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ON ABELIAN SANDPILE MODEL OF A DIRECTED MULTI-GRAPH

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ABELIAN SANDPILE MODEL OF A DIRECTED
MULTI-GRAPH

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Dedication

... I was not able to write anything about it for several months and I wish I would have waited for several more months. However if I had waited long enough I probably never would have written anything at all since there is a tendency when you really begin to learn something about a thing not to want to write about it but rather to keep on learning about it always and at no time, unless you are very egotistical, which, of course, accounts for many books, will you be able to say: now I know all about this and will write about it. Certainly I do not say that now; every year I know there is more to learn ...

-[Ernest Hemingway, from, *Death in the Afternoon*]

This project is dedicated to my late Dad and Mum, Mr. and Mrs. Opollo, my sister Joyce Anita and to the family of my Uncle, Tobias Muga.

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Abstract

In this project, we study the combinatorics, algebraic and algebraic geometry of an Abelian Sandpile Model (ASM). We aim to explore the Algebraic properties of the Abelian Sandpile Model (ASM). We also aim to study the algebraic geometry of the Abelian Sandpile Model (ASM).

In this model, we consider a directed multi graph with a sink vertex which is accessible from every other vertex and associates with it a commutative monoid M , a commutative semigroup S and a commutative Group G which finally turns out to be the sandpile monoid, sandpile semigroup and sandpile group respectively. We observe that the sandpile group is the unique minimal ideal of the sandpile monoid [21]. We study the combinatorial structure of our model and the connections between the algebraic structure of the sandpile monoid, the sandpile semigroup and the sandpile group.

Outline

The first chapter sets the stage: it surveys some of the origins of the Sandpile Model and the facts that led to the Model being given the name Sandpile.

Chapter 2 is on the literature review and also highlights the various mathematical aspects for which we study the Abelian Sandpile Model.

The third chapter covers the introduction and basics about the combinatorics, algebra (basically the commutative algebra) [where I have presupposed the basic knowledge of groups, rings, fields and abstract vector spaces] and some introduction to algebraic geometry. The chapter includes intertwine of these concepts with those of the Sandpile Model.

Chapter 4 is on the Abelian Sandpile Model. Here we define the concepts related to the Abelian Sandpile Model and we also discuss some of the properties of the model.

In chapter five, we look at some of the applications of the model with respect to Algebra and algebraic geometry.

INTRODUCTION

Definition 1.0.1. *Abelian Sandpile Model* is a system defined on a set of some given number of sites on a graph that are labeled by integers. Each site i is assigned some integer value h_i called the height of the sandpile at site i where the order of firing does not affect the final outcome and two processes namely, addition operation and relaxation are applied [14].

Such sites are supposed to hold a given number of sand grains at a given time. The process of the Abelian Sandpile Model involves addition of grains of sand to a particular site, which if the number of sand grains is equal to or exceed the number of sand grains that particular site is supposed to hold then firing occurs stabilizing the system. In this case, the order of firing the unstable vertices does not affect the final outcome.

1.1 Origin of the model

The inspiration behind the name given to this model originated from the following example, consider a scenario of a teenager at the beach letting grains of sand drop one onto another randomly to form a pile which we call sandpile. At the beginning, the pile is tiny and the motion of the sand grains is evident in the physical properties of the individual sand grains. As the sandpile increases there is a trickling down effect which we call toppling, at this point, the system is far out of balance and so its behaviour can only be understood from a holistic

description of the properties of the entire pile. These avalanches form a dynamic of their own and thus the sandpile is a complex system.

Therefore generally the Abelian Sandpile Model is purely a game in which grains of sand are placed on the vertices of a finite graph and their behaviour observed.

1.2 Background

The Abelian Sandpile Model is a diffusion process on a finite directed multi-graph that has been used to describe the phenomenon of a self organized criticality. The model was defined on square grids with cells that randomly, with breaks in between, received sand grains. These had a maximum capacity of 3 sand grains; once this capacity was exceeded, the sand would topple into adjacent cells or fall off the edge of the grid. These square grids are a type of graph, with the cells as the vertices and edges connecting adjacent cells and allowing sand grains to pass from one cell to a neighbouring cell [12].

Since its appearance, this model has been studied intensely, in the theoretical physics, theoretical Computer Science and in the mathematics literature, for example in the algebraic graph theory. This model brings to our attention the fact that a simple dynamics can lead to the emergence of very complex structures and drive the system towards a stationary state which shares several properties of equilibrium systems at the critical point. For example, the power law decay of cluster sizes and of correlations of the height-variables.

In this project, we study the Abelian Sandpile Model purely as a game in which one adds grains of sand on the vertices of a graph Γ which are supposed to hold only a specified number of sand grains. Thus if a vertex receives so much sand, it fires to each of its neighbouring vertex which may not or may become unstable making the process of firing to continue. We specify one vertex which is accessible to absorb all the sand grains fired into it, meaning that the firing process always stops.

The main objectives of this project are (1) Study the Abelian Sandpile Model for a directed multi graph. (2) A motivation for studying algebraic geometry for Abelian Sandpile Model for further study.

LITERATURE REVIEW

The Abelian Sandpile Model is a diffusion process on graphs which has been studied by many authors. In 1987, Bak et. al, [2] discovered a lattice model of what they called self-organized criticality, using the example of a Sandpile. They argued that in many natural phenomena, the dissipative dynamics of the system is such that it drives the system to a critical state, thereby leading to ubiquitous power law behaviours. They considered the process of successive random placement of sand grains onto the pile to which each of the placements may have no effect or may cause a cascading reaction what we call avalanches. The self-organized criticality has no real or precise meaning but it can be taken to describe a system that naturally evolves to a stable state whose instabilities are defined by power law.

In 1991, Manna [22] discovered that for an Abelian Sandpile Model, the final stable height configuration of the system is so much independent of the sequence in which the sand grains are added to reach this stable configuration as compared to other Sandpile Models where the stability of a height configuration depends on the local slope.

In the year 1995, Middleton and Tang [23] discovered that the self-organized criticality state of an Abelian Sandpile Model is built from the long range correlations that establish a delicate balance between internal avalanches and the avalanches that touch the boundaries of the system i.e. the system is open to the outside. For example consider a case where there are incoming grains of sand until at that particular vertex the number of sand grains

is either equal to or exceeds the number of sand grains, that particular vertex is supposed to hold then there exists outgoing grains of sand which normally occurs at the boundaries of the system for the system to be stable. Suppose we increase the size of the vertex, the effect at the boundaries must be felt for the system to realize self organized criticality. So basically increasing the size of the vertex only increases the time over which the self organized criticality establishes itself.

Some years later, Deepak Dhar (1999) [13] generalized the model and discovered the Abelian Group Structure of addition operators, in it and called it the Abelian Sandpile Model (ASM) which is the simplest and most popular. The principal geometric object governing the dynamics of the recurrent set of configurations is in the stationary state.

In 2004, Didier Sornette [11], discovered that the four different quantities of an abelian sandpile model are interdependent and are related to each other by scaling laws. For example, the size i.e number of toppling grains of an avalanche is proportional to its surface, the average duration (t) of an avalanche grows faster with its typical radius (r) than linearly e.t.c. Where these quantities are:

- size (s): the total number of toppling in the avalanche,
- area (a): the number of distinct sites that toppled,
- lifetime (t): the duration of the avalanche, and
- radius (r): the maximum distance of a toppled site from the origin.

And an avalanche is a cascade of toppling of a number of sites created by the addition of a sand grain.

In 2010, Lionel Levine and James Propp [21], considered finite connected graph Γ for which they associated an abelian group $S(G)$ of which they discovered is an isomorphism invariant of the graph and reflects certain combinatorial information about the graph. To define the group, they singled out one vertex of Γ as the sink and ignored the sand that fell in there. The operation of addition followed by stabilization gave the set M of all stable sandpile on Γ the structure of a commutative monoid. An ideal of M is a subset $J \subset M$

satisfying $\sigma J \subset J$ for all $\sigma \in M$. The sandpile group $S(G)$ is the minimal ideal of M (i.e. the intersection of all ideals). The minimal ideal of a finite commutative monoid is always a group. $S(G)$ is independent of the choice of sink up to isomorphism. Hence the problem of finding the identity element of this group is very interesting.

In 2011, D. Perkinson, J. Perlman and J. Wilmes [9] generalized the duality theorem between elements of the sandpile group of directed multi graphs and the graphs' super stable configurations.

In 2014, R. Cori, D. Rossin, and B. Salvy [27] in their paper Polynomial Ideals for Sand piles and their Gröbner Bases, associated a toppling ideal to a directed multi graph, encoding configurations with monomials and toppling with binomials.

2.1 Mathematical aspects of the Abelian Sandpile Model

2.1.1 Algebraic aspects of the Abelian Sandpile Model

We study some of the algebraic structures of an Abelian Sandpile Model by associating sand grains to the vertices of the graph. We think of a sandpile as a nonnegative integer weighting on the n vertices of a graph, inform of a vector $h \in \mathbb{Z}^n$.

We study some features of the abelian group $S(G)$ associated to the Abelian Sandpile Model (ASM). In particular we define scalar function, invariant under toppling.

2.1.2 Geometric Aspects of the Abelian Sandpile Model

In this project, we consider problems in sandpile model from a geometric viewpoint, either by viewing groups as geometric objects, or by finding suitable geometric objects a group acts on [8]. The first idea is made precise by means of the Cayley graph, i.e. check Example 4.2.20 whose vertices correspond to group elements and edges correspond to right multiplication in the group where multiplication in this case represents point wise addition.

2.1.3 Algebraic Geometry of the Abelian Sandpile Model

We also study some of the affine properties of the Abelian Sandpile Model where we consider sand piles as points in affine space, and toppling and sand addition as affine transformations.

GENERAL PRELIMINARIES

We establish some terminology that will be used in the project.

3.1 Combinatorics

Definition 3.1.1 ([6]). *Combinatorics is a branch of mathematics that studies, usually finite or countable collections of objects that satisfy specified criteria.*

For example, objects like graphs, hyper-graphs, partitions or partially ordered sets.

Other than the objects of study, combinatorics is also characterized by methods such as counting arguments, induction, inclusion-exclusion, the probabilistic method e.t.c. Hence Combinatorics is a branch of mathematics that deals with combinations and permutations. One of the oldest and most accessible parts of combinatorics is graph theory among others, thus graph theory is an area of combinatorics.

In this project we do not consider graph to mean a visual representation of a function or a data set, but a structure consisting of vertices which are connected by edges.

In combinatorics, Graph theory is but the study of graphs which are mathematical structures that are used to model pairwise relations between objects. In this case, graph is but sets of vertices that are connected by edges. For instance, weighted graph represent structures in which pairwise connections have some values which are numerical.

Definition 3.1.2. A graph Γ is a diagram consisting of small circles called vertices V and curves called the edges E where each curve connects two of the circles together.

So a graph Γ is a set with two types of elements, namely vertices and edges abbreviated $\Gamma(V, E)$ that connect pairs of vertices. The set of edges is usually defined as a set of two-element subsets of the set of vertices denoted by $E(u, v)$.

3.1.1 Types of Graphs

Definition 3.1.3. Directed graph or a digraph is a graph where each of its edges have a specific orientation.

So when the edges of a graph are endowed with direction, then they are called directed edges. This leads to the notion of a directed graph, normally referred to as a digraph. Therefore a digraph is a graph whose edges have directions as shown in the following example.

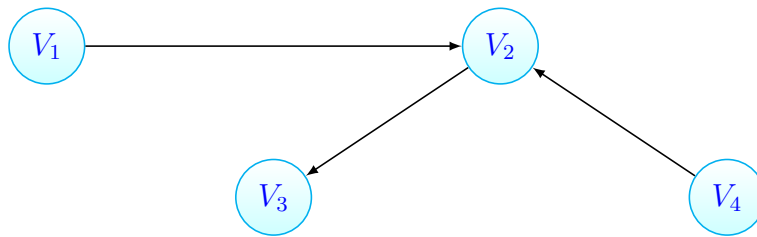


Figure 3.1: A directed graph

Definition 3.1.4. A multi graph is a graph whose edges have repeated elements.

In such a case, an edge is a multi edge if there is another edge sharing the same end vertices and so a multi graph is a graph with multiple edges and multiplicity of an edge of a graph, is the number of multiple edges sharing the same end vertices; like most of the time we would talk about the multiplicity of a graph, the maximum multiplicity of its edges etc. The diagram below shows a multi graph.

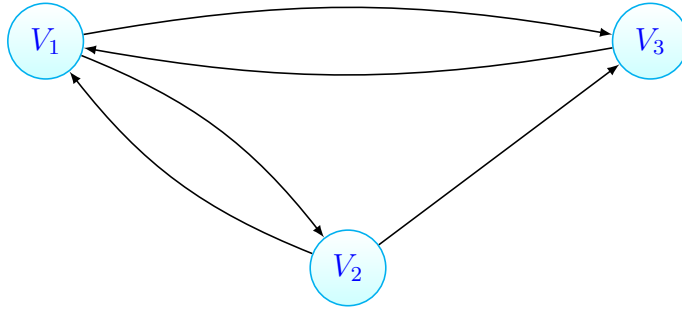


Figure 3.2: A multi-graph

Definition 3.1.5. *To say a graph has loops, we mean that the graph has edges which connect a vertex to itself.*

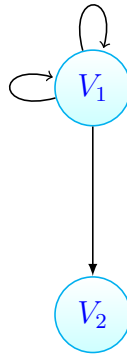


Figure 3.3: A Graph with loops

Definition 3.1.6. *The order of a graph is the number of its vertices, i.e. denoted as $|V|$.*

For instance the order of the digraph above is 4. That is to say $|V| = 4$.

Definition 3.1.7. *The size of a graph is the number of its edges, i.e. denoted as $|E|$.*

Thus the graph with loops drawn above is of size 3. Hence $|E| = 4$

Definition 3.1.8. *A graph is a simple graph if it has no multiple edges or loops.*

When stated without any qualification, a graph is almost always assumed to be simple. An example of a graph drawn below is a simple graph with vertex set $V = (1, 2, 3, 4, 5, 6)$ and edge set $E = \{(1, 2)(1, 4)(2, 3)(2, 4)(3, 5)(4, 5)(5, 6)\}$.

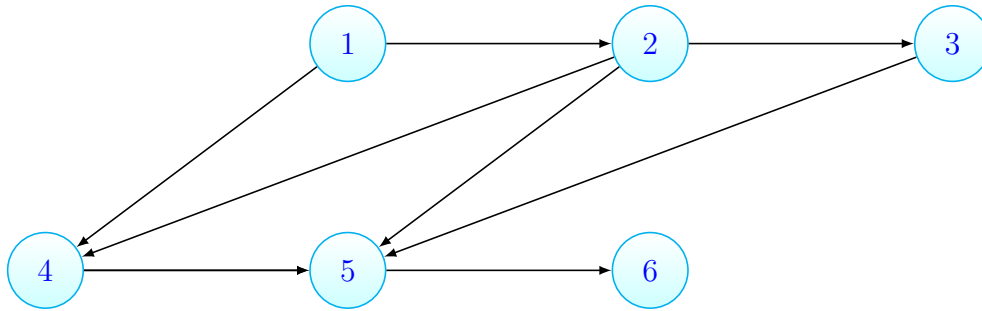


Figure 3.4: Simple Graph

A label may be used on either an edge or a vertex to either uniquely identify it or otherwise indicate meaning. Graphs with labelled edges or vertices are known as labelled graphs. Those without labels are referred to as unlabelled graphs. More specifically, graphs with labelled vertices only are vertex-labelled; those with labelled edges only are edge-labelled.

Definition 3.1.9. *A graph is finite if it has a finite number of vertices or edges or both.*

When stated without any qualification, a graph is usually assumed to be finite otherwise a graph is infinite, meaning that the graph in question has infinitely many vertices or edges or both.

Definition 3.1.10. *The degree of a vertex of a graph is the number of edges incident to the vertex, normally denoted as $\text{deg}(v)$.*

The diagram below illustrates a graph whose vertices are labelled by degree.

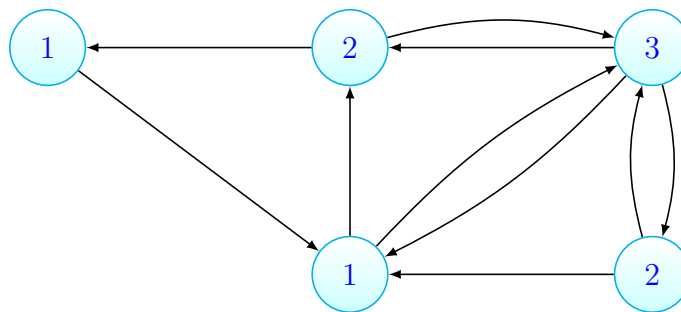


Figure 3.5: A graph whose vertices are labelled by degree

In graph theory, the degree matrix is a diagonal matrix which contains information about the degree of each vertex. i.e. the number of edges attached to each vertex.

Two vertices say u and v are called adjacent vertices if there is an edge existing between them. Hence the adjacency matrix is a way or a means of representing which vertices of a graph are adjacent to which other vertices.

Definition 3.1.11 ([6]). *Let Γ be a multi graph, with vertex set V and edge set E . Consider a graph H such that $V(H) \subseteq V(\Gamma)$ and $E(H) \subseteq E(\Gamma)$. Furthermore, if $e \in E(H)$ and $i(e) = \{u, v\}$, then $u, v \in V(H)$. Under these conditions, H is called a subgraph of Γ .*

Sandpile Graphs

Definition 3.1.12. *A sandpile graph is a finite, directed multi graph with a global sink.*

Suppose that $\Gamma = (V, E)$ is a sandpile graph with a sink vertex say s , then sandpile is a map

$$\sigma : V \rightarrow \mathbb{N}$$

such that $\sigma(v)$ equals the number of sand grains at vertex v

Definition 3.1.13. *Out degree of a sandpile graph is the total number of sand grains that a vertex gives to its neighbouring vertices.*

Definition 3.1.14. *Weight of a sandpile graph is the integer value assigned on the edges of the graph.*

Definition 3.1.15. *Let Γ denote a graph of any given order, Laplacian matrix is a matrix representation of Γ which is given by $L = D - A$.*

The D above denotes the degree matrix which is the diagonal matrix of out degrees of the vertices and A denotes the adjacency matrix which is the weight of the edges from one vertex to another. If there is no edge, the Adjacency matrix is taken to be 0 [7]. The Reduced Laplacian denoted by \tilde{L} is the sub matrix of the Laplacian matrix generated by deleting the row and column corresponding to the sink vertex.

3.2 Algebra

Algebra is a branch of mathematics in which symbols, usually letters of the alphabet represent numbers or members of a specified set, where these symbols are used to represent quantities and to express general relationship that hold for all the members of the set hence a precise definition in line with this project is as given below.

Definition 3.2.1. *Algebra is a set together with a pair of binary operations defined on a set.*

So one can simply say that Algebra is a branch of mathematics that deals with or studies abstract formal structures like sets, groups, rings, fields e.t.c.

Definition 3.2.2. *A binary operation $*$ on a set S is a function which sends elements of the cartesian product $S \times S$ to S*

$$\begin{aligned} * : S \times S &\rightarrow S \\ (s_1 s_2) &\mapsto s_1 + s_2 \end{aligned}$$

In other words, a binary operation $*$ on a set S is a calculation that combines for example, two elements of the set S to produce another element that belongs to the same set S . Since the result of performing the operation on a pair of elements of S is again an element of S , we say that the binary operation $*$ is closed. The binary operation $*$ is well defined on all of $S \times S$ since for instance, we do not consider division of real numbers because one can not divide a real number say k by a zero. i.e. $k/0$ is not defined for any real number k

Definition 3.2.3. *Magma is a set M matched with an operation \cdot which sends any two elements $a, b \in M$ to another element $a \cdot b \in M$. i.e. $a \cdot b \in M$ for all $a, b \in M$*

3.2.1 Semi Groups

Definition 3.2.4. *A semi-group is a magma where the operation is associative.*

So a semi-group is just a set equipped with an associative binary operation which may or may not have an identity element.

3.2.2 Monoids

Definition 3.2.5. *Monoids are semi-groups with identity elements.*

Note that, a monoid satisfies all the axioms of a group with the exception of having inverses. A monoid with inverses is the same thing as a group. A monoid whose operation is commutative is called a commutative monoid (or, less commonly, an Abelian monoid).

3.2.3 Groups

Definition 3.2.6. *Groups are Monoid with inverse elements.*

In particular, a group $(G, *)$ is a set G , closed under a binary operation $*$, such that the following axioms are satisfied:

- Closure

For all a, b in G , the result of the operation $a * b \in G$.

- Associativity

For all a, b and c in G , the equation $(a * b) * c = a * (b * c)$ holds.

- Identity element

There exists an element e in G , such that for all elements $a \in G$, the equation $e * a = a * e = a$ holds.

- Inverse element

For each $a \in G$, there exists an element $b \in G$ such that $a * b = b * a = e$ where e is the identity element.

Definition 3.2.7. *Abelian groups are groups whose binary operation is commutative.*

A group G with binary operation $*$ is Abelian if for any $a, b \in G$, we have that $a*b = b*a$. In other words, the binary operation $*$ is commutative.

3.2.4 Rings

Definition 3.2.8. *A ring $R = (R, +, \cdot)$ is an abelian group with respect to closure, associative $*$ and distributive law satisfied.*

For a ring with unity R , there exist 1 i.e. $1 \neq 0$ in R such that for all a in R , $a1 = 1a = a$ and a commutative ring R with unity is called an integral domain if the ring R satisfies the cancellation law:

for all $a, b, c \in R$, with $c \neq 0$, $ca = cb \Rightarrow a = b$ or we can refer to this property as no divisors of zero: this implies that, for all a, b in R , $ab = 0 \Rightarrow a = 0$ or $b = 0$

A commutative ring R with unity is called a field if there is the existence of inverses:

For all $a \neq 0$ in R , there exists a^{-1} in R such that $aa^{-1} = 1$ or perhaps write $a \cdot 1 \div a = 1$

Definition 3.2.9. *Free Abelian Group is an abelian group with a basis.*

A free abelian group is thus a set with associative, commutative and invertible binary operation and its basis is a subset of its element such that every element of the group can be written in only one way, and that is, as a linear combination of basis elements with integer coefficients, finitely many of which are nonzero.

Let X be any finite set and $\mathbb{N}X = \{\sum a_x x : a_x \in \mathbb{N}\}$ for all $x \in X$ be the free Abelian group on X restricted to nonnegative coefficients. We define $deg(a) = \sum a_x$ and $a \geq b$ if $a_x \geq b_x$ for all $x \in X$ and $a, b \in \mathbb{N}X$. Then we have the following definition.

Definition 3.2.10. *The support of a is given by $supp(a) = \{x \in X : a_x \neq 0\}$.*

Definition 3.2.11. *A lattice in \mathbb{R} is an n dimensional additive free group over \mathbb{Z} which generates \mathbb{R}^n over \mathbb{R} .*

Example 3.2.12. *The following is an example of a lattice $L \subset \mathbb{R}^2$ which is generated by $k_1 = (1, 2), k_2 = (4, 1)$. Hence $L = \{\alpha k_1 + \beta k_2 : \alpha, \beta \in \mathbb{Z}\}$*

Definition 3.2.13 ([1]). *An ideal I of a ring A is a subset of A which is an additive subgroup and is such that*

$$AI \subset I$$

and we write $x \in A$ and $y \in I \Rightarrow xy \in I$.

In other words, A nonempty subset I of a ring A is called an ideal, which we denote by $I \triangleleft A$, if for all $x, y \in I, x + y \in I$ and $-x \in I$ which is equivalent to writing for all $x, y \in I, x - y \in I$ and finally if $x \in A$ and $y \in I \Rightarrow xy \in I$

3.3 Algebra and Geometry

The link between Algebra and Geometry is established by Gauss' Fundamental Theorem of Algebra which states that a polynomial in one variable over \mathbb{C} , an algebraic object, is determined up to a scalar factor by the set of its roots (with multiplicities), a geometric object. This link between Algebra and Geometry is further extended to certain ideals of polynomials in many variables as a consequence of the Fundamental Theorem of Algebra since it holds for any algebraically closed field. Algebraic geometers basically study loci of polynomials and more precisely they study geometric objects called algebraic varieties. An algebraic variety is a geometric object that locally resembles the zero locus of a collection of polynomials [19].

Let Γ be a finite directed multi graph with a vertex v which is accessible i.e each vertex $v_i \in V$ apart from the vertex v has a directed path to v . This graph has an associated commutative sandpile group. The sandpile group can be obtained from the sandpile semi-group by the use of graphs. For example if we consider a graph with three vertices with one of them being a sink then we can obtain the sandpile group associated with this graph

on a cartesian plane with either of the vertices being each of the axis of the cartesian plane respectively.

Also the study of these groups is made precise by the use of Cayley tables, as you will realize, the vertices of this table correspond to the elements of that group and the edges correspond to the operation of the group. For sure Geometry makes things simpler, from drawing of graphs to obtaining the sandpile groups associated with such graphs to studying their structures and properties through Cayley tables.

A polynomial $f \in k[x_1, \dots, x_n]$ actually defines a function $f : k^n \rightarrow k$; the value of f at a point $(a_1, \dots, a_n) \in k^n$ is obtained by substituting the a_i for the x_i in f . The function which is defined by f is called a polynomial function on the n - dimensional vector space k^n over k with values in k . In this case, these polynomials are continuous functions from k^n to k so their zero sets are closed. k^n is usually called the affine n - space over k .

ABELIAN SANDPILE MODEL

4.1 Abelian Sandpile

Definition 4.1.1. *This is the process in which the order of firing vertices that exceed their capacity do not matter, the resulting final stable configuration is always the same.*

Let Γ be a finite directed multi graph, for each pair of vertices say $v_1, v_2 \in V$ and a non negative integer $E(v_1, v_2)$, which is the number of arrows from v_1 to v_2 , we give the following definitions with respect to the Abelian Sandpile Model (ASM). We assume our graph Γ , is loop less and is strongly connected i.e there is a directed path from any vertex v_1 to any other vertex v_2 .

4.1.1 Configurations

Definition 4.1.2. *A Configuration/state on Γ is a sequence of non negative integers say $h = (h_1, \dots, h_n)$ mapping from V to $\mathbb{N} = \{1, 2, \dots\}$ by assigning to each site a natural number $h_i \geq 1$, the number of sand grains at site i .*

We denote the set of all configurations by $\chi_G = \{0, 1, \dots\}$, therefore a configuration is an element of $\mathbb{N}^{\tilde{V}}$.

A divisor on Γ is an element in the free Abelian group on all of the vertices i.e. It is an

element of the $\mathbb{Z}V$ but in this case we think of the divisors as assigning negative integers to the vertices signifying debt.

Definition 4.1.3. *Suppose h is a sandpile configuration on a given graph Γ then the support of h denoted as $\text{supp}(h)$ is the set of all $v \in V$ which hold at least one grain of sand.*

Definition 4.1.4. *A configuration is a stable configuration if for all the ordinary vertices $i, 0 \leq h_i < \text{deg } i$, where deg denotes the out-degree.*

We denote the set of stable configurations by $\Omega_G = \{0, 1, \dots, \text{deg}_G - 1\}$ while a configuration is said to be unstable if $h_i \geq \text{deg } i$ for an ordinary vertex i which topples sending one or more grains through each edge leaving i .

Definition 4.1.5. *Recurrent configurations are configurations which appears infinitely often as the process of addition of sand followed by stabilization is repeated indefinitely.*

Consider an experiment where grains of sand are dropped one at a time onto a graph, pausing to allow the configuration to stabilize between drops. Some configurations appear only once in this process, we call them transient configurations. That is, for most graphs, once sand is dropped on the graph, no addition of sand then followed by stabilization will result in a graph empty of sand.

Consider the simple set R . First of all we know that $|R| = N + 1$ where

$$N + 1 = \det(\Delta)$$

We define the matrix Δ as, $\Delta_{ii} = 2, \Delta_{ij} = -1$ for $|i - j| = 1$ where $i, j \in \{1, \dots, N\}$.

R is the set of recurrent configurations and from above the number of the recurrent configurations gives the order of G , where G is the abelian Sandpile group. This is due to the facts that if there are any two recurrent configurations, say h_1 and h_2 then an element $g \in G$ is such that $h_2 = gh_1$, so we conclude that $|R| = |G| = \det(\Delta)$

A configuration h is recurrent if it is stable and if given any configuration h_1 , there exists a configuration h_2 such that $h = \text{stab}(h_1 + h_2)$. Hence we conclude with the following proposition:

Proposition 4.1.6. *A configuration h is recurrent if and only if there exist a nonnegative configuration h_1 such that $h = (h_1 + h^{\max})$ [9].*

In fact, these recurrent configurations form a submonoid of a monoid. To be precise, they form a group called the Sandpile group, denoted by $S(G)$.

Remark 4.1.7.

1. A configuration h is recurrent if there is a nonempty sequence of operations B_i leading from h to itself.
2. The recurrent configurations can all be reached one from the other.
3. The number of recurrent configurations is: $(n + 1)^{n-1}$ [6].

Definition 4.1.8. *Maximal configuration is the maximal element of Ω_G denoted by h^{\max} .*

The maximal configuration is obtained by $h^{\max}(i) = \text{deg}i - 1$ for all $i \in V$. Suppose a configuration is maximal then it is stable and so no legal firing are required. Otherwise if we fire either of the vertices then we obtain a negative configuration on the fired vertex. But if we fire the vertices simultaneously, we obtain a non-negative configuration: the zero configuration. Clearly h^{\max} is recurrent. We show the fact that h^{\max} is always recurrent. Assume that h^{\max} is any configuration h . Then a stable configuration h is recurrent implies there exists some stable configuration h_1 such that $h = (h_1 + h^{\max})^\circ$. Therefore h_1 must be e so that $h = (e + h^{\max})^\circ$ which implies that $h = (h^{\max})^\circ$ i.e. $h = h^{\max}$ hence we conclude that h^{\max} is always recurrent.

Definition 4.1.9. *Burning configuration is a nonnegative integer-linear combination of the rows of the reduced Laplacian matrix with nonnegative entries such that every vertex has a path from some vertex in its support.*

The corresponding burning script gives the integer-linear combination needed to obtain the burning configuration and so for example, if b is the burning configuration, σ is its script, and \tilde{L} is the reduced Laplacian, then $\sigma\tilde{L} = b$.

The following are equivalent for a configuration c with burning configuration b having script σ :

1. c is recurrent;
2. $c + b$ stabilizes to c ;
3. the firing vector for the stabilization of $c + b$ is σ .

The minimal burning configuration is one with the minimal script (its components are smaller than the components of any other script for a burning configuration). In order for us to obtain the minimal burning configuration, we settle for b equal to the sum of the columns of $\tilde{\Delta}$ where $b \geq 0$. But we compute $b + \tilde{\Delta}(v)$ if $b_v < 0$ for some $v \in \tilde{V}$, then we repeat the process until $b \geq 0$. Hence obtained the minimal burning configuration and the burning script σ_b records the columns of $\tilde{\Delta}$ that are used to obtain b .

Theorem 4.1.10. *The collection of recurrent configurations of Γ form a group under a stable addition [9].*

First of all, we note that the collection of all stable configurations forms a commutative monoid with addition which we define as point wise addition followed by stabilization. The identity element of this monoid is the all zero configuration and this monoid becomes a group when the associated graph is a directed acyclic graph. By Proposition 4.1.6, the Sandpile group is formed by systematically adding sand to h^{max} then toppling and stabilizing. Therefore considering a connected multi graph with a common sink as shown below, every Abelian group is actually the Sandpile group for some graph. For example:

The out-degree matrix is, $D = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$

adjacency matrix, $A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$

and so the Laplacian matrix is $L = \begin{pmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{pmatrix}$ $\det L = 16$

is the order of the sandpile group which is associated with the following graph.

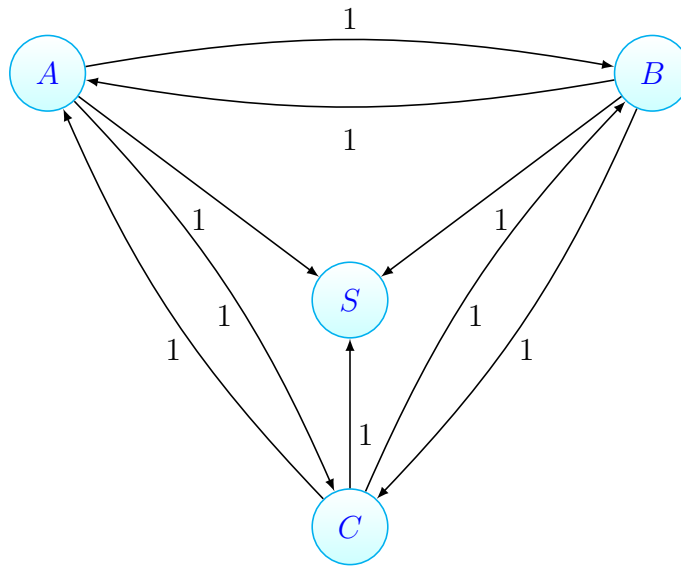


Figure 4.1: Sandpile group of order 16.

4.1.2 Toppling

Toppling is the cascading reaction of the sand particles when the slope of the pile exceeds a specific threshold value thereby collapsing transferring sand into the adjacent sites. A sequence or series of toppling as a result of adding a single particle of sand to the pile is

what we call an Avalanche. If $h \in \chi_G$ and $h_i \geq \deg_i$ then i topples sending one grain along each edge incident with which can be written as $h_j \rightarrow h_j - \Delta_{ij}$ and $j \in V$ where Δ_{ij} is the toppling matrix. The toppling matrix is the graph Laplacian:

$$(\Delta_G)_{ij} = \begin{cases} \deg_{(G)} i - a_{ii}, & \text{if } i = j \\ -a_{ij}, & \text{if } i \neq j \quad i, j \in V \\ 0 & \text{otherwise.} \end{cases} \quad (4.1.1)$$

Therefore the toppling rule corresponding to the toppling matrix Δ_{ij} in other words is the mapping T_x given by $T_x : \mathbb{N} \rightarrow \mathbb{N}$ defined by

$$T_x(h_j) = \begin{cases} h_j - \Delta_{ij}, & \text{if } h_i \geq \deg_G i \\ h_i = h_i & \text{otherwise.} \end{cases} \quad (4.1.2)$$

The toppling rule commutes on unstable configurations which means that for $x, z \in V$ and h such that $h(x) \geq \Delta_{x,x}$ and $h_z \geq \Delta_{z,z}$, we have that

$$T_x \circ T_z(h) = T_z \circ T_x(h) \quad (4.1.3)$$

The toppling rule is a single variable, labeled by the site index i such that $h_i < \bar{h}_i$. But when the inequality fails causing an instability in the height configuration at i , this instability is relaxed with a constant shift $h_j \rightarrow h_j - \Delta_{ij}$.

If we choose some enumeration $\{x_1, x_2, \dots, x_n\}$ of the set V , the toppling transformation is the mapping $T_\Delta : \mathbb{N} \rightarrow \Omega_G$ defined by

$$T_\Delta(h) = \prod_{i=1}^N T_{x_i}(h) \quad (4.1.4)$$

Remark 4.1.11. *It is not clear whether $T_\Delta(h) \in \Omega_G$ together with (4.1.4) defines the toppling transformation uniquely. Could it be that \mathbb{N} in (4.1.4) is not finite. But this cannot*

happen due to the presence of dissipative sites i.e. for any unstable configuration h there are (x_1, \dots, x_N) such that $\prod_{i=1}^N T_{x_i}(h)$ is stable. Is the N – tuple, $(x_1, \dots, x_N) \in V$ unique up to permutations? This brings us to the idea of the abelian property which actually shows that equation (3.4) defines a transformation properly from unstable configurations to stable configurations.

Theorem 4.1.12 ([30]). *The operator T_Δ is well defined.*

Proof. We want to show that the order of performing the topplings does not matter at all, because the same sites are normally toppled the same number of times yielding the same final configuration.

Let h be an unstable configuration and also let

$$T_{x_N} \circ \dots \circ T_{x_k} \circ T_{x_1}$$

and

$$T_{y_w} \circ \dots \circ T_{y_2} \circ T_{y_1}$$

be toppling sequences that are both stable. If both the toppling sequences are minimal such that

$$T_{x_i} \circ \dots \circ T_{x_2} \circ T_{x_1}$$

and

$$T_{y_j} \circ \dots \circ T_{y_2} \circ T_{y_1}$$

are not stable, for all $i < N$ and $j < w$. We show that $N = w$ and the sequences

$$x_1, \dots, x_N$$

and

$$y_1, \dots, y_w$$

are permutations of each other. Assume that N is minimal with the property that there exists a sequence

$$x_1, \dots, x_N$$

such that the toppling sequence

$$T_{x_N} \circ \dots \circ T_{x_2} \circ T_{x_1}$$

is stable; we can always use induction with respect to N . Let

$$y_1, \dots, y_w$$

be a sequence so that

$$T_{y_w} \circ \dots \circ T_{y_2} \circ T_{y_1}$$

is stable. Since $h(x_1) \geq \deg x_1$ then x_1 must appear at least once in the sequence

$$y_1, y_2, \dots, y_w.$$

Suppose we choose p to be minimal so that $y_p = x_1$ then

$$T_{y_w} \circ \dots \circ T_{y_{p+1}} \circ T_{x_1} \circ T_{y_{p-1}} \circ \dots \circ T_{y_2} \circ T_{y_1}$$

and

$$T_{y_w} \circ \dots \circ T_{y_{p+1}} \circ T_{y_{p-1}} \circ T_{x_1} \circ \dots \circ T_{y_2} \circ T_{y_1}$$

are the same. In order for this argument to be clear, define

$$h' = T_{y_{p-2}} \circ \dots \circ T_{y_2} \circ T_{y_1}.$$

Then x_1 has not been toppled at this point, therefore, we also have that

$$h'(x_1) \geq \Delta_{x_1, x_1}$$

and so we have that

$$h'(y_{p-1}) \geq \Delta_{y_{p-1}, y_{p-1}}$$

and so we can interchange T_{x_1} and $T_{y_{p-1}}$. Arguing in this manner repeatedly we conclude that

$$T_{y_w} \circ \dots \circ T_{y_{p+1}} \circ T_{y_p} \circ T_{y_{p-1}} \circ \dots \circ T_{y_2} \circ T_{y_i}$$

and

$$T_{y_w} \circ \dots \circ T_{y_{p+1}} \circ T_{y_{p-1}} \circ \dots \circ T_{y_1} \circ T_{x_1}$$

are the same stable configuration hence the proof. \square

Definition 4.1.13. *Toppling invariants are scalar functions over the space of all the semi-group configurations of the sandpile whose values are equal to the configurations equivalent under toppling.*

Given a toppling matrix we can obtain a minimal set of the toppling invariants. See [4].

4.1.3 Addition Operators

Definition 4.1.14. *Let h be a stable configuration, sand addition operator, b_i , is an addition operator such that the stable configuration $b_i(h)$ is the configuration realized after adding a sand grain to vertex i and then stabilizing. That is,*

$$b_i(h) = (h + \delta_i)^\circ.$$

Let h^i denote the configuration obtained from h by adding one grain to site i for $i \in V$

then the addition operator

$$b_i : \Omega \rightarrow \Omega$$

defined by

$$b_i(h) = S(h + \delta_i)$$

for $i \in \{1, \dots, N\}$ represents the effect of adding a grain to the stable site and letting the system to topple until a new stable configurations is obtained i.e. $b_i(h)$ is the stable result of an addition of grain at site i .

By the abelian property discussed above, the composition of addition operators is commutative, i.e. for all $i, j \in \{1, \dots, N\}$ we have that

$$b_i b_j = b_j b_i.$$

Denote by $R(h)$ the stable configuration obtained from the relaxation of the configuration h , so $R(h) \in S$ and

$$h \in S \Leftrightarrow z = R(h).$$

Suppose we take two configurations h and h' we can introduce the configuration $h + h'$ which has at each vertex i the height $h_i + h'_i$. Let e^i be the configuration which has non vanishing height only at the site i where it has height 1, that is

$$e_j^{(i)} = \delta_{i,j},$$

[19] then each configuration h can be obtained by depositing h_i particles at the vertex i . Hence

$$h_i e^i = \{e^i + e^i + \dots + e^i\}, h_i$$

times, so that summing on every vertex i , we obtain our h as

$$h = \sum_{i \in V} h_i e^i$$

4.1.4 Markov Chain

Definition 4.1.15 ([25]). *Markov Chain is defined as the process of adding a grain of sand followed by toppling on the space of stable configurations with a given equilibrium measure.*

Markov Chain gives the impression that even after a large amount of sand has been added, the system eventually reaches a stationary state.

Consider the operation K_i : assume h is a stable configuration then add a grain of sand to h_i after which perform topplings until a stable configuration is reached.

Example 4.1.16. *Consider Figure 4.1, if we take vertex C to be vertex K , then*

$$K_3(1, 2, 2) = (0, 2, 1) \rightarrow (1, 3, 2) \rightarrow (2, 0, 3) \rightarrow (3, 1, 0) \rightarrow (0, 2, 1),$$

is the Markov chain, while if we choose i at random, and then perform K_i defines the behaviour of the Markov chain.

In the theory of discrete-time Markov Chains, a state h is called recurrent if starting from the state h the process will almost surely (with probability 1) return to h in a positive number of steps.

Proposition 4.1.17. *A stable sandpile configuration h is recurrent in the sense that it is accessible from every configuration if and only if h is a recurrent state of the Sandpile Markov Chain.*

This is immediate from the following more general observation.

Let M be a finite Markov Chain with a set M of states. Let X be the transition digraph of M (edges indicate positive transition probability). Suppose there is a state h_0 that is

accessible from all states. Then a state $h \in M$ is recurrent in the sense of almost sure return if and only if h is accessible from all states.

Proof. Assume first that h is not accessible from all states and therefore not accessible from h_0 . Then with positive probability, the process, started from h , will reach h_0 without passing through h ; and then it can never return to h . Conversely, if h is accessible from all states then for some $t > 0$ and $p > 0$, the process, started from any state, will visit h within t steps with probability p . So the probability that no visit occurs within kt steps is $\leq (1-p)^k$ which goes to 0 as $k \rightarrow \infty$. See [20]

Having seen that in whatever way one organizes the activity of stabilization, one always end up with the same stable configuration. Thus the configuration at time n of this process is given by

$$h_n = S(h_0 + \sum_{i=1}^n \delta_{X_i}),$$

where X_i are the places we add a grain of sand and S is the stabilization. This defines the Markov chain on the space Ω where it is reducible because of the presence of the transient configurations. After stabilization, restricting to the recurrent configurations the Markov chain is irreducible. i.e every element of R can be reached from every other element of R .

□

4.1.5 Parking Functions

Definition 4.1.18 ([26]). *A parking function is a sequence of nonnegative integers, $u = u_1, u_2, \dots, u_n$ such that there exists a permutation $a = a_1, a_2, \dots, a_n$ satisfying: for all i we have that $u_i < a_i$.*

Example 4.1.19. *Let $u = \{1, 5, 2, 3, 5, 7, 0\}$ is a parking function if we use the permutation $a = \{5, 7, 3, 4, 6, 8, 1\}$ but if we use the permutation $b = \{2, 3, 1, 4, 1, 6, 5\}$, u ceases to be a parking function.*

Bijection between parking functions and recurrent configurations

Proposition 4.1.20. *The configuration $(u_1, \dots, u_i, \dots, u_n)$ is recurrent if and only if $(n - 1 - u_1, \dots, n - 1 - u_i, \dots, n - 1 - u_n)$ is a parking function.*

Proof: See Dhar's criteria, [2].

Consequence: The number of parking functions of length n is: $(n + 1)^{n-1}$. c.f. [26].

4.1.6 The Abelian Structure

Recall: Suppose we denote an operator which adds a grain of sand at vertex i by a_i , then if h is some given configuration,

$$a_i h = h + e^i.$$

If h is not stable at the vertex say j then

$$t_j a_i h = a_i t_j h.$$

Definition 4.1.21. *Let a_i be the addition of a grain of sand at vertex i which is stabilized, then $a_i h = (e^i + h)^\circ = \text{stab}(e^i + h) = R(e^i + h)$ does not depend on the order of the unstable vertices chosen for each toppling. Since our toppling commute, the stable configuration $a_i h$ is independent from the sequence of toppling. Hence the term Abelian Sandpile.*

Let α and β be two unstable sites of an unstable configuration h , toppling site α still leaves site β unstable and after the toppling of site β we get the same final stable configuration irrespective of which site is toppled first. Reasoning in a similar manner as above, if we topple site α followed by addition of a grain of sand at site β we obtain the same result for site β of the reversed ordered operation. Therefore for two operations a_i and a_j we obtain the configurations

$$a_i a_j h = a_j a_i h = R(e^i + e^j + h).$$

Thus the idea of abelianness is as a result of the property that the rule whether to topple a site only depends on the height at that site, and not on its neighbours.

Given two configurations h and h' we can always define an abelian composition $h \oplus h'$ of the height variables followed by stabilization. Thus for a configuration h , multiplication by a positive integer $m \in \mathbb{N}$ is defined as

$$mh = h \bigoplus \dots \bigoplus h$$

m times

4.2 Abelian Sandpile Model

For us to describe the abelian sandpile model, we begin with a sandpile graph which we denote by Γ . A sandpile graph (Γ) is a directed multi-graph with a vertex s that is accessible from every vertex apart from itself where s is the sink vertex and for example s is accessible from some vertex v means there is a sequence of directed edges that start from v and end at s . In this case, by saying our graph is multi-graph we mean that for every edge our graph possess, we assign a nonnegative integer weight.

The Sink

The Sink is one of the vertices of the Sandpile graph that collects the grains falling off the ordinary vertices and never topples. The presence of a sink vertex on a sandpile graph assures us that the period or the process of the above activity normally stops signifying stability of the sandpile graph. The sink is accessible from every non sink vertex under study and since it never fires, then every configuration will stabilize after a given finite number of vertex-firings. The resulting stabilization is independent of the order in which unstable vertices are fired. Thus, each configuration stabilizes to a unique stable configuration.

If a sink s has a directed path going into it from every other vertex, then s is called

a global sink. In our examples of graph, s is global because both v_1 and v_2 have an edge directed into it. Note that if a global sink exists it must be unique [21]. Suppose both s and s' are global sinks, then s must have a directed edge to s' , but this contradicts the definition of a sink.

The number of sand grains in the sink is not taken into account, i.e we assume them in our operations of the system, thus two configurations h and h' which differ only in position n are considered as equal; we write $u = v$ if

$$u_i = v_i$$

for all $i < n$. This translates the fact that the sink collects all grains of sand getting out of the system.

We define the Abelian Sandpile Model on a finite, connected multi-graph $G = (V, E)$ that has a distinguished vertex s , called the sink. We write $\tilde{V} = V \setminus \{s\}$. We assume G has no loop-edges and write $a_{xy} = a_{yx}$ for the number of edges between x and y in G , where $x, y \in V$. Also our graph is directed meaning $\vec{G} = (V, \vec{E})$ is obtained from G by replacing each edge by an arrow or by directed edges in each direction. A directed edge $e \in \vec{E}$ is written as $e = [e_-, e_+]$ to specify an oriented edge by its tail and head.

4.2.1 The Sandpile Monoid

Definition 4.2.1. *The Sandpile monoid is defined as the set of stable states under the operation of point wise addition and stabilization where this operation is denoted by \oplus and the all zero state is the identity element of the Sandpile monoid.*

Let $\Gamma = (V, E)$ be a graph with an accessible sink vertex s then $\tilde{\Gamma} = (\tilde{V}, \tilde{E})$ denotes the sub graph of Γ obtained from the ordinary vertices only. Thus the sandpile monoid is the commutative monoid generated by $\{a_x : x \in \tilde{V}\}$ subjected to the set $R = \{deg(x)a_x = \sum_{y \in \tilde{V}} b_{x,y}a_x : x \in \tilde{V}\}$.

Thus M is a sandpile monoid because if h_1, h_2, h_3 are sandpile configurations then

$$((h_1 + h_2)^\circ + h_3)^\circ = (h_1 + h_2 + h_3)^\circ = (h_1 + (h_2 + h_3)^\circ)^\circ$$

the asociative binary operation holds and

$$(h_1 + 0)^\circ = h_1$$

the identity element exists.

Notice that if the set of all configurations is denoted by H , then $\sigma : H \rightarrow M$ is a homomorphism. That is for any $h, h' \in H$, $\sigma(h+h') = \sigma(h) \oplus \sigma(h')$. In particular, $\sigma : H \rightarrow M$ is surjective since for $h \in M$ we have that $\sigma(h) = h$. As mentioned earlier, the order of the sandpile monoid is given by $|M| = \prod_{i \in V} \text{deg}(i)$.

4.2.2 The Sandpile Semi Group

Definition 4.2.2. *The sub-semigroup generated by the non-zero states of the Sandpile monoid is the Sandpile semigroup.*

We say that the set S is an ideal of M because $S = M \setminus \{0\}$, but we know that $H \setminus \{0\}$ is an ideal in the free commutative monoid H . See [20].

Definition 4.2.3. *A configuration h of the sandpile semigroup is said to be idempotent if $h^2 = h$.*

Every finite sandpile semigroup has at least an idempotent configuration.

4.2.3 The Sandpile Group

Definition 4.2.4. *A stable configuration is recurrent or critical if it is accessible from every state.*

We say that a configuration w is recurrent if it is stable and given any configuration k there is a configuration p so that $w = stab(k + p)$, where $stab(k + p)$ simply means the stabilization of $k + p$. A stable state which is not recurrent is called transient state. The set of all the recurrent configurations forms the Sandpile group which actually satisfies the axioms of a group as we demonstrate in a while.

We capture the structure of our graph in a matrix form by the use of a graph Laplacian which is an $n \times n$ matrix. Refer to the graph in Figure 4.5 the out degree of both vertices is 2 while our sink vertex has out degree 0, these out degrees sits in the diagonal of D then we compute the Adjacency matrix by considering the weight of the edges from one vertex to another as defined earlier and subtract. This gives us

$$\Delta = D - A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ 0 & 0 & 0 \end{pmatrix}.$$

We notice that the rows of the Laplacian matrix sums to zero. This tells us that the amount of sand that topples off an unstable vertex is the same amount gained by its adjacencies [24].

We obtain the reduced Laplacian of Γ by deleting row and column corresponding to the sink.

$$\text{E.g. } \tilde{\Delta} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}.$$

Notice again that we can always recover our original Laplacian matrix from the reduce Laplacian matrix by just adding a row of zeros to the bottom of $\tilde{\Delta}$ and adding a new column with values that ensure all rows sums to zero. Thus we state the following theorem

Theorem 4.2.5 ([13]). *The determinant of the reduced Laplacian matrix gives the order of the sandpile group.*

Proof. Suppose two grains of sand are added to a vertex i so that it exceeds its capacity by the two grains of sand added, then it will topple and give a grain of sand each to his neighbours hence

$$a_i^2 = a_{i-1}a_{i+1}$$

Note that, the toppling of sand on a vertex v is equivalent to subtracting the corresponding row of the reduced Laplacian from the configuration h . Therefore using the toppling matrix discussed in §4.1.2 where in this case the toppling matrix satisfies $\Delta_{ii} > 0$ and $\Delta_{ij} > 0$ for $i \neq j$, we summarize it as

$$a_i^{\Delta_{ii}} = \prod_{j \in V, j \neq i} a_j^{-\Delta_{ij}} \quad (4.2.1)$$

for all $i \in V$, where toppling at site j for instance, implies that

$$h_j \geq \text{outdeg}(j)$$

then

$$h_i \rightarrow h_i - \Delta_{ij}$$

for all i .

If the toppling matrix is brought to act on the set of the recurrent configurations, we can always take the right hand side of (4.2.1) which is a site, to the left to obtain

$$\prod_{j \in V} a_j^{\Delta_{ij}} = e$$

for all $i \in V$. By commutativity and for all $n : V \rightarrow \mathbb{Z}$,

$$\prod_{i \in V} \prod_{j \in V} a_j^{n_i \Delta_{ij}} = e$$

Using $\Delta_{ij} = \Delta_{ji}$ and the definition $(\Delta n)_i = \sum_{j \in V} \Delta_{i,j} n_j$, to make work easier and things look simpler, we obtain

$$\prod_{i \in V} a_i^{(\Delta n)_i} = e$$

Conversely, if $\prod_{i \in V} a_i^{m_i}$ then there exists $n : V \rightarrow \mathbb{Z}$ such that $m_i = (\Delta n)_i$. This relation means that the only obvious additions on the recurrent set are the multiples of the matrix Δ . Suppose that

$$\prod_{x \in V} a_x^{m_x} = e$$

is acted upon a recurrent configuration h to give

$$\prod_{x \in V} a_x^{m_x^+} h = \prod_{x \in V} a_x^{m_x^-} h,$$

where

$$m = m^+ - m^- \in \mathbb{Z}V.$$

This means that the addition of m^+ or m^- leads to the same final stable configuration, say β . But there exists k^+, k^- non negative integer valued functions on V such that

$$h + m^+ - \Delta k^+ = \beta = h + m^- - \Delta k^-$$

Collecting like terms to obtain

$$m = m^+ - m^- = \Delta(k^+ - k^-)$$

Let W, G be groups and a homeomorphism Θ be such that $\Theta : W \rightarrow G$. Define $W = \{n : V \rightarrow \mathbb{Z}\} = \mathbb{Z}^V$ where our group operation is point wise addition, hence

$$n \mapsto \prod_{i \in V} a_i^{n_i}$$

So as we have seen, the

$$\ker(\Theta) = \Delta\mathbb{Z}^V = \{\Delta n : n \in \mathbb{Z}^V\}$$

Hence

$$G \cong \mathbb{Z}^V / \Delta\mathbb{Z}^V$$

Therefore we have that

$$|R| = |G| = |\mathbb{Z}^V / \Delta\mathbb{Z}^V| = \det(\Delta)$$

□

As mentioned earlier, this is a consequence of the fact that if C and C' are any two recurrent configurations, then there is an element $g \in G$ such that $C' = gC$. See details in [14].

Result/ Observation: The product of the out degree gives the order of the sandpile monoid. The product of the maximal configuration gives the order of the transient elements. Therefore to get the order of the sandpile group associated with the graph under study, we subtract the order of transient elements from the order of the sandpile monoid.

Theorem 4.2.6. *Let Γ be a directed multi graph of the form*

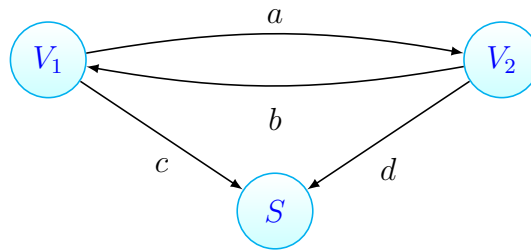


Figure 4.2: Sandpile graph 1

where v_1, v_2 are vertices and s is the sink vertex, then the order of the sandpile group associated with the above graph is given as $cb + ad + cd$.

Proof. We show that given any graph of the above type then we can always get the sandpile group associated with such a graph simply as the sum of the products of the configurations that goes to the sink vertex from the two non sink vertices, the cycle generated by V_1 through V_2 and the cycle generated by V_2 through V_1 . The out degree for V_1 is $a + c$, we can call it h_1 and the out degree for V_2 is $b + d$, we also call it h_2 . Suppose a grain of sand is added to vertex V_1 then the total configuration at V_1 becomes $h_1 + 1 > h_1$ hence it will topple to give a grain of sand each to its neighbours who in this case are the vertex V_2 and the sink vertex. We forget about all the grains of sand that fall into the sink vertex. Now $h_1 + 1$ represents the configuration obtained from h_1 after adding one grain of sand to site $V_1 \in V$ and so the addition operator $b_1 : \Omega \rightarrow \Omega$ is defined by $b_1(h) = S(h_1 + 1)$ thus toppling at site V_1 means that $h_1 + 1 > \text{outdeg}(V_1)$ hence

$$h_2 \rightarrow h_2 - \Delta_{1,2}$$

for all V_1 . Therefore our toppling matrix

$$\Delta_{1,2} = \begin{pmatrix} a + c & -a \\ -b & b + d \end{pmatrix}$$

gives the toppling rule

$$T_V(h_2) = h_2 - \Delta_{1,2}$$

which transforms vertex V_1 whose previous height configuration was unstable to

$$T_\Delta(h_1) = \prod_{i=1}^n T_{V_i}(h)$$

So V_1 is now stable because $T_\Delta : \mathbb{N} \rightarrow \Omega$. The order of all the stable configurations Ω as a result of the operation b_1 becomes

$$ab + cd + ad + cb$$

and since $|R| = \det \Delta$ as seen earlier, the order of the sandpile group associated with the above graph is

$$cb + ad + cd$$

while the number of the transient elements are ab . □

Corollary 4.2.7. *Let Γ be a directed multi graph of the form where v_1, v_2 are vertices and*

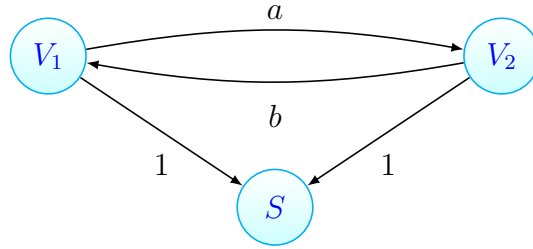


Figure 4.3: Sandpile graph 2

s is the sink vertex, then the order of the sandpile group associated with the above graph is given as $a + b + 1$.

Proof. The total out degrees for V_1 and V_2 are a and b respectively. Suppose V_1 is unstable, i.e. configuration at V_1 is say h_1 then since V_1 is unstable then it implies that $h_1 \geq a$. Firing at vertex V_1 occurs sending grains of sand to the neighbouring vertices, that is, vertex V_2 and the sink vertex thus the toppling matrix satisfies $\Delta_{ii} > 0$ and also Δ_{ij} for $i \neq j$. By this, we mean that toppling at V_1 , h_1 must decrease while the other site V_2 must increase height thus there is no creation of new sand piles in the toppling process. We obtain the order of the sandpile group associated with the above graph by obtaining the

$$\det(\Delta) = a + b + 1.$$

□

Example 4.2.8. *Suppose in the above graph, $a = 1, b = 1$ then the order of the sandpile group associated with this graph would be $1 + 1 + 1 = 3$. If $a = 2, b = 1$ then the order of the*

sandpile group associated with this graph is $2 + 1 + 1$. Thus for

$$a = x, b = y \Rightarrow |S(G)| = x + y + 1.$$

Notice that the revolution of the non sink vertices gives the order of the semi group associated with such a graph. Therefore we can obtain the order of the sandpile monoid associated with the same graph as the sum of the order of the sandpile group to the order of the sandpile semigroup. We can always choose a site at random in the above graph and increase its height configuration by 1 while the height configuration at the other site left unperturbed, then for all $a, b \in V$ where $a = b$, the number of sand grains that go to the sink vertex until the configuration is stable will always be the constant number 4 while for all $a, b \in V$ where $a \neq b$, the number of sand grains that go to the sink vertex until the configuration is stable will always be the constant number 5. This defines the following Lemma.

Lemma 4.2.9 ([21]). *The size of an avalanche is the sum of the number of times each vertex fires.*

From the above Lemma, we give the following definition

Definition 4.2.10. *Odometer of a sandpile β is a function on vertices defined by say $u(v)$ is equal to the number of times v topples during the stabilization of β .*

Definition 4.2.11. *A nonempty subset I of a monoid M is called a sandpile ideal, which we denote by $I \triangleleft M$, if for all $x, y \in I, x + y \in I$ and $-x \in I$ which is equivalent to writing for all $x, y \in I, x - y \in I$ and finally if $x \in M$ and $y \in I \Rightarrow xy \in I$*

Let $S(G)$ be a Sandpile group and S be the Sandpile Semigroup. Then $S(G)$ is a Sandpile Ideal of S since $S(G)$ is a subset of S and has the property that for all $a \in S(G)$ and $s \in S \Rightarrow sa \in S(G)$.

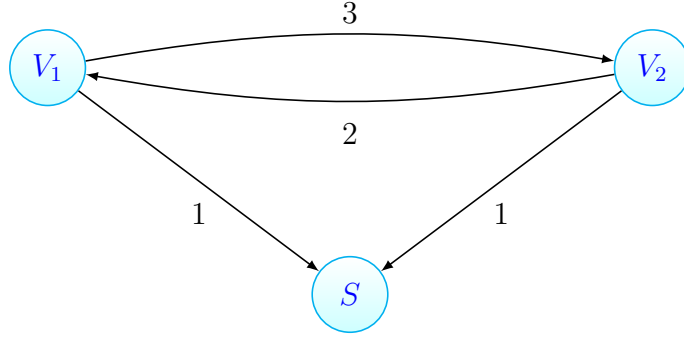


Figure 4.4: Illustrate an ideal of a sandpile group.

Example 4.2.12. In Figure 4.4, the Sandpile Monoid M is the set

$$M = \{(0, 0), (0, 1), (1, 0), (0, 2), (1, 2), (1, 1), (2, 0), (2, 1), (2, 2), (3, 0), (3, 1), (3, 2)\}.$$

The Sandpile Semigroup S is the set

$$S = \{(0, 1), (1, 0), (0, 2), (1, 2), (1, 1), (2, 0), (2, 1), (2, 2), (3, 0), (3, 1), (3, 2)\}$$

The Sandpile Group $S(G)$ is the set of the recurrent configurations

$$\{(2, 0), (2, 1), (2, 2), (3, 0), (3, 1), (3, 2)\}.$$

Then $S(G)$ is an ideal of S since for all $a \in S(G)$,

$$s \in S \Rightarrow sa \in S(G).$$

The following example illustrates a sample of how we obtain the sandpile ideal set.

$(2,0) + (1,1)=(3,1)$	$(2,0) + (0,1)=(2,1)$	$(2,0) + (0,2)=(2,2)$	$(2,0) + (1,0)=(3,0)$
$(2,1) + (1,1)=(3,2)$	$(2,1) + (0,1)=(2,2)$	$(2,1) + (0,2)=(2,0)$	$(2,1) + (1,0)=(3,1)$
$(2,2) + (1,1)=(3,0)$	$(2,2) + (0,1)=(2,0)$	$(2,2) + (0,2)=(2,1)$	$(2,2) + (1,0)=(3,2)$
$(3,0) + (1,1)=(2,1)$	$(3,0) + (0,1)=(3,1)$	$(3,0) + (0,2)=(3,2)$	$(3,0) + (1,0)=(2,0)$
$(3,1) + (1,1)=(2,2)$	$(3,1) + (0,1)=(3,2)$	$(3,1) + (0,2)=(3,0)$	$(3,1) + (1,0)=(2,1)$
$(3,2) + (1,1)=(2,0)$	$(3,2) + (0,1)=(3,0)$	$(3,2) + (0,2)=(3,1)$	$(3,2) + (1,0)=(2,2)$

Each of the sandpile ideal set turns out to be the sandpile group associated to this graph.

I is an additive subgroup of A and therefore we can form the quotient group

$$S/G = \{G + a : a \in S\},$$

which is actually the group of cosets of I in which we can also define addition by, suppose $a, b \in G$ then

$$(G + a) + (G + b) = G + (a + b).$$

The Laplacian lattice, $L \subset \mathbb{Z}V$, is the image of Δ . While the reduced Laplacian lattice, $\tilde{L} \subset \tilde{\mathbb{Z}}V$, is the image of $\tilde{\Delta}$. The critical group [5] for G is given by

$$C(G) = \tilde{\mathbb{Z}}V / \tilde{L}.$$

And the sandpile group for the graph Γ is given by

$$G = \tilde{\mathbb{Z}}V / \text{Row}(\tilde{\Delta}),$$

that is, the quotient group obtained after modding out by the integer row span, or image, of the reduced Laplacian of Γ [21]. Hence there is an isomorphism between the elements of the

sandpile group and the critical group $C(G)$. Therefore we conclude by giving the following theorem:

Theorem 4.2.13. *There is an isomorphism of Abelian groups $S(G) \rightarrow C(G)$, $c \mapsto c + \tilde{L}$.*

Thus, each element of \tilde{ZV} is equivalent to a unique recurrent element modulo the reduced Laplacian lattice. The identity of the sandpile group is the recurrent configuration in \tilde{L} . It can be calculated as

$$\varepsilon = ((h^{\max} - (2h^{\max})^\circ) + h^{\max})^\circ.$$

Since $h^{\max} - (2h^{\max})^\circ \geq 0$, then ε is always recurrent.[5] This is because a configuration h is recurrent if and only if a configuration $a \geq 0$ exists such that

$$h = (a + h^{\max})^\circ.$$

Theorem 4.2.14. *Let $S(G)$ be a Sandpile group and $a \in G$, then*

$$H = \{a^n : n \in \mathbb{N}\}$$

is a Sandpile subgroup of G and is the smallest Sandpile subgroup of G that contains a i.e. every subgroup containing a contains H .

A subset H of a Sandpile group G is a sandpile subgroup of G if and only if

1. H is closed under the binary operation of G
2. The identity element e of G is in H
3. For all $a \in H$ it is true that $a^{-1} \in H$. See [17].

So because $a^r a^k = a^{r+k}$ for r and $k \in \mathbb{N}$ then the operation in G of two elements of H is in H . Therefore H is closed under the operation of G . $a^0 = e$, then $e \in H$ and for $a^r \in H$, $a^{-1} \in H$ and therefore all the three conditions above are satisfied and H is a sandpile

subgroup of G . Since any sandpile subgroup of G containing a must contain H , so H is the smallest subgroup of G containing a .

Example 4.2.15. Consider the graph in Example 4.2.12 above.

This graph has an associated Sandpile group G of order six which consists of the following elements $\{(2, 0), (2, 1), (2, 2), (3, 0), (3, 1), (3, 2)\}$, let H be a subgroup of G with three elements namely $H = \{(2, 0), (2, 1), (2, 2)\}$, then H is actually the Sandpile subgroup of G since it satisfies the above three axioms and it is an easy exercise.

4.2.4 Generators of a sandpile group

Let G be a sandpile group, and an element $a \in G$, if order of the sandpile group G is equal to n , then G is a cyclic sandpile group if for some $a \in G, G = \{e, a, \dots, a^{n-1}\}$. For a practical example consider Example 4.2.15, the order of G denoted $\circ(G)$ is equal to 6 hence G is cyclic because there exists an element $(3, 1) \in G$ such that

$$G = \{a^6, a^4, a^2, a^3, a^1, a^5\}$$

in ascending order.

Theorem 4.2.16. Order of an element a^m is equal to the number of elements n of the sandpile group G divided by the greatest common divisor of m, n i.e $\circ(a^m) = n \div \gcd(m, n)$.

Therefore we say that a^m is a generator of G if and only if

$$\circ(a^m) = n \Leftrightarrow \gcd(m, n) = 1.$$

Thus from the above example, $|G| = n = 6$, hence $m = 1, 5$ and so our sandpile group in example (2.2) has two generators namely $\{(3, 1), (3, 2)\}$.

This leads us to a **corollary** [2]: If a is a generator of a finite cyclic sandpile group G of order n , then the other generators of G are the elements of the form a^r , where r is relatively

prime to n . The following illustration gives an explicit explanation. Suppose $a = (3, 1)$ is the generator of the cyclic sandpile group G then

$$\{a^1 = (3, 1), a^2 = (2, 2), a^3 = (3, 0), a^4 = (2, 1), a^5 = (3, 2), a^6 = (2, 0)\}$$

are elements in G . In this case $|G| = 6$ and numbers relatively prime to 6 are 1 and 5, i.e. numbers relatively prime to 6 are two, in other words, the number of the total items that are relatively prime to the order of the Sandpile Group gives the number of generators such a Sandpile group has, hence the sandpile group G has two generators $a^1 = (3, 1)$ and $a^5 = (3, 2)$. Alternatively if we first choose $a = (3, 2)$ to be the generator of the sandpile group G then the following are elements in G ,

$$\{a^1 = (3, 2), a^2 = (2, 1), a^3 = (3, 0), a^4 = (2, 2), a^5 = (3, 1), a^6 = (2, 0)\}$$

Thus $a^1 = (3, 2)$ and $a^5 = (3, 1)$ are the generators of the cyclic sandpile group G . Another thing about the numbers relatively prime to the Sandpile group is that they are actually the exponents to which we obtain our generators.

We study the canonical decomposition of $S(G)$ as a product of cyclic groups $S(G) \cong \mathbb{Z}_{d_1} \times \mathbb{Z}_{d_2} \times \dots \times \mathbb{Z}_{d_g}$ [28], where g is the least number of generators of $S(G)$ and d_i is a multiple of d_{i+1} and $d_i = \frac{w_{i-1}}{w_i}$ where w_i is the greatest common divisor of the determinant of $(N - i) \times (N - i)$ sub matrices of the toppling matrix Δ . Ideally we reduce the Laplacian matrix by some operations on the rows and columns to its Smith normal form, which is a diagonal matrix with integers $(d_1, \dots, d_{n-1}, 0)$ on the diagonal; then the group is $\bigoplus_{i=1}^{n-1} \mathbb{Z}_{d_i}$.

Recall that any finite abelian group G can be expressed as a product of cyclic groups as stated below:

Theorem 4.2.17 (The Fundamental Theorem of Finite Abelian Groups (FTFAG), [17]).

Let G be any finite abelian group of order greater than or equal to 2, G is isomorphic to a direct product of 1 or more cyclic groups whose orders are powers of primes: i.e. $G \cong$

$$\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \times \dots \times \mathbb{Z}_{p_n}.$$

The structure of $S(G)$ is determined in terms of the toppling matrix Δ as seen earlier. Let h_j be some fixed configuration in the set of sandpile semigroup and define an operator

$$w_i = \prod_{j=1}^N a_j^{A_{ji}}$$

for $1 \leq i \leq g$ to act on the configuration h_j , then w_i will produce a new configuration which is equivalent under toppling, to the configuration $(h_j + A_{ji})$. This is because if we topple some site say k then if site h is a neighbouring site to site k it becomes affected as say $h \equiv h_i$ is transformed into $h' \equiv (h_i - \Delta_{ik})$. If the g -uple corresponding to h_j is (I_1, I_2, \dots, I_g) where (I_i) is the minimal set of these toppling invariants, see [28] then $w_i(h_j)$ has toppling invariants i.e $I_k = I'_k + \delta_{ik}$. If this process is repeated sufficiently many times on h_j , then all the toppling invariants corresponding to h_j are obtained. Hence there is a one to one correspondence between the number of the g -uples (I_1, I_2, \dots, I_g) and the total number of the recurrent configurations. Therefore we conclude that the total number of recurrent configurations is equal to the number of the g -uples (I_1, I_2, \dots, I_g) .

Mark you, the operators a_j can be expressed in terms of the operators w_i as

$$a_j = \prod_{i=1}^g w_i^{(A^{-1})_{ij}}$$

for $1 \leq j \leq N$. Therefore the operators w generate the whole set $S(G)$ and hence $S(G)$ has a canonical decomposition as a product of cyclic groups.

We define a binary operation of addition on the set of the recurrent configurations by point wise addition followed by stabilization and since the invariants I_i provides an additive representation of the sandpile group $S(G)$, \oplus is a group law on R with the identity an easy exercise to obtain. Therefore

$$S(G) \cong \mathbb{Z}_{d_1} \times \mathbb{Z}_{d_2} \times \dots \times \mathbb{Z}_{d_g}.$$

We ask ourselves that suppose Γ_G is the sand pile graph associated to $\mathbb{Z}_2 \times \mathbb{Z}_3$, are the graphs of \mathbb{Z}_2 and \mathbb{Z}_3 sub graphs of Γ_G ?

Yes, the graphs of \mathbb{Z}_2 and \mathbb{Z}_3 are sub graphs of Γ_G . How about the Laplacian matrices?

The leading diagonal of the reduced Laplacian matrix of Γ_G is equal to the leading diagonal of the reduced laplacian matrix associated with \mathbb{Z}_2 multiplied by the leading diagonal of the reduced laplacian matrix associated with \mathbb{Z}_3 . The other diagonal tells us the number of transient configurations in a sandpile graph. The determinant of the reduced laplacian matrix associated with \mathbb{Z}_2 multiplied by the determinant of the reduced laplacian matrix associated with \mathbb{Z}_3 gives the product of the other diagonal of the reduced laplacian matrix of Γ_G . In other words, the determinant of each of the reduced laplacian matrices associated with \mathbb{Z}_3 and \mathbb{Z}_2 sits in the other diagonal of the reduced laplacian matrix of Γ_G respectively.

4.2.5 Automorphisms

Definition 4.2.18. *An automorphism of a group is actually an isomorphism of a group with itself. Let G be a sandpile group, then the mapping $\phi : G \rightarrow G$ which is one to one and onto is an automorphism.*

Theorem 4.2.19 ([17]). *Let $S(G)$ be a cyclic sandpile group with generator a , if the order of $S(G)$ is infinite, then $S(G)$ is isomorphic to $\langle \mathbb{Z}, + \rangle$, If $S(G)$ has finite order say n , then $S(G)$ is isomorphic to $\langle \mathbb{Z}_n, +_n \rangle$.*

The Abelian Sandpile Group $S(G)$, is defined uniquely up to isomorphism as seen earlier where we defined it through the Laplacian Matrix L of Γ . Recall that, if $V = \{v_1, \dots, v_n\}$ then the Laplacian matrix $L = (l_{ij})$ of size $n \times n$ is defined by its entries as

$$l_{ii} = \text{out deg}(v_i), l_{ij} = e(v_i, v_j),$$

if $i \neq j$. Delete the row corresponding to the sink vertex and also the column corresponding to the sink vertex from matrix L and call the resulting $(n - 1) \times (n - 1)$ matrix \tilde{L} . \tilde{L} is the

sublattice of \mathbb{Z}^{n-1} spanned by the rows of \tilde{L} hence the sandpile group is given by

$$S(G) = \mathbb{Z}^{n-1}/\tilde{L}$$

Suppose that $\acute{L} \subset \mathbb{Z}^n$ is the lattice spanned by the rows of L , then \acute{L} is a sublattice of $\mathbb{Z}_0^n = (a_1, \dots, a_n) : a_1 + \dots + a_n = 0$. Therefore we say that

$$S(G) = \mathbb{Z}_0^n/\acute{L}$$

Thus

$$S(G) = \mathbb{Z}^{n-1}/\tilde{L} \cong \mathbb{Z}_0^n/\acute{L}$$

Let $\Gamma = (V, E, s)$, we can label the vertices with $\{1, 2, \dots, n+1\}$ where $n+1$ is the sink vertex, we thus obtain the exact sequence for $S(G)$ associated with the sandpile graph Γ as shown [8]

$$0 \longrightarrow \mathbb{Z}^n \xrightarrow{\tilde{\Delta}} \mathbb{Z}^n \longrightarrow S(G) \longrightarrow 0$$

For if we consider the graph in Example 4.2.20, then we can form an exact sequence as follows:

$$0 \xrightarrow{k} \mathbb{Z}^4 \xrightarrow{\tilde{\Delta}} \mathbb{Z}^4 \xrightarrow{g} S(G) \xrightarrow{w} 0$$

We can check that $\ker(k) = \ker(\tilde{\Delta}) = \text{img}(k) = \text{img}(w) = 0$ and $\ker(g) = \ker(w) = \text{img}(\tilde{\Delta}) = \text{img}(g)$ Similarly, for the graph in Example ??, the exact sequence is given by

$$0 \xrightarrow{k} \mathbb{Z}^{12} \xrightarrow{\tilde{\Delta}} \mathbb{Z}^{12} \xrightarrow{g} S(G) \xrightarrow{w} 0$$

We study the structure of the Sandpile Group by considering the set R of recurrent configurations. We can define the inverse operator say a_i^{-1} for all i because each configuration in a cycle has exactly one incoming arrow corresponding to the operator a_i , [25].

We look at some of the fascinating group structure of the recurrent set R . Let us consider $N = 2$ for the sake of simplicity, where N is the number of non sink vertices. Define the operation \oplus on R by $(h_1 \oplus h_2) = S(h_1 + h_2)$ where the ordinary $+$ means pointwise addition. One can always pick at random the weights on the edges of the graph. So we consider an example where we pick weight 1 on the edges of our graph as shown below.

Example 4.2.20. Consider the graph drawn below

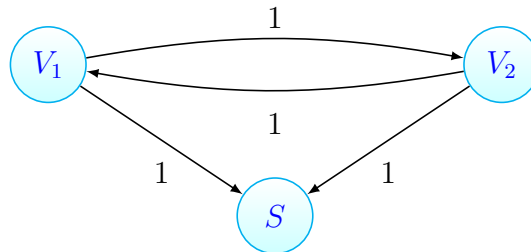


Figure 4.5: The abelian structure of the sandpile group

We obtain the reduced Laplacian matrix associated with the above graph as discussed in definition 3.4. Notice that the out degrees for both v_1 and v_2 equals 2 hence our $h^{\max} = (1, 1)$ and as proved earlier, the determinant of the reduced Laplacian matrix is equal to the order of the sandpile group associated with the above graph. Thus $(2 \times 2) - (-1 \times -1) = 3$ gives the order of the sandpile group.

This gives rise to the following table with three elements which completely satisfies the axioms of a group. We therefore seek to find an abelian group isomorphic to this sandpile group which turns out to be \mathbb{Z}_3 .

\oplus	(0,1)	(1,0)	(1,1)
(0,1)	(1,0)	(1,1)	(0,1)
(1,0)	(1,1)	(0,1)	(1,0)
(1,1)	(0,1)	(1,0)	(1,1)

Thus a Cayley table as shown above is of an Abelian group for example (R, \oplus) is an

Abelian group with the identity element $(1, 1)$. Notice that we can always define \oplus on the whole of the monoid set Ω , while (Ω, \oplus) is not a group but a monoid.

A general case for a directed multi graph $\Gamma = (V, E, s)$, see the figure above, the order of the sandpile group associated with this graph is given by either $c(b + d) + ad$ or $cb + d(a + c)$ for all $a, b, c, d \in \mathbb{N}$ as proved in Theorem 4.2.6.

Sandpile linear algebra

We briefly explain the linear algebra which is associated with the Abelian Sandpile Model.

- A configuration on a graph $\Gamma = (V, E)$ is a vector.
- A toppling consists of subtracting the vector Δ_i such that $\Delta_{i,i} = d_i \cdot \Delta_{i,j}$ is equal to the number of edges joining x_i and x_j .
- The stable addition is a vector addition in $\mathbb{N}\tilde{V}$ which is followed by stabilization.
- Two configurations are equivalent if one can be obtained from the other by adding a linear combination of Δ_i , for example, two stable configurations can be confirmed to be equivalent under toppling. We define an equivalence relation on the commutative semigroup by declaring that two configurations say, h_i and h'_i are equivalent if there exists some integers d_j such that $h'_i = h_i - \sum_j \Delta_{i,j}d_j$ for all i . This equivalence is called equivalence under toppling.
- This defines an equivalence relation and we have:

Theorem 4.2.21. *Any class contains exactly one recurrent configuration.*

The set of all configurations is a super lattice whose fundamental cell is the set R , the rows of Δ are the principal vectors of the super lattice and $\det \Delta$ is the volume of the fundamental cell, that is the number of stable recurrent configurations $|R|$.

4.3 Algebraic Geometry of the Abelian Sandpile Model

Algebra is but written Geometry; Geometry is but drawn Algebra. By Sophie Germain (1776 – 1831)

Fix k an algebraically closed field, this means that, every nonconstant polynomial $f \in k[x]$, has a root $x \in k$ i.e. $f(x) = 0$. Since any $f \in k[x_1, \dots, x_n]$ determines a function $k^n \rightarrow k$, we think of it as a set with a weaker structure and not as a vector space, hence elements of $k[x_1, \dots, x_n]$ can be understood as k valued functions on it. Therefore we denote this set differently as \mathbb{A}^n or as \mathbb{A}_k^n that moment we feel that we should not forget about our field k and thus we refer to \mathbb{A}^n as the affine n space over k .

4.3.1 Affine Spaces

Definition 4.3.1. *An Affine n - space over k is simply the set $A_k^n = \{(a_1, \dots, a_n) : a_i \in k\}$.*

Let $f \in k[x_1, \dots, x_n]$, then the zeroes of f is the set

$$Z(f) = \{p \in A^n : f(p) = 0\}$$

For example, consider a function

$$f = (x^2y) \in \mathbb{C}[x, y]$$

then the set

$$Z(f) = \{x, 0\} \cup \{0, y\}.$$

Suppose T is a subset of $k[x_1, \dots, x_n]$, $T = \{f_1, \dots, f_m\}$ where $f_i \in k[\bar{x}]$,

$$Z(T) = \{p \in A^n : f_i(p) = 0, i = 1, 2, \dots, m\}$$

which is the set of common zeroes in T . Let $T = \{f_1, \dots, f_m\}$ and I be an ideal generated

by T , then

$$I = \{a_1 f_1 + \dots + a_m f_m : a_i \in k[\bar{x}]\}.$$

Thus we can easily say that $Z(T) = Z(I)$, so we show that $Z(T) \subseteq Z(I)$ and $Z(I) \subseteq Z(T)$. Let's assume that $T = \{f_1, f_2\}$, let $x \in Z(T)$ then $f_1(x) = f_2(x) = 0$, $g \in I = \langle f_1, f_2 \rangle \Rightarrow g = a_1 f_1 + a_2 f_2 : a_1, a_2 \in k[\bar{x}]$, so $g(x) = a_1 x f_1(x) + a_2 x f_2(x) = 0$ by definition of $Z(T)$ hence $Z(T) \subseteq Z(I)$. Conversely, let $y \in Z(I)$, since $f_1 \in I$ and $f_2 \in I$ then the two functions vanish at y , thus $f_1(y) = 0$ and $f_2(y) = 0$ therefore $y \in Z(T)$ hence $Z(I) \subseteq Z(T)$.

Definition 4.3.2. *A subset Y of A_k^n is an algebraic set if there exists a subset T of a ring of polynomials $k[x_1, \dots, x_n]$ such that $Y = Z(T)$.*

Let A_1, \dots, A_n be algebraic sets, then

1. $A_1 \cup A_2$ is also an algebraic set.
2. $\bigcap_{i \in I} A_i$ is an algebraic set.
3. \emptyset and the whole space, A^n are algebraic sets.

The irreducible algebraic sets are called algebraic varieties i.e. irreducible in the sense that they can not be decomposed into union of two proper algebraic subsets. The affine varieties can be given the natural topology called the Zariski Topology by declaring that the algebraic sets are closed sets. This is possible since the set of all complements of affine algebraic sets satisfies the four axioms defining topology.

Consider any set $Y \subset k^n$; we therefore define the set

$$I(Y) = \{f \in k[x_1, \dots, x_n] : f(a_1, \dots, a_n) = 0\}$$

for all $(a_1, \dots, a_n) \in Y$ as an ideal.

Pick a graph with n vertices, we consider sand piles as some points in the affine n -space, A^n . Affine Transformation is the addition of sand particles to a particular vertex.

Definition 4.3.3 ([24]). *An affine transformation is a map between affine spaces and consists of a linear transformation, followed by a translation: $x \mapsto Ax + b$.*

The process of adding a grain of sand to a vertex on a graph is what we call a translation so we write $A = I$. In this project, we have defined sandpiles only over positive integers hence $h = Ix + b$ becomes our affine transformation. So we translate the sandpile via the rows of the reduced Laplacian until it is in the stable region. Considering this process, we are concerned about the order in which toppling the sand grains on the vertices might change the long-term behavior of the system. However, the toppling order turns out to be completely inconsequential.

If we consider graphs with only two vertices and a sink vertex and concentration on the non sink vertices. We visualize this group as lattice points in the affine plane. Choosing an arbitrary lattice point, we proceed to translate the sandpile by subtracting rows of the reduced Laplacian until we reach a sandpile in the stable region. **Note that:** If we begin at a sandpile with a particular shape and stabilize, we find that the shape associated to the stable sandpile matches the shape with which we started. The shapes denote the equivalence classes of the group.

So definitely we are considering a graph with three vertices where one of the vertices is an accessible sink. We want to visualize if the size of the sandpile group is equal to the determinant of its associated reduced Laplacian matrix in the affine space, off course we shall talk of the affine plane. So given Figure 4.4 and as defined earlier, a configuration is stable if for each vertex v we have any amount of sand say h such that $0 \leq h < outdeg(v)$. Therefore the possible stable configurations will be 4×3 which is basically the product of the out degrees of the vertices. So we plot all the possible stable configurations in the affine plane as shown below.

But some of these stable configurations are not recurrent. So we experiment by choosing several sand piles plotted and stabilizing following the rules of stabilization. We realize that the ones in dots we never reach them hence they are not recurrent. Notice that the off-diagonals of the reduced Laplacian gives the exact dimension of the square of transient configurations where the minus sign just signifies that this the the total number of sand

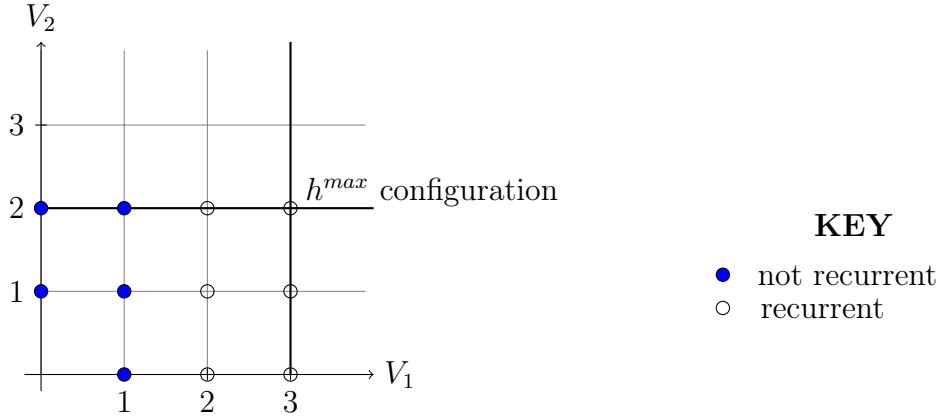


Figure 4.6: Affine plane

grains given to the other vertex and thus the order of the recurrent configurations can be counted directly from the plane by working out the determinant of the reduced Laplacian. i.e $(4 \times 3) - (2 \times 3)$.

Sandpile stabilization is unique as the solution to a system of linear inequalities.

Recall Figure 4.5 where $\tilde{L} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$

and let $c = [c_1, c_2]$ be a column vector which contains the coefficients of the columns of the reduced Laplacian that stabilize a sandpile h . Also, let d denote the vector of out-degrees of the vertices, i.e. the vector containing the diagonal entries of the reduced Laplacian. Then $h - \tilde{L}^T c \leq h^{\max}$ and $h - \tilde{L}^T c \geq 0$ respectively to mean that we must subtract off some linear combination of the rows of the reduced Laplacian until amount of sand falls below the out degree of the vertex, d . This is so because the number of sand grains left on vertex, v_i must be less than the out-degree of the vertex, so that the final result must not be negative. In this example, these inequalities explicitly become the two systems of inequalities:

$$\begin{aligned}
l_1 &= h_1 - 2c_1 + c_2 \leq 1 \\
l_2 &= h_1 + c_1 - 2c_2 \leq 1 \\
l_3 &= h_2 - 2c_1 + c_2 \geq 0 \\
l_4 &= h_2 + c_1 - 2c_2 \geq 0
\end{aligned}$$

Let $n = h^{\max}$, Γ be a graph with an accessible sink vertex denoted by s , then the full Laplacian of Γ is the mapping of groups $\eta : \mathbb{Z}V \rightarrow \mathbb{Z}V$ given on the vertices V defined by $\eta(V) = \text{outdeg}(v) - \sum_{u \in v} \text{wt}(v, u)u$. The reduced Laplacian of Γ is the mapping of groups $\tilde{\eta} : \tilde{\mathbb{Z}}V \rightarrow \tilde{\mathbb{Z}}V$ which is defined on non-sink vertices V by $\tilde{\eta}(\tilde{v}) = \text{outdeg}(v) - \sum_{u \in v} \text{wt}(v, u)u$ summing over \tilde{v} only. So by fixing an ordering say, v_1, \dots, v_{n+1} on the non-sink vertices of Γ , we identify \mathbb{Z}^v with \mathbb{Z}^{n+1} . Let $\tilde{L} \subset \mathbb{Z}^n$ symbolize the column-span of \tilde{L}^t which is the transpose of the reduced Laplacian, label vertex i with the indeterminate x_i , let $C[\Gamma_s] = C[x_1, \dots, x_n]$ then the Sandpile ideal or the toppling ideal is the lattice ideal for \tilde{L} defined by

$$I = I(\Gamma_s) = \{x^u - x^v : u - v \in \tilde{L}\} \subset C(\Gamma_s).$$

4.3.2 Toppling Ideals

Let G be a sand pile graph, associate to a configuration $h = (h_1, h_2, \dots, h_n) \in \mathbb{N}^n$, then we can form a monomial $x_h = x_1^{h_1} x_2^{h_2} \dots x_n^{h_n} \in \mathbb{Q}[x_1, \dots, x_n]$. Also associate to a toppling matrix Δ_i the binomial i.e a polynomial with utmost two terms (say, $ax^\alpha + bx^\beta$), $T(x_i) = x_i^{d_i} - \prod_j x_j^{e_{i,j}}$. So basically the addition of two configurations translates into the multiplication of the corresponding monomials while toppling vertex i in h translates into the division of x_h by $x_i^{d_i}$ followed by multiplication by $\prod_j x_j^{e_{i,j}}$ [25].

Recall: Given a vector $\beta \in \mathbb{Z}^n$, then $\beta = \beta^+ - \beta^-$ where we only consider the non negative integers β^+ and β^- thus for each i , we have that $\beta_i = \beta_i^+$ or $\beta_i = -\beta_i^-$.

Lemma 4.3.4 ([26]). *Let α, β, \dots be in \mathbb{Z}^n and \sim be the transitive closure of the relations: $u + \alpha^- = v + \alpha^+$, $u + \beta^- = v + \beta^+$, $\dots \in \mathbb{N}^n$. Then $u \sim v$ if and only if the binomial $\prod x_j^{u_j} -$*

$\prod x_j^{v_j}$ belongs to the ideal generated by the polynomials: $\prod x_j^{\alpha_j^+} - \prod x_j^{\alpha_j^-}$, $\prod x_j^{\beta_j^+} - \prod x_j^{\beta_j^-}$ in $\mathbb{Q}[x_1, \dots, x_n]$.

Definition 4.3.5. The toppling ideal I_G is generated by $x^b - 1$ and the toppling polynomials $T(x_i)$ for $i \in \{1, \dots, n\}$ where b is any burning configuration.[9]

The toppling ideal is a binomial ideal since it is generated by binomials. Suppose $J = T(x_i), i \in \{1, \dots, n\} + (x^b - 1)$ then $J \subseteq I_G$ hence I_G is the saturation of J with respect to the ideal (x_1, \dots, x_n) . It is sufficient to show that J is saturated with respect to that ideal I_G . See[9]

Proposition 4.3.6. Two configurations u and v are equivalent if and only if $x_u - x_v \in I_G$ or equivalently $u - v \in \Delta$.

This is a consequence of Lemma 4.26 since introducing Δ_n the required transitive closure is obtain.

A non homogenous polynomial $p = (x_1, \dots, x_n)$ can be homogenized by introducing an additional variable say x_0 and defining the homogenous polynomial $p^h = (x_0, \dots, x_n) = x_0^d p(\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0})$ where d is the degree of the polynomial p . Thus given any polynomial $p = x_3^3 + x_1x_2 + 7$, the homogenized polynomial becomes $p^h = x_3^3 + x_0x_1x_2 + 7x_0^3$.

Definition 4.3.7. Suppose $f \in \mathbb{C}[x_1, \dots, x_n]$, chose x_{n+1} to be an additional variable, then define the homogenization of f as $f^h = x_{n+1}^{deg f} f(\frac{x_1}{x_{n+1}}, \dots, \frac{x_n}{x_{n+1}})$. If $I \subseteq \mathbb{C}[x_1, \dots, x_n]$ is an ideal, the homogenization of I with respect to x_{n+1} is the ideal $I^h = (f^h : f \in I)$.

So studying the graph given in example 4.33, the toppling ideal is $f = [x_1^2 - x_2^3 - 1]$ and $f^h = x_0x_1^2 - x_2^3 - x_0x_1x_2$

APPLICATIONS

In this section, we study how the Abelian Sandpile Model can be used to understand certain abelian networks. The Abelian Sandpile Model exhibits self organized criticality. Grains of sand are added on a vertex, one at a time until the amount of sand exceeds a specific threshold value at which that site collapses transferring sand particles into the adjacent sites. If this causes the amount of sand particles the neighbouring sites to exceed their threshold, an avalanche of sand starts. The sand particles gets out of the system through the sink vertex, on a network without boundaries, so one can add boundary nodes during avalanche or at the end of avalanche and let sand dissipate gradually over time.

Definition 5.0.8. *Network is a system with complex structures where self organization occurs.*

Examples of Network includes internet, Neural network, world wide web, social networks etc.

Definition 5.0.9 ([3]). *An abelian network on a directed multi graph $\Gamma(V, E, s)$ is a collection of sand grains $W = (P_v : v \in V)$ indexed by the vertices of Γ , such that each P_v is abelian.*

An Abelian network is the collection of communicating processors as a single entity. These systems are categorized by the abelian property which states that, changing the order of certain interactions has no effect on the final state of the system. So our aim is to identify

explicitly what these various systems have in common and exhibit them as special cases of the abelian network.

5.1 Sandpile Networks

The sandpile network is driven by adding sand grains drop by drop to a randomly chosen nodes where in each node is stored in an integer value, the height of the sandpile columns. After a specified density, the nodes will have some threshold value for which if it exceeds the critical height, it is rendered unstable, it then topples and the sand grains flow from the toppled node to its adjacent nodes by links.

5.2 Toppling Networks

When the configuration of a particular site, equals or exceeds the specified threshold, toppling occurs sending the sand grains to the adjacent nodes. Since there is no rigid boundary for a network, the grains of sand are then distributed among the surrounding sites of the network. Dissipation of sand particles to those sites is made with an appropriate dissipation factor. If the toppling of one site causes the adjacent sites to become unstable then the process continues until there is no unstable node present in the system.

5.3 Sink Networks

Consider a sandpile network with a vertex which never sends any messages. Every vertex of Γ has a directed path to the sink. This implies that any finite input of sand grains into the system will produce a finite number of toppling.

So each processor in an abelian network performs the minimum amount of work possible to remove all the messages from the network.

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