sigma^2 + 11\sigma^4 + 6c^2\sigma^2 \\
&= \mu^4 + 6\sigma^2\mu^2 - 4\sigma^2\mu + 11\sigma^4 + \left(6c^2 - 6c + 1\right)\sigma^2 \text{ as shown is } (6.29)
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\mu_4 &= E\left(X-\mu\right)^4 \\
&= E\left(X^4\right) - 4\mu E\left(X^3\right) + 6\mu^2 E\left(X^2\right) - 3\mu^4 \\
&= \mu^4 + 6\sigma^2\mu^2 - 4\sigma^2\mu + 11\sigma^4 + \left(6c^2 - 6c + 1\right)\sigma^2 \\
&- 4\mu\left(\mu^3 + 3\sigma^2\mu + \left(2c-1\right)\sigma^2\right) + 6\mu^2\left(\sigma^2 + \mu^2\right) - 3\mu^4 \\
&= \mu^4 + 6\sigma^2\mu^2 - 4\sigma^2\mu + 11\sigma^4 + \left(6c^2 - 6c + 1\right)\sigma^2 - 4\mu^4 \\
&- 12\sigma^2\mu^2 - 8\sigma^4 + 4\mu\sigma^2 + 6\mu^2\sigma^2 + 6\mu^4 - 3\mu^4 \\
&= 3\sigma^4 + \left(6c^2 - 6c + 1\right)\sigma^2 \text{ as in } (6.30)
\end{aligned}
$$

Remark :

$$G'(1) = \frac{\alpha}{1-\beta} = \mu$$

$$G''(1) = \frac{\alpha+\beta}{1-\beta} G'(1)$$

$$G'''(1) = \frac{\alpha(\alpha+\beta)(\alpha+2\beta)}{(1-\beta)^3} = \frac{(\alpha+2\beta)}{1-\beta} G''(1)$$

$$G^{iv}(1) = \frac{\alpha(\alpha+\beta)(\alpha+2\beta)(\alpha+3\beta)}{(1-\beta)^4} = \frac{(\alpha+3\beta)}{1-\beta} G'''(1)$$

Therefore

$$
\begin{aligned}
G^{(r+1)}(1) &= \frac{(\alpha+r\beta)}{1-\beta} G^{(r)}(1) \text{ for } r = 0,1,2,\ldots \\
&= \left[\frac{\alpha}{1-\beta} + \frac{r\beta}{1-\beta}\right] G^{(r)}(1) \\
&= \left[\mu + r\frac{\beta}{1-\beta}\right] G^{(r)}(1)
\end{aligned}
$$

but

$$\mu = \frac{\alpha}{1-\beta} \text{ and } \sigma^2 = \frac{\alpha}{(1-\beta)^2}$$

Therefore

$$c = \frac{\sigma^2}{\mu} = \frac{1}{1-\beta} \text{ and } \beta = 1 - \frac{\mu}{\sigma^2}$$
$$\Rightarrow \frac{\beta}{1-\beta} = \frac{\sigma^2}{\mu} - 1 = c - 1$$

In general

$$G^{(r+1)}(1) = [\mu + r(c-1)]\, G^{(r)}(1) \text{ for } r = 0, 1, 2, \ldots \text{ and } c = \frac{\sigma^2}{\mu} \tag{6.45}$$

$$r = 0 \qquad \Rightarrow G'(1) = \mu$$
$$r = 1 \qquad \Rightarrow G''(1) = [\mu + c - 1]\, G'(1)$$

$$\begin{aligned} G''(1) &= [\mu + c - 1]\, G'(1) = [\mu + c - 1]\, \mu \\ &= \mu^2 + \mu c - \mu = \mu^2 + \sigma^2 - \mu \end{aligned}$$

$$r = 2 \qquad \Rightarrow G'''(1) = [\mu + 2(c-1)]\, G''(1)$$

$$\begin{aligned} G'''(1) &= [\mu + 2(c-1)]\, G''(1) \\ &= (\mu + 2c - 2)\left(\mu^2 + \sigma^2 - \mu\right) \\ &= 2\mu - 2c\mu - 2\sigma^2 - 3\mu^2 + \mu^3 + 2c\sigma^2 + 2c\mu^2 + \sigma^2\mu \\ &= \mu^3 + 2\mu - 2\sigma^2 - 2\sigma^2 - 3\mu^2 + 2c\sigma^2 + 2\sigma^2\mu + \sigma^2\mu \\ &= \mu^3 - 3\mu^2 + \left(2\sigma^2 + \sigma^2 + 2\right)\mu + (-2 - 2 + 2c)\,\sigma^2 \\ &= \mu^3 - 3\mu^2 + \left(3\sigma^2 + 2\right)\mu + (2c - 4)\,\sigma^2 \\ &= \mu^3 - 3\mu^2 + \left(3\sigma^2 + 2\right)\mu + 2(c-2)\,\sigma^2 \end{aligned}$$

$$r = 3 \qquad \Rightarrow G^{iv}(1) = [\mu + 3c - 3]\, G'''(1)$$

$$\begin{aligned} G^{iv}(1) &= (\mu + 3c - 3)\left(\mu^3 - 3\mu^2 + \left(3\sigma^2 + 2\right)\mu + 2(c-2)\,\sigma^2\right) \\ &= (\mu + 3c - 3)\left(2\mu - 4\sigma^2 - 3\mu^2 + \mu^3 + 2c\sigma^2 + 3\sigma^2\mu\right) \\ &= 6c\mu - 6\mu + 12\sigma^2 + 11\mu^2 - 6\mu^3 + \mu^4 - 18c\sigma^2 - 9c\mu^2 + 3c\mu^3 \\ &- 13\sigma^2\mu + 11c\sigma^2\mu + 6c^2\sigma^2 + 3\sigma^2\mu^2 \\ &= \mu^4 - 6\mu^3 + \left(6\sigma^2 + 11\right)\mu^2 - \left(22\sigma^2 + 6\right)\mu + 11\sigma^4 + 6\left(c^2 - 3c + 3\right)\sigma^2 \end{aligned}$$

Chapter 7

Sum of Independent Random Variables

7.1 Introduction

The chapter entails determining the the distribution of a sum of independent random variables in terms of the distributions of the individual constituents. Only sums of discrete random variables will be considered. Also, random variables whose values are integers will be considered and their distribution functions are then defined on these integers.

Let

$$S_N = X_1 + X_2 + X_3 + \cdots + X_N$$

Where $X_i's$ are independent and identically distributed random variables. The problem is to find the distribution of S_N when,

i) N is fixed.

ii) N is also a random variable independent of $X_i's$.

In both cases $X_i's$ are inflated power series random variables. Thus we wish to determine convolutions and compound distributions for independent and identically inflated power series random variables.

7.2 Convolutions

7.2.1 Convolutions in general

Definition

Let $\{a_k\}$ and $\{b_k\}$ be any two numerical sequences. The new sequence $\{c_k\}$ defined by

$$\begin{aligned} c_k &= \sum_{r=0}^{k} a_r b_{k-r} \\ &= a_0 b_k + a_1 b_{k-1} + \cdots + a_{k-1} b_1 + a_k b_0 \end{aligned}$$

is called the convolution of $\{a_k\}$ and $\{b_k\}$ and will be denoted by

$$\{c_k\} = \{a_k\} * \{b_k\}$$

as given by Feller (1968).

In terms of probability generating function we have the following theorem

Theorem 7.1

If $\{a_k\}$ and $\{b_k\}$ are sequences with generating functions $A(s)$ and $B(s)$, and $\{c_k\}$ is their convolution, then the generating function $C(s)$ is the product of $A(s)$ and $B(s)$.

Proof :

If

$$A(s) = \sum_{k=0}^{\infty} a_k s^k$$

$$B(s) = \sum_{k=0}^{\infty} b_k s^k$$

Then

$$\begin{aligned} A(s)\,B(s) &= \left(a_0 + a_1 s + a_2 s^2 + \cdots\right)\left(b_0 + b_1 s + b_2 s^2 + \cdots\right) \\ &= a_0 b_0 + (a_0 b_1 + a_1 b_0)\,s + (a_0 b_2 + a_1 b_1 + a_2 b_0)\,s^2 + \cdots \\ &= c_0 + c_1 s + c_2 s^2 + c_3 s^3 + \cdots \\ &= C(s) \end{aligned}$$

We can extend the notion of convolution to many sequences for example.

Let $\{a_k\}, \{b_k\}, \{c_k\}, \{d_k\}, \ldots$be any sequences. We can form the convolution $\{a_k\} * \{b_k\}$, and then the convolution of this new equation with $\{c_k\}$,etc. The generating function of $\{a_k\} * \{b_k\} * \{c_k\} * \{d_k\}$ is $A(s)\,B(s)\,C(s)\,D(s)$, and this fact shows that the order in which the convolutions are performed is immaterial. For example, $\{a_k\} * \{b_k\} * \{c_k\} = \{c_k\} * \{b_k\} * \{a_k\}$ and $(\{a_k\} * \{b_k\}) * \{c_k\} = \{a_k\} * (\{b_k\} * \{c_k\})$. This shows that, convolution is an associative and commutative operation.

7.2.2 Convolution in random variables

Let X and Y be non-negative independent integral-valued random variables with probability distributions

$$p_i = \Pr(X = i) \qquad \text{and} \qquad q_j = \Pr(Y = j)$$

Further, let $Z = X + Y$ such that $r_k = \Pr(Z = k)$, The event $Z = k$ is the union of mutually exclusive events that is $(X = i,\ Y = j)$. Then, the distribution $r_k = \Pr(Z = k)$ is given by

$$\begin{aligned} r_k &= \Pr(X + Y = k) \\ &= \sum_{i=0}^{k} \Pr(X = i)\Pr(Y = k - i) \end{aligned}$$

and denoted by

$$\{r_k\} = \{p_i\} * \{q_j\}$$

Therefore in general the probability distribution of the sum of two or more independent random variables is the convolution of their individual distributions. That is, the probability mass function of a sum of random variables is the convolution of their corresponding probability mass function.

The pgf Technique

Let

$$H(s) = \text{ the pgf of } S_N$$
$$G_i(s) = \text{the pgf of } X_i$$

Then

$$\begin{aligned} H(s) &= E\left[S^{X_1+X_2+X_3+\cdots+X_N}\right] \\ &= \prod_{i=1}^{N} E\left[S^{X_i}\right] \\ &= \prod_{i=1}^{N} G_i(s) \end{aligned}$$

For independent and identically distributed (i.i.d) random variables $G_i(s) = G(s)$. Hence,

$$H(s) = [G(s)]^N$$

Remark : Convolution of distribution of random variables doesn't have to come from the same distribution.

7.2.3 Special cases

Poisson Distribution

If X and Y are independent Poisson random variables with parameters λ_1 and λ_2, then $Z = X + Y$ is a Poisson random variable with parameter $\lambda_1 + \lambda_2$. It follows that

$$\{p(k; \lambda_1 + \lambda_2)\} = \{p(i; \lambda_1)\} * \{p(j; \lambda_2)\}$$

Proof : By discrete convolution formula, $Z = X + Y$ has the probability mass function

$$\begin{aligned} \Pr(Z = k) &= \sum_{i=0}^{k} \Pr(X = i) \cdot \Pr(Y = k - i) \\ &= \sum_{i=0}^{k} e^{-\lambda_1} \frac{\lambda_1^i}{i!} e^{-\lambda_2} \frac{\lambda_2^{k-i}}{(k-i)!} \\ &= e^{-(\lambda_1+\lambda_2)} \sum_{i=0}^{k} \frac{\lambda_1^i}{i!} \frac{\lambda_2^{k-i}}{(k-i)!} \text{ using binomial formula we have} \\ &= e^{-(\lambda_1+\lambda_2)} \frac{(\lambda_1 + \lambda_2)^k}{k!}, k = 0, 1, 2, \ldots \end{aligned}$$

which is also the probability mass function for the Poisson distribution with parameter $\lambda_1 + \lambda_2$.

In general, Let $X_1, X_2, ..., X_k$ be independent Poisson random variables where X_i has a Poisson distribution with parameters (λ_i) for $i = 1, 2, ..., k$. Then, $X_1 + X_2 + \cdots + X_k$ has a Poisson distribution with parameter $(\lambda_1 + \lambda_2 + \cdots + \lambda_k)$. Thus

$$\{p(k; \lambda_i)\}^{k*} = \{p(i; \lambda_i)\}^{(k-1)*} * \{p(k; \lambda_k)\}$$

and the mean and variance is given by

$$E(S_k) = Var(S_k) = \lambda$$

Remark : Note that the rate parameters λ_i add

Binomial distribution

Let X and Y be independent binomial random variables where $X\ \tilde{}\, b(n_1, p)$ and $Y\tilde{}\, b(n_2, p)$. Then, $Z = X + Y$ has a binomial distribution with $n_1 + n_2$ trials and probability of success p. It follows that

$$\{b(k; n_1 + n_2, p)\} = \{b(i; n_1, p)\} * \{b(j; n_2, p)\}$$

Proof : By the discrete convolution formula, $Z = X + Y$ has probability mass function.

$$\begin{aligned}
\Pr(Z = k) &= \sum_{i=0}^{k} \Pr(X = i) \cdot \Pr(\ Y = k - i) \\
&= \sum_{i=0}^{k} \binom{n_1}{i} p^i (1-p)^{n_1 - i} \binom{n_2}{k-i} p^{k-i} (1-p)^{n_2 - (k-i)} \\
&= p^k (1-p)^{n_1 + n_2 - k} \sum_{i=0}^{k} \binom{n_1}{i} \binom{n_2}{k-i} \qquad (*)
\end{aligned}$$

but

$$\begin{aligned}
\sum_{k=0}^{\infty} c_k s^k &= \sum_{k=0}^{\infty} \sum_{i=0}^{k} \binom{n_1}{i} \binom{n_2}{k-i} s^k \\
&= \sum_{k=0}^{\infty} \sum_{i=0}^{\infty} \binom{n_1}{i} \binom{n_2}{k-i} s^k \\
&= \sum_{i=0}^{\infty} \left\{ \binom{n_1}{i} s^i \sum_{k=i}^{\infty} \binom{n_2}{k-i} s^{k-i} \right\} \\
&= \sum_{i=0}^{\infty} \left\{ \binom{n_1}{i} s^i (1+s)^{n_2} \right\} \\
&= (1+s)^{n_2} \sum_{i=0}^{\infty} \binom{n_1}{i} s^i \\
&= (1+s)^{n_2} (1+s)^{n_1} = (1+s)^{n_1 + n_2}
\end{aligned}$$

Hence

$$\sum_{i=0}^{k}\binom{n_1}{i}\binom{n_2}{k-i}=\binom{n_1+n_2}{k}$$

Replacing this value in $(*)$ above we get

$$\begin{aligned}\Pr(Z=k) &= p^k(1-p)^{n_1+n_2-k}\sum_{i=0}^{k}\binom{n_1}{i}\binom{n_2}{k-i}\\ &= \binom{n_1+n_2}{k}p^k(1-p)^{n_1+n_2-k}\end{aligned}$$

which is also the probability mass function for the binomial distribution with parameters(n_1+n_2) and p.

In general, Let $X_1, X_2, ..., X_k$ be independent binomial random variables where X_i has a

Binomial distribution with parameters (n_i, p) for $i = 1, 2, ..., k$. Then $X_1 + X_2 + \cdots X_k$ has a Binomial distribution with parameters $(n_1 + n_2 + \cdots + n_k)$ and p. Thus

$$\{b(k;n_i,p)\}^{k*}=\{b(i;n_i,p)\}^{(k-1)*}*\{b(k;n_k,p)\}$$

and the mean and variance is given by

$$\begin{aligned}E(S_k) &= np\\ Var(S_k) &= npq\end{aligned}$$

Remark : Note that the sample sizes add but the success probability remains the same.

Negative Binomial distribution

Let X and Y be independent Negative binomial random variables where X and Y is Negative Binomial with parameters (α_1, p) and (α_2, p) respectively. Then, $Z = X+Y$ has a Negative binomial distribution with $\alpha_1 + \alpha_2$ and probability of success p. It follows that

$$\{NB(k;\alpha_1+\alpha_2,p)\}=\{NB(i;\alpha_1,p)\}*\{NB(j;\alpha_2,p)\}$$

Proof : By the discrete convolution formula, $Z = X + Y$ has probability distribution.

$$\begin{aligned}\Pr(Z=k) &= \sum_{i=0}^{k}\Pr(X=i)\cdot\Pr(Y=k-i)\\ &= \sum_{i=0}^{k}\binom{\alpha_1+i-1}{i}p^{\alpha_1}(1-p)^i\binom{\alpha_2+(k-i)-1}{k-i}p^{\alpha_2}(1-p)^{k-i}\\ &= p^{\alpha_1+\alpha_2}(1-p)^k\sum_{i=0}^{k}\binom{\alpha_1+i-1}{i}\binom{\alpha_2+(k-i)-1}{k-i} \qquad (**)\end{aligned}$$

but

$$\begin{aligned}
\sum_{k=0}^{\infty} c_k s^k &= \sum_{k=0}^{\infty}\sum_{i=0}^{k}\binom{\alpha_1+i-1}{i}\binom{\alpha_2+(k-i)-1}{k-i}s^k \\
&= \sum_{k=0}^{\infty}\sum_{i=0}^{\infty}\binom{\alpha_1+i-1}{i}\binom{\alpha_2+(k-i)-1}{k-i}s^k \\
&= \sum_{i=0}^{\infty}\left\{\binom{\alpha_1+i-1}{i}s^i\sum_{k=i}^{\infty}\binom{\alpha_2+(k-i)-1}{k-i}s^{k-i}\right\} \\
&= \sum_{i=0}^{\infty}\left\{\binom{\alpha_1+i-1}{i}s^i\sum_{k=i}^{\infty}\binom{-\alpha_2}{k-i}(-s)^{k-i}\right\} \\
&= \sum_{i=0}^{\infty}\left\{\binom{\alpha_1+i-1}{i}s^i(1-s)^{-\alpha_2}\right\} \\
&= (1-s)^{-\alpha_2}\sum_{i=0}^{\infty}\binom{\alpha_1+i-1}{i}s^i \\
&= (1-s)^{-\alpha_2}\sum_{i=0}^{\infty}\binom{-\alpha_1}{i}(-s)^i \\
&= (1-s)^{-\alpha_2}(1-s)^{-\alpha_1} = (1-s)^{-(\alpha_1+\alpha_2)}
\end{aligned}$$

Hence

$$\sum_{i=0}^{k}\binom{\alpha_1+i-1}{i}\binom{\alpha_2+(k-i)-1}{k-i} = \binom{\alpha_1+\alpha_2+k-1}{k}$$

Replacing this value in $(**)$ above we get

$$\begin{aligned}
\Pr(Z=k) &= p^{\alpha_1+\alpha_2}(1-p)^k\sum_{i=0}^{k}\binom{\alpha_1+i-1}{i}\binom{\alpha_2+(k-i)-1}{k-i} \\
&= \binom{\alpha_1+\alpha_2+k-1}{k}p^{\alpha_1+\alpha_2}(1-p)^k
\end{aligned}$$

which is also the probability mass function for the negative binomial distribution with parameters $(\alpha_1+\alpha_2)$ and p.

In general, Let $X_1, X_2, ..., X_k$ be independent Negative binomial random variables where X_i has a Negative binomial distribution with parameters (α_i, p) for $i = 1, 2, ..., k$. Then

$X_1+X_2+\cdots X_k$ has a Negative binomial distribution with parameter $(\alpha_1+\alpha_2+\cdots\cdot+\alpha_k)$ and p. Thus

$$\{NB(k;\alpha_i,p)\}^{k*} = \{NB(i;\alpha_i,p)\}^{(k-1)*} * \{NB(k;\alpha_k,p)\}$$

and the mean and variance is given by

$$\begin{aligned}
E(S_k) &= \frac{\alpha q}{p} \\
Var(S_k) &= \frac{\alpha q}{p^2}
\end{aligned}$$

Geometric Distribution

Let X_i, $i = 1, 2$ be two independent Geometric random variables. where X_i, $i = 1, 2$ is Geometric with parameters (p).

$$\Pr(X_i = x) = p(1-p)^k : k = 0, 1, 2, \ldots, i = 1, 2$$

with the corresponding generating function.

$$p\sum_{i=1}^{2}\{(1-p)s\}^k = \frac{p}{1-(1-p)s}$$

Then,

$$Z = \sum_{i=1}^{2} X_i$$

has the probability distribution

Using the Generating function technique. Since X_i, $i = 1, 2$ are independent with $G(x_1)$ and $G(x_2)$, then

$$G(z) = G(x_1)G(x_2)$$

Then the Generating function would characterize distribution.

Therefore

$$\begin{aligned} G(z) &= \prod_{i=1}^{2}\left(\frac{p}{1-(1-p)s}\right) \\ &= \prod_{i=1}^{2} G(x_i) \text{ and since } G(x_i) = G(x) \text{ is i.i.d random variables} \\ &= [G(x)]^2 \end{aligned}$$

Which is the Generating function of negative binomial distribution with parameter $(2, p)$

In general, Let $X_1, X_2, ..., X_k$ be independent Geometric random variables where X_i has a Geometric distribution with parameters p for $i = 1, 2, ..., k$. Then $X_1 + X_2 + \cdots X_k$ has a Negative binomial distribution with parameter (k, p).

7.3 Compound Distributions

If a probability distribution is altered by allowing one of its parameters to behave as a random variable, the resulting distribution is said to be compound. An important compound distribution is that of the sum of random variables.

Let $X_1, X_2, X_3, \ldots, X_N$ be independently and identically distributed random variables. Let N be also a random variable independent of the $X_i's$.

If

$$S_N = X_1 + X_2 + X_3 + \cdots + X_N$$

then now N and S_N are two random variables to be studied.

Given the distributions of $X_i's$ and N, the problem is to find the pgf, probability distribution, mean and variance of S_N.

7.3.1 Review of Bivariate conditional and Marginal Distribution

If X and Y are two discrete random variables and $f(x,y)$ is their probability mass function (pmf), then the marginal distribution of X is

$$f_1(x) = \sum_y f(x,y)$$

and that of Y is

$$f_2(y) = \sum_x f(x,y)$$

The condition distribution of Y given X is

$$f(y|x) = \frac{f(x,y)}{f_1(x)}$$

while that of X given Y is

$$f(x|y) = \frac{f(x,y)}{f_2(x)}$$

Hence,

$$f(x,y) = f(x|y)\, f_2(x) = f(y|x)\, f_1(x)$$

Next, let $\phi(x,y)$ be a function of X and Y. We should note that since X and Y are random variables, their functions are also random variables.

Therefore,

$$E[\phi(X,Y)] = \sum_x \sum_y \phi(x,y)\, f(x,y)$$

For example, if

$$\phi(X,Y) = X^r Y^s$$

then

$$E[X^r Y^s] = \sum_x \sum_y x^r y^s f(x,y)$$

In particular, if $r = 0$ and $s = 1$, then

$$E(Y) = \sum_x \sum_y y f(x,y).$$

If $r = 1$ and $s = 1$, then

$$E(XY) = \sum_x \sum_y xy f(x,y)$$

Let us now prove the following

Theorem 7.2

For any two random variables X and Y

(*i*)

$$E(Y) = EE(Y|X)$$

(*ii*)

$$E(XY) = E[XE(Y|X)]$$

(iii)

$$VarY = E\left[Var\left(Y|X\right)\right] + Var\left[E\left(Y|X\right)\right]$$

Proof :

(i)

$$\begin{aligned} E\left(Y|X\right) &= \sum_y yf\left(y|x\right) \\ &= \sum_y y\frac{f\left(x,y\right)}{f_1\left(x\right)} \end{aligned}$$

which is a function of x, say $u\left(x\right)$;

That is, let $u\left(X\right) = E\left(Y|X\right)$

Therefore

$$\begin{aligned} EE\left(Y|X\right) &= E\left[u\left(X\right)\right] \\ &= \sum_x u\left(x\right)f_1\left(x\right) \\ &= \sum_x \left[\sum_y y\frac{f\left(x,y\right)}{f_1\left(x\right)}\right] f_1\left(x\right) \\ &= \sum_x \sum_y yf\left(x,y\right) \\ &= E\left(Y\right) \end{aligned}$$

(ii)

$$\begin{aligned} E\left[XE\left(Y|X\right)\right] &= E\left[Xu\left(X\right)\right] \\ &= \sum_x xu\left(x\right)f_1\left(x\right) \\ &= \sum_x x\left[\sum_y y\frac{f\left(x,y\right)}{f_1\left(x\right)}\right] f_1\left(x\right) \\ &= \sum_x \sum_y xyf\left(x,y\right) \\ &= E\left[XY\right] \end{aligned}$$

(iii)

$$\begin{aligned} VarY &= E\left(Y^2\right) - \left[E\left(Y\right)\right]^2 \\ &= EE\left(Y^2|X\right) - \left[EE\left(Y|X\right)\right]^2 \\ &= E\left[E\left(Y^2|X\right)\right] - \left\{E\left[u\left(X\right)\right]\right\}^2 \\ &= E\left[E\left(Y^2|X\right) - \left(E\left(Y|X\right)\right)^2 + \left(E\left(Y|X\right)\right)^2\right] - \left\{E\left[u\left(X\right)\right]\right\}^2 \\ &= E\left[Var\left(Y|X\right) + \left(E\left(Y|X\right)\right)^2\right] - \left\{E\left[u\left(X\right)\right]\right\}^2 \\ &= E\left[Var\left(Y|X\right)\right] + E\left(E\left(Y|X\right)\right)^2 - \left\{E\left[u\left(X\right)\right]\right\}^2 \\ &= E\left[Var\left(Y|X\right)\right] + E\left[u\left(X\right)\right]^2 - \left\{E\left[u\left(X\right)\right]\right\}^2 \\ &= E\left[Var\left(Y|X\right)\right] + Var\left[u\left(X\right)\right] \end{aligned}$$

Hence

$$
\begin{aligned}
VarY &= E\left[Var\left(Y|X\right)\right] + Var\left[u\left(X\right)\right] \\
&= E\left[Var\left(Y|X\right)\right] + Var\left[E\left(Y|X\right)\right]
\end{aligned}
$$

7.3.2 Conditional Expectation of a Random Sum of i.i.d random variables

Consider the two variables N and S_N. Setting $X = N$ and $Y = S_N$ in the ***Theorem* 7.2** we get,

(i)

$$
\begin{aligned}
E\left[S_N\right] &= EE\left[S_N|N\right] \\
&= EE\left[X_1 + X_2 + X_3 + \cdots + X_N\right] \\
&= E\left[NE\left(X_i\right)\right] \\
&= E\left(N\right)E\left(X_i\right)
\end{aligned}
$$

(ii)

$$
\begin{aligned}
VarS_N &= VarE\left(S_N|N\right) + EVar\left(S_N|N\right) \\
&= Var\left\{E\left[X_1 + X_2 + \cdots + X_N\right]\right\} + EVar\left[X_1 + X_2 + \cdots + X_N\right] \\
&= Var\left\{NE\left(X_i\right)\right\} + E\left\{NVarX_i\right\} \\
&= \left[E\left(X_i\right)\right]^2 VarN + VarX_iE\left(N\right)
\end{aligned}
$$

7.3.3 The probability generating function technique

Let $\{X_i\}$ be a sequence of independent and identically distributed random variables with common pgf.

$$G\left(s\right) = E\left(S^{X_i}\right)$$

and let

$$S_N = X_1 + X_2 + X_3 + \cdots + X_N$$

where N is a random variable independent of the $X_i's$ with pgf

$$F\left(s\right) = E\left(S^N\right)$$

We denote the pgf of S_N by $H\left(s\right)$ and it is given by,

$$
\begin{aligned}
H\left(s\right) &= E\left(S^{S_N}\right) \\
&= EE\left(S^{S_N|N}\right) \text{ using part } (i) \text{ of theorem } (7.2) \\
&= EE\left(S^{X_1+X_2+X_3+\cdots+X_N}\right) \\
&= E\prod_{i=1}^{N} E\left(S^{X_i}\right), \text{ because of independence of } X_i's \\
&= E\left[E\left(S^{X_i}\right)\right]^N, \text{ since } X_i's \text{ are identical} \\
&= E\left[G\left(s\right)\right]^N
\end{aligned}
$$

Therefore
$$H(s) = F[G(s)]$$

Therefore to obtain mean and variance using pgf technique, we find the first and second derivative of $H(s)$ with respect to s to obtain

$$H'(s) = G'(s)F'[G(s)]$$

and
$$H''(s) = [G'(s)]^2 F''(G(s)) + G''(s)F'[G(s)]$$

setting $s = 1$, we have
$$H(1) = F[G(1)] = F(1) = 1$$
$$H'(1) = G'(1)F'(1)$$
$$H''(1) = [G'(1)]^2 F''(1) + G''(1)F'(1)$$

Therefore,
$$E[S_N] = H'(1) = E(N)E(X_i)$$

$$\begin{aligned}
Var[S_N] &= H''(1) + H'(1) - [H'(1)]^2 \\
&= [G'(1)]^2 F''(1) + G''(1)F'(1) + G'(1)F'(1) - [G'(1)]^2[F'(1)]^2 \\
&= [G'(1)]^2 \left[F''(1) + (F'(1))^2\right] + [G''(1) - G'(1)]F'(1) \\
&= [G'(1)]^2[VarN - F'(1)] + \left[VarX + [G'(1)]^2\right]F'(1) \\
&= [G'(1)]^2 VarN + (VarX)F'(1) \\
&= [E(X)]^2 VarN + (VarX)E(N)
\end{aligned}$$

7.3.4 Special cases of Compound Power Series Distributions

a) Compound Poisson Distribution

Suppose
$$S_N = X_1 + X_2 + X_3 + \cdots + X_N$$

Where $X_i's$ are independent random variables, with N being a Poisson random variable. Then S_N is said to have a Compound Poisson Distribution. Suppose N is Poisson with parameter λ. Then the pgf of N is given by,

$$F(s) = e^{\lambda(s-1)}$$

and the pgf of S_N is given by,

$$H(s) = F[G_X(s)] = e^{\lambda[G_X(s)-1]}$$

Hence, the mean and the variance of S_N is given by

$$E[S_N] = E(N)E(X_i) = \lambda E(X_i)$$

and

$$\begin{aligned} Var\left[S_N\right] &= E\left[N\right] VarX_i + \left[E\left(X\right)\right]^2 VarN \\ &= \lambda VarX_i + \lambda \left[E\left(X_i\right)\right]^2 \\ &= \lambda \left[VarX_i + \left[E\left(X_i\right)\right]^2\right] \end{aligned}$$

Case (i) if X_i is Bernoulli with parameter $\Pr\{X_i = 1\} = p$ and $\Pr\{X_i = 0\} = q$, Then,

$$G_X(s) = q + ps \quad \text{where } q = 1 - p$$

Therefore

$$\begin{aligned} H(s) &= e^{\lambda[G_X(s)-1]} \\ &= e^{\lambda[q+ps-1]} = e^{\lambda[1-p+ps-1]} \\ &= e^{\lambda p(s-1)} \end{aligned}$$

Which is the pgf of Poisson-Bernoulli distribution.

Thus, the sum S_N is Poisson with parameter λp

The mean and the variance is given by

$$E\left[S_N\right] = E\left(N\right) E\left(X_i\right) = \lambda p$$

and

$$\begin{aligned} Var\left[S_N\right] &= \lambda \left[VarX_i + \left[E\left(X_i\right)\right]^2\right] \\ &= \lambda \left[p\left(1-p\right) + p^2\right] \\ &= \lambda \left[p - p^2 + p^2\right] = \lambda p \end{aligned}$$

Case (ii) if X_i is Zero truncated geometric Distribution with parameter p, then

$$G_X(s) = \frac{ps}{1-qs} \quad \text{where } q = 1 - p$$

Therefore

$$\begin{aligned} H(s) &= e^{\lambda[G_X(s)-1]} \\ &= e^{\lambda\left[\frac{ps}{1-qs}-1\right]} = e^{\lambda\left[\frac{ps-1+qs}{1-qs}\right]} \\ &= e^{\lambda\left[\frac{ps-1+s-ps}{1-qs}\right]} = e^{\lambda\left[\frac{s-1}{1-qs}\right]} \end{aligned}$$

Which is the pgf of Poisson-Zero truncated Geometric Distribution. This distribution is called Polya-Aeppli Distribution.

The mean and the variance is given by

$$E\left[S_N\right] = E\left(N\right) E\left(X_i\right) = \lambda \frac{1}{p} = \frac{\lambda}{p}$$

and

$$\begin{aligned} Var\left[S_N\right] &= \lambda \left[VarX_i + \left[E\left(X_i\right)\right]^2\right] \\ &= \lambda \left[\frac{1-p}{p^2} + \frac{1}{p^2}\right] \\ &= \lambda \left[\frac{2-p}{p^2}\right] = \lambda \left[\frac{1+q}{p^2}\right] \end{aligned}$$

Case (iii) if X_i is Zero-Inflated Binomial Distribution with parameters (n, p, ρ), then

$$G_X(s) = \rho + (1-\rho)[1-p+ps]^n$$

Therefore

$$\begin{aligned} H(s) &= e^{\lambda[G_X(s)-1]} \\ &= e^{\lambda\{\rho+(1-\rho)[1-p+ps]^n-1\}} \end{aligned}$$

Which is the pgf of Poisson-Zero-Inflated Binomial Distribution.

The mean and the variance is given by

$$E[S_N] = E(N)E(X_i) = \lambda np(1-\rho)$$

and

$$\begin{aligned} Var[S_N] &= \lambda\left[VarX_i + [E(X_i)]^2\right] \\ &= \lambda\left\{(1-\rho)\left[p^2n^2\rho + npq\right] + [np(1-\rho)]^2\right\} \\ &= \lambda(1-\rho)\left[p^2n^2\rho + npq + n^2p^2(1-\rho)\right] \\ &= \lambda(1-\rho)\left[npq + n^2p^2\right] \end{aligned}$$

Case (iv) if X_i is Zero-Inflated Negative Binomial Distribution with parameters (α, p, ρ), then

$$G_X(s) = \rho + (1-\rho)\left\{\frac{p}{1-(1-p)s}\right\}^\alpha$$

Therefore

$$\begin{aligned} H(s) &= e^{\lambda[G_X(s)-1]} \\ &= \exp\left\{\lambda\left(\rho + (1-\rho)\left[\frac{p}{1-(1-p)s}\right]^\alpha - 1\right)\right\} \end{aligned}$$

Which is the pgf of Poisson-Zero-Inflated Negative Binomial Distribution.

The mean and the variance is given by

$$E[S_N] = E(N)E(X_i) = (1-\rho)\lambda\alpha\frac{q}{p}$$

and

$$\begin{aligned} Var[S_N] &= \lambda\left[VarX_i + [E(X_i)]^2\right] \\ &= \lambda\left\{(1-\rho)\left[\frac{\alpha^2q^2}{p^2}\rho + \frac{\alpha q}{p^2}\right] + \left[(1-\rho)\alpha\frac{q}{p}\right]^2\right\} \\ &= \lambda(1-\rho)\left\{\frac{\alpha q + \alpha^2q^2}{p^2}\right\} \\ &= (1-\rho)\lambda p^{-2}\left(\alpha q + \alpha^2q^2\right) \end{aligned}$$

Case (v) if X_i is Zero-Modified Logarithmic Series Distribution with parameters (p, ρ), then

$$G_X(s) = \rho + (1-\rho)\frac{\ln[1-ps]}{\ln(1-p)}$$

Therefore

$$\begin{aligned} H(s) &= e^{\lambda[G_X(s)-1]} \\ &= \exp\left\{\lambda\left[\rho + (1-\rho)\frac{\ln[1-ps]}{\ln(1-p)} - 1\right]\right\} \end{aligned}$$

Which is the pgf of Poisson-Zero-Modified Logarithmic Series Distribution.

The mean and the variance is given by

$$E[S_N] = E(N)E(X_i) = \frac{\lambda(1-\rho)p}{-(1-p)\log(1-p)}$$

and

$$\begin{aligned} Var[S_N] &= \lambda\left[VarX_i + [E(X_i)]^2\right] \\ &= \lambda\left\{-(1-\rho)\left\{\frac{p\log(1-p)+(1-\rho)p^2}{(1-p)^2[\log(1-p)]^2}\right\} + \left[\frac{-(1-\rho)p}{(1-p)\log(1-p)}\right]^2\right\} \\ &= \lambda\left\{-(1-\rho)\left[\frac{p\log(1-p)+(1-\rho)p^2}{(1-p)^2[\log(1-p)]^2}\right] + \frac{(1-\rho)^2p^2}{(1-p)^2[\log(1-p)]^2}\right\} \\ &= -(1-\rho)\lambda\left\{\frac{p\log(1-p)+(1-\rho)p^2-(1-\rho)p^2}{(1-p)^2[\log(1-p)]^2}\right\} \\ &= -\frac{(1-\rho)\lambda p}{(1-p)^2\log(1-p)} \end{aligned}$$

Case (vi) if X_i is Zero-inflated Poisson distribution with parameter (λ, ρ)

$$G_X(s) = \rho + (1-\rho)e^{\theta(s-1)}$$

Therefore

$$\begin{aligned} H(s) &= e^{\lambda[G_X(s)-1]} \\ &= e^{\lambda\left[\rho+(1-\rho)e^{\theta(s-1)}-1\right]} \end{aligned}$$

Which is the pgf of Poisson-Zero-Inflated Poisson Distribution.

The mean and the variance is given by

$$E[S_N] = E(N)E(X_i) = \lambda^2(1-\rho)$$

and

$$\begin{aligned} Var[S_N] &= \lambda\left[VarX_i + [E(X_i)]^2\right] \\ &= \lambda\left[(1-\rho)\lambda(1+\rho\lambda) + \lambda^2(1-\rho)^2\right] \\ &= \lambda^2(1-\rho)[1+\rho\lambda+\lambda-\lambda\rho] \\ &= \lambda^2(1-\rho)[1+\lambda] \end{aligned}$$

b) Compound Binomial Distribution

Suppose

$$S_N = X_1 + X_2 + X_3 + \cdots + X_N$$

Where $X_i^{\prime}s$ are independent random variables, with N being a Binomial random variable. Then S_N is said to have a Compound Binomial Distribution. Suppose N is Binomial with parameters (n, p). Then the pgf of N is given by,

$$F(s) = [1 - p + ps]^n$$

and the pgf of S_N is given by,

$$H(s) = F[G_X(s)] = \{1 - p + p[G_X(s)]\}^n$$

Hence, the mean and the variance of S_N is given by

$$E[S_N] = E(N)E(X_i) = npE(X_i)$$

and

$$\begin{aligned} Var[S_N] &= E[N]\,VarX_i + [E(X)]^2\,VarN \\ &= npVarX_i + np(1-p)[E(X_i)]^2 \\ &= np\left[VarX_i + (1-p)[E(X_i)]^2\right] \end{aligned}$$

Case (i) if X_i is Zero-Inflated Poisson Distribution with parameters (λ, ρ), then

$$G_X(s) = \rho + (1-\rho)e^{\lambda(s-1)}$$

Therefore

$$\begin{aligned} H(s) &= \{1 - p + p[G_X(s)]\}^n \\ &= \left\{1 - p + p\left[\rho + (1-\rho)e^{\lambda(s-1)}\right]\right\}^n \end{aligned}$$

Which is the pgf of Binomial-Zero-Inflated Poisson Distribution.

The mean and the variance is given by

$$E[S_N] = E(N)E(X_i) = np\lambda(1-\rho)$$

and

$$\begin{aligned} Var[S_N] &= E[N]\,VarX_i + [E(X)]^2\,VarN \\ &= np\left[VarX_i + (1-p)[E(X_i)]^2\right] \\ &= np\left\{(1-\rho)\lambda(1+\rho\lambda) + (1-p)[(1-\rho)\lambda]^2\right\} \\ &= np(1-\rho)\lambda\{(1+\rho\lambda) + (1-p)(1-\rho)\lambda\} \\ &= np(1-\rho)\lambda\{1 + \rho\lambda + \lambda - \rho\lambda - \lambda p + \lambda p\rho\} \\ &= \lambda np(1-\rho)\{1 + \lambda(1 - p + p\rho)\} \end{aligned}$$

Case (ii) if X_i is Zero-Inflated Negative Binomial Distribution with parameters (α, p, ρ), then

$$G_X(s) = \rho + (1-\rho)\left\{\frac{p}{1-(1-p)s}\right\}^{\alpha}$$

Therefore

$$\begin{aligned} H(s) &= \{1-p+p[G_X(s)]\}^n \\ &= \left\{1-p+p\left[\rho+(1-\rho)\left(\frac{p}{1-(1-p)s}\right)^{\alpha}\right]\right\}^n \end{aligned}$$

Which is the pgf of Binomial-Zero-Inflated Negative Binomial Distribution.

The mean and the variance is given by

$$E[S_N] = E(N)E(X_i) = n\alpha q(1-\rho)$$

and

$$\begin{aligned} Var[S_N] &= E[N]\,VarX_i + [E(X_i)]^2\,VarN \\ &= np\left[VarX_i + (1-p)[E(X_i)]^2\right] \\ &= np\left\{(1-\rho)\left[\frac{\alpha^2 q^2}{p^2}\rho + \frac{\alpha q}{p^2}\right] + (1-p)\left[\alpha\frac{q}{p}(1-\rho)\right]^2\right\} \\ &= np(1-\rho)\left\{\frac{\alpha^2 q^2}{p^2}\rho + \frac{\alpha q}{p^2} + \frac{\alpha^2 q^2}{p^2}(1-\rho)^2\right\} \\ &= np(1-\rho)\left\{\frac{\alpha^2 q^2}{p^2}\rho + \frac{\alpha q}{p^2} + \frac{\alpha^2 q^2}{p^2} - 2\frac{\alpha^2 q^2}{p^2}\rho + \frac{\alpha^2 q^2}{p^2}\rho^2\right\} \\ &= n(1-\rho)\left\{\frac{\alpha q}{p} + \frac{\alpha^2 q^2}{p}\left(1-\rho+\rho^2\right)\right\} \end{aligned}$$

Case (iii) if X_i is Zero-Modified Logarithmic Series Distribution with parameters (p, ρ), then

$$G_X(s) = \rho + (1-\rho)\frac{\ln[1-ps]}{\ln(1-p)}$$

Therefore

$$\begin{aligned} H(s) &= \{1-p+p[G_X(s)]\}^n \\ &= \left\{1-p+p\left[\rho+(1-\rho)\frac{\ln[1-ps]}{\ln(1-p)}\right]\right\}^n \end{aligned}$$

Which is the pgf of Binomial-Zero-Modified Logarithmic Series Distribution.

The mean and the variance is given by

$$\begin{aligned} E[S_N] &= E(N)E(X_i) = npE(X_i) \\ &= \frac{(1-\rho)np^2}{-(1-p)\log(1-p)} \end{aligned}$$

and

$$
\begin{aligned}
Var\left[S_N\right] &= E\left[N\right] VarX_i + \left[E\left(X_i\right)\right]^2 VarN \\
&= np\left[VarX_i + (1-p)\left[E\left(X_i\right)\right]^2\right] \\
&= np\left(1-\rho\right)\left\{-\left[\frac{p\log(1-p)+(1-\rho)\,p^2}{(1-p)^2\left[\log(1-p)\right]^2}\right] + \left(\frac{(1-\rho)\,p}{-(1-p)\log(1-p)}\right)^2\right\} \\
&= -np\left(1-\rho\right)\left\{\frac{p\log(1-p)+(1-\rho)\,p^2-(1-\rho)^2 p^2}{(1-p)^2\left[\log(1-p)\right]^2}\right\}
\end{aligned}
$$

Case (iv) if X_i is Zero-Inflated Binomial Distribution with parameters (n, p, ρ), then

$$
G_X\left(s\right) = \rho + (1-\rho)\left[1-p+ps\right]^n
$$

Therefore

$$
\begin{aligned}
H\left(s\right) &= \left\{1-p+p\left[G_X\left(s\right)\right]\right\}^n \\
&= 1-p+p\left\{\rho+(1-\rho)\left[1-p+ps\right]^n\right\}
\end{aligned}
$$

Which is the pgf of Binomial-Zero-Inflated Binomial Distribution.

The mean and the variance is given by

$$
\begin{aligned}
E\left[S_N\right] &= E\left(N\right)E\left(X_i\right) = npE\left(X_i\right) \\
&= n^2p^2\left(1-\rho\right)
\end{aligned}
$$

and

$$
\begin{aligned}
Var\left[S_N\right] &= E\left[N\right] VarX_i + \left[E\left(X_i\right)\right]^2 VarN \\
&= np\left\{(1-\rho)\left[p^2n^2\rho+npq\right] + (1-p)\left[n^2p^2\left(1-\rho\right)^2\right]\right\} \\
&= np\left(1-\rho\right)\left\{npq + n^2p^2\left[1-\rho+\rho^2\right]\right\}
\end{aligned}
$$

c) Compound Negative Binomial Distribution

Suppose

$$
S_N = X_1 + X_2 + X_3 + \cdots + X_N
$$

Where $X_i's$ are independent random variables, with N being a Negative Binomial random variable. Then S_N is said to have a Compound Negative Binomial Distribution. Suppose N is Negative Binomial with parameters (α, p). Then the pgf of N is given by,

$$
F\left(s\right) = \left(\frac{p}{1-(1-p)\,s}\right)^{\alpha}
$$

and the pgf of S_N is given by,

$$
H\left(s\right) = F\left[G_X\left(s\right)\right] = \left(\frac{p}{\left[1-(1-p)\,G_X\left(s\right)\right]}\right)^{\alpha}
$$

Hence, the mean and the variance of S_N is given by

$$
E\left[S_N\right] = E\left(N\right)E\left(X_i\right) = \frac{\alpha\left(1-p\right)}{p}E\left(X_i\right)
$$

and

$$
\begin{aligned}
Var\left[S_N\right] &= E\left[N\right] VarX_i + \left[E\left(X\right)\right]^2 VarN \\
&= \frac{\alpha\left(1-p\right)}{p} VarX_i + \frac{\alpha\left(1-p\right)}{p^2}\left[E\left(X_i\right)\right]^2 \\
&= \frac{\alpha\left(1-p\right)}{p}\left[VarX_i + \frac{1}{p}\left[E\left(X_i\right)\right]^2\right]
\end{aligned}
$$

Case (i) if X_i is Zero-Modified Logarithmic Series Distribution with parameters (p,ρ), then

$$
G_X\left(s\right) = \rho + \left(1-\rho\right)\frac{\ln\left[1-ps\right]}{\ln\left(1-p\right)}
$$

Therefore

$$
\begin{aligned}
H\left(s\right) = F\left[G_X\left(s\right)\right] &= \left(\frac{p}{\left(1-\left(1-p\right)G_X\left(s\right)\right)}\right)^{\alpha} \\
&= \left\{\frac{p}{\left[1-\left(1-p\right)\left(\rho+\left(1-\rho\right)\frac{\ln[1-ps]}{\ln(1-p)}\right)\right]}\right\}^{\alpha}
\end{aligned}
$$

Which is the pgf of Negative Binomial-Zero-Modified Logarithmic Series Distribution.

The mean and the variance is given by

$$
E\left[S_N\right] = E\left(N\right)E\left(X_i\right) = -\frac{\left(1-\rho\right)\alpha}{\log(1-p)}
$$

and

$$
\begin{aligned}
Var\left[S_N\right] &= E\left[N\right] VarX_i + \left[E\left(X_i\right)\right]^2 VarN \\
&= \frac{\alpha\left(1-p\right)}{p}\left[VarX_i + \frac{1}{p}\left(E\left(X_i\right)\right)^2\right] \\
&= \frac{\alpha\left(1-p\right)}{p}\left\{\begin{array}{c} -\left(1-\rho\right)\left[\frac{p\log(1-p)+(1-\rho)p^2}{(1-p)^2[\log(1-p)]^2}\right] \\ +\frac{1}{p}\left(\frac{(1-\rho)p}{-(1-p)\log(1-p)}\right)^2 \end{array}\right\} \\
&= -\left(1-\rho\right)\alpha\left\{\frac{\log(1-p)+\left(1-\rho\right)p-\left(1-\rho\right)}{\left(1-p\right)\left[\log(1-p)\right]^2}\right\}
\end{aligned}
$$

Case (ii) if X_i is Zero-Inflated Poisson Distribution with parameters (λ,ρ), then

$$
G_X\left(s\right) = \rho + \left(1-\rho\right)e^{\lambda(s-1)}
$$

Therefore

$$
\begin{aligned}
H\left(s\right) = F\left[G_X\left(s\right)\right] &= \left(\frac{p}{\left(1-\left(1-p\right)G_X\left(s\right)\right)}\right)^{\alpha} \\
&= \left\{\frac{p}{\left[1-\left(1-p\right)\left[\rho+\left(1-\rho\right)e^{\lambda(s-1)}\right]\right]}\right\}^{\alpha}
\end{aligned}
$$

Which is the pgf of Negative Binomial-Zero-Inflated Poisson Distribution.

The mean and the variance is given by

$$
\begin{aligned}
E\left[S_N\right] &= E\left(N\right)E\left(X_i\right) = \frac{\alpha\left(1-p\right)}{p}E\left(X_i\right) \\
&= \frac{\lambda\alpha\left(1-p\right)\left(1-\rho\right)}{p}
\end{aligned}
$$

and

$$
\begin{aligned}
Var\left[S_N\right] &= E\left[N\right]VarX_i + \left[E\left(X\right)\right]^2 VarN \\
&= \frac{\alpha\left(1-p\right)}{p}VarX_i + \frac{\alpha\left(1-p\right)}{p^2}\left[E\left(X_i\right)\right]^2 \\
&= \frac{\alpha\left(1-p\right)}{p}\left[\left(1-\rho\right)\lambda\left(1+\rho\lambda\right) + \frac{1}{p}\left[\lambda\left(1-\rho\right)\right]^2\right] \\
&= \frac{\alpha\left(1-p\right)\left(1-\rho\right)\lambda}{p}\left[\left(1+\rho\lambda\right) + \frac{\lambda\left(1-\rho\right)}{p}\right]
\end{aligned}
$$

Case (iii) if X_i is Zero-Inflated Binomial Distribution with parameters (n, p, ρ), then

$$
G_X\left(s\right) = \rho + \left(1-\rho\right)\left[1-p+ps\right]^n
$$

Therefore

$$
\begin{aligned}
H\left(s\right) = F\left[G_X\left(s\right)\right] &= \left(\frac{p}{\left(1-\left(1-p\right)G_X\left(s\right)\right)}\right)^{\alpha} \\
&= \left\{\frac{p}{\left(1-\left(1-p\right)\left[\rho+\left(1-\rho\right)\left(1-p+ps\right)^n\right]\right)}\right\}^{\alpha}
\end{aligned}
$$

Which is the pgf of Negative Binomial-Zero-Inflated Binomial Distribution.

The mean and the variance is given by

$$
\begin{aligned}
E\left[S_N\right] &= E\left(N\right)E\left(X_i\right) = \frac{\alpha\left(1-p\right)}{p}E\left(X_i\right) \\
&= \alpha\left(1-p\right)\left(1-\rho\right)n
\end{aligned}
$$

and

$$
\begin{aligned}
Var\left[S_N\right] &= E\left[N\right]VarX_i + \left[E\left(X_i\right)\right]^2 VarN \\
&= \frac{\alpha\left(1-p\right)}{p}VarX_i + \frac{\alpha\left(1-p\right)}{p^2}\left[E\left(X_i\right)\right]^2 \\
&= \frac{\alpha\left(1-p\right)}{p}\left[\left(1-\rho\right)\left\{p^2n^2\rho + npq\right\} + \frac{1}{p}\left[np\left(1-\rho\right)\right]^2\right] \\
&= \frac{\alpha n^2\left(1-p\right)\left(1-\rho\right)}{p}\left[p^2\rho + npq + 1 - \rho\right]
\end{aligned}
$$

Case (iv) if X_i is Zero-Inflated Negative Binomial Distribution with parameters (α, p, ρ), then

$$
G_X\left(s\right) = \rho + \left(1-\rho\right)\left\{\frac{p}{1-\left(1-p\right)s}\right\}^{\alpha}
$$

Therefore

$$H(s) = F[G_X(s)] = \left(\frac{p}{(1-(1-p)G_X(s))}\right)^{\alpha}$$
$$= \left\{\frac{p}{\left(1-(1-p)\left[\rho+(1-\rho)\left\{\frac{p}{1-(1-p)s}\right\}^{\alpha}\right]\right)}\right\}^{\alpha}$$

Which is the pgf of Negative Binomial-Zero-Inflated Negative Binomial Distribution.

The mean and the variance is given by

$$E[S_N] = E(N)E(X_i) = \frac{\alpha(1-p)}{p}E(X_i)$$
$$= (1-\rho)\frac{\alpha^2 q^2}{p^2}$$

and

$$Var[S_N] = \frac{\alpha(1-p)}{p}\left[VarX_i + \frac{1}{p}[E(X_i)]^2\right]$$
$$= \frac{\alpha(1-p)}{p}\left\{(1-\rho)\left[\frac{\alpha^2 q^2}{p^2}\rho + \frac{\alpha q}{p^2}\right] + \frac{1}{p}\left[\alpha\frac{q}{p}(1-\rho)\right]^2\right\}$$
$$= (1-\rho)\frac{\alpha q}{p}\left\{\frac{\alpha^2 q^2}{p^2}\left(\rho + \frac{1}{p} - \frac{1}{p}\rho\right) + \frac{\alpha q}{p^2}\right\}$$

d) Compound Geometric Distribution

Suppose

$$S_N = X_1 + X_2 + X_3 + \cdots + X_N$$

Where $X_i's$ are independent random variables, with N being a Geometric random variable. Then S_N is said to have a Compound Geometric Distribution. Suppose N is Geometric with parameter p.Then the pgf of N is given by,

$$F(s) = \frac{p}{(1-(1-p)s)}$$

and the pgf of S_N is given by,

$$H(s) = F[G_X(s)] = \frac{p}{(1-(1-p)G_X(s))}$$

Hence, the mean and the variance of S_N is given by

$$E[S_N] = E(N)E(X_i) = \frac{(1-p)}{p}E(X_i)$$

and

$$Var[S_N] = E[N]VarX_i + [E(X)]^2 VarN$$
$$= \frac{(1-p)}{p}VarX_i + \frac{(1-p)}{p^2}[E(X_i)]^2$$
$$= \frac{(1-p)}{p}\left[VarX_i + \frac{1}{p}[E(X_i)]^2\right]$$

Case (i) if X_i is Zero-Modified Logarithmic Series Distribution with parameters (p, ρ), then

$$G_X(s) = \rho + (1-\rho)\frac{\ln[1-ps]}{\ln(1-p)}$$

Therefore

$$\begin{aligned} H(s) = F[G_X(s)] &= \frac{p}{\{1-(1-p)G_X(s)\}} \\ &= \frac{p}{\left\{1-(1-p)\left(\rho+(1-\rho)\frac{\ln[1-ps]}{\ln(1-p)}\right)\right\}} \end{aligned}$$

Which is the pgf of Geometric-Zero-Modified Logarithmic Series Distribution.

The mean and the variance is given by

$$E[S_N] = E(N)E(X_i) = -\frac{(1-\rho)}{\log(1-p)}$$

and

$$\begin{aligned} Var[S_N] &= E[N]\,VarX_i + [E(X_i)]^2\,VarN \\ &= \frac{(1-p)}{p}\left[VarX_i + \frac{1}{p}(E(X_i))^2\right] \\ &= \frac{(1-p)}{p}\left\{-(1-\rho)\left[\frac{p\log(1-p)+(1-\rho)p^2}{(1-p)^2[\log(1-p)]^2}\right] + \frac{1}{p}\left(\frac{(1-\rho)p}{-(1-p)\log(1-p)}\right)^2\right\} \\ &= -(1-\rho)\left\{\frac{\log(1-p)+(1-\rho)p-(1-\rho)}{(1-p)[\log(1-p)]^2}\right\} \end{aligned}$$

Case (ii) if X_i is Zero-Inflated Poisson Distribution with parameters (λ, ρ), then

$$G_X(s) = \rho + (1-\rho)e^{\lambda(s-1)}$$

Therefore

$$\begin{aligned} H(s) = F[G_X(s)] &= \frac{p}{\{1-(1-p)G_X(s)\}} \\ &= \frac{p}{\{1-(1-p)[\rho+(1-\rho)e^{\lambda(s-1)}]\}} \end{aligned}$$

Which is the pgf of Geometric-Zero-Inflated Poisson Distribution.

The mean and the variance is given by

$$\begin{aligned} E[S_N] = E(N)E(X_i) &= \frac{(1-p)}{p}E(X_i) \\ &= \frac{\lambda(1-p)(1-\rho)}{p} \end{aligned}$$

and

$$
\begin{aligned}
Var\left[S_N\right] &= E\left[N\right]VarX_i + \left[E\left(X\right)\right]^2 VarN \\
&= \frac{(1-p)}{p}VarX_i + \frac{(1-p)}{p^2}\left[E\left(X_i\right)\right]^2 \\
&= \frac{(1-p)}{p}\left[(1-\rho)\lambda(1+\rho\lambda) + \frac{1}{p}\left[\lambda(1-\rho)\right]^2\right] \\
&= \frac{(1-p)(1-\rho)\lambda}{p}\left[(1+\rho\lambda) + \frac{\lambda(1-\rho)}{p}\right]
\end{aligned}
$$

Case (iii) if X_i is Zero-Inflated Binomial Distribution with parameters (n, p, ρ), then

$$
G_X(s) = \rho + (1-\rho)\left[1-p+ps\right]^n
$$

Therefore

$$
\begin{aligned}
H(s) = F\left[G_X(s)\right] &= \frac{p}{\{1-(1-p)G_X(s)\}} \\
&= \frac{p}{\{1-(1-p)\left[\rho+(1-\rho)(1-p+ps)^n\right]\}}
\end{aligned}
$$

Which is the pgf of Geometric-Zero-Inflated Binomial Distribution.
The mean and the variance is given by

$$
\begin{aligned}
E\left[S_N\right] = E(N)E(X_i) &= \frac{(1-p)}{p}E(X_i) \\
&= (1-p)(1-\rho)n
\end{aligned}
$$

and

$$
\begin{aligned}
Var\left[S_N\right] &= E\left[N\right]VarX_i + \left[E\left(X_i\right)\right]^2 VarN \\
&= \frac{(1-p)}{p}VarX_i + \frac{(1-p)}{p^2}\left[E\left(X_i\right)\right]^2 \\
&= \frac{(1-p)}{p}\left\{(1-\rho)\left[p^2n^2\rho + npq\right] + \frac{1}{p}\left[np(1-\rho)\right]^2\right\} \\
&= \frac{n^2(1-p)(1-\rho)}{p}\left[p^2\rho + npq + 1 - \rho\right]
\end{aligned}
$$

e) Compound Logarithmic Series Distribution

Suppose

$$
S_N = X_1 + X_2 + X_3 + \cdots + X_N
$$

Where $X_i's$ are independent random variables, with N being a Logarithmic Series random variable. Then S_N is said to have a Compound Logarithmic Series Distribution. Suppose N is Logarithmic Series Distribution with parameter π.Then the pgf of N is given by,

$$
F(s) = \frac{\ln\left[1-ps\right]}{\ln(1-p)}
$$

and the pgf of S_N is given by,

$$H(s) = F[G_X(s)] = \frac{\ln[1 - pG_X(s)]}{\ln(1-p)}$$

Hence, the mean and the variance of S_N is given by

$$E[S_N] = E(N)E(X_i) = \frac{p}{-(1-p)\log(1-p)} E(X_i)$$

and

$$\begin{aligned} Var[S_N] &= E[N]\, VarX_i + [E(X)]^2 VarN \\ &= \frac{p}{-(1-p)\log(1-p)} VarX_i - \frac{[p + \log(1-p)]}{[(1-p)\log(1-p)]^2} [E(X_i)]^2 \\ &= \frac{p}{-(1-p)\log(1-p)} \left[VarX_i + \frac{[p+\log(1-p)]}{(1-p)\log(1-p)} [E(X_i)]^2 \right] \end{aligned}$$

Case (i) if X_i is Zero-Inflated Poisson Distribution with parameters (λ, ρ), then

$$G_X(s) = \rho + (1-\rho)e^{\lambda(s-1)}$$

Therefore

$$H(s) = F[G_X(s)] = \frac{\ln\left[1 - p\left(\rho + (1-\rho)e^{\lambda(s-1)}\right)\right]}{\ln(1-p)}$$

Which is the pgf of Logarithmic-Zero-Inflated Poisson Distribution.
The mean and the variance is given by

$$E[S_N] = E(N)E(X_i) = \frac{\lambda(1-\rho)p}{-(1-p)\log(1-p)}$$

and

$$\begin{aligned} Var[S_N] &= E[N]\, VarX_i + [E(X)]^2 VarN \\ &= \frac{p}{-(1-p)\log(1-p)} VarX_i - \frac{[p + \log(1-p)]}{[(1-p)\log(1-p)]^2} [E(X_i)]^2 \\ &= \frac{\lambda(1-\rho)p}{-(1-p)\log(1-p)} \left[(1+\rho\lambda) + \frac{[p+\log(1-p)]}{(1-p)\log(1-p)} \lambda(1-\rho) \right] \end{aligned}$$

Case (ii) if X_i is Zero-Inflated Binomial Distribution with parameters (n, p, ρ), then

$$G_X(s) = \rho + (1-\rho)[1 - p + ps]^n$$

Therefore

$$H(s) = F[G_X(s)] = \frac{\ln[1 - p\{\rho + (1-\rho)[1-p+ps]^n\}]}{\ln(1-p)}$$

Which is the pgf of Logarithmic-Zero-Inflated Binomial Distribution.

The mean and the variance is given by

$$E\left[S_N\right] = E\left(N\right)E\left(X_i\right) = \frac{np^2\left(1-\rho\right)}{-\left(1-p\right)\log(1-p)}$$

and

$$\begin{aligned} Var\left[S_N\right] &= E\left[N\right]VarX_i + \left[E\left(X\right)\right]^2 VarN \\ &= \frac{p}{-\left(1-p\right)\log(1-p)}VarX_i - \frac{\left[p+\log(1-p)\right]}{\left[\left(1-p\right)\log(1-p)\right]^2}\left[E\left(X_i\right)\right]^2 \\ &= \frac{p}{-\left(1-p\right)\log(1-p)}\left\{ \begin{array}{c} \left(1-\rho\right)\left[p^2n^2\rho+npq\right] \\ +\frac{\left[p+\log(1-p)\right]}{\left(1-p\right)\log(1-p)}\left[np\left(1-\rho\right)\right]^2 \end{array} \right\} \\ &= \frac{p\left(1-\rho\right)}{-\left(1-p\right)\log(1-p)}\left\{p^2n^2\rho+npq+\frac{\left[p+\log(1-p)\right]}{\left(1-p\right)\log(1-p)}n^2p^2\left(1-\rho\right)\right\} \end{aligned}$$

Case (iii) if X_i is Zero-Inflated Negative Binomial Distribution with parameters (α, p, ρ), then

$$G_X\left(s\right) = \rho + \left(1-\rho\right)\left\{\frac{p}{1-\left(1-p\right)s}\right\}^{\alpha}$$

Therefore

$$H\left(s\right) = F\left[G_X\left(s\right)\right] = \frac{\ln\left\{1-p\left[\rho+\left(1-\rho\right)\left(\frac{p}{1-(1-p)s}\right)^{\alpha}\right]\right\}}{\ln\left(1-p\right)}$$

Which is the pgf of Logarithmic-Zero-Inflated Negative Binomial Distribution.

The mean and the variance is given by

$$E\left[S_N\right] = E\left(N\right)E\left(X_i\right) = \frac{\alpha q\left(1-\rho\right)}{-\left(1-p\right)\log(1-p)}$$

and

$$\begin{aligned} Var\left[S_N\right] &= E\left[N\right]VarX_i + \left[E\left(X_i\right)\right]^2 VarN \\ &= \frac{\left(1-\rho\right)}{-\left(1-p\right)\log(1-p)}\left\{\frac{\alpha^2q^2}{p}\rho+\frac{\alpha q}{p}+\frac{\left[p+\log(1-p)\right]}{\left(1-p\right)\log(1-p)}\left[\frac{\alpha^2q^2}{p^2}-\frac{\alpha^2q^2}{p^2}\rho\right]\right\} \end{aligned}$$

7.3.5 Special cases of Compound Inflated Power Series Distributions

a) Compound Zero-Inflated Poisson Distribution

Let

$$S_N = X_1 + X_2 + X_3 + \cdots + X_N$$

Where $X_i's$ are independent random variables, with N being a Zero-Inflated Poisson random variable. Then S_N is said to have a Compound Zero-Inflated Poisson Distribution. Suppose N is Zero-Inflated Poisson with parameters (λ, ρ). Then the pgf of N is given by,

$$F\left(s\right) = \rho + \left(1-\rho\right)e^{\lambda(s-1)}$$

and the pgf of S_N is given by,

$$H(s) = F[G_X(s)] = \rho + (1-\rho) e^{\lambda[G_X(s)-1]}$$

Hence the mean and variance is given by,

$$\begin{aligned} E[S_N] &= E(N) E(X_i) \\ &= (1-\rho)\lambda E(X_i) \end{aligned}$$

and

$$\begin{aligned} Var[S_N] &= E[N] Var X_i + [E(X)]^2 Var N \\ &= (1-\rho)\lambda Var X_i + (1-\rho)\lambda(1+\rho\lambda)[E(X)]^2 \\ &= (1-\rho)\lambda\left\{Var X_i + (1+\rho\lambda)[E(X)]^2\right\} \end{aligned}$$

Case (i) if X is Zero truncated geometric Distribution with parameter p, then

$$G_X(s) = \frac{ps}{1-qs} \quad \text{where } q = 1-p$$

Therefore

$$\begin{aligned} H(s) = F[G_X(s)] &= \rho + (1-\rho) e^{\lambda\left[\frac{ps}{1-qs}-1\right]} \\ &= \rho + (1-\rho) e^{\lambda\left[\frac{s-1}{1-qs}\right]} \end{aligned}$$

which is the pgf of Zero-Inflated Poisson-zero truncated Geometric distribution

Hence the mean and variance is given by,

$$\begin{aligned} E[S_N] &= E(N) E(X_i) \\ &= \frac{(1-\rho)\lambda}{p} \end{aligned}$$

and

$$\begin{aligned} Var[S_N] &= E[N] Var X_i + [E(X)]^2 Var N \\ &= (1-\rho)\lambda Var X_i + (1-\rho)\lambda(1+\rho\lambda)[E(X)]^2 \\ &= (1-\rho)\lambda\left\{\frac{1-p}{p^2} + \frac{(1+\rho\lambda)}{P^2}\right\} \\ &= (1-\rho)\lambda\left\{\frac{1-p+(1+\rho\lambda)}{p^2}\right\} \end{aligned}$$

Case (ii) if X_i is Binomial Distribution with parameters (n, p), then

$$G_X(s) = [1-p+ps]^n$$

Therefore

$$H(s) = F[G_X(s)] = \rho + (1-\rho) e^{\lambda[(1-p+ps)^n-1]}$$

which is the pgf of Zero-Inflated Poisson-Binomial Distribution.

Hence the mean and variance is given by,

$$\begin{aligned} E\left[S_N\right] &= E\left(N\right)E\left(X_i\right) \\ &= \left(1-\rho\right)\lambda np \end{aligned}$$

and

$$\begin{aligned} Var\left[S_N\right] &= E\left[N\right]VarX_i + \left[E\left(X\right)\right]^2 VarN \\ &= \left(1-\rho\right)\lambda VarX_i + \left(1-\rho\right)\lambda\left(1+\rho\lambda\right)\left[E\left(X\right)\right]^2 \\ &= \left(1-\rho\right)\lambda np\left\{1-p+\left(1+\rho\lambda\right)np\right\} \end{aligned}$$

Case (iii) if X_i is Negative Binomial Distribution with parameters (α, p), then

$$G_X\left(s\right) = \left\{\frac{p}{1-\left(1-p\right)s}\right\}^{\alpha}$$

Therefore

$$H\left(s\right) = F\left[G_X\left(s\right)\right] = \rho + \left(1-\rho\right)e^{\lambda\left[\left\{\frac{p}{1-(1-p)s}\right\}^{\alpha}-1\right]}$$

Which is the pgf of Zero-Inflated Poisson-Negative Binomial Distribution.

Hence the mean and variance is given by,

$$\begin{aligned} E\left[S_N\right] &= E\left(N\right)E\left(X_i\right) \\ &= \left(1-\rho\right)\lambda\alpha\frac{q}{p} \end{aligned}$$

and

$$\begin{aligned} Var\left[S_N\right] &= E\left[N\right]VarX_i + \left[E\left(X_i\right)\right]^2 VarN \\ &= \left(1-\rho\right)\lambda\frac{\alpha q}{p^2} + \left(1-\rho\right)\lambda\left(1+\rho\lambda\right)\frac{\alpha^2 q^2}{p^2} \\ &= \left(1-\rho\right)\lambda\left\{\frac{\alpha q}{p^2} + \left(1+\rho\lambda\right)\frac{\alpha^2 q^2}{p^2}\right\} \end{aligned}$$

Case (iv) if X_i is Logarithmic Series Distribution with parameter (p), then

$$G_X\left(s\right) = \frac{\ln\left[1-ps\right]}{\ln\left(1-p\right)}$$

Therefore

$$H\left(s\right) = F\left[G_X\left(s\right)\right] = \rho + \left(1-\rho\right)e^{\lambda\left[\frac{\ln[1-ps]}{\ln(1-p)}-1\right]}$$

Which is the pgf of Zero-Inflated Poisson-Logarithmic Series Distribution.

Hence the mean and variance is given by,

$$\begin{aligned} E\left[S_N\right] &= E\left(N\right)E\left(X_i\right) \\ &= \frac{\left(1-\rho\right)\lambda p}{-\left(1-p\right)\log(1-p)} \end{aligned}$$

and

$$
\begin{aligned}
Var\left[S_N\right] &= E\left[N\right] VarX_i + \left[E\left(X\right)\right]^2 VarN \\
&= \left(1-\rho\right)\lambda VarX_i + \left(1-\rho\right)\lambda\left(1+\rho\lambda\right)\left[E\left(X\right)\right]^2 \\
&= \left(1-\rho\right)\lambda\left\{\frac{-\left[p^2+p\log(1-p)\right]}{\left[(1-p)\log(1-p)\right]^2} + \left(1+\rho\lambda\right)\left[\frac{p}{-\left(1-p\right)\log(1-p)}\right]^2\right\} \\
&= \left(1-\rho\right)\lambda\left\{\frac{-\left[p^2+p\log(1-p)\right]+\left(1+\rho\lambda\right)p^2}{\left(1-p\right)^2\left[\log(1-p)\right]^2}\right\}
\end{aligned}
$$

Case (v) if X_i is Poisson Distribution with parameters λ, then

$$G_X\left(s\right) = e^{\lambda[s-1]}$$

Therefore

$$H\left(s\right) = F\left[G_X\left(s\right)\right] = \rho + \left(1-\rho\right)e^{\lambda\left[e^{\lambda(s-1)}-1\right]}$$

which is the pgf of Zero-Inflated Poisson-Poisson Distribution.

Hence the mean and variance is given by,

$$
\begin{aligned}
E\left[S_N\right] &= E\left(N\right)E\left(X_i\right) \\
&= \left(1-\rho\right)\lambda^2
\end{aligned}
$$

and

$$
\begin{aligned}
Var\left[S_N\right] &= E\left[N\right] VarX_i + \left[E\left(X\right)\right]^2 VarN \\
&= \left(1-\rho\right)\lambda^2 + \left(1-\rho\right)\lambda^3\left(1+\rho\lambda\right) \\
&= \left(1-\rho\right)\lambda^2\left\{1+\lambda\left(1+\rho\lambda\right)\right\}
\end{aligned}
$$

b) Compound Zero-Inflated Binomial Distribution

Suppose

$$S_N = X_1 + X_2 + X_3 + \cdots + X_N$$

Where $X_i's$ are independent random variables, with N being a Zero-Inflated Binomial random variable. Then S_N is said to have a Compound Zero-Inflated Binomial Distribution. Suppose N is Zero-Inflated Binomial with parameter (n, p, ρ). Then the pgf of N is given by,

$$F\left(s\right) = \rho + \left(1-\rho\right)\left[1-p+ps\right]^n$$

and the pgf of S_N is given by,

$$H\left(s\right) = F\left[G_X\left(s\right)\right] = \rho + \left(1-\rho\right)\left[1-p+pG_X\left(s\right)\right]^n$$

Hence the mean and variance is given by,

$$
\begin{aligned}
E\left[S_N\right] &= E\left(N\right)E\left(X_i\right) \\
&= \left(1-\rho\right)npE\left(X_i\right)
\end{aligned}
$$

and

$$
\begin{aligned}
Var\left[S_N\right] &= E\left[N\right]VarX_i + \left[E\left(X\right)\right]^2 VarN \\
&= \left(1-\rho\right)npVarX_i + \left(1-\rho\right)np\left\{np\rho + \left(1-p\right)\right\}\left[E\left(X\right)\right]^2 \\
&= \left(1-\rho\right)np\left\{VarX_i + \left[np\rho + \left(1-p\right)\right]\left[E\left(X\right)\right]^2\right\}
\end{aligned}
$$

Case (i) if X_i is Poisson Distribution with parameter (λ), then

$$G_X\left(s\right) = e^{\lambda(s-1)}$$

Therefore

$$H\left(s\right) = F\left[G_X\left(s\right)\right] = \rho + \left(1-\rho\right)\left[1-p+pe^{\lambda(s-1)}\right]^n$$

Which is the pgf of Zero-Inflated Binomial-Poisson Distribution.

The mean and the variance is given by

$$
\begin{aligned}
E\left[S_N\right] &= E\left(N\right)E\left(X_i\right) \\
&= \left(1-\rho\right)np\lambda
\end{aligned}
$$

and

$$
\begin{aligned}
Var\left[S_N\right] &= E\left[N\right]VarX_i + \left[E\left(X\right)\right]^2 VarN \\
&= \left(1-\rho\right)npVarX_i + \left(1-\rho\right)np\left\{np\rho + \left(1-p\right)\right\}\left[E\left(X\right)\right]^2 \\
&= \left(1-\rho\right)np\lambda\left\{1 + \lambda np\rho + \lambda\left(1-p\right)\right\}
\end{aligned}
$$

Case (ii) if X_i is Negative Binomial Distribution with parameters (α, p), then

$$G_X\left(s\right) = \left\{\frac{p}{1-\left(1-p\right)s}\right\}^{\alpha}$$

Therefore

$$H\left(s\right) = F\left[G_X\left(s\right)\right] = \rho + \left(1-\rho\right)\left[1-p+p\left\{\frac{p}{1-\left(1-p\right)s}\right\}^{\alpha}\right]^n$$

Which is the pgf of Zero-Inflated Binomial-negative Binomial Distribution

Hence the mean and variance is given by,

$$
\begin{aligned}
E\left[S_N\right] &= E\left(N\right)E\left(X_i\right) \\
&= \left(1-\rho\right)n\alpha q
\end{aligned}
$$

and

$$
\begin{aligned}
Var\left[S_N\right] &= E\left[N\right]VarX_i + \left[E\left(X_i\right)\right]^2 VarN \\
&= \left(1-\rho\right)np\left\{\alpha\frac{q}{p^2} + \left[np\rho + \left(1-p\right)\right]\left[\alpha\frac{q}{p}\right]^2\right\} \\
&= \left(1-\rho\right)n\left\{\alpha\frac{q}{p} + \left[np\rho + q\right]\frac{\alpha^2 q^2}{p}\right\}
\end{aligned}
$$

Case (iii) if X_i is Logarithmic Series Distribution with parameter (p), then

$$G_X(s) = \frac{\ln[1-ps]}{\ln(1-p)}$$

Therefore

$$H(s) = F[G_X(s)] = \rho + (1-\rho)\left[1-p+p\frac{\ln[1-ps]}{\ln(1-p)}\right]^n$$

Which is the pgf of Zero-Inflated Binomial-Logarithmic Series Distribution.

Hence the mean and variance is given by,

$$\begin{aligned} E[S_N] &= E(N)E(X_i) \\ &= \frac{(1-\rho)np^2}{-(1-p)\log(1-p)} \end{aligned}$$

and

$$\begin{aligned} Var[S_N] &= E[N]\,VarX_i + [E(X)]^2\,VarN \\ &= (1-\rho)\,npVarX_i + (1-\rho)\,np\{np\rho + (1-p)\}[E(X)]^2 \\ &= (1-\rho)\,np\left\{\begin{array}{c} -\left[\frac{p^2+p\log(1-p)}{(1-p)^2[\log(1-p)]^2}\right] \\ +[np\rho+(1-p)]\left[\frac{p}{-(1-p)\log(1-p)}\right]^2 \end{array}\right\} \\ &= -(1-\rho)\,np^2\left\{\frac{\log(1-p) - np^2\rho + p^2}{(1-p)^2[\log(1-p)]^2}\right\} \end{aligned}$$

Case (iv) if X_i is Binomial Distribution with parameters (n,p), then

$$G_X(s) = [1-p+ps]^n$$

Therefore

$$H(s) = F[G_X(s)] = \rho + (1-\rho)[1-p+p(1-p+ps)^n]^n$$

Which is the pgf of Zero-Inflated Binomial-Binomial Distribution.

Hence the mean and variance is given by,

$$E[S_N] = (1-\rho)\,n^2p^2$$

and

$$Var[S_N] = (1-\rho)\,n^2p^2\{q + np(np\rho + q)\}$$

c) Compound Zero-Inflated Negative Binomial Distribution

Suppose

$$S_N = X_1 + X_2 + X_3 + \cdots + X_N$$

Where $X_i's$ are independent random variables, with N being a Zero-Inflated Negative Binomial random variable. Then S_N is said to have a Compound Zero-Inflated

Negative Binomial Distribution. Suppose N is Zero-Inflated Negative Binomial with parameter (α, p, ρ). Then the pgf of N is given by,

$$F(s) = \rho + (1-\rho)\left\{\frac{p}{1-(1-p)s}\right\}^{\alpha}$$

and the pgf of S_N is given by,

$$H(s) = F[G_X(s)] = \rho + (1-\rho)\left\{\frac{p}{1-(1-p)G_X(s)}\right\}^{\alpha}$$

Hence the mean and variance is given by,

$$\begin{aligned} E[S_N] &= E(N)E(X_i) \\ &= (1-\rho)\alpha\frac{(1-p)}{p}E(X_i) \end{aligned}$$

and

$$\begin{aligned} Var[S_N] &= E[N]VarX_i + [E(X)]^2 VarN \\ &= (1-\rho)\alpha\frac{(1-p)}{p}VarX_i + (1-\rho)\frac{\alpha(1-p)}{p^2}[\alpha(1-p)\rho+1][E(X)]^2 \\ &= (1-\rho)\alpha\frac{(1-p)}{p}\left\{VarX_i + \frac{1}{p}[\alpha(1-p)\rho+1][E(X)]^2\right\} \end{aligned}$$

Case (i) if X_i is Logarithmic Series Distribution with parameter (p), then

$$G_X(s) = \frac{\ln[1-ps]}{\ln(1-p)}$$

Therefore

$$H(s) = F[G_X(s)] = \rho + (1-\rho)\left\{\frac{p}{1-(1-p)\frac{\ln[1-ps]}{\ln(1-p)}}\right\}^{\alpha}$$

Which is the pgf of Zero-Inflated Negative Binomial-Logarithmic Series Distribution.

Hence the mean and variance is given by,

$$\begin{aligned} E[S_N] &= E(N)E(X_i) \\ &= \frac{(1-\rho)\alpha}{-\log(1-p)} \end{aligned}$$

and

$$\begin{aligned} Var[S_N] &= E[N]VarX_i + [E(X)]^2 VarN \\ &= (1-\rho)\alpha\frac{(1-p)}{p}VarX_i + (1-\rho)\frac{\alpha(1-p)}{p^2}[\alpha(1-p)\rho+1][E(X)]^2 \\ &= -(1-\rho)\alpha\left\{\frac{p+\log(1-p)-[\alpha(1-p)\rho+1]}{(1-p)[\log(1-p)]^2}\right\} \end{aligned}$$

Case (ii) if X_i is Poisson Distribution with parameter λ, then

$$G_X(s) = e^{\lambda(s-1)}$$

Therefore

$$H(s) = F[G_X(s)] = \rho + (1-\rho)\left\{\frac{p}{1-(1-p)e^{\lambda(s-1)}}\right\}^{\alpha}$$

Which is the pgf of Zero-Inflated Negative Binomial-Poisson Distribution.

Hence the mean and variance is given by,

$$\begin{aligned} E[S_N] &= E(N)E(X_i) \\ &= (1-\rho)\lambda\alpha\frac{(1-p)}{p} \end{aligned}$$

and

$$\begin{aligned} Var[S_N] &= E[N]VarX_i + [E(X)]^2 VarN \\ &= (1-\rho)\alpha\frac{(1-p)}{p}VarX_i + (1-\rho)\frac{\alpha(1-p)}{p^2}[\alpha(1-p)\rho+1][E(X)]^2 \\ &= (1-\rho)\alpha\frac{(1-p)\lambda}{p}\left\{1+\frac{\lambda}{p}[\alpha(1-p)\rho+1]\right\} \end{aligned}$$

Case (iii) if X_i is Binomial Distribution with parameters (n,p), then

$$G_X(s) = [1-p+ps]^n$$

Therefore

$$H(s) = F[G_X(s)] = \rho + (1-\rho)\left\{\frac{p}{1-(1-p)[1-p+ps]^n}\right\}^{\alpha}$$

Which is the pgf of Zero-Inflated Negative Binomial-Binomial Distribution.

Hence the mean and variance is given by,

$$\begin{aligned} E[S_N] &= E(N)E(X_i) \\ &= (1-\rho)\alpha np \end{aligned}$$

and

$$\begin{aligned} Var[S_N] &= E[N]VarX_i + [E(X)]^2 VarN \\ &= (1-\rho)\alpha\frac{(1-p)}{p}VarX_i + (1-\rho)\frac{\alpha(1-p)}{p^2}[\alpha(1-p)\rho+1][E(X)]^2 \\ &= (1-\rho)\alpha n(1-p)\{q+\alpha n\rho(1-p)+n\} \end{aligned}$$

Case (iv) if X_i is Negative Binomial Distribution with parameters (α, p), then

$$G_X(s) = \left\{\frac{p}{1-(1-p)s}\right\}^{\alpha}$$

Therefore

$$H(s) = F[G_X(s)] = \rho + (1-\rho)\left\{\frac{p}{1-(1-p)\left[\frac{p}{1-(1-p)s}\right]^{\alpha}}\right\}^{\alpha}$$

Which is the pgf of Zero-Inflated Negative binomial-Negative Binomial Distribution.
Hence the mean and variance is given by,

$$\begin{aligned} E[S_N] &= E(N)\,E(X_i) \\ &= (1-\rho)\,\alpha^2\frac{(1-p)^2}{p^2} \end{aligned}$$

and

$$\begin{aligned} Var[S_N] &= (1-\rho)\,\alpha\frac{(1-p)}{p}\left\{VarX_i + \frac{1}{p}[\alpha(1-p)\rho+1][E(X)]^2\right\} \\ &= (1-\rho)\,\alpha\frac{(1-p)}{p}\left\{\frac{\alpha q}{p^2} + \frac{\alpha^3q^2}{p^3}\rho - \frac{\alpha^3q^2}{p^2}\rho + \frac{\alpha^2q^2}{p^3}\right\} \\ &= (1-\rho)\,\alpha\frac{(1-p)}{p^3}\left\{\alpha q\left(1-\alpha^2q\rho\right) + \frac{\alpha^2q^2}{p}(\alpha\rho+1)\right\} \end{aligned}$$

d) Compound Zero-Modified Logarithmic Series Distribution

Suppose

$$S_N = X_1 + X_2 + X_3 + \cdots + X_N$$

Where $X_i's$ are independent random variables, with N being a Zero-Modified Logarithmic Series random variable. Then S_N is said to have a Compound Zero-Modified Logarithmic Series Distribution. Suppose N is Zero-Modified Logarithmic Series Distribution with parameter π.Then the pgf of N is given by,

$$F(s) = \rho + (1-\rho)\frac{\ln[1-ps]}{\ln(1-p)}$$

and the pgf of S_N is given by,

$$H(s) = F[G_X(s)] = \rho + (1-\rho)\frac{\ln\{1-pG_X(s)\}}{\ln(1-p)}$$

Hence the mean and variance is given by,

$$\begin{aligned} E[S_N] &= E(N)\,E(X_i) \\ &= \frac{-(1-\rho)\,p}{(1-p)\log(1-p)}E(X_i) \end{aligned}$$

and

$$
\begin{aligned}
Var\left[S_N\right] &= E\left[N\right]VarX_i + \left[E\left(X\right)\right]^2 VarN \\
&= \frac{-\left(1-\rho\right)p}{\left(1-p\right)\log(1-p)}VarX_i - \left(1-\rho\right)\left\{\frac{p\log(1-p)+p^2\left(1+\rho\right)}{\left[\left(1-p\right)\log(1-p)\right]^2}\right\}\left[E\left(X\right)\right]^2 \\
&= \frac{-\left(1-\rho\right)p}{\left(1-p\right)\log(1-p)}\left\{VarX_i + \frac{\log(1-p)+p\left(1+\rho\right)}{\left(1-p\right)\log(1-p)}\left[E\left(X\right)\right]^2\right\}
\end{aligned}
$$

Case (i) if X_i is Poisson Distribution with parameter λ, then

$$G_X\left(s\right) = e^{\lambda(s-1)}$$

Therefore

$$H\left(s\right) = F\left[G_X\left(s\right)\right] = \rho + \left(1-\rho\right)\frac{\ln\left\{1-pe^{\lambda(s-1)}\right\}}{\ln\left(1-p\right)}$$

Which is the pgf of Zero-Modified Logarithmic Series-Poisson Distribution.
Hence the mean and variance is given by,

$$
\begin{aligned}
E\left[S_N\right] &= E\left(N\right)E\left(X_i\right) \\
&= \frac{-\left(1-\rho\right)p\lambda}{\left(1-p\right)\log(1-p)}
\end{aligned}
$$

and

$$
\begin{aligned}
Var\left[S_N\right] &= E\left[N\right]VarX_i + \left[E\left(X\right)\right]^2 VarN \\
&= \frac{-\left(1-\rho\right)p}{\left(1-p\right)\log(1-p)}VarX_i - \left(1-\rho\right)\left\{\frac{p\log(1-p)+p^2\left(1+\rho\right)}{\left[\left(1-p\right)\log(1-p)\right]^2}\right\}\left[E\left(X\right)\right]^2 \\
&= \frac{-\left(1-\rho\right)p\lambda}{\left(1-p\right)\log(1-p)}\left\{1+\lambda\frac{\log(1-p)+p\left(1+\rho\right)}{\left(1-p\right)\log(1-p)}\right\}
\end{aligned}
$$

Case (ii) if X_i is Binomial Distribution with parameters $\left(n,p\right)$, then

$$G_X\left(s\right) = \left[1-p+ps\right]^n$$

Therefore

$$H\left(s\right) = F\left[G_X\left(s\right)\right] = \rho + \left(1-\rho\right)\frac{\ln\left\{1-p\left[1-p+ps\right]^n\right\}}{\ln\left(1-p\right)}$$

Which is the pgf of Zero-Modified Logarithmic Series-Binomial Distribution
.Hence the mean and variance is given by,

$$
\begin{aligned}
E\left[S_N\right] &= E\left(N\right)E\left(X_i\right) \\
&= \frac{-\left(1-\rho\right)np^2}{\left(1-p\right)\log(1-p)}
\end{aligned}
$$

and

$$
\begin{aligned}
Var\left[S_N\right] &= E\left[N\right] VarX_i + \left[E\left(X\right)\right]^2 VarN \\
&= \frac{-\left(1-\rho\right)p}{\left(1-p\right)\log(1-p)} VarX_i - \left(1-\rho\right)\left\{\frac{p\log(1-p)+p^2\left(1+\rho\right)}{\left[\left(1-p\right)\log(1-p)\right]^2}\right\}\left[E\left(X\right)\right]^2 \\
&= \frac{-\left(1-\rho\right)np^2}{\left(1-p\right)\log(1-p)}\left\{q+\frac{np\log(1-p)+np^2\left(1+\rho\right)}{\left(1-p\right)\log(1-p)}\right\}
\end{aligned}
$$

Case (iii) if X_i is Negative Binomial Distribution with parameters (α, p), then

$$
G_X\left(s\right) = \left\{\frac{p}{1-\left(1-p\right)s}\right\}^\alpha
$$

Therefore

$$
H\left(s\right) = F\left[G_X\left(s\right)\right] = \rho + \left(1-\rho\right)\frac{\ln\left\{1-p\left[\frac{p}{1-(1-p)s}\right]^\alpha\right\}}{\ln\left(1-p\right)}
$$

Which is the pgf of Zero-Modified Logarithmic Series-Negative Binomial Distribution. Hence the mean and variance is given by,

$$
\begin{aligned}
E\left[S_N\right] &= E\left(N\right)E\left(X_i\right) \\
&= \frac{-\left(1-\rho\right)\alpha}{\log(1-p)}
\end{aligned}
$$

and

$$
\begin{aligned}
Var\left[S_N\right] &= E\left[N\right] VarX_i + \left[E\left(X_i\right)\right]^2 VarN \\
&= \frac{-\left(1-\rho\right)}{\log(1-p)}\frac{\alpha}{p} - \left(1-\rho\right)\frac{\alpha^2}{p}\left\{\frac{\log(1-p)+p\left(1+\rho\right)}{\left[\log(1-p)\right]^2}\right\} \\
&= \frac{-\left(1-\rho\right)}{\log(1-p)}\frac{\alpha}{p}\left[1+\alpha\frac{\log(1-p)+p\left(1+\rho\right)}{\log(1-p)}\right]
\end{aligned}
$$

Case (iv) if X_i is Logarithmic Series Distribution with parameter (p), then

$$
G_X\left(s\right) = \frac{\ln\left[1-ps\right]}{\ln\left(1-p\right)}
$$

Therefore

$$
H\left(s\right) = F\left[G_X\left(s\right)\right] = \rho + \left(1-\rho\right)\frac{\ln\left\{1-p\frac{\ln[1-ps]}{\ln(1-p)}\right\}}{\ln\left(1-p\right)}
$$

Which is the pgf of Zero-Modified Logarithmic Series-Logarithmic Series Distribution. Hence the mean and variance is given by,

$$
\begin{aligned}
E\left[S_N\right] &= E\left(N\right)E\left(X_i\right) \\
&= \frac{\left(1-\rho\right)p^2}{\left[\left(1-p\right)\log(1-p)\right]^2}
\end{aligned}
$$

and

$$
\begin{aligned}
Var\left[S_N\right] &= \frac{-\left(1-\rho\right)p}{\left(1-p\right)\log(1-p)}\left\{VarX_i + \frac{\log(1-p)+p\left(1+\rho\right)}{\left(1-p\right)\log(1-p)}\left[E\left(X\right)\right]^2\right\} \\
&= \frac{-\left(1-\rho\right)p}{\left[\left(1-p\right)\log(1-p)\right]^2}\left\{\frac{-\left[p^2+p\log(1-p)\right]}{\left(1-p\right)\log(1-p)} + \frac{p^2\left(\log(1-p)+p\left(1+\rho\right)\right)}{\left[\left(1-p\right)\log(1-p)\right]^2}\right\}
\end{aligned}
$$

Chapter 8

Applications of Inflated Power Series Distributions to Migration

8.1 Introduction

In certain applications involving discrete data, it is common to encounter many zeroes than predicted by models based on standard assumptions. The problem of a high proportion of observations at some of the support has received great attention of the practitioners and the researchers in data analysis and modeling as applied in various fields that includes; econometrics, demography, medical, public health, epidemiology, biology, demography and in many other fields. In this chapter, we review zero-inflated and one-inflated power series distribution models as applied in describing migration at various levels.

8.2 An Inflated Power Series Distribution for Modelling Rural Out-Migration at Household Level

8.2.1 Introduction

Analysis of migration data at various level has momentous implication for regional planning as well as for formulation of housing policies (Rossi, 1955; Pryor,1975). A number of attempts has been made during the past few decades to study the migration phenomena at macro-level (Friedlander and Roshier, 1966; Lee,1966; Greenwoood, 1971; Muller, 1967). They have generally adopted a macro-level approach by operating on highly aggregate data for countries, states' districts and nations as a whole. These studies might not provide the adequate explanation for the tremendous regional and local heterogeneity planning, especially in developing counties. Recently, micro-level research on both residential mobility and migration has played a decisive role in development of the theory of migration (Dejong and Gardner, 1981; Speare etal., 1975). The need for collection and analysis of migration data at the household level is based on the fact that the household is the basic socio-economic unit for integrated rural development. The number of migrants from a household has important bearing on the economic and cultural characteristics of the household. Household with at least one migrant are more prone to have new ideas than household having no migrants (Yadava and Sing, 1991).

Generally in rural area, the occurrence of migration from household can be classified as (i) Adult males ($\geqslant$15 years) who migrate singly to a place, leaving their wives and children in their village homes, (ii) Individuals who migrate with their wives and children, and (iii) Males who migrate with their wives, children and some members of their households, as identified by Yadava and Singh (1991). The impact of these three types of migrants on the sociocultural and economic characteristics of the household are usually different. It is obvious that migrations of many members of a household especially females is more likely to affect the economic status and the sociocultural outlook of the household. This consideration underscores the importance of concentrating attention on the pattern of distribution of total number of migrants from a household.

A good number of studies have been done and several models have been proposed to study the pattern of rural male (≥ 15) out migration (Hossain, 2000; Iwunor, 1995; Singh, 1992; Sharma, 1987; 1985; Yadava, Tripathi and singh, 1994). However, this models were not appropriate to fit the distribution of total number of migrants due to the following limitations, Firstly, the prior distribution of males aged fifteen years and over is not known. Secondly, the model do not take into account those households from where the wives, children and other members of the household migrate i.e. migration in clusters and thirdly, the distribution of living children to a couple is not known. Taking these limitations into account. Consequently, several attempts have been made to describe the distribution of households according to the total number of migrants under different assumptions (Yadava and Singh, 1991;Janardan, 1973).

Yadava and Singh (1991) proposed a model that describes the variation in the total number of migrants from a household. Their model is based on the following assumptions: (i) Migration from a household occurs in clusters (groups), (ii) Migration from a household is a rare event, (iii) The risks of a cluster of migrants vary from household to household. They assumed that the number of clusters migrating from a household follows the Poisson distribution, while the number of migrants in a cluster follows the one inflated zero truncated geometric distribution.

Inuwor (2004) studied a model that takes into account zero observation. He assumed the Poisson distribution for the number of clusters migrating, and that the number of migrants in a cluster follows each of the members of the class of one-Inflated power series distributions namely: the binomial, the Poisson, the negative binomial, the geometric, the log-series, and the mis-recorded Poisson. At least one person is expected to migrate in household is exposed to the risk of migration thus, the use of the one-inflated distributions. This is justified by the need to reduce the risk of underestimation of the probability that one person migrates in households are exposed to the risk of migration. Hence the use of zero-truncated distributions as proposed by Yadava and Singh (1991) is not justifiable since the zeros are real zeros are real and observable as there is the possibility that nobody migrates in a cluster in a household.

A review on estimation of parameters of the Inflated Geometric Distribution for modelling rural out-migration according to the number migrants and a comparison of Poisson-one-Inflated power series distribution for modelling rural out-migration at the household level as done by Inuwor (2004) will be reviewed. In each case the description of the model, the method of estimating the parameters, application and conclusion will be derived.

8.2.2 Model for the Distribution of Households According to the Number of Migrants

The distribution have been proposed on the basis of the following assumptions:

(i) At any point in time, at least one member of each household has a chance of α of migrating out and a chance of $1-\alpha$ of not migrating.

(ii) The pattern of migration from each household follows the geometric distribution with parameter p representing the probability of a single individual migrating from a household.

Let X represent the number of male rural out-migrant from a household, then X follows the inflated geometric distribution with probability density function:

$$P(X=x)=\begin{cases} 1-\alpha+\alpha p & for \quad x=0 \\ \alpha q^x p & for \quad x=1,2,3,\ldots \end{cases}$$

where $p+q=1$.

The probability of x members migrating from a household is more than the probability of $(x+1)$ members migrating from a household (for $x=1,2,3,\ldots$), Thus X is a decreasing function. The use of this model is further justified by the fact that migration is selective of age and other socio-demographic characteristics. Adult males aged fifteen years and over tend to migrate singly. The chance that the entire members of a household migrate decreases with increase in the household size.

Estimation of Parameters

The proposed Inflated geometric distribution involves two parameters α and p to be estimated from the observed distribution of migrants from the households. The two parameters would be estimated using by the method based on Moments, Maximum Likelihood Function and the method of the mean-zero-frequency, i.e., Let x_1, x_2, ..., x_n denote a random sample of size n from the population, also let n_0 denote the number of zero observations and n the total number of observations.

Estimation of the Parameters Using Maximum Likelihood Function From chapter five subsection 5.4.5, the maximum likelihood estimators of α and p when they exist are given by

$$\hat{\rho}=\frac{n_0-n+n\theta}{n\theta}$$

$$\hat{\theta}=1-\frac{1}{\bar{x}}$$

as given in (5.43) and (5.45) respectively. Where $\bar{x}$ is the mean of positive observations.

Reparameterize by writing $\hat{\theta}=1-\hat{p}$, $\hat{\rho}=1-\alpha$ to obtain the estimates of α and p as

$$\alpha=\frac{n_0-n}{n(p-1)} \tag{8.1}$$

and

$$p = \frac{1}{\bar{x}} \tag{8.2}$$

Solving for of α and p by substituting the values of $\bar{x},\ n$ and n_0 in (8.1) and (8.2), for the three types of villages, i.e.,

Semi urban $\quad n_0 = 1042,\ n = 1171,\ \bar{x} = 0.160\,55$

Remote $\quad n_0 = 872,\ n = 1135,\ \bar{x} = 0.345\,37$

growth centre $\quad n_0 = 978,\ n = 1208,\ \bar{x} = 0.290\,56$

to obtain the maximum likelihood estimators. The estimators obtained is as given in the table below:

Table 8.1: Estimators based on method of Maximum Likelihood Function

Parameters	Semi-urban	Remote	Growth Centre
$\hat{p}$	0.686 17	0.670 92	0.655 27
$\hat{\alpha}$	0.351 03	0.704 14	0.552 31

The mean-zero-frequency method of estimation The mean-zero-frequency method of estimation is based on the zero relative frequency of the data set which is equated to the probability of zero under the assumed distribution (see Kemp and Kemp, 1988). So, the resulting system of equations is

$$1 - \alpha + \alpha p = f_0 \tag{8.3}$$

$$\alpha \frac{1-p}{p} = \bar{x} \tag{8.4}$$

Where $f_0 = \frac{n_0}{n}$ is the proportion of zero observations in the sample

$\bar{x}$ is the observed mean of the distribution in the sample

Solving equation (8.3) and (8.4) simultaneously to obtain p and α we get,

$$\hat{\alpha} = \frac{n - n_0}{n\,(1-p)} \tag{8.5}$$

and

$$\hat{p} = \frac{\alpha}{\bar{x}}\,(1-p) \tag{**}$$

but $\alpha = \frac{n-n_0}{n(1-p)}$ therefore equation $(**)$ becomes

$$\hat{p} = \frac{n - n_0}{n\bar{x}} \tag{8.6}$$

Solving for of α and p by substituting the values of $\bar{x},\ n$ and n_0 in (8.5) and (8.6) ,for the three types of villages, i.e.,

Semi urban $\quad n_0 = 1042,\ n = 1171,\ \bar{x} = 0.160\,55$

Remote $\quad n_0 = 872,\ n = 1135,\ \bar{x} = 0.345\,37$

growth centre $\quad n_0 = 978,\ n = 1208,\ \bar{x} = 0.290\,56$

to obtain the estimators. The estimators obtained by the method of mean-zero-frequency is as given in the table below:

Table 8.2: Estimators based on mean-zero-frequency method

Parameters	Semi-urban	Remote	Growth Centre
$\hat{p}$	0.686 17	0.670 92	0.655 27
$\hat{\alpha}$	0.351 03	0.704 14	0.552 31

Remark **:** The use of the mean-zero-frequency method is the same as the maximum likelihood estimates for Zero-inflated Geometric distribution.

Estimation of the Parameters Based on the method of Moments The moment estimators of $\hat{\theta}$ and $\hat{\rho}$ as obtained in chapter five subsection 5.4.5, equation (5.38) and (5.39) respectively are given as

$$\hat{\theta} = \frac{\sum_{i=1}^{n} x_i^2 - \bar{x}n}{\bar{x}n + \sum_{i=1}^{n} x_i^2}$$

and

$$\hat{\rho} = 1 - \frac{\bar{x}\left\{\bar{x}n + \sum_{i=1}^{n} x_i^2\right\}}{\left\{\sum_{i=1}^{n} x_i^2 - \bar{x}n\right\}} + \bar{x}$$

Reparameterize by writing $\hat{\theta} = 1 - \hat{p}$, $1 - \hat{\rho} = \hat{\alpha}$ to obtain the moment estimators of α and p as

$$\hat{p} = 1 - \frac{\sum_{i=1}^{n} x_i^2 - \bar{x}n}{\bar{x}n + \sum_{i=1}^{n} x_i^2} \tag{8.7}$$

and

$$\hat{\alpha} = \frac{\bar{x}\left\{\bar{x}n + \sum_{i=1}^{n} x_i^2\right\}}{\left\{\sum_{i=1}^{n} x_i^2 - \bar{x}n\right\}} - \bar{x} \tag{8.8}$$

Solving for of α and p by substituting the values of $\bar{x}$, n and $\sum_{i=1}^{n} x_i^2$ in (8.7) and (8.8) ,for the three types of villages, i.e.,

Semi urban $\sum_{i=1}^{n} x_i^2 = 392,\ n = 1171,\ \bar{x} = 0.160\,55$
Remote $\sum_{i=1}^{n} x_i^2 = 770,\ n = 1135,\ \bar{x} = 0.345\,37$
growth centre $\sum_{i=1}^{n} x_i^2 = 737,\ n = 1208,\ \bar{x} = 0.290\,56$
to obtain the moment estimators as given in the table below:

Table 8.3: Estimators based on method of moments

Parameters	Semi-urban	Remote	Growth Centre
$\hat{p}$	0.648 28	0.674 70	0.645 22
$\hat{\alpha}$	0.295 91	0.716 33	0.528 43

Application

Using the estimates obtained in table 8.2 and table 8.3 are used to fit the inflated geometric distribution to the same data used by Sharma (1985). The observed number of households according to the number of male migrants aged fifteen and over from a household in the three villages types, the expected number and χ^2 statistics are computed to test for the goodness of fit. The results for the parameters obtained, based on the method of the mean-zero-frequency and maximum likelihood function are presented in table 8.4. The corresponding results obtained by method of moments are displayed in table 8.5.

Table 8.4: *The distribution of the observed and expected number of households according to the number of rural out-migrants per household in the three types of villages. (Based on the method of the mean-zero-frequency and maximum likelihood function)*

Number of migrants	Number of Households					
	Semi-urban		Remote		Growth Centre	
	Obs.	Exp.	Obs.	Exp.	Obs.	Exp.
0	1042	1042	872	872	978	978
1	95	88.52	176	176.45	154	150.71
2	19	27.78	59	58.07	47	51.96
3	10		18	19.11	18	17.91
4	2		6		9	
5	2	12.70	4		1	
6	0		0	9.37	0	9.42
7	1		0		0	
8+	0		0		1	
Total	1171	1171	1135	1135	1135	1135
$\hat{\alpha}$		0.351 03		0.704 14		0.552 31
$\hat{p}$		0.686 17		0.670 92		0.655 27
χ^2		3.6659		0.1229		0.8108
df		1		2		2

Table 8.5: *The distribution of the observed and expected number of households according to the number of rural out-migrants per household in the three types of villages.* (*Based on the method of moments*)

Number of migrants	Number of Households					
	Semi-urban		Remote		Growth Centre	
	Obs.	Exp.	Obs.	Exp.	Obs.	Exp.
0	1042	1049.13	872	870.52	978	981.53
1	95	79.04	176	178.44	154	146.12
2	19	27.83	59	58.05	47	51.84
3	10	9.80	18	18.88	18	18.39
4	2		6		9	
5	2		4		1	
6	0	5.20	0	9.11	0	10.12
7	1		0		0	
8+	0		0		1	
Total	1171	1171	1135	1135	1135	1135
$\hat{\alpha}$		0.295 91		0.716 33		0.528 43
$\hat{p}$		0.648 28		0.674 70		0.645 22
χ^2		6.0845		0.1794		0.9743
df		2		2		2

Table 8.4 and 8.5 shows that the tendency to migrate by members of a household is higher(0.704 14 and 0.716 33 respectively) in remote rural areas, moderate(0.55231 and 0.52843 respectively) for residents in growth centres and relatively low as 0.351 03 and 0.295 91 respectively for residents in semi-urban areas. In table 8.4 the chance of individual members of households actually migrating is higher for residents in the semi-urban areas who are already adapted to city life and must have established links in the cities. The insignificant values of the χ^2 at the 5% level attests to the goodness of the fit for values obtained in table 8.4 and 8.5 for Remote and Growth centres. But for table 8.5 Semi-urban the χ^2 is significant at the 5% level this may be due to the bias in the estimators. That is, they sometimes fail to take into account all relevant information

in the sample. However, the goodness of fit obtained using the proposed method of estimation (the method of the mean-zero-frequency and maximum likelihood function) are not too different from those obtained by the method of moments except for the semi-urban, as evidenced from the χ^2 values.

Conclusion

The findings show that estimates of the parameters of the inflated geometric distribution for rural out-migration obtained using the mean-zero frequency method (which is the same as the maximum likelihood estimates) are different from those obtained using the method of moments. The goodness of the fit of the model is almost the same for the three methods except for the method of Moments that is biased and sometimes fail to take into account all relevant information in the sample. The maximum likelihood method however provides more efficient results. It has a higher probability of being close to the quantities to be estimated and are more often unbiased.

8.2.3 Model for the Distribution of Household According to the Total Number of Migrants

The Distribution to describe the variation in the households according to the total number of out migrants is derived under the following assumptions:

(i) Migration from a household occurs in clusters (groups),

(ii) Migration from a household is a rare event,

(iii) The risks of a cluster of migrants vary from household to household.

Let Z_i $i = 1, 2, \ldots, N$ denote the number of migrants from ith cluster in a household and let N denote the number of clusters of potential migrants in a household. Then,

$$X = Z_1 + Z_2 + Z_3 + \cdots + Z_N$$

is the total number of migrants from a household.

Define

$$g(s) = E\left(S^{Z_i}\right) = \sum_{z=0}^{\infty} P_{Z_i}(z)\, s^z$$

as the probability generating function(pgf) of Z_i where $P_{Z_i}(z)$ is the probability mass function (pmf) of Z_i.

Also,

$$h(s) = E\left(S^N\right) = \sum_{z=0}^{\infty} P_N(n)\, s^n$$

is the pgf of N. Where $P_N(n)$ is the pmf of N.

Then, the pgf of X is given as (Feller, 1968).

$$\begin{aligned} G_X(s) &= E\left(S^X\right) \\ &= EE\left(S^{Z_1+Z_2+Z_3+\cdots+Z_N}\right) \\ &= E\prod_{i=1}^{N} E\left(S^{Z_i}\right), \text{ because of independence} \\ &= E\left[E\left(S^{Z_i}\right)\right]^N, \text{ since } Z_i's \text{ are identical} \\ &= E\left[g(s)\right]^N \end{aligned}$$

Therefore

$$G_X(s) = h(g(s)) \tag{8.9}$$

from which the pmf of X can be derived. Specifically, $P_X(x)$ is the coefficient of the s^x in the expansion of $G_X(s)$ as a power series in s.

The mean and the variance of the number of migrants in a household are respectively given as

$$\begin{aligned} E[X] &= G'_X(1) \\ Var[X] &= G''_X(1) \; + G'_X(1) - \left[G'_X(1)\right]^2 \end{aligned}$$

where $G'_X(1)$ and $G''_X(1)$ are respectively the first and second derivatives of $G_X(s)$ at $s = 1$.

Assuming that N follows the Poisson distribution with pmf given as

$$P_N(n) = \frac{e^{-\theta}\theta^n}{n!}, \; n = 0, 1, 2, \ldots, \infty$$

The pgf of N is given as

$$\begin{aligned} h(s) &= \sum_{z=0}^{\infty} P_N(n)\, s^n = \sum_{z=0}^{\infty} \frac{e^{-\theta}(\theta s)^n}{n!} \\ &= e^{-\theta}\sum_{z=0}^{\infty}\frac{(\theta s)^n}{n!} \qquad \text{but } \; e^{\theta s} = \sum_{z=0}^{\infty}\frac{(\theta s)^n}{n!} \\ &= e^{\theta(s-1)} \end{aligned} \tag{8.10}$$

Where θ is the mean number of clusters of migrants per household/the average number of clusters per household.

Using equation (8.9), the pmf of X are derived assuming that Z follows:

(*i*) The one-inflated Poisson distribution.

(*ii*) The one-inflated log-series distribution.

(*iii*) The one-inflated geometric distribution.

(*iv*) The one-inflated negative binomial distribution.

(*v*) The mis-recorded Poisson distribution.

(*vi*) The one-inflated binomial distribution.

The resulting mixed distribution are presented below.

The Poisson-one-Inflated Poisson Distribution

$$\Pr(Z=z) = \begin{cases} \omega e^{-\lambda} & z=0 \\ (1-\omega)+\omega\lambda e^{-\lambda} & z=1 \\ \frac{\omega\lambda^z e^{-\lambda}}{z!} & z=2,3,4,\ldots,\infty \end{cases}$$

where λ is the average number of persons migrating from a cluster.

The pgf of Z_i is given as

$$\begin{aligned} g(s) &= \sum_{z=0}^{\infty} p_z s^z \\ &= p_0 + p_1 s + \sum_{z=2}^{\infty} p_z s^z \\ &= \omega e^{-\lambda} + \left[(1-\omega)+\omega\lambda e^{-\lambda}\right] s + \omega \sum_{z=2}^{\infty} \frac{(\lambda s)^z e^{-\lambda}}{z!} \\ &= \omega e^{-\lambda} + \left[(1-\omega)+\omega\lambda e^{-\lambda}\right] s + \omega e^{-\lambda}\left\{e^{\lambda s} - \lambda s - 1\right\} \\ &= \omega e^{-\lambda} + (1-\omega)s + \omega\lambda e^{-\lambda}s + \omega e^{-\lambda(1-s)} - \omega\lambda e^{-\lambda}s - \omega e^{-\lambda} \\ &= (1-\omega)s + \omega e^{\lambda(s-1)} \end{aligned} \tag{8.11}$$

Therefore substituting equations (8.10) and (8.11) into (8.9) gives the pgf of X as

$$G_X(s) = \exp\left[\theta(1-\omega)s + \theta\omega e^{\lambda(s-1)} - \theta\right] \tag{8.12}$$

The first and the second derivatives of $G_X(s)$ w.r.t s is given by

$$\begin{aligned} G'_X(s) &= \theta(1-\omega) + \theta\omega\lambda e^{\lambda(s-1)} \exp\left[\theta(1-\omega)s + \theta\omega e^{\lambda(s-1)} - \theta\right] \\ G''_X(s) &= \theta\omega\lambda^2 e^{\lambda(s-1)} \exp\left[\theta(1-\omega)s + \theta\omega e^{\lambda(s-1)} - \theta\right] \\ &\quad + \left[\theta(1-\omega) + \theta\omega\lambda e^{\lambda(s-1)}\right]^2 \exp\left[\theta(1-\omega)s + \theta\omega e^{\lambda(s-1)} - \theta\right] \end{aligned}$$

setting $s=1$

$$\begin{aligned} G'_X(1) &= \theta(1-\omega) + \theta\omega\lambda = \theta[1-\omega+\omega\lambda] \\ G''_X(1) &= \theta\omega\lambda^2 + [\theta(1-\omega)+\theta\omega\lambda]^2 \end{aligned}$$

Therefore the mean and the variance is given by

$$E(X) = G'_X(1) = \theta[1-\omega+\lambda\omega]$$

$$\begin{aligned} Var(X) &= G''_X(1) + G'_X(1) - [G'_X(1)]^2 \\ &= \theta\omega\lambda^2 + [\theta(1-\omega)+\theta\omega\lambda]^2 + \theta[1-\omega+\lambda\omega] - [\theta(1-\omega+\lambda\omega)]^2 \\ &= \theta\omega\lambda^2 + \theta[1-\omega+\lambda\omega] \end{aligned}$$

The probability density function of X is obtained by extracting the coefficients of s^x in (8.12) as follows,

$$\begin{aligned}
G_X(s) &= \exp\left[\theta(1-\omega)s + \theta\omega e^{\lambda(s-1)} - \theta\right] \\
&= e^{-\theta}e^{\theta(1-\omega)s}e^{\theta\omega e^{\lambda(s-1)}} \\
&= e^{-\theta}e^{\theta(1-\omega)s}\sum_{i=0}^{\infty}\frac{(\theta\omega)^i}{i!}e^{\lambda i(s-1)} \\
&= e^{-\theta}e^{\theta(1-\omega)s}\sum_{i=0}^{\infty}\frac{\left(\theta\omega e^{-\lambda}\right)^i}{i!}e^{\lambda is} \\
&= e^{-\theta}e^{\theta(1-\omega)s}\sum_{i=0}^{\infty}\left\{\frac{\left(\theta\omega e^{-\lambda}\right)^i}{i!}\sum_{r=0}^{\infty}\frac{(\lambda is)^r}{r!}\right\} \\
&= e^{-\theta}e^{\theta(1-\omega)s}\sum_{i=0}^{\infty}\sum_{r=0}^{\infty}\frac{\left(\theta\omega e^{-\lambda}\right)^i}{i!}\frac{(\lambda i)^r s^r}{r!} \\
&= e^{-\theta}e^{\theta(1-\omega)s}\sum_{r=0}^{\infty}\left\{\sum_{i=0}^{\infty}\frac{\left(\theta\omega e^{-\lambda}\right)^i}{i!}\frac{(\lambda i)^r}{r!}\right\}s^r
\end{aligned}$$

$$G_X(s) = e^{-\theta}\sum_{j=0}^{\infty}\frac{[\theta(1-\omega)s]^j}{j!}\left[\sum_{r=0}^{\infty}\left\{\sum_{i=0}^{\infty}\frac{\left(\theta\omega e^{-\lambda}\right)^i}{i!}\frac{(\lambda i)^r}{r!}\right\}s^r\right] \tag{8.13}$$

let

$$\phi(r) = \sum_{i=0}^{\infty}\frac{\left(\theta\omega e^{-\lambda}\right)^i}{i!}\frac{(\lambda i)^r}{r!}$$

Therefore (8.13) becomes

$$\begin{aligned}
G_X(s) &= e^{-\theta}\left\{\sum_{j=0}^{\infty}\frac{[\theta(1-\omega)s]^j}{j!}\right\}\left\{\sum_{r=0}^{\infty}\phi(r)s^r\right\} \\
&= e^{-\theta}\left\{1+\theta(1-\omega)s+\frac{[\theta(1-\omega)]^2s^2}{2!}+\cdots\right\}\left\{\phi(0)+\phi(1)s+\phi(2)s^2+\cdots\right\} \\
&= e^{-\theta}\left\{\begin{array}{c} 1\cdot\phi(0)+[1\cdot\phi(1)+\theta(1-\omega)\cdot\phi(0)]s \\ +\left[1\cdot\phi(2)+\theta(1-\omega)\cdot\phi(1)+\frac{[\theta(1-\omega)]^2}{2!}\cdot\phi(0)\right]s^2+\cdots \end{array}\right\}
\end{aligned}$$

Hence,

$$p_0 = e^{-\theta}\cdot\phi(0) = e^{-\theta}$$

$$p_1 = e^{-\theta}\left\{1\cdot\phi(1)+\theta(1-\omega)\cdot\phi(0)\right\}$$

$$p_2 = e^{-\theta}\left\{1\cdot\phi(2)+\theta(1-\omega)\cdot\phi(1)+\frac{[\theta(1-\omega)]^2}{2!}\cdot\phi(0)\right\}$$

$$\vdots$$

$$p_x = e^{-\theta} \left\{ \begin{array}{c} 1 \cdot \phi(x) + \frac{\theta(1-\omega)}{1!} \cdot \phi(x-1) + \frac{[\theta(1-\omega)]^2}{2!} \cdot \phi(x-2) \\ + \cdots + \frac{[\theta(1-\omega)]^x}{x!} \cdot \phi(0) \end{array} \right\}$$

$$= e^{-\theta} \sum_{r=o}^{x} \frac{[\theta(1-\omega)]^r}{r!} \phi(x-r)$$

$$= e^{-\theta} \sum_{r=o}^{x} \left\{ \frac{[\theta(1-\omega)]^r}{r!} \sum_{i=0}^{\infty} \frac{\left(\theta\omega e^{-\lambda}\right)^i}{i!} \frac{(\lambda i)^r}{r!} \right\}$$

Thus the pdf of X is given by

$$P_X(x) = e^{-\theta} \sum_{r=o}^{x} \left\{ \frac{[\theta(1-\omega)]^r}{r!} \sum_{i=0}^{\infty} \frac{\left(\theta\omega e^{-\lambda}\right)^i}{i!} \frac{(\lambda i)^r}{r!} \right\}, x = 0, 1, 2, \ldots, \infty$$

Alternatively, since

$$\sum_{r=o}^{x} \frac{[\theta(1-\omega)]^r}{r!} = \sum_{r=o}^{x} \frac{[\theta(1-\omega)]^{x-r}}{(x-r)!} \tag{8.14}$$

We can re-write $P_X(x)$ as

$$P_X(x) = \begin{cases} e^{-\theta\left(1-\omega e^{-\lambda}\right)} & x = 0 \\ e^{-\theta} \sum_{r=o}^{x} \frac{[\theta(1-\omega)]^{x-r}}{(x-r)!} \sum_{i=o}^{\infty} \frac{\left(\theta\omega e^{-\lambda}\right)^i}{i!} \frac{(\lambda i)^r}{r!} & x = 1, 2, \ldots, \infty \end{cases} \tag{8.15}$$

as given by Iwunor (2004).

The estimating equations for the parameters θ, ω and λ are

$$e^{-\hat{\theta}\left(1-\hat{\omega}e^{-\lambda}\right)} = f_0 \tag{8.16}$$

$$f_0 \left[\hat{\theta}(1-\hat{\omega}) + \hat{\theta}\hat{\omega}\hat{\lambda}e^{-\lambda}\right] = f_1 \tag{8.17}$$

$$\hat{\theta}\left[(1-\hat{\omega}) + \hat{\lambda}\hat{\omega}\right] = \bar{X} \tag{8.18}$$

Where

f_0 is the proportion of zero observations
f_1 is the proportion of one observations
$\bar{X}$ is the observed mean of the Distribution

To estimate for the parameters θ, ω and λ divide (8.16) and (8.17) by (8.18) to get,

$$\frac{1-\hat{\omega}e^{-\lambda}}{1-\hat{\omega}+\hat{\lambda}\hat{\omega}} = \frac{-\ln f_0}{\bar{X}} \tag{8.19}$$

$$\frac{1-\hat{\omega}+\hat{\omega}\hat{\lambda}e^{-\lambda}}{1-\hat{\omega}+\hat{\lambda}\hat{\omega}} = \frac{f_1}{\bar{X}f_0} \tag{8.20}$$

From (8.19) and (8.20) solve for ω and λ.

By making ω the subject of the formula in (8.19) we obtain,

$$
\begin{aligned}
1-\hat{\omega}e^{-\lambda} &= \frac{-\ln f_0}{\bar{X}}\left(1-\hat{\omega}+\hat{\lambda}\hat{\omega}\right)\\
1-\hat{\omega}e^{-\lambda} &= -\hat{\lambda}\hat{\omega}\frac{\ln f_0}{\bar{X}}-\frac{\ln f_0}{\bar{X}}+\hat{\omega}\frac{\ln f_0}{\bar{X}}\\
1+\frac{\ln f_0}{\bar{X}} &= \hat{\omega}\left(e^{-\lambda}-\hat{\lambda}\frac{\ln f_0}{\bar{X}}+\frac{\ln f_0}{\bar{X}}\right)\\
\hat{\omega} &= \frac{1+\frac{\ln f_0}{\bar{X}}}{e^{-\lambda}-\hat{\lambda}\frac{\ln f_0}{\bar{X}}+\frac{\ln f_0}{\bar{X}}} \qquad (8.21)
\end{aligned}
$$

and substitute the value of ω in (8.20) to obtain

$$
\begin{aligned}
1-\frac{f_1}{\bar{X}f_0} &= \hat{\omega}\left(1-\hat{\lambda}e^{-\lambda}-\frac{f_1}{\bar{X}f_0}+\hat{\lambda}\frac{f_1}{\bar{X}f_0}\right)\\
1-\frac{f_1}{\bar{X}f_0} &= \frac{1+\frac{\ln f_0}{\bar{X}}}{e^{-\lambda}-\hat{\lambda}\frac{\ln f_0}{\bar{X}}+\frac{\ln f_0}{\bar{X}}}\left(1-\hat{\lambda}e^{-\lambda}-\frac{f_1}{\bar{X}f_0}+\hat{\lambda}\frac{f_1}{\bar{X}f_0}\right)\\
1-\frac{f_1}{\bar{X}f_0} &= e^{-\lambda}\left(1-\frac{f_1}{\bar{X}f_0}\right)+\hat{\lambda}e^{-\lambda}\left(1+\frac{\ln f_0}{\bar{X}}\right)-\hat{\lambda}\left(\frac{\ln f_0}{\bar{X}}+\frac{f_1}{\bar{X}f_0}\right) \qquad (8.22)
\end{aligned}
$$

Then solve for λ in equation (8.22).

Substitute the value of λ in (8.21) to obtain ω. Then substitute the value of ω and λ obtained in any of the estimating equations to obtain the value of $\hat{\theta}$

Semi urban

The proportion of zero observations	$f_0 = 0.889\,84$
The proportion of one observations	$f_1 = 8.112\,7\times 10^{-2}$
The observed mean of the Distribution	$\bar{X} = 0.160\,55$

Substituting the values of $f_0 = 0.889\,84$, $f_1 = 8.112\,7\times 10^{-2}$ and $\bar{X} = 0.160\,55$ in equations (8.18), (8.21) and (8.22) we obtain

$$
\hat{\theta} = \frac{188}{1171\left(\lambda\omega-\omega+1\right)} \qquad (8.23)
$$

$$
\hat{\omega} = \frac{0.273\,01}{0.726\,99\lambda+e^{-1.0\lambda}-0.726\,99} \qquad (8.24)
$$

$$
\begin{aligned}
\lambda\left(\frac{1171}{188}\ln\frac{1042}{1171}+\frac{111\,245}{195\,896}\right)-\lambda e^{-\lambda}\left(\frac{1171}{188}\ln\frac{1042}{1171}+1\right)-\frac{84\,651}{195\,896}e^{-\lambda} &= -\frac{84\,651}{195\,896}\\
-0.159\,11\lambda-0.432\,12e^{-\lambda}-0.273\,01\lambda e^{-\lambda} &= -0.432\,12\\
\lambda+\frac{0.432\,12}{0.159\,11}e^{-\lambda}+\frac{0.273\,01\lambda e^{-\lambda}}{0.159\,11} &= \frac{0.432\,12}{0.159\,11}\\
\lambda+2.715\,9e^{-\lambda}+1.715\,9\lambda e^{-\lambda} &= 2.715\,9 \qquad (8.25)
\end{aligned}
$$

Solving for λ in (8.25) we obtain $\lambda = 1.6522$

Substitute the value of λ obtained in (8.24) to obtain ω. That is

$$\hat{\omega} = \left[\frac{0.27301}{0.72699\lambda + e^{-1.0\lambda} - 0.72699}\right]_{\lambda=1.6522} = 0.41007$$

Then substituting the value of ω and λ obtained in (8.23) to obtain the value of θ

$$\hat{\theta} = \left[\frac{188}{1171\left(\lambda\omega - \omega + 1\right)}\right]_{\lambda=1.6522,\omega=0.41007} = 0.12667$$

Therefore the solution to the parameters are $\hat{\omega} = 0.41007$, $\hat{\lambda} = 1.6522$, $\hat{\theta} = 0.12667$

Remote

The proportion of zero observations	$f_0 = 0.76828$
The proportion of one observations	$f_1 = 0.15507$
The observed mean of the Distribution	$\bar{X} = 0.34537$

Substituting the values of $f_0 = 0.76828$, $f_1 = 0.15507$ and $\bar{X} = 0.34537$ in the equation (8.18), (8.21) and (8.22) we obtain

$$\hat{\theta} = \frac{392}{1135\left(1 - \hat{\omega} + \hat{\lambda}\hat{\omega}\right)} \tag{8.26}$$

$$\hat{\omega} = \frac{0.23677}{0.76323\lambda + e^{-\lambda} - 0.76323} \tag{8.27}$$

$$\begin{aligned}
\lambda\left(\frac{1135}{392}\ln\frac{872}{1135} + \frac{12485}{21364}\right) - \lambda e^{-\lambda}\left(\frac{1135}{392}\ln\frac{872}{1135} + 1\right) - \frac{8879}{21364}e^{-\lambda} &= -0.41561 \\
-0.17883\lambda - 0.41561e^{-\lambda} - 0.23677\lambda e^{-\lambda} &= -0.41561 \\
\frac{0.17883}{0.23677}\lambda + \frac{0.41561}{0.23677}e^{-\lambda} + \lambda e^{-\lambda} &= \frac{0.41561}{0.23677} \\
0.75529\lambda + 1.7553e^{-\lambda} + \lambda e^{-\lambda} &= 1.7553
\end{aligned} \tag{8.28}$$

Solving for λ in (8.28) we obtain $\lambda = 0.84650$

Substitute the value of λ in (8.27) to obtain ω. That is

$$\hat{\omega} = \left[\frac{0.23677}{0.76323\lambda + e^{-\lambda} - 0.76323}\right]_{\lambda=0.84650} = 0.75947$$

Then substituting the value of ω and λ in (8.26) to obtain the value of θ

$$\hat{\theta} = \left[\frac{392}{1135\left(1 - \hat{\omega} + \hat{\lambda}\hat{\omega}\right)}\right]_{\lambda=0.84650,\omega=0.75947} = 0.39095$$

Therefore the solution to the parameters are $\hat{\omega} = 0.75947$, $\hat{\lambda} = 0.84650$, $\hat{\theta} = 0.39095$

Growth centre

The proportion of zero observations	$f_0 = 0.809\,6$
The proportion of one observations	$f_1 = 0.127\,48$
The observed mean of the Distribution	$\bar{X} = 0.290\,56$

Substituting the values of $f_0 = 0.809\,6,\ f_1 = 0.127\,48$ and $\bar{X} = 0.290\,56$ in the equation (8.18), (8.21) and (8.22) we obtain

$$\hat{\theta} = \frac{351}{1208\left(\lambda\omega - \omega + 1\right)} \tag{8.29}$$

$$\hat{\omega} = \frac{0.273\,09}{0.726\,91\lambda + e^{-1.0\lambda} - 0.726\,91} \tag{8.30}$$

$$\frac{78\,623}{171\,639} = \frac{78\,623}{171\,639}e^{-\lambda} + \lambda e^{-\lambda}\left(\frac{1208}{351}\ln\frac{489}{604} + 1\right) - \lambda\left(\frac{1208}{351}\ln\frac{489}{604} + \frac{93\,016}{171\,639}\right)$$

$$0.458\,07 = 0.184\,98\lambda + 0.458\,07e^{-\lambda} + 0.273\,09\lambda e^{-\lambda}$$

$$1.\,677\,4 = 0.677\,36\lambda + 1.\,677\,4e^{-\lambda} + \lambda e^{-\lambda} \tag{8.31}$$

Solving for λ in (8.31) we obtain $\lambda = 1.\,180\,9$

Substitute the value of λ in (8.30) to obtain ω. That is

$$\hat{\omega} = \left[\frac{0.273\,09}{0.726\,91\lambda + e^{-1.0\lambda} - 0.726\,91}\right]_{\lambda=1.\,180\,9} = 0.622\,78$$

Then substituting the value of ω and λ in (8.29) to obtain the value of θ

$$\hat{\theta} = \left[\frac{351}{1208\left(\lambda\omega - \omega + 1\right)}\right]_{\lambda=1.\,180\,9,\omega=0.622\,78} = 0.261\,14$$

Therefore the solution to the parameters are $\hat{\omega} = 0.622\,78,\ \hat{\lambda} = 1.\,180\,9,\ \hat{\theta} = 0.261\,14$

The Poisson-one-Inflated Log-series Distribution

$$\Pr\left(Z = z\right) = \begin{cases} \left(1-\omega\right) + \omega\alpha p & z = 1 \\ \frac{\omega\alpha p^z}{z} & z = 2, 3, 4, \ldots, \infty \end{cases}$$

Where $p = 1 - q$ is the probability of a person migrating from a cluster. $\alpha = -\left[\ln\left(1-p\right)\right]^{-1}$

The pgf of Z_i is given as

$$\begin{aligned} g\left(s\right) &= \sum_{z=0}^{\infty} p_z s^z = \sum_{z=1}^{\infty} p_z s^z \\ &= p_1 s + \sum_{z=2}^{\infty} p_z s^z \\ &= \left[\left(1-\omega\right) + \omega\alpha p\right] s + \omega\alpha\sum_{z=2}^{\infty}\frac{\left(ps\right)^z}{z} \\ &= \left[\left(1-\omega\right) + \omega\alpha p\right] s + \omega\alpha\left\{-\ln\left(1-ps\right) - ps\right\} \\ &= \left(1-\omega\right)s + \omega\alpha ps - \omega\alpha\ln\left(1-ps\right) - \omega\alpha ps \end{aligned}$$

Therefore $g(s)$ is given by

$$g(s) = (1-\omega)\,s - \omega\alpha \ln(1-ps) \tag{8.32}$$

Therefore substituting equations (8.10) and (8.32) into (8.9) gives the pgf of X as

$$G_X(s) = \exp\left[\theta(1-\omega)\,s - \omega\theta\alpha\ln(1-ps) - \theta\right] \tag{8.33}$$

The first and the second derivatives of $G_X(s)$ w.r.t s is given by

$$\begin{aligned}
G'_X(s) &= \theta(1-\omega) + \frac{\omega\theta\alpha p}{1-ps}\exp\left[\theta(1-\omega)\,s - \omega\theta\alpha\ln(1-ps) - \theta\right] \\
G''_X(s) &= \left[\theta(1-\omega) + \frac{\omega\theta\alpha p}{1-ps}\right]^2 \exp\left[\theta(1-\omega)\,s - \omega\theta\alpha\ln(1-ps) - \theta\right] \\
&\quad + \frac{\omega\theta\alpha p^2}{(1-ps)^2}\exp\left[\theta(1-\omega)\,s - \omega\theta\alpha\ln(1-ps) - \theta\right]
\end{aligned}$$

setting $s = 1$

$$\begin{aligned}
G'_X(1) &= \theta(1-\omega) + \omega\theta\alpha pq^{-1} \\
G''_X(1) &= \left[\theta(1-\omega) + \omega\theta\alpha pq^{-1}\right]^2 - \omega\theta\alpha p^2 q^{-2}
\end{aligned}$$

Thus the mean and variance is given by

$$E(X) = G'_X(1) = \theta(1-\omega) + \omega\theta\alpha pq^{-1}$$

$$\begin{aligned}
Var(X) &= G''_X(1) \quad + G'_X(1) - \left[G'_X(1)\right]^2 \\
&= \left[\theta(1-\omega) + \omega\theta\alpha pq^{-1}\right]^2 + \omega\theta\alpha p^2 q^{-2} + \theta(1-\omega) \\
&\quad + \omega\theta\alpha pq^{-1} - \left[\theta(1-\omega) + \omega\theta\alpha pq^{-1}\right]^2 \\
&= \theta(1-\omega) + \omega\theta\alpha pq^{-1} + \omega\theta\alpha p^2 q^{-2}
\end{aligned}$$

The probability density function of X is obtained by extracting the coefficients of s^x in (8.33) as follows,

$$\begin{aligned}
G_X(s) &= \exp\left[\theta(1-\omega)\,s - \omega\theta\alpha\ln(1-ps) - \theta\right] \\
&= e^{-\theta} e^{\theta(1-\omega)s} e^{-\omega\theta\alpha\ln(1-ps)} \\
&= e^{-\theta} e^{\theta(1-\omega)s}(1-ps)^{-\omega\theta\alpha} \\
&= e^{-\theta}\sum_{r=0}^{\infty}\left\{\frac{[\theta(1-\omega)]^r}{r!}\binom{\beta+r-1}{r}p^r\right\}s^r \\
&= e^{-\theta}\left\{1 + [\theta(1-\omega)\cdot\beta p]\,s + \left[\frac{[\theta(1-\omega)]^2}{2!}\binom{\beta+1}{2}p^2\right]s^2 + \cdots\right\}
\end{aligned}$$

Hence,

$$p_0 = e^{-\theta}\cdot 1$$

$$p_1 = e^{-\theta}\left\{1 + [\theta(1-\omega)\cdot\beta p]\right\}$$

$$p_2 = e^{-\theta}\left\{1+\theta(1-\omega)\cdot\beta p+\frac{[\theta(1-\omega)]^2}{2!}\binom{\beta+1}{2}p^2\right\}$$

$$\vdots$$

$$\begin{aligned} p_x &= e^{-\theta}\left\{1+\theta(1-\omega)\beta p+\frac{[\theta(1-\omega)]^2}{2!}\binom{\beta+1}{2}p^2+\cdots+\frac{[\theta(1-\omega)]^x}{x!}\binom{\beta+x-1}{x}p^x\right\} \\ &= e^{-\theta}\sum_{r=o}^{x}\frac{[\theta(1-\omega)]^r}{r!}\binom{\beta+r-1}{r}p^r \\ &= e^{-\theta}\sum_{r=o}^{x}\frac{[\theta(1-\omega)]^r}{r!}\binom{\beta+r-1}{r}p^r \end{aligned}$$

Thus the pdf of X is given by

$$P_X(x) = e^{-\theta}\sum_{r=o}^{x}\frac{[\theta(1-\omega)]^r}{r!}\binom{\beta+r-1}{r}p^r, x=1,2,\ldots,\infty$$

Alternatively, from equation (8.14) we can re-write $P_X(x)$ as

$$P_X(x) = \begin{cases} e^{-\theta} & x=0 \\ e^{-\theta}\sum_{r=o}^{x}\binom{\beta+r-1}{r}p^r\frac{[\theta(1-\omega)]^{x-r}}{(x-r)!} & x=1,2,\ldots,\infty \end{cases} \tag{8.34}$$

as given by Iwunor (2004).

where $\beta=\theta\omega\alpha$

The estimating equations for the parameters $\theta,\omega,$ and p are

$$e^{-\hat{\theta}} = f_0 \tag{8.35}$$

$$f_0\left[\hat{\theta}(1-\hat{\omega})+\hat{\omega}\hat{\theta}\hat{p}\hat{\alpha}\right] = f_1 \tag{8.36}$$

$$\hat{\theta}(1-\hat{\omega})+\hat{\omega}\hat{\theta}\hat{p}\hat{\alpha}\hat{q}^{-1} = \bar{X} \tag{8.37}$$

To estimate for the parameters θ,ω and λ we find the value of $\hat{\theta}$

$$\hat{\theta} = -\ln f_0$$

and substitute in (8.36) and (8.37) to obtain

$$1-\hat{\omega}+\hat{\omega}\hat{p}\hat{\alpha} = \frac{f_1}{(-\ln f_0)f_0} \tag{8.38}$$

$$(1-p)-\hat{\omega}(1-p)+\hat{\omega}\hat{p}\hat{\alpha}+\frac{\bar{X}}{-\ln f_0}p = \frac{\bar{X}}{-\ln f_0} \tag{8.39}$$

From (8.38) and (8.39) solve for ω and p.

By making ω the subject of the formula in (8.38) we obtain,

$$1-\hat{\omega}+\hat{\omega}\hat{p}\hat{\alpha}=\frac{f_1}{(-\ln f_0)\, f_0}$$
$$\hat{\omega}=\frac{f_1}{(-\ln f_0)\, f_0\,(\hat{p}\hat{\alpha}-1)}-\frac{1}{(\hat{p}\hat{\alpha}-1)} \tag{8.40}$$

and substitute the value of ω in (8.39) to obtain

$$\frac{\bar{X}}{-\ln f_0}-(1-p)-\frac{\bar{X}}{-\ln f_0}p=\hat{\omega}\,(-(1-p)+\hat{p}\hat{\alpha})$$
$$\frac{\bar{X}}{-\ln f_0}-(1-p)-\frac{\bar{X}}{-\ln f_0}p=\left(\frac{f_1}{(-\ln f_0)\, f_0\,(\hat{p}\hat{\alpha}-1)}-\frac{1}{(\hat{p}\hat{\alpha}-1)}\right)\{\hat{p}\hat{\alpha}-(1-p)\}$$

$$\frac{\bar{X}}{-\ln f_0}=\frac{\hat{p}\alpha f_1}{(-\ln f_0)\, f_0\,(\hat{p}\hat{\alpha}-1)}-\frac{\hat{p}\alpha}{(\hat{p}\hat{\alpha}-1)}-\frac{f_1\,(1-p)}{(-\ln f_0)\, f_0\,(\hat{p}\hat{\alpha}-1)}$$
$$+\frac{1\,(1-p)}{(\hat{p}\hat{\alpha}-1)}+(1-p)+\frac{\bar{X}}{-\ln f_0}p \tag{8.41}$$

Then solve for p.

Substitute the value of p in (8.40) to obtain ω.

Semi urban

The proportion of zero observations	$f_0 = 0.889\,84$
The proportion of one observations	$f_1 = 8.112\,7\times 10^{-2}$
The observed mean of the Distribution	$\bar{X} = 0.160\,55$

Substituting the values of $f_0 = 0.889\,84$, $f_1 = 8.112\,7\times 10^{-2}$, $\alpha=\frac{1}{-\ln(1-p)}$ and $\bar{X}=0.160\,55$ in equations (8.35), (8.40) and (8.41) we obtain

$$\hat{\theta}=-\ln 0.889\,84=0.116\,72$$
$$\hat{\omega}=-\frac{0.218\,87}{-\frac{p}{\ln(1-p)}-1} \tag{8.42}$$

$$1.\,375\,5=0.375\,53p+0.218\,87\frac{1-p}{-\frac{p}{\ln(1-p)}-1}+0.218\,87\frac{p}{(\ln(1-p))\left(-\frac{p}{\ln(1-p)}-1\right)}+1$$
$$0.\,375\,5=0.375\,53p+0.218\,87\frac{1-p}{-\frac{p}{\ln(1-p)}-1}+0.218\,87\frac{p}{(\ln(1-p))\left(-\frac{p}{\ln(1-p)}-1\right)} \tag{8.43}$$

Solving for $\hat{p}$ in (8.43) we obtain $\hat{p}=0.554\,07$

Substitute the value of $\hat{p}$ in (8.42) to obtain ω. That is

$$\hat{\omega}=\left[-\frac{0.218\,87}{-\frac{p}{\ln(1-p)}-1}\right]_{p=0.554\,07}=0.697\,21$$

Therefore the solution to the parameters are $\hat{\omega}=0.697\,21$, $\hat{p}=0.554\,07$, $\hat{\theta}=0.116\,72$

Remote

The proportion of zero observations	$f_0 = 0.768\,28$
The proportion of one observations	$f_1 = 0.155\,07$
The observed mean of the Distribution	$\bar{X} = 0.345\,37$

Substituting the values of $f_0 = 0.768\,28,\ f_1 = 0.155\,07\ \alpha = \frac{1}{-\ln(1-p)}$ and $\bar{X} = 0.345\,37$ in equations (8.35), (8.40) and (8.41) we obtain

$$\hat{\theta} = -\ln 0.768\,28 = 0.263\,60$$

$$\hat{\omega} = -\frac{0.234\,31}{-\frac{p}{\ln(1-p)} - 1} \tag{8.44}$$

$$1.\,310\,2 = 0.310\,23p + 0.234\,31\frac{1-p}{-\frac{p}{\ln(1-p)} - 1} + 0.234\,31\frac{p}{(\ln(1-p))\left(-\frac{p}{\ln(1-p)} - 1\right)} + 1$$

$$0.\,310\,2 = 0.310\,23p + 0.234\,31\frac{1-p}{-\frac{p}{\ln(1-p)} - 1} + 0.234\,31\frac{p}{(\ln(1-p))\left(-\frac{p}{\ln(1-p)} - 1\right)} \tag{8.45}$$

Solving for $\hat{p}$ in (8.45) we obtain $p = 0.343\,29$

Substitute the value of $\hat{p}$ in (8.44) to obtain ω. That is

$$\hat{\omega} = \left[-\frac{0.234\,31}{-\frac{p}{\ln(1-p)} - 1}\right] = 1.\,275\,9$$

Therefore the solution to the parameters are $\hat{\omega} = 1.\,275\,9,\ \hat{p} = 0.343\,29,\ \hat{\theta} = 0.263\,60$

Growth centre

The proportion of zero observations	$f_0 = 0.809\,6$
The proportion of one observations	$f_1 = 0.127\,48$
The observed mean of the Distribution	$\bar{X} = 0.290\,56$

Substituting the values of $f_0 = 0.809\,6,\ f_1 = 0.127\,48,\ \alpha = \frac{1}{-\ln(1-p)}$ and $\bar{X} = 0.290\,56$ in equations (8.35), (8.40) and (8.41) we obtain

$$\hat{\theta} = -\ln\frac{978}{1208} = 0.211\,21$$

$$\hat{\omega} = -\frac{0.254\,47}{-\frac{p}{\ln(1-p)} - 1} \tag{8.46}$$

$$1.3757 = 0.37570p + 0.25447\frac{1-p}{-\frac{p}{\ln(1-p)} - 1} + 0.25447\frac{p}{(\ln(1-p))\left(-\frac{p}{\ln(1-p)} - 1\right)} + 1$$

$$0.3757 = 0.37570p + 0.25447\frac{1-p}{-\frac{p}{\ln(1-p)} - 1} + 0.25447\frac{p}{(\ln(1-p))\left(-\frac{p}{\ln(1-p)} - 1\right)} \tag{8.47}$$

Solution is: Solving for $\hat{p}$ in (8.47) we obtain $p = 0.44216$

Substitute the value of $\hat{p}$ in (8.46) to obtain ω. That is

$$\hat{\omega} = \left[-\frac{0.25447}{-\frac{p}{\ln(1-p)} - 1}\right]_{p=0.44216} = 1.0495$$

Therefore the solution to the parameters are $\hat{\omega} = 1.0495$, $\hat{p} = 0.44216$, $\hat{\theta} = 0.21121$

The Poisson-one-Inflated Geometric Distribution

$$\Pr(Z = z) = \begin{cases} \omega p & z = 0 \\ (1-\omega) + \omega q p & z = 1 \\ \omega p q^z & z = 2, 3, 4, \ldots, \infty \end{cases} \tag{8.48}$$

Where $p = 1 - q$ is the probability of a person migrating from a cluster.

The pgf of Z_i is given as

$$\begin{aligned} g(s) &= \sum_{z=0}^{\infty} p_z s^z \\ &= p_0 + p_1 s + \sum_{z=2}^{\infty} p_z s^z \\ &= \omega p + [(1-\omega) + \omega q p]\, s + \omega p \sum_{z=2}^{\infty} (qs)^z \\ &= \omega p + [(1-\omega) + \omega q p]\, s + \omega p \left[(1-qs)^{-1} - qs - 1\right] \\ &= (1-\omega)\, s + \omega p (1-qs)^{-1} \end{aligned}$$

Therefore $g(s)$ is given by

$$g(s) = (1-\omega)\, s + \omega p (1-qs)^{-1} \tag{8.49}$$

Therefore substituting equations (8.10) and (8.49) into (8.9) gives the pgf of X as

$$G_X(s) = \exp\left[\theta(1-\omega)\, s + \omega\theta p (1-qs)^{-1} - \theta\right] \tag{8.50}$$

The first and the second derivatives of $G_X(s)$ w.r.t s is given by

$$\begin{aligned} G'_X(s) &= \theta(1-\omega) + \omega\theta q p (1-qs)^{-2} \exp\left[\theta(1-\omega)\, s + \omega\theta p (1-qs)^{-1} - \theta\right] \\ G''_X(s) &= \left[\theta(1-\omega) + \omega\theta q p (1-qs)^{-2}\right]^2 \exp\left[\theta(1-\omega)\, s + \omega\theta p (1-qs)^{-1} - \theta\right] \\ &\quad + 2\omega\theta q^2 p (1-qs)^{-3} \exp\left[\theta(1-\omega)\, s + \omega\theta p (1-qs)^{-1} - \theta\right] \end{aligned}$$

setting $s=1$

$$G'_X(1) = \theta(1-\omega) + \omega\theta q p^{-1}$$
$$G''_X(1) = \left[\theta(1-\omega) + \omega\theta q p^{-1}\right]^2 + 2\omega\theta q^2 p^{-2}$$

Thus the mean and variance is given by,

$$E(X) = G'_X(1) = \theta(1-\omega) + \omega\theta q p^{-1} \tag{8.51}$$

$$\begin{aligned}
Var(X) &= G''_X(1) \quad + G'_X(1) - \left[G'_X(1)\right]^2 \\
&= \left[\theta(1-\omega) + \omega\theta q p^{-1}\right]^2 + 2\omega\theta q^2 p^{-2} + \theta(1-\omega) \\
&+ \omega\theta q p^{-1} - \left[\theta(1-\omega) + \omega\theta q p^{-1}\right]^2 \\
&= \theta(1-\omega) + \omega\theta q^2 p^{-2} + \omega\theta q p^{-2}(q+p) \\
&= \theta(1-\omega) + \omega\theta q p^{-2}(1+q) \\
&= \theta(1-\omega) + \omega\theta q p^{-2}(2-p)
\end{aligned} \tag{8.52}$$

The probability density function of X is obtained by extracting the coefficients of s^x in (8.50) as follows,

$$\begin{aligned}
G_X(s) &= \exp\left[\theta(1-\omega)s + \omega\theta p(1-qs)^{-1} - \theta\right] \\
&= e^{-\theta} e^{\theta(1-\omega)s} e^{\omega\theta p(1-qs)^{-1}} \\
&= e^{-\theta} e^{\theta(1-\omega)s} \sum_{i=o}^{\infty} \frac{(\omega\theta)^i}{i!} \left(\frac{p}{1-qs}\right)^i \\
&= e^{-\theta} e^{\theta(1-\omega)s} \sum_{i=o}^{\infty} \frac{(\omega\theta)^i}{i!} \left\{\sum_{r=o}^{\infty} \binom{i+r-1}{r} p^i (qs)^r\right\} \\
&= e^{-\theta} e^{\theta(1-\omega)s} \sum_{i=o}^{\infty} \sum_{r=o}^{\infty} \frac{(\omega\theta)^i}{i!} \binom{i+r-1}{r} p^i q^r s^r \\
&= e^{-\theta} e^{\theta(1-\omega)s} \sum_{r=o}^{\infty} \left\{\sum_{i=o}^{\infty} \frac{(\omega\theta)^i}{i!} \binom{i+r-1}{r} p^i q^r\right\} s^r
\end{aligned}$$

$$G_X(s) = \sum_{j=0}^{\infty} \frac{\left[\theta(1-\omega)s\right]^j}{j!} \sum_{r=o}^{\infty} \left\{\sum_{i=o}^{\infty} \frac{(\omega\theta)^i}{i!} \binom{i+r-1}{r} p^i q^r\right\} s^r \tag{8.53}$$

let

$$\phi(r) = \sum_{i=o}^{\infty} \frac{(\omega\theta)^i}{i!} \binom{i+r-1}{r} p^i q^r$$

Therefore (8.53) becomes

$$\begin{aligned}
G_X(s) &= e^{-\theta} \left\{\sum_{j=0}^{\infty} \frac{\left[\theta(1-\omega)s\right]^j}{j!}\right\} \left\{\sum_{r=0}^{\infty} \phi(r) s^r\right\} \\
&= e^{-\theta} \left\{1 + \theta(1-\omega)s + \frac{\left[\theta(1-\omega)\right]^2 s^2}{2!} + \cdots\right\} \left\{\phi(0) + \phi(1)s + \phi(2)s^2 + \cdots\right\} \\
&= e^{-\theta} \left\{\begin{array}{c} 1\cdot\phi(0) + \left[1\cdot\phi(1) + \theta(1-\omega)\cdot\phi(0)\right]s \\ + \left[1\cdot\phi(2) + \theta(1-\omega)\cdot\phi(1) + \frac{\left[\theta(1-\omega)\right]^2}{2!}\cdot\phi(0)\right]s^2 + \cdots \end{array}\right\}
\end{aligned}$$

Hence,

$$p_0 = e^{-\theta} \cdot \phi(0) = e^{-\theta}$$

$$p_1 = e^{-\theta}\{1 \cdot \phi(1) + \theta(1-\omega) \cdot \phi(0)\}$$

$$p_2 = e^{-\theta}\left\{1 \cdot \phi(2) + \theta(1-\omega) \cdot \phi(1) + \frac{[\theta(1-\omega)]^2}{2!} \cdot \phi(0)\right\}$$

$$\vdots$$

$$p_x = e^{-\theta}\left\{\phi(x) + \frac{\theta(1-\omega)}{1!}\phi(x-1) + \frac{[\theta(1-\omega)]^2}{2!}\phi(x-2) + \cdots + \frac{[\theta(1-\omega)]^x}{x!}\phi(0)\right\}$$

$$= e^{-\theta}\sum_{r=o}^{x}\frac{[\theta(1-\omega)]^r}{r!}\phi(x-r)$$

$$= e^{-\theta}\sum_{r=o}^{x}\left\{\frac{[\theta(1-\omega)]^r}{r!}\sum_{i=o}^{\infty}\frac{(\omega\theta)^i}{i!}\binom{i+r-1}{r}p^i q^r\right\}$$

Thus the pdf of X is given by

$$P_X(x) = e^{-\theta}\sum_{r=o}^{x}\left\{\frac{[\theta(1-\omega)]^r}{r!}\sum_{i=o}^{\infty}\binom{i+r-1}{r}\frac{(\omega\theta)^i}{i!}p^i q^r\right\}, x = 0,1,2,\ldots,\infty$$

Alternatively, from equation (8.14) we can re-write $P_X(x)$ as

$$P_X(x) = \begin{cases} e^{-\theta(1-\omega p)} & x = 0 \\ e^{-\theta}\sum_{r=o}^{x}\frac{[\theta(1-\omega)]^{x-r}}{(x-r)!}\sum_{i=o}^{\infty}\binom{i+r-1}{r}\frac{(\omega\theta)^i}{i!}p^i q^r & x = 1,2,\ldots,\infty \end{cases} \quad (8.54)$$

as given by Iwunor (2004)

The estimating equations for the parameters $\theta, \omega,$ and p are

$$e^{-\hat{\theta}(1-\hat{\omega}\hat{p})} = f_0 \quad (8.55)$$

$$f_0\left[\hat{\theta}(1-\hat{\omega}) + \hat{\omega}\hat{\theta}\hat{p}\hat{q}\right] = f_1 \quad (8.56)$$

$$\hat{\theta}(1-\hat{\omega}) + \hat{\omega}\hat{\theta}\hat{q}\hat{p}^{-1} = \bar{X} \quad (8.57)$$

Unlike the distribution proposed by Yadava and Singh (1991), this distribution incorporates the possibility that nobody migrates from a cluster in a household.

To estimate for the parameters θ, ω and p divide (8.55) and (8.56) by (8.57) to get,

$$\frac{1-\hat{\omega}\hat{p}}{1-\hat{\omega}+\frac{\hat{\omega}(1-\hat{p})}{\hat{p}}} = \frac{-\ln f_0}{\bar{X}} \quad (8.58)$$

$$\frac{1-\hat{\omega}+\hat{\omega}\hat{p}(1-\hat{p})}{1-\hat{\omega}+\frac{\hat{\omega}(1-p)}{p}} = \frac{f_1}{\bar{X}f_0} \quad (8.59)$$

From (8.58) and (8.59) solve for ω and p.

By making ω the subject of the formula in (8.58) we obtain,

$$
\begin{aligned}
1-\hat{\omega}\hat{p} &= \frac{-\ln f_0}{\bar{X}}\left(1-\hat{\omega}+\frac{\hat{\omega}\,(1-\hat{p})}{\hat{p}}\right)\\
1+\frac{\ln f_0}{\bar{X}} &= \hat{\omega}\frac{\ln f_0}{\bar{X}}-\frac{\ln f_0}{\bar{X}}\frac{\hat{\omega}\,(1-\hat{p})}{\hat{p}}+\hat{\omega}\hat{p}\\
1+\frac{\ln f_0}{\bar{X}} &= \hat{\omega}\left(\frac{\ln f_0}{\bar{X}}-\frac{\ln f_0}{\bar{X}}\frac{(1-\hat{p})}{\hat{p}}+\hat{p}\right)\\
\hat{\omega} &= \frac{1+\frac{\ln f_0}{\bar{X}}}{\frac{\ln f_0}{\bar{X}}-\frac{\ln f_0}{\bar{X}}\frac{(1-\hat{p})}{\hat{p}}+\hat{p}} \qquad (8.60)
\end{aligned}
$$

and substitute the value of ω in (8.59) to obtain

$$
\begin{aligned}
1-\hat{\omega}+\hat{\omega}\hat{p}\,(1-\hat{p}) &= \frac{f_1}{\bar{X}f_0}\left(1-\hat{\omega}+\frac{\hat{\omega}\,(1-p)}{p}\right)\\
1-\frac{f_1}{\bar{X}f_0} &= \hat{\omega}\left(1-\hat{p}\,(1-\hat{p})-\frac{f_1}{\bar{X}f_0}+\frac{f_1}{\bar{X}f_0}\frac{(1-p)}{p}\right)\\
\frac{f_1}{\bar{X}f_0}-1 &= -\hat{p}\,(1-\hat{p})+\frac{f_1}{\bar{X}f_0}\frac{(1-p)}{p}-\frac{\ln f_0}{\bar{X}}\hat{p}\,(1-\hat{p})\\
&\quad+\frac{\ln f_0}{\bar{X}}\frac{(1-\hat{p})}{\hat{p}}-\hat{p}+\frac{f_1}{\bar{X}f_0}\hat{p} \qquad (8.61)
\end{aligned}
$$

Then solve for $\hat{p}$.

Substitute the value of $\hat{p}$ in (8.60) to obtain $\hat{\omega}$. Then substitute the value of $\hat{\omega}$ and $\hat{p}$ in any of the estimating equations to obtain the value of $\hat{\theta}$

Semi urban

The proportion of zero observations	$f_0 = 0.889\,84$
The proportion of one observations	$f_1 = 8.112\,7\times 10^{-2}$
The observed mean of the Distribution	$\bar{X} = 0.160\,55$

Substituting the values of $f_0 = 0.889\,84$, $f_1 = 8.112\,7\times 10^{-2}$ and $\bar{X} = 0.160\,55$ in equations (8.55), (8.60) and (8.61) we obtain

$$\hat{\theta} = \frac{0.116\,72}{1-\hat{\omega}\hat{p}} \qquad (8.62)$$

$$\hat{\omega} = \frac{0.273\,01}{p+\frac{0.726\,99}{p}(1-p)-0.726\,99} \qquad (8.63)$$

$$-0.432\,12 = -0.432\,12p-0.273\,01p\,(1-p)-\frac{0.159\,11}{p}(1-p) \qquad (8.64)$$

Solving for $\hat{p}$ in (8.64) we obtain $\hat{p} = 0.582\,80$

Substitute the value of p in (8.63) to obtain ω. That is

$$\hat{\omega} = \left[\frac{0.273\,01}{p+\frac{0.726\,99}{p}(1-p)-0.726\,99}\right]_{p=0.582\,80} = 0.725\,65$$

Then substituting the value of $\hat{\omega}$ and $\hat{p}$ in (8.62) to obtain the value of $\hat{\theta}$

$$\hat{\theta} = \left[\frac{0.116\,72}{1-\hat{\omega}\hat{p}}\right]_{p=0.582\,80,\omega=0.725\,65} = 0.202\,26$$

Therefore the solution to the parameters are $\hat{\omega} = 0.725\,65$, $\hat{p} = 0.582\,80$, $\hat{\theta} = 0.202\,26$

Remote

The proportion of zero observations	$f_0 = 0.768\,28$
The proportion of one observations	$f_1 = 0.155\,07$
The observed mean of the Distribution	$\bar{X} = 0.345\,37$

Substituting the values of $f_0 = 0.768\,28$, $f_1 = \frac{176}{1135}$ and $\bar{X} = 0.345\,37$ in equations (8.55), (8.60) and (8.61) we obtain

$$\hat{\theta} = \frac{0.263\,60}{1-\hat{\omega}\hat{p}} \tag{8.65}$$

$$\hat{\omega} = \frac{0.236\,77}{p + \frac{0.763\,23}{p}(1-p) - 0.763\,23} \tag{8.66}$$

$$-0.415\,61 = -0.415\,61p - 0.236\,77p\,(1-p) - \frac{0.178\,83}{p}(1-p) \tag{8.67}$$

Solving for $\hat{p}$ in (8.67) we obtain $\hat{p} = 0.755\,16$

Substitute the value of $\hat{p}$ in (8.66) to obtain ω. That is

$$\hat{\omega} = \left[\frac{0.236\,77}{p + \frac{0.763\,23}{p}(1-p) - 0.763\,23}\right]_{p=0.755\,16} = 0.989\,07$$

Then substituting the value of $\hat{\omega}$ and $\hat{p}$ in (8.65) to obtain the value of $\hat{\theta}$

$$\hat{\theta} = \left[\frac{0.263\,60}{1-\hat{\omega}\hat{p}}\right]_{p=0.755\,16,\omega=0.989\,07} = 1.041\,5$$

Therefore the solution to the parameters are $\hat{\omega} = 0.989\,07$, $\hat{p} = 0.755\,16$, $\hat{\theta} = 1.041\,5$

Growth centre

The proportion of zero observations	$f_0 = 0.809\,6$
The proportion of one observations	$f_1 = 0.127\,48$
The observed mean of the Distribution	$\bar{X} = 0.290\,56$

Substituting the values of $f_0 = 0.809\,6$, $f_1 = 0.127\,48$, and $\bar{X} = 0.290\,56$ in equations (8.55), (8.60) and (8.61) we obtain

$$\hat{\theta} = \frac{0.211\,21}{1-\hat{\omega}\hat{p}} \tag{8.68}$$

$$\hat{\omega} = \frac{0.273\,09}{p + \frac{0.726\,91}{p}(1-p) - 0.726\,91} \tag{8.69}$$

$$-0.458\,07 = -0.458\,07p - 0.273\,09p(1-p) - \frac{0.184\,98}{p}(1-p) \tag{8.70}$$

Solving for $\hat{p}$ in (8.70) we obtain $\hat{p} = 0.677\,36$

Substitute the value of $\hat{p}$ in (8.69) to obtain ω. That is

$$\hat{\omega} = \left[\frac{0.273\,09}{p + \frac{0.726\,91}{p}(1-p) - 0.726\,91}\right]_{p=0.677\,36} = 0.920\,45$$

Then substituting the value of $\hat{\omega}$ and $\hat{p}$ in (8.68) to obtain the value of $\hat{\theta}$

$$\hat{\theta} = \left[\frac{0.211\,21}{1 - \hat{\omega}\hat{p}}\right]_{p=0.677\,36, \omega=0.920\,45} = 0.560\,95$$

Therefore the solution to the parameters are $\hat{\omega} = 0.920\,45$, $\hat{p} = 0.677\,36$, $\hat{\theta} = 0.560\,95$

The Poisson-one-Inflated Negative Binomial Distribution

$$\Pr(Z = z) = \begin{cases} \omega p^m & z = 0 \\ (1-\omega) + \omega m q p^m & z = 1 \\ \omega \binom{m+z-1}{z} p^m q^z & z = 2, 3, 4, \ldots, \infty \end{cases} \tag{8.71}$$

Where $p = 1 - q$ is the probability of a person migrating from a cluster.

The pgf of Z_i is given as

$$\begin{aligned} g(s) &= \sum_{z=0}^{\infty} p_z s^z \\ &= p_0 + p_1 s + \sum_{z=2}^{\infty} p_z s^z \\ &= \omega p^m + [(1-\omega) + \omega m q p^m] s + \omega p^m \sum_{z=2}^{\infty} \binom{m+z-1}{z} (qs)^z \\ &= \omega p^m + [(1-\omega) + \omega m q p^m] s + \omega p^m \left\{(1-qs)^{-m} - mqs - 1\right\} \\ &= \omega p^m + (1-\omega) s + \omega m q p^m s + \omega p^m (1-qs)^{-m} - \omega m q p^m s - \omega p^m \\ &= (1-\omega) s + \omega p^m (1-qs)^{-m} \end{aligned}$$

Therefore

$$g(s) = (1-\omega) s + \omega p^m (1-qs)^{-m} \tag{8.72}$$

Therefore substituting equations (8.10) and (8.72) into (8.9) gives the pgf of X as

$$G_X(s) = \exp\left[\theta(1-\omega) s + \omega\theta p^m (1-qs)^{-m} - \theta\right] \tag{8.73}$$

The first and the second derivatives of $G_X(s)$ w.r.t s is given by

$$G'_X(s) = \theta(1-\omega) + \omega\theta m q p^m (1-qs)^{-m-1} \exp\left[\theta(1-\omega) s + \omega\theta p^m (1-qs)^{-m} - \theta\right]$$

$$\begin{aligned} G''_X(s) = &\left[\theta(1-\omega) + \omega\theta m q p^m (1-qs)^{-m-1}\right]^2 \exp\left[\theta(1-\omega) s + \omega\theta p^m (1-qs)^{-m} - \theta\right] \\ &+ \omega\theta m(m+1) q^2 p^m (1-qs)^{-(m+2)} \exp\left[\theta(1-\omega) s + \omega\theta p^m (1-qs)^{-m} - \theta\right] \end{aligned}$$

setting $s=1$

$$G'_X(1)=\theta(1-\omega)+\omega\theta mqp^{-1}$$
$$G''_X(1)=\left[\theta(1-\omega)+\omega\theta mqp^{-1}\right]^2+\omega\theta m(m+1)q^2p^{-2}$$

The mean and variance of X is given by

$$E(X)=G'_X(1)=\theta(1-\omega)+\omega\theta mqp^{-1} \tag{8.74}$$

$$\begin{aligned}
Var(X)&=G''_X(1)\;\;+G'_X(1)-\left[G'_X(1)\right]^2\\
&=\left[\theta(1-\omega)+\omega\theta m\frac{q}{p}\right]^2+\omega\theta m(m+1)\left(\frac{q}{p}\right)^2+\theta(1-\omega)\\
&+\omega\theta m\frac{q}{p}-\left[\theta(1-\omega)+\omega\theta m\frac{q}{p}\right]^2\\
&=\theta(1-\omega)+\omega\theta m^2q^2p^{-2}+\omega\theta mqp^{-2}(q+p)\\
&=\theta(1-\omega)+m\omega\theta qp^{-2}(1+mq)
\end{aligned} \tag{8.75}$$

The probability density function of X is obtained by extracting the coefficients of s^x in (8.73) as follows,

$$\begin{aligned}
G_X(s)&=\exp\left[\theta(1-\omega)s+\omega\theta p^m(1-qs)^{-m}-\theta\right]\\
&=e^{-\theta}e^{\theta(1-\omega)s}e^{\omega\theta p^m(1-qs)^{-m}}\\
&=e^{-\theta}e^{\theta(1-\omega)s}\sum_{i=o}^{\infty}\frac{(\omega\theta)^i}{i!}p^{mi}(1-qs)^{-mi}\\
&=e^{-\theta}e^{\theta(1-\omega)s}\sum_{i=o}^{\infty}\frac{(\omega\theta)^i}{i!}\left\{\sum_{r=o}^{\infty}\binom{mi+r-1}{r}p^{mi}(qs)^r\right\}\\
&=e^{-\theta}e^{\theta(1-\omega)s}\sum_{i=o}^{\infty}\sum_{r=o}^{\infty}\frac{(\omega\theta)^i}{i!}\binom{mi+r-1}{r}p^{mi}q^rs^r\\
&=e^{-\theta}e^{\theta(1-\omega)s}\sum_{r=o}^{\infty}\left\{\sum_{i=o}^{\infty}\frac{(\omega\theta)^i}{i!}\binom{mi+r-1}{r}p^{mi}q^r\right\}s^r
\end{aligned}$$

$$G_X(s)=\sum_{j=0}^{\infty}\frac{[\theta(1-\omega)s]^j}{j!}\sum_{r=o}^{\infty}\left\{\sum_{i=o}^{\infty}\frac{(\omega\theta)^i}{i!}\binom{mi+r-1}{r}p^{mi}q^r\right\}s^r \tag{8.76}$$

let

$$\phi(r)=\sum_{i=o}^{\infty}\frac{(\omega\theta)^i}{i!}\binom{mi+r-1}{r}p^{mi}q^r$$

Therefore (8.76) becomes

$$\begin{aligned}
G_X(s)&=e^{-\theta}\left\{\sum_{j=0}^{\infty}\frac{[\theta(1-\omega)s]^j}{j!}\right\}\left\{\sum_{r=0}^{\infty}\phi(r)s^r\right\}\\
&=e^{-\theta}\left\{1+\theta(1-\omega)s+\frac{[\theta(1-\omega)]^2s^2}{2!}+\cdots\right\}\left\{\phi(0)+\phi(1)s+\phi(2)s^2+\cdots\right\}\\
&=e^{-\theta}\left\{\begin{array}{c}1\cdot\phi(0)+[1\cdot\phi(1)+\theta(1-\omega)\cdot\phi(0)]s\\+\left[1\cdot\phi(2)+\theta(1-\omega)\cdot\phi(1)+\frac{[\theta(1-\omega)]^2}{2!}\cdot\phi(0)\right]s^2+\cdots\end{array}\right\}
\end{aligned}$$

Hence,

$$p_0 = e^{-\theta} \cdot \phi(0) = e^{-\theta}$$

$$p_1 = e^{-\theta} \{1 \cdot \phi(1) + \theta(1-\omega) \cdot \phi(0)\}$$

$$p_2 = e^{-\theta} \left\{ 1 \cdot \phi(2) + \theta(1-\omega) \cdot \phi(1) + \frac{[\theta(1-\omega)]^2}{2!} \cdot \phi(0) \right\}$$

$$\vdots$$

$$\begin{aligned} p_x &= e^{-\theta} \left\{ \begin{array}{c} 1 \cdot \phi(x) + \frac{\theta(1-\omega)}{1!} \cdot \phi(x-1) + \frac{[\theta(1-\omega)]^2}{2!} \cdot \phi(x-2) \\ + \cdots + \frac{[\theta(1-\omega)]^x}{x!} \cdot \phi(0) \end{array} \right\} \\ &= e^{-\theta} \sum_{r=o}^{x} \frac{[\theta(1-\omega)]^r}{r!} \phi(x-r) \\ &= e^{-\theta} \sum_{r=o}^{x} \left\{ \frac{[\theta(1-\omega)]^r}{r!} \sum_{i=o}^{\infty} \frac{(\omega\theta)^i}{i!} \binom{mi+r-1}{r} p^{mi} q^r \right\} \end{aligned}$$

Thus the pdf of X is given by

$$P_X(x) = e^{-\theta} \sum_{r=o}^{x} \left\{ \frac{[\theta(1-\omega)]^r}{r!} \sum_{i=o}^{\infty} \frac{(\omega\theta)^i}{i!} \binom{mi+r-1}{r} p^{mi} q^r \right\}, x = 0, 1, 2, \ldots, \infty$$

Alternatively, from equation (8.14) we can re-write $P_X(x)$ as

$$P_X(x) = \begin{cases} e^{-\theta(1-\omega p^m)} & x = 0 \\ e^{-\theta} \sum_{r=o}^{x} \frac{[\theta(1-\omega)]^{x-r}}{(x-r)!} \sum_{i=o}^{\infty} \binom{mi+r-1}{r} \frac{(\omega\theta p^m)^i}{i!} q^r & x = 1, 2, \ldots, \infty \end{cases} \tag{8.77}$$

as given by Iwunor (2004)

The estimating equations for the parameters θ, ω, m and p are

$$f_0 = e^{-\hat{\theta}\left(1-\hat{\omega}\hat{p}^{\hat{m}}\right)} \tag{8.78}$$

$$f_1 = f_0 \left[\hat{\theta}(1-\hat{\omega}) + \hat{m}\hat{\omega}\hat{\theta}\hat{q}\hat{p}^{\hat{m}} \right] \tag{8.79}$$

$$\hat{\theta}(1-\hat{\omega}) + \hat{m}\hat{\omega}\hat{\theta}\hat{q}\hat{p}^{-1} = \bar{X} \tag{8.80}$$

$$\hat{\theta}(1-\hat{\omega}) + \hat{m}\hat{\omega}\hat{\theta}\hat{q}\hat{p}^{-2}(1+mq) = \sigma^2 \tag{8.81}$$

Where σ^2 is the observed variance of the distribution

To estimate for the parameters θ, ω, m and p

$$\left(1-\hat{\omega}\hat{p}^{\hat{m}}\right) = -\ln f_0 \tag{8.82}$$

$$\left[1-\hat{\omega}+\hat{m}\hat{\omega}(1-\hat{p})\hat{p}^{\hat{m}}\right] = \frac{f_1}{f_0} \tag{8.83}$$

$$(1-\hat{\omega}) + \hat{m}\hat{\omega}\frac{1-\hat{p}}{\hat{p}} = \bar{X} \tag{8.84}$$

$$1 - \hat{\omega} + \hat{m}\hat{\omega}\frac{1-\hat{p}}{\hat{p}^2}\left[1 + m\left(1-\hat{p}\right)\right] = \sigma^2 \tag{8.85}$$

Divide (8.83), (8.84) (8.85) by (8.82) to get

$$\hat{\omega}\left(-1 + \hat{m}\left(1-\hat{p}\right)\hat{p}^{\hat{m}} - \frac{1}{f_0}\frac{f_1}{\ln f_0}\hat{p}^{\hat{m}}\right) = -\frac{1}{f_0}\frac{f_1}{\ln f_0} - 1 \tag{8.86}$$

$$\hat{\omega}\left(-1 + \hat{m}\frac{1-\hat{p}}{\hat{p}} - \frac{\bar{X}}{\ln f_0}\hat{p}^{\hat{m}}\right) = -\frac{\bar{X}}{\ln f_0} - 1 \tag{8.87}$$

$$\hat{\omega}\left(-1 + \hat{m}\frac{1-\hat{p}}{\hat{p}^2}\left[1 + m\left(1-\hat{p}\right)\right] - \frac{\sigma^2}{\ln f_0}\hat{p}^{\hat{m}}\right) = -\frac{\sigma^2}{\ln f_0} - 1 \tag{8.88}$$

Divide (8.87) and (8.88) by (8.86) to obtain

$$-1 + \hat{m}\frac{1-\hat{p}}{\hat{p}} - \frac{\bar{X}}{\ln f_0}\hat{p}^{\hat{m}} = \frac{\left(-\frac{\bar{X}}{\ln f_0} - 1\right)}{-\frac{1}{f_0}\frac{f_1}{\ln f_0} - 1}\left(-1 + \hat{m}\left(1-\hat{p}\right)\hat{p}^{\hat{m}} - \frac{1}{f_0}\frac{f_1}{\ln f_0}\hat{p}^{\hat{m}}\right) \tag{8.89}$$

$$-1+\hat{m}\frac{1-\hat{p}}{\hat{p}^2}\left(1 + m\left(1-\hat{p}\right)\right) - \frac{\sigma^2}{\ln f_0}\hat{p}^{\hat{m}} = \frac{\left(-\frac{\sigma^2}{\ln f_0} - 1\right)}{-\frac{1}{f_0}\frac{f_1}{\ln f_0} - 1}\left(-1 + \hat{m}\left(1-\hat{p}\right)\hat{p}^{\hat{m}} - \frac{1}{f_0}\frac{f_1}{\ln f_0}\hat{p}^{\hat{m}}\right) \tag{8.90}$$

Solving equation (8.89) and (8.90) simultaneously to obtain $\hat{m}$ and $\hat{p}$

The Poisson-Misrecorded Poisson Distribution

The mis-recorded Poisson distribution takes into account errors in reporting the number of migrants in a cluster (Johnson et.al., 1992)

$$\Pr\left(Z = z\right) = \begin{cases} e^{-\lambda}\left(1 + \lambda\phi\right) & z = 0 \\ \lambda e^{-\lambda}\left(1 - \phi\right) & z = 1 \\ \frac{\lambda^z e^{-\lambda}}{z!} & z = 2, 3, 4, \ldots, \infty \end{cases} \tag{8.91}$$

where λ is the average number of persons migrating from a cluster, ϕ is the probability that one migrant recorded in a cluster is not reported. The pgf of Z_i is given as

$$\begin{aligned} g\left(s\right) &= \sum_{z=0}^{\infty} p_z s^z \\ &= p_0 + p_1 s + \sum_{z=2}^{\infty} p_z s^z \\ &= e^{-\lambda}\left(1 + \lambda\phi\right) + \lambda e^{-\lambda}\left(1-\phi\right)s + \sum_{z=2}^{\infty}\frac{\left(\lambda s\right)^z e^{-\lambda}}{z!} \\ &= e^{-\lambda}\left(1 + \lambda\phi\right) + \lambda s e^{-\lambda}\left(1-\phi\right) + \left\{e^{\lambda(s-1)} - \lambda s e^{-\lambda} - e^{-\lambda}\right\} \\ &= e^{-\lambda} + \lambda\phi e^{-\lambda} + \lambda s e^{-\lambda} - \lambda\phi s e^{-\lambda} + e^{\lambda(s-1)} - \lambda s e^{-\lambda} - e^{-\lambda} \\ &= \lambda\phi e^{-\lambda} - \lambda\phi s e^{-\lambda} + e^{\lambda(s-1)} \end{aligned} \tag{8.92}$$

Therefore substituting equations (8.10) and (8.92) into (8.9) gives the pgf of X as

$$G_X(s) = \exp\left[\lambda\theta\phi e^{-\lambda} - \lambda\theta\phi e^{-\lambda}s + \theta e^{\lambda(s-1)} - \theta\right] \tag{8.93}$$

The probability density function of X is obtained by extracting the coefficients of s^x in (8.93) as follows,

$$\begin{aligned}
G_X(s) &= \exp\left[\lambda\theta\phi e^{-\lambda} - \lambda\theta\phi e^{-\lambda}s + \theta e^{\lambda(s-1)} - \theta\right] \\
&= e^{-\theta\left(1-\lambda\phi e^{-\lambda}\right)} e^{-\lambda\theta\phi e^{-\lambda}s} e^{\theta e^{\lambda(s-1)}} \\
&= e^{-\theta\left(1-\lambda\phi e^{-\lambda}\right)} e^{-\lambda\theta\phi e^{-\lambda}s} \sum_{i=o}^{\infty} \frac{\left(\theta e^{-\lambda}\right)^i}{i!} e^{\lambda i s} \\
&= e^{-\theta\left(1-\lambda\phi e^{-\lambda}\right)} e^{-\lambda\theta\phi e^{-\lambda}s} \sum_{i=0}^{\infty} \left\{ \frac{\left(\theta e^{-\lambda}\right)^i}{i!} \sum_{r=0}^{\infty} \frac{(\lambda i s)^r}{r!} \right\} \\
&= e^{-\theta\left(1-\lambda\phi e^{-\lambda}\right)} e^{-\lambda\theta\phi e^{-\lambda}s} \sum_{i=0}^{\infty} \sum_{r=0}^{\infty} \frac{\left(\theta e^{-\lambda}\right)^i}{i!} \frac{(\lambda i s)^r}{r!} \\
&= e^{-\theta\left(1-\lambda\phi e^{-\lambda}\right)} e^{-\lambda\theta\phi e^{-\lambda}s} \sum_{r=0}^{\infty} \left\{ \sum_{i=0}^{\infty} \frac{\left(\theta e^{-\lambda}\right)^i}{i!} \frac{(\lambda i)^r}{r!} \right\} s^r
\end{aligned}$$

$$G_X(s) = e^{-\theta\left(1-\lambda\phi e^{-\lambda}\right)} \sum_{j=0}^{\infty} \frac{\left[-\lambda\theta\phi e^{-\lambda}s\right]^j}{j!} \left[\sum_{r=0}^{\infty} \left\{ \sum_{i=0}^{\infty} \frac{\left(\theta e^{-\lambda}\right)^i}{i!} \frac{(\lambda i)^r}{r!} \right\} s^r \right] \tag{8.94}$$

let

$$\phi(r) = \sum_{i=0}^{\infty} \frac{\left(\theta e^{-\lambda}\right)^i}{i!} \frac{(\lambda i)^r}{r!}$$

Therefore (8.94) becomes

$$\begin{aligned}
G_X(s) &= e^{-\theta\left(1-\lambda\phi e^{-\lambda}\right)} \left\{ \sum_{j=0}^{\infty} \frac{\left[-\lambda\theta\phi e^{-\lambda}s\right]^j}{j!} \right\} \left\{ \sum_{r=0}^{\infty} \phi(r) s^r \right\} \\
&= e^{-\theta\left(1-\lambda\phi e^{-\lambda}\right)} \left\{ 1 - \lambda\theta\phi e^{-\lambda}s + \frac{\left[\lambda\theta\phi e^{-\lambda}\right]^2 s^2}{2!} + \cdots \right\} \left\{ \begin{array}{c} \phi(0) + \phi(1)s \\ +\phi(2)s^2 + \cdots \end{array} \right\} \\
&= e^{-\theta\left(1-\lambda\phi e^{-\lambda}\right)} \left\{ \begin{array}{c} 1\cdot\phi(0) + \left[1\cdot\phi(1) - \lambda\theta\phi e^{-\lambda}\cdot\phi(0)\right]s \\ + \left[1\cdot\phi(2) - \lambda\theta\phi e^{-\lambda}\cdot\phi(1) + \frac{\left[\lambda\theta\phi e^{-\lambda}\right]^2}{2!}\cdot\phi(0)\right]s^2 + \cdots \end{array} \right\}
\end{aligned}$$

Hence,

$$p_0 = e^{-\theta\left(1-\lambda\phi e^{-\lambda}\right)}\cdot\phi(0) = e^{-\theta\left(1-\lambda\phi e^{-\lambda}\right)}$$

$$p_1 = e^{-\theta\left(1-\lambda\phi e^{-\lambda}\right)}\left\{1\cdot\phi(1) - \lambda\theta\phi e^{-\lambda}\cdot\phi(0)\right\}$$

$$p_2 = e^{-\theta\left(1-\lambda\phi e^{-\lambda}\right)}\left\{1\cdot\phi(2) - \lambda\theta\phi e^{-\lambda}\cdot\phi(1) + \frac{\left[\lambda\theta\phi e^{-\lambda}\right]^2}{2!}\cdot\phi(0)\right\}$$

$$\vdots$$

$$p_x = e^{-\theta\left(1-\lambda\phi e^{-\lambda}\right)} \left\{ \begin{array}{c} 1\cdot\phi(x) - \lambda\theta\phi e^{-\lambda}\cdot\phi(x-1) + \frac{\left[\lambda\theta\phi e^{-\lambda}\right]^2}{2!}\cdot\phi(x-2) \\ +\cdots+\frac{\left[-\lambda\theta\phi e^{-\lambda}\right]^x}{x!}\cdot\phi(0) \end{array} \right\}$$

$$= e^{-\theta\left(1-\lambda\phi e^{-\lambda}\right)} \sum_{r=o}^{x} \frac{\left[-\lambda\theta\phi e^{-\lambda}\right]^r}{r!}\phi(x-r)$$

$$= e^{-\theta\left(1-\lambda\phi e^{-\lambda}\right)} \sum_{r=o}^{x} \left\{ \frac{\left[-\lambda\theta\phi e^{-\lambda}\right]^r}{r!} \sum_{i=0}^{\infty} \frac{\left(\theta e^{-\lambda}\right)^i}{i!}\frac{(\lambda i)^r}{r!} \right\}$$

Thus the pdf of X is given by

$$P_X(x) = A\sum_{r=o}^{x} \left\{ \frac{\left[-\lambda\theta\phi e^{-\lambda}\right]^r}{r!} \sum_{i=0}^{\infty} \frac{\left(\theta e^{-\lambda}\right)^i}{i!}\frac{(\lambda i)^r}{r!} \right\}, x = 0,1,2,\ldots,\infty$$

Alternatively, from equation (8.14) we can re-write $P_X(x)$ as

$$P_X(x) = \begin{cases} Ae^{\theta e^{-\lambda}} & x=0 \\ A\sum_{r=o}^{x}\frac{\left(-\lambda\theta\phi e^{-\lambda}\right)^{x-r}}{(x-r)!}\sum_{i=o}^{\infty}\frac{\left(\theta e^{-\lambda}\right)^i}{i!}\frac{(\lambda i)^r}{r!} & x=1,2,\ldots,\infty \end{cases} \quad (8.95)$$

as given by Iwunor (2004)

Where $A = \exp\left[-\theta\left(1-\lambda\phi e^{-\lambda}\right)\right]$

$$E(X) = \lambda\theta\left[1-\phi e^{-\lambda}\right] \quad (8.96)$$

$$Var(X) = \lambda\theta\left(1+\lambda-\phi e^{-\lambda}\right) + \phi^2 e^{-2\lambda}\left(\theta^2\lambda^2-1\right) \quad (8.97)$$

The estimating equations for the parameters θ, ϕ and λ are given as

$$\hat{A}e^{\hat{\theta}e^{-\hat{\lambda}}} = f_0 \quad (8.98)$$

$$f_0\left[\hat{\theta}\hat{\lambda}e^{-\hat{\lambda}}\left(1-\hat{\phi}\right)\right] = f_1 \quad (8.99)$$

$$\hat{\theta}\hat{\lambda}\left(1-\hat{\phi}e^{-\hat{\lambda}}\right) = \bar{X} \quad (8.100)$$

To estimate for the parameters θ, ϕ and λ

$$\theta\left(1-\lambda\phi e^{-\lambda}-e^{-\hat{\lambda}}\right) = -\ln f_0 \quad (8.101)$$

$$\hat{\theta}\left(\hat{\lambda}e^{-\hat{\lambda}}-\hat{\phi}\hat{\lambda}e^{-\hat{\lambda}}\right) = \frac{f_1}{f_0} \quad (8.102)$$

$$\hat{\theta}\left(\hat{\lambda}-\hat{\phi}\hat{\lambda}e^{-\hat{\lambda}}\right) = \bar{X} \quad (8.103)$$

Divide (8.101) and (8.102) by (8.103) to obtain

$$\frac{1-\lambda\phi e^{-\lambda}-e^{-\hat{\lambda}}}{\hat{\lambda}-\hat{\phi}\hat{\lambda}e^{-\hat{\lambda}}} = \frac{-\ln f_0}{\bar{X}} \quad (8.104)$$

$$\frac{\hat{\lambda}e^{-\hat{\lambda}}-\hat{\phi}\hat{\lambda}e^{-\hat{\lambda}}}{\hat{\lambda}-\hat{\phi}\hat{\lambda}e^{-\hat{\lambda}}} = \frac{f_1}{f_0\bar{X}} \quad (8.105)$$

Making ϕ the subject of the formula in (8.104) and substitute the value of ϕ to obtain

$$\phi = \frac{-\frac{\ln f_0}{\bar{X}}\hat{\lambda} + e^{-\hat{\lambda}} - 1}{-\lambda e^{-\lambda} - \frac{\ln f_0}{\bar{X}}\hat{\lambda} e^{-\hat{\lambda}}} \tag{8.106}$$

$$\frac{\lambda}{e^{\lambda}} - \frac{\lambda}{e^{2\lambda}} - \frac{\lambda^2}{e^{2\lambda}} + \frac{1}{\bar{X}}\frac{\lambda^2}{e^{\lambda}}\ln f_0 - \frac{1}{\bar{X}}\frac{\lambda}{e^{\lambda} f_0} f_1 - \frac{1}{\bar{X}}\lambda^2 \frac{\ln f_0}{e^{2\lambda}} + \frac{1}{\bar{X}}\frac{\lambda}{f_0}\frac{f_1}{e^{2\lambda}} + \frac{1}{\bar{X}}\frac{\lambda^2}{e^{\lambda} f_0} f_1 = 0$$

Solve for λ and substitute the value in (8.106) to obtain ϕ.

To obtain θ, substitute the values of λ and ϕ in any of the estimating equation above and solve for θ

The Poissson-one-Inflated Binomial Distribution

$$\Pr(Z = z) = \begin{cases} \omega q^n & z = 0 \\ (1-\omega) + \omega n p q^{n-1} & z = 1 \\ \omega \binom{n}{z} p^z q^{n-z} & z = 2, 3, 4, \ldots, n \end{cases} \tag{8.107}$$

where n is the cluster size, p is the probability of a person migrating from a cluster, $p + q = 1$. The pgf of Z_i is given as

$$\begin{aligned} g(s) &= \sum_{z=0}^{\infty} p_z s^z \\ &= p_0 + p_1 s + \sum_{z=2}^{\infty} p_z s^z \\ &= \omega q^n + (1-\omega) s + \omega n p q^{n-1} s + \omega \sum_{z=2}^{\infty} \binom{n}{z} (ps)^z q^{n-z} \\ &= \omega q^n + (1-\omega) s + \omega n p q^{n-1} s + \omega \left[(q + ps)^n - npq^{n-1} s - q^n \right] \\ &= (1-\omega) s + \omega (q + ps)^n \end{aligned} \tag{8.109}$$

substituting equations (8.4) and (8.109) into (8.1) gives the pgf of X as

$$G_X(s) = \exp\left[\theta (1-\omega) s + \omega\theta (q + ps)^n - \theta\right] \tag{8.110}$$

The first and the second derivatives of $G_X(s)$ w.r.t s is given by

$$\begin{aligned} G'_X(s) &= \theta(1-\omega) + \omega\theta np (q+ps)^{n-1} e^{[\theta(1-\omega)s + \omega\theta(q+ps)^n - \theta]} \\ G''_X(s) &= \omega\theta n(n-1) p^2 (q+ps)^{n-2} e^{[\theta(1-\omega)s + \omega\theta(q+ps)^n - \theta]} \\ &\quad + \left[\theta(1-\omega) + \omega\theta np (q+ps)^{n-1}\right]^2 e^{[\theta(1-\omega)s + \omega\theta(q+ps)^n - \theta]} \end{aligned}$$

setting $s = 1$

$$\begin{aligned} G'_X(1) &= \theta(1-\omega) + \omega\theta np \\ G''_X(1) &= \omega\theta n(n-1) p^2 + \left[\theta(1-\omega) + \omega\theta np\right]^2 \end{aligned}$$

Therefore the mean and the variance is given by

$$E(X) = G'_X(1) = \theta(1-\omega) + \omega\theta np \tag{8.111}$$

$$\begin{aligned}
Var(X) &= G''_X(1) \ \ + G'_X(1) - [G'_X(1)]^2 \\
&= \omega\theta n(n-1)p^2 + [\theta(1-\omega) + \omega\theta np]^2 + \theta(1-\omega) \\
&+ \omega\theta np - [\theta(1-\omega) + \omega\theta np]^2 \qquad (8.1) \\
&= \omega\theta n(n-1)p^2 + \theta(1-\omega) + \omega\theta np = \theta(1-\omega) + \omega\theta np(1-p) + \omega\theta n^2p^2 \\
&= \theta(1-\omega) + n\omega\theta pq + \omega\theta n^2p^2 \qquad (8.112)
\end{aligned}$$

The probability density function of X is obtained by extracting the coefficients of s^x in (8.110) as follows,

$$\begin{aligned}
G_X(s) &= \exp\left[\theta(1-\omega)s + \omega\theta(q+ps)^n - \theta\right] \\
&= e^{\theta(1-\omega)s}e^{\omega\theta(q+ps)^n}e^{-\theta} \\
&= e^{-\theta}e^{\theta(1-\omega)s}\sum_{i=0}^{\infty}\frac{(\omega\theta)^i}{i!}(q+ps)^{ni} \\
&= e^{-\theta}e^{\theta(1-\omega)s}\sum_{i=0}^{\infty}\left\{\frac{(\omega\theta)^i}{i!}\sum_{r=0}^{ni}\binom{ni}{r}(ps)^r q^{ni-r}\right\} \\
&= e^{-\theta}e^{\theta(1-\omega)s}\sum_{i=0}^{\infty}\left\{\frac{(\omega\theta)^i}{i!}\sum_{r=0}^{ni}\binom{ni}{r}(ps)^r q^{ni-r}\right\} \\
&= e^{-\theta}e^{\theta(1-\omega)s}\sum_{i=0}^{\infty}\sum_{r=0}^{ni}\frac{(\omega\theta)^i}{i!}\binom{ni}{r}p^r q^{ni-r}s^r \\
&= e^{-\theta}e^{\theta(1-\omega)s}\sum_{r=0}^{\infty}\left\{\sum_{i=0}^{\infty}\frac{(\omega\theta)^i}{i!}\binom{ni}{r}q^{ni-r}\right\}p^r s^r
\end{aligned}$$

$$G_X(s) = e^{-\theta}\sum_{j=0}^{\infty}\frac{[\theta(1-\omega)s]^j}{j!}\left[\sum_{r=0}^{\infty}\left\{\sum_{i=0}^{\infty}\frac{(\omega\theta)^i}{i!}\binom{ni}{r}p^r q^{ni-r}\right\}s^r\right] \qquad (8.113)$$

let

$$\phi(r) = \sum_{i=0}^{\infty}\frac{(\omega\theta)^i}{i!}\binom{ni}{r}p^r q^{ni-r}$$

Therefore (8.113) becomes

$$\begin{aligned}
G_X(s) &= e^{-\theta}\left\{\sum_{j=0}^{\infty}\frac{[\theta(1-\omega)s]^j}{j!}\right\}\left\{\sum_{r=0}^{\infty}\phi(r)s^r\right\} \\
&= e^{-\theta}\left\{1 + \theta(1-\omega)s + \frac{[\theta(1-\omega)]^2s^2}{2!} + \cdots\right\}\left\{\phi(0) + \phi(1)s + \phi(2)s^2 + \cdots\right\} \\
&= e^{-\theta}\left\{\begin{array}{c} 1\cdot\phi(0) + [1\cdot\phi(1) + \theta(1-\omega)\cdot\phi(0)]s \\ +\left[1\cdot\phi(2) + \theta(1-\omega)\cdot\phi(1) + \frac{[\theta(1-\omega)]^2}{2!}\cdot\phi(0)\right]s^2 + \cdots \end{array}\right\}
\end{aligned}$$

Hence,

$$p_0 = e^{-\theta}\cdot\phi(0) = e^{-\theta}$$

$$p_1 = e^{-\theta}\left\{1\cdot\phi(1) + \theta(1-\omega)\cdot\phi(0)\right\}$$

$$p_2 = e^{-\theta}\left\{1\cdot\phi(2)+\theta(1-\omega)\cdot\phi(1)+\frac{[\theta(1-\omega)]^2}{2!}\cdot\phi(0)\right\}$$

$$\vdots$$

$$\begin{aligned}
p_x &= e^{-\theta}\left\{\begin{array}{c} 1\cdot\phi(x)+\frac{\theta(1-\omega)}{1!}\cdot\phi(x-1)+\frac{[\theta(1-\omega)]^2}{2!}\cdot\phi(x-2)\\ +\cdots+\frac{[\theta(1-\omega)]^x}{x!}\cdot\phi(0)\end{array}\right\}\\
&= e^{-\theta}\sum_{r=o}^{x}\frac{[\theta(1-\omega)]^r}{r!}\phi(x-r)\\
&= e^{-\theta}\sum_{r=o}^{x}\left\{\frac{[\theta(1-\omega)]^r}{r!}\sum_{i=0}^{\infty}\frac{(\omega\theta)^i}{i!}\binom{ni}{r}p^r q^{ni-r}\right\}
\end{aligned}$$

Thus the pdf of X is given by

$$P_X(x) = e^{-\theta}\sum_{r=o}^{x}\left\{\frac{[\theta(1-\omega)]^r}{r!}\sum_{i=0}^{\infty}\binom{ni}{r}\frac{(\omega\theta)^i}{i!}p^r q^{ni-r}\right\}, x=0,1,2,\ldots,n$$

Alternatively, from equation (8.14) we can re-write $P_X(x)$ as

$$P_X(x) = \left\{\begin{array}{ll} e^{-\theta(1-\omega q^n)} & x=0\\ e^{-\theta}\sum_{r=o}^{x}\frac{[\theta(1-\omega)]^{x-r}}{(x-r)!}\sum_{i=o}^{\infty}\binom{ni}{r}\frac{(\omega\theta)^i}{i!}p^r q^{ni-r} & x=1,2,\ldots,n\end{array}\right. \quad (8.114)$$

as given by Iwunor (2004).

The estimating equations for the parameters θ, ω and p are

$$f_0 = e^{-\hat{\theta}(1-\hat{\omega}\hat{q}^n)} \quad (8.115)$$

$$f_1 = f_0\left[\hat{\theta}(1-\hat{\omega})+n\hat{\omega}\hat{\theta}\hat{p}\hat{q}^{n-1}\right] \quad (8.116)$$

$$\hat{\theta}(1-\hat{\omega})+n\hat{\omega}\hat{\theta}\hat{p} = \bar{X} \quad (8.117)$$

To estimate for the parameters θ, ω and p

$$\hat{\theta}(1-\hat{\omega}(1-p)^n) = -\ln f_0 \quad (8.118)$$

$$\hat{\theta}\left(1-\hat{\omega}+n\hat{\omega}\hat{p}(1-p)^{n-1}\right) = \frac{f_1}{f_0} \quad (8.119)$$

$$\hat{\theta}(1-\hat{\omega}+n\hat{\omega}\hat{p}) = \bar{X} \quad (8.120)$$

Divide (8.118) and (8.119) by (8.120) to get,

$$\frac{1}{\bar{X}}np\omega\ln f_0-\frac{1}{\bar{X}}\omega\ln f_0-\omega(1-p)^n = -\frac{1}{\bar{X}}\ln f_0-1 \quad (8.121)$$

$$\frac{1}{\bar{X}}\frac{\omega}{f_0}f_1-\omega-\frac{1}{\bar{X}}np\frac{\omega}{f_0}f_1+np\frac{\omega}{1-p}(1-p)^n = \frac{1}{\bar{X}f_0}f_1-1 \quad (8.122)$$

Make ω the subject of the formula in (8.121) and substitute in (8.122)

$$\omega = \frac{-\frac{1}{X}\ln f_0 - 1}{\frac{1}{X}np\ln f_0 - \frac{1}{X}\ln f_0 - (1-p)^n} \tag{8.123}$$

$$\left\{ \begin{array}{c} \frac{1}{X}np\ln f_0 - (1-p)^n - \frac{1}{Xf_0}f_1 + \frac{1}{X}n\frac{p}{f_0}f_1 + \frac{1}{Xf_0}f_1(1-p)^n \\ -n\frac{p}{1-p}(1-p)^n - \frac{1}{X}np\frac{\ln f_0}{1-p}(1-p)^n \end{array} \right\} = -1 \tag{8.125}$$

Solve for p in (8.125) and substitute in (8.123) to obtain ω

To obtain θ, substitute the values of p and ω in any of the three estimating equation and solve for θ

A summary of the difference between the estimated Parameters and Iwunor's (2004) **Parameters**

The Poisson-One-Inflated Poisson Distribution									
	Semi-urban			Remote			Growth Centre		
Par.	Est.	Iwunor	Diff.	Est.	Iwunor	Diff.	Est.	Iwunor	Diff.
$\hat{\theta}$	0.1267	0.1193	0.0074	0.3910	0.2762	0.1148	0.2611	0.2194	0.0417
$\hat{\omega}$	0.4101	0.2423	0.1678	0.7595	0.2925	0.4670	0.6228	0.2991	0.3237
$\hat{\lambda}$	1.6522	2.4242	−0.772	0.8465	1.8566	−1.0101	1.1809	2.0858	−0.9049

The Poisson-One-Inflated Geometric Distribution									
	Semi-urban			Remote			Growth Centre		
Par.	Est.	Iwunor	Diff.	Est.	Iwunor	Diff.	Est.	Iwunor	Diff.
$\hat{\theta}$	0.2023	0.3410	−0.1387	1.0415	0.6912	0.3503	0.5610	0.6226	−0.0616
$\hat{\omega}$	0.7257	0.9332	−0.2075	0.9891	0.8966	0.0925	0.9205	0.9486	−0.0281
$\hat{p}$	0.5828	0.7046	−0.1218	0.7552	0.6900	0.0652	0.6774	0.6966	−0.0192

The Poisson-One-Inflated Log-series Distribution									
	Semi-urban			Remote			Growth Centre		
Par.	Est.	Iwunor	Diff.	Est.	Iwunor	Diff.	Est.	Iwunor	Diff.
$\hat{\theta}$	0.1167	0.1168	−0.0001						
$\hat{\omega}$	0.6972	0.4684	0.2288						
$\hat{p}$	0.5541	0.6483	−0.0942						

Where

- Est.=Estimated
  - Par.=Parameters
  - Diff.=Difference

Application

The estimates derived from the equations with explicit solutions are applied by fitting the various distributions and testing their adequacy for each of the village types; Semi-urban, Remote and Growth centre, using the data contained in Sharma (1985). Table 8.6-8.8 show the distribution of observed and expected frequencies of the number of households according to the the total number of Migrants from a household and the χ^2 values for the different types of village, based on each of the distributions.

Table 8.6: Observed and Expected Number of Households according to the Number of Migrants and Type of Village

The Poisson-One-Inflated Poisson Distribution

Number of migrants	Number of Households					
	Semi-urban		Remote		Growth Centre	
	Obs.	Exp.	Obs.	Exp.	Obs.	Exp.
0	1042	1042	872	872	978	978
1	95	95	176	176	154	154.01
2	19	18.49	59	57.55	47	46.17
3	10	9.22	18	20.45	18	19.40
4	2		6		9	
5	2		4		1	
6	0	6.29	0	9	0	10.42
7	1		0		0	
8	0		0		1	
Total	1171	1171	1135	1135	1135	1135
$\hat{\theta}$		0.126 67		0.390 95		0.261 14
$\hat{\omega}$		0.410 07		0.759 47		0.622 78
$\hat{\lambda}$		1. 652 2		0.846 50		1. 180 9
χ^2		0.3446		0.4412		0.1482
df		1		1		1

Table 8.7: Observed and Expected Number of Households according to the Number of Migrants and Type of Village

The Poisson-One-Inflated Log-series Distribution

Number of migrants	Number of Households					
	Semi-urban		Remote		Growth Centre	
	Obs.	Exp.	Obs.	Exp.	Obs.	Exp.
0	1042	1042				
1	95	95				
2	19	20.45				
3	10	7.55				
4	2					
5	2					
6	0	6				
7	1					
8	0					
Total	1171	1171				
$\hat{\theta}$		0.116 72				
$\hat{\omega}$		0.697 21				
$\hat{p}$		0.554 07				
χ^2		1.0645				
df		1				

Table 8.8: Observed and Expected Number of Households according to the Number of Migrants and Type of Village
The Poisson-One-Inflated Geometric Distribution

Number of migrants	Number of Households					
	Semi-urban		Remote		Growth Centre	
	Obs.	Exp.	Obs.	Exp.	Obs.	Exp.
0	1042	1042	872	872	978	978
1	95	95.01	176	176	154	154
2	19	19.84	59	58.43	47	47.73
3	10	8.02	18	19.36	18	17.73
4	2		6		9	
5	2		4		1	
6	0	6.13	0	9.21	0	10.54
7	1		0		0	
8	0		0		1	
Total	1171	1171	1135	1135	1208	1208
$\hat{\theta}$		0.202 26		1. 041 5		0.560 95
$\hat{\omega}$		0.725 65		0.989 07		0.920 45
$\hat{p}$		0.582 80		0.755 16		0.677 36
χ^2		0.7327		0.1689		0.0354
df		1		1		1

Table 8.9: A summary of $\bar{X}$, $\frac{\bar{X}}{\hat{\theta}}$ and $\hat{\theta}$			
	Semi-urban	Remote	Growth Centre
$\bar{X}$	0.1605	0.3453	0.2906
The Poisson-One-Inflated Poisson Distribution			
$\hat{\theta}$	0.12667	0.39095	0.26114
$\frac{\bar{X}}{\hat{\theta}}$	1.26744	0.88342	1.112671
The Poisson-One-Inflated Geometric Distribution			
$\hat{\theta}$	0.20226	1.0415	0.56095
$\frac{\bar{X}}{\hat{\theta}}$	0.79376	0.33161	0.51798
The Poisson-One-Inflated Log-series Distribution			
$\hat{\theta}$	0.11672		
$\frac{\bar{X}}{\hat{\theta}}$	1.37548		

Where

- The mean number of clusters of migrants per household $\hat{\theta}$
- The average number of migrants per cluster $\frac{\bar{X}}{\hat{\theta}}$
- The average number of migrants per household $\bar{X}$

The tables shows that the values of χ^2 are insignificant at 5% and 1% level for all the fitted distributions. On the basis of χ^2 test is noted that the Poisson-one-inflated Poisson distribution provides a superior fit than the two other distributions for the same degrees of freedom, in modelling out-migration from Semi-urban villages. In the case of household residing in remote and Growth centres villages, the Poisson-one-Inflated geometric distribution provides a superior fit compared to the Poisson-one-inflated Poisson distribution.

From table 8.9, it is found that in all the sets of the model the average number of clusters per household is greater for remote villages, moderate for growth centre and smaller in semi-urban villages, but the average number of migrants per cluster is smaller for remote villages in comparison to growth-centre that is moderate and greater for semi-urban villages. This might be attributed to the fact that from the remote households males migrate singly in different cluster leaving their wives and children in the village while in growth-centre and semi-urban villages males, mostly well educated and employed in white collar jobs, migrate with their wives and children in less number of cluster. Since remaining persons from semi-urban villages may commute to city for their livelihood. Lastly, the average number of migrants per household from remote villages is higher (0.35) than the growth centre (0.29) and semi-urban (0.16).

8.2.4 Conclusion

In order to capture the event that at least one person migrates in a household, we have fitted the mixture of the Poisson and some one-inflated distributions; i.e., one-inflated Poisson distribution, one-inflated log-series distribution and one-inflated geometric distribution. The results of the fit shows that distributions that take into consideration variation in the probability of a person migrating in a cluster in a household (the geometric which is a special case of the negative binomial distribution) performed better in modelling out-migration from remote and growth centres. While Poisson-one-Poisson distribution demonstrated a satisfactory fit for modelling out-migration from Semi-urban areas.

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