PUBLIC SERVICE FACILITIES:
OPTIMAL PRODUCTION AND FACTOR UTILISATION

by

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ABSTRACT

Optimality conditions are derived for organizations operating under socialistic principles. The analysis includes the possibilities of advertising, joint factors of production, and demand interdependence for the spectrum of facilities provided in the public service. Operational aspects of a public versus a profit maximizing enterprise are contrasted and the difficulties in attaining efficiency of operation examined.

1. Introduction

Economic systems in western economies typically have two distinct sectors, a public or government sector and a private, capitalist sector. While the latter has received significant attention in both theoretical and empirical research this is not true of the former sector. A theoretical structure encompassing the goals and objectives of public service facilities reflecting supply and demand conditions for these services is lacking for the socialistic sector.

This analysis formalizes such a theoretical structure Overall budgetary allocation for various categories of public services is normally a political process and this is taken into account.

Section 2 considers the economic problem facing public service facilities from an analytic viewpoint while Section 3 discusses the resultant optimality conditions derived in Section 2.

2. Theoretical Analysis of Public Service Facilities

2.1) Introduction

In modern societies many services are allocated through some centralized governmental authority which may be either local or national in scope. Examples include public health facilities, public education, police protection, road works, park facilities and postal services. A purview of such economic goods which may fall within the range of possible health services alone are quite diverse; encompassing the various family planning services, public health education, maternal health care services, immunization programs, emergency treatments, geriatric care, etc. Since monetary resources for the total funding of such projects must necessarily be limited an obvious question arises as to how best to allocate funds to the various competing programs as well as what factor content should be utilized.

Complicating features of a multiple service program are the existence of joint factors of production and the possibility of advertising for the various services offerred.

Most public service facilities have the characteristic that there is an overall administrative organization coordinating the activities of several or many services. This is true for as small a microeconomic unit as a hospital, where administrative functions such as admittance and discharging, purchasing billing, filing, etc. are normally centrally located and commonly utlized by the personnel of all of the individual service facilities. Also it is true of such a macroeconomic unit as a nationally coordinated family planning program. Overall administrative functions coordinate the various factor flows and services directly reaching the recipients. The existence of joint factors of production arises in that such administrative functions cannot be assigned exclusively to one service or another (e.g., as in assigning costs of administration proportional to specific service expenditures or output value as the organisational format is not of such a one to one nature.

The advantage of advertising one or more of the services offered is that demand can possibly be influenced so that overall effectiveness of the program can be increased. The question then arises as to what is the optimal amount of advertising.

2.2) A Two-service Model

We consider first the case when there are only two types of services offered.[1] Denote these services by X_1 and X_2 respectively. It is presumed that the administrating authority has some overall functional evaluation of such services, denoted by $h(X_1,X_2)$ where $h(X_1,X_2)$ is strictly convex in X_1 and X_2 with $h_1,h_2>0$. That is, marginal utility is positive but incrementally decreasing at all levels of output for both services.[2]

Supply of each service requires productive inputs. Specifically there exist administrative factors consisting of capital and labor, denoted by B_K and B_L, which are joint to both services,

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1. In relationship to the Kenyan context the author is currently evaluating the Family Planning and Maternal Health Care Programmes for the Ministry of Health, Research and Evaluation Unit. This particular model being part of the theoretical foundation of this research.

2. Differentiability of all functions utilized in this analysis is assumed throughout.

factors consisting of capital and labor, denoted by K_i, L_i for $i = 1,2$, and advertising costs specific to each service, A_i for $i=1,2$. Therefore it is assumed that the relationship between productive outputs (public services) and factor inputs are expressed by production functions, denoted by

2.2.1) $$X_i = f_i(K_i, L_i, A_i, B_k, B_L) \text{ for } i=1,2.$$

The usual assumption of strict convexity with respect to the arguments of these production functions is made here with the exception of advertising. Advertising cannot be expected to increase output in and of itself so

$$\frac{\partial f_i}{\partial A_i} = 0 \text{ for } i=1,2.$$

If we consider the service outlets as representing a market it is reasonable to presume that a supply and demand analysis is applicable. As such equations (2.2.1) represent the supply functions. If for each level of advertising expenditures people potentially serviced by the facility are rank ordered according to their willingness to pay for the alternative services demand curves can be constructed as

2.2.2) $X_i = d_i\,(P_i, A_i)$ for $i=1,2$; here P_i stands for price of service i. It is assumed that advertising has a positive impact upon demand, $\frac{\partial d_i}{\partial A_i} > 0$ while by construction the demand curve; will necessarily be downward sloping, $\frac{\partial d_i}{\partial P_i} < 0$. It is additionally assumed that d_i for $i=1,2$ are both strictly convex.[3] Construction of demand curves in this manner does not preclude the possibility of a negative equilibrium price. The usual definitional switchover of a used turning into a supplier of a good utilized in most economic analysis is not applicable in a service industry, particularly those in the health services where typically a doctor treats a patient. Thus a negative price, i.e. a subsidy, is plausible.

The administrative authority must take into account the total funds at his disposal and attempt to allocate them optimally. Total funds available include funds gathered externally, denoted by C , and revenues gained through the program, $P_1X_1 + P_2X_2$. As capital and labor at service outlets will typically be similar we denote their costs per unit by r and w respectively. Administrative capital and labor will typically have different costs as these factors will represent different commodities and personnel, e.g., bureaucrats instead of doctors. Denote these costs per unit by s and γ respectively. The budgetary restriction can then be expressed as:

$$2.2.3)\quad C + P_1X_1 + P_2X_2 \geq r(k_1 + k_2) + sB_k + w(L_1 + L_2) + \gamma B_L + A_1 + A_2 .$$

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3. Strict convexity with respect to price is an empirical question. If strict convexity is not maintained the equilibrium conditions yet to be derived will still be maintained at the optimum, however there will exist a set of suboptimum to choose from. Corner-type solutions are plausible for this particular problem only if provision of one service is extremely inefficient and/or evaluation of this service is very low. Inter-connection of demand functions via prices is temporarily eschewed until section 2-3.

The optimization problem facing the central authority is thus to

I) Maximize $h(X_1,X_2)$

subject to

$$X_1 = f_1(k_1,L_1,A_1,B_k,B_L)$$

$$X_2 = f_2(k_2,L_2,A_2,B_k,B_L)$$

$$X_1 = d_1(P_1,A_1)$$

$$X_2 = d_2(P_2,A_2)$$

$$C + P_1X_1 + P_2X_2 \geq r(k_1 + K_2) + sB_k + w(L_1+L)\ \gamma B_L + A_1 + A_2.$$

$$X_1,X_2,k_1,k_2,L_2,B_k,B_L,A_1,A_2 \geq 0.$$

Problem (I) is equivalent to (assuming that a solution utilizing nonnegative factor quantities will be found):

(II) Maximize $h(X_1,X_2)$

subject to

$$f_1(k_1,L_1,A_1,B_k,B_1) - X_1 \geq 0$$

$$F_2(k_2,L_2,A_2,B_k,B_L) - X_2 \geq 0$$

$$d_1(P_1,A_1) - X_1 \geq 0$$

$$d_2(P_2,A_2) - X_2 \geq 0$$

$$C + P_1X_1 + P_2X_2 - r(k_1 + k_2) - sB_k - w(L_1 + L_2) - \gamma B_L - A_1 - A_2 \geq 0.$$

From Problem (II) we can immediately form the Lagrangian, $\mathcal{L}$, the maximization of which yields our solution.

2.2.4) $\pounds(X_1,X_2,K_1,K_2,L_1,L_2,B_K,B_L,A_1,A_2,P_1,P_2,\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5)$

$$= h(X_1,X_2) + \lambda_1[f_1(K_1,L_1,A_1,B_K,B_L) - X_1]$$
$$+ \lambda_2[f_2(K_2,L_2,A_2,B_K,B_L) - X_2]$$
$$+ \lambda_3[d_1(P_1,A_1) - X_1]$$
$$+ \lambda_4[d_2(P_2,A_2) - X_2]$$
$$+ \lambda_5[C + P_1X_1 + P_2X_2 - r(K_1 + K_2) - sB_K - w(L_1 + L_2)$$
$$+\gamma(B_L) - A_1 - A_2].$$

Given an optimal distribution of resoruces, as will be subsequently determined from this Lagrangian, the following interpretations are applicable to the respective Lagrangian multipliers.

λ_1,λ_2 : The marginal value of an increase in output of the respective services or the marginal value of unitary increase in technology of the respective services.

λ_3,λ_4 : Private utility of the marginal individual using the alternative health services.

λ_5 : Marginal social value of an increase in subsidy funds. An overall optimal distribution within the context of government programs will equate this value with other projects funded by the government.

The necessary first order conditions are therefore (assuming an interior solution):

2.2.5) $\frac{\partial \pounds}{\partial X_1} = \frac{2h}{2X_1} - \lambda_1 - \lambda_3 + P_1\lambda_5 = 0.$

2.2.6) $\frac{\partial \pounds}{\partial X_2} = \frac{2h}{2X_2} - \lambda_2 - \lambda_4 + P_1\lambda_5 = 0 .$

2.2.7) $\frac{\partial \pounds}{\partial K_1} = \lambda_1 \frac{2f_1}{2K_1} - \lambda_5 r = 0$

2.2.8) $\frac{\partial £}{\partial K_2} = \lambda_2 \frac{\partial f_2}{2K_2} - \lambda_5 r = 0.$

2.2.9) $\frac{\partial £}{\partial L_1} = \lambda_1 \frac{\partial f_1}{\partial L_1} - \lambda_5 w = 0 .$

2.2.10) $\frac{\partial £}{\partial L_2} = \lambda_2 \frac{\partial f_2}{\partial L_2} - \lambda_5 w = 0 .$

2.2.11) $\frac{\partial £}{\partial B_K} = \lambda_1 \frac{\partial f_2}{\partial B_K} + \lambda_2 \frac{\partial f_2}{\partial B_K} - \lambda_5 S = 0 .$

2.2.12) $\frac{\partial £}{\partial B_L} = \lambda_1 \frac{\partial f_1}{\partial B_L} + \lambda_2 \frac{\partial f_2}{\partial B_K} - \lambda_5 r = 0 .$

2.2.13) $\frac{\partial f}{\partial A_1} = \lambda_2 \frac{\partial f_1}{\partial A_1} + \lambda_3 \frac{\partial d_1}{\partial A_1} - \lambda_5 = 0 .$

2.2.14) $\frac{\partial £}{\partial A_2} = \lambda_2 \frac{\partial f_2}{\partial A_2} + \lambda_4 \frac{\partial d_2}{\partial A_2} - \lambda_5 = 0 .$

2.2.15) $\frac{\partial £}{\partial P_1} = \lambda_3 \frac{\partial d_1}{\partial P_1} + \lambda_5 X_1 = 0 .$

2.2.16) $\frac{\partial £}{\partial P_2} = \lambda_4 \frac{\partial d_2}{\partial P_2} + \lambda_5 X_2 = 0 .$

2.2.17) $\frac{\partial £}{\partial \lambda_1} = f_1 - X_1 = 0 .$

2.2.18) $\frac{\partial £}{\partial X_2} = f_2 - X_2 = 0 .$

2.2.19) $\frac{\partial £}{\partial \lambda_3} = d_1 - X_1 = 0 .$

2.2.20) $\frac{\partial £}{\partial \lambda_4} = d_2 - X_2 = 0 .$

2.2.21) $\frac{\partial f}{\partial \lambda_5} = C+P_1X_1+P_2X_2-r(K_1+K_2)-SB_K-w(L_1+L_2)$

$-\gamma B_K-A_1-A_2 = 0 .$

Interpretation of equations (2.2.17) through (2.2.21) can be made directly. Equations (2.2.17) and (2.2.18) state that it is optimal to utilise your productive processes efficiently and that the supply curve is the production function. Equations (2.2.19) and (2.2.20) state that it is optimal for the administering authority to charge a price in accordance with demand conditions. Specifically a price should always be charged so that there is no excess demand or supply condition present in either of the services being offerred. While seemingly an obvious condition many public service facilities charge negligible if any fees for heavily demanded services resulting in long queues and excess demand. Such conditions represent suboptimal behaviour. Equation (2.2.21) states that all funds and revenues accrued should be spent.

Rearranging equations (2.2.7) (2.2.8), (2.2.9) and (2.2.10) we have

2.2.22) $$\frac{\lambda_1}{\lambda_5} \frac{\partial f_1}{\partial k_1} = r .$$

2.2.23) $$\frac{\lambda_2}{\lambda_5} \frac{\partial f_2}{\partial k_2} = r .$$

2.2.24) $$\frac{\lambda_1}{\lambda_5} \frac{\partial f_1}{\partial L_1} = w$$

2.2.25) $$\frac{\lambda_2}{\lambda 5} \frac{\partial f_2}{\partial k_2} = w .$$

These equations imply that

2.2.26) $$\frac{\frac{\partial f_i}{\partial k_i}}{\frac{\partial f_i}{\partial L_i}} = \frac{r}{w} \quad \text{for } i = 1,2 .$$

The ratio of marginal factor products should be equal to the ratio of factor prices.

Equivalently defining g_i as factor gain

2.2.27) $$g_i = \frac{r}{\frac{\partial f_i}{\partial k_i}} = \frac{w}{\frac{\partial f_i}{\partial L_i}} \quad \text{for } i = 1,2 .$$

Factor gain should be equalized within a productive process.

From equations (2.2.7) and (2.2.8)

2.2.28) $$\lambda_1 = g_1 \lambda_5 ,$$

2.2.29) $$\lambda_2 = g_2 \lambda_5 .$$

and from equations (2.2.15) and (2.2.16)

2.2.30) $$\lambda_3 = \frac{-X_1}{\frac{\partial d_1}{\partial P_1}} \lambda_5 ,$$

2.2.31) $$\lambda_4 = \frac{-X_2}{\frac{\partial d_2}{\partial P_2}} \lambda_5 .$$

Substituting equations (2.2.28), (2.2.29), (2.2.30) and (2.2.31) into equations (2.2.5) and (2.2.6) we obtaining

2.2.32) $$- g_1\lambda_5 - \left(\frac{-X_1}{\frac{\partial d_1}{\partial P_1}} \right)\lambda_5 + P_1\lambda_5 = 0$$

which implies

2.2.33) $$\frac{P_1 - g_1 - \left(\frac{X_1}{\frac{\partial d_1}{\partial P_1}}\right)}{\frac{\partial h}{\partial X_1}} = \frac{1}{\lambda_5} .$$

Similarly we obtain

2.2.34) $$\frac{P_2 - g_2 - \left(\frac{X_2}{\frac{\partial d_1}{\partial P_2}}\right)}{\frac{\partial h}{\partial X_2}} = \frac{1}{\lambda_5} .$$

Combining equations (2.2.33) and (2.2.34) we obtain

2.2.35)

2.2.35) $$\frac{P_1 - g_1 - \left(\frac{X_1}{\frac{\partial d_1}{\partial P_1}}\right)}{\frac{\partial h}{\partial X_1}} = \frac{P_2 - g_2 - \left(\frac{X_2}{\frac{\partial d_2}{\partial P_2}}\right)}{\frac{\partial h}{\partial X_2}} .$$

Optimal production requires that the price minus factor gain minus the elasticity of demand all divided by marginal utility be equalized for the two services.

Substituting equations (2.2.28), (2.2.29), (2.2.30) and (2.2.31) into equations (2.2.13) and (2.2.14):

2.2.36) $$g_1 \frac{\partial f_1}{\partial A_1} + \left(\frac{-X_1}{\frac{\partial d_1}{\partial P_1}}\right) \lambda_5 \frac{\partial d_1}{\partial A_1} - \lambda_5 = 0$$

and

2.2.37) $$g_2\lambda_5 \frac{\partial f_2}{\partial A_2} + \frac{-X_2}{\frac{\partial d_2}{\partial P_2}} \lambda_5 \frac{\partial d_2}{\partial A_2} - \lambda_5 = 0 .$$

Noting $\frac{\partial f_i}{\partial A_i} = 0$ for i = 1,2 and eliminating λ_5 :

2.2.38) $$\frac{\frac{\partial d_i}{\partial P_i}}{X_i} = \frac{1}{\frac{\partial d_i}{\partial A_i}} \quad \text{for } i = 1,2.$$

Thus optimal resources allocation implies that the elasticity of demand for either service in equilibrium equals the inverse of the marginal increase in demand arising from advertising expenditure.

Substituting equations (2.2.28) and (2.2.29) into equations (2.2.11) and (2.2.12) while eliminating λ_5:

2.2.39) $$g_1 \frac{\partial f_1}{\partial B_L} + g_2 \frac{\partial f_2}{\partial B_k} - S = 0 \quad \text{and}$$

2.2.40) $$g_1 \frac{\partial f_1}{\partial B_L} + g_2 \frac{\partial f_2}{\partial B_L} - \gamma = 0 .$$

Rewriting:

2.2.41) $$g_1 \frac{\partial f_1}{\partial B_k} + g_2 \frac{\partial f_2}{\partial B_k} = s ,$$

and

2.2.42) $$g_1 \frac{\partial f_1}{\partial B_L} + g_1 \frac{\partial f_2}{} = Y .$$

Optimal resource allocation requires that the sum of joint factor processes times factor gain be set equal to joint factor cost.

This exhausts the comparative statics (excluding the λ_i) of this model.

2.3) A Two-Service Model with Interdependent Demands

The problem considered in this section is identical to that considered in the previous section, except now we allow interdependencies of demand. Typically an administrator in a public service program will be allocating a spectrum of services which are highly interdependent.

In a family planning program an increase in price charged for prenatal care would presumably increase demand for contraceptives. In a road maintenance program an increase in tolls on one trunk route into a metropolitan area would presumably increase demand on other like trunk routes.

Utilizing the same definitions and assumptions as in the previous section the demand functions formerly represented by equations (2.2.2.) are changed to

2.3.1) $X_i = d_i(P_i, P_j, A_i)$ for $i,j = 1,2,\ i \neq j$.

Here $\frac{\partial d_i}{\partial P_i} < 0$, $\frac{\partial d_i}{\partial A_i} > 0$.

In general it does not seem reasonable to place any conditions on the signs of $\frac{\partial d_i}{\partial P_j}$ as the services offered may be either complements or substitutes in consumption. However in an individual application it might be reasonable to infer some prior sign to these derivatives.

The optimization problem facing the administrating authority is thus to:

(III) Maximize $h(X_1,X_2)$

subject to

$$X_1 = f_1(K_1,L_1,A_1,B_K,B_L)$$

$$X_2 = f_2(K_2,L_2,A_2,B_K,B_L)$$

$$X_1 = d_1(P_1,P_2,A_1)$$

$$X_2 = d_2(P_1,P_2,A_2)$$

$$C + P_1X_1 + P_2X_2 \geq r(K_1+K_2) + sB_K + w(L_1+L_2) + \gamma B_L + A_1 + A_2$$

$$X_1,X_2,K_1,K_2,L_1,L_2,B_K,B_L,A_1,A_2 \geq 0$$

Assuming an interior solution problem (III) is equivalent to

(IV) Maximize $h(X_1,X_2)$

subject to

$$f_1(K_1,L_1,A_1,B_K,B_L) - X_1 \geq 0$$

$$f_2(K_2,L_2,A_2,B_K,B_L) - X_2 \geq 0$$

$$d_1(P_1,P_2,A_1) - X_1 \geq 0$$

$$d_2(P_1,P_2,A_2) - X_2 \geq 0$$

$$C + P_1X_1 + P_2X_2 - r(K_1+K_2) - sB_K - w(L_1 + L_2) - \gamma B_L - A_1 - A_2 \geq 0.$$

The Lagrangian is:

2.3.2) $$\mathcal{L} = h(X_1,X_2) + \lambda_1[f_1(K_1,L_1,A_1,B_K,B_L) - X_1] + \lambda_2[f_2(K_2,L_2,A_2,B_K,B_L) - X_2] + \lambda_3[d_1(P_1,P_2,A_1) - X_1] + \lambda_4[d_2(P_1,P_2,A_2) - X_2] + \lambda_5[C + P_1X_1 + P_2X_2 - r(K_1+K_2) - sB_K - W(L_1+L_2) - vB_L - A_1 - A_2].$$

The interpretations of the lagrangian multipliers are identical to those given in section 2.2.
The necessary first order conditions are therefore:

2.3.3) $$\frac{\partial \mathcal{L}}{\partial X_1} = \frac{\partial h}{\partial X_1} - \lambda_1 - \lambda_3 + P_1\lambda_5 = 0.$$

2.3.4) $$\frac{\partial \mathcal{L}}{\partial X_2} = \frac{\partial h}{\partial X_2} - \lambda_2 - \lambda_4 + P_2\lambda_5 = 0$$

2.3.5) $$\frac{\partial \mathcal{L}}{\partial K_1} = \lambda_1\frac{\partial f_1}{\partial K_1} - \lambda_5 r = 0.$$

2.3.6) $$\frac{\partial \mathcal{L}}{\partial K_2} = \lambda_2\frac{\partial f_2}{\partial K_2} - \lambda_5 r = 0.$$

2.3.7) $$\frac{\partial \mathcal{L}}{\partial L_1} = \lambda_1\frac{\partial f_1}{\partial L_1} - \lambda_5 w = 0.$$

2.3.8) $$\frac{\partial \mathcal{L}}{\partial L_2} = \lambda_2\frac{\partial f_2}{\partial L_2} - \lambda_5 w = 0.$$

2.3.9) $$\frac{\partial \mathcal{L}}{\partial B_K} = \lambda_1\frac{\partial f_1}{\partial B_K} + \lambda_2\frac{\partial f_2}{\partial B_K} - \lambda_5 s = 0.$$

2.3.10) $$\frac{\partial \mathcal{L}}{\partial B_L} = \lambda_1\frac{\partial f_1}{\partial B_L} + \lambda_2\frac{\partial f_2}{\partial B_L}$$

2.3.11) $$\frac{\partial \mathcal{L}}{\partial A_1} = \lambda_1\frac{\partial f_1}{\partial A_1} + \lambda_3\frac{\partial d_1}{\partial A_1} - \lambda_5 = 0.$$

2.3.12) $$\frac{\partial \mathcal{L}}{\partial A_2} = \lambda_2\frac{\partial f_2}{\partial A_2} + \lambda_4\frac{\partial d_2}{\partial A_2} - \lambda_5 = 0.$$

2.3.13) $\frac{\partial £}{\partial P_1} = \lambda_3 \frac{\partial d_1}{\partial P_1} + \lambda_4 \frac{\partial d_2}{\partial P_1} + \lambda_5 X_1 = 0.$

2.3.14) $\frac{\partial £}{\partial P_2} = \lambda_3 \frac{\partial d_1}{\partial P_2} + \lambda_4 \frac{\partial d_2}{\partial P_2} + \lambda_5 X_2 = 0.$

2.3.15) $\frac{\partial £}{\partial \lambda_1} = f_1 - X_1 = 0.$

2.3.16) $\frac{\partial £}{\lambda_2} = f_2 - X_2 = 0.$

2.3.17) $\frac{\partial £}{\partial \lambda_3} = d_1 - X_1 = 0.$

2.3.18) $\frac{\partial £}{\partial \lambda_4} = d_2 - X_2 = 0.$

2.3.19) $\frac{\partial £}{\partial \lambda_5} = C+P_1X_1+ P_2X_2 - r(K_1+K_2)-sB_K-w(L_1+L_2)-\nu B_L-A_1-A_2 = 0.$

The interpretation of equations (2.3.15) through (2.3.19) is ~~identical~~ to that given for equations (2.2.17) through (2.2.21).

Equations (2.3.5) through (2.3.8) reduce to the same equilibrium condition as equation (2.2.26). Defining g_i as in equation (2.2.27) the effect of price interdependency is felt in each of the remaining optimality conditions.

Equating and then rearranging equations (2.3.13) and (2.3.14) we find:

2.3.20) $$\lambda_3 = \frac{\left(\frac{1}{X_1}\frac{\partial d_2}{\partial P_1} - \frac{1}{X_2}\frac{\partial d_2}{\partial P_2}\right)}{\left(\frac{1}{X_1}\frac{\partial d_1}{\partial P_1} - \frac{1}{X_2}\frac{\partial d_1}{\partial P_2}\right)}\lambda_4$$

$= d\,\lambda_4$; defining d ,

or

2.3.21) $$\lambda_4 = \frac{\lambda 3}{d}$$

Substituting back into equation (2.3.13):

2.3.22)

$$\lambda 3 = \frac{-X_1}{\frac{\partial d_1}{\partial P_1} + \frac{1}{d}\frac{\partial d_2}{\partial P_1}} \lambda 5 ,$$

2.3.23)

$$\lambda 4 = \frac{-X_2}{\frac{\partial d_2}{\partial P_2} + \frac{1}{d}\frac{\partial d_1}{\partial P_2}} \lambda 5 .$$

Substituting equations (2.2.28), (2.2.29), (2.3.22), and (2.3.23) into equations (2.3.3) and (2.3.4), simplifying and then equating we obtain as an equilibrium condition

2.3.24)

$$\frac{P_1 - g_1 - \frac{X_1}{\frac{\partial d_1}{\partial P_1} + \frac{1}{d}\frac{\partial d_2}{\partial P_1}}}{\frac{\partial h}{\partial X_1}} = \frac{P_2 - g_2 - \frac{X_2}{\frac{\partial d_2}{\partial P_2} + \frac{1}{d}\frac{\partial d_1}{\partial P_2}}}{\frac{\partial h}{\partial X_2}} .$$

This equation can be compared directly with that correspondingly obtained in the independence of demand case, equation (2.2.35). The difference enters into the third term in the numerator. As the cross partials $\frac{\partial d_2}{\partial P_1}$ and $\frac{\partial d_1}{\partial P_2}$ approach zero the limit is clearly equation (2.2.35).

Noting $\frac{\partial f_i}{\partial A_i} = 0$, substituting equations (2.3.22) and (2.3.23) into equations (2.3.11) and (2.3.12) and rearranging we obtain as optimality conditions

(2.3.25)

$$\frac{\partial d_1}{\partial A_1} = \frac{\frac{\partial d_1}{\partial P_1} + \frac{1}{\partial d} \frac{\partial d_2}{\partial P_1}}{X_1}$$

and

(2.3.26)

$$\frac{\partial d_2}{\partial A_2} = \frac{\frac{\partial d_2}{\partial P_2} + d \frac{\partial d_1}{\partial P_2}}{X_2} .$$

Optimality conditions regarding joint factors of production are identical to those obtained in the independence of demand case; i.e., equations (2.2.41) and (2.2.42).

2.4) A Multiple-Service Model

This section considers a multiple service model, where there can be many unifunctional and joint factors of production as well as the possibility of advertising for each of the services. As long as the demand equations are strictly independent of one another optimality conditions are directly analagous with those derived for the Two-Service Model of section 2.2. When demand interdependencies are allowed in a multiple-service model the complications involve are generalizations to those encountered in the Two-Service Model with Demand Interdependence as in Section 2.3. A generalization to a multiple-service model with demand interdependencies is found in section 2.5.

Let $X_1, \ldots, X_n$ be measures of the service outputs with corresponding convex production functions $f_1, \ldots, f_n$ where the unifunctional factors of production (having positive marginal products) are $y_{11}, y_{12}, \ldots, y_{1p}, y_{21}, \ldots, y_{2p}, \ldots, y_{n1}, \ldots, y_{np}$, joint factors of production are $z_1, \ldots, z_n$ and advertising expenditures $A_1, \ldots, A_n$. Respective prices of outputs are $P_1, \ldots, P_n$, of univariate factors are $r_1, \ldots, r_p$, and of joint factors are $\gamma_1, \ldots, \gamma_c$. Demand for X_i is given by $d_i(P_i, A_i)$ $i = 1, \ldots, n$.

Assuming a convex utility function, $h(X_1, \ldots, X_n)$, the optimization problem facing the central authority is thus to

(V) Maximize $h(X_1,\dots\dots,X_n)$

subject to

$$X_i = f_i(y_{i1},\dots\dots,y_{ip},z_1,\dots\dots,z_q,A_i) \quad \forall\, i=1,\dots\dots,n.$$

$$X_i = d_i(P_i,A_i) \quad \forall\, i = 1,\dots\dots,n.$$

$$C + \sum_{i=1}^{n} P_i X_i \geq \sum_{i=1}^{n} \sum_{j=1}^{p} r_j y_{ij} + \sum_{k=1}^{q} r_k z_k + \sum_{i=1}^{n} A_i .$$

$$X_i, y_{ij}, z_k, A_i \geq 0 \qquad 1 \leq i \leq n \ ; \ 1 \leq j \leq p \ ; \ 1 \leq k \leq q.$$

Assuming an interior solution problem (V) is equivalent to:

(VI) Maximize $h(X_1,\dots\dots,X_n)$

subject to:

$$f_i(y_{i1},\dots\dots,y_{ip},z_1,\dots\dots,z_q,A_i) - X_i = 0 \quad \forall i=1,\dots\dots,n$$

$$d_i(P_i,A_i) - X_i = 0 \quad \forall i=1,\dots\dots,n .$$

$$C + \sum_{i=1}^{n} P_i X_i - \sum_{i=1}^{n} \sum_{j=1}^{p} r_j y_{ij} - \sum_{k=1}^{q} r_k z_k - \sum_{i=1}^{n} A_i \geq 0.$$

The corresponding Lagrangian is:

2.4.1)
$$\mathcal{L} = h(X_1,\dots\dots,X_n) + \sum_{i=1}^{n} \lambda_i \left[f_i(y_{i1},\dots\dots,y_{ip},z_1,\dots\dots,z_q,A_i) - X_i \right]$$
$$+ \sum_{i=1}^{n} \alpha_i \left[d_i(P_i,A_i) - X_i \right]$$
$$+ \beta \Big(C + \sum_{i=1}^{n} P_i X_i - \sum_{i=1}^{n} \sum_{j=1}^{p} r_j y_{ij}$$
$$- \sum_{k=1}^{q} \gamma_k z_k - \sum_{i=1}^{n} A_i \Big)$$

The necessary first order conditions are therefore:

2.4.2) $$\frac{\partial \pounds}{\partial X_i} = \frac{\partial h}{\partial X_i} - \lambda_i - \alpha_i \quad P_i \beta = 0 \quad \forall i = 1, \ldots, n .$$

2.4.3) $$\frac{\partial \pounds}{\partial y_{ij}} = \lambda_i \frac{\partial f_i}{\partial y_{ij}} - \beta r_j = 0 \qquad \forall i=1, \ldots, n \text{ and } j=1, \ldots, p .$$

2.4.4) $$\frac{\partial \pounds}{\partial z_k} = \sum_{i=1}^{n} \lambda_i \frac{\partial f_i}{\partial z_k} - \beta r_k = 0 \quad \forall k=1, \ldots, q .$$

2.4.5) $$\frac{\partial \pounds}{\partial A_i} = \lambda_i \frac{\partial f}{\partial A_i} + \alpha_i \frac{\partial d_i}{\partial A_i} - \beta = 0 \quad \forall i=1, \ldots, n .$$

2.4.6) $$\frac{\partial \pounds}{\partial P_i} = \alpha_i \frac{\partial d_i}{\partial P_i} + \beta X_i = 0 \qquad \forall i=1, \ldots, n .$$

2.4.7) $$\frac{\partial \pounds}{\partial \lambda_i} = f_i - X_i = 0 \qquad i=1, \ldots, n .$$

2.4.8) $$\frac{\partial \pounds}{\partial \alpha_i} = d_i - X_i = 0 \qquad i=1, \ldots, n .$$

2.4.9) $$\frac{\partial \pounds}{\partial \beta} = C + \sum_{i=1}^{n} P_i X_i - \sum_{i=1}^{n} \sum_{j=1}^{p} r_j y_{ij} - \sum_{k=1}^{q} r_k z_k - \sum_{i=1}^{n} A_i = 0 .$$

Interpretation of equations (2.4.7) is that it is optimal to produce efficiently and that the supply curves are the production functions, of equations (2.4.8) is that prices of services should be equal to demand conditions, and of equations (2.4.9) that all resources received should be spent.

Rearranging equations (2.4.3):

2.4.10)

$$\frac{\dfrac{\partial f_i}{\partial y_{ij}}}{\dfrac{\partial f_i}{\partial y_{ik}}} = \frac{r_i}{r_k} . \qquad \forall i=1,\ldots,n ;\ j=1,\ldots,p ;\ k=1,\ldots,c .$$

The ratio of marginal factor products should be equal to the ratio of factor prices.

Factor gain, g_j, is defined as

2.4.11)

$$g = \frac{r_i}{\dfrac{\partial f_j}{\partial X_i}} . \qquad \forall i=1,\ldots,p ;\ \forall j=1,\ldots,n .$$

Factor gain should be equalized within a productive process.

From equations (2.4.3):

2.4.12)

$$\lambda_i = g_i \beta \qquad \forall i=1,\ldots,n .$$

From equations (2.4.6):

2.4.13)

$$\alpha_i = \frac{-\partial X_i}{\dfrac{\partial d_i}{\partial p_i}} \beta \qquad \forall i=1,\ldots,n .$$

Substituting equations (2.4.12) and (2.4.13) and rearranging in a manner similar to the derivation of equation (2.2.35) we find:

2.4.14)

$$\frac{P_i - g_i - \dfrac{X_i}{\dfrac{\partial d_i}{\partial P_i}}}{\dfrac{\partial h}{\partial X_i}} = \frac{P_j - g_j - \dfrac{X_j}{\dfrac{\partial d_j}{\partial P_j}}}{\dfrac{\partial h}{\partial X_j}}$$

$i,j,i=1,\ldots,n \;;\; i \neq j.$

Optimal production requires that price minus factor gain minus elasticity all divided by marginal utility be equated for all services.

Noting $\dfrac{\partial f_i}{\partial A_i} = 0 \quad \forall i=1,\ldots,n$, eliminating β, and rearranging from equations (2.4.5):

2.4.15)

$$\frac{\dfrac{\partial d_i}{\partial P_i}}{X_i} = \frac{1}{\dfrac{\partial d_i}{\partial A_i}} \qquad \forall i=1,\ldots,n .$$

or

$$\frac{\partial d_i}{\partial P_i} \cdot \frac{\partial d_i}{\partial A_i} = X_i \qquad \forall i=1,\ldots,n .$$

The partial derivatives of the demand curve with respect to price and advertising when multiplied should equal output.

Using the concept of factor gains in equations 2.4.4), we find:

2.4.16)

$$\sum_{i=1}^{n} g_i \frac{\partial f_i}{\partial z_i} = \gamma_j .$$

Optimality requires that the sum of factor gains times joint factor marginal physical products should be set equal to joint factor cost.

As can be readily seen these optimality conditions are direct generalizations of those in the two service model considered in Section 2.2.

2.5) A Multiple-Service Model with Demand Interdependence

The model econsidered here is identical to that analyzed in Section 2.4 except interdependence of demand is included. The results derived here are the multivariate extensions of analysis undertaken in Section 2.3. Definitions and assumptions are identical to those undertaken in the previous section excepting demand for X_i is given by $d_i(P_1, \ldots, P_n, A_i)$ for all $i=1,\ldots,n$.

The optimization problem is then:

(VII) Maximize $h(X_1,\ldots,X_n)$

subject to:

$$X_i = f_i(y_{i1},\ldots,y_{ip},z_1,\ldots z_q,A_i) \quad \forall i=1,\ldots,n \; .$$

$$X_i = d_i(P_1,\ldots,P_n,A_i) \quad \forall i=1,\ldots,n \; .$$

$$C + \sum_{i=1}^{n} P_i X_i \geq \sum_{i=1}^{n} \sum_{j=1}^{p} r_j y_{ij} + \sum_{k=1}^{q} \gamma_k z_k + \sum_{i=1}^{n} A_i \; .$$

$$X_i, y_{ij}, z_k, A_i \geq 0 \; \forall 1 \leq i \leq n \; ; \; 1 \leq j \leq p \; ; \; 1 \leq k \leq q \; .$$

Assuming an interior solutions Problem (VI) is equivalent to:

(VIII) Maximize $h(X_1,\ldots,X_n)$

subject to:

$$f_i(y_{i1},\ldots,y_{ip},z_1,\ldots,z_q,A_i) - X_i = 0 \quad \forall i=1,\ldots,n \; .$$

$$d_i(P_1,\ldots,P_n,A_i) - X_i = 0 \quad \forall i=1,\ldots,n \; .$$

$$C + \sum_{i=1}^{n} P_i X_i - \sum_{i=1}^{n} \sum_{j=1}^{p} r_j y_{ij} - \sum_{k=1}^{q} r_k z_k - \sum_{i=1}^{n} A_i = 0.$$

The corresponding Lagrangian is:

2.5.1) $$\pounds = h(X_1,\ldots,X_n) + \sum_{i=1}^{n} \lambda_i \,[\, f_i(y_{i1},\ldots,y_{ip},z_1,\ldots,z_q A_i) - X_i \,]$$

$$+ \sum_{i=1}^{n} \alpha_i \,[\, d_i(_1,\ldots,P_n) - X_i \,]$$

$$+ \beta \,(C + \sum_{i=1}^{n} P_i X_i - \sum_{i=1}^{n} \sum_{j=1}^{P} r_j y_{ij} - \sum_{k=1}^{q} \gamma_k z_k - \sum_{i=1}^{n} A_i) .$$

The necessary first order conditions are therefore:

2.5.2) $$\frac{\partial \pounds}{\partial X_i} = \frac{\partial h}{\partial X_i} - \lambda_i - \alpha_i + P_i \beta = 0$$ for all $i=1,\ldots,n$.

2.5.3) $$\frac{\partial \pounds}{\partial y_{ij}} = \lambda_i \frac{\partial f_i}{\partial y_{ij}} - \beta r_j = 0$$ for all $i=1,\ldots,n$ and $j=1,\ldots,p$.

2.5.4) $$\frac{\partial \pounds}{\partial z_k} = \sum_{i=1}^{n} \lambda_i \frac{\partial f_i}{\partial z_k} - \beta \gamma_k = 0$$ for all $k=1,\ldots,q$.

2.5.5) $$\frac{\partial \pounds}{\partial A_i} = \lambda_i \frac{\partial f}{\partial A_i} + \alpha_i \frac{\partial d_i}{\partial A_i} - \beta = 0$$ for all $i=1,\ldots,n$.

2.5.6) $$\frac{\partial \pounds}{\partial P_i} = \sum_{j=1}^{n} \alpha_j \frac{\partial d_j}{\partial P_i} + \beta X_i = 0$$ for all $i=1,\ldots,n$.

2.5.7) $$\frac{\partial \pounds}{\partial \lambda_i} = f_i - X_i = 0$$ for all $i=1,\ldots,n$.

2.5.8) $$\frac{\partial \pounds}{\partial \alpha_i} = d_i - X_i = 0$$ for all $i=1,\ldots,n$.

2.5.9) $$\frac{\partial \pounds}{\partial \beta} = C + \sum_{i=1}^{n} P_i X_i - \sum_{i=1}^{n} \sum_{j=1}^{P} r_j y_{ij} - \sum_{k=1}^{q} \gamma_k z_k - \sum_{i=1}^{n} A_i .$$

Interpretation of equations (2.5.7), (2.5.8), and (2.5.9) are identical to those of equations (2.4.7), (2.4.8) and (2.4.9).

Rearranging equations (2.5.3) and (2.5.4) yields conclusions identical to equations (2.4.10) and (2.4.16) respectively. Interdependence of demand does not affect the equilibrium conditions of both unifunctional and joint factors of production.

From equations (2.5.6) we can find the α_i as functions of β. Wtiting out equations (2.5.6):

2.5.10)
$$\alpha_1 \frac{2d_1}{2P_1} + \ldots + \alpha_n \frac{2d_1}{2P_1} = \beta X_1$$
$$\vdots$$
$$\alpha_1 \frac{2d_1}{2P_n} + \ldots + \alpha_n \frac{2d_1}{2P_n} = \beta X_n .$$

Dividing the ith equation by X_i, for all $i=1,\ldots,n$ yields:

2.5.11)
$$\alpha_1 \frac{1}{X_1}\frac{2d_1}{2P_1} + \ldots + \alpha_n \frac{1}{X_1}\frac{2d_n}{2P_1} = \beta$$
$$\vdots$$
$$\alpha_1 \frac{1}{X_n}\frac{2d_1}{2P_n} + \ldots + \alpha_n \frac{1}{X_1}\frac{2d_n}{2P_n} = \beta .$$

Equations (2.5.11) can be written in matrix form as:

2.5.12)
$$\begin{bmatrix} \frac{1}{X_1}\frac{2d_1}{2P_1} & \ldots & \frac{1}{X_1}\frac{2d_n}{2P_1} \\ \vdots & & \vdots \\ \frac{1}{X_n}\frac{2d_1}{2P_n} & \ldots & \frac{1}{X_n}\frac{2d_n}{2P_n} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} = \begin{bmatrix} \beta \\ \vdots \\ \beta \end{bmatrix}$$

which is equivalent to

2.5.13) $F\alpha = \beta^*$, this defining F_{nxn} α_{nx1} , and

$^*_{nx1}$

Assuming F has an inverse then

2.5.14) $\alpha = F^{-1} \beta^*$

and

2.5.15) $\alpha_i = \sum_{j=1}^{n} [F_{ij}]^{-1} \beta$

$= \beta \sum_{j=1}^{n} [F_{ij}]^{-1}$

$= \beta F_i^{-1}$

defining F_i^{-1} as the ith row sum.

Substituting equation (2.5.15) and defining g_i as in equations (2.4.12) and also substituting into equations (2.5.2):

2.5.16) $\frac{\partial h}{\partial X_i} - g_i\beta - F_i^{-1}\beta + P_i\beta = 0$ for all i=1,....,n .

Equations (2.5.16) imply that under optimal allocations:

2.5.17) $\frac{P_i - g_i - F_i^{-1}}{\frac{2h}{2X_i}} = \frac{P_j - g_j - F_j^{-1}}{\frac{2h}{2X_j}}$ for all i,j=1, ...,n , $i \neq j$.

Advertising expenditures are also affected by interdependence of demand conditions. Noting $\frac{\partial f}{\partial A_i} = 0$ for all i=1,....,n and substituting into equations (2.5.5) from (2.5.15):

2.5.18)
$$\beta F_i^{-1} \frac{\partial d_i}{\partial A_i} - \beta = 0$$

which implies the optimality condition

2.5.19)
$$\frac{\partial d_i}{\partial A_i} = \frac{1}{F^{-1}} .$$

Similarly optimal price are affected by interdependence of demand. Substituting equations (2.5.15) into equations (2.5.6):

2.5.20)
$$\sum_{j=1}^{n} \beta F_j^{-1} \frac{\partial d_j}{\partial P_i} + \beta X_i = 0 \qquad \text{for all } i=1,\ldots,n .$$

which implies

2.5.21)
$$X_i = - \sum_{j=1}^{n} F_j^{-1} \frac{\partial d_j}{\partial P_i} \qquad \text{for all } i=1,\ldots,n .$$

Equations 2.5.21 exhaust the comparative statics of multiservice allocation under demand interdependence. Interdependence of demand for services does not affect factor demands. However the optimal prices charged, advertising expenditures and quantities offered are most definitely affected as can be seen by equation systems (2.5.17), (2.5.19), and (2.5.21); moreover the overall effect upon any single variable of demand interdependence is not 'obvious', as the matrix inverse F^{-1} needs to be computed, which in itself is dependent upon optimal outputs and partial derivatives of demand with respect to prices.

<u>Conclusions</u>

As can readily be ascertained the only similarity between optimality conditions of a typical government organization and a profit maximizing firm both providing several services or products is that the ratio of marginal factor products should be equal to the ratio of factor prices for unifunctional factors. The remaining optimality conditions facing the administrator of a public service program are entirely unique to his own individual administrative task. These have been derived in sections 2.3, 2.4 and 2.5. Such conditions also are far from intuitively obvious. A question which can be readily asked is whether any such organization will, as a natural consequence of administrative decisions, tend to gravitate towards an optimal productive organization? Whereas in market competition natural market forces, as a consequence of resulting profits and losses, will tend to select and have remaining profit maximizing firms satisfying the necessary optimality conditions for profit maximization such a motivational and selective process seems to be almost entirely lacking in the case of a multiple service providing agency. Beyond exhortations to individual workers to accomplish more in their duties 'obvious' indicators towards improvement, capable of administrative recognition and action, seem to be entirely lacking. Also natural selection is rarely at work, especially for public services (the chance of a government supplied service being eliminated is rather small in most societies), because typically either legislative acts prevent competition (which in the long run could be expected to direct provision of these services to profit maximization) or else market conditions simply preclude competitive firms being profitable.

Given a situation where optimality of operations is unlikely to be met spontaneously through either market forces or administrative decree what can be done to improve the allocative process? It appears necessary to approach optimality econometric investigation must first be made into the relevant supply and demand relations for these services and subsequently optimality conditions regarding resource purchase and use, outputs prices, and distributions of outputs be imposed by administrative decree upon the organization.

The unlikelihood of a public service industry operating efficiently due to ignorance of optimality conditions covering operations can be used to account in part for the apparently lower statistical productivity of employees in the public sector as compared to those in the private sector. Also it can be used to explain a lower rate of technical progress in the public sector as opposed to the private sector; it is difficult to achieve progress when explicit goals in terms of optimality conditions are difficult to ascertain and achieve.

The concept of factor gains, the ratio of specific factor costs to marginal factor productivity, for each of the productive processes plays an extremely important part in the optimality conditions. In a profit maximization context under perfect competition. factor gains should optimally be one for all factors and productive processes. This corresponds to all firms operating at the minimum point on their long-run U-shaped average cost curves where the production function is homogeneous of degree one. From a socialistic context this condition on individual service units is lacking, and instructions to service units to operate in such a manner will result in suboptimal behavior. Optimal socialistic behavior implies that service units will typically operate at capacities where the production function is not homogeneous of degree one. As such the sum of value marginal products will not add up to the value of services; implying that some factors must necessarily be 'exploited". Satisfaction of the optimality conditions stated in this analysis however results in exploitation of factors in favor of the public good.