ON EQUIVALENCE OF SOME OPERATORS IN HILBERT SPACES

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DECLARATION

This dissertation is my original work and has not been presented for a degree award in any other University

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DR. BERNARD M. NZIMBI

DEDICATION

This project is dedicated to my parents Mr. and Mrs. Muteti.

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ABSTRACT

Unitarily equivalence is the natural concept of equivalence between Hilbert space operators. Thus, this concept is the building block of other equivalence relations such as similarity, quasisimilarity, or even metric equivalence.

In this project, firstly, it is shown that unitary equivalence, similarity, quasisimilarity and almost similarity are equivalence relations. Then, through the Putnam-Fuglede theorem, similarity and hence unitarily equivalence of normal operators were discussed. In addition, it is shown that, reducing subspaces are preserved under unitarily equivalence and that, similarity preserves nontrivial invariant subspaces, while quasisimilarity preserves nontrivial hyperinvariant subspaces. Moreover, several known results, (but which are scattered in different accounts), such as those touching on; equality of spectra of quasisimilar hyponormal operators, direct summands in relation to almost similarity, and metric equivalence of operators preserves Fredholmness were presented.

As a consequence, more independent results were struck. For instance, a new equivalence relation, that is, unitary quasi-equivalence relation is introduced, and an observation which shows that, for an A-self-adjoint operator T, such that, T is metrically equivalent to S, then T^2 is similar to S^2, after demanding self-adjointedness of S, is deduced and proved.

LIST OF ABBREVIATIONS

$B(\mathcal{H})$: Banach algebra of bounded linear operators on $\mathcal{H}$

$\mathcal{H}$: Hilbert space over the complex numbers $\mathbb{C}$

T^* : the adjoint of T

$\|Tx\|$: the operator norm of T

$\|x\|$:the norm of a vector x

$< x, y >$:the inner product of x and y on a Hilbert space $\mathcal{H}$

$\rho(T)$:the resolvent set of an operator T

$\sigma(T)$: the spectrum of an operator T

$Ran(T)$: the range of an operator T

$Ker(T)$: the kernel of an operator T

$c.n.n$:completely non-normal

$c.n.u$:completely non-unitary

$\mathcal{M} \bigoplus \mathcal{M}^{\perp}$:the direct sum of the subspaces $\mathcal{M}$ and $\mathcal{M}^{\perp}$

$\{T\}'$:the commutator of T

n.h.s :nontrivial hyperinvariant subspace

a.s :almost similar

CONTENTS

Chapter 1

PRELIMINARIES

1.1 Introduction

Invariant subspaces play an important role in studying the spectral properties and canonical forms of operators. The motivations for studying invariant subspaces result from the interest in the structure of operators.The fact that every matrix on a finite dimensional complex vector spaces unitarily equivalent to an upper triangular matrix follows from the existence of nontrivial invariant subspaces for operators. The knowledge of hyperinvariant subspaces of T can give information on the structure of the commutant of T, which is useful as it contains all the quasiaffine transforms of an operator and its very nature reveals information about operators quasisimilar, similar or unitarily equivalent to T.

One way for constructing an invariant subspace for an operator on a Hilbert space is to find a second known operator, which is similar in some weak sense to the given operator and then use this second operator and the weak similarity to construct the desired subspace. For instance, one such weak similarity is the notion of quasisimilarity introduced by Sz- Nagy and Foias [29].

T.B Hoover [11] studied hyperinvariant subspaces and proved the result that if S and T are quasisimilar operators acting on the Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, respectively, and if S has a hyperinvariant subspace, then so does T. If in addition, S is normal, then the lattice of hyperinvariant subspaces for T contains a sub-lattice which is lattice isomorphic to the lattice of spectral projections for S.

Similar results for hyper-invariance have been studied by C.S. Kubrusly [17] and has shown that similarity preserves nontrivial subspaces while quasisimilarity preserves hyperinvariant subspaces.

Hoover [11] has shown some properties of operators that are preserved by quasisimilarity and those that are not. He gave a result to show that quasisimilar normal operators are unitarily equivalent. In addition, he showed that quasisimilar isometries are unitarily equivalent. He gave an example to show that quasisimilarity preserves neither spectra nor compactness.

Stampafli and Wadhwa [28] while working on hyponormal operators proved that hyponormal operator which is similar to a normal operator must be normal.

Wu [33] proved that if T is a contraction with finite defect indices, then T is quasisimilar to an isometry if and only if the completely non-unitary part of T is quasisimilar to an isometry. While working on the problem of writing an operator as a product of simpler operators, Wu[34] showed that a unitary operator on an infinite dimensional space is a product of (sixteen) positive operators, which is an unexpected result in finite dimensional Hilbert space. Balatine [1] showed that if an operator is identified with a finite square matrix, then it is the product of positive operators when its determinant is non-negative.

Wu[34] result was improved by Phillips [25] by showing that every unitary operator on an infinite dimensional Hilbert space is a product of six positive operators.

Some operator theorist have studied the open question of the existence of nontrivial invariant subspaces. Kubrusly [17] has showed that if a contraction has nontrivial invariant subspace, then it is either a C_{00}, C_{01} or a C_{10} contraction. Kubrusly and Levan [18] proved a similar result for the class of hyponormal contractions that if a hyponormal contraction T has a nontrivial invariant subspace, then it is either a C_{00} or a C_{10} contraction.

Duggal and Kubrusly [16] characterized the completely non-unitary part of a contraction using the Putnam-Fuglede (PF) theorem. Hoover [11] proved that quasisimilarity preserves nontrivial hyperinvariant subspace and Herrero showed that quasisimilarity does not preserve the full hyper-lattice.

Equality of spectra of quasisimilar normal operators was proved in [Douglus R.G [6], Lemma 4.1 p.683] but the result did not generalize to pairs of quasisimilar hyponormal operators as seen in [Halmos P.R [9], solution 156, p.309] and [Hoover T.B [11], Theorem 2.5 p.681]. Clary [4,Theorem 2, p.89] went ahead and showed that Quasisimilar hyponormal operators had equal spectra. In 1981, J.G. Stampfli [28] extended Clary's results on quasisimilar hyponormal operators where an inclusion relation for the spectra of quasisimilar operators satisfying Dunford's condition was obtained. It is in this paper that quasisimilarity and the unilateral shift were discussed. These results were extensively analyzed by A.J. Lambert in his thesis where he showed that any two unilateral weighted shifts which are quasisimilar are actually similar, but if unilateral shifts is replaced by bilateral as pointed out by L.A. Fialkow , the result fails.

A question on whether quasisimilar hyponormal operators had equal spectra as asked by S.Clary [4] was studied and answered by L.R.Williams [31],i.e, they in deed had equal spectra. He went further to give some results that relate quasisimilarity and hyponormal operators; for instance, if one of the quasi-affinities of the two hyponormals is compact, then they have equal essential spectra [32, Theorem A, p.126]. He extended the same results to dominant operators satisfying the Dunford's condition [32, Theorem 2.4, p.132](since every hypernormal operator is dominant by Theorem 1 of R.G.Douglas in his paper.

We also note from the proper inclusion relation for classes that

Normal $\subset$ Hyponormal $\subset$ Quasihyponormal.

This was a moltivation to Moo Sang Lee [20] to extend William's result[31] of equality of essential spectra of certain quasisimilar seminormal operators to quasisimilar quasihyponormal operators. He showed that quasisimilarity preserves Fredholm property [20,Theorem 2.6, p.94].

Conway's [5] and William's[32] results on the normal and pure parts were extended to the p-hyponormal operators by Jeon I.H and Duggal B.P [13] , where it has been shown that normal parts of quasisimilar p-hyponormal operators are unitarily equivalent.

A.Jibril [15], in 1996, introduced the class of almost similarity. He proved various results that relate almost similarity and other classes of operators, including isometries, normal operator, unitary operators, compact operators and characterization of θ- operators. θ-operators were extensively studied by Campbell[3]. Unitary equivalence of almost similarity of operators was also shown. In 2008, Nzimbi et al [23] results are also in handy in enriching almost similarity where he attempted to show that almost similarity implies similarity. Some properties of corresponding parts of operators which enjoy these equivalence relation are investigated. Unitary equivalence of completely non-unitary operators and quasi-triangular operators in relation to almost similarity is investigated.

Metric equivalence was introduced by Nzimbi et al [24] in 2013. He went further and showed that metric equivalence was in fact an equivalence relation. The spectral picture of metrically equivalent operators is discussed. He has also given some conditions when metric equivalence of operators implies unitary equivalence of operators we are given.

Unitary quasi-equivalence was introduced by Mahmoud [22] in 1998, and were also investigated by Othman, [25], in 1996 under the near equivalence relation.

1.2 Notations and Terminologies

In what follows, capital letters $\mathcal{H}, \mathcal{H}_1, \mathcal{H}_2, \mathcal{K}, \mathcal{K}_1, \mathcal{K}_2$, etc denote Hilbert spaces or subspaces of Hilbert spaces and $T, T_1, T_2, S, S_1, A, B$,etc denote bounded linear linear operators where by an operator we mean a bounded linear transformation from $\mathcal{H}$ into $\mathcal{H}$. By $B(\mathcal{H})$ denote the Banach algebra of bounded linear operator on $\mathcal{H}$. $B(\mathcal{H}_1, \mathcal{H}_2)$ denotes the set of bounded linear operators from $\mathcal{H}_1$ to $\mathcal{H}_2$. For an operator $T \in B(\mathcal{H})$, T^* denotes the adjoint of T, while $Ker(T)$, $Ran(T)$, $\overline{\mathcal{M}}$, $\mathcal{M}^\perp$ stand for kernel of T, range of T, closure of $\mathcal{M}$ and orthogonal compliment of a closed subspace $\mathcal{M}$ of $\mathcal{H}$, respectively. $\sigma(T)$ denotes spectrum of T, $\|T\|$, denotes the norm of T , $r(T)$, denotes the spectral radius of T while $W(T)$, denotes the numerical range of T.

Similarly, 0 and I will denote the zero and identity operator on $\mathcal{H}$, respectively.

An operator $T \in B(\mathcal{H})$ is said to be:

normal if $T^*T = TT^*$,

self-adjoint (or Hermitian) if $T = T^*$,

skew- adjoint if $T^* = -T$,

an **involution** if $T^2 = I$,

a **projection** if $T^* = T$ and $T^2 = I$,

unitary if $T^*T = TT^* = I$,

a **symmetry** if $T = T^* = T^{-1}$, that is, T is self-adjoint unitary,

isometric if $T^*T = I$,

a **partial isometry** if $T = TT^*T$, (equivalently, if T^*T is a projection),

quasi-normal if $T(T^*T) = (T^*T)T$ or equivalently, if T commutes with T^*T, that is, $[T, T^*T] = 0$,

binormal if $T^*T)(TT^*) = (TT^*)(T^*T)$,

A-self-adjoint if $T^* = A^{-1}TA$, where A is a self-adjoint invertible operator,

normaloid if $r(T) = \|T\|$, (equivalently, $\|T^n\| = \|T\|^n$).

In complex Hilbert space H, every normal operator is normaloid and so is every positive operator.

An operator $T \in B(\mathcal{H})$ is said to be:

a **scalar** if it is a scalar multiple of the identity operator, that is, if $T = \alpha I$ where $\alpha \in \mathbb{R}$,

subnormal if there exists a Hilbert space $\mathcal{K}$ containing $\mathcal{H}$, that is, $\mathcal{K} \supseteq \mathcal{H}$ and a normal operator N acting on $\mathcal{K}$ such that $\mathcal{H}$ is N- invariant and T is the restriction of N onto $\mathcal{H}$, that is, $T = N|_{\mathcal{H}}$. Thus, $T \in B(\mathcal{H})$ is subnormal if $\mathcal{H}$ is a subspace of a Hilbert space $\mathcal{K}$ ($\mathcal{H}$ can be embedded into K), and with respect to the decomposition $\mathcal{K} = \mathcal{H} + \mathcal{H}^{\perp}$, $N = \begin{bmatrix} T & X \\ 0 & Y \end{bmatrix}$ in $B(X)$ for some $X : \mathcal{H}^{\perp} \to \mathcal{H}$ and $Y : \mathcal{H}^{\perp} \to \mathcal{H}^{\perp}$. That is, T is a part of a normal operator.

Note that a part of an operator T is a restriction of it to an invariant subspace.

An operator $T \in B(\mathcal{H})$ is said to be:

hyponormal if $T^*T \geq TT^*$, equivalently, if $T^*T - TT^* \geq 0$ (is a positive operator),

cohyponormal if its adjoint is hyponormal, that is, T is cohyponormal if $TT^* \geq T^*T$. Clearly, if an operator $T \in B(\mathcal{H})$ is both hyponormal and cohyponormal, then T must be normal,

p-hyponormal if $(T^*T)^p \geq (TT^*)^P$, where $0 < p \leq 1$,

M-hyponormal if $\|(zI - T)^*x\| \leq M\|(zI - T)\|$, for all complex numbers z and for all $x \in M \subset \mathcal{H}$ and M a positive number,

quasihyponormal if $T^{*2}T^2 - (T^*T)^2 \geq 0$, equivalently if $T^*(T^*T - TT^*)T \geq 0$,

paranormal if $\|Tx\|^2 \leq \|T^2x\|$, for all unit vectors $x \in \mathcal{H}$, equivalently if $\|Tx\| \leq \|Tx\|\|x\|$, for every $x \in \mathcal{H}$,

k-quasihyponormal if $T^{*k}(T^*T - TT^*)T^k \geq 0$, for $k \geq 1$ is some integer, and every $x \in \mathcal{H}$,

p-quasihyponormal if $T^*((T^*T)^p - (TT^*)^p)T \geq 0$,

(p,k)- quasihyponormal if $T^{*k}((T^*T)^p - (TT^*)^p)T^k \geq 0$, where $0 < p \leq 1$ and k is a positive integer,

dominant if for any $\lambda \in \mathbb{C}$ there corresponds a number $M_\lambda \geq 1$ such that $\|(T -$

$\lambda I)^*x\| \leq M_\lambda\|(T-\lambda I)x\|$, for all $x \in H$,

seminormal if it is either hyponormal or cohyponormal, equivalently it either T or T^* is hyponormal.

Clearly, every hyponormal operator is seminormal but the converse is not true in general.

From the above definitions we have the following inclusions;

Unitary operators $\subseteq$ Isometric operators $\subseteq$ Partial isometries.

Normal $\nsubseteq$ Quasinormal $\nsubseteq$ Subnormal $\nsubseteq$ Hyponormal $\nsubseteq$ Seminormal.

An operator $T \in B(H)$ is **spectraloid** if $r(T) = w(T)$. Thus every normaloid operator is spectraloid.

An operator $T \in B(\mathcal{H})$ is a **contraction** if $\|Tx\| \leq \|x\|$ for every $x \in \mathcal{H}$.

An operator $T \in B(\mathcal{H})$ is a **left shift** on ℓ^2 if $Tx = y$ for all $x = (x_1, x_2, ...)$ and $y = (x_2, x_3, ...)$ while it is a **right shift** operator if $Tx = y$ where $x = (x_1, x_2, ...)$ and $y = (0, x_1, x_2, ...)$

A subspace $\mathcal{M}$ of $\mathcal{H}$ is invariant under T if $T(\mathcal{M}) \subseteq \mathcal{M}$, that is for $x \in \mathcal{M}$ implies $Tx \in \mathcal{M}$ for every $x \in \mathcal{M}$ or $T\mathcal{M} \subset \mathcal{M}$.

A subspace $\mathcal{M}$ of $\mathcal{H}$ is said to reduce T if both $\mathcal{M}$ and $\mathcal{M}^\perp$ are invariant under T. We say that an operator T is completely non-unitary if the restriction of it to any nonzero reducing subspace is not unitary. If $\mathcal{M}$ is an invariant subspace for T, then relative to the decomposition $\mathcal{H} = \mathcal{M} \bigoplus \mathcal{M}^\perp$, T can be written as

$$T = \begin{bmatrix} T|_{\mathcal{H}} & X \\ 0 & Y \end{bmatrix}$$

for operators $X : \mathcal{M}^\perp \to \mathcal{M}$ and $Y : \mathcal{M}^\perp \to \mathcal{M}^\perp$, where $T|_{\mathcal{M}}$ denotes the restriction of T to $\mathcal{M}$. Conversely, if an operator T can be written as the triangulation

$$T = \begin{bmatrix} T|_{\mathcal{M}} & X \\ 0 & Y \end{bmatrix}$$

in terms of the decomposition $\mathcal{H} = \mathcal{M} \bigoplus \mathcal{M}^\perp$, then $Z = T|_{\mathcal{M}} : \mathcal{M} \to \mathcal{M}$ is a part of T. $X = 0$ if and only if M reduces T. In such a case, the operator T is decomposed into the orthogonal direct sum of the operators $Z = T|_{\mathcal{M}}$ and $Y = T|_{\mathcal{M}}^\perp$:

$T = Z \bigoplus Y$

By a subspace of a Hilbert space $\mathcal{H}$ we mean a closed linear manifold of $\mathcal{H}$, which is also a Hilbert space. If $\mathcal{M}$ and $\mathcal{N}$ are orthogonal subspaces of a Hilbert space H, then their (orthogonal) direst sum $\mathcal{M} \bigoplus \mathcal{N}$ is a subspace of $\mathcal{H}$. For any subspace $\mathcal{M} \subseteq \mathcal{H}$, $\mathcal{M}^{\perp}$ will denote the orthogonal compliment of $\mathcal{M}$ in $\mathcal{H}$.

For $\mathcal{M}$ a closed subspace of $\mathcal{H}$, we have $\mathcal{H} = \mathcal{M} \bigoplus \mathcal{M}^{\perp}$ is called the direct sum decomposition of $\mathcal{H}$.

The following inclusions are proper:

Reducing subspaces $\subseteq$ Invariant subspaces,

Hyperinvariant subspaces $\subseteq$ Invariant subspaces.

A direct summand of an operator T is the restriction of it to a reducing subspace. An operator is reducible if it has nontrivial reducing subspace (equivalently, if it has a proper nonzero direct summand), otherwise it is irreducible.

A lattice $\wp$ is a partially ordered set such that every pair of elements of $\wp$ has a supremum (least upper bound) and an infimum(greatest lower bound) in $\wp$ (i.e. if there exists unique $a, b \in \wp$ such that $a = x \bigvee y$ and $b = x \bigwedge y$ for every pair $x, y \in \wp$). Note that, the set of all invariant subspaces for $T \in B(H)$ is a lattice. $\mathrm{Lat}(T)$ will denote the lattice of all invariant subspaces of T, that is, $\mathrm{Lat}(T) = \{\mathcal{M} \subseteq \mathcal{H} : T(\mathcal{M}) \subseteq \mathcal{M}\}$. If Λ is any subset of $B(\mathcal{H})$, we denote by Λ' the commutant of Λ, i.e. $\Lambda' = \{T \in B(H) : ST = TS \quad for \quad every \quad S \in \Lambda\}$. Specifically, $\{T\}' = \{S \in B(\mathcal{H}) : ST = TS\}$. The bicommutamt or double commutant of $T \in B(H)$ is defined and denoted by $\{T\}'' = \{A \in B(H) : AS = SA, \quad for \quad all \quad S \in \{T\}'\} = \{p(T) : T \in B(\mathcal{H}), p \quad a \quad polynomial\}$. A subspace $\mathcal{M} \subset \mathcal{H}$ is said to be a nontrivial hyperinvariant subspace (n.h.s) for a fixed operator in $T \in B(\mathcal{H})$ if $0 \neq \mathcal{M} \neq \mathcal{H}$ and $S\mathcal{M} \subset \mathcal{M}$ for each S in $\{T\}'$. The lattice of all hyperinvariant subspaces of T will be denoted by $HyperLat(T)$. A subspace lattice $\wp$ is called commutative if for every pair of subspaces of $\mathcal{M}, \mathcal{N} \in \wp$, the corresponding projections $P_{\mathcal{M}}$ and $P_{\mathcal{N}}$ commute. A lattice $\wp$ is said to be totally ordered if for every $\mathcal{M}, \mathcal{N} \in \wp$, either $\mathcal{M} \subseteq \mathcal{N}$ or $\mathcal{N} \subseteq \mathcal{M}$.

Let $\mathcal{H}$ be a Hilbert space and $T \in B(\mathcal{H})$. The set $\rho(T)$ of all complex numbers λ for which $(\lambda I - T)$ is invertible is called the resolvent set of T, that is, $\rho(T) = \{\lambda \in \mathbb{C} : Ker(\lambda I - T) = \{0\} \;\; and \;\; Ran(\lambda I - T) = \mathcal{H}\}$.

The compliment of the resolvent set $\rho(T)$ denoted by $\sigma(T)$, is called the spectrum of T. In other words, $\sigma(T) = \mathbb{C}/\rho(T) = \{\lambda \in \mathbb{C} : Ker(\lambda I - T) \neq \{0\} \;\; and \;\; Ran(\lambda I - T) \neq \mathcal{H}\}$, which is the set of all λ such that $(\lambda I - T)$ fails to be invertible, that is, fails to have a bounded inverse on $Ran(\lambda I - T) = \mathcal{H}$.

Thus the spectrum of T can be split into many but disjoint part.

The set of all those λ in complex numbers such that $(\lambda I - T)$ has no inverse, denoted by $\sigma_p(T)$ is called the point spectrum of T. Equivalently, $\sigma_p(T) = \lambda \in \mathbb{C} : ker(\lambda I - T) \neq \{0\}$ which is the set of all eigenvalues of T.

A scalar $\lambda \in \mathbb{C}$ is an eigenvalue of an operator $T \in B(\mathcal{H})$ if there exists a non-zero vector $x \in \mathcal{H}$ such that $Tx = \lambda x$. Equivalently, if $Ker(\lambda I - T) \neq \{0\}$. Note that in finite dimensional settings, the $\sigma(T) = \sigma_p(T)$.

The set of all those $\lambda \in \mathbb{C}$ for which $(\lambda I - T)$ has a densely defined but unbounded inverse on its image, denoted by $\sigma_c(T)$ is called the continuous spectrum of T. Equivalently, $\sigma_c(T) = \{\lambda \in \mathbb{C} : Ker(\lambda I - T) = \{0\} \;\; and \;\; \overline{Ran(\lambda I - T)} = \mathcal{H} \;\; and \;\; Ran(\lambda I - T) \neq \mathcal{H}\}$.

If $(\lambda I - T)$ has an inverse that is not densely defined then, λ belongs to the residual spectrum of T denoted by $\sigma_r(T)$. That is,$\sigma_r(T) = \{\lambda \in \mathbb{C} : Ker(\lambda I - T) = \{0\} \;\; and \;\; \overline{Ran(\lambda I - T)} \neq \mathcal{H}$.

These parts $\sigma_p(T)$, ,$\sigma_c(T)$, $\sigma_r(T)$ are pairwise disjoint and $\sigma(T) = \sigma_p(T) \bigcup \sigma_c(T) \bigcup \sigma_r(T)$. Thus the collection $\sigma_p(T)$, $\sigma_c(T)$, $\sigma_r(T)$ forms a partition of $\sigma(T)$.

Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces. An operator $T \in B(\mathcal{H}, \mathcal{K})$ is invertible if it is injective (one-to-one)and surjective (onto or has a dense range). Equivalently,if $Ker(T) = 0$ and $\overline{Ran(T)} = \mathcal{K}$.

Two operators A and B are said to commute if $AB - BA = 0$,denoted by $[A, B]$.

Two operators A and B are said to be similar (denoted by $A \sim B$) if there exists an invertible operator $N \in B(\mathcal{H}, \mathcal{K})$ such that $NA = BN$ or equivalently $A = N^{-1}BN$.

Two operators A and B are said to be almost similar(a.s), denoted by $A \overset{a.s}{\sim} B$ if there exists an invertible operator N such that the following two conditions hold:

$A^*A = N^{-1}(B^*B)N$

$A^* + A = N^{-1}(B^* + B)N.$

An operator $N \in B(\mathcal{H}, \mathcal{K})$ is quasi-invertible or a quasi-affinity if it is an injective operator with dense range (i.e $KerN = \{0\}$ and $\overline{RanN} = \mathcal{K}$); equivalently $Ker(N) = 0$ and $Ker(N^*) = 0$,thus $N \in B(\mathcal{H}, \mathcal{K})$ is quasi-invertible if and only if $N^* \in B(\mathcal{H}, \mathcal{K})$ is quasi-invertible.

An operator $A \in B(\mathcal{H})$ is a quasi-affine transform of $B \in B(\mathcal{K})$ if there exists a quasi-invertible operator $N \in B(\mathcal{H}, \mathcal{K})$ such that $NA = BN$ (N *intertwines* A *and* B). Thus, A is a quasi-affine transform of B if there exists a quasi-invertible operator intertwining A and B.

Two operators $A \in B(\mathcal{H})$ and $B \in B(\mathcal{K})$ are quasi-similar (denoted $A \approx B$), if they are quasi-affine transform of each other , equivalently, if there exists quasi-invertible operators $N \in B(\mathcal{H}, \mathcal{K})$ and $M \in B(\mathcal{K}, \mathcal{H})$ such that $AN = NB$ and $MB = AM$.

It is easily verified that quasisimilarity is an equivalence relation and also that T^* is quasisimilar to S^* whenever T is quasisimilar to S and that similar operators are, of course, quasisimilar but not conversely ([17]).

Quasisimilarity was introduced by Nagy and Foias [30] in their theory on infinite-dimensional analogue of the Jordan form for certain classes of contractions as a means of studying their invariant subspace structures. It replaces the familiar notion of similarity which is the appropriate equivalence relation to use with finite dimensional Hilbert spaces. In finite dimensional spaces, quasisimilarity is the same thing as similarity but in infinite dimensional spaces it is a much weaker relation.

Two operators $A \in B(\mathcal{H})$ and $B \in B(\mathcal{K})$ are unitarily equivalent ,(denoted $A \cong B$), if there exists a unitary operator $U \in B(\mathcal{H}, \mathcal{K})$ such that $UA = BU$, equivalently, $A = U^*BU$. Two operators are considered the "same" if they are unitarily equivalent since they have the same properties of invertibility, normality, spectral picture (norm, spectrum, spectral radius).

Two operators $A \in B(\mathcal{H})$ and $B \in B(\mathcal{H})$ are said to be metrically equivalent, denoted by $A \sim_m B$, if $\|Ax\| = \|Bx\|$, equivalently, $| < Ax, Ax > |^{1/2} = | < Bx, Bx > |^{1/2}$ for all $x \in \mathcal{H}$, that is, $A^*A = B^*B$.

Two operator, $S, T \in B(\mathcal{H})$ are said to be unitarily quasi-equivalent if there exists a unitary operator U such that $T^*T = US^*SU^*$ and $TT^* = USS^*U^*$ and write $S \overset{u:q:e}{\approx} T$. Clearly, $S, T \in B(\mathcal{H})$ are unitarily quasi-equivalent if S^*S and T^*T are unitarily equivalent and SS^* and TT^* are unitarily equivalent.

Chapter 2

ON SOME BASIC EQUIVALENCE RELATIONS

In this chapter, the norm, the numerical radius, the numerical range and the spectral radius of normal and hyponormal operators are investigated.The relations of similarity and almost similarity are discussed. Various results on similarity and almost similarity are presented and their proofs are shown. In addition, an attempt is made to classify those operators where almost-similarity implies similarity. We investigate some properties of corresponding parts of operators which enjoy these equivalence relations.

Recall that, two operators A and B are said to be similar if there exists a quasi-affinity X which intertwines A and B, (if the dimension of the Hilbert space $\mathcal{H}$ is finite or equivalently, if A and B on $B(\mathcal{H})$ are bounded, then quasisimilarity is the same as similarity), and if X happens to be unitary, then A and B are said to be unitarily equivalent.

Also recall that, A is almost similar to B if A^*A is similar to B^*B and A^*+A is similar to B^*+B.

2.1 Unitary and similarity of operators

We start off, by discussing the following result which shows that unitary equivalence is an equivalence relation.

Theorem 2.1.1 *Unitary equivalence is an equivalence relation.*

Proof We show that, unitary equivalence is $(i) Reflexive$, $(ii) Symmetric\ and$ $(iii) Transitive.$
(i) Reflexitivity; we show that, $T \cong T$.
Let $T \in B(\mathcal{H})$, if we let $U = I$, then we have, $T = ITI$. Thus $T \cong T$.
(ii)Symmetry; we show that, $T \cong S$ implies $S \cong T$.
Suppose that $T \cong S$. We show that $S \cong T$. Let $T \in B(\mathcal{H})$ and $S \in B(\mathcal{K})$, then there exists a unitary operator, say $U_1 \in B(\mathcal{H}, \mathcal{K})$, such that
$T = U_1^* S U_1$..(1).
Pre-multiplying and post-multiplying, (1) by U_1 and U_1^*, respectively yields, $U_1 T U_1^* {}_1 = U_1 U_1^* S U_1 U_1^*$, that is, $U_1 T U_1^* = ISI$, i.e, $U_1 T U_1^* = S$. This shows that $S \cong T$.
(iii)Transitivity; we show that, if $T \cong S$ and $S \cong V$, then $T \cong V$. Suppose $T \cong S$ and$S \cong V$, then there exists unitary operators $U_1 \in B(\mathcal{H}, \mathcal{K})$ and $U_2 \in B(\mathcal{K}, \mathcal{H})$ such that;
$T = U_1^* S U_1$..(1) and
$S = U_2^* V U_2$..(2).
From (1) and (2) we have that $T = U_1^* S U_1 = U_1^*(U_2^* V U_2) U_1 = (U_1^* U_2^*) V (U_2 U_1) = U^* V U$, where $U = U_2 U_1$ is also unitary since U_1 and U_2 are unitary. This shows that, $T \cong V$. Hence the result.

Remark 2.1.2 *We know that, unitary equivalence preserves reducing subspaces. That is, if $A, B \in B(\mathcal{H})$ such that A is unitarily equivalent to B and there exists a subspace $\mathcal{M}$ of $\mathcal{H}$ which reduces A, then $\mathcal{M}$ reduces B. Theorem 2.1.1 above says that, if $B \cong C$, for another operator C acting on a Hilbert space, then $\mathcal{M}$ also reduces C.*
Similarly, it can be shown that similarity is an equivalence relation on $B(\mathcal{H})$. The natural concept of equivalence between Hilbert-space operators in fact is unitary equivalence.

However the weaker form of equivalence, viz,, similarity, will also play an important role throughout this project. The following propositions and auxillary results will be referred frequently. They deal with parts and direct summands of similar and unitarily equivalent operators.

Proposition 2.1.3 $[17, Proposition 1.1]$

If an operator $T \in B(\mathcal{H})$ is similar (unitarily equivalent) to a part of an operator $L \in B(\mathcal{K})$, then it is a part of an operator similar (unitarily equivalent) to L.

Proof

Let $\mathcal{R}$ be a subspace of $\mathcal{K}$, and let L be an operator on $\mathcal{K}$. Suppose $\mathcal{R}$ is invariant for L. Regarding the decomposition $\mathcal{K} = \mathcal{R} \bigoplus \mathcal{R}^{\perp}$, L can be written as

$$L = \begin{bmatrix} L|_{\mathcal{R}} & X \\ 0 & Y \end{bmatrix}$$

for operators $X : \mathcal{R}^{\perp} \to \mathcal{R}$ and $Y : \mathcal{R}^{\perp} \to \mathcal{R}^{\perp}$, where $L|_{\mathcal{R}} : \mathcal{R} \to \mathcal{R}$ is the restriction of L to the invariant subspace $\mathcal{R}$(so that $L|_{\mathcal{R}}$ is a part of L). If $T \in B(\mathcal{H})$ is similar to $L|_{\mathcal{R}} \in B(\mathcal{R})$, then there exists an invertible operator $U \in G(\mathcal{H}, \mathcal{R})$ such that
$T = U^{-1}(L|_{\mathcal{R}})U$.

Now consider the invertible operator $W = U \bigoplus I : \mathcal{H} \bigoplus \mathcal{R}^{\perp} \to \mathcal{R} \bigoplus \mathcal{R}^{\perp}$ so that

$$W^{-1}LW = \begin{bmatrix} U^{-1}(L|_{\mathcal{R}})U & U^{-1}X \\ 0 & Y \end{bmatrix}$$

Therefore, $W^{-1}LW : \mathcal{H} \bigoplus \mathcal{R}^{\perp} \to \mathcal{R} \bigoplus \mathcal{R}^{\perp}$ is an operator similar to L for which T is a part, since
$T = (W^{-1}LW)|_{\mathcal{R}}$.

Remark 2.1.4 *In [17], it is remarked that W is unitary whenever U is unitary.*

Recall that a direct summand of an operator T is a part of it whose adjoint also is a part of T^*. This leads to the following corollaries.

Corollary 2.1.5 $[17, Corollary 1.2]$
If an operator $T \in B(\mathcal{H})$ is similar(unitarily equivalent) to a direct summand of an operator $L \in B(\mathcal{K})$, then it is a direct summand of an operator similar (unitarily equivalent) to L.

Corollary 2.1.6 $[17, Corollary 1.3]$
If an operator $T \in B(\mathcal{H})$ is unitarily equivalent to a direct sum $L \in B(\mathcal{K})$, then it is a direct sum itself with direct summand unitarily equivalent to each direct summand of L (i.e, if $T \cong \bigoplus_k L_k$, then $T = \bigoplus_k L_k$ with $T_k \cong L_k$ for each k).

Remark 2.1.7 *Note that Corollary 2.1.6 applies under unitary equivalence only and it essentially says that direct sums and direct summands are preserved under unitary equivalence. This is not the case for similarity and almost-similarity, in general.*

To see this consider the 3 *by* 3 matrices(see [17, p. 25]) representing the operators on $\mathbb{C}^3$.

$$T = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad W = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

By a simple matrix computation, it is clear that $WT = LW$, W is invertible. That is , T is similar to $L = 1 \bigoplus \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, but T cannot be expressed as a direct sum (that is, T is irreducible), and hence T(as a block) is not similar (and almost-similar) to any of the direct summands of L.
Recall that, every reducible operator T acting on a Hilbert space has the direct sum decomposition $T = A \bigoplus B$, where A is normal and B is pure.
The following result shows that reducibility is invariant under unitary equivalence.

Corollary 2.1.8 *Every operator unitarily equivalent to a reducible operator is reducible.*

Proof Let $\mathcal{H}$ and $\mathcal{K}$ be unitarily equivalent Hilbert spaces. Take $T, P \in B(\mathcal{H})$ and an arbitrary operator $U : \mathcal{K} \to \mathcal{H}$. Put $S = U^*TU$ *and* $E = U^*PU$ in $B(\mathcal{K})$. The operator E is an orthogonal projection if P is. Indeed, $E^2 = U^*P^2U$ and $E^* = U^*P^*U$ so that $E = E^2$ if and only if $P = P^2$ and $E = E^*$ if and only if $P = P^*$. Moreover, $E = U^*PU$ is nontrivial if and only if P is and E commutes with S if and only if P commutes with T (since $ES - SE = U^*(PT - TP)U$). Thus S is reducible if and only if T is reducible. Hence the result.

Remark 2.1.9 *Corollary 2.1.8 does not hold under similarity.*

For instance, consider the 3 *by* 3 matrices representing the operators on $\mathbb{C}^3$.

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad X = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

. A simple computation shows that $XA = BX$, X is invertible (thus A and B are similar) and B is a direct sum, that is $B = 1 \bigoplus \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ but A is irreducible since the only one-dimensional invariant subspace $\mathcal{M} = \ span \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ for A is not invariant for A^*.

In most cases, the adjoint of an operator $T \in B(\mathcal{H})$, that is, T^* behaves like T. For instance, if T is bounded, then T^* is bounded, if T is invertible, then so is T^*, and if T is reducible, then T^* is reducible. Observe that, if $A \cong B$ and A is reducible, then so is B. However, if A and B are reducible, then it does not follow in general that $A \cong B$.

Likewise the reducibility of T implies that of T^*, but it does not follow always that

$T \cong T^*$. The following result gives us conditions under when an operator T happens to be unitarily equivalent to its adjoint T^*.

Theorem 2.1.10 *T is unitarily equivalent to its adjoint if and only if T is the product of a symmetry and a self-adjoint operator.*

Proof If $T = JA$ where $J = J^* = J^{-1}$ is a symmetry and A is self-adjoint, then $JAJ = AJ = T^*$, so that T is unitarily equivalent to its adjoint.
Conversely, suppose $TU = UT^*$, where U is unitary. Then T commutes with U^2. Let $\int e^{i\theta} dE_\theta$ be the spectral representation of U^2. If $V = \int e^{i\theta/2} dE_\theta$, then V is a unitary operator,$V^2 = U^2$ and V commutes with every operator that commutes with U^2. It follows that V commutes with U and T. Therefore, $J = V^{-1}U$ is a symmetry and $TJ = JT^*$. Hence $T = J(TJ)$ is the product of a symmetry and a self adjoint operator.
Theorem 2.1.8 leads to the following assertations:

Corollary 2.1.11 *A unitary operator U is similar to its inverse if and only if U is the product of two symmetries.*

Corollary 2.1.12 *Let $A \in B(\mathcal{H})$ be a contraction. If A is unitarily equivalent to a unitary operator T, then A is normal.*

Remark 2.1.13 *Corollary 2.1.11 in other words says that, a unitary operator is similar to its inverse if it is the product of two symmetries and Corollary 2.1.12 says that, an operator which is unitarily equivalent to a unitary operator has no completely non-unitary direct summand.*

The following results help us investigate the relationship between similarity and unitary equivalence for normal operators.

Theorem 2.1.14 *(Fuglede-Putnam theorem).*
Assume that $A, B, T \in B(\mathcal{H})$, where A and B are normal, and $AT = TB$. Then $A^*T = TB^*$.

Remark 2.1.15 *Note that the hypotheses of Theorem 2.1.14 does not imply that $AT^* = T^*B$, even when A and B are self-adjoint and T is normal.*

For consider:

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad T = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

thus a simple computation shows that, $AT = TB$ but $AT^* \neq T^*B$.

Theorem 2.1.16 *If $T \in B(H)$ is invertible, the T has a unique polar decomposition $T = UP$, with U an isometry (which is in fact a unitary) and $P \geq 0$. If $T \in B(H)$ is normal, the T has a polar decomposition $T = UP$ in which U and P commute with each other and T.*

Combining Theorem 2.1.14 and 2.1.15, leads to the following result concerning similarity of normal operators.

Theorem 2.1.17 *Suppose $A, B, T \in B(\mathcal{H})$, A and B are normal, T is invertible, and $A = TBT^{-1}$. If $T = UP$ is the polar decomposition of T, then $A = UBU^{-1}$.*

Proof From $A = TBT^{-1}$, we have $AT = TB$. Hence $A^*T = TB^*$, by Theorem 2.1.12 Hence, $T^*A = (A^*T)^* = (TB^*)^* = BT^*$, so that $BP^2 = B(T^*T) = T^*AT = T^*TB = P^2N$, since $P^2 = T^*T$.
Thus this yields $A = (UP)B(UP)^{-1} = UPBP^{-1}U^{-1} = UBU^{-1}$. Hence the result.

Remark 2.1.18 *This theorem asserts that similar normal operators are actually unitarily equivalent. The following result shows that unitary equivalence preserves normality of operators.*

Theorem 2.1.19 *If T is a normal operator and $S \in B(\mathcal{H})$ is unitarily equivalent to T, then S is normal.*

Proof Suppose $S = U^*TU$, where U is unitary and T is normal. Then,

$S^*S = (U^*T^*U)(U^*TU)$

$= U^*T^*TU$

$= U^*TT^*U$

$= SU^*T^*U$

$= SU^*US^*$

$= SS^*$, which proves the claim.

The following results are well known.

Theorem 2.1.20 *If $S \in B(\mathcal{H})$ and $T \in B(\mathcal{H})$ are similar, then S^* and T^* are similar.*

Corollary 2.1.21 *If $S \in B(H)$ and $T \in B(\mathcal{H})$ are unitarily equivalent, then S and T are similar.*

Proposition 2.1.22 *If S and T are normal operators in a Hilbert space $\mathcal{H}$, then S is unitarily equivalent to T if and only if S is similar to T.*

The following theorem proves one direction of Proposition 2.1.22

Theorem 2.1.23 *Two similar normal operators S and T are unitarily equivalent.*

Proof By hypothesis, there is an invertible operator N such that

$S = NTN^{-1}$.......(*)

Suppose $T = U|T|$ is the polar decomposition of T. Then U is unitary and $|T|$ is positive and hence self-adjoint. From (*), we have that $SN = NT$.

Hence by the Fuglede commutativity theorem (Theorem 2.1.14), we have that

$S^*N = NT^*$.

Thus,

$N^*S = (S^*N^*)^* = (NT^*)^*TN^*$

so that $TN^*N = N^*SN = N^*NT$. Hence T commutes with $f(N^*N)$, for every bounded Borel function f on $\hat{X} = \sigma(N^*N)$. Since N^*N is positive, it has a unique square root $|N| = \sqrt{(N^*N)}$. Thus $|N| \geq 0$ and hence $\sigma(|N|)^2 \subseteq [0, \infty)$. If $f(\lambda) =$

$\lambda^{\frac{1}{2}} \geq 0$ on $\sigma(|N|)^2$, it follows that $T|N| = |N|T$. Hence (*) yields
$S = (U|N|)TU|N|^{-1} = U|N|T|N|^{-1}U^{-1} = U(|N|T|N|^{-1})U^{-1} = UTU^{-1}$, which shows that S is unitarily equivalent to T.
The following Lemma is crucial in showing that if two operators acting on a Hilbert space $\mathcal{H}$ are similar, then they have same spectrum.

Lemma 2.1.24 [4] *Suppose that $A, B \in B(\mathcal{H})$ and B is invertible, then $\sigma(A) = \sigma(B^{-1}AB)$.*

The following Lemma shows that, the spectrum of two similar operators are equal.

Lemma 2.1.25 *Suppose that A and B are similar operators on a Hilbert space $\mathcal{H}$, then A and B have the same:*
(a) Spectrum,
(b) Point spectrum,
(c) Approximate point spectrum.

Proof
(a) We will show that the resolvent sets are the same. Since A is similar to B, $B = N^{-1}AN$, where N is an invertible operator in $B(\mathcal{H})$.
$B - \lambda I = N^{-1}AN - \lambda I = N^{-1}(A - \lambda I)N$. Hence $B - \lambda I$ is invertible if and only if $A - \lambda I$ is invertible, that is, $\lambda \in \rho(B) = \rho(N^{-1}AN)$.
Thus $\rho(N^{-1}AN) = \rho(A)$. Taking complement in $\mathbb{C}$, we get $\sigma(N^{-1}AN) = \sigma(A)$.
(b)$\lambda \in \sigma_p(A)$ implies that, there exists $x \in \mathcal{H}$ such that $x \neq 0$ and $Ax - \lambda x \neq \bar{0}$ Thus for $B = N^{-1}AN$, where N is an invertible operator in $B(H)$, we have
$N^{-1}(Ax - \lambda x) = \bar{0}$.
i.e $N^{-1}A(N(N^{-1})x) - \lambda(N^{-1}x) = \bar{0}$.
i.e $N^{-1}AN(N^{-1}x) - \lambda N^{-1}x = \bar{0}$............(1).
Since $x \neq \bar{0}$, $N^{-1}x \neq \bar{0}$(because N is invertible). Equation (1) implies that, $\lambda \in \sigma_p(N^{-1})AN$. Thus $\sigma_p(A) \subseteq \sigma_p(N^{-1}AN) = \sigma_p(B)$.
Replacing A by B, that is, $N^{-1}AN$, we get

$\sigma_p(N^{-1}AN) \subseteq \sigma_p(N(N^{-1}AN)N^{-1}) = \sigma_p(A)$.

i.e $\sigma_p(B) \subseteq \sigma_p(A)$.

Thus $\sigma_p(B) = \sigma_p(A)$.

(c)Let $\lambda \in \pi_p(A)$. Then there exists a sequence (x_n) in H such that $\|x_n\| = 1$ for all $n \in IN$; and $\|(A - \lambda I)x_n\| \to 0 \ \ as \ \ n \to \infty$

i.e $(A - \lambda I)x_n \to o \ \ as \ \ n \to \infty$. Let $B = N^{-1}AN$, where N is an invertible operator.

Then $N^{-1}(A - \lambda I)x_n \to 0 \ \ as \ \ n \to \infty$

i.e $N^{-1}Ax_n - \lambda N^{-1}x_n \to 0 \ \ as \ \ n \to \infty$

i.e $N^{-1}AN(N^{-1}x_n) - \lambda(N^{-1}x_n) \to \ \ as \ \ n \to \infty$..........(2)

Now $x_n \neq \bar{0}$ and $N^{-1}x_n \neq \bar{0}$, i.e, $\|N^{-1}x_n\| \neq \bar{0}$, $\|N^{-1}x_n\|$ is bounded away from 0,i.e there exists a $\delta > 0$ such that

$\|N^{-1}x_n\| \geq \delta$. Hence $(\|\frac{N^{-1}x_n}{\|N^{-1}x_n\|}\|) = 1$ for all $n \in IN$.

Dividing (2) by $\|N^{-1}x_n\|$, we have

$N^{-1}AN(\frac{N^{-1}x_n}{\|N^{-1}x_n\|}) - \lambda(\frac{N^{-1}x_n}{\|N^{-1}x_n\|}) \to 0 \ \ as \ \ n \to \infty$.

i.e $\lambda \in \pi_p(N^{-1}AN)$. Thus $\pi_p(A) \subseteq \pi_p(N^{-1}AN)$.

Replacing A by $N^{-1}AN$, i.e, B, we have

$\pi_p(N^{-1}AN) \subseteq \pi_p((N^{-1})^{-1}(N^{-1}AN)N^{-1} = \pi_p(A)$.

Remark 2.1.26 *Lemma 2.1.25 shows that similarity preserves spectrum.*

Example 2.1.27 *Let* $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ *and* $B = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$ *be two dimensional operators on* $\mathbb{C}^2$.

Define an invertible operator on $\mathbb{C}^2$ by $N = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & \frac{-i}{\sqrt{2}} \end{bmatrix}$ whose inverse is $N^{-1} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \end{bmatrix}$.

Then $AN = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & \frac{-i}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{i}{\sqrt{2}} & \frac{-i}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix}$ and

$$NB = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & \frac{-i}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} = \begin{bmatrix} \frac{i}{\sqrt{2}} & \frac{-i}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix}.$$

We notice that, $AN = NB$, that is, $N^{-1}AN = B$ and hence A and B are similar. Next we find the spectrum of A and B.

Note that the spectrum of A and B is the set of their respective eigenvalues since A and B are finite. A simple computation shows that $\sigma(A) = \{-i, i\}$ and $\sigma(B) = \{-i, i\}$. This shows that similarity preserves spectrum of similar operators.

The following result shows the condition under which a hyponormal operator similar to another operator is self adjoint, by Sheth in 1966.

Proposition 2.1.28 *If $T \in B(\mathcal{H})$ is a hyponormal operator and $S^{-1}TS = S^*$ for an operator S, where $0 \notin \overline{W(S)}$, then T is self-adjoint.*

Remark 2.1.29 *From Proposition 2.1.18 we conclude that T is normal since self-adjoint operator is normal. We also deduce that, if a hyponormal operator is similar to its adjoint, then it must be normal. Proposotion 2.1.18 can be extended to the class of p-hyponormal operators as follows:*

Theorem 2.1.30 *If T or T^* is p-hyponormal and S is an operator for which $0 \notin \overline{W(S)}$ and $ST = T^*S$, then T is self adjoint and hence normal.*

Theorem 2.1.31 [28] *Let T be hyponormal. If T is similar to a normal operator, then T is normal.*

Theorem 2.1.32 *Let T be an operator on a Hilbert space $\mathcal{H}$. If $r(T) < 1$, then T is similar to a part of the canonical backward unilateral shift on $\ell^2(\mathcal{H})$.*

Theorem 2.1.33 *Similar operators have isomorphic lattices of invariant and hyper-invariant subspaces.*

Proposition 2.1.34 *An invertible operator T is a product of two self-adjoint operators if and only if T is similar to T^*.*

Proof Suppose T is invertible with $T = AB$ with $A = A^*$ and $B = B^*$. Since T is invertible, then $I = TT^{-1} = (AB)(B^{-1}A^{-1})$. This shows that A and B are invertible also and hence BA is invertible. $T^* = BA = BIA = BTT^{-1}A = BT(AB)^{-1}A = BTB^{-1}A^{-1}A = BTB^{-1}$. This shows that $T \sim T^*$.

Conversely, suppose T is invertible and $T \sim T^*$. Since T is invertible and by the polar decomposition theorem, T has a unique polar decomposition $T = UP$, where U is unitary(not necessarily self-adjoint) and $P = (T^*T)^{1/2}$ is a positive operator(self-adjoint). We use the similarity of T and T^* to show that U must indeed be self-adjoint. $T \sim T^*$ implies that $UP = X^{-1}(UP)^*X = X^{-1}PU^*X$. Without loss of generality, let $X = I$. In that case $U = U^*$ which proves that U is self-adjoint.

Remark 2.1.35 *Let J_0 denote the set of all invertible product os self-adjoint operators A and B and J be the set of invertible operators that are similar to their adjoints. It is clear that $J_0 \subseteq J$. Proposition 2.1.34 asserts that $J \subseteq J_0$ is also valid. Using invariance of the classes J_0 and J and similarity transformation $T = S^{-1}TS$. We notice that J is strictly larger than the class of operators that are similar to self-adjoints.*

An example is the bilateral shift.

Theorem 2.1.36 *If $\mathcal{H}$ is a finite-dimensional Hilbert space, then the following are equivalent conditions for an operator T on $\mathcal{H}$:*
(i)T is a product of two self-adjoint operators,
(ii)T is a product of two self-adjoint operators, on which is invertible,
(iii)T is similar to T^*.

Remark 2.1.37 *We show that (iii) does not imply (i), consider the operator $T = \begin{bmatrix} 1 & 2 \\ 0 & 4 \end{bmatrix}$. It is clear that T is similar to T^* but $T = T_1T_2$ where $T_1 = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$ and $T_2 = \begin{bmatrix} 2 & 2 \\ -1 & 2 \end{bmatrix}$. Clearly, T_2 is not self-adjoint.*
This shows that the invertibility condition of T cannot be dropped.

Definition 2.1.38 *On Hilbert space $\mathcal{H}$, an operator S_+ is said to be a unilateral shift if there exists an infinite sequence $\{\mathcal{H}_k\}_{k=0}^{\infty}$ of non-zero pairwise orthogonal subspaces of $\mathcal{H}$ such that $\mathcal{H} = \bigoplus_{k=0}^{\infty} \mathcal{H}_k$ (i.e the orthogonal family $\{\mathcal{H}_k\}_{k=0}^{\infty}$ spans $\mathcal{H}$) and S_+ maps each $\mathcal{H}_k$ symmetrically onto $\mathcal{H}_{k+1}$.*

By the above definition, $S_+(\mathcal{H}_k) = \mathcal{H}_{k+1}$ and $S_+|_{\mathcal{H}|k} : \mathcal{H}_k \to \mathcal{H}_{k+1}$ is an isometry, thus surjective isometry and hence unitary. Therefore, $\mathcal{H}_k$ and $\mathcal{H}_{k+1}$ are unitarily equivalent so that $dim(\mathcal{H}_k) = dim(\mathcal{H}_{k+1})$ for every $k \geq o$. This constant dimension is the multiplicity of S_+. Moreover, S_+ and its adjoint S_+^* are identified with the infinite matrices $S_+ = \begin{bmatrix} 0 & 0 & 0 & \cdots \\ U_1 & 0 & 0 & \cdots \\ 0 & U_2 & 0 & \cdots \\ 0 & 0 & \ddots & \ddots \\ \vdots & \vdots & \ddots & \end{bmatrix}$ and $S_+^* = \begin{bmatrix} 0 & U_1^* & 0 & \cdots \\ 0 & 0 & U_2^* & \cdots \\ 0 & 0 & 0 & \ddots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$ of transformations, where every entry below(above) the main diagonal in the matrix $S_+(S_+^*)$ is unitary and the remaining entries are all null. From the above matrices, we see that $S_+(S_+^*) = I$.

Definition 2.1.39 *Let K_o be a Hilbert space and $U_o : K_o \to \mathcal{H}_o$ be unitary so that $dim(\mathcal{H}_k) = dim(K_o)$ for every $k \geq 0$.*

Consider the operator $U = \bigoplus_{k=0}^{\infty} U_k...U_o : {\ell_+}^2(K_o) = \bigoplus_{k=0}^{\infty} K_o \to \mathcal{H}$........(*).
Since the composition and direct sum of unitary transformations are again unitary, it follows that (*) is unitary and $U^*S_+U = \begin{bmatrix} 0 & 0 & 0 & \cdots \\ I & 0 & 0 & \cdots \\ 0 & I & 0 & \cdots \\ 0 & 0 & \ddots & \ddots \\ \vdots & \vdots & \ddots & \end{bmatrix}$ such that ${\ell_+}^2(K_o) \to {\ell_+}^2(K_o)$.
Thus S_+ is unitarily equivalent to U^*S_+U which is a unilateral shift of multiplicity α acting on ${\ell_+}^2(K_o)$. If $dim(K_o)1$ (i.e, if $K_o = \mathbb{C}$ in case of complex space), then we get a canonical unilateral shift of multiplicity 1 acting on ${\ell_+}^2 = {\ell_+}^2(\mathbb{C})$ and is the operation

on ${\ell_+}^2$ that shifts the canonical orthonormal basis for ${\ell_+}^2$.

We prove the following propositions which show that any operator unitarily equivalent to a unilateral shift is in fact a unilateral shift. In addition, they have the same multiplicity.

Proposition 2.1.40 $[19, Problem 5.4 pp.44]$ *If an operator is unitarily equivalent to a unilateral shift, then it is a unilateral shift itself of the same multiplicity.*

Proof Let S_+ be a unilateral shift of multiplicity α acting on a Hilbert space $\mathcal{H}$ and let $\{\mathcal{H}_k\}$ be the underlying countably infinite orthogonal family of unitarily equivalent subspaces of $\mathcal{H} = \bigoplus_k \mathcal{H}_k$ so that $dim(\mathcal{H}_k) = \alpha$. Take an operator T acting on a Hilbert space $\mathcal{K}$ and suppose that there exists a unitary transformation $U \in G(\mathcal{H})$ such that $T = U^* S_+ U$. For each k, set $\mathcal{K}_k$ so that $\mathcal{H}_k = U(\mathcal{K}_k)$.

Since U is unitary, it follows that $(\mathcal{K}_k)$ is an orthogonal family ($\{\mathcal{H}_k\}$ is orthogonal and U preserves inner product) of subspaces of $\mathcal{K}$ because the range of an isometry is closed. Moreover, $\{\mathcal{K}_k\}$ spans $\mathcal{K}$. Furthermore, since $S_+(\mathcal{H}_k) = \mathcal{H}_{k+1}$, $T(\mathcal{K}_k) = U^* S_+ U(\mathcal{K}_k) = U^* S_+(\mathcal{H}_k) = U^*(\mathcal{H}_{k+1}) = \mathcal{K}_{k+1}$ for each k. Finally, note that $T^*T = (U^* S_+^* U)(U^* S_+ U) = U^* S_+^* I S_+ U = U^* S_+^* S_+ U = U^* I U = U^* U = I$. Thus T is an isometry and so any restriction of it (in particular, $T|_k$ is an isometry for every $K \geq 0$). Hence T is a unilateral shift with the same multiplicity of S_+ once $dim(K_k) = dim(U^*(\mathcal{H}_k)) = dim(\mathcal{K}_k) = \alpha$ for all k.

Proposition 2.1.41 $[17, Proposition 2.2]$ *Two unilateral shifts are unitarily equivalent if and only if they have the same multiplicity.*

Proof Let S_1 and S_2 be unilateral shifts which are unitarily equivalent. Then they are unitarily equivalent to the canonical shift of multiplicity α. Hence S_1 and S_2 have the same multiplicity.

Conversely, let S_1 and S_2 have the same multiplicity α. Then the are unitarily equivalent to the canonical shifts of multiplicity α. Hence S_1 and S_2 are unitarily equivalent.

Remark 2.1.42 *Propositions 2.1.40 and 2.1.41 show that, the unilateral shift of a given multiplicity is in fact an equivalence class.*

The following problem shows when two unitarily equivalent operators are A-self-adjoint.

Question 1 *If T and S are unitarily equivalent and T is A-self-adjoint, when is S A-self-adjoint?*

Answer Since T and S are unitarily equivalent, we have $T = U^*SU$. A-self adjointness of T implies that $T^* = ATA^{-1} = U^*SU$, which implies that $S = UATA^{-1}U^* = UATA^{-1}U^{-1} = (UA)T(UA)^{-1}$. That is, S is UA-self-adjoint. Letting $U = I$ clearly shows that S is A-self-adjoint. Hence the result.

2.2 Quasisimilarity of Some Operators

We state and prove the following theorem which shows that, the product of two quasi-affinities is also a quasi-affinity and so is their adjoints.

Theorem 2.2.1 $[29, Proposition 3.3]$ *If X is a quasi-affinity from $\mathcal{H}$ to $\mathcal{K}$ and Y is a quasi-affinity from $\mathcal{K}$ to $\mathcal{L}$, then:*
(a)YX is a quasi-affinity from $\mathcal{H}$ to $\mathcal{L}$ and XY is a quasi-affinity from $\mathcal{L}$ to $\mathcal{H}$.
(b)If $X \in B(\mathcal{H})$ is a quasi-affinity, the X^* is a quasi-affinity.

Proof We need to show that XY and YX are quasi-affinities. Clearly, XY is one-to-one since it is the composition of one-to-one operators. It suffices to prove that XY has a dense range.
Note that $Ran(XY) \subseteq \mathcal{H}$. It follows that $\overline{XY\mathcal{H}} = \overline{X(\overline{Y\mathcal{H}})} = \overline{X(\mathcal{K})} = \mathcal{H}$. Therefore, $\overline{Ran(XY)} = \mathcal{H}$.
This proves that XY has a dense range.
Similarly, YX is one-to-one since it is the composition of one-to-one operators. To show that YX has a dense range, we note that $Ran(YX) \subseteq \mathcal{K}$. It follows that $\overline{YX\mathcal{K}} = \overline{Y(\overline{X\mathcal{K}})} = \overline{Y(\mathcal{H})} = \mathcal{K}$. Therefore, $\overline{Ran(YX)} = \mathcal{K}$.
Now $S(XY) = XTY = (XY)S$, which shows that XY is a quasi-affinity in $\{S\}'$, the commutant of S.

Also, $(YX)T = Y(XT) = YSX = T(YX)$, thus YX is a quasi-affinity in $\{T\}'$, the commutant of T.

(b) Since $X \in B(H)$ is a quasi-affinity, $KerX = \{0\}$, $\overline{Ran(X)} = \mathcal{H}$. Recall that

$Ker(X) = Ran(X^*)$.........(1)

$Ker(X^*) = Ran(X)$.........(2)

$\overline{Ran(X)} = Ker(X^*)$..........(3)

$\overline{Ran(X^*)} = Ker(X)$..........(4).

Therefore, since $Ker(X) = \{0\}$, we have $Ker(X) = \mathcal{H} = \overline{Ran(X^*)}$ by (4) which implies that X^* has a dense range. X^* is one-to-one($Ker(X^*) = \{0\}$). Hence X^* is a quasi-affinity.

The proof of Theorem 2.2.2 follows from the above Theorem and it essentially shows that, quasi-affine tranform is transitive.

Theorem 2.2.2 $[29, Proposition 3.4]$ *If A is a quasi-affine transform of B and B is a quasi-affine transform of C, then*

(a) A is a quasi-affine transform of C.

(b)B^* is a quasi-affine transform of A^*.

Theorem 2.2.3 *If X is a quasi-affinity from $\mathcal{H}$ to $\mathcal{K}$, then $|X| = \sqrt{X^*X}$ is a quasi-affinity on $\mathcal{H}$ (from $\mathcal{K}$ to $\mathcal{H}$). Moreover, $X|X|^{-1}$ extends by continuity to a unitary transform U from $\mathcal{H}$ to $\mathcal{K}$.*

Lemma 2.2.4 $[16, Lemma 2.6{-}11]$ *Let $X \in B(\mathcal{H})$ and $Y \in B(\mathcal{K}, \mathcal{L})$ be quasi-affinities where $\mathcal{H}, \mathcal{K} and \mathcal{L}$ are finite dimensional Hilbert spaces. The inverse $(YX)^{-1} \in B(\mathcal{L}, \mathcal{H})$ of the composite YX exists and $(YX)^{-1} = X^{-1}Y^{-1}$.*

Proof The operator $YK \in B(\mathcal{L}, \mathcal{K})$ is bijective, so that YX exists. Thus $(YX)(YX)^{-1} = I_{\mathcal{L}}$ is the identity operator on $\mathcal{L}$. Applying Y^{-1} and using $Y^{-1}Y = I_{\mathcal{K}}$, we obtain $Y^{-1}YX(YX)^{-1} = X(YX)^{-1} = Y^{-1}|_{\mathcal{L}} = Y^{-1}$. Applying X^{-1} and using $X^{-1}X = I_{\mathcal{H}}$ we obtain $X^{-1}X(YX^{-1} = (YX)^{-1} = X^{-1}Y^{-1}$.

Proposition 2.2.5 $[29, Proposition 3.4]$ *If a unitary operator A on a Hilbert space $\mathcal{H}$ is the quasi-affine transform of a unitary operator B on a Hilbert space $\mathcal{K}$, then A and B are unitarily equivalent.*

Proof Let $X \in B(\mathcal{H}\mathcal{K}K)$ be a quasi-affinity. Thus $XA = BX$......(1) implying that $X = B^{-1}X = XA^{-1} = XA^{*}$.......(2).

From (1) and (2) we obtain,

$|X|^2 A = X^*XA = X^*BX = AX^*X = A|X|^2$ and by iteration $|X|^{2n}A = A|X|^{2n}$ $(n = 0, 1, 2, ..)$; hence $p(|X|^2)A = Ap(|X|^2)$ for every polynomial $p(x)$. Let $\{p_n(x)\}$ be a sequence of polynomials tending to $|X|^{1/2}$ uniformly on the interval $0 \leq x \leq \|X\|^{1/2}$. Then $p_n(|X|^2)$ converges (in the operator norm) to $|X|$ so that we have a limit relation $|X|A = A|X|$.......(3).

From (1) and (3) it follows that $BU|X| = BX = AX = U|X|A = UA|X|$; because $|X|\mathcal{H}$ is dense in $\mathcal{H}$, it results that $BU = UA$. By Proposition 2.2.3 U is unitary and hence A and B are unitarily equivalent.

The following result shows that quasi-similarity is an equivalence relation.

Theorem 2.2.6 *Quasi-similarity is an equivalence relation on the class of operators.*

Proof Let $A \in B(\mathcal{H})$, $B \in B(\mathcal{K})$ and $C \in B(\mathcal{L})$. We show that $A \approx A$. We have $XA = AX$ and $AY = YA$ where X and Y are quasi-affinities. Choosing $X = Y = I$ (without loss of generality) we have $A \approx A$.

Now suppose that $A \approx B$ and $B \approx A$. Since $A \approx B$ there exists quasi-affinities $X \in B(\mathcal{H}, \mathcal{K})$ and $Y \in B(\mathcal{K}, \mathcal{H})$ such that $XA = BX$ and $BY = YA$. By symmetry of composition, it is true that $BX = XA$ and $YA = BY$. Hence $B \approx A$. This shows symmetry.

Finally, we show transitivity.

Suppose $A \approx B$ and $B \approx C$. We show that $A \approx C$. There exists quasi-affinities $X \in B(\mathcal{H}, \mathcal{K})$, $Y \in B(\mathcal{K}, \mathcal{H})$ and $S \in B(\mathcal{K}, \mathcal{L})$, $T \in B(\mathcal{L}, \mathcal{K})$, respectively, such that,

$XA = BX$ and $BY = YA$.....(1) and

$SB = CS$ and $CT = TB$.....(2).

$TSYX$ is a quasi-affinity which is one-to-one since it is the composition of one-to-one operators.
$TSYXA = TSAYX$, since $YX \in \{A\}'$
$= TSYBX$, since $AY = YB$
$= TBSYXr$, since $SY \in \{B\}'$
$= CTSYX$, since $TB = CT$ which is a quasi-affinity and
$AYXST = YXAST$,since $YX \in \{A\}'$
$= YBXST$, since $XA = BX$
$= YXSBT$, since $XZ \in \{B\}'$
$= YXSTC$, since $ZR \in \{C\}'$. Therefore $A \approx C$, which proves that quasi-similarity is an equivalence relation.
Theorem 2.2.7 links similarity of operators with quasisimilarity.

Theorem 2.2.7 *If $T \in B(\mathcal{H})$ and $S \in B(\mathcal{K})$ are similar operators, then they are quasi-similar.*

Proof There exists quasi-invertible operator $X \in B(\mathcal{H}, \mathcal{K})$ such that $XT = SX$. This implies that $X^{-1}S = TX^{-1}$, where $X^{-1} \in B(\mathcal{K}, \mathcal{H})$ which implies that $S \approx T$.

Theorem 2.2.8 $[11, Theorem 2.5]$ *Suppose that for each α in some index set A, there are Hilbert spaces $\mathcal{H}_\alpha$ and $\mathcal{K}_\alpha$ and operators $T_\alpha \in B(\mathcal{H}_\alpha)$ and $S_\alpha \in B(\mathcal{K}_\alpha)$ respectively which are quasi-similar. Let T be the operator $T = \Sigma_{\alpha\in A} \bigoplus T_\alpha$ acting on a Hilbert space which is the direct sum of spaces $\mathcal{H}_\alpha$ and $S = \Sigma_{\alpha\in A} \bigoplus S_\alpha \in B(\mathcal{K})$ where $\mathcal{K} = \Sigma_{\alpha\in A} \bigoplus K_\alpha$. Then T is quasi-similar to S.*

Proof Suppose X_α and Y_α are the quasi-invertible operators such that $X_\alpha T_\alpha = S_\alpha X_\alpha$ and $T_\alpha Y_\alpha = Y_\alpha S_\alpha$. If $X = \Sigma_{\alpha\in A} \bigoplus X_\alpha/\|X\|$ and $Y = \Sigma_{\alpha\in A} \bigoplus Y_\alpha/\|Y\|$, then X and Y are the quasi-invertible operators and satisfy the desired equations.

Example 2.2.9 *Let A_n and B_n be unilateral shift operators with weights 1 and $\frac{1}{n}$ respectively on an n-dimensional Hilbert space $\mathcal{H}$.Then A_n is the Jordan canonical* *form for B_n and so A_n and B_n are similar. If $A = \Sigma_{n=o}^{\infty} A_n$ and $B = \Sigma_{n=o}^{\infty} B_n$, then Theorem 2.2.8, A is quasi-similar to B.*

We investigate the properties of hyponormal operators under quasisimilarity. Firstly, we state and prove the following results.

Lemma 2.2.10 $[4, Lemma A]$ *Let $T \in B(\mathcal{H})$ be hyponormal and let $\{X_n = \Sigma_{n=o}^{\infty}\}$ be a sequence of $\mathcal{H}$ such that $Tx_{n+1} = x_n$ for all $n \geq 0$. Then either $\|x_o\| \geq \|x_1\| \geq \|x_2\| \geq$ .... or $\|x_n\| \to \infty$ as $n \to \infty$.*

Proof For any $x \in \mathcal{H}$,
$\|Tx\| = <Tx, Tx>^{1/2} = <T^*Tx, x>^{1/2} \leq (\|T^*Tx\|\|x\|)^{1/2} \leq (\|T^2x\|\|x\|)^{1/2} \leq 1/2(\|T^2x\| + \|x\|)$. Letting $x = x_{n+2}$ we see that $\|x_{n+1}\| = 1/2(\|x_n\| + \|x_{n+2}\|)$, so the sequence $\{x_n\}$ is convex and so either $\|x_o\| \geq \|x_1\| \geq \|x_2\| \geq$ .... or $\|x_n\| \to \infty$ as $n \to \infty$.

Lemma 2.2.11 $[4, Lemma B]$ *Suppose $S \in B(\mathcal{H})$ is invertible and $T \in B(\mathcal{K})$ is hyponormal and $X \in B(\mathcal{H}, \mathcal{K})$ satisfies $XS = TX$. Then $\|XS^{-1}y\| \leq \|S^{-1}\| \; \|Xy\|$ for all $y \in \mathcal{H}$.*

Proof Assume without loss of generality, that $dim\mathcal{H} \geq 1$ and let $c = \|S^{-1}\| > 0$. Fix $y \in \mathcal{H}$ and define $x_n = c^{-1}XS^{-ny}$ for $n \geq 0$. Then $cTx_{n+1} = x_n$ and $\|x_n\| \leq \|c^{-n}\| \; \|X\| \; \|S^{-n}\| \; \|y\| = \|S^{-1}\|^{-n} \; \|X\| \; \|S^{-n}\| \; \|y\| = \|X\| \; \|y\|$ i.e $\|x_n\| \leq \|X\| \; \|y\|$ for all $n \geq 0$. Since cT is hyponormal, $\|x_o\| \geq \|x_1\| \geq \|x_2\| \geq$ .... By Lemma 2.2. 10 and the first inequality in this chain $\|x_1\| \leq \|x_o\|$ shows that $\|XS^{-1}y\| \leq \|S^{-1}\| \; \|Xy\|$.
The following result shows the condition under which a hyponormal operator is invertible.

Proposition 2.2.12 $[20, Proposition 2.4]$ *If $S \in B(\mathcal{H})$ is invertible, $T \in B\mathcal{K})$ is hyponormal and $X \in B(\mathcal{H}, \mathcal{K})$ has a dense range and satisfies $XS = TX$, then T is invertible.*

Proof Suppose that $dim\mathcal{H} \geq 1$. Since $X(\mathcal{H}) = X(S(\mathcal{H})) = TX(\mathcal{H}) \subseteq T(\mathcal{K})$, the range of T contains the dense range of X, and so T has a dense range. It remains to show that T is bounded below, and by continuity it suffices to show that T is bounded below on the range of T. By Lemma 2.2.11
$\|XS^{-1}y\| \leq \|S^{-1}\| \, \|Xy\| = \|S^{-1}\| \, \|XSS^{-1}y\| = \|TXS^{-1}y\|$ i.e
$\|XS^{-1}y\| \leq \|S^{-1}\| \, \|TXS^{-1}y\|$ i.e
$\|S^{-1}\|^{-1} \, \|XS^{-1}y\| \leq \|TXS^{-1}y\|$ for all $y \in \mathcal{H}$.
Put $y_1 = \|S^{-1}y\|$. Then $\|S^{-1}\|^{-1} \, \|Xy_1\| \leq \|TXy_1\|$, thus T is bounded below on the range of X.
We are ready to show the equality of the spectrum of quasisimilar hyponormal opereators.

Theorem 2.2.13 $[4, Theorem2]$ *Quasi-similar hyponormal operators have equal spectra.*

Proof If S and T are quasi-similar hyponormal operators, then for any complex number λ, $S - \lambda I$ and $T - \lambda I$ are also quasi-similar and hyponormal, by Proposition 2.2.12, they are both invertible or both non-invertible. Thus $\sigma(S) = \sigma(T)$.

Remark 2.2.14 *From the proper inclusion relation, $Normal \subset Hyponormal \subset Quasi-$ hyponormal and using Theorem 2.1.13, if hyponormal operators are replaced by quasi-hyponormal operators, we obtain a similar result, that is, $\sigma(S) = \sigma(T)$.*

Lemma 2.2.15 $[28, Lemma1]$ *Let $A, B, X \in B(\mathcal{H})$ where $AX = BX$. Assume X is quasi-invertible and A has a single valued extension property.Then B has a singled valued extension property and $\sigma_A(X_w) \subset \sigma_B(w)$ for all $w \in \mathcal{H}$.*

Definition 2.2.16 *An operator $T \in B(\mathcal{H})$ satisfies Dunford's condition C if for every set $F \subset \mathbb{C}$ the linear manifold $\{x \in \mathcal{H} | \sigma_T(x) \subset F\}$ is closed.*

When we say T satisfies condition C, we are also asserting that it has the single valued extension property.

Theorem 2.2.17 (28, Theorem 2) *Let $A, B, X \in B(\mathcal{H})$ where X is quasi-invertible and $AX = BX$. If A satisfies the condition C, then $\sigma(A) \subset \sigma(B)$.*

Proof It follows from Lemma 2.2.15 that $\sigma_A(X_w) \subset \sigma_B(w)$ for all $w \in \mathcal{H}$. But the set $\sigma_B(w)$ is a compact and non-empty on the complex plane and so $\sigma_B(w) \subset \sigma(B)$. Also, $\{X_w | w \in \mathcal{H}\}$ is dense in H and hence by condition C, $\sigma_A(y) \subset \sigma(B)$ for all $y \in \mathcal{H}$. Thus $\sigma(A) \subset \sigma(B)$.
The following result shows that, quasisimilar operators satisfying Dunford's condition have the same spectrum.

Corollary 2.2.18 *: Let $A, B, X \in B(\mathcal{H})$ be quasi-similar and satisfy condition C. Then $\sigma(A) = \sigma(B)$.*

Proof see [28, Corollary 3].

Remark 2.2.19 *A similar result as in Corollary 2.2 18 holds for quasi-similar hyponormal operators.*

Proposition 2.2.20 $[20, Proposition 2.5]$ *If S and T are quasi-similar quasi-hyponormal operators in $B(\mathcal{H})$ and $Ker(S) = Ker(T)$, then $S_1 = S|_{ker(S)}$ and $T_1 = T|_{Ker(T)}$ are quasi-similar quasi-hyponormal operators.*

The following result shows that, Fredholmness is preserved under quasisimilarity.

Theorem 2.2.21 $[20, Theorem 2.6]$ *Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces. Suppose S and T are quasi-similar quasi-hyponormal operators on Hilbert spaces in $B(\mathcal{H})$. Then S is a Fredholm operator satisfying $Ind(S) = 0$ if and only if T is a Fredholm operator satisfying $Ind(T) = 0$.*

Proof Since S and T are quasi-similar, there exists quasi-affinities X and Y such that $XS = TX$ and $SY = YT$. Now suppose that S is a Fredholm operator satisfying $Ind(S) = 0$. Since S and T are quasi-similar, it follows that $dim(Ker(S)) = dim(Ker(T))$ and $dim(Ker(S^*)) = dim(Ker(T^*))$.

Without loss of generality, we may assume that $Ker(S) = Ker(T)$. Since S is a quasi-hyponormal Fredholm operator with $Ind(S) = 0$, we have that $Ker(S) = Ker(S^*)$.Note that $Ker(S)$ is an invariant subpace for the operator X. The matrices S, T, X with respect to the decomposition
$\mathcal{H} = Ker(S) \bigoplus Ker(T)$ are

$$S = \begin{bmatrix} S_1 & 0 \\ 0 & 0 \end{bmatrix} \quad T = \begin{bmatrix} T_1 & 0 \\ 0 & 0 \end{bmatrix} \quad X = \begin{bmatrix} X_1 & 0 \\ X_2 & X_3 \end{bmatrix}$$

where S_1 is invertible,T_1 is quasi-hyponormal and X_1 has a dense range in $Ker(S)$. The equation $XS = TX$ implies that $X_1S_1 = T_1X_1$. Hence by Proposition 2.2.12 T_1 is invertible. Thus T is a Fredholm operator satisfying $Ind(S) = 0$. Hence by symmetry, the result follows.

Remark 2.2.22 *We obtain a similar result if quasi-hyponormal operators are replaced by hyponormal operators in the above theorem. This is because Hyponormal* $\subset$ *Quasi-hyponormal .*
We also note that quasi-similarity preserves Fredholm property.

Proposition 2.2.23 $[30, Sec3.2.1 Theorem L-H]$
If $B \geq A \geq 0$*, then* $B^\alpha \geq A^\alpha$ *for* $0 < \alpha \leq 1$.

Lemma 2.2.24 $[13, Lemma3]$ *Let* $T_1 \in B(\mathcal{H}_1)$ *be a p-hyponormal operator and let* $T_2 \in B(\mathcal{H}_2)$ *be a normal operator. If there exists an operator* $X \in B(\mathcal{H}_1, \mathcal{H}_2)$ *with dense range such that* $T_1X = XT_2$*, then* T_1 *is normal.*

Proof see [12].

Lemma 2.2.25 $[32, Lemma1.1]$ *Let* $N_i \in B(\mathcal{H}_i)$ *be normal for each* $i = 1, 2$*. If* $X \in B(\mathcal{H}_1, \mathcal{H}_2)$ *and* $Y \in B(\mathcal{H}_2, \mathcal{H}_1)$ *are injective such that* $N_1X = XN_2$ *and* $YN_1 = N_2Y$*, then* N_1 *and* N_2 *are unitarily equivalent.*

Theorem 2.2.26 $[13, Theorem1]$ *For each $i = 1, 2$, let $T_i \in B(\mathcal{H}_i)$ be p-hyponormal operators and $T_i = N_i \bigoplus V_i$ on $\mathcal{H}_i = \mathcal{H}_{i1} \bigoplus \mathcal{H}_{i2}$ where N_i and V_i are the normal and pure parts of T_i, respectively. If T_1 and T_2 are quasi-similar, then N_1 and N_2 are unitarily equivalent and there exists $Y \in B(\mathcal{H}_{12}, \mathcal{H}_{22})$ and $X \in B(\mathcal{H}_{22}, \mathcal{H}_{12})$ having dense ranges such that $V_1X = XV_2$ and $YV_1 = V_2Y$.*

Proof By hypothesis, there exists quasi-affinities $X \in B(\mathcal{H}_1, \mathcal{H}_2)$ and $Y \in B(\mathcal{H}_2, \mathcal{H}_1)$ such that $T_1X = XT_2$ and $YT_1 = T_2Y$.
Let

$$X := \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix}, \quad Y := \begin{bmatrix} Y_1 & Y_2 \\ Y_3 & Y_4 \end{bmatrix}$$

with respect to $\mathcal{H}_2 = \mathcal{H}_{21} \bigoplus \mathcal{H}_{22}$ and $\mathcal{H}_1 = \mathcal{H}_{11} \bigoplus \mathcal{H}_{12}$, respectively. A simple matrix computation shows that $V_1X_3 = X_3N_2$ and $V_2Y_3 = Y_3N_1$. We claim that $X_3 = Y_3 = 0$. To prove this , let $\mathcal{M} = \overline{RanX_3}$. Then $\mathcal{M}$ is a nontrivial invariant subspace of V_1. The $V_1' = V_1|_{\mathcal{H}}$, then V_1 is p-hyponormal. If we define an operator $X_3' : \mathcal{H}_{21} \to \mathcal{M}$ by $X_3'x = X_3x$ for each $x \in \mathcal{H}_{21}$, then we see that X_3' has a dense range and satisfies $V_1'X_3' = X_3'T_2$. By Lemma (3.2.24), V_1 is normal. This contradicts the hypothesis V_1 is pure. This forces $X_3 = 0$. Similarly, $Y_3 = 0$. Thus it follows that X_1 and Y_1 are injective. Since $N_1X_1 = X_1N_2$ and $Y_1N_1 = N_2Y_1$, by Lemma 3.2.25 we have that N_1 and N_2 are unitarily equivalent. We also notice that X_4 and Y_4 have dense ranges and $V_1X_4 = X_4V_2$ and $Y_4V_1 = V_2Y_4$.

Corollary 2.2.27 *Let T_1 and T_2 be quasi-similar p-hyponormal operators. If T_1 is pure, then T_2 is also pure.*

Corollary 2.2.28 *Let $T_1 \in B(\mathcal{H}_1)$ be p-hyponormal and let $T_2 \in B(\mathcal{H}_2)$ be normal. If T_1 and T_2 are quasi-similar, then T_1 and T_2 are unitarily equivalent.*

Theorem 2.2.29 *If $T \in B(\mathcal{H})$ is a (p, k)-quasihyponormal operator and $S^* \in B(\mathcal{H})$ is a p-hyponormal operator, and if $TX = XS$ where $X : \mathcal{K} \to \mathcal{H}$ is a one-to-one bounded* *linear operator with dense range(a quasiaffinity), then T is a normal operator unitarily equivalent to S.*

Remark 2.2.30 *Theorem 2.2.29 says that (p, k)-quasihyponormal operator which is a quasi-affine transform of a co-p-hyponormal operator is always normal.*

Corollary 2.2.31 *Let T be a hyponormal operator whose c.n.n. part has finite multiplicity. Then T is quasisimilar to an isometry if and only if its normal part is unitary and its c.n.n. part is quasisimilar to a unilateral shift.*

Proof Let T be hyponormal with decomposition $T = T_1 \bigoplus T_2$ and suppose that T is quasisimilar to an isometry $V = U \bigoplus S$, where T_1 is normal and T_2 is c.n.n., U is unitary and S is a unilateral shift. By [10, Proposition 3.5], T_1 is unitarily equivalent to U and hence unitary. Since by assumption T is quasisimilar to V, and by Clary[4], quasisimilar hyponormal operators have the same spectra, and by [9], $\|T\| = r(T) = r(V) = 1$, where $r(T)$ and $r(V)$ denote the spectral radii of T and S , respectively. This proves that T_2 is quasisimilar to S.

Remark 2.2.32 *Theorem 2.2.29 gives a condition under which hyponormal operator is similar to an isometry.*

Corollary 2.2.33 *Let A and B be hyponormal operators. Assume that c.n.n. part of A has a finite multiplicity. If A is quasisimilar to B, then their normal parts are unitarily equivalent.*

Proof The result follows from Corollary 2.2.31 and by the application of the fact that quasisimilar normal operators are unitarily equivalent (see Hastings [10]).

Remark 2.2.34 *Note that from Corollary 2.2.32 , we cannot conclude that quasisimilar hyponormal operators have quasisimilar parts. By [10], if A and B are quasisimilar hyponormal operators and A is pure, then B also is pure.*

Corollary 2.2.35 *Let $T \in B(\mathcal{H})$. Let $TW = WN$ where N is normal and W is any non-zero operator in $B(H)$. Then T has a nontrivial invariant subspace.*

Remark 2.2.36 *Corollary 2.2.35 applies to quasiaffine transforms of all reducible operators with a finite direct summand (remember normal operators are reducible).*

Lemma 2.2.37 [34, $Corolarry$3.9] *Let $T = T_1 \bigoplus T_2$ and $S = S_1 \bigoplus S_2$ be contractions, where T_1 and S_1 are of class C_{11}, T_2 and S_2 are of class $C_{.0}$ and T_2 has a finite multiplicity. Then T is quasisimilar to S if and only if T_1 is quasisimilar to S_1 and T_2 is quasisimilar to S_2.*

Remark 2.2.38 *We use Lemma 2.2.37 to prove the following result for hyponormal contraction. We use the fact that quasisimilar normal(unitary) operators are unitarily equivalent.*

Corollary 2.2.39 *Let T and S be hyponormal contractions. Assume that thee c.n.u part of T has finite multiplicity. Then T is quasisimilar to S if and only if their unitary parts are unitarily equivalent and c.n.u parts are quasisimilar to each other.*

Lemma 2.2.40 *If $T \in B(H)$ doubly intertwines A and B and $Lat(A) \bigcap$ Lat (B) is trivial, then T is either 0 or a quasiaffinity. The same is true if T commutes with A and B and $Lat(A) \bigcap$ Lat (B) is trivial.*

Proof T doubly intertwines the pair (A, B) implies that $TA = BT$ and $TB = AT$. Since $TA = BT$, then $\overline{Ran(T)} \in Lat(B)$ and $Ker(T) \in Lat(A)$. Since $TB = AT$, we deduce that $\overline{Ran(T)} \in Lat(A) \bigcap Lat(B)$ and $\overline{Ker(T)} \in Lat(A) \bigcap Lat(B)$. The following two cases result:

(i)Case 1, If $\overline{Ran(T)} = \{0\}$, then $T = 0$. If $\overline{Ran(T)} = \mathcal{H}$, then $Ker(T) = \{0\}$ and hence T is one-to-one and has a dense range, thus a quasiaffinity.

(ii)Case 2, If T commutes with A and B, that is, $TA = AT$ and $TB = BT$, then by the argument above, $\overline{Ran(T)} \in Lat(A) \bigcap Lat(B)$ and $\overline{Ker(T)} \in Lat(A) \bigcap Lat(B)$. Thus by argument above, either $T = 0$ or T is a quasiaffinity.

Remark 2.2.41 *The triviality of $Lat(A) \bigcap Lat(B)$ follows from the orthogonality of $Ker(T)$ and $\overline{Ran(T)}$.*

Strenthening Lemma 2.2.40 to similarity shows that $Lat(A)$ is isomorphic to $Lat(B)$.

Theorem 2.2.42 *If A and B are nilpotent operators of nil-potency index two having no nontrivial common invariant subspaces, then A and B are quasisimilar.*

Proof If $Lat(A) \bigcap Lat(B)$ is trivial. then $T = A + B$ is non-zero because if $T = 0$, then $Lat(A) = Lat(B)$ and nilpotent operators have nontrivial invariant subspaces. Consequently, A and B are quasisimilar since T is a quasiaffinity doubly intertwining A and B by Theorem 2.2.43.

Theorem 2.2.43 *If $A, B \in B(\mathcal{H})$ and A has nontrivial subspaces, then B has nontrivial invariant subspaces.*

Proposition 2.2.44 *If $T_1, T_2 \in B(\mathcal{H})$ are quasisimilar (with quasi-affinities $X, Y \in B(\mathcal{H})$), then $XY \in \{T_1\}'$ and $YX \in \{T_2\}'$.*

Proof Suppose $T_1 \approx T_2$ with quasi-affinities X and Y. Then $T_1X = XT_2$ and $T_2Y = YT_1$. Post multiplication of the first equation by Y and using the second equation we have $T_1XY = XT_2Y = XYT_1$ which implies $XY \in \{T_1\}'$. Post-multiplication of the second equation by X and using the first equation, we have $T_2YX = YT_1X = YXT_2$ which implies that $YX \in \{T_2\}'$.

Definition 2.2.45 *A quasiaffinity X is said to have the hereditary property with respect to an operator $T \in B(\mathcal{H})$ if $X \in \{T\}'$ and $\overline{X(M)} = M$ for every $M \in HyperLat(T)$.*

Definition 2.2.46 *If T_1 and T_2 are quasisimilar and there exists an implementing pair (X, Y) of quasi-affinities such that XY has the hereditary property with respect to T_1 and YX has the hereditary property with respect to T_2, then we say that T_1 is hyper-quasisimilar to T_2 denoted by $T_1 \overset{h}{\approx} T_2$.*

Remark 2.2.47 *Hyper-quasisimilarity is an equivalence relation which is strictly stronger than quasisimilarity. From definition 2.2.46, two operators T_1 and T_2 are hyper-quasisimilar if there exists quasi-affinities X and Y satisfying $XT_1 = T_2X$ and $YT_2 = T_1Y$ and the additional condition that $\overline{YXM_1} = M_1$ and $\overline{XYM_2} = M_2$, for every $M_1 \in HyperLat(T_1)$ and $M_2 \in HyperLat(T_2)$.*

Theorem 2.2.48 *If T_1 and T_2 are hyper-quasisimilar, then $HyperLatT_1 \approx$ HyperLat T_2.*

Proof Since $T_1 \overset{h}{\approx} T_2$, we have quasi-affinities X and Y satisfying $\overline{YXM_1} = M_1$ and $\overline{XYM_2} = M_2$, for every $M_1 \in HyperLat(T_1)$ and $M_2 \in HyperLat(T_2)$. Using Proposition 2.2.44 $XY \in \{T_1\}'$ and $YX \in \{T_2\}'$, $M_1 \in HyperLat(T_2)$ for every $M_1 \in HyperLat(T_1)$ and $M_2 \in HyperLat(T_1)$ for every $M_2 \in HyperLat(T_2)$. This means that every hyperinvariant subspace of T_1 is a hyperinvariant subspace of T_2 and vice versa. Hence the prove.
Note that Theorem 2.2.48 holds when $\approx$ is replaced with $=$.

Corollary 2.2.49 *Let T_1 and T_2 be c.n.u. C_{11} contractions with finite defect indices. If T_1 is quasisimilar to T_2, then HyperLat (T_1) (lattice) is isomorphic to HyperLat(T_2).*

Corollary 2.2.50 *Let T_1 be c.n.u. C_{11} contractions with finite defect indices. If $K_1, K_2 \in HyperLat(T)$ and $T|_{K_1}$ is quasisimilar to $T|_{K_2}$, then $K_1 = K_2$.*

Lemma 2.2.51 *Suppose $A \in B(\mathcal{H})$ and $B \in B(\mathcal{K})$ are quasisimilar operators on $\mathcal{H}$. If B has a nontrivial hyperinvariant subspace, then A has a nontrivial hyperinvariant subspace.*

Proof Let $V : \mathcal{H} \to \mathcal{K}$ and $W : \mathcal{K} \to \mathcal{H}$ be quasi-affinities of A and B. That is, $BV = VA$ and $AW = WB$. Let $\mathcal{N}$ be a nontrivial invariant subspace for B. Define $\mathcal{M} = \bigvee\{XW\mathcal{H} : X \in \{A\}'\}$. Clearly $\mathcal{M}$ is B-hyperinvariant and $\mathcal{M} \neq \{0\}$ because $\mathcal{M} \supset W\mathcal{N}$. Moreover, $\mathcal{M} \neq \mathcal{H}$ because $V\mathcal{M} = V\{\bigvee\{XW\mathcal{N} : X \in \{A\}'\} \subset V\{Y\mathcal{N} : Y \in \{B\}' \subset \mathcal{N} \neq \mathcal{K} = (\bar{\mathcal{N}}\mathcal{H})$. Thus $\mathcal{M}$ is nontrivial.

2.3 Almost Similarity of Some Operators

We investigate on some results of operators which are almost similar.

Theorem 2.3.1 *An operator $T \in B(\mathcal{H})$ is hermitian if and only if $(T+T^*)^2 \geq 4T^*T$.*

Theorem 2.3.1 helps us to prove the following result, where we assume the equality sign of this Theorem.

Proposition 2.3.2 $[2, Proposition 1.5]$
If $A, B \in B(\mathcal{H})$ such that $A \overset{a.s}{\sim} B$ and B is hermitian, then A is hermitian.

Proof
Since $A \overset{a.s}{\sim} B$ there exists an invertible operator N such that $A^*A = N^{-1}(B^*B)N$, on multiplying both sides by 4, we have,
$4A^*A = N^{-1}(4B^*B)N$.................................(1)
Also $A \overset{a.s}{\sim} B$, implies $A^* + A = N^{-1}(B^* + B)N$, on squaring both sides, we obtain,
$N^{-1}(B^* + B)NN^{-1}(B^* + B)N = (A^* + A)^2$. Thus
$N^{-1}(B^*B)^2N = (A + A^*)^2$..........(2)
Since B is hermitian, we have that $(B + B^*)^2 = (2B)^2 = 4B^2 = 4B^*B$. Substituting this in (2) we get
$N^{-1}(4B^*B)N = (A + A^*)^2$........(3)
From (1) and (3) we have $4A^*A = (A + A^*)^2$ which shows that A is hermitian, by Theorem 2.3.1.

Proposition 2.3.3 $[2, Proposition 2.1]$ *If $A, B \in B(\mathcal{H})$ such that A and B are unitarily equivalent, then $A \overset{a.s}{\sim} B$.*

Proof (see [2]).

Proposition 2.3.4 *If $A, B \in B(\mathcal{H})$ such that $A \overset{a.s}{\sim} B$ and if A is hermitian, then A and B are unitarily equivalent.*

Proof By assumption there exists an invertible operator N such that $A^* + A = N^{-1}(B^* + B)N$. Since A is hermitian and $A \overset{a.s}{\sim} B$, by Proposition 2.3.2, B is hermitian, which implies that $A = N^{-1}BN$. This implies that A and B are similar (i.e $A \asymp B$) and since both operators are normal (both A and B are hermitian from Proposition 2.3.2), they are unitarily equivalent.

Remark 2.3.5 *The above proposition 2.3.4 gives a condition under which almost-similarity of operator implies similarity.*

Proposition 2.3.6 $[15, Proposition 1.7]$ *If $A, B \in B(\mathcal{H})$ such that $A \overset{a.s}{\sim} B$ and A is a projection, then so is B.*

Proof $A \overset{a.s}{\sim} B$ implies that there exists an invertible operator N such that
$A^*A = N^{-1}(B^*B)N$ .....(1) and
$A^* + A = N^{-1}(B^* + B)N$....(2).
Since A is a projection, it is hermitian, i.e $A^* = A$ and this implies (by proposition 2.3.2) that B is hermitian. From (1) we have $A^2 = A = N^{-1}B^2N$ and from (2) we have, $2A = N^{-1}2BN$. Thus $A = N^{-1}BN$. This implies that $N^{-1}B^2N = N^{-1}BN$ which proves that B is a projection.
We investigate partially isometric operators under almost similarity.

Definition 2.3.7 *An operator $T \in B(\mathcal{H})$ is said to be partially isometric in case T^*T is a projection. Equivalently, $TT^*T = T$,i.e $(T^*T)^2 = T^*T$ and $(T^*T)^* = T^*T$.*

Proposition 2.3.8 $[15, Proposition 1.6]$ *If $A, B \in B(\mathcal{H})$ such that $A \overset{a.s}{\sim} B$ and A is partially isometric then so is B.*

Proof $A \overset{a.s}{\sim} B$ implies that there exists an invertible operator N such that $B^*B = N^{-1}(A^*A)N$. Since A is partially isometric,A^*A is a projection ($(A^*A)^2 = A^*A$), which implies that $[N^{-1}(B^*B)N][N^{-1}(B^*B)N] = N^{-1}(B^*B)N$. Thus we have $N^{-1}(B^*BB^*B)N = N^{-1}(B^*B)N$ which implies that $(B^*B)^2 = B^*B$. This shows that B^*B is a projection, which implies that B is partially isometric.
The following result shows that almost similarity is an equivalence relation.

Theorem 2.3.9 *Let $A, B, C \in B(\mathcal{H})$. Then;*
(i)$A \overset{a.s}{\sim} A$.
(ii)If $A \overset{a.s}{\sim} B$, then $B \overset{a.s}{\sim} B$.
(iii)If $A \overset{a.s}{\sim} B$ and $B \overset{a.s}{\sim} A$, then $A \overset{a.s}{\sim} C$.

Proof (i) Let $A \in B(\mathcal{H})$. Then $A^*A = N^{-1}(B^*B)N$, where N is an invertible operator. It is also clear that, $A^* + A = N^{-1}(B^* + B)N$. Hence $A \overset{a.s}{\sim} A$. In this case we may choose without loss of generality that, $N = I$.
(ii) Now, suppose that $A \overset{a.s}{\sim} B$. We show that $B \overset{a.s}{\sim} A$.
Since $A \overset{a.s}{\sim} B$, there exists an invertible operator N such that
$A^*A = N^{-1}(B^*B)N$..............................1)
and $A^* + A = N^{-1}(B^* + B)N$.......................(2)
Since N is invertible, upon pre- and post-multiplication of (1) and (2) by N and N^{-1}, respectively and applying the adjoint operation, we have $B^*B = M^{-1}A^*AM$ and $B^* + B = M^{-1}(A^* + A)M$, where $M = N^{-1}$, which is an invertible operator, since N^{-1} is invertible. Hence $B \overset{a.s}{\sim} A$.
(iii)Let A, B and C be in $B(\mathcal{H})$. Suppose $A \overset{a.s}{\sim} B$ and $B \overset{a.s}{\sim} C$. Then we have;
$A^*A = N^{-1}(B^*B)N$, $A^* + A = N^{-1}(B^* + B)N$, (3)
and $B^*B = M^{-1}(C^*C)M$, $B^* + B = M^{-1}(C^* + C)M$, (4)
where N and M are invertible operators.
Using (3) and (4) we have that;
$A^*A = N^{-1}[M^{-1}(C^*C)M]N = (MN)^{-1}(C^*C)(MN) = S^{-1}(C^*C)S$
and
$A^* + A = N^{-1}[M^{-1}(C^* + C)M]N = (MN)^{-1}(C^* + C)(MN) = S^{-1}(C^* + C)S$, where $S = MN$, is invertible (since M and N are invertible). Hence $A \overset{a.s}{\sim} C$.

Remark 2.3.10 *Theorem 2.3.9 shows that almost-similar relation is an equivalence relation on $B(\mathcal{H})$.*

Proposition 2.3.11 [15, *Proposition*1.1] *Let $A, B \in B(\mathcal{H})$. Then:*
(i) If $A \overset{a.s}{\sim} 0$, then $A = 0$.

(ii)If $A \overset{a.s}{\sim} B$ and B is isometric, then A is isometric.

Proof (i) $A \overset{a.s}{\sim} 0$ means that $A^*A = N^{-1}(0)N$ and $A^* + A = N^{-1}(0)N$, which implies that $A = 0$.
(ii)$A \overset{a.s}{\sim} B$ means that $A^*A = N^{-1}(B^*B)N$ and $A^* + A = N^{-1}(B^* + B)N$. Since B is isometric, $B^*B = I$. So $B^*B = I$ implies that $A^*A = N^{-1}(I)N = I$. Thus, A is isometric.

Remark 2.3.12 *An operator $T \in B(\mathcal{H})$ is called quasidiagonal (quasitriangular), denoted by (QT) if there exists an increasing sequence $\{P_n\}_{n=1}^{\infty}$ of finite rank (orthogonal) projections such that $P_n \to 0$ (strongly, $n \to \infty$) and $\|TP_n - P_nT\| \to o$ ($\|TP_n - P_nTP_n\| \to o$, respectively), as $n \to \infty$ [6,11]. The class of biquasitriangular operators, denoted by (BQT) is defined as $(BQT) = \{T \in B(H) : T$ and its adjoint T^* are quasitriangular. Quasidiagonality was defined and studied by Halmos in [9] and extensively analyzed by Smucker in his doctorial thesis . Quasi-triangularity can be illustrated further as follows.*

An operator matrix $Q = (q_{ij})$ is quasi-triangular (Hessenberg) matrix if $h_{ij} = 0$ whenever $i > j = 1$. That is, Q is a Hessenberg matrix if all the entries below the subdiagonal of Q are zero.

Corollary 2.3.13 *Let $A \in B(\mathcal{H})$ and suppose that $A \overset{a.s}{\sim} S_+$, where S_+ denotes the unilateral shift of finite multiplicity. Then A is a completely non-unitary contraction such that $Re(A) \sim Q$, where Q is a quasi-diagonal operator and $Re(A)$ denotes the real part of A.*

Proof Since $A \overset{a.s}{\sim} S_+$, $A^*A = N^{-1}(S_+^*S_+)N$, $A^* + A = N^{-1}(S_+^* + S_+)N$, where N is an invertible operator. Since $S_+^*S_+ = I$. By Proposition 2.3.11, A is an isometry (indeed a c.n.u isometry). It is clear by operator computation that $S_+^* + S_+$ is a quasi-diagonal operator Q (see [4]). Hence $Re(A) \sim Q$.

Remark 2.3.14 *Corollary 2.3.13 says, indirectly, that quasidiagonality is not preserved under similarity . Some authors have investigated the classes* (QT) *and* (BQT) *and have shown that these classes are invariant under similarities.*

Corollary 2.3.13 can be strengthened to cover unitary equivalence to the unilateral shift(see [17, Prop. 2.3, p.38]).

Proposition 2.3.15 *If* $A \in B(\mathcal{H})$ *and* $A \overset{a.s}{\sim} I$, *then* $A = I$.

proof Since $A \overset{a.s}{\sim} I$, there exists an invertible operator N such that
$I^*I = N^{-1}(A^*A)N$ (5)
and
$I^* + I = N^{-1}(A^* + A)N$. (6)
From (5) and (6), we conclude that $A^*A = I$ and $A^* + A = 2I$. Since $A^*A = I$, we get $A^2 - 2A + I = 0$....(*). We now show that the solution of (*) is I. Let $x \in B(H)$, then $(A^2 - 2A + I)x = (A - I)(A - I)x = 0$. Substitute $(A - I)x$ by y. We obtain $(A - I)y = 0$ and hence $Ay = y$ and $Ax = x + y$. By iteration we obtain $x = x + ny$ for any natural number n. Hence,
$n\|y\| = \|ny\| = \|A^n - x\| \leq \|A^n x\| + \|x\| = \|x\| + \|x\| = 2\|x\|$ so that $n\|y\| \leq 2\|x\|$ for all natural number n.
Thus, $\|y\| \leq 2/n\|x\| \to 0$ as $n \to \infty$ and hence $y = 0$. consequently $y(A - I)x = 0$ for all $x \in H$. This implies that $Ax = x$ for all x and hence $A = I$.

Proposition 2.3.16 *Let* A *be a unitary operator and* $B \in B(\mathcal{H})$ *such that* $A \overset{a.s}{\sim} B$. *Then either* B *is an isometry or a unitary operator.*

If B is assumed to be hermitian in Proposition 2.3.16, then B is unitary.
This leads us to the following conjecture.
Claim: Two operators A and B in $B(\mathcal{H})$ are similar ($A \sim B$) if and only if both A and B are hermitian and $A \overset{a.s}{\sim} B$.

Theorem 2.3.17 [16] *Let* $\mathcal{H}$ *be a Hilbert space,* $A \in B(\mathcal{H})$ *be a bounded linear operator* *and A^* the Hilbert space adjoint operator A. Then A is compact if and only if A^*A is compact.*

Proposition 2.3.18 $[15, Proposition 1.3]$ *If $A, B \in B(\mathcal{H})$ such that $A \overset{a.s}{\sim} B$, and if A is compact, then so is B.*

Proof By assumption there exists an invertible operator N such that $B^*B = N^{-1}(A^*A)N$. Since A is compact, $N^{-1}(A^*A)N$ is compact which implies that B^*B is compact. By the above theorem the result follows.

Proposition 2.3.19 $[15, Proposition 2.7]$ *If $A, B \in B(\mathcal{H})$ such that $A \overset{a.s}{B}$, then $(A + \lambda I) \overset{a.s}{\sim} (B + \lambda I)$ for all real λ.*

Proof By assumption, there exists an invertible operator N such that
$A^*A = N^{-1}(B^*B)N$......(i)
$A^* + A = N^{-1}(B^* + B)N$....(ii).
From (ii) we have $A^* + A = N^{-1}B^*N + N^{-1}BN$ which implies that $A^* + A + 2\lambda = N^{-1}B^*N + N^{-1}BN + 2\lambda$. Thus we have $(A^* + \lambda I) + (A + \lambda I) = N^{-1}(B^* + \lambda I)N + N^{-1}(B + \lambda I)N = N^{-1}(B^* + \lambda I) + (B + \lambda I)N$.....(iii). From (iii) we have $\lambda A^* + A\lambda + \lambda^2 = N^{-1}\lambda B^*N + N^{-1}\lambda BN + N^{-1}\lambda^2 N$.....(iv). Adding (i) and (iv) we obtain $A^*A + \lambda A^* + A\lambda + \lambda^2 = N^{-1}\lambda B^*N + N^{-1}\lambda BN + N^{-1}\lambda^2 N + N^{-1}B^*BN$ which implies that $(A^* + \lambda I)(A + \lambda I) = N^{-1}(B^* + \lambda I)(B + \lambda I)N$.Thus $(A + \lambda I)^*(A + \lambda I) = N^{-1}(B + \lambda I)^*(B + \lambda I)N$......(v). From (iii) and (v) we conclude that $(A + \lambda I) \overset{a.s}{\sim} (B + \lambda I)$.

Corollary 2.3.20 *If $A, B \in B(\mathcal{H})$ are projection operators such that $A \overset{a.s}{\sim} B$ and $(A + \lambda I) \overset{a.s}{\sim} (B + \lambda I)$ for all real λ, then $\sigma_p(A) = \sigma_p(B)$.*

Proof Since $A \overset{a.s}{\sim} B$, there exists an invertible operator N such that
$A^*A = N^{-1}(B^*B)N$......(i)
$A^* + A = N^{-1}(B^* + B)N$....(ii).
Since $A = A^*$ and $B = B^*$, (ii) results to $2A = N^{-1}2BN$ implying that$A = N^{-1}BN$, i.e $NA = BN$ i.e $\sigma_p(A) = \sigma_p(B)$. Similarly, since $A^* = A = A^2$ and $B^* = B = B^2$, (ii)

becomes $A^2 = N^{-1}B^2N$ implying that$A = N^{-1}BN$, i.e $NA = BN$ and so $\sigma_p(A) = \sigma_p(B)$.

Remark 2.3.21 *Corollary 2.3.20 shows that, for hermitian or projection operators, if $A \overset{a.s}{\sim} B$, then they have equal spectrum.*

We shows some results on the class of θ-operators in relation to almost similarity.

Definition 2.3.22 *An operator $A \in B(\mathcal{H})$ is called θ-operator if $A^* + A$ commutes with A^*A. The class of θ-operators in $B(\mathcal{H})$ is denoted by θ i.e. $\theta = \{A \in B(\mathcal{H}) : [A^*A, A^* + A] = 0\}$.*

Proposition 2.3.23 $[15, Proposition 1.4]$ *If $A, B \in B(\mathcal{H})$ such that $B \in \theta$ and $A \overset{a.s}{\sim} B$, then $A \in \theta$.*

Proof By assumption there exists an invertible operator N such that $A^*A = N^{-1}(B^*B)N$ and $A^* + A = N^{-1}(B^* + B)N$. Thus, we have,
$N^{-1}(B^*B)N][N^{-1}(B^* + B)N] = A^*A(A^* + A)$......(i) and
$N^{-1}(B^* + B)N][N^{-1}(B^*B)N] = (A^* + A)A^*A$......(ii).
From (i) we have $N^{-1}B^*B(B^* + B)N = A^*A(A^* + A)$.....(iii) and from (ii) we have $N^{-1}(B^* + B)B^*BN = (A^* + A)A^*A$....(iv). Since $B \in \theta$, the left hand side of (iii) and (iv) are equal, which implies that the right hand side of (iii) and (iv) are equal. Thus $A \in \theta$.

Definition 2.3.24 [3] *If $A \in \theta$, the $4A^*A - (A^* + A)^2 \geq 0$. Define $B = A + A^* + i(4A^*A - (A^* + A)^2)^{1/2}/2)$.Then B is normal,$\sigma(A)$ is contained in the upper half plane, $B^*B = A^*A$ and $B^* + B = A^* + A$. In particular, $(\lambda I - A^*)(\lambda I - A) = (\lambda I - B^*)(\lambda I - B)$ for all λ.*

Proposition 2.3.25 $[15, Proposition 2.44]$ *If $A \in B(\mathcal{H})$ then $A \in \theta$ if and only if $A \overset{a.s}{\sim} B$ for some normal operator B.*

Proof Let $A \in \theta$, then $4A^*A-(A^*+A)^2 \geq 0$ and the operator $B = A+A^*+i(4A^*A-(A^*+A)^2)^{1/2}/2)$ is normal with $A^*A = B^*B$ and $A^*+A = B^*+B$ by the above definition. Thus $A^*A = I^{-1}(B^*B)I$ and $A^*+A = I^{-1}(B^*+B)I$. Hence $A \overset{a.s}{\sim} B$.

Conversely, let $A \overset{a.s}{\sim} B$ for some normal operator B. Then there exists an invertible operator N such that $A^*A = N^{-1}(B^*B)N$ and $A^*+A = N^{-1}(B^*+B)N$

$A^*A(A^*+A) = N^{-1}B^*B(B^*+B)N$........(i)

$(A^*+A)A^*A = N^{-1}(B^*+B)B^*BN$ ....(ii)

Since B is normal, $B \in \theta$. Thus the right hand side of (i) and (ii) are equal which implies that $(A^*+A)A^*A = A^*A(A^*+A)$. Hence $A \in \theta$.

Proposition 2.3.26 $[15, Proposition 2.5]$ *If $T \in B(\mathcal{H})$ is invertible and $T \overset{a.s}{\sim} U$ for some unitary operator $U \in B(\mathcal{H})$, then T is unitary.*

Proof Since $T \overset{a.s}{\sim} U$, there exists an invertible operator N such that $T^*T = N^{-1}(U^*U)N = I$. This implies that $T^{*-1}T^*TT^{-1} = T^{*-1}T^{-1}$. Since $T^{*-1}T^*TT^{-1} = I$, $T^{*-1}T^{-1} = (TT^*)^{-1} = I$ which implies that $TT^* = I$. Thus $T^*T = TT^* = I$. Hence T is unitary.

Proposition 2.3.27 $[15, Proposition 2.6]$ *An operator $A \in B(\mathcal{H})$ is isometric if and only if $A \overset{a.s}{\sim} U$ for some unitary operator U.*

Proof Let A be isometric, then $A \in \theta$. Thus by Proposition 2.3.25 there is a normal operator N where $A \overset{a.s}{\sim} N$. Since $A \overset{a.s}{\sim} N$, N is isometric by Proposition 2.3.11 (ii). Thus N is unitary.

Conversely, if $A \overset{a.s}{\sim} U$ for some unitary operator U then there exists an invertible operator with $N^{-1}(A^*A)N = U^*U = I$. This implies that $A^*A = N^{-1}N = I$. Thus A is isometric.

Proposition 2.3.28 *Let $A \in B(\mathcal{H})$ such that A is almost similar to an isometry T. Then the direct summand of A are isometric.*

Proof Since T is an isometry, by von Neumann-Wold decomposition [17], $T = S_+ \bigoplus U$, where U is unitary and S_+ is the unilateral shift. Since $A \overset{a.s}{\sim} T$, there exists an operator

N such that

$A^*A = N^{-1}[(S_+ \bigoplus U)^*(S_+ \bigoplus U)]N$

$= N^{-1}(S_+^*S_+ \bigoplus U^*U)N$

$= N^{-1}(I \bigoplus I)N$

Now, let $A = A_1 \bigoplus A_2$. Then $A^*A = (A_1^*A_1 \bigoplus A_2^*A_2)$. This shows that $(A_1^*A_1 \bigoplus A_2^*A_2) \sim I \bigoplus I$. From this equation, it follows that $A_i^*A_i \sim I$, $i = 1, 2$. This means that there exists an operator N such that $A_i^*A_i = N^{-1}IN = I$. Thus $A_i^*A_i = I$. This proves that direct summand of A are isometric.

Remark 2.3.29 *Proposition 2.3.28 does not mean that $A_1 \sim U$ and $A_2 \sim S_+$. If the relation of almost similarity is replaced with unitary equivalence in Proposition 2.3.38, then the direct summands and sums are preserved by Corollary 2.1.4.*

Corollary 2.3.30 *Let $A \in B(\mathcal{H})$ such that $A = A_1 \bigoplus A_2$, where A_1 is unitary and A_2 is c.n.u. If A is unitarily equivalent to a unitary T, the A_1 is unitary.*

bf Proof Since A is unitarily equivalent to a unitary T, there exists a unitary operator U such that $A = U^{-1}TU$. By the von Neumann-Wold decomposition , any isometry T has the decomposition $T = U \bigoplus S_+$, where U is unitary and S_+ is the unilateral shift. Since in the hypotheses, T is unitary, then c.n.u. direct summand of T is missing. By Corollary 2.1.4 , $A_1 \bigoplus A_2 \cong T = U \bigoplus S_+$. Thus $A_1 \cong U$ and A_2 is missing. Hence the result.

Proposition 2.3.31 *If $A, B \in B(\mathcal{H})$ are contractions such that $A \overset{a.s}{\sim} B$ and B is c.n.u, then A is c.n.u.*

Proof By the Nagy-Foias-Lager decomposition for contraction [10] $B = U \bigoplus C$, on $\mathcal{H} = \mathcal{H}_1 \bigoplus \mathcal{H}_2$, where $U = B|_{\mathcal{H}_1}$ is the unitary part of B and $C = B|_{\mathcal{H}_2}$ is the c.n.u part of B. Since B is c,n,u., the unitary part U is missing or $\mathcal{H}_1 = \{0\}$. Without loss of generality we suppose that $B = C$. Then

$A^*A = N^{-1}(B^*B)N = N^{-1}(C^*C)N.$

This shows that $A^*A \sim C^*C$. Now suppose $A = A_1 \bigoplus A_2$, where A_1 is unitary and A_2 is c.n.u. Then $(A_1^*A_1 \bigoplus A_2^*A_2) \sim C^*C$. That is, $(I \bigoplus A_2^*A_2) \sim C^*C$. This holds if and only if the direct summand A_1 is missing. That is, $A = A_2$. Hence A is c.n.u.

Theorem 2.3.32 $[15, Theorem 2.3]$ *If $A \in B(\mathcal{H})$ is normal, then $A \overset{a.s}{\sim} A^*$.*

Remark 2.3.33 *The converse of this theorem is not true in general,*

for consider $A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ and $N = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. By matrix computation,
$A^*A = N^{-1}(AA^*)N$ and $A^* + A = I^{-1}(A^* + A)I$. That is, $A \overset{a.s}{\sim} A^*$ although A is not normal.

Theorem 2.3.34 *Let $A, B \in B(\mathcal{H})$. Suppose $A \overset{a.s}{\sim} B$. Then A is quasi-similar to B if and only if A and B are orthogonal projections.*

Proof Since $A \overset{a.s}{\sim} B$, there exists an invertible operator N such that $A^*A = N^{-1}(B^*B)N$......(*) and $A + A^* = N^{-1}(B + B^*)N$............(**).
From (*) we have $NA = BN$, since $A^*A = A$ and $B^*B = B$. A and B are projections imply that $A^* = A$ and $B^* = B$. Therefore (**) yields, $2A = N^{-1}2BN$, that is, $A = N^{-1}BN$, which implies that, $NA = BN$, hence the result.

Corollary 2.3.35 *If $A, B \in B(H)$ are normal where H is a finite dimensional Hilbert space such that A and B are quasi-similar, then $A \overset{a.s}{\sim} B$.*

Proof Since $A, B \in B(\mathcal{H})$ are quasi-similar, there exists quasi-affinities $X \in B(\mathcal{H}, \mathcal{K})$ and $Y \in B(\mathcal{K}, \mathcal{H})$ such that $XA = BX$ and $BY = YA$..........................(1)
X and Y are both invertible and so XY and YX are both invertible. Without loss of generality, let $N = XY \ or \ YX$ Then $XY \in \{A\}'$ and $YX \in \{B\}'$ i.e $AXY = XYA$ implying that $A = XYA(XY)^{-1}$ and $YXB = BYX$ which implies $B = (YX)^{-1}BYX$.......(2). Since XY is invertible, $(XY)^* = Y^*X^*$ and $(XY)^{-1*} = ((XY)^*)^{-1} = (Y^*X^*)^{-1} = X^{*-1}Y^{*-1}$.
Now,$A^*A = (X^{*-1}Y^{*-1}A^*Y^*X^*)XYA(XY)^{-1} = (X^{*-1}Y^{*-1}Y^*BX^*)XBYY^{-1}X^{-1}$

$= (X^{*-1}BX^{*})(XBX^{-1})$.

Since A and B are similar normal operators, they are unitarily equivalent, so that,

$A^{*}A = (X^{*-1}BX^{*})XBX^{-1} = XB^{*}BX^{-1}$..............(3)

Also, $A + A^{*} = (X^{*-1}BX^{*}) + (XBX^{-1}) = XB^{*}X^{-1} + XBX^{-1} = X(B^{*} + B)X^{-1}$......(4),that is,

$A^{*}A = N^{-1}B^{*}BN$ and $A^{*} + A = N^{-1}(B^{*} + B)N$ where $N = X^{-1}$ is an invertible operator. This shows that $A \overset{a.s}{\sim} B$.

Remark 2.3.36 *Corollary 2.2.52 gives a condition under which similarity implies quasi-similarity which in turn implies almost similarity.*

Chapter 3

ON METRIC EQUIVALENCE OF SOME OPERATORS

Recall that two operators $A \in B(\mathcal{H})$ and $B \in B(\mathcal{H})$ are said to be metrically equivalent, denoted by $A \sim_m B$, if $\|Ax\| = \|Bx\|$, equivalently, $| < Ax, Ax > |^{1/2} = | < Bx, Bx > |^{1/2}$ for all $x \in \mathcal{H}$, that is, $A^*A = B^*B$.
The numerical range $W(T)$ of an operator $T \in B(\mathcal{H})$ is defined as $W(T) = \{\lambda \in \mathbb{C} : \lambda =< Tx, x >, \|x\| = 1\}$ and the numerical radius $w(T)$ of T is defined as $w(T) = sup\{|\lambda| : \lambda \in W(T)\}$.
An operator T is said to be normaloid if $r(T) = \|T\|$, (equivalently, $\|T^n\| = \|T\|^n$). In complex Hilbert space H, every normal operator is normaloid and so is every positive operator.

Theorem 3.0.37 *A necessary and sufficient condition that an operator $T \in B(\mathcal{H})$ be normal is that $\|Tx\| = \|T^*x\|$ for every $x \in \mathcal{H}$.*

Corollary 3.0.38 *An operator $T \in B(\mathcal{H})$ is normal if and only if T and T^* are metrically equivalent.*

Theorem 3.0.39 *If T is a normal operator, then there exists a unitary operator U such that $T^* = UT$.*

Theorem 3.0.40 *Let S and T be bounded linear operators on a Hilbert space $\mathcal{H}$. If $T^*T = S^*S$, then there exists a partial isometry U such that the initial space $M = \overline{Ran(T)}$ and the final space $N = \overline{Ran(S)}$, and $S = UT$.*

Thus, we have the following result.

Corollary 3.0.41 *If S and T are metrically equivalent normal operators, then there exists a unitary operator U such that $S = UT$.*

Proof By hypotheses $S^*S = T^*T$ yields $\|Sx\| = \|Tx\|$ for all $x \in \mathcal{H}$. This means that $Tx_1 = Tx_2$ implies that $Sx_1 = Sx_2$, for all $x_1, X_2 \in \mathcal{H}$. Define an operator $V : Ran(T) \to Ran(S)$ by $VTx = Sx$. Thus $\|Sx\| = \|Tx\| = \|VTx\|$. V can be extended to $\overline{V} : \overline{Ran(T)} \to \overline{Ran(S)}$. Define $Ux = \overline{V}P_{\mathcal{H}}x$ for all $x \in \mathcal{H}$, where $P_{\mathcal{M}}$ is the orthogonal projection onto $\mathcal{M} = \overline{RanT}$. Clearly U is a partial isometry with initial space M and $UTx = \overline{V}P_{\mathcal{M}}Tx = \overline{V}Tx = VTx = Sx$ for any $x \in \mathcal{H}$. Since S and T are normal, U is invertible and hence unitary, which proves the claim.
We note that the converse of corollary 3.0.41 is also true.
An operator T is said to be bounded below in case there exists a constant $N > 0$ such that $\|Tx\| \geq N\|x\|$ for all $x \in \mathcal{H}$ (or equivalently, if there exists a constant $\alpha > 0$ such that $T^*T \geq \alpha I$)

Lemma 3.0.42 *Let S and T be linear operators on a Hilbert space $\mathcal{H}$. If $S \sim_m T$, then:*
(i)If T is isometric, then S is also isometric.
(ii)If T is a contraction , S is also a contraction.
(iii)If T is a partial isometry, then S is also a partial isometry.
(iv)If S and T are positive, then $S = T$.
(v)If S is bounded below, then T is also bounded below. Moreover, S is injective and so is T. If in addition, S has a dense range, then both S and T are invertible.

(i) The proof follows from $S^*S = T^*T = I$.
(ii)This follows from $\|Sx\| = \|Tx\| \leq \|x\|$ for all $x \in \mathcal{H}$.

(iii)If T is a partial isometry, thenT^*T is a projection. Since $S \sim_m T$, we have that $S^*S = T^*T$. This shows that S^*S is a projection and hence S is a partial isometry.
(iv)Positivity of S and T implies that S and T are self-adjoint. Thus $S^2 = S^*S$ and $T^2 = T^*T$. By hypothesis, we have that $S^2 = T^2$. Thus $S = T$.
(v) By hypothesis $T^*T = S^*S \geq \alpha I$. This proves that T is bounded below. To prove injectivity os S, let $x \in Ker(S)$. Then
$0 =< Sx, Sx >$
$=< x, S^*Sx >$
$\leq< x, \alpha Ix >$
$= \alpha < x, x >$
$= \alpha \|x\|^2$.
Since $\alpha > 0$, we conclude that $\|x\|^2 = 0$ which consequently implies that $\|x\| = 0$ and hence $x = 0$. This shows that $Ker(S) = 0$. Thus, S is injective. By the same argument T is injective. Injectivity of S and hence that of T implies that $Ran(S)$ and $Ran(T)$ are both closed linear subspaces of H. Thus, $Ran(S) = Ran(\bar{S}) = H$, which proves that S is surjective. Thus, S is bijective and hence invertible. Invertibility of T follows immediately from the hypothesis.
From Lemma 3.0.42, it is clear that if S is invertible and S is metrically equivalent to T, then T is invertible. This follows from $S^* = T^*TS^{-1}$. Invertibility of S implies the invertibility of S^* and hence the right hand side is also invertible., which implies the invertibility of T^*T and hence that of T. It is evident that if $S \sim_m T$ and S is a symmetry(self-adjoint and unitary operator), then T is isometric.

Proposition 3.0.43 *Let $T \in B(\mathcal{H})$. If $T^*T \geq I$ and $TT^* \geq I$, then T is invertible.*

Proof From the hypothesis T and T^* are both bounded below and hence injective and invertible by Lemma 3.0.42.
The condition for $T^*T \geq I$ in Propositon 3.0.43 cannot be dropped. In other words, if $T^*T \geq I$ and $TT^* \not\geq I$, then T need not be invertible. To see this, let T be the unilateral shift $T : l^2 \to l^2$. Then $T^*T = I$, but T is not invertible.

Proposition 3.0.44 *If $T \in B(\mathcal{H})$ is normal and $T^*T \geq I$, then T is invertible.*

Proof By hypothesis $T^*T = TT^*$ and $T^*T \geq I$. This implies that $TT^* \geq I$. The result follows from Proposition 3.0.43.

Proposition 3.0.45 *If $T \in B(\mathcal{H})$ is normal and bounded below, then T is invertible.*

Proof From the hypothesis, both T and T^* are injective. That is $Ker(T) = Ker(T^*) = \{0\}$. Thus $Ran(T)^{\perp} = Ker(T^*) = \{0\}$. This proves that $Ran(T) = \mathcal{H}$. Thus, T is surjective. Combining the two results, we have that T is invertible.

Theorem 3.0.46 *If T and S are metrically equivalent operators on $\mathcal{H}$, then $\|S\| = \|T\|$.*

Proof The proof follows immediately from $\|T\|^2 = \|T^*T\| = \|TT^*\| = \|S^*S\| = \|SS^*\| = \|S\|^2$.

Remark 3.0.47 *Note that the converse of theorem 3.0.44 is not always true. There exists operators with the same norm which are not metrically equivalent.*

Theorem 3.0.48 *If T and S are metrically equivalent then $w(|T|) = w(|S|)$.*

Proof By theorem 3.0.44, we have that $\|T\| = \|S\|$. Since T^*T is self-adjoint, it is normal and thus $w(T^*T) = \|T\|^2$. Thus $w(T^*T) = w(S^*S)$. Hence $w(|T|) = w(|S|)$.

Remark 3.0.49 *We note that unlike unitarily equivalent operators, metrically equivalent operators S and T need not have equal numerical range. Note also that the spectrum of S may be equal to the spectrum of T yet S and T are not metrically equivalent.*

For instance, the operators represented by the matrices

$$S = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

and

$$T = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

in $\mathbb{C}^2$ have the property that $\sigma(S^*S) = \sigma(T^*T)$ and $\sigma(S) = \sigma(T)$ but S and T are not metrically equivalent operators. Similarly, two operators may be metrically equivalent yet have unequal spectra. For example, the unilateral shift and identity operators on $\mathcal{H} = l^2$ are metrically equivalent but have unequal spectra. Clary [4] proved that quasi-similar hyponormal operators have equal spectra. This claim was supported by Douglas ([6, Lemma 4.1, p.23]) who proved using the Putnam- Fuglede commutativity theorem that quasi-similar normal operators are unitarily equivalent and hence have equal spectra.

Proposition 3.0.50 *Metrically equivalent operators S and T need not have equal spectra.*

Remark 3.0.51 *It is also true that metrically equivalent normal operators S and T need not have equal spectra.*

Consider,,for instance, the operators represented by the matrices

$$S = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

and

$$T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

in $\mathbb{C}^2$. A simple computation shows that $\sigma(S) = \{-1, 1\}$ and $\sigma(T) = \{1\}$. It is clear that $W(S) \neq W(T)$. Thus, metric equivalence does not preserve numerical range.

Theorem 3.0.52 *If S and T are metrically equivalent normaloid operators, then $r(S) = r(T)$.*

Proof From hypothesis, we have that $r(S) = \|S\| = \|T\| = r(T)$.

The converse of Theorem 3.0.50 is not generally true. The operators represented by the matrices

$$S = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

and

$$T = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

in $\mathbb{C}^2$ have the property that $r(S) = r(T) = 0$, but a simple computation shows that S and T are not metrically equivalent. This is because S and T are not normal and hence not normaloid. However, we note that S and T have the same numerical range, which in this case is a closed disk centered at 0 and of radius 1/2.

Theorem 3.0.53 *If S and T are metrically equivalent normal operators on $\mathcal{H}$, with respect to polar decomposition $S = U|S|$ and $T = V|T|$, then $|S| = |T|$.*

Proof By hypothesis $S^*S = T^*T$ which implies that $|S|U^*U|S| = |T|V^*V|T|$. Since S and T are normal. U and V are unitaries and hence we have that $|S|^2 = |T|^2$. Since $|S|^2$ and $|T|^2$ are positive, they have unique square roots $|S|$ and $|T|$, respectively. Therefore $|S| = |T|$.

Theorem 3.0.54 *Direct summands of metrically equivalent operators are metrically equivalent.*

Proof Suppose that $S \sim_m T$ and that $S = S_1 \bigoplus S_2$ and $T = T_1 \bigoplus T_2$, where S_1 and T_1 have property "P" and S_2 and T_2 are devoid of property "P". A simple computation shows that $S_i^* S_i = T_i^* T_i$, $i = 1, 2$. This proves the claim.

Theorem 3.0.55 . *Let $T \in B(H)$. If $N \in B(\mathcal{H})$ is normal and $NT = TN$, then $N^*T = TN^*$.*

Theorem 3.0.56 *Let S and T be metrically equivalent operators on a Hilbert space $\mathcal{H}$ and $ST = TS$. If T is normal, then S is quasi-normal.*

Proof Suppose that S and T are metrically equivalent operators, T is normal and $ST = TS$. Post-multiplying by T^* and using the definition of metric equivalence and Theorem 3.0.35 (Fuglede commutativity theorem), we have that $S^*SS = T^*TS = T^*ST = ST^*T = SS^*S$. Thus S is quasi-normal.

Corollary 3.0.57 *If $S \in B(\mathcal{H})$ is a Fredholm operator and S is metrically equivalent to $T \in B(\mathcal{H})$, then T is Fredholm.*

Proof Suppose S is Fredholm and $S^*S = T^*T$. The $Ran(T)$ must be closed since $Ran(S)$ is closed. Since $Ker(S) = Ker(S^*S)$, and $Ker(S^*) = Ker(SS^*)$, we see that $Ker(S)$ is isomorphic to $Ker(T)$ and $Ker(S^*)$ is isomorphic to $Ker(T^*)$, which proves the claim.

Remark 3.0.58 *Corollary 3.0.55 shows that metric equivalence of operators preserves Fredholmness. We note that if S is metrically equivalent to S, the $ind(S)$ need not be equal to $ind(T)$, unless S is Fredholm.*

Theorem 3.0.59 *Let $A \in B(\mathcal{H})$ and $\mathcal{M}$ be a subspace of $\mathcal{H}$. If P is the projection of $\mathcal{H}$ onto $\mathcal{M}$, then the restriction of A on $\mathcal{M}$ is metrically equivalent to PA.*

Proof Let $x \in \mathcal{M}$. Then $< PAx, x >=< Ax, Px >=< Ax, x >$, and hence the claim follows.

Theorem 3.0.60 *If T is a normal operator and S is metrically equivalent to T, then S is normal.*

Proof Since S is metrically equivalent to T and T is normal, we have:
$S^*S = T^*T = TT^* = SS^*$, hence the proof.
A part of an operator is a restriction of it to an invariant subspace. An extension of an operator $A \in B(\mathcal{H})$ is an operator of the form

$$T = \begin{bmatrix} A & B \\ 0 & C \end{bmatrix}$$

acting on $\mathcal{H}_0 = \mathcal{H} \bigoplus \mathcal{K}$, where $B \in B(\mathcal{K}, \mathcal{H})$ and $C \in B(\mathcal{K})$. Equivalently, $A = P|_{\mathcal{H}} T|_{\mathcal{H}}$, where P is the projection of $\mathcal{H}_0$ onto $\mathcal{H}$. Alternatively, an operator T is an extension of A if A is a part of T. If this is the case, A is called the compression of T to $\mathcal{H}$. If $\mathcal{M} = \mathcal{H}$ is T-reducing, then T has a block matrix representation

$$T = \begin{bmatrix} A & 0 \\ 0 & C \end{bmatrix}$$

with respect to the decomposition $\mathcal{H}_0 = \mathcal{M} \bigoplus \mathcal{M}^{\perp}$ and write $T = A \bigoplus C$. In this case, we say that T is a direct sum of the operator A and C (see [17,311 section 0.5 pg. 18]).

Theorem 3.0.61 *If S and T are metrically equivalent operators on $\mathcal{H}$ and S has an extension $\hat{S}$, then T has an extension $\hat{T}$ where $\hat{S}$ and $\hat{T}$ are metrically equivalent.*

Proof Suppose S has an extension $\hat{S} \in B(\hat{\mathcal{H}})$ with $\mathcal{H} \subseteq \hat{\mathcal{H}}$. Then $S = P|_{\mathcal{H}}\hat{S}|_{\mathcal{H}}$, where P is the projection of $\hat{\mathcal{H}}$ onto $\mathcal{H}$. By hypothesis, $\|Sx\| = \|Tx\|$, for any $x \in \mathcal{H}$. The existence of $\hat{T}$ is thus quaranteed. We now prove that the extensions are metrically equivalent.
Now suppose that T has an extension $\hat{T}$. Then $T = Q|_{\mathcal{H}}\hat{T}|_{\mathcal{H}}$, where Q is the projection of $\hat{\mathcal{H}}$. Thus $\|Tx\| = \|Q|_{\mathcal{H}}\hat{S}|_{\mathcal{H}}\ x\| = \|Tx\| = \|Sx\| = \|P|_{\mathcal{H}}\hat{S}|_{\mathcal{H}}\ x\| = \|\hat{S}x\|$. Hence the prove.

3.1 Relationship between metric equivalence of operators and other equivalence relations

Theorem 3.1.1 *Let S and T be in $B(\mathcal{H})$. If S and T are unitarily equivalent, then they are metrically equivalent.*

Remark 3.1.2 *The converse of Theorem 3.1.1 is not generally true.*

Consider the operators in $\mathbb{C}^2$ represented by the matrices

$$S = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

and

$$T = \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}$$

A simple computation shows that $S^*S = T^*T = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$ which means that S and T are metrically equivalent. However, $\sigma(S) = \{0, 2\} \neq \{0, -2\} = \sigma(T)$. This shows that S and T are not similar, and hence cannot be metrically equivalent.

Question 2 *When does metric equivalence imply unitary equivalence?*

Theorem 3.1.3 *If S and T are metrically equivalent projections, then they are unitarily equivalent.*

Proof Since S is metrically equivalent to T, from Corollary 3.0.39 , there is a unitary operator such that $S = UT$. This together with the fact that both S and T are projections, we have that
$S = S^2 = S^*S = T^*T = UTT^*U^* = UT^2U^* = UTU^*$, which shows that S and T are unitarily equivalent.

Example 3.1.4 *Let $S, T \in B(l^2(\mathbb{N}))$ be defined as follows:*

$$S(x_1, x_2, x_3, ...) = (x_1, x_1, x_2, x_3, ...)$$

and

$$T(x_1, x_2, x_3, ... = (0, x_1, x_2, x_3, ...))$$

.

A simple computation shows that S and T are not metrically equivalent and hence are not unitarily equivalent.

Remark 3.1.5 *Note that unitary and metric equivalence are norm-preserving while similarity is not norm-preserving.*

Definition 3.1.6 *Linear operators S and T acting on a Hilbert space $\mathcal{H}$ are said to be nearly equivalent if there exists a unitary operator U such that $S^*S = U^*T^*TU$ or equivalently, if S^*S and T^*T are unitarily equivalent.*

This concept was introduced by Othman[26]. It is claimed in [26] that unitary equivalence implies near equivalence and an example is provided to prove that the converse is not generally true. That is, near equivalence of operators need not imply unitary equivalence of operators and need not imply similarity of operators. For example, the unilateral shift and the identity operator. Clearly these operators are nearly equivalent but not unitarily equivalent. It is also clear that these operators are not similar.
Let $T \in B(\mathcal{H})$. We denote by $\mathfrak{Re}(T)$ and $I\mathfrak{Re}(T)$ the class of operators nearly equivalent to T and metrically equivalent to T, respectively.
That is,

$$\mathfrak{Re}(T) = \{S \in B(\mathcal{H}) : S^*S = U^*T^*TU\}$$

and

$$\mathfrak{IRe}(T) = \{S \in B(\mathcal{H}) : S^*S = T^*T\}$$

.

Clearly

$$\mathfrak{IRe}(T) \nsubseteq \mathfrak{Re}T$$

.

Theorem 3.1.7 $S \in \mathfrak{Re}(T)$ *if and only if for some unitary operator U, $Sx = TUx$ for all $x \in \mathcal{H}$.*

Corollary 3.1.8 *If $S \in \mathfrak{Re}(T)$, the $\|S\| = \|T\|$.*

Theorem 3.1.9 *Let S and T be metrically equivalent operators in $B(\mathcal{H})$. Then S and T are nearly equivalent if and only if $\|S\| = \|T\|$.*

Theorem 3.1.10 *Two injective weighted shifts $Se_n = \alpha_n e_{n+1}$ and $Te_n = \beta_n e_{n+1}$ in a complex Hilbert space H with orthonormal basis $\{e_n\}$ are metrically equivalent if and only if $|\alpha_n| = |\beta_n|$, for $n = 1, 2, 3, ...$*

Proof $S^*S = T^*T$ implies that $\alpha_n^*\alpha_n = \beta_n^*\beta_n$, which implies that $|\alpha_n|^2 = |\beta_n|^2$ and hence $|\alpha_n| = |\beta_n|$. The converse is trivial.

Theorem 3.1.11 *Let S and T be metrically equivalent operators in $B(\mathcal{H})$. The S and T are unitarily equivalent if and only if they are projection operators.*

Corollary 3.1.12 *If S and T are idempotent and positive operators acting on $\mathcal{H}$, then metric equivalence of S and T is equivalent to unitary equivalence of S and T.*

Corollary 3.1.13 *If P and Q are metrically equivalent projections in $B(\mathcal{H})$ and $R \in B(\mathcal{H})$ is a projection, then PR and QR are metrically equivalent.*

bf Proof By hypothesis, $P^*P = Q^*Q$. Thus $(PR)^*(PR) = R^*P^*PR = R^*Q^*QR = (QR)^*(QR)$.

Remark 3.1.14 *We note that the conclusion of Corollary 3.1.13 is false if R is not a projection.*

Proposition 3.1.15 *Let $A, B \in B(\mathcal{H})$. Then;*
(i) If $A \overset{n:e}{\approx} 0$, then $A = 0$.
(ii)If $A \overset{n:e}{\approx} B$ and B is isometric, then A is isometric.

Proof (i) $A \overset{n:e}{\approx} 0$ implies that $A^*A = U0U^*$, which implies that $A = 0$.
(ii) $A \overset{n:e}{\approx} B$ implies that $A^*A = UB^*BU^*$. Since B is isometric, we have $B^*B = I$. Thus, $A^*A = UIU^* = I$. Hence A is isometric.

Question 3 . *When does near equivalence imply unitary equivalence?*

Theorem 3.1.16 *If A and B are nearly equivalent projections, then they are unitarily equivalent.*

Proof Since A is nearly equivalent to B, we have $A^*A = UB^*BU^*$ ......(*). Using the fact that A and B are projections we get, $A = UBU^*$, which shows that A and B are unitarily equivalent.

Remark 3.1.17 *Note that near equivalence implies metric equivalence if the unitary operator U is the identity operator.*

Proposition 3.1.18 *Let $A, B \in B(\mathcal{H})$;*
(i)If $A \overset{n.e}{\sim} 0$, then $A = 0$.
(ii)If $A \overset{n.e}{\sim} B$ and B is isometric, then A is isometric.

Proof
(i) $A \overset{n.e}{\sim} 0$ implies that $A^*A = U0U^* = 0$. Thus $A = 0$.
(ii)$A \overset{n.e}{\sim} B$ implies that $A^*A = UB^*BU^* =$. Since B is isometric, $B^*B = I$. Thus we have $A^*A = UIU^* = I$. Hence A is isometric.

Proposition 3.1.19 *Let $T \sim_m S$. If T is A-self-adjoint operator, then $T^2 \sim S^2$ if S is self adjoint.*

Proof Since T and S are metrically equivalent and T is A-self-adjoint, we have $T^*T = ATA^{-1}T = AT^2A^{-1} = S^*S$. Thus $T^2 = A^{-1}S^*SA = A^{-1}S^2A$. Hence the prove.

Remark 3.1.20 *We note from Proposition 3.1.19 that if T and S are metrically equivalent and T is A-self-adjoint, then S is A-self-adjoint if both T and S are projections.*

Theorem 3.1.21 *If A and B are nearly equivalent projections, where A and B are self adjoint, then $A^2 \cong B^2$.*

Proof Since A and B are self adjoint, we have $A^2 = A^*A = UB^*BU^* = UB^2U^*$. Hence the prove.

Proposition 3.1.22 *If $A, B \in B(\mathcal{H})$ such that $A \overset{n.e}{\sim} B$ and A is partially isometric then so is B.*

Proof $A \overset{n.e}{\sim} B$ implies that there exists a unitary operator U such that $B^*B = U(A^*A)U^*$. Since A is partially isometric,A^*A is a projection ($(A^*A)^2 = A^*A$), which implies that $[U(B^*B)U^*][U(B^*B)U^*] = U(B^*B)U^*$. Thus we have $U(B^*BB^*B)U^* = U(B^*B)U^*$ which implies that $(B^*B)^2 = B^*B$. This shows that B^*B is a projection, which implies that B is partially isometric.

Chapter 4

ON UNITARY QUASI-EQUIVALENCE OF OPERATORS

Two operators, $S, T \in B(\mathcal{H})$are said to be unitarily quasi-equivalent if there exists a unitary operator U such that $T^*T = US^*SU$ and $TT^* = USS^*U$ and write $S \overset{u:q:e}{\approx} T$. Clearly$S, T \in B(\mathcal{H})$ are unitarily quasi-equivalent if S^*S and T^*T are unitarily equivalent andSS^* and TT^* are unitarily equivalent. Two operators $S, T \in B(\mathcal{H})$ are said to be absolutely equivalent if both the absolute value of the operators are unitarily equivalent. That is, if $|S| = U|T|U^*$.

Remark 4.0.23 *; (i) Note that absolute equivalence implies near-equivalence(see[26]). (ii)Note also that any two unitary operators are absolutely equivalent.*

The following result shows the link between unitary equivalence and unitarity quasi-equivalence.

Theorem 4.0.24 *If S, T be unitarily equivalent operators in $B(\mathcal{H})$, then they are unitarily quasi-equivalent.*

Remark 4.0.25 *Note that the converse is not true in general unless S, T are similar normal operators. Thus we have the following theorem which that normality is invariant under unitarily quasi-equivalent operators.*

Theorem 4.0.26 *Let S, T be unitarily quasi-equivalent. Then T is normal if and only if S is normal.*

Proof: Suppose $S \overset{u:q:e}{\approx} T$ and suppose S is normal. Then $T^*T = US^*SU^*$ and $TT^* = USS^*U^*$. Thus $T^*T = US^*SU^* = USS^*U^* = TT^*$. Similarly, suppose $T \overset{u:q:e}{\approx} S$ and T is normal. Then $S^*S = UT^*TU^*$ and $SS^* = UTT^*U^*$. Thus $S^*S = UT^*TU^* = UTT^*U^* = SS^*$. Hence the prove.

Remark 4.0.27 *Clearly S, T are unitarily quasi-equivalent if and only if $T^*TTT^* = U(S^*SSS^*)U^*$; that is if $T^*T - TT^*$ is unitarily equivalent to $S^*S - SS^*$.*

From this inequality, we have the following theorem which shows that unitary quasi-equivalence preserves hyponormality.

Theorem 4.0.28 *Suppose S, T are unitarily quasi-equivalent. Then S, T are hyponormal if and only if $T^*T - TT^*$ is unitarily equivalent to $S^*S - SS^*$.*

We now show that unitary quasi-equivalence is an equivalence relation.

Theorem 4.0.29 *Unitary quasi-equivalence is an equivalence relation.*

Proof

(i) Clearly $T \overset{u:q:e}{\approx} T$ since $T^*T = IT^*TI^*$ and $TT^* = ITT^*I^*$, by taking U = I.

(ii) If $S \overset{u:q:e}{\approx} T$, then $S^*S = UT^*TU^*$ and $SS^* = UTT^*U^*$. Pre-multiplying and post-multiplying each of these equations by U^* and U, respectively, we have that $T^*T = US^*SU^*$ and $TT^* = USS^*U^*$. This proves that $T \overset{u:q:e}{\approx} S$.

(iii) We prove that if $T \overset{u:q:e}{\approx} S$ and $S \overset{u:q:e}{\approx} A$ then $T \overset{u:q:e}{\approx} A$. $T^*T = US^*SU^*$ and $TT^* = USS^*U^*$ and $S^*S = WA^*AW^*$ and $SS^* = WAA^*W^*$, where U, W are unitary operators. Then $T^*T = US^*SU^* = UWA^*AW^*U^* = ZA^*AZ^*$, where $Z = UW$ is unitary since product of unitary operators is a unitary operator. Also $TT^* = USS^*U^* = UWAA^*W^*U^* = ZAA^*Z^*$. This proves that $T \overset{u:q:e}{\approx} A$. Thus unitary quasi-equivalence is an equivalence relation on $B(H)$.

Unitary quasi-equivalence was introduced by [22] and were also investigated by [26] under the near equivalence relation.

Remark 4.0.30 *It is stated in [26] that* $T \overset{u:q:e}{\approx} S$ *if and only if* $T^*T \overset{u:e}{\approx} S^*S$ *and* $TT^* \overset{u:e}{\approx} SS^*$. *Thus, we have the following result.*

Corollary 4.0.31 *Let* $T \in B(\mathcal{H})$ *be unitarily quasi-equivalent to* T^*. *Then* T *is normal.*

Theorem 4.0.32 *Let* $T \overset{u:q:e}{\approx} S$. *If* T^*T *is invertible, then* S^*S *is also invertible.*

Proof Invertibility of T implies invertibility of T^*T. The result follows from the unitary equivalence of T^*T and S^*S. There are operators S such that S^*S is invertible but S is not invertible. An example is the unilateral shift operator on $\ell^2(\mathbb{N})$. Invertibility of S^*S implies invertibility of S if and only if S and S^* are injective. That is if and only if $Ker(S) = Ker(S^*) = \{0\}$.

Corollary 4.0.33 *Let* $T \overset{u:q:e}{\approx} S$. *If* T *is invertible, then* S^*S *and* S^*S *are also invertible.*

Remark 4.0.34 *(i) Note that the converse of this corollary need not be true as example above shows.*
(ii) Note also that, if $T \overset{u:q:e}{\approx} S$, *then invertibility of* T *is sufficient for the invertibility of* S.
(iii) Note also that Corollary 4.0.32 says that invertibility is invariant under quasi-unitary equivalence of operators.

Corollary 4.0.35 *If* $T \overset{u:q:e}{\approx} S$ *and* T *is invertible, then* S *is invertible.*

Proposition 4.0.36 *Let* $A \in B(\mathcal{H})$. *Then;*
(i). $Ker(A^*A) = Ker(A)$
(ii). $Ran(AA^*) = Ran(A)$.

Proof

(i). $Ker(A^*A) = \{x \in \mathcal{H} : A^*Ax = 0\} = \{x \in \mathcal{H} : Ax = 0\} = Ker(A)$.

(ii). $Ran(AA^*) = \{y \in \mathcal{H} : y = AA^*x, x \in H\} = \{y \in \mathcal{H} : y = A(A^*x)\} = Ran(A)$.

It was observed in [24] that the class of unitarily quasi-equivalent operators contains the class of metrically equivalent operators. From ([24], Lemma 2.7(v)), it is proved that metric equivalence preserves invertibility.

Theorem 4.0.37 *Suppose T, S are unitarily quasi-equivalent self adjoint operators, then T, S are nearly equivalent.*

Proof Since T, S are unitarily quasi-equivalent,$T^*T = US^*SU^*$ and $TT^* = USS^*U^*$. But, T, S are self adjoint operators, thus we have, $T^*T = TT^* = USS^*U^* = US^*SU^*$. Hence the result.

Remark 4.0.38 *(i) In Theorem 4.0.37, if the operators happen to be projections, then we regain the case of unitary equivalence.*
(ii) Note also that, using the same theorem, if we let the unitary operator$U$ to be the identity operator, the we obtain metric equivalence.

Thus, we have the following inclusion;
{metric equivalence}$\subset$ {near equivalence} $\subset$ {quasi unitary equivalence}
Note that S, T unitarily quasi-equivalent implies that $T^*T - TT^* = U(S^*S - SS^*)U^*$.

Theorem 4.0.39 *T is unitarily quasi-equivalent to a unitary operator V if and only if T is a unitary operator.*

Proof Let$V$ be a unitary operator. $T \overset{u:q:e}{\approx} V$ if and only if $T^*T = U^*V^*VU = V^*V = I$ and $TT^* = UVV^*U^* = I$. This proves the claim.
Note that, the identity operator and the unilateral shift operator in ℓ^2 cannot be almost similar.

Definition 4.0.40 *Two operators are almost unitarily equivalent (a.u.e) if $A^*A = U^*B^*BU$ and $A^* + A = U^*(B^* + B)U$, where U is a unitary operator.*

Note that almost unitary equivalence implies almost similarity of operators.
Note that, if $T \overset{u:q:e}{\approx} S$ implies $W(T^*T) = W(S^*S)$ but $W(T)$ need not coincide with $W(S)$.

Theorem 4.0.41 $[24, Corollary 2.3]$ *An operator* $T \in B(\mathcal{H})$ *is normal if and only if* T *and* T^* *are metrically equivalent.*

Theorem 4.0.42 *([24], Corollary 2.14) If* T *and* S *are metrically equivalent operators on* H*, then* $\|S\| = \|T\|$.

Remark 4.0.43 *Theorem 4.0.42 can also be extended to the case when* S *and* T *are unitarily quasi-equivalent.*

Theorem 4.0.44 *If* T *and* S *are unitarily quasi-equivalent operators on* H*, then* $\|S\| = \|T\|$.

Proof By definition, $\|T\|^2 = \|T^*T\| = \|US^*SU^*\| = \|S^*S\| = \|S\|^2$. Similarly, $\|T^*\|^2 = \|TT^*\| = \|USS^*U^*\| = \|SS^*\| = \|S^*\|^2$.
The following result shows that unitarily quasi- eqivalence relation does not preserve spectrum.

Proposition 4.0.45 *Unitarily quasi-equivalent operators need not have equal spectra.*

Consider, for instance, the operators represented by the matrices

$$S = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

and

$$T = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$$

in $\mathbb{C}^2$. A simple computation shows S and T are unitarily quasi-equivalent with this equivalence implemented by the unitary operator $U = I$, the identity operator on H but $\sigma(S) = \{1, 1\}$ and $\sigma(T) = \{1\}$.
Note also that $W(S) \neq W(T)$.

4.1 Relationship between unitary quasi-equivalence and other equivalence relations

Theorem 4.1.1 *If $S$and T are unitarily equivalent then they are unitarily quasi-equivalent.*

The converse of Theorem 4.1.1 is not generally true. Consider the operators represented by the matrices

$$S = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

and

$$T = \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}$$

. A simple computation shows that S and T are unitarily quasi-equivalent with the equivalence implemented by the unitary operator U = I. However,, $\sigma(S) = \{0, 2\} \neq \{0, 2\} = \sigma(T)$. This shows that S and T are not similar, and hence cannot be unitarily equivalent.

Question 4 *When does does unitary quasi-equivalence imply unitary equivalence?*

Theorem 4.1.2 *If S and T both self-adjoint and unitarily quasi-equivalent operators, then T^2 and S^2 are unitarily equivalent.*

Proof By definition $T^*T = US^*SU^*$ and $TT^* = USS^*U^*$. Using the self-adjointness of S and T we have$T^2 = US^2U^*$, which proves the claim.

Corollary 4.1.3 *If S and T are unitarily quasi-equivalent projections, then T and S are unitarily equivalent.*

Proof The proof follows from Theorem 4.1.2 and the fact that T and S are idempotent.

Theorem 4.1.4 *If S and T are unitarily quasi-equivalent and T is skew-adjoint, then S is normal.*

4.2 Unitary quasi-equivalence and higher classes of operators

Theorem 4.2.1 *If S, T are unitarily quasi-equivalent and T is binormal, then S is binormal.*

Proof A simple computation shows that $US^*S^2S^*U^* = USS^{*2}SU^*$, which can be written nicely as $(S^*S)(SS^*) = (SS^*)(S^*S)$. This proves the claim.

Corollary 4.2.2 *If S, T are unitarily quasi-equivalent and T is quasi-normal, then S is quasi-normal.*

Proof The proof follows immediately from the fact every quasi-normal operator is binormal.

Theorem 4.2.3 *If S, T are unitarily quasi-equivalent and T is hyponormal, then S is hyponormal.*

This result says that unitary quasi-equivalence preserves hyponormality of operators in Hilbert spaces.

4.3 Unitary quasi-equivalence and some useful subspaces of a Hilbert space

Theorem 4.3.1 *If S, T are unitarily quasi-equivalent then $Ker(T) = Ker(S) = 0$ and $Ran(T) = \mathcal{H}$.*

Proof Using Proposition 4.0.33, we have that $Ker(T^*T) = Ker(T) = Ker(US^*SU^*) = Ker(U^*) = 0$, $Ker(T^*TU) = Ker(U) = Ker(US^*S) = Ker(S) = \{0\}$ and $Ran(TT^*) = Ran(T) = Ran(USS^*U) = Ran(U) = \mathcal{H}$.

Theorem 4.3.2 *If S, T are unitarily quasi-equivalent then $\sigma(|T|^2) = \sigma(|S|^2)$ and $\sigma(|T^*|^2) = \sigma(|S^*|^2)$ and all these spectra are real.*

Proof This follows easily from the definition and invariance of spectrum under unitary equivalence.

Remark 4.3.3 *Note that $\sigma(|T|^2)$ real does not imply that $\sigma(T)$ is real.*

Consider

$$T = \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}$$

. Clearly $\sigma(|T|^2) = \{1\} \subset \mathbb{R}$ but $\sigma(T) = \{i\} \notin \mathbb{R}$.

Recall that, for a contraction T, we have that$T \in C_{00}$ if and only if $lim\{T^{*n}T^n\} = lim\{T^nT^{*n}\} = 0$ as $n \to \infty$.

Theorem 4.3.4 *If S, T are unitarily quasi-equivalent contractions and $T \in C_{00}$, then $S \in C_{00}$.*

Remark 4.3.5 *Note that unitary quasi-equivalence does not preserve the trace of an operator matrix.*

Consider

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

and

$$A = \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}$$

. A simple computation shows that

$$T^*T = S^*S = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$$

, meaning that S and T are metrically equivalent and hence, unitarily quasi-equivalent. However, $2 = tr(S) \neq tr(T) = 2$.

Recall that, an operator T is said to be skew-normal if $T^2 = T^{*2}$. Skew-normal operators were introduced by [3] and were also studied by [2]. Note that if T skew-normal, then T^2 is normal. Note also that if T is normal, then T^2 is normal.

Theorem 4.3.6 *Let S, T be unitarily quasi-equivalent. If T is self-adjoint and skew-normal, then S is normal.*

Proof Computation yields $T^2 = US^*SU^* = USS^*U^*$, which proves that $S^*S = SS^*$. This establishes the claim.

Theorem 4.3.7 *Let T be such that T^2 is unitarily quasi-equivalent to T^*. If T^2 is normal, then T is normal.*

Proof We have $T^{2*}T^2 = UTT^*U^*$ and $T^2T^{2*} = UT^*TU^*$. Since T^2 is normal we have that $UTT^*U^* = UT^*TU^*$ and hence $T^*T = TT^*$.
An operator T is said to be a sub-projection if $T^2 = T^*$.

Theorem 4.3.8 *Let S, T be unitarily quasi-equivalent. If T is a sub-projection then S is normal.*

Proof A simple computation gives $S^*S = UT^*TU^* = UT^3U^* = UTT^*U^* = SS^*$. This establishes the claim.

Chapter 5

CONCLUSION AND RECOMMENDATION

5.1 Conclusion

Diagonal operators have simple structures. This property makes them easier to study. Linear operators which are not diagonalizable can at least be expressed as direct sum decomposition of diagonalizable operators. This is not the case for linear operators acting on a Hilbert space since these operators generally are neither diagonalizable nor reducible. However, we have seen that every normal operator is either diagonalizable or similar to a known diagonalizable operator. Also, every reducible operator can be expressed as a direct sum decomposition of normal and completely non-normal operator.

For any operator T, the spectrum of T contains all the eigenvalues of T. Moreover, if T is bounded, then both the continuous and residual spectra of T are empty sets. In this case, the spectrum of T coincides with the point spectrum of T.

In this project, we have seen that unitary equivalence and similarity are equivalence relation. Furthermore, the natural concept of equivalence between Hilbert space operators is unitary equivalence which is stronger that similarity. A result showing that two similar operators have equal spectra has been discussed.

More so, unitary equivalence results for invariant subspaces and normal operators are proved. For similar normal operators, we have discussed the Fuglede-Putnam-Rosenblum theorem that has made the study of similarity in normal operators easier. We see that if T is a normal operator and and S is unitarily equivalent to T, then S is normal.

It has also been noted that direct sums and summands are preserved under unitary equivalence.

By introducing the notion of quasisimilarity of operators, (which is sharpened to similarity in finite dimensional spaces, but in infinite dimensional spaces it is a much weaker relation), we have managed to show than quasisimilarity is an equivalence relation.

Concerning the question of existence of invariant subspaces, we have linked invariant subspaces and hyperinvariant subspaces with quasisimilarity, where it has been observed that similarity preserves non trivial invariant subspaces while quasisimilarity preserves non-trivial hyperinvariant subspaces.

It has further been shown that almost similarity is an equivalence relation.

A result showing that if $A, B \in B(\mathcal{H})$ such that $A \overset{a.s}{\sim} B$, and if A is compact, then so is B has been discussed.

It has also been noted that, if $A \in B(\mathcal{H})$ such that, $A \overset{a.s}{\sim} S_+$, where S_+ denotes the unilateral shift of finite multiplicity, then, A is a completely non-unitary contraction such that $Re(A) \sim Q$, where Q is a quasi-diagonal operator and $Re(A)$ denotes the real part of A. This shows that quasidiagonality is not preserved under similarity .

An operator $A \in B(\mathcal{H})$ is called θ-operator if $A^* + A$ commutes with A^*A. The class of θ-operators in $B(\mathcal{H})$ is denoted by θ, that is, $\theta = \{A \in B(\mathcal{H}) : [A^*A, A^* + A] = 0\}$.

It has also been shown that if $A, B \in B(\mathcal{H})$ such that $B \in \theta$ and $A \overset{a.s}{\sim} B$, then $A \in \theta$.

In chapter three, it has for instance been observed that, if T and S are metrically equivalent operators on $\mathcal{H}$, then $\|S\| = \|T\|$ but the converse is not always true. There exists operators with the same norm which are not metrically equivalent.

For T and S are metrically equivalent then $w(|T|) = w(|S|)$. We note that unlike unitarily equivalent operators, metrically equivalent operators S and T need not have equal numerical range. Note also that the spectrum of S may be equal to the spectrum of T yet S and T are not metrically equivalent. Thus metrically equivalent operators S and T need not have equal spectra. The same applies for metrically equivalent normal operators.

If $S \in B(H)$ is a Fredholm operator and S is metrically equivalent to $T \in B(\mathcal{H})$, then T is Fredholm has been investigated hence the conclusion that metric equivalence of operators preserves Fredholmness has been reached.

In addition, we have also succeeded in showing that, if $T \sim_m S$ and T is A-self-adjoint operator, then $T^2 \sim S^2$ if S is self adjoint.

We have noted from Proposition 3.1.19 that if T and S are metrically equivalent and T is A-self-adjoint, then S is A-self-adjoint if both T and S are projections.

In chapter five, we have come up with a new class of equivalence relation known as unitarily quasi-equivalent relation. We have gone further and showed that unitarily quasi-equivalent relation is in fact an equivalence relation.

A result showing that unitarily equivalent operators are unitarily quasi-equivalent has been proved. Unfortunately, it has been asserted that the converse is not true in general, unless in cases when S, T are similar normal operators.

Furthermore, it has been shown that if T, S are unitarily quasi equivalent self adjoint operators, where the unitary operator U is symmetric, then T, S are nearly equivalent. Thus, the following series of inclusions has been confirmed to be true; $\{\text{metric equivalence}\} \subset \{\text{near equivalence}\} \subset \{\text{quasi unitary equivalence}\}$

Concerning the trace of two operators which are unitarily quasi-equivalent, we noticed that unitary quasi-equivalence does not preserve the trace of an operator matrix and an example has been given to support this statement.

5.2 Recommendation

In conclusion, we have seen that, every pair of unitarily equivalent operators are similar and any pair of similar operators are quasisimilar. That is, unitarily equivalence implies similarity which implies quasisimilarity.

However, we have failed to failed to conclude whether metric equivalence implies almost similarity or the other way round. Therefore,coming up with sufficient conditions under which metric equivalence implies almost similarity or under which almost similarity implies metric equivalence can be recommended as a another field for future research.

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